Efimov K-theory

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Preface

These notes grew out of the research seminar on Efimov K-theory organized by Marc Hoyois at the University of Regensburg in the winter term 2026/27. They are organized by topic rather than by talk and include material from the workshop *Dualizable categories and continuous K-theory* (MPIM Bonn, September 2024) and from lectures by Dustin Clausen (July 2023).

I thank Marc Hoyois for organizing the seminar and all the speakers whose lectures contributed to these notes.

Generative AI has been used to convert lecture notes into LaTeX, reorganize the material, and draft additional exposition and proofs based on the literature. This includes material not covered in the lectures, as well as completions of arguments that were only sketched or omitted there. The source notes below distinguish lecture-based material from additions from the literature. Both kinds of material have received AI assistance. These remain working notes, and some of the added arguments have not yet been checked in detail.

Sources by chapter

Chapters 1–7 and Appendix A contain the material written so far. Chapters 8–11 are planned and are visibly marked as such. The following paragraphs record the sources of the written material. References for individual results and proofs also appear throughout the text.

Chapter 1

This chapter draws on Marc Hoyois's first seminar talk (13 October 2026) and Dustin Clausen's first three lectures (17–19 July 2023).

Chapter 2

Marc Hoyois's first seminar talk (13 October 2026) contributes to Sections 2.3 and 2.6. Akhil Mathew's first Bonn lecture (9 September 2024) is the source for Section 2.5 and parts of Sections 2.2 to 2.4 and 2.6.

Section 2.1 and most of Sections 2.2 to 2.4 were added from the literature, using [NS18; Cal+20; Lur09; Efi24]. The characterization of semiorthogonal decompositions in terms of adjoints (Proposition 2.23) comes from Jacopo Mazzei's seminar talk (17 November 2026), supplemented by [Efi24, Remark 2.75]. The proof of Tamme's excision theorem (Theorem 2.41) was added from [Tam18,

Proposition 10 and Theorem 15], using lax pullbacks. The explanation of Theorem 2.40 was added from the stated non-finitary universal property of K-theory.

Chapter 3

This chapter is based on Marc Hoyois's second seminar talk (20 October 2026). Akhil Mathew's second Bonn lecture (9 September 2024) contributes to Sections 3.2 and 3.4. Dustin Clausen's first two lectures (17–18 July 2023) contribute to Sections 3.3 and 3.4.

The introductory exposition was expanded with AI assistance. The example on continuous posets (Example 3.7) was added from the literature, following Johnstone–Joyal and Gierz et al. [JJ82; Gie+03]. The proof of Proposition 3.22 was added using [Efi24]. The proof of Theorem 3.20 was revised using countable directed subposets, with the indexing lemma in Appendix A.

Chapter 4

This chapter is based on Jiapeng Liu's seminar talk (3 November 2026). The converse implication in Proposition 4.9 and the proofs of Proposition 4.14, corollaries 4.12 and 4.17, and lemma 4.20 were added using [Efi24; KNP24]. A proof of (AB6) for sets and animae (Example 4.19) was added when preparing these notes. The interpolation results (Lemma 4.7 and proposition 4.8) were also added from the literature, following [KNP24]. Section 4.5 is based on Anton Zakrewski's seminar talk (10 November 2026) and his notes for it, following [Efi24, Section 2.8]; the second proof of Proposition 4.25 was added using compact exhaustions. The alternative compact-morphism definitions are collected in Remark 4.10 after the criteria establishing their agreement.

Chapter 5

Max Glaser's seminar talk (27 October 2026), Dustin Clausen's second lecture (18 July 2023), and Akhil Mathew's third Bonn lecture (10 September 2024) contribute to Sections 5.1 to 5.3. The proofs and references were supplemented from [Nee92; BS17; BS22; KNP24]. The proofs of Propositions 5.2 and 5.13 were added when preparing these notes. The latter supplies the Hilbert-space models for the compact-operator and S_{0+} kernels, including their symmetric monoidal structures; the identification of the latter with liquid rigidification is still quoted from Clausen.

The material on semiorthogonal decompositions and gluing in Section 5.4 comes from Jacopo Mazzei's seminar talk (17 November 2026), following [Efi24, Propositions 2.76–2.78]; parts of the proof of Proposition 5.26 were completed when preparing these notes. The Tate-extension material comes from Alexander Efimov's fourth Bonn lecture (10 September 2024); the proofs of Proposition 5.28 and Example 5.30 were completed using [Efi24; Hen17; Cal+20].

Chapter 6

This chapter draws on Max Glaser's seminar talk (27 October 2026) and Dustin Clausen's second and third lectures (18–19 July 2023). Much of the treatment of rigidity and nuclearity was added from [GR17; Ram24b; Efi25; Sch26; And23]. The operator-algebra presentation and the Hochschild-homology obstruction in Example 6.21 were supplied from Efimov's paper. The formula in Example 6.23 remains explicitly attributed to the unpublished lecture notes.

Chapter 7

Sections 7.1 to 7.3 are based on Anton Zakrewski's seminar talk (10 November 2026) and his notes for it. Section 7.4 is based on Jacopo Mazzei's seminar talk (17 November 2026). Both follow [Efi24, Sections 2.9–2.12], which was also used to complete the proofs. Proposition 7.3, Example 7.20, Corollary 7.22, Remark 7.17 and part (2) of Proposition 7.21 were added from [Efi24], and the proof of Proposition 7.9 uses [Ram24a, Proposition A.26]. The categorical facts preceding Proposition 7.14 use [Lur17, Section 1.1.4]; the explanations of coherence, adjunction units and conservativity of the core were added when checking these proofs. Remark 7.6 identifies the continuous construction with the small Calkin construction in the compactly generated case, following [Efi24, Proposition 2.59].

Appendix A

Marc Hoyois's second seminar talk (20 October 2026) contributes to Sections A.3 and A.5. Sections A.1, A.2 and A.4 and most of Sections A.3 and A.5 were added from the literature, using [Lur09; Lur17; KS74; RV22].

1 Introduction and overview

These notes are about a recent extension, due to Efimov [Efi24], of the domain of definition of algebraic K-theory: from small stable categories to a class of *large* stable categories, the *dualizable* ones. The history of algebraic K-theory can be read as a sequence of such extensions,

$$\begin{array}{ccccc} \mathrm{Vect}(X) & \rightsquigarrow & \mathrm{Perf}(X) & \rightsquigarrow & \mathrm{Nuc}(X) \\ \text{exact categories} & & \text{small stable categories} & & \text{dualizable categories,} \end{array}$$

and the guiding principle behind each of them is the same:

the input of K-theory should already have the properties that K-theory is supposed to have.

In this introductory chapter we explain this principle, motivate why one would want to go beyond small stable categories, and give an overview of the main definitions and results that will be developed in detail in the remaining chapters. The statements in this chapter are an overview. We give references to proofs in the written chapters or in the literature; the later chapters explicitly marked as planned do not yet contain proofs.

Convention 1.1. Throughout, *category* means ∞-category, and K denotes *nonconnective* algebraic K-theory, regarded as a functor $K\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Sp}$ on small idempotent complete stable categories. If C is a compactly generated stable category, we often abuse notation and write $K(C) := K(C^{\omega})$.

1.1 Why change the input of K-theory?

To illustrate the guiding principle, let us look at the first expansion: from vector bundles to perfect complexes.

Pushforwards and the Riemann–Roch theorem

A crucial ingredient in Grothendieck's relative version of the Riemann–Roch theorem [BS58] is that, for every proper map $f\colon X \to Y$ of smooth quasi-projective varieties, there is a pushforward map

$$f_*\colon K(\mathrm{Vect}(X)) \to K(\mathrm{Vect}(Y)).$$

At the level of vector bundles, however, it is not at all obvious where such a map should come from: the pushforward $f_*\mathcal{E}$ of a vector bundle is in general neither a vector bundle nor concentrated in a

single degree, and one is forced to use the *derived* pushforward together with resolutions. If instead we allow perfect complexes as input, the map simply comes from the functor

$$f_*\colon \mathrm{Perf}(X) \to \mathrm{Perf}(Y),$$

and in this generality also the annoying assumption of quasi-projectivity disappears. In other words: K-theory admits proper pushforwards because the *categories* $\mathrm{Perf}(-)$ admit proper pushforwards.

Further examples

The same pattern appears everywhere once one works with stable categories:

- *Descent.* The functor $X \mapsto K(\mathrm{Perf}(X))$ satisfies Zariski (and even Nisnevich) descent by the work of Thomason–Trobaugh [TT90], which fails for $X \mapsto K(\mathrm{Vect}(X))$ in general. The categorical reason is that for a quasi-compact open $U \subseteq X$ with closed complement Z, the restriction functor $\mathrm{Perf}(X) \to \mathrm{Perf}(U)$ is a localization (up to idempotent completion) with kernel $\mathrm{Perf}_Z(X)$, the perfect complexes supported on Z. K-theory sends such sequences to fiber sequences, and descent follows formally.
- *Additive inverses.* The cofiber sequence $x \to 0 \to \Sigma x$ gives $[x] + [\Sigma x] = 0$ in $K_0(C)$ after imposing the cofiber relations. At the spectrum level, additivity implies that suspension induces multiplication by -1 on $K(C)$. This does not make x and Σx inverses under direct sum in the category itself.
- *Projective bundle formulas.* For a commutative ring R, there is a semiorthogonal decomposition (in the sense of Section 2.4)

  $$\mathrm{Perf}(\mathbb{P}^1_R) \simeq \langle \mathrm{Perf}(R) \otimes \mathcal{O}(-1),\ \mathrm{Perf}(R) \otimes \mathcal{O} \rangle,$$

  and K-theory turns this into a splitting $K(\mathbb{P}^1_R) \simeq K(R) \oplus K(R)$.

In each of these cases, a property of K-theory that is hard to establish by hand becomes a formal consequence of a structural property of the input category, combined with the single fundamental property of K-theory: it is a *localizing invariant*, i.e. it sends short exact sequences of small stable categories to fiber sequences of spectra (Definition 2.30).

1.2 Where small stable categories fall short

Why should one want to go further, beyond small stable categories? Let us give two reasons, which we will see resolved by Efimov's theory later in this chapter.

K-theory of analytic spaces

Suppose we want to develop a theory of algebraic K-theory for *analytic* spaces, such as formal schemes, rigid analytic varieties, or complex analytic spaces, with good descent properties and a Grothendieck–Riemann–Roch theorem. Then the naive approach of taking $K(\mathrm{Perf}(-))$ does not

work. The basic reason is that in analytic geometry, restriction to an open subset does not give rise to a localization in the categorical sense: for instance, if $U \subseteq X$ is a rational open subset of an affinoid adic space, the restriction functor on perfect complexes is in general not a Verdier quotient. Hence the Thomason–Trobaugh argument sketched above breaks down at its very first step.

The K-theory of the compact operators

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space, let $B = B(\mathcal{H})$ be the C*-algebra of bounded operators, and let $\mathcal{K} = \mathcal{K}(\mathcal{H}) \subseteq B$ be the closed two-sided ideal of compact operators. Since $\mathcal{K}$ is a non-unital ring, its algebraic K-theory is *defined* using excision:

$$K(\mathcal{K}) := \operatorname{fib}\big(K(B) \to K(B/\mathcal{K})\big).$$

A theorem of Suslin–Wodzicki [SW92] shows that this is independent of the choice of B: we have $K(\mathcal{K}) \simeq \operatorname{fib}(K(R) \to K(R/\mathcal{K}))$ for every unital ring R containing $\mathcal{K}$ as a two-sided ideal. They also proved *Karoubi's conjecture*, which in particular implies that

$$K(\mathcal{K}) \simeq \mathrm{KU}$$

is the periodic complex topological K-theory spectrum. This is a beautiful statement, but it raises a question: K-theory is supposed to be an invariant of categories, so *where is the category* whose K-theory is $K(\mathcal{K})$? We will answer this question in Example 1.14 below.

1.3 From small to large categories

The first step is to reformulate K-theory of small categories in terms of large categories. Every small stable category C_0 gives rise to a compactly generated presentable stable category $\operatorname{Ind}(C_0)$, and if C_0 is idempotent complete, we can recover it as $C_0 \simeq \operatorname{Ind}(C_0)^\omega$. (Our nonconnective K-theory is extended to a small stable category by first taking its idempotent completion. In contrast, connective K-theory need not be invariant under idempotent completion: the map on K_0 can fail to be surjective, although the positive K-groups are unchanged.) This gives an equivalence of categories

$$\mathrm{Cat}_{\mathrm{perf}} \xrightarrow{\sim} \mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{st}} = \left\{ \begin{array}{c} \text{compactly generated categories in } \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}, \\ \text{strongly continuous functors} \end{array} \right\}, \qquad C_0 \mapsto \operatorname{Ind}(C_0),$$

where a functor $F\colon C \to D$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is called *strongly continuous* if its right adjoint again preserves colimits. Between compactly generated stable categories, this is equivalent to F preserving compact objects. So K-theory may equally well be regarded as a functor on compactly generated categories and strongly continuous functors.

From this perspective, the obvious next question is: *which presentable stable categories, beyond the compactly generated ones, admit a reasonable K-theory?* The answer is given by the following theorem, which provides many equivalent characterizations of a single class of categories.

Theorem 1.2 (Lurie, [Lur18, Proposition D.7.3.1])**.** *Consider $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. The following are equivalent:*

(1) *There exists a localization $\varphi\colon D \to E$ in $\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{st}}$ such that $C = \ker(\varphi)$;*

(2) *C is dualizable in* $(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}, \otimes)$*;*

(3) *The functor* $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ *has a left adjoint ("compactly assembled");*

(4) *C is a retract in* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ *of a compactly generated category;*

(5) *(Efimov, [Efi24, Theorem 5.2])* $C \otimes (-)\colon \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ *preserves fully faithful functors ("flat");*

(6) *(Clausen–Scholze, see [Efi24, Proposition 2.51]) arbitrary products distribute over filtered colimits ("AB6").*

The equivalence of (1)–(4) will be proved in Theorem 3.16, the axiom (AB6) is discussed in Chapter 4, and flatness in Chapter 10. Characterization (1) is the one most relevant for K-theory: a dualizable category is precisely something that can be described as the "difference" of two compactly generated categories. Characterization (3) is the one that is easiest to work with in practice. In a compactly generated category, the presentation of an object x as the formal colimit of all compact objects mapping to it is the *finest* presentation of x: it maps uniquely into every ind-object $X \in \mathrm{Ind}(C)$ with $\mathrm{colim}\, X \simeq x$, and as a consequence maps out of x into filtered colimits can be computed at the level of ind-objects. Characterization (3) says precisely that every object x of C still admits such a finest presentation $\hat{y}(x) \in \mathrm{Ind}(C)$, whose terms are however no longer required to be compact. This heuristic is explained in more detail at the beginning of Chapter 3.

In particular, compactly generated categories are dualizable, with dual

$$\mathrm{Ind}(C_0)^{\vee} = \mathrm{Ind}(C_0^{\mathrm{op}}).$$

But there are many more examples.

Example 1.3. The following categories are dualizable but in general not compactly generated:

- Keller's example of a localization of a compactly generated category with a non-compactly generated kernel (Example 2.20);
- the category $\mathrm{Shv}(X; \mathrm{Sp})$ of sheaves of spectra on a locally compact Hausdorff space X (see Chapter 9); for instance $\mathrm{Shv}(\mathbb{R}^n; \mathrm{Sp})$ with $n \geq 1$ has no nonzero compact objects at all (for sheaves of abelian groups on a non-compact connected manifold this is due to Neeman [Nee01, Theorem 0.1]; for sheaves of spectra it follows from Efimov's description of the compact objects [Efi24, Proposition 10.17]);
- the categories of nuclear modules arising in analytic geometry: $\mathrm{Nuc}(\mathbb{Z}_p^{\square})$, $\mathrm{Nuc}(\mathbb{R}^{p\text{-liq}})$, $\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})$ [Sch26; CS26].

The nuclear examples are self-dual because they are *rigid*. Sheaf categories on compact Hausdorff spaces are rigid as well. For general locally compact Hausdorff spaces they are locally rigid, rather than rigid; their self-duality is provided by categorical Verdier duality, see [KNP24, Section 4.4] and Example 6.13.

Definition 1.4 (cf. [GR17, Chapter 1, Definition 9.1.2])**.** A *rigid category* is $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ such that for every finite set I, the multiplication functor $\mu\colon C^{\otimes I} \to C$ admits a right adjoint in $\mathrm{Mod}_{C^{\otimes I}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$.

Example 1.5. Let C_0 be a small idempotent complete symmetric monoidal stable category in which every object is dualizable. Then $\mathrm{Ind}(C_0)$ is rigid. For instance, $\mathrm{D}(R) = \mathrm{Ind}(\mathrm{Perf}(R))$ is rigid for every commutative ring R.

Proposition 1.6 ([GR17, Chapter 1, Section 9.2.1])**.** *If C is rigid, then C is self-dual via*

$$C \otimes C \xrightarrow{\mu} C \xrightarrow{\Gamma} \mathrm{Sp},$$

where $\Gamma = \hom_C(\mathbb{1}, -)$.

Rigid categories, trace-class morphisms and nuclear modules are discussed in detail in Sections 6.1 and 6.2.

Remark 1.7. There is a useful dictionary between the compactly generated and the dualizable world, which will be developed in Chapter 4: compact *objects* are replaced by compact *morphisms*. A dualizable category is generated under colimits by sequential colimits $\mathrm{colim}(x_0 \to x_1 \to x_2 \to \cdots)$ along compact morphisms, and in fact its ω_1-compact objects are precisely of this form. In a symmetric monoidal setting, the analogy continues: just as a compactly generated category with compact unit is rigid if its compact objects are precisely its dualizable objects, a dualizable symmetric monoidal category is rigid if its compact morphisms are precisely its trace-class morphisms.

The morphisms between dualizable categories which we will consider are the strongly continuous functors.

Definition 1.8. We denote by

$$\mathrm{Pr}^{\mathrm{dual}} \subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

the subcategory consisting of the dualizable categories and the strongly continuous functors, i.e. those functors whose right adjoint again preserves colimits (see Definition 3.11).

By the above discussion, Ind-completion defines a fully faithful functor

$$\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{perf}} \hookrightarrow \mathrm{Pr}^{\mathrm{dual}}.$$

The category $\mathrm{Pr}^{\mathrm{dual}}$ has excellent formal properties: it is closed under tensor products in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, the inclusion into $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ creates colimits, and $\mathrm{Pr}^{\mathrm{dual}}$ admits all limits and internal homs, although these are more subtle (see Chapter 7).

1.4 Continuous K-theory

We would now like to extend the K-theory functor from $\mathrm{Cat}_{\mathrm{perf}}$ to $\mathrm{Pr}^{\mathrm{dual}}$. Characterization (1) of Theorem 1.2 suggests the following: if C is dualizable, write it as the kernel of a localization $D \to E$ of compactly generated categories along strongly continuous functors, and define

$$K^{\mathrm{cont}}(C) := \mathrm{fib}\left(K(D^\omega) \to K(E^\omega)\right).$$

If C is itself compactly generated, then $C^\omega \to D^\omega \to E^\omega$ is a short exact sequence of small stable categories by the Thomason–Neeman localization theorem (Theorem 2.17), so this gives back

$K(C^\omega)$. But it is not clear that this is well-defined in general: we would like to have a *canonical* way of writing C as the fiber of a map between compactly generated categories.

Such a canonical presentation is provided by characterization (3). Let $C \in \mathrm{Pr}^{\mathrm{dual}}$ and let κ be a regular cardinal which is large enough for the left adjoint $\hat{y}$ of $\mathrm{colim}\colon \mathrm{Ind}(C^\kappa) \to C$ to exist; in fact, $\kappa = \omega_1$ always suffices (Theorem 3.20). We get a localization sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ with strongly continuous functors

$$C \overset{\hat{y}}{\hookrightarrow} \mathrm{Ind}(C^\kappa) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)),$$

where the *(continuous) Calkin category* $\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)$ is defined as the category of compact objects of the quotient $\mathrm{Ind}(C^\kappa)/\hat{y}(C)$.

Definition 1.9. Let F be a localizing invariant. Then we define

$$F^{\mathrm{cont}}(C) := \mathrm{fib}\left(F(C^\kappa) \to F(\mathrm{Calk}^{\mathrm{cont}}_\kappa(C))\right)$$

for $\kappa = \omega_1$. When F is K-theory, $K^{\mathrm{cont}}(C)$ is called the *continuous K-theory* of C.

For many localizing invariants, such as K-theory, the category C^{ω_1} is invisible: it admits countable coproducts, so by an Eilenberg swindle we have $K(C^{\omega_1}) = 0$, and hence

$$K^{\mathrm{cont}}(C) \simeq \Omega K(\mathrm{Calk}^{\mathrm{cont}}_{\omega_1}(C)).$$

At first sight this looks rather crazy: in order to compute $K^{\mathrm{cont}}_0(C)$, we have to compute K_1 of a huge and inexplicit category. Fortunately, this is not how one should think about continuous K-theory in practice. The following theorem gives a more conceptual characterization, which also makes much more flexible computational techniques available.

Theorem 1.10 (Efimov, [Efi24, Theorem 8.10])**.** *Restriction along* Ind *induces an equivalence of categories*

$$\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Pr}^{\mathrm{dual}}, \mathrm{Sp}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}_{\mathrm{perf}}, \mathrm{Sp}),$$

with inverse given by $F \mapsto F^{\mathrm{cont}}$. Moreover, F commutes with filtered colimits if and only if F^{cont} does.

Here, on the left-hand side, a localizing invariant is a functor $\mathrm{Pr}^{\mathrm{dual}} \to \mathrm{Sp}$ that sends short exact sequences of dualizable categories, i.e. localization sequences in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ whose functors are strongly continuous, to fiber sequences. The key input in the proof is that the construction $C \mapsto \mathrm{Calk}^{\mathrm{cont}}_{\omega_1}(C)$ sends short exact sequences of dualizable categories to short exact sequences of small categories. In particular, K^{cont} is the *unique* localizing invariant of dualizable categories extending K-theory, which makes the definition independent of all choices involved. A treatment of this theorem is planned for Chapter 8; for now we use the cited external result.

Example 1.11. The continuous extension of THH admits a very clean description: it is the *categorical trace*

$$\mathrm{THH}\colon \mathrm{Pr}^{\mathrm{dual}} \to \mathrm{Sp}, \qquad C \mapsto \mathrm{tr}(\mathrm{id}_C) \in \mathrm{End}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(\mathrm{Sp}) = \mathrm{Sp},$$

also known as the dimension of C as a dualizable object of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. That this is a localizing invariant is a formal property of traces.

1.5 First computations

Let us now see Efimov's theory in action, and in particular come back to the two problems raised in Section 1.2.

Sheaves on locally compact Hausdorff spaces

The first computation shows that continuous K-theory is a genuinely new invariant: the category $\mathrm{Shv}(\mathbb{R}^n; \mathrm{Sp})$, for $n \geq 1$, has no nonzero compact objects, so the only K-theory one could have assigned to it naively is zero.

Theorem 1.12 (Efimov, [Efi24, Theorem 10.10])**.** *Let C be dualizable and X a locally compact Hausdorff space. Then*

$$K^{\mathrm{cont}}(\mathrm{Shv}(X; C)) \simeq \Gamma_c(X; K^{\mathrm{cont}}(C)).$$

Orientation for the proof. Closed excision for sheaves and continuity under inverse limits of compact Hausdorff spaces are important inputs. Continuity here means that the contravariant invariant sends a cofiltered limit of spaces to a filtered colimit of spectra. Identifying the resulting invariant with compactly supported cohomology requires an additional argument, supplied in [Efi24, Section 10, especially Theorem 10.10]. We do not reproduce that proof here.

Example 1.13. Taking $X = \mathbb{R}^n$, we get $K^{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}^n; C)) \simeq \Omega^n K^{\mathrm{cont}}(C)$. In particular, the higher K-groups of C can be recovered as Grothendieck groups:

$$K_n^{\mathrm{cont}}(C) \cong K_0^{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}^n; C)).$$

Idempotent algebras and the compact operators

A large supply of dualizable categories comes from *idempotent* maps of ring spectra $A \to B$, i.e. maps for which $B \otimes_A B \to B$ is an equivalence. Equivalently, the base change functor $- \otimes_A B \colon \mathrm{D}(A) \to \mathrm{D}(B)$ is a localization (a *homological epimorphism*, see Definition 2.16). Its kernel

$$\mathrm{D}(A \to B) := \ker\big(\mathrm{D}(A) \to \mathrm{D}(B)\big)$$

is then dualizable by Theorem 1.2(1), and

$$K^{\mathrm{cont}}(\mathrm{D}(A \to B)) \simeq \mathrm{fib}\big(K(A) \to K(B)\big).$$

In fact, *every* dualizable category is of this form (Proposition 5.2): the Calkin sequence above exhibits C as the kernel of $\mathrm{Ind}(C^\kappa) \to \mathrm{Ind}(\mathrm{Calk}_\kappa^{\mathrm{cont}}(C))$, and for κ large enough one can replace C^κ by a subcategory with a single generator.

Example 1.14. By a theorem of Wodzicki, if A is a unital C^*-algebra and $I \subseteq A$ a closed two-sided ideal, then $A \to A/I$ is idempotent. Applying this to $A = B(\mathcal{H})$ and $I = \mathcal{K}$, we get a dualizable category $\mathrm{D}(\mathcal{K}) := \mathrm{D}(B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K})$, and

$$K^{\mathrm{cont}}(\mathrm{D}(\mathcal{K})) \simeq \mathrm{fib}\big(K(B(\mathcal{H})) \to K(B(\mathcal{H})/\mathcal{K})\big) = K(\mathcal{K}) \simeq \mathrm{KU}\,.$$

So this is the category we were looking for in Section 1.2! Since $K(B(\mathcal{H})) = 0$ by an Eilenberg swindle, $K^{\mathrm{cont}}(\mathrm{D}(\mathcal{K}))$ is the loop spectrum of the K-theory of the *Calkin algebra* $B(\mathcal{H})/\mathcal{K}$; this is the origin of the name "Calkin category".

According to Clausen [Cla23a], there is an intrinsic model of $\mathrm{D}(\mathcal{K})$ using sequential colimits of separable complex Hilbert spaces along compact operators. The construction and its exact-category conventions are given in Proposition 5.13, which also proves that the completed Hilbert tensor product makes this model a commutative algebra in $\mathrm{Pr}^{\mathrm{dual}}$. The lecture interprets the resulting multiplicative structure on continuous K-theory through this model.

Nuclear modules and K-theory of analytic spaces

For a commutative ring R with a finitely generated ideal I, the category $\mathrm{Nuc}^{CS}(R^\wedge_I)$ of nuclear modules in the sense of Clausen–Scholze provides an input for continuous K-theory of the formal completion. Here $R^\wedge_I$ denotes derived completion. An operator-algebra presentation analogous to the compact-operator example is given in Example 6.21. The category $\mathrm{Nuc}^{CS}(R^\wedge_I)$ can be obtained from the category of solid $R^\wedge_I$-modules by a "rigidification" procedure, see Section 6.2. Efimov also introduced a variant $\mathrm{Nuc}(R^\wedge_I)$, which contains $\mathrm{Nuc}^{CS}(R^\wedge_I)$ as a full subcategory with the same compact objects [Efi25, Corollary 7.6]. The comparison induces an isomorphism on continuous K-theory [Efi25, Corollary 7.11]. We keep the notation Nuc^{CS} for the Clausen–Scholze category and Nuc for Efimov's variant; equality of compact objects alone would not imply this K-theoretic comparison. Efimov computed the continuous K-theory:

Theorem 1.15 (Efimov, [Efi25, Corollary 6.3])**.** *Let R be a commutative ring, and $I = (f_1, \dots, f_m) \subseteq R$. Then*

$$K^{\mathrm{cont}}(\mathrm{Nuc}(R^\wedge_I)) = \lim_{n\to\infty} K(R/\!\!/(f_1^n, \dots, f_m^n)).$$

The same formula holds for $\mathrm{Nuc}^{CS}(R^\wedge_I)$ [Efi25, Corollary 7.10], and if R is Noetherian, the right-hand side is $\lim_n K(R/I^n)$.

The Clausen–Scholze construction also applies to complete analytic Huber pairs. Rational localizations then have the categorical localization properties needed for descent. In the following theorem, Nuc denotes the nuclear category associated with the analytic ring of the Huber pair, as in Andreychev, not an identification with Efimov's variant for formal completions. The descent theorem is an additional result; it does not follow just from the calculation for formal completions.

Theorem 1.16 (Andreychev, [And23, Satz 4.2 and Satz 5.12])**.** *Let (A, A^+) be a complete sheafy analytic Huber pair. The assignments $U \mapsto \mathrm{Nuc}(\mathcal{O}(U))$ and $U \mapsto K^{\mathrm{cont}}(\mathrm{Nuc}(\mathcal{O}(U)))$, for $U \subseteq \mathrm{Spa}(A, A^+)$ rational open, satisfy descent for covers by rational open subsets.*

A related result is the pro-descent theorem of Kerz–Strunk–Tamme [KST18, Theorem A]. One consequence, after taking inverse limits, is the following: if $f\colon X \to Y$ is a proper map of Noetherian schemes which is an isomorphism outside a closed subscheme $Z \subseteq Y$, then the square

$$\begin{array}{ccc} K(Y) & \longrightarrow & \lim_n K(Z_n) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ K(X) & \longrightarrow & \lim_n K((f^{-1}Z)_n) \end{array}$$

is a pullback, where Z_n and $(f^{-1}Z)_n$ denote the n-th infinitesimal thickenings. The pro-descent theorem is stronger than this inverse-limit statement. The nuclear-module viewpoint suggests interpreting the right-hand terms through formal completions and comparing their punctured generic fibers. This is motivation, not a proof here: it requires localization and formal-gluing results beyond the rational descent theorem above.

Outlook: archimedean analytic geometry

The same ideas apply in archimedean functional analysis. There is a canonical rigid category $\mathrm{D}^{\mathrm{an}}(\mathbb{C})$ (obtained for instance by rigidifying the category of liquid $\mathbb{C}$-modules, independently of the choice of p), and Clausen–Scholze have announced results relating the continuous K-theory of complex analytic spaces built from it to topological K-theory and to (Hodge-filtered) de Rham cohomology. We will not discuss this further in these notes.

1.6 Overview of the notes

The written portion of these notes follows this route:

(1) Chapter 2 develops stable localization sequences, ordinary semiorthogonal decompositions, small Calkin categories and localizing invariants. Tamme's excision theorem is proved using lax pullbacks.

(2) Chapter 3 introduces compact assembly and dualizability, proves Lurie's characterization, and establishes ω_1-presentability.

(3) Chapter 4 develops compact morphisms, interpolation, compact exhaustions and the axiom (AB6), and deduces that dualizability is inherited by images of strongly continuous functors.

(4) Chapter 5 treats homological epimorphisms, almost modules and continuous posets, followed by gluing and the more specialized theory of Tate extensions.

(5) Chapter 6 develops rigidity, trace-class morphisms and nuclear categories, including the solid and liquid examples.

(6) Chapter 7 studies colimits and finite limits in $\mathrm{Pr}^{\mathrm{dual}}$ and the continuous Calkin categories.

The remaining chapters are planned: Chapter 8 will construct continuous invariants; Chapter 9 will treat sheaf applications; and Chapters 10 and 11 will develop the further structural results. Statements quoted above from these subjects currently rely on the cited literature.

Appendix A is a reference appendix. For a first reading, the main prerequisites are the universal property of Ind, the adjoint functor theorem, and the tensor product of presentable categories. The countable-indexing and mates arguments can be consulted when they are used.

2 Stable categories and localization sequences

In this chapter we collect the basic theory of localization sequences, both for small stable categories and for presentable stable categories, and introduce localizing invariants. Throughout, we denote by $\mathrm{Cat}_{\mathrm{st}}$ the category of small stable categories and exact functors, and by $\mathrm{Cat}_{\mathrm{perf}} \subseteq \mathrm{Cat}_{\mathrm{st}}$ the full subcategory of idempotent complete ones (this category is denoted $\mathrm{Cat}^{\mathrm{perf}}$ in [BGT13; Efi24]). For a stable category B we write $\hom_B(x, y) \in \mathrm{Sp}$ for the mapping spectra. A full subcategory of a stable category is called *stable* if it contains 0 and is closed under fibers and cofibers, and *thick* if it is moreover closed under retracts.

2.1 Verdier sequences

The basic construction of this chapter is the quotient of a small stable category B by a stable subcategory A, obtained by forcing the objects of A to become zero. In the setting of stable categories this is simply a localization in the sense of Dwyer–Kan, and the following theorem shows that it behaves just like Verdier's classical quotient of triangulated categories: it is again stable, it has the expected universal property, and its mapping spectra are computed by a colimit over A. We then introduce the corresponding notion of exact sequence, the Verdier sequences.

Theorem 2.1 ([NS18, Theorem I.3.3])**.** *Let B be a small stable category and $A \subseteq B$ a full stable subcategory. Let W_A be the class of morphisms of B whose cofiber lies in A, and let $q\colon B \to B/A := B[W_A^{-1}]$ be the localization. Then:*

(1) *The category B/A is stable and q is exact.*

(2) *For every stable category E, restriction along q induces an equivalence between $\mathrm{Fun}^{\mathrm{ex}}(B/A, E)$ and the full subcategory of $\mathrm{Fun}^{\mathrm{ex}}(B, E)$ spanned by the functors that vanish on A.*

(3) *For $x, y \in B$, the category $A_{/y}$ is filtered and*

$$\mathrm{Map}_{B/A}(qx, qy) \simeq \mathrm{colim}_{(a \to y) \in A_{/y}} \mathrm{Map}_B\big(x, \mathrm{cofib}(a \to y)\big).$$

The category B/A is called the Verdier quotient *of B by A.*

Note that q is essentially surjective, being a localization. Moreover, part (2) says that $B \to B/A$ is the cofiber of $A \hookrightarrow B$ in $\mathrm{Cat}_{\mathrm{st}}$. Indeed, for a stable category E, exact functors out of the pushout

$0 \sqcup_A B$ in $\mathrm{Cat}_{\mathrm{st}}$ are given by

$$\mathrm{Fun}^{\mathrm{ex}}(B, E) \times_{\mathrm{Fun}^{\mathrm{ex}}(A,E)} \{0\},$$

the category of exact functors $F\colon B \to E$ together with an equivalence $F|_A \simeq 0$. Since 0 is a zero object of $\mathrm{Fun}^{\mathrm{ex}}(A, E)$, the space of such equivalences is either empty or contractible, so this is the full subcategory of $\mathrm{Fun}^{\mathrm{ex}}(B, E)$ spanned by the functors vanishing on A.

Lemma 2.2. *In the situation of Theorem 2.1, the kernel of $q\colon B \to B/A$ is the thick closure of A in B, i.e. the full subcategory of retracts of objects of A. In particular, $B/A \simeq B/\bar{A}$, where $\bar{A}$ is the thick closure of A.*

Proof. Since q is exact, its kernel is a thick subcategory of B, and it contains A since $0 \to a$ lies in W_A for $a \in A$. Conversely, let $x \in B$ with $q(x) \simeq 0$. Then id_{qx} is nullhomotopic, so by Theorem 2.1(3) (applied with $y = x$) there exists $a \to x$ with $a \in A$ such that the canonical map $x \to c := \mathrm{cofib}(a \to x)$ is nullhomotopic. Since $a \simeq \mathrm{fib}(x \to c)$ and the fiber of a nullhomotopic map $x \to c$ is $x \oplus \Omega c$, it follows that x is a retract of a. This shows that the kernel of q consists of retracts of objects of A, and every retract of an object of A lies in the thick subcategory $\ker(q)$. The last statement follows from the universal property Theorem 2.1(2), since an exact functor vanishes on A if and only if it vanishes on $\bar{A}$. □

Remark 2.3. Let $F\colon B \to C$ be a functor into a pointed category C. Since the constant functor 0 with value a zero object is a terminal object of $\mathrm{Fun}(B, C)$, the space of natural transformations $F \to 0$ is contractible. Hence the space of equivalences $F \simeq 0$ is either empty or contractible, and it is non-empty if and only if F sends every object to a zero object. So for a composable pair $A \xrightarrow{i} B \xrightarrow{p} C$ of functors between pointed categories, a nullhomotopy $pi \simeq 0$ is unique if it exists, and its existence is a *property* of the pair (i, p). We will therefore write "$pi \simeq 0$" as a condition throughout, without specifying a nullhomotopy.

Definition 2.4. A sequence $A \xrightarrow{i} B \xrightarrow{p} C$ in $\mathrm{Cat}_{\mathrm{st}}$ with $pi \simeq 0$ is called a *Verdier sequence* if it is both a fiber sequence and a cofiber sequence in $\mathrm{Cat}_{\mathrm{st}}$. Explicitly, this means that i is fully faithful with essential image $\ker(p)$, and that the induced functor $B/A \to C$ is an equivalence. In this case p is called a *Verdier projection*.

By Lemma 2.2, for every full stable subcategory $A \subseteq B$ the sequence $\bar{A} \hookrightarrow B \to B/A$ is a Verdier sequence. See [Cal+20, Lemma A.1.8 and Proposition A.1.9] for a detailed discussion of Verdier sequences.

2.2 Idempotent completion and short exact sequences

Every small stable category A admits an idempotent completion $A \hookrightarrow A^{\mathrm{idem}}$, which is again stable [Lur17, Corollary 1.1.3.7]; it is fully faithful and every object of A^{idem} is a retract of an object of A, and $A \mapsto A^{\mathrm{idem}}$ is left adjoint to the inclusion $\mathrm{Cat}_{\mathrm{perf}} \subseteq \mathrm{Cat}_{\mathrm{st}}$ (see Section A.3). In particular, cofibers in $\mathrm{Cat}_{\mathrm{perf}}$ are computed as $(B/A)^{\mathrm{idem}}$.

Definition 2.5. A sequence $A \xrightarrow{i} B \xrightarrow{p} C$ in $\mathrm{Cat}_{\mathrm{perf}}$ with $pi \simeq 0$ is called a *short exact sequence* (or *Karoubi sequence*) if it is both a fiber sequence and a cofiber sequence in $\mathrm{Cat}_{\mathrm{perf}}$. Explicitly, this

means that i is fully faithful with essential image $\ker(p)$, and that the induced functor $(B/A)^{\mathrm{idem}} \to C$ is an equivalence. In this case p is called a *Karoubi projection*.

We follow [Efi24] in calling these short exact sequences; the term Karoubi sequence is used in [Cal+20].

Lemma 2.6. *A sequence $A \xrightarrow{i} B \xrightarrow{p} C$ in $\mathrm{Cat}_{\mathrm{perf}}$ with $pi \simeq 0$ is short exact if and only if i is fully faithful with essential image $\ker(p)$, the induced functor $B/A \to C$ is fully faithful, and every object of C is a retract of an object in the image of p.*

Proof. The functor $\bar{p}\colon B/A \to C$ factors as $B/A \hookrightarrow (B/A)^{\mathrm{idem}} \to C$. If $\bar{p}$ is fully faithful, then so is $(B/A)^{\mathrm{idem}} \to C$, since mapping spaces between retracts of objects are retracts of mapping spaces between those objects. If moreover every object of C is a retract of an object $p(b) = \bar{p}(qb)$, then $(B/A)^{\mathrm{idem}} \to C$ is essentially surjective since $(B/A)^{\mathrm{idem}}$ is idempotent complete. Conversely, if $(B/A)^{\mathrm{idem}} \to C$ is an equivalence, then $\bar{p}$ is fully faithful and every object of C is a retract of an object of B/A, and hence of an object $p(b)$, as q is essentially surjective. □

Remark 2.7. Every Verdier sequence between idempotent complete categories is short exact. The converse fails whenever $A \subseteq B$ is a thick subcategory such that the Verdier quotient B/A is not idempotent complete: then $A \to B \to (B/A)^{\mathrm{idem}}$ is short exact but not a Verdier sequence.

Remark 2.8. Let $A \subseteq B$ be a full stable subcategory with quotient functor $q\colon B \to B/A$. Applying Theorem 2.1(3) to all shifts of x, and using that the colimit is filtered and hence commutes with cofibers, we obtain the following formula for the mapping spectra:

$$\begin{aligned}\hom_{B/A}(qx, qy) &\simeq \operatorname{colim}_{(a \to y) \in A_{/y}} \hom_B\big(x, \operatorname{cofib}(a \to y)\big) \\ &\simeq \operatorname{cofib}\Big(\operatorname{colim}_{(a \to y) \in A_{/y}} \hom_B(x, a) \to \hom_B(x, y)\Big).\end{aligned}$$

Since $B/A \to (B/A)^{\mathrm{idem}}$ is fully faithful, the same formula computes mapping spectra in the idempotent complete quotient.

Remark 2.9. In a short exact sequence $A \to B \xrightarrow{p} C$, the functor p need not be essentially surjective: in general, objects of C are only retracts of objects in the image of p. It is possible to say precisely which objects lift. By Thomason's classification of dense subcategories [Tho97], the essential image of the dense subcategory $B/A \subseteq C$ consists precisely of those objects $c \in C$ whose class $[c] \in K_0(C)$ lies in the image of $K_0(B) \to K_0(C)$. For example, $c \oplus \Sigma c$ always lifts to B, since $[c \oplus \Sigma c] = [c] - [c] = 0$.

Example 2.10 (Thomason–Trobaugh, [TT90])**.** Let X be a qcqs scheme and $U \subseteq X$ a quasi-compact open subscheme with closed complement Z. Then

$$\mathrm{Perf}_Z(X) \hookrightarrow \mathrm{Perf}(X) \to \mathrm{Perf}(U)$$

is a short exact sequence, where $\mathrm{Perf}_Z(X)$ denotes the perfect complexes on X whose restriction to U vanishes. In general, the restriction functor $\mathrm{Perf}(X) \to \mathrm{Perf}(U)$ is not essentially surjective; by Remark 2.9, a perfect complex on U extends to X if and only if its class in $K_0(U)$ extends.

2.3 Localization sequences of presentable stable categories

We now turn to large categories. Recall that $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ denotes the category of presentable stable categories and colimit-preserving functors. A colimit-preserving functor $F\colon D \to E$ between presentable categories has a right adjoint F^R (Theorem A.19); recall that F is called *strongly continuous* if F^R again preserves colimits. We denote by $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the subcategory with the same objects and strongly continuous functors.

Definition 2.11. A sequence $K \xrightarrow{i} D \xrightarrow{p} E$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ with $pi \simeq 0$ is called a *localization sequence* if i is fully faithful with essential image $\ker(p)$ and p is a *localization*, i.e. its right adjoint p^R is fully faithful. It is called *strongly continuous* if moreover i and p are strongly continuous.

For a full subcategory $K \subseteq D$ we write $K^\perp = \{x \in D \mid \mathrm{Map}_D(k, x) \simeq * \text{ for all } k \in K\}$ for its right orthogonal.

Proposition 2.12.

(1) *Let D be a presentable stable category and $K \subseteq D$ a full stable subcategory which is closed under colimits and presentable. Then the inclusion $i\colon K \hookrightarrow D$ has a right adjoint i^R, the full subcategory $K^\perp \subseteq D$ is presentable, its inclusion admits a left adjoint $L\colon D \to K^\perp$ given by*

$$L(x) = \mathrm{cofib}(ii^R x \to x),$$

and $K \xrightarrow{i} D \xrightarrow{L} K^\perp$ is a localization sequence. It is both a fiber sequence and a cofiber sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

(2) *Conversely, let $K \xrightarrow{i} D \xrightarrow{p} E$ be a localization sequence. Then p^R induces an equivalence $E \simeq K^\perp$ under which p corresponds to L, and there is a natural cofiber sequence*

$$ii^R \to \mathrm{id}_D \to p^R p.$$

Proof. (1) The inclusion i preserves colimits, so it has a right adjoint by Theorem A.19. Define $L(x) = \mathrm{cofib}(ii^R x \to x)$, using the counit. For $k \in K$, the map $\hom_D(k, ii^R x) \to \hom_D(k, x)$ is an equivalence since i is fully faithful, so $\hom_D(k, L(x)) \simeq 0$; that is, $L(x) \in K^\perp$. For $y \in K^\perp$, applying $\hom_D(-, y)$ to the cofiber sequence $ii^R x \to x \to L(x)$ shows that $\hom_D(L(x), y) \to \hom_D(x, y)$ is an equivalence, since $\hom_D(ii^R x, y) \simeq 0$. Hence $L\colon D \to K^\perp$ is left adjoint to the inclusion $K^\perp \subseteq D$, with unit $x \to L(x)$. The functor L is accessible since i^R is (being a right adjoint between presentable categories), so $K^\perp$ is presentable [Lur09, Section 5.5.4]. Since its right adjoint is fully faithful, L is a localization, and $L(x) \simeq 0$ if and only if $ii^R x \to x$ is an equivalence, if and only if $x \in K$. This shows that $K \to D \to K^\perp$ is a localization sequence. It is a fiber sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ since limits in Pr^{L} are computed in $\widehat{\mathrm{Cat}}$. For the cofiber sequence, let $F\colon D \to E$ be a colimit-preserving functor to a presentable stable category E. Since L is a localization, restriction along L identifies $\mathrm{Fun}^{\mathrm{colim}}(K^\perp, E)$ with the full subcategory of $\mathrm{Fun}^{\mathrm{colim}}(D, E)$ of functors inverting the maps $x \to x'$ with $L(x) \simeq L(x')$. Since L is exact, these are precisely the maps whose fiber lies in $\ker(L) = K$, and an exact functor inverts all such maps if and only if it vanishes on K. This is precisely the universal property of the cofiber of i in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

(2) Since p^R is fully faithful and $\mathrm{Map}_D(k, p^R e) \simeq \mathrm{Map}_E(pk, e) \simeq *$ for $k \in K$, the functor p^R lands in $K^\perp$. Conversely, let $x \in K^\perp$ and let $f \to x \to p^R p(x)$ be the fiber sequence of the unit. As $pp^R \simeq \mathrm{id}$, we have $p(f) \simeq 0$, so $f \in K$. Then $\hom_D(f, x) \simeq 0$ and $\hom_D(f, p^R px) \simeq \hom_E(pf, px) \simeq 0$, so $\hom_D(f, f) \simeq 0$ and hence $f \simeq 0$. So $x \simeq p^R p(x)$, which shows that p^R induces an equivalence $E \simeq K^\perp$. Under this equivalence, p is identified with the left adjoint L of the inclusion $K^\perp \subseteq D$, and the unit $x \to p^R p(x) = L(x)$ is the cofiber of $ii^R x \to x$ by (1). □

Lemma 2.13. *Let $K \xrightarrow{i} D \xrightarrow{p} E$ be a localization sequence. Then i^R preserves colimits if and only if p^R preserves colimits. In particular, the sequence is strongly continuous as soon as i or p is strongly continuous.*

Proof. Since i is fully faithful and preserves colimits, i^R preserves colimits if and only if ii^R does: indeed, for a diagram (x_j) the map $i(\mathrm{colim}_j\, i^R x_j) \simeq \mathrm{colim}_j\, ii^R x_j \to ii^R(\mathrm{colim}_j\, x_j)$ is the image under i of the comparison map for i^R. Similarly, p^R preserves colimits if and only if $p^R p$ does: if $p^R p$ preserves colimits and (e_j) is a diagram in E, then $\mathrm{colim}_j\, e_j \simeq p(\mathrm{colim}_j\, p^R e_j)$, so $p^R(\mathrm{colim}_j\, e_j) \simeq p^R p(\mathrm{colim}_j\, p^R e_j) \simeq \mathrm{colim}_j\, p^R pp^R e_j \simeq \mathrm{colim}_j\, p^R e_j$. Finally, by Proposition 2.12(2) there is a cofiber sequence of exact functors $ii^R \to \mathrm{id} \to p^R p$, so ii^R preserves colimits if and only if $p^R p$ does. □

Recall that a presentable stable category E is *compactly generated* if $E \simeq \mathrm{Ind}(E^\omega)$, equivalently if there is a set G of compact objects of E such that an object $e \in E$ with $\hom_E(g, e) \simeq 0$ for all $g \in G$ is zero.

Lemma 2.14. *Let E be a presentable stable category and $G \subseteq E^\omega$ a set of compact objects such that $\hom_E(g, e) \simeq 0$ for all $g \in G$ implies $e \simeq 0$. Then E^ω is the thick closure of G, and $E \simeq \mathrm{Ind}(E^\omega)$.*

Proof. Let $T \subseteq E^\omega$ be the thick closure of G; it is essentially small. The functor $\mathrm{Ind}(T) \to E$ extending the inclusion is fully faithful since T consists of compact objects, and its essential image is closed under colimits since T is closed under finite colimits [Lur09, Section 5.3.5]. Its right adjoint is conservative: if $\hom_E(t, e) \simeq 0$ for all $t \in T$ then $e \simeq 0$ by assumption. Hence $\mathrm{Ind}(T) \simeq E$. Since T is idempotent complete, the compact objects of $\mathrm{Ind}(T)$ are precisely the objects of T (Theorem A.17(1)), so $E^\omega = T$. □

Lemma 2.15. *Let $K \xrightarrow{i} D \xrightarrow{p} E$ be a strongly continuous localization sequence. Then i and p preserve compact objects, and $K^\omega = K \cap D^\omega$. If D is compactly generated, then so is E, and E^ω is the thick closure of $p(D^\omega)$.*

Proof. A colimit-preserving functor whose right adjoint preserves filtered colimits preserves compact objects. If $k \in K$ is compact in D, then it is compact in K since i is fully faithful and preserves colimits. If D is compactly generated and $e \in E$ satisfies $\hom_E(p(d), e) \simeq \hom_D(d, p^R e) \simeq 0$ for all $d \in D^\omega$, then $p^R e \simeq 0$ and hence $e \simeq pp^R e \simeq 0$. The claim follows from Lemma 2.14. □

Definition 2.16 ([Efi24, Definition 1.9])**.** A functor $f\colon A \to B$ in $\mathrm{Cat}_{\mathrm{perf}}$ is called a *homological epimorphism* if $\mathrm{Ind}(f)\colon \mathrm{Ind}(A) \to \mathrm{Ind}(B)$ is a localization, i.e. its right adjoint is fully faithful.

Theorem 2.17 (Thomason–Neeman)**.**

(1) *If* $A \xrightarrow{i} B \xrightarrow{p} C$ *is a short exact sequence in* $\mathrm{Cat}_{\mathrm{perf}}$*, then* $\mathrm{Ind}(A) \to \mathrm{Ind}(B) \to \mathrm{Ind}(C)$ *is a strongly continuous localization sequence. In particular, every Karoubi projection is a homological epimorphism.*

(2) *Conversely, if* $K \xrightarrow{i} D \xrightarrow{p} E$ *is a strongly continuous localization sequence in which* K *and* D *are compactly generated, then* E *is compactly generated and* $K^\omega \to D^\omega \to E^\omega$ *is a short exact sequence.*

Proof. (1) The functor $\mathrm{Ind}(i)$ is fully faithful, as mapping spaces between ind-objects are computed by the formula of Construction A.21. Its essential image $K \subseteq \mathrm{Ind}(B)$ is closed under colimits and presentable, so by Proposition 2.12(1) we obtain a localization sequence $K \to \mathrm{Ind}(B) \xrightarrow{L} K^\perp$ which is a cofiber sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. For every presentable stable category E, we have natural equivalences

$$\begin{aligned}\mathrm{Fun}^{\mathrm{colim}}(K^\perp, E) &\simeq \mathrm{Fun}^{\mathrm{colim}}_{K\mapsto 0}(\mathrm{Ind}(B), E) \simeq \mathrm{Fun}^{\mathrm{ex}}_{A\mapsto 0}(B, E)\\ &\simeq \mathrm{Fun}^{\mathrm{ex}}(B/A, E) \simeq \mathrm{Fun}^{\mathrm{ex}}((B/A)^{\mathrm{idem}}, E) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(C), E),\end{aligned}$$

using Theorem A.15(3), Theorem 2.1(2), the fact that E is idempotent complete, and $(B/A)^{\mathrm{idem}} \simeq C$. Under these equivalences, precomposition with L corresponds to precomposition with $\mathrm{Ind}(p)$. By the Yoneda lemma there is an equivalence $K^\perp \simeq \mathrm{Ind}(C)$ under which L corresponds to $\mathrm{Ind}(p)$. Hence $\mathrm{Ind}(A) \to \mathrm{Ind}(B) \to \mathrm{Ind}(C)$ is a localization sequence. Since $\mathrm{Ind}(i)$ preserves compact objects, its right adjoint preserves filtered colimits, hence all colimits; so the sequence is strongly continuous by Lemma 2.13.

(2) Put $A = K^\omega$ and $B = D^\omega$. By Lemma 2.15, i restricts to a fully faithful functor $A \to B$ whose image $A = K \cap B$ is a thick subcategory of B. By Lemma 2.2, the sequence $A \to B \to (B/A)^{\mathrm{idem}}$ is short exact, so by (1) we obtain a localization sequence $\mathrm{Ind}(A) \to \mathrm{Ind}(B) \to \mathrm{Ind}((B/A)^{\mathrm{idem}})$. As $K \simeq \mathrm{Ind}(A)$ and $D \simeq \mathrm{Ind}(B)$, both E and $\mathrm{Ind}((B/A)^{\mathrm{idem}})$ are identified with $K^\perp$ compatibly with the functors from D, by Proposition 2.12(2). Hence $E \simeq \mathrm{Ind}((B/A)^{\mathrm{idem}})$ is compactly generated with $E^\omega \simeq (B/A)^{\mathrm{idem}}$, and the claim follows. □

The distinctions can now be summarized as follows. In the first three rows, the first functor identifies the left-hand category with the kernel of the second.

Notion	Quotient condition
Verdier sequence	A, B, C are small stable categories and $B/A \simeq C$.
Short exact sequence	A, B, C are small idempotent complete stable categories and $(B/A)^{\mathrm{idem}} \simeq C$.
Localization sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$	K, D, E are presentable stable categories and $D/K \simeq E$; the right adjoint of $D \to E$ is fully faithful.
Homological epimorphism	An exact functor $A \to B$ in $\mathrm{Cat}_{\mathrm{perf}}$ for which $\mathrm{Ind}(A) \to \mathrm{Ind}(B)$ is a localization; no compact-generation condition on its kernel is imposed.

Passing through Ind turns a short exact sequence into a localization sequence. For a localization between compactly generated categories, going back to a short exact sequence of compact objects requires the kernel to be compactly generated as in Theorem 2.17; strong continuity alone does not supply that hypothesis.

In the language of Definition 2.16: a functor $f\colon A \to B$ in $\mathrm{Cat}_{\mathrm{perf}}$ is a homological epimorphism if and only if $\ker(\mathrm{Ind}(f)) \to \mathrm{Ind}(A) \to \mathrm{Ind}(B)$ is a strongly continuous localization sequence, and it is a Karoubi projection if and only if moreover $\ker(\mathrm{Ind}(f))$ is compactly generated by objects of A. In general, the kernel of a homological epimorphism need not be compactly generated, see Example 2.20 below; it is, however, always a dualizable category (Chapter 3).

Theorem 2.17 gives a description of idempotent complete quotients in terms of large categories. Recall from Corollary A.31 that $\mathrm{Ind}(B) \simeq \mathrm{Fun}^{\mathrm{ex}}(B^{\mathrm{op}}, \mathrm{Sp})$ for $B \in \mathrm{Cat}_{\mathrm{perf}}$.

Corollary 2.18. *Let $i\colon A \hookrightarrow B$ be a fully faithful functor in* $\mathrm{Cat}_{\mathrm{perf}}$. *Under the identification* $\mathrm{Ind}(-) \simeq \mathrm{Fun}^{\mathrm{ex}}((-)^{\mathrm{op}}, \mathrm{Sp})$, *the right adjoint* $\mathrm{Ind}(i)^R\colon \mathrm{Ind}(B) \to \mathrm{Ind}(A)$ *is given by restriction along i, and*

$$(B/A)^{\mathrm{idem}} \simeq \ker\left(\mathrm{Ind}(i)^R\right)^{\omega},$$

where $\ker(\mathrm{Ind}(i)^R) \subseteq \mathrm{Fun}^{\mathrm{ex}}(B^{\mathrm{op}}, \mathrm{Sp})$ *is the full subcategory of exact functors vanishing on A. For the description of exact presheaves on B/A by this vanishing condition, see* Synthetic Category Theory*, Corollary 19.6.8.*

Proof. For $a \in A$ and $F \in \mathrm{Ind}(B)$ we have $\hom_{\mathrm{Ind}(A)}(a, \mathrm{Ind}(i)^R F) \simeq \hom_{\mathrm{Ind}(B)}(ia, F) \simeq F(ia)$ by the Yoneda lemma, so $\mathrm{Ind}(i)^R(F) \simeq F \circ i^{\mathrm{op}}$. Since A is idempotent complete and i is fully faithful, the essential image of i is a thick subcategory, so $A \to B \to (B/A)^{\mathrm{idem}}$ is short exact by Lemma 2.2. By Theorem 2.17(1), $\mathrm{Ind}(A) \to \mathrm{Ind}(B) \to \mathrm{Ind}((B/A)^{\mathrm{idem}})$ is a localization sequence, so by Proposition 2.12(2) we have $\mathrm{Ind}((B/A)^{\mathrm{idem}}) \simeq \mathrm{Ind}(A)^{\perp}$. Finally, $F \in \mathrm{Ind}(A)^{\perp}$ if and only if $\hom(a, \mathrm{Ind}(i)^R F) \simeq \hom(ia, F) \simeq 0$ for all $a \in A$, if and only if $\mathrm{Ind}(i)^R F \simeq 0$. Taking compact objects gives the claim. □

Remark 2.19. The functor Ind induces an equivalence of categories

$$\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{perf}} \rightleftarrows \mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{st}} : (-)^{\omega},$$

where $\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{st}}$ denotes the category of compactly generated presentable stable categories and colimit-preserving functors that preserve compact objects (equivalently, strongly continuous functors), see Theorem A.17 and Lemma 2.14. For a functor $f\colon A \to B$ in $\mathrm{Cat}_{\mathrm{perf}}$:

- f is fully faithful if and only if $\mathrm{Ind}(f)$ is fully faithful (by the formula for mapping spaces in Construction A.21), if and only if $\mathrm{Ind}(f)$ is a monomorphism [Efi24, Proposition 1.13];
- If f is a Karoubi projection, then $\mathrm{Ind}(f)$ is a localization (Theorem 2.17), or equivalently $\mathrm{Ind}(f)$ is an epimorphism [Efi24, Proposition 1.13]. The converse does not hold, see Example 2.20.

Example 2.20 (Keller, [Kel94])**.** Let A be a ring and let $0 \neq I \subseteq \mathrm{Jac}(A)$ be a Tor-unital ideal. Then $A/I \otimes^L_A A/I \simeq A/I$, so the restriction functor $\mathrm{Mod}_{A/I} \to \mathrm{Mod}_A$ is fully faithful and $f\colon \mathrm{Perf}_A \to \mathrm{Perf}_{A/I}$ is a homological epimorphism. The category

$$\ker(\mathrm{Mod}_A \to \mathrm{Mod}_{A/I})$$

contains no nonzero compact objects. Note that $\mathrm{Mod}_A \simeq \mathrm{Ind}(\mathrm{Perf}_A)$ and $\mathrm{Mod}_{A/I} \simeq \mathrm{Ind}(\mathrm{Perf}_{A/I})$. So the kernel of f is zero, whereas the large kernel contains the nonzero module I, since $(A/I) \otimes^L_A I \simeq$

0. This gives a counterexample to the converse of the second statement of Remark 2.19: $\mathrm{Ind}(f)$ is a localization, but f is not a Karoubi projection.

For example, consider $A = k[t, t^{1/2}, t^{1/4}, \dots]_{(t,t^{1/2},t^{1/4},\dots)}$, for k a field. Let $\mathfrak{m}$ be the maximal ideal. Then the map $A \to A/\mathfrak{m}$ is an example.

2.4 Semiorthogonal decompositions

The simplest localization sequences are the split ones. In this case the middle category E is glued from two subcategories C and D between which there are no maps in one direction, and every object of E is uniquely an extension of an object of C by an object of D. Such a splitting is called a semiorthogonal decomposition. We follow Efimov's convention: in $E = \langle C_1, \dots, C_n \rangle$ there are no nonzero maps from C_i to C_j for $i > j$. We first treat the presentable setting, which is the one relevant for dualizable categories, and then explain the analogous notion for small stable categories and its relation to short exact sequences.

Definition 2.21 ([Efi24, Definition 2.74])**.** A *semiorthogonal decomposition* of a presentable stable category E is a collection of full presentable stable subcategories $C_1, \dots, C_n \subseteq E$ such that

(1) the inclusion functors $C_i \hookrightarrow E$ are strongly continuous;

(2) for $i > j$, $x \in C_i$ and $y \in C_j$ we have $\hom_E(x, y) \simeq 0$;

(3) the smallest full stable subcategory of E containing $C_1, \dots, C_n$ is E itself.

In this case we write $E = \langle C_1, \dots, C_n \rangle$.

Proposition 2.22. *Let $E = \langle C, D \rangle$ be a semiorthogonal decomposition, with inclusions $i_C \colon C \hookrightarrow E$ and $i_D \colon D \hookrightarrow E$. Then:*

(1) *Every $x \in E$ fits into a cofiber sequence $d \to x \to c$ with $d \in D$ and $c \in C$, which is unique up to unique equivalence and functorial in x.*

(2) *The functor i_C admits a left adjoint i_C^L and i_D admits a right adjoint i_D^R, given by $x \mapsto c$ and $x \mapsto d$ respectively; there is a natural cofiber sequence $i_D i_D^R \to \mathrm{id}_E \to i_C i_C^L$, and $i_C^L i_D \simeq 0$.*

(3) *The sequence $D \xrightarrow{i_D} E \xrightarrow{i_C^L} C$ is a strongly continuous localization sequence, split by i_C.*

Proof. Note first that $\hom_E(d, c) \simeq 0$ for $d \in D$ and $c \in C$. Suppose $d \to x \to c$ is a cofiber sequence with $d \in D$ and $c \in C$. For $d' \in D$ we then have $\hom_E(d', d) \xrightarrow{\simeq} \hom_E(d', x)$, since $\hom_E(d', c) \simeq 0$; so $d \to x$ exhibits d as the value at x of a right adjoint to i_D. Dually, for $c' \in C$ we have $\hom_E(c, c') \xrightarrow{\simeq} \hom_E(x, c')$, so $x \to c$ exhibits c as the value at x of a left adjoint of i_C. In particular, such a sequence is unique up to unique equivalence and functorial in x, wherever it exists.

Let $E' \subseteq E$ be the full subcategory of those x that admit such a cofiber sequence. It contains C (take $0 \to c \to c$) and D (take $d \to d \to 0$) and is closed under shifts. It is closed under cofibers: a map $x \to x'$ in E' induces, by the universal properties above, a map of cofiber sequences, and passing to cofibers gives a cofiber sequence $\mathrm{cofib}(d \to d') \to \mathrm{cofib}(x \to x') \to \mathrm{cofib}(c \to c')$ with first term

in D and last term in C, as these are stable subcategories. Hence E' is a stable subcategory containing C and D, so $E' = E$ by assumption (3). This proves (1), and by [Lur09, Proposition 5.2.4.2] also the existence of the adjoints in (2), together with the cofiber sequence $i_D i_D^R \to \mathrm{id} \to i_C i_C^L$. Applying it to $x = i_D(d)$ gives $i_C^L i_D(d) \simeq 0$.

For (3), i_C^L is a localization since its right adjoint i_C is fully faithful, and $i_C^L(x) \simeq 0$ if and only if $i_D i_D^R x \to x$ is an equivalence, if and only if x lies in the essential image of D. The right adjoints i_D^R and i_C preserve colimits by assumption (1) and since $C \subseteq E$ is closed under colimits, so the sequence is strongly continuous. Finally, $i_C^L i_C \simeq \mathrm{id}_C$ as i_C is fully faithful, so i_C splits i_C^L. □

Conversely, a semiorthogonal decomposition can be recognized from the adjoints of the inclusion functors. In particular, the projections onto the two pieces are themselves strongly continuous.

Proposition 2.23 ([Efi24, Remark 2.75])**.** *Let $i_1 \colon C \to E$ and $i_2 \colon D \to E$ be strongly continuous functors between presentable stable categories. Then i_1 and i_2 are fully faithful and exhibit $E = \langle i_1(C), i_2(D)\rangle$ as a semiorthogonal decomposition if and only if the following conditions hold:*

(1) *i_1 admits a left adjoint i_1^L and i_2 admits a right adjoint i_2^R;*

(2) *the counit $i_1^L i_1 \to \mathrm{id}_C$ and the unit $\mathrm{id}_D \to i_2^R i_2$ are equivalences;*

(3) *$i_1^L i_2 \simeq 0$, or equivalently $i_2^R i_1 \simeq 0$;*

(4) *the sequence $i_2 i_2^R \to \mathrm{id}_E \to i_1 i_1^L$, formed by the counit and the unit, is a cofiber sequence.*

In this case, the functors i_1^L and i_2^R are strongly continuous.

The composite in (4) admits a unique nullhomotopy, as $\mathrm{Map}(i_2 i_2^R, i_1 i_1^L) \simeq \mathrm{Map}(i_1^L i_2 i_2^R, i_1^L) \simeq *$ by (3); so the condition in (4) makes sense. The equivalence of the two formulations of (3) follows from $\hom_D(d, i_2^R i_1 c) \simeq \hom_E(i_2 d, i_1 c) \simeq \hom_C(i_1^L i_2 d, c)$.

Proof. Suppose first that i_1 and i_2 are fully faithful and $E = \langle i_1(C), i_2(D)\rangle$. Then (1), (3) and (4) are Proposition 2.22(2), applied to the essential images, and (2) holds as i_1 and i_2 are fully faithful.

Conversely, assume (1)–(4). By (2), the functors i_1 and i_2 are fully faithful. For $c \in C$ and $d \in D$ we have $\hom_E(i_2 d, i_1 c) \simeq \hom_C(i_1^L i_2 d, c) \simeq 0$ by (3), so the semiorthogonality condition holds. For $e \in E$, the cofiber sequence $i_2 i_2^R e \to e \to i_1 i_1^L e$ of (4) shows that e lies in the stable subcategory generated by the essential images of i_1 and i_2. Finally, the inclusions are strongly continuous by assumption, and their essential images are closed under colimits, as i_1 and i_2 preserve colimits.

It remains to show that i_1^L and i_2^R are strongly continuous. The right adjoint of i_1^L is i_1, which preserves colimits. The right adjoint of $i_1 i_1^L$ is $i_1 i_1^R$, which preserves colimits since i_1 is strongly continuous. Passing to right adjoints is an equivalence between the stable category of colimit-preserving endofunctors of E and the opposite of the stable category of accessible limit-preserving endofunctors, so it carries the fiber sequence $i_2 i_2^R \to \mathrm{id}_E \to i_1 i_1^L$ to a cofiber sequence $i_1 i_1^R \to \mathrm{id}_E \to (i_2 i_2^R)^R$. Hence $(i_2 i_2^R)^R$ preserves colimits. Since i_2 is fully faithful, we have

$$\hom_D(i_2^R e, d) \simeq \hom_E(i_2 i_2^R e, i_2 d) \simeq \hom_E(e, (i_2 i_2^R)^R i_2 d),$$

so $(i_2 i_2^R)^R \circ i_2$ is right adjoint to i_2^R and preserves colimits. □

Remark 2.24. Definition 2.21 makes sense for any stable category E and full stable subcategories $C_1, \dots, C_n \subseteq E$, omitting condition (1). No presentability or size assumption is needed. Parts (1) and (2) of Proposition 2.22 hold in this setting with the same proof, and the sequence $D \xrightarrow{i_D} E \xrightarrow{i_C^L} C$ is a Verdier sequence: i_C^L is a localization with kernel D, and it inverts precisely the maps whose cofiber lies in D. If E is idempotent complete, then so are C and D, being thick subcategories of E (as kernels of the exact functors i_D^R and i_C^L), and the sequence is short exact. (In the talk, the definition was given simultaneously for presentable stable categories, for dualizable categories and for small stable categories.)

Conversely, short exact sequences admitting adjoints give rise to semiorthogonal decompositions.

Proposition 2.25. *Let $A \xrightarrow{i} B \xrightarrow{p} C$ be a short exact sequence in $\mathrm{Cat}_{\mathrm{perf}}$. Then i admits a right adjoint i^R if and only if p admits a right adjoint p^R. In this case:*

(1) *The functor p^R is fully faithful, with essential image the right orthogonal $A^\perp = \{x \in B \mid \hom_B(ia, x) \simeq 0 \text{ for all } a \in A\}$;*

(2) *There is a semiorthogonal decomposition $B = \langle p^R(C),\, i(A)\rangle$;*

(3) *The sequence $C \xrightarrow{p^R} B \xrightarrow{i^R} A$ is again short exact.*

Proof. Assume first that i^R exists, and for $b \in B$ let $L(b) := \mathrm{cofib}(ii^R b \to b)$. As in the proof of Proposition 2.12(1), $L(b) \in A^\perp$ and $L\colon B \to A^\perp$ is left adjoint to the inclusion. Moreover, $L(b) \simeq 0$ if and only if $b \simeq ii^R b$ lies in A, so L inverts precisely the maps whose cofiber lies in A. Since L is a reflective localization, it exhibits $A^\perp$ as the localization of B at these maps [Lur09, Proposition 5.2.7.12], so $B/A \simeq A^\perp$. As $A^\perp$ is closed under retracts in the idempotent complete category B, it is idempotent complete, so $C \simeq (B/A)^{\mathrm{idem}} \simeq A^\perp$, with p corresponding to L. Hence p admits the fully faithful right adjoint $A^\perp \hookrightarrow B$.

Conversely, assume that p^R exists. For $a \in A$ and $c \in C$ we have $\hom_B(ia, p^R c) \simeq \hom_C(pia, c) \simeq 0$, so p^R lands in $A^\perp$. Every $x \in A^\perp$ is local with respect to the maps $s\colon b' \to b$ with $\mathrm{cofib}(s) \in A$, i.e. $s^*\colon \hom_B(b, x) \to \hom_B(b', x)$ is an equivalence. Hence $\hom_{B/A}(qb, qx) \simeq \hom_B(b, x)$ for all $b \in B$ and $x \in A^\perp$ (for instance by Remark 2.8, applied to B^{op}). Applying this to $x = p^R c$, we find that the composite

$$\hom_B(b, p^R c) \to \hom_C(pb, pp^R c) \to \hom_C(pb, c),$$

whose second map is induced by the counit, is the adjunction equivalence. Its first map is an equivalence as well: since the functor $B/A \to C$ induced by p is fully faithful, it is identified with the equivalence $\hom_B(b, p^R c) \simeq \hom_{B/A}(qb, qp^R c)$. Hence so is the second map, i.e. the counit $pp^R c \to c$ induces an equivalence on $\hom_C(pb, -)$ for all $b \in B$. Since every object of C is a retract of an object of the form pb, the counit is an equivalence, i.e. p^R is fully faithful. For $b \in B$, let $f \to b \to p^R pb$ be the fiber sequence of the unit. Then $p(f) \simeq 0$, so f lies in A, and for $a \in A$ we get $\hom_B(ia, f) \simeq \hom_B(ia, b)$ since $\hom_B(ia, p^R pb) \simeq 0$. Hence $b \mapsto f$ is right adjoint to i.

It remains to prove (1)–(3). For (1), it remains to show that every $x \in A^\perp$ lies in the image of p^R. The fiber f of the unit $x \to p^R px$ lies in A, so $\hom_B(f, x) \simeq 0$ and $\hom_B(f, p^R px) \simeq 0$, whence $\hom_B(f, f) \simeq 0$ and $f \simeq 0$. For (2), we have $\hom_B(ia, p^R c) \simeq 0$, and every $b \in B$ fits into the

cofiber sequence $ii^R b \to b \to p^R p b$. For (3), the kernel of i^R is $A^\perp = p^R(C)$ by (1), and i^R is a coreflective localization since its left adjoint i is fully faithful; applying the first paragraph to opposite categories, i^R exhibits A as the Verdier quotient $B/p^R(C)$. □

2.5 Calkin categories

For $A \in \mathrm{Cat}_{\mathrm{perf}}$ and a regular cardinal κ, we write $\mathrm{Ind}(A)^\kappa$ for the full subcategory of κ-compact objects of $\mathrm{Ind}(A)$. It is essentially small and stable, and it contains $A = \mathrm{Ind}(A)^\omega$ as a thick subcategory.

Definition 2.26 ([Efi24, Section 2.9])**.** Let $A \in \mathrm{Cat}_{\mathrm{perf}}$ and let κ be an uncountable regular cardinal. The *κ-Calkin category* of A is

$$\mathrm{Calk}_\kappa(A) := \left(\mathrm{Ind}(A)^\kappa / A\right)^{\mathrm{idem}},$$

so that there is a short exact sequence $A \hookrightarrow \mathrm{Ind}(A)^\kappa \to \mathrm{Calk}_\kappa(A)$.

Remark 2.27. The mapping spectra in $\mathrm{Calk}_\kappa(A)$ can be computed using Remark 2.8. Let $x = \text{“colim”}_{i\in I} x_i$ and $y = \text{“colim”}_{j\in J} y_j$ be objects of $\mathrm{Ind}(A)^\kappa$, with I and J filtered. The functor $J \to A_{/y}$ sending j to $y_j \to y$ is cofinal, and $\hom_{\mathrm{Ind}(A)}(x, y_j) \simeq \lim_i \hom_A(x_i, y_j)$. Hence

$$\hom_{\mathrm{Calk}_\kappa(A)}(x, y) \simeq \mathrm{cofib}\left(\mathrm{colim}_j \lim_i \hom_A(x_i, y_j) \to \lim_i \mathrm{colim}_j \hom_A(x_i, y_j)\right).$$

So the Calkin category measures the failure of the limit over i to commute with the filtered colimit over j.

To justify the cofinality assertion, fix $a \to y$ with $a \in A$. The comma category $(a \to y)/J$ is the category of elements of the functor

$$j \mapsto \mathrm{fib}_{a\to y}\left(\mathrm{Map}_A(a, y_j) \to \mathrm{Map}_{\mathrm{Ind}(A)}(a, y)\right).$$

Its groupoid completion is the colimit of this functor. Since a is compact and filtered colimits of animae commute with finite limits, this colimit is the fiber of the identity of $\mathrm{Map}_{\mathrm{Ind}(A)}(a, y)$ over $a \to y$, hence is contractible. The cofinality criterion therefore applies.

Example 2.28. Let R be an $\mathbb{E}_\infty$-ring and $A = \mathrm{Perf}_R$, so that $\mathrm{Ind}(A) \simeq \mathrm{Mod}_R$. For $M \in \mathrm{Mod}_R$ and a perfect R-module P we have $\hom_R(M, P) \simeq \hom_R(M, R) \otimes_R P$, since P is dualizable. Writing $M' \in \mathrm{Mod}_R$ as a filtered colimit of perfect modules, Remark 2.8 gives

$$\hom_{\mathrm{Calk}_\kappa(\mathrm{Perf}_R)}(M, M') \simeq \mathrm{cofib}\left(\hom_R(M, R) \otimes_R M' \to \hom_R(M, M')\right)$$

for $M, M' \in \mathrm{Mod}_R^\kappa$. For instance, if $R = k$ is a field and V is a k-vector space of dimension $< \kappa$, the map $V^\vee \otimes_k V \to \mathrm{End}_k(V)$ is injective with image the finite rank endomorphisms, so

$$\mathrm{End}_{\mathrm{Calk}_\kappa(\mathrm{Perf}_k)}(V) \simeq \mathrm{End}_k(V)/\{\text{finite rank endomorphisms}\}.$$

This is the algebraic analogue of the Calkin algebra $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ of bounded operators modulo compact operators on a Hilbert space, which explains the name.

2.6 Localizing invariants

We start with some recollections on the functor $K\colon \mathrm{Cat}_{\mathrm{st}} \to \mathrm{Sp}$.

Definition 2.29. The *Grothendieck group* $K_0(A)$ of a small stable category A is the free abelian group on the equivalence classes $[x]$ of objects $x \in A$, modulo the relations $[x] = [x'] + [x'']$ for every cofiber sequence $x' \to x \to x''$ in A.

Nonconnective algebraic K-theory is a spectrum-valued refinement $K\colon \mathrm{Cat}_{\mathrm{st}} \to \mathrm{Sp}$ of this construction, satisfying $\pi_0 K(A) \cong K_0(A)$ for A idempotent complete. Its fundamental property is that it is a localizing invariant in the following sense.

Definition 2.30. Let $\mathcal{E}$ be a stable category. A functor $F\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathcal{E}$ is called a *localizing invariant* if $F(0) \simeq 0$ and F sends short exact sequences (Definition 2.5) to fiber sequences. We extend F to $\mathrm{Cat}_{\mathrm{st}}$ by setting $F(A) := F(A^{\mathrm{idem}})$. We denote by $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}_{\mathrm{st}}, \mathcal{E})$ the full subcategory of localizing invariants.

By Theorem 2.17, a sequence $A \to B \to C$ in $\mathrm{Cat}_{\mathrm{perf}}$ is short exact if and only if $\mathrm{Ind}(A) \to \mathrm{Ind}(B) \to \mathrm{Ind}(C)$ is a strongly continuous localization sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Lemma 2.31. *Let $\mathcal{E}$ be stable and let $F\colon \mathrm{Cat}_{\mathrm{st}} \to \mathcal{E}$ satisfy $F(0) \simeq 0$. Consider the following conditions:*

(1) *For every Verdier sequence $A \hookrightarrow B \to B/A$ in the sense of Definition 2.4, F takes this sequence to a fiber sequence.*

(2) *Each sequence $A \hookrightarrow B \to C$ with $A \to B$ fully faithful and inducing an equivalence $(B/A)^{\mathrm{idem}} \xrightarrow{\sim} C^{\mathrm{idem}}$ is sent to a fiber sequence under F.*

Then (2) implies (1), and (2) holds if and only if F satisfies (1) and inverts the maps $A \to A^{\mathrm{idem}}$. In this case, F is (the extension of) a localizing invariant in the sense of Definition 2.30.

Proof. The sequences in (1) are among those of (2). Suppose that F satisfies (1) and inverts idempotent completions. For a sequence as in (2), the fully faithful functor $A^{\mathrm{idem}} \hookrightarrow B^{\mathrm{idem}}$ has thick image, so

$$A^{\mathrm{idem}} \hookrightarrow B^{\mathrm{idem}} \to B^{\mathrm{idem}}/A^{\mathrm{idem}}$$

is a Verdier sequence. Moreover, $(B^{\mathrm{idem}}/A^{\mathrm{idem}})^{\mathrm{idem}} \simeq (B/A)^{\mathrm{idem}} \simeq C^{\mathrm{idem}}$, by the universal property of the idempotent complete quotient. Applying (1) and idempotent invariance therefore gives (2). Conversely, (2) applied to $0 \to A \to A^{\mathrm{idem}}$ shows that $F(A) \to F(A^{\mathrm{idem}})$ is an isomorphism. □

Remark 2.32. Condition (1) of Lemma 2.31 alone does not imply that F inverts $A \to A^{\mathrm{idem}}$. For instance, connective algebraic K-theory satisfies (1) (see the proof of [Cal+20, Corollary 4.4.15]), but $K_0(A) \to K_0(A^{\mathrm{idem}})$ is in general not surjective. This is why localizing invariants are defined on idempotent complete categories.

Remark 2.33. One frequently additionally requires that $\mathcal{E}$ is presentable and that F preserves filtered colimits (or at least κ-filtered colimits for some regular cardinal κ). Algebraic K-theory and THH are examples of localizing invariants preserving filtered colimits. Topological cyclic homology TC is localizing, but does not in general preserve filtered colimits.

Example 2.34. Let $B = \langle C, D \rangle$ be a semiorthogonal decomposition in $\mathrm{Cat}_{\mathrm{perf}}$ (Remark 2.24). Then every localizing invariant F sends the short exact sequence $D \to B \to C$ to a fiber sequence $F(D) \to F(B) \to F(C)$, which is split by $F(i_C)$. Hence

$$F(B) \simeq F(C) \oplus F(D).$$

Functors $F\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathcal{E}$ with this property are called *additive*.

Example 2.35. Let $A \in \mathrm{Cat}_{\mathrm{perf}}$ and let F be a localizing invariant. The short exact sequence $A \to \mathrm{Ind}(A)^{\omega_1} \to \mathrm{Calk}_{\omega_1}(A)$ of Definition 2.26 gives a fiber sequence

$$F(A) \to F(\mathrm{Ind}(A)^{\omega_1}) \to F(\mathrm{Calk}_{\omega_1}(A)).$$

Since $\mathrm{Ind}(A)^{\omega_1}$ admits countable coproducts, the middle term vanishes by an Eilenberg swindle: for $T = \bigoplus_{\mathbb{N}}\colon \mathrm{Ind}(A)^{\omega_1} \to \mathrm{Ind}(A)^{\omega_1}$ we have $\mathrm{id} \oplus T \simeq T$, and additivity gives $F(\mathrm{id}) + F(T) = F(T)$, so $F(\mathrm{id}) = 0$. Hence

$$F(A) \simeq \Omega F(\mathrm{Calk}_{\omega_1}(A)).$$

This is the formula that will be used to extend F to dualizable categories, see Definition 1.9.

The most fundamental localizing invariant is nonconnective algebraic K-theory, as made precise by the following universal property.

Theorem 2.36. *Nonconnective algebraic K-theory* $K\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Sp}$ *is the initial localizing invariant equipped with a natural transformation* $(-)^{\simeq} \to \Omega^{\infty} K(-)$.

Remark 2.37. In the literature, Theorem 2.36 is usually stated for localizing invariants preserving filtered colimits, see [BGT13] and also [Bar16]. The version without this condition is explicitly stated in [Ram24c, page 115, footnote 19]. That footnote does not supply a proof of the additional generality.

Definition 2.38 ([BGT13])**.** The category of *localizing motives* $\mathrm{Mot}_{\mathrm{loc}}$ is the presentable stable category equipped with a localizing invariant

$$U_{\mathrm{loc}}\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Mot}_{\mathrm{loc}}$$

preserving filtered colimits, characterized as follows: for every presentable stable category $\mathcal{E}$, precomposition with U_{loc} induces an equivalence between colimit-preserving functors $\mathrm{Mot}_{\mathrm{loc}} \to \mathcal{E}$ and localizing invariants $\mathrm{Cat}_{\mathrm{perf}} \to \mathcal{E}$ preserving filtered colimits. We fix a universe for the small categories in $\mathrm{Cat}_{\mathrm{perf}}$ and use a larger universe for functor categories; these size conventions are suppressed in the notation.

Although $\mathrm{Mot}_{\mathrm{loc}}$ is defined by a universal property, it can be constructed explicitly as a localization of a category of spectral presheaves on a suitable small subcategory of $\mathrm{Cat}_{\mathrm{perf}}$. This is the *finitary* category of motives. It is not asserted to classify all the localizing invariants of Definition 2.30.

Theorem 2.39 ([BGT13, Theorem 9.8])**.** *For all* $A \in \mathrm{Cat}_{\mathrm{perf}}$ *there is a natural equivalence*

$$\hom_{\mathrm{Mot}_{\mathrm{loc}}}\left(U_{\mathrm{loc}}(\mathrm{Sp}^{\omega}), U_{\mathrm{loc}}(A)\right) \simeq K(A).$$

Marc formulated the universal property of K-theory as follows.

Theorem 2.40 (Theorem/definition, cf. [BGT13])**.** *K is the unit for the Day convolution operad structure on* $\mathrm{Fun}_{\mathrm{loc}}(\mathrm{Cat}_{\mathrm{perf}}, \mathrm{Sp})$*, where localizing invariants are not required to preserve filtered colimits.*

Explanation from Theorem 2.36. The Day convolution unit on the full functor category is $Y(A) = \Sigma^\infty_+ \mathrm{Map}(\mathrm{Sp}^\omega, A) \simeq \Sigma^\infty_+(A^\simeq)$. By Theorem 2.36, its canonical map to K induces an isomorphism on mapping animae into every localizing invariant. The same assertion applies in any variable of a Day multimorphism: with all other input categories fixed, its target has the form $A \mapsto H(A \otimes B_1 \otimes \cdots \otimes B_r)$, which is localizing because tensoring small idempotent complete stable categories preserves exact sequences. Taking the ends that express naturality in the remaining variables therefore shows that replacing a Y-input by K induces an isomorphism on multimorphism animae. This is the operadic unit property for K. This argument uses the non-finitary universal property, not the universal property of $\mathrm{Mot}_{\mathrm{loc}}$. □

Equivalently, one can work on $\mathrm{Cat}_{\mathrm{st}}$ by extending each invariant along idempotent completion; in that formulation the unlocalized Day unit is corepresented by the finite spectra. No claim that the non-finitary category of invariants is presentable is needed here.

Theorem 2.41 (Tamme, before Efimov, [Tam18, Theorem 18])**.** *Consider a square Q in* $\mathrm{Cat}_{\mathrm{st}}$ *of the form*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow[g]{} & D. \end{array}$$

Assume that $\mathrm{Ind}(Q)$ *is cartesian, and that* $\mathrm{Ind}(f)$ *and* $\mathrm{Ind}(g)$ *are localizations. Then every localizing invariant F sends Q to a cartesian square.*

Proof. Write $h\colon B \to D$ for the right vertical functor and form the lax pullback $P = B \times^{\rightarrow}_D C$. Its objects are triples (b, c, φ) with $\varphi\colon h(b) \to g(c)$. There is a split exact sequence $C \to P \to B$, and hence $F(P) \simeq F(C) \oplus F(B)$, by [Tam18, Proposition 10]. The hypotheses make Q an excisive square in the sense of [Tam18, Definition 14]; in fact only $\mathrm{Ind}(g)$ needs to be a localization. By [Tam18, Theorem 15], the functor $(b, c, \varphi) \mapsto \mathrm{cofib}(\varphi)$ fits into an exact sequence

$$A \to P \to D.$$

Here and in the split sequence, we may pass to idempotent completions before applying F. Under the splitting, the map $F(P) \to F(D)$ is $(-F(h), F(g))$: on $(b, 0, h(b) \to 0)$ it is suspension followed by h, and on $(0, c, 0 \to g(c))$ it is g. Thus $F(A)$ is the fiber of $F(B) \oplus F(C) \xrightarrow{-F(h)+F(g)} F(D)$, compatibly with the maps of Q. This is precisely the assertion that $F(Q)$ is cartesian. □

Warning 2.42. One cannot replace the lax-pullback argument by simply applying F to the common kernel of f and g. The hypotheses say that $\mathrm{Ind}(f)$ and $\mathrm{Ind}(g)$ are localizations; they do not say that f and g are Karoubi projections. Their small kernels therefore need not give short exact sequences to which F applies.

Remark 2.43. The proof above is Tamme's lax-pullback argument and predates continuous K-theory. It applies to localizing invariants valued in any stable category, without assuming preservation of

filtered colimits. A later proof using continuous invariants identifies the kernels of $\mathrm{Ind}(f)$ and $\mathrm{Ind}(g)$ and applies the continuous extension to the resulting localization sequences. See [LT19] for a generalization to arbitrary pullbacks of ring spectra.

Example 2.44 (Milnor patching, [Mil71])**.** Consider a pullback square of $\mathbb{E}_1$-rings:

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

and assume that $A' \otimes_A A' \xrightarrow{\sim} A'$. Then the induced square

$$\begin{array}{ccc} \mathrm{Mod}_A & \longrightarrow & \mathrm{Mod}_{A'} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_B & \longrightarrow & \mathrm{Mod}_{B'} \end{array}$$

is also a pullback square, and $\mathrm{Mod}_B \to \mathrm{Mod}_{B'}$ is a localization [Tam18, Lemma 23 and Theorem 26]. In particular, we deduce:

Theorem 2.45 ([Tam18, Theorem 28]; see also [LT19])**.** *In the above situation, there is a pullback square*

$$\begin{array}{ccc} K(A) & \longrightarrow & K(A') \\ \downarrow & & \downarrow \\ K(B) & \longrightarrow & K(B'). \end{array}$$

Example 2.46. Given a map of rings $A \to B$ that sends an ideal $I \subseteq A$ isomorphically to the induced ideal $IB \subseteq B$, the induced square

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/IB \end{array}$$

is a pullback of $\mathbb{E}_1$-rings.

3 Compactly assembled and dualizable categories

In Chapter 1 we saw that compact generation is too restrictive for many purposes: for instance, the category $\mathrm{Shv}(\mathbb{R}^n; \mathrm{Sp})$, for $n \geq 1$, has no nonzero compact objects at all. The goal of this chapter is to introduce the class of categories that replaces the compactly generated ones: the *dualizable* categories, or, in the unstable setting, the *compactly assembled* categories.

To motivate the definition, let us first ask what makes compact generation so useful. Let C be a category with filtered colimits, and let us call an ind-object $X \in \mathrm{Ind}(C)$ together with an equivalence $\mathrm{colim}\, X \simeq x$ a *presentation* of $x \in C$. Every object x has a trivial presentation, namely $y(x)$; by the adjunction $\mathrm{colim} \dashv y$, every presentation of x maps uniquely to $y(x)$, compatibly with the identifications of the colimits with x. So $y(x)$ is the *terminal*, or coarsest, presentation of x. If $C = \mathrm{Ind}(C_0)$ is compactly generated, then x has a second canonical presentation

$$\hat{y}(x) = \text{“colim”}_{(c \to x) \in (C_0)_{/x}} c,$$

and its usefulness comes from the fact that it maps into every other presentation: since every map $c \to \mathrm{colim}_j z_j$ from a compact object factors through some stage z_j, we have

$$\mathrm{Map}_C\left(x, \mathrm{colim}_j z_j\right) \simeq \mathrm{Map}_{\mathrm{Ind}(C)}\left(\hat{y}(x), \text{“colim”}_j z_j\right)$$

for every filtered diagram $(z_j)_j$ in C. In other words, $\hat{y}(x)$ is the *initial*, or finest, presentation of x. This formula says that maps out of x into filtered colimits are computed by the formal object $\hat{y}(x)$, even if x itself is not compact.

A category is *compactly assembled* if every object admits such a finest presentation, i.e. if $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ admits a left adjoint $\hat{y}$. Unlike in the compactly generated case, the terms of the ind-object $\hat{y}(x)$ are no longer required to be compact. A presentable stable category is compactly assembled if and only if it is dualizable (Theorem 3.16), and this is the class of categories to which K-theory will be extended.

3.1 Compactly assembled categories

We now make this precise. In this section C is a category with filtered colimits, not necessarily stable; the stable case is the subject of Section 3.2. Besides the definition, the key notion is that of a *compact morphism*, which in compactly assembled categories takes over the role played by compact objects in compactly generated ones. We also show that the compactly assembled categories are

precisely the retracts of compactly generated ones (Theorem 3.8), and introduce the corresponding class of strongly continuous functors.

Definition 3.1. A category C is called *compactly assembled* if it is accessible, admits filtered colimits, and the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ admits a left adjoint $\hat{y}\colon C \hookrightarrow \mathrm{Ind}(C)$.

Note that $\hat{y}$ is automatically fully faithful, since its double right adjoint $y\colon C \hookrightarrow \mathrm{Ind}(C)$ is fully faithful (Lemma A.26).

Since y is fully faithful, the counit $\mathrm{colim}\, y \to \mathrm{id}$ is invertible. Inverting it gives a map $\mathrm{id} \to \mathrm{colim}\, y$, which under the adjunction $\hat{y} \dashv \mathrm{colim}$ corresponds to a natural transformation

$$\hat{y} \to y.$$

By construction, applying colim to $\hat{y}(x) \to y(x)$ gives (up to the identifications $\mathrm{colim}\, \hat{y}(x) \simeq x \simeq \mathrm{colim}\, y(x)$) the identity of x.

Lemma 3.2. *Let C be compactly assembled and $x \in C$. Then x is compact if and only if the map $\hat{y}(x) \to y(x)$ is an equivalence.*

Proof. Consider the natural transformation

$$\mathrm{Map}_{\mathrm{Ind}(C)}(y(x), -) \to \mathrm{Map}_C(x, \mathrm{colim}(-))$$

of functors $\mathrm{Ind}(C) \to \mathrm{An}$ given by applying colim and using $\mathrm{colim}\, y(x) \simeq x$. Composing with the adjunction equivalence $\mathrm{Map}_C(x, \mathrm{colim}(-)) \simeq \mathrm{Map}_{\mathrm{Ind}(C)}(\hat{y}(x), -)$, it corresponds by the Yoneda lemma to the map $\hat{y}(x) \to y(x)$, by the description of this map given above. Hence $\hat{y}(x) \to y(x)$ is an equivalence if and only if this transformation is an equivalence. Evaluated at "colim"$_j z_j$, it is the canonical map

$$\mathrm{colim}_j \mathrm{Map}_C(x, z_j) \simeq \mathrm{Map}_{\mathrm{Ind}(C)}\left(y(x), \text{“colim”}_j z_j\right) \to \mathrm{Map}_C\left(x, \mathrm{colim}_j z_j\right),$$

where the first equivalence holds as $y(x)$ is compact in $\mathrm{Ind}(C)$. This is an equivalence for all filtered diagrams (z_j) if and only if x is compact. □

Compact objects of C are those x for which maps out of x into a filtered colimit factor through some stage of the colimit. The following relative version of this notion is the basic tool for studying compactly assembled categories.

Definition 3.3. Let C be a category with filtered colimits. A morphism $f\colon x \to x'$ in C is called *compact* if the natural transformation

$$f^*\colon \mathrm{Map}_C\left(x', \mathrm{colim}(-)\right) \to \mathrm{Map}_C\left(x, \mathrm{colim}(-)\right)$$

of functors $\mathrm{Ind}(C) \to \mathrm{An}$ factors through the natural transformation

$$\mathrm{Map}_{\mathrm{Ind}(C)}\left(y(x), -\right) \to \mathrm{Map}_C\left(x, \mathrm{colim}(-)\right)$$

given by applying colim and using $\mathrm{colim}\, y(x) \simeq x$.

Evaluated at an ind-object $Z = \text{“colim”}_j z_j$, the second transformation is the canonical map $\operatorname{colim}_j \operatorname{Map}_C(x, z_j) \to \operatorname{Map}_C(x, \operatorname{colim}_j z_j)$. So informally, f is compact if for every map $g\colon x' \to \operatorname{colim}_j z_j$ the composite gf factors through some stage z_j, in a way which is natural in the filtered diagram $(z_j)_j$. By definition, x is compact if and only if the second transformation is an equivalence; in that case every morphism $x \to x'$ and every morphism $x'' \to x$ is compact.

Remark 3.4. Compact morphisms form a two-sided ideal: if $f\colon x \to x'$ is compact and $k\colon w \to x$ and $h\colon x' \to x''$ are arbitrary, then hfk is compact. Indeed, given a factorization $\theta\colon \operatorname{Map}_C(x', \operatorname{colim}(-)) \to \operatorname{Map}_{\operatorname{Ind}(C)}(y(x), -)$ of f^*, the composite $y(k)^* \circ \theta \circ h^*$ is a factorization of $(hfk)^* = k^* f^* h^*$.

Lemma 3.5. *Let C be compactly assembled and let $f\colon x \to x'$ be a morphism in C.*

(1) *The morphism f is compact if and only if $y(f)\colon y(x) \to y(x')$ factors through the map $\hat{y}(x') \to y(x')$.*

(2) *The identity id_x is a compact morphism if and only if x is a compact object.*

Proof. (1) Write $\varepsilon\colon \hat{y}(x') \to y(x')$, and recall that $\operatorname{colim}(\varepsilon)$ is the identity of x'. By the adjunction $\hat{y} \dashv \operatorname{colim}$, we have $\operatorname{Map}_C(x', \operatorname{colim}(-)) \simeq \operatorname{Map}_{\operatorname{Ind}(C)}(\hat{y}(x'), -)$, so by the Yoneda lemma, natural transformations $\theta\colon \operatorname{Map}_C(x', \operatorname{colim}(-)) \to \operatorname{Map}_{\operatorname{Ind}(C)}(y(x), -)$ correspond to maps $g\colon y(x) \to \hat{y}(x')$, via $\theta = g^*$. The composite of g^* with the transformation $\operatorname{Map}_{\operatorname{Ind}(C)}(y(x), -) \to \operatorname{Map}_C(x, \operatorname{colim}(-))$ is precomposition with $\operatorname{colim}(g)\colon x \to x'$. Hence f is compact if and only if there is a map $g\colon y(x) \to \hat{y}(x')$ with $\operatorname{colim}(g) \simeq f$. If $\varepsilon g \simeq y(f)$, then $\operatorname{colim}(g) \simeq \operatorname{colim}(\varepsilon g) \simeq f$. Conversely, if $\operatorname{colim}(g) \simeq f$, then εg and $y(f)$ are maps $y(x) \to y(x')$ with the same image under colim; since $\operatorname{colim}\colon \operatorname{Map}_{\operatorname{Ind}(C)}(y(x), y(x')) \to \operatorname{Map}_C(x, x')$ is inverse to the equivalence induced by the fully faithful functor y, we get $\varepsilon g \simeq y(f)$.

(2) By (1), id_x is compact if and only if $\varepsilon\colon \hat{y}(x) \to y(x)$ admits a section g. In that case, $g\varepsilon\colon \hat{y}(x) \to \hat{y}(x)$ corresponds under the adjunction to $\operatorname{colim}(g)\colon x \to x$, which is the identity since $\operatorname{colim}(\varepsilon g) = \mathrm{id}_x$ and $\operatorname{colim}(\varepsilon) = \mathrm{id}_x$. Hence $g\varepsilon \simeq \mathrm{id}$, so ε is an equivalence and x is compact by Lemma 3.2. The converse is clear. □

Example 3.6. If C is accessible and compactly generated, then C is compactly assembled. Indeed, if $C = \operatorname{Ind}(C_0)$ with C_0 small, then

$$\hat{y} = \operatorname{Ind}(y_{C_0})\colon \operatorname{Ind}(C_0) \to \operatorname{Ind}(\operatorname{Ind}(C_0)),$$

where $y_{C_0}\colon C_0 \hookrightarrow \operatorname{Ind}(C_0) = C$ is the Yoneda embedding. When C is cocomplete, we may take $C_0 = C^\omega$ by *Stable Homotopy Theory and Higher Algebra*, Proposition 23.3.7.

The following example explains the terminology, and will be generalized in Chapter 9.

Example 3.7 (Continuous posets, [JJ82; Gie+03])**.** Let P be a partially ordered set admitting directed joins, regarded as a category. Then $\operatorname{Ind}(P)$ is the poset of *ideals* of P, i.e. non-empty directed downward closed subsets $I \subseteq P$, ordered by inclusion, and $\operatorname{colim}\colon \operatorname{Ind}(P) \to P$ sends I to its join $\bigvee I$. The functor y sends x to the principal ideal $\downarrow x$, which is the largest ideal with join $\leq x$. A left adjoint $\hat{y}$, if it exists, sends x to the smallest ideal I with $x \leq \bigvee I$.

Say that y is *way below* x, written $y \ll x$, if $y \in I$ for every ideal I with $x \leq \bigvee I$; taking $I = \downarrow x$ shows that $y \ll x$ implies $y \leq x$. Then $\hat{y}$ exists if and only if for every $x \in P$ the set $\Downarrow x = \{y \in P \mid y \ll x\}$

is directed with join x, in which case $\hat{y}(x) = \Downarrow x$; we prove this in Proposition 5.19. Posets with this property are classically called *continuous*. An element x is compact if and only if $x \ll x$, in accordance with Lemma 3.2. For example:

(1) In the interval $[0, 1]$, we have $y \ll x$ if and only if $y < x$ or $y = 0$. So $[0, 1]$ is continuous, and its only compact element is 0.

(2) Let X be a locally compact Hausdorff space and let $\mathrm{Open}(X)$ be the poset of open subsets. Then $U \ll V$ if and only if there exists a compact subset $K \subseteq X$ with $U \subseteq K \subseteq V$, and $\mathrm{Open}(X)$ is continuous. The compact elements are the compact open subsets; for $X = \mathbb{R}^n$ the only one is $\emptyset$.

(3) The power set of a set S is continuous, and in fact *algebraic*, i.e. compactly generated: the compact elements are the finite subsets, and every subset is the directed union of its finite subsets.

Note how example (2) foreshadows the fact that $\mathrm{Shv}(\mathbb{R}^n; \mathrm{Sp})$ is dualizable but has no nonzero compact objects. Continuous posets are discussed in more detail, with proofs, in Section 5.3.

The following theorem gives an alternative characterization of compactly assembled categories, which shows that they are precisely the retracts of compactly generated ones.

Theorem 3.8 (Lurie, [Lur18, Definition 21.1.2.1 and Theorem 21.1.2.10])**.** *Let C be a category with filtered colimits. Then the following are equivalent:*

(1) *C is compactly assembled;*

(2) *There exists a small category D such that C is a retract of* $\mathrm{Ind}(D)$ *as a category with filtered colimits.*

Proof. (1) $\implies$ (2): We have an adjunction between $\hat{y} \colon C \hookrightarrow \mathrm{Ind}(C)$ and $\mathrm{colim} \colon \mathrm{Ind}(C) \to C$. Since $\hat{y}$ is fully faithful, this exhibits C as a retract of $\mathrm{Ind}(C)$, with both functors preserving filtered colimits. To ensure we may pick D small, write $C = \mathrm{Ind}_\kappa(C^\kappa)$, with C^κ small (Theorem A.17). Since $\mathrm{Ind}(C) = \bigcup_\lambda \mathrm{Ind}(C^\lambda)$ (Construction A.21) and C^κ is small, we may find some regular cardinal λ such that $\hat{y}(C^\kappa) \subseteq \mathrm{Ind}(C^\lambda)$. Since the inclusion $\mathrm{Ind}(C^\lambda) \subseteq \mathrm{Ind}(C)$ preserves filtered colimits, it follows that in fact $\hat{y}(C) \subseteq \mathrm{Ind}(C^\lambda)$. So it follows that C is already a retract of $\mathrm{Ind}(C^\lambda)$, giving the claim.

(2) $\implies$ (1): Assume we have $u \colon C \hookrightarrow \mathrm{Ind}(D)$ with retraction $v \colon \mathrm{Ind}(D) \to C$, and assume u and v preserve filtered colimits. It follows that C is accessible: we have

$$C = \lim\big(\cdots \xrightarrow{uv} \mathrm{Ind}(D) \xrightarrow{uv} \mathrm{Ind}(D)\big),$$

and the subcategory $\mathrm{Acc} \subseteq \widehat{\mathrm{Cat}}$ of accessible categories and accessible functors is closed under limits in $\widehat{\mathrm{Cat}}$ (Theorem A.18).

To show the existence of $\hat{y}$, consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ind}(C) & \xrightarrow{\mathrm{colim}_C} & C \\ {\scriptstyle \mathrm{Ind}(u)}\downarrow\uparrow{\scriptstyle \mathrm{Ind}(v)} & & {\scriptstyle u}\downarrow\uparrow{\scriptstyle v} \\ \mathrm{Ind}(\mathrm{Ind}(D)) & \xrightarrow[\mathrm{colim}_{\mathrm{Ind}(D)}]{} & \mathrm{Ind}(D). \end{array}$$

It follows that colim_C is a retract of $\mathrm{colim}_{\mathrm{Ind}(D)}$. So we are done in light of Lemma A.22 in the appendix. □

Definition 3.9. Let C and D be compactly assembled. A functor $f\colon C \to D$ is called *strongly continuous* if it preserves filtered colimits and commutes with $\hat{y}$, meaning that the mate transformation

$$\hat{y}_D f \to \mathrm{Ind}(f)\hat{y}_C$$

of the commutative square

$$\begin{array}{ccc} \mathrm{Ind}(C) & \xrightarrow{\mathrm{colim}_C} & C \\ {\scriptstyle \mathrm{Ind}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Ind}(D) & \xrightarrow[\mathrm{colim}_D]{} & D \end{array}$$

is a natural isomorphism.

Remark 3.10. Assume f admits a right adjoint $g\colon D \to C$. Then the condition that f is strongly continuous is equivalent to g preserving filtered colimits, see Proposition A.27.

3.2 Dualizable categories

We now specialize to presentable stable categories. Recall from Section A.5 the tensor product on Pr^{L}, and that compactly generated categories are dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ by Corollary A.32.

Definition 3.11. A *dualizable category* is a dualizable object of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. We denote by $\mathrm{Pr}^{\mathrm{dual}} \subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the subcategory whose objects are the dualizable categories and whose morphisms are the internal left adjoints, i.e. those functors in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ whose right adjoint also lies in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, i.e. preserves colimits.

Explicitly, a presentable stable category C is dualizable if there exist a presentable stable category $C^\vee$ and colimit-preserving functors

$$\mathrm{ev}\colon C \otimes C^\vee \to \mathrm{Sp}, \qquad \mathrm{coev}\colon \mathrm{Sp} \to C^\vee \otimes C,$$

satisfying the triangle identities: the composites

$$C \xrightarrow{C\otimes\mathrm{coev}} C \otimes C^\vee \otimes C \xrightarrow{\mathrm{ev}\otimes C} C \qquad \text{and} \qquad C^\vee \xrightarrow{\mathrm{coev}\otimes C^\vee} C^\vee \otimes C \otimes C^\vee \xrightarrow{C^\vee\otimes\mathrm{ev}} C^\vee$$

are equivalent to the identity. Since $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is closed, the dual is then necessarily given by

$$C^\vee \simeq \mathrm{Fun}^{\mathrm{colim}}(C, \mathrm{Sp}),$$

the internal hom from C to the unit, with ev the evaluation functor.

Example 3.12. Let A be an $\mathbb{E}_1$-ring and let Mod_A denote the category of right A-modules, so that $\mathrm{Mod}_A \simeq \mathrm{Ind}(\mathrm{Perf}_A)$. Then Mod_A is dualizable with dual $\mathrm{Mod}_{A^{\mathrm{op}}}$, the category of left A-modules. The evaluation and coevaluation are given by

$$\mathrm{ev}\colon \mathrm{Mod}_A \otimes \mathrm{Mod}_{A^{\mathrm{op}}} \to \mathrm{Sp}, \quad (M,N) \mapsto M \otimes_A N, \qquad \mathrm{coev}\colon \mathrm{Sp} \to \mathrm{Mod}_{A^{\mathrm{op}}} \otimes \mathrm{Mod}_A \simeq \mathrm{BMod}_{A,A}, \quad \mathbb{S} \mapsto A,$$

where the coevaluation sends the sphere spectrum to A regarded as an A-bimodule. This is a special case of Corollary A.32, using the equivalence $\mathrm{Perf}_A^{\mathrm{op}} \simeq \mathrm{Perf}_{A^{\mathrm{op}}}$, $P \mapsto \hom_A(P, A)$.

Remark 3.13. Note that a right adjoint of a functor between stable categories automatically preserves finite colimits, so it preserves all colimits if and only if it preserves filtered colimits. So the morphisms in $\mathrm{Pr}^{\mathrm{dual}}$ are precisely the strongly continuous functors.

Remark 3.14. By Corollary A.32, every compactly generated category $C = \mathrm{Ind}(C_0)$ is dualizable. Moreover, every colimit-preserving functor preserving compact objects is strongly continuous, so a morphism in $\mathrm{Pr}^{\mathrm{dual}}$ by Remark 3.13. So we get a fully faithful inclusion

$$\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{dual}}.$$

Remark 3.15. Dualizable objects of a symmetric monoidal category are closed under tensor products, and the tensor product of two strongly continuous functors is again strongly continuous: since $- \otimes D$ is a functor of $(\infty, 2)$-categories, it preserves adjunctions, so the right adjoint of $f \otimes D$ is $f^R \otimes D$, which again preserves colimits. Hence the symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ restricts to $\mathrm{Pr}^{\mathrm{dual}}$. Moreover, the inclusion $\mathrm{Pr}^{\mathrm{dual}} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ preserves colimits (Proposition 7.2), and $\mathrm{Pr}^{\mathrm{dual}}$ admits all limits, although these are *not* computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$; see Section 7.4 for finite limits.

The following theorem is the main result of this chapter: it shows that dualizable categories are precisely the compactly assembled presentable stable categories.

Theorem 3.16 (Lurie, [Lur18, Proposition D.7.3.1])**.** *Let $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Then the following are equivalent:*

(1) *C is dualizable;*

(2) *C is compactly assembled;*

(3) *There exists a localization $D \to E$ in $\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{st}}$ such that $C = \ker(D \to E)$;*

(4) *C has the left lifting property with respect to any localization in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$: for every localization $D \to E$ and any $C \to E$, there exists a lift $C \to D$:*

$$\begin{array}{ccc} & & D \\ & \nearrow & \downarrow \\ C & \longrightarrow & E. \end{array}$$

The proof first connects compact assembly, dualizability and lifting through localizations. The remaining condition gives a concrete presentation: a compactly assembled category embeds strongly continuously in a compactly generated category with compactly generated quotient.

Proof. (2) $\implies$ (1): Every compactly assembled category is a retract of a compactly generated one. Remark 3.14 shows that compactly generated categories are dualizable, and since dualizable objects are closed under retracts we are done.

(1) $\implies$ (4): Assume C is dualizable, and consider a cofiber sequence

$$K \hookrightarrow D \to E$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Tensoring with $C^\vee$ preserves colimits, so gives another cofiber sequence

$$C^\vee \otimes K \to C^\vee \otimes D \to C^\vee \otimes E.$$

In particular, the functor $C^\vee \otimes D \to C^\vee \otimes E$ is again a localization, hence essentially surjective. It follows that the functor $\mathrm{Sp} \to C^\vee \otimes E$ classifying $C \to E$ lifts to a functor $\mathrm{Sp} \to C^\vee \otimes D$ classifying a lift $C \to D$.

(4) $\implies$ (2): Let $C_0 \subseteq C$ be a small dense stable subcategory, and consider the adjunction

$$y\colon C \rightleftarrows \mathrm{Ind}(C_0)\ :\mathrm{colim}\,.$$

Since y is fully faithful, it follows that $\mathrm{colim}\colon \mathrm{Ind}(C_0) \to C$ is a localization, and so by (4) this admits a section in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. It follows that C is a retract of $\mathrm{Ind}(C_0)$, hence it is compactly assembled.

(3) $\implies$ (2): Consider a fiber sequence

$$C \hookrightarrow D \to E.$$

Since $D \to E$ preserves compact objects, its right adjoint preserves colimits. By Lemma 2.13, the kernel inclusion $C \to D$ also has a colimit-preserving right adjoint $D \to C$. This right adjoint is a retraction, so C is a retract of D in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

(2) $\implies$ (3): As before, we may find some λ such that $\mathrm{colim}\colon \mathrm{Ind}(C^\lambda) \to C$ admits a left adjoint $C \to \mathrm{Ind}(C^\lambda)$. Then consider the cofiber sequence

$$C \xhookrightarrow{\hat{y}} \mathrm{Ind}(C^\lambda) \to \mathrm{cofib}(\hat{y}) \simeq \mathrm{fib}(\mathrm{colim}).$$

The localization functor $\mathrm{Ind}(C^\lambda) \to \mathrm{cofib}(\hat{y})$ admits a right adjoint in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and so it preserves compact objects. It in particular follows that $\mathrm{cofib}(\hat{y})$ is compactly generated, i.e. $\mathrm{cofib}(\hat{y}) \simeq \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\lambda(C))$ where $\mathrm{Calk}^{\mathrm{cont}}_\lambda(C) := \mathrm{cofib}(\hat{y})^\omega$ is the continuous λ-Calkin category. So this gives (3). □

There are several further useful characterizations. Recall that a presentable category C is *κ-compactly generated* if $\mathrm{Ind}_\kappa(C^\kappa) \simeq C$.

Proposition 3.17. *Let $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Then the following are equivalent:*

(1) *C is dualizable;*

(2) *The functor* $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ *preserves small products;*

(3) *For every regular cardinal κ such that C is κ-compactly generated, the functor* $\mathrm{colim}\colon \mathrm{Ind}(C^\kappa) \to C$ *admits a left adjoint;*

(4) *There exist a regular cardinal κ such that C is κ-compactly generated and the functor* $\mathrm{colim}\colon \mathrm{Ind}(C^\kappa) \to C$ *admits a left adjoint;*

(5) *There exists a fully faithful strongly continuous functor $C \hookrightarrow D$ into a compactly generated category D.*

Condition (2) is one formulation of Grothendieck's axiom (AB6), see Chapter 4.

Proof. (1) $\implies$ (2): By Theorem 3.16, colim admits a left adjoint $\hat{y}$, hence preserves all limits.

(2) $\implies$ (3): Let κ be such that C is κ-compactly generated, let $j\colon C^\kappa \hookrightarrow C$ be the inclusion and $F = \mathrm{Ind}(j)\colon \mathrm{Ind}(C^\kappa) \hookrightarrow \mathrm{Ind}(C)$. The functor F is fully faithful. Its right adjoint is restriction of

presheaves along j: for $X \in \mathrm{Ind}(C)$, the presheaf $d \mapsto \mathrm{Map}_{\mathrm{Ind}(C)}(y(d), X)$ on C^κ preserves finite limits, so belongs to $\mathrm{Ind}(C^\kappa)$. Here we use the description of the Ind-completion of a category with finite colimits as finite-limit-preserving presheaves. This constructs F^R without a presentability assumption on the large category $\mathrm{Ind}(C)$. Since representables are compact, this restriction functor preserves filtered colimits. We claim that

$$\mathrm{colim}_C \simeq \mathrm{colim}_\kappa \circ F^R,$$

where $\mathrm{colim}_\kappa \colon \mathrm{Ind}(C^\kappa) \to C$ denotes the colimit functor. Both sides preserve filtered colimits, so it suffices to compare them on objects $y(c)$ with $c \in C$, naturally in c. For $d \in C^\kappa$ we have $\mathrm{Map}(y(d), F^R y(c)) \simeq \mathrm{Map}_C(d, c)$, so $F^R y(c)$ is the ind-object "colim"$_{(d \to c) \in (C^\kappa)_{/c}} d$; note that $(C^\kappa)_{/c}$ is κ-filtered with colimit c, as $C \simeq \mathrm{Ind}_\kappa(C^\kappa)$. Hence $\mathrm{colim}_\kappa F^R y(c) \simeq c \simeq \mathrm{colim}_C y(c)$, naturally in c.

Now let $(X_s)_{s \in S}$ be a family of objects of $\mathrm{Ind}(C^\kappa)$. Since F^R preserves products and $F^R F \simeq \mathrm{id}$, we have $\prod_s X_s \simeq F^R(\prod_s F X_s)$, and hence

$$\mathrm{colim}_\kappa \left(\prod_s X_s \right) \simeq \mathrm{colim}_C \left(\prod_s F X_s \right) \simeq \prod_s \mathrm{colim}_C(F X_s) \simeq \prod_s \mathrm{colim}_\kappa(X_s),$$

using (2). So colim_κ preserves products. It is moreover an exact functor between stable categories, so it preserves finite limits, and hence all small limits. Since it also preserves filtered colimits, it admits a left adjoint by the adjoint functor theorem (Theorem A.19).

(3) $\implies$ (4): Every presentable category is κ-compactly generated for some κ.

(4) $\implies$ (5): Let $\hat{y} \colon C \to \mathrm{Ind}(C^\kappa)$ be the left adjoint of colim_κ, and let F be as in the proof of (2) $\implies$ (3). Since $\mathrm{colim}_C \circ F$ preserves filtered colimits and restricts to the inclusion on C^κ, we have $\mathrm{colim}_C \circ F \simeq \mathrm{colim}_\kappa$, and hence

$$\mathrm{Map}_{\mathrm{Ind}(C^\kappa)}(X, F^R y(c)) \simeq \mathrm{Map}_{\mathrm{Ind}(C)}(FX, y(c)) \simeq \mathrm{Map}_C(\mathrm{colim}_C FX, c) \simeq \mathrm{Map}_C(\mathrm{colim}_\kappa X, c).$$

So colim_κ admits the right adjoint $F^R \circ y$, which is fully faithful since $\mathrm{colim}_\kappa F^R y(c) \simeq c$ as shown above. Hence $\hat{y}$ is fully faithful by Lemma A.26. Its right adjoint colim_κ preserves colimits, so $\hat{y}$ is strongly continuous, and $\mathrm{Ind}(C^\kappa)$ is compactly generated.

(5) $\implies$ (1): Let $i \colon C \hookrightarrow D$ be as in (5). Then $i^R i \simeq \mathrm{id}_C$ and i^R preserves colimits, so C is a retract of D in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Since D is dualizable (Corollary A.32) and dualizable objects are closed under retracts, C is dualizable. □

Remark 3.18. Further characterizations of dualizable categories will be given later in these notes:

- C is equivalent to $\ker(\mathrm{Mod}_A \to \mathrm{Mod}_B)$ for a homological epimorphism of $\mathbb{E}_1$-rings $A \to B$ (Chapter 5);
- C is generated under colimits by *compactly exhaustible* objects, i.e. sequential colimits along compact morphisms (Proposition 4.14);
- C satisfies (AB6), i.e. filtered colimits distribute over products in C (Proposition 4.21);
- C is *flat*, i.e. $C \otimes -$ preserves fully faithful functors in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (Chapter 10).

3.3 Localization sequences of dualizable categories

Localization sequences are the main tool for computing continuous K-theory, so we need to understand how dualizability behaves in them. One direction is easy: dualizability passes from the middle term of a strongly continuous localization sequence to the two outer terms. The converse fails; the gluing criterion and a counterexample are developed in Section 5.4, after the compact-morphism criteria needed for their proofs.

Proposition 3.19. *Let $K \xrightarrow{i} D \xrightarrow{p} E$ be a strongly continuous localization sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (Definition 2.11). If D is dualizable, then so are K and E, and the sequence is a sequence in $\mathrm{Pr}^{\mathrm{dual}}$.*

Proof. Since i and p^R are fully faithful, we have $i^R i \simeq \mathrm{id}_K$ and $pp^R \simeq \mathrm{id}_E$. By strong continuity, i^R and p^R preserve colimits, so K and E are retracts of D in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, hence dualizable. The functors i and p are strongly continuous by assumption. □

3.4 ω_1-presentability

A striking consequence of compact assembly is that it bounds the size of the generators: every compactly assembled category is generated by its ω_1-compact objects, which moreover admit an explicit description. Recall the natural transformation $\hat{y} \to y$ from Section 3.1.

Theorem 3.20 (Efimov, [Efi24, Corollary 2.13 and Proposition 2.16])**.** *Let C be compactly assembled.*

(1) *C is ω_1-compactly generated.*

(2) *An object x of C is ω_1-compact if and only if it can be written as*

$$x = \mathrm{colim}\left(\cdots \to x_n \to x_{n+1} \to \dots\right)$$

where each map $y(x_n) \to y(x_{n+1})$ factors as

$$y(x_n) \to \hat{y}(x_{n+1}) \to y(x_{n+1}).$$

The proof has two steps. First, interleaving identifies the finest presentation of a sequential colimit with a countable colimit of representables, giving the criterion in (2). Then countable directed pieces of an arbitrary presentation give the ω_1-filtered presentation required in (1).

Proof. We first prove (2). For $\Longleftarrow$, write $x = \mathrm{colim}_n x_n$. Then $\hat{y}(x) = \mathrm{colim}_n \hat{y}(x_n) = \mathrm{colim}_n y(x_n)$, where the last equality holds by considering

$$\begin{array}{ccccccc} \dots & \longrightarrow & y(x_n) & \longrightarrow & y(x_{n+1}) & \longrightarrow & \dots \\ & & \uparrow & \searrow & \uparrow & & \\ \dots & \longrightarrow & \hat{y}(x_n) & \longrightarrow & \hat{y}(x_{n+1}) & \longrightarrow & \dots \end{array}$$

filled with the given maps $y(x_n) \to \hat{y}(x_{n+1})$. The top triangle commutes by construction. The fact that we can also make the bottom triangle commute in a way that makes the total square agree with

the canonical one follows from the adjunction $\hat{y} \dashv \operatorname{colim}$: after applying the adjunction, this square looks like

$$\begin{array}{ccc} x_n & \longrightarrow & x_{n+1} \\ \| & & \| \\ x_n & \longrightarrow & x_{n+1}. \end{array}$$

Now, each $y(x_n)$ is compact in $\mathrm{Ind}(C)$, and in particular ω_1-compact, so also the colimit $\hat{y}(x) = \operatorname{colim}_n y(x_n)$ is ω_1-compact. But since $\hat{y}$ is fully faithful and preserves filtered colimits, this implies that x is already ω_1-compact in C.

For $\Longrightarrow$, consider $x \in C$.

Observation: We may write $\hat{y}(x) = \operatorname{colim}_{i \in I} y(x_i)$ for I a filtered poset. (Any filtered category admits a cofinal map from a filtered poset, see Theorem A.8.) By applying $\operatorname{colim}\colon \mathrm{Ind}(C) \to C$, we get $x = \operatorname{colim}_{i \in I} x_i$. Since $\hat{y}$ preserves colimits, we get $\hat{y}(x) = \operatorname{colim}_{i \in I} \hat{y}(x_i)$. Since each $y(x_i)$ is compact, it follows that for each $i \in I$ there exists some $j \geq i$ such that the map $y(x_i) \to y(x_j)$ factors through $\hat{y}(x_j) \to y(x_j)$.

Now, assume x is ω_1-compact. Then also $\hat{y}(x)$ is ω_1-compact in $\mathrm{Ind}(C)$: by adjunction this follows because its right adjoint colim preserves ω_1-filtered colimits. Thus we may write $\hat{y}(x) = \operatorname{colim}_n y(x_n)$ by Proposition A.13 (applied to $\mathrm{Ind}(C^\lambda) \subseteq \mathrm{Ind}(C)$ for λ large enough that $\hat{y}(x) \in \mathrm{Ind}(C^\lambda)$, see Construction A.21). By the previous observation we see that for each n there exists some $k \geq n$ such that the map $y(x_n) \to y(x_k)$ factors through $\hat{y}(x_k)$. By reindexing, we may assume $k = n + 1$, giving the claim.

Finally, we prove (1). As in the proof of Theorem 3.8, choose a regular cardinal λ such that $\hat{y}(C) \subseteq \mathrm{Ind}(C^\lambda)$, with C^λ essentially small. For $x \in C$, choose a presentation $\hat{y}(x) = \operatorname{colim}_{i \in I} y(x_i)$ with I a small filtered poset and $x_i \in C^\lambda$. By the observation above, choose for each i an element $s(i) \geq i$ such that $y(x_i) \to y(x_{s(i)})$ factors through $\hat{y}(x_{s(i)})$.

Let Σ_s be the poset of nonempty countable directed subsets $Q \subseteq I$ closed under s, ordered by inclusion. Every countable subset of I is contained in such a Q: start with that subset and one element of I, and repeatedly add the chosen successors and upper bounds for every pair. The union after countably many steps has the required properties. Applying this to countable unions shows that Σ_s is ω_1-filtered and cofinal in the poset of all countable directed subsets. By Proposition A.9,

$$x \simeq \operatorname{colim}_{Q \in \Sigma_s} \operatorname{colim}_{i \in Q} x_i.$$

For each Q, enumerate its elements and choose a cofinal sequence $i_0 \leq i_1 \leq \cdots$ in Q with $i_{n+1} \geq s(i_n)$, by successively taking upper bounds with the next enumerated element. The transition $y(x_{i_n}) \to y(x_{i_{n+1}})$ factors through $\hat{y}(x_{i_{n+1}})$, using the chosen factorization at $s(i_n)$ and naturality of $\hat{y} \to y$. Thus $\operatorname{colim}_{i \in Q} x_i$ is ω_1-compact by (2).

This writes every object as an ω_1-filtered colimit of ω_1-compact objects. These objects may be chosen from the essentially small collection of colimits of sequences in C^λ. Every ω_1-compact object is a retract of one of them, by factoring its identity through this presentation. Since C is locally small, the collection of these retracts is essentially small as well. Consequently C is ω_1-compactly generated. □

For dualizable categories, there is an alternative and perhaps more intuitive argument for part (1) of Theorem 3.20, using the presentation as a kernel.

Alternative proof of Theorem 3.20(1) for C dualizable. By Theorem 3.16, we may write $C = \ker(L\colon D \to E)$ for a localization L between compactly generated categories which preserves compact objects. Since $C \subseteq D$ is closed under colimits, every object of C which is ω_1-compact in D is ω_1-compact in C.

Let $x \in C$, and write $x = \operatorname{colim}_{p\in P} d_p$ in D with P a filtered partially ordered set and $d_p \in D^\omega$ (Theorem A.8). Then $\operatorname{colim}_p L(d_p) \simeq L(x) \simeq 0$, and each $L(d_p)$ is compact in E. Hence for every $p \in P$ there is some $p' \geq p$ such that the map $L(d_p) \to L(d_{p'})$ is nullhomotopic. Let $\Sigma' \subseteq \Sigma$ be the set of countable filtered subsets $Q \subseteq P$ such that for every $q \in Q$ there is some $q' \in Q$ with $q' \geq q$ and $L(d_q) \to L(d_{q'})$ nullhomotopic. Arguing as in the proof of Proposition A.9, but alternately adding upper bounds and such witnesses q', one sees that every countable collection of elements of Σ has an upper bound in Σ'. Hence Σ' is ω_1-filtered and cofinal in Σ, so by Proposition A.9 and Lemma A.7 we get

$$x \simeq \operatorname{colim}_{Q\in\Sigma'} x_Q, \qquad x_Q := \operatorname{colim}_{q\in Q} d_q.$$

For $Q \in \Sigma'$ we have $L(x_Q) \simeq \operatorname{colim}_{q\in Q} L(d_q) \simeq 0$, since every transition map is eventually nullhomotopic, so $x_Q \in C$. Moreover, x_Q is a countable colimit of compact objects, hence ω_1-compact in D (Corollary A.4), and hence in C. So every object of C is an ω_1-filtered colimit of ω_1-compact objects. □

Remark 3.21. By Lemma 3.5, the condition on the transition maps in Theorem 3.20(2) says precisely that the maps $x_n \to x_{n+1}$ are compact morphisms. So the ω_1-compact objects of a compactly assembled category are precisely the sequential colimits along compact morphisms, and every object is an ω_1-filtered colimit of such. In Chapter 4 we will see that, conversely, a presentable stable category is dualizable if and only if it is generated under colimits by such sequential colimits. While compact *objects* are the right tool to study compactly generated categories, compact *morphisms* are the right tool to study dualizable ones.

As a first illustration of this principle, strongly continuous functors between dualizable categories can be detected using compact morphisms, just like functors between compactly generated categories are strongly continuous if and only if they preserve compact objects.

Proposition 3.22. *Let $F\colon C \to D$ be a colimit-preserving functor between dualizable categories. Then F is strongly continuous if and only if F preserves compact morphisms.*

Proof. Write $\varepsilon\colon \hat{y} \to y$ for the natural transformation from Section 3.1, and let $\beta\colon \hat{y}_D F \to \operatorname{Ind}(F)\hat{y}_C$ be the mate from Definition 3.9. By Proposition A.27, F is strongly continuous if and only if β is an equivalence. We will repeatedly use the following observation: a map $\hat{y}_D(z) \to Y$ in $\operatorname{Ind}(D)$ is determined, via the adjunction $\hat{y}_D \dashv \operatorname{colim}$, by its image under colim (composed with the identification $z \simeq \operatorname{colim} \hat{y}_D(z)$). Applying colim to ε_{Fx}, to $\operatorname{Ind}(F)(\varepsilon_x) \circ \beta_x$, and to β_x yields the identity of Fx in each case (up to the canonical identifications), by construction of ε and of the mate β. Hence

$$\varepsilon_{Fx} \simeq \operatorname{Ind}(F)(\varepsilon_x) \circ \beta_x\colon \hat{y}_D(Fx) \to y(Fx). \tag{3.1}$$

Suppose first that F is strongly continuous, and let $f\colon x \to x'$ be a compact morphism in C. By Lemma 3.5, there is $g\colon y(x) \to \hat{y}_C(x')$ with $\varepsilon_{x'} g \simeq y(f)$. Then $g' := \beta_{x'}^{-1} \circ \operatorname{Ind}(F)(g)\colon y(Fx) \to \hat{y}_D(Fx')$ satisfies

$$\varepsilon_{Fx'} \circ g' \simeq \operatorname{Ind}(F)(\varepsilon_{x'}) \circ \operatorname{Ind}(F)(g) \simeq \operatorname{Ind}(F)(y(f)) = y(Ff)$$

by (3.1), so Ff is compact by Lemma 3.5.

Conversely, suppose that F preserves compact morphisms. Both functors $\hat{y}_D F$ and $\mathrm{Ind}(F)\hat{y}_C$ preserve colimits: $\hat{y}_C$ and $\hat{y}_D$ are left adjoints, and $\mathrm{Ind}(F)$ is an exact functor between stable categories which preserves filtered colimits. Since C is generated under colimits by its ω_1-compact objects (Theorem 3.20(1)), it suffices to show that β_x is an equivalence for x ω_1-compact. By Theorem 3.20(2) and Lemma 3.5, we may write $x \simeq \mathrm{colim}(x_0 \to x_1 \to \cdots)$ with compact transition maps, and the proof of " $\Longleftarrow$ " in Theorem 3.20(2) shows that the canonical map

$$\hat{y}_C(x) \simeq \mathrm{colim}_n\, \hat{y}_C(x_n) \xrightarrow{\mathrm{colim}_n\, \varepsilon_{x_n}} \mathrm{colim}_n\, y(x_n)$$

is an equivalence in $\mathrm{Ind}(C)$. Since F preserves colimits and compact morphisms, $Fx \simeq \mathrm{colim}_n\, Fx_n$ with compact transition maps, so likewise $\hat{y}_D(Fx) \to \mathrm{colim}_n\, y(Fx_n)$ is an equivalence. Now consider the composite

$$\hat{y}_D(Fx) \xrightarrow{\beta_x} \mathrm{Ind}(F)(\hat{y}_C(x)) \xrightarrow{\simeq} \mathrm{Ind}(F)\big(\mathrm{colim}_n\, y(x_n)\big) \simeq \mathrm{colim}_n\, y(Fx_n).$$

Applying colim to it gives the identity of Fx (up to the canonical identifications), so by the observation above it agrees with the canonical equivalence $\hat{y}_D(Fx) \to \mathrm{colim}_n\, y(Fx_n)$. Hence β_x is an equivalence. □

4 Compact morphisms and the axiom (AB6)

In compactly generated categories, compact objects control everything: they generate the category, and maps out of them into filtered colimits factor through a finite stage. In dualizable categories there may be no nonzero compact objects at all, and their role is taken over by *compact morphisms*, which we introduced in Section 3.1. We develop their factorization properties and use them to characterize dualizability.

The argument proceeds from individual maps to whole categories: the lifting criteria lead to interpolation, compact exhaustions give generators, and (AB6) packages the result as a commutation property of products and filtered colimits. The final section uses these criteria to prove automatic dualizability.

Throughout, for a category C with filtered colimits we write $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ for the colimit functor and $\varepsilon\colon \hat{y} \to y$ for the natural transformation of Section 3.1, whenever C is compactly assembled. (In the talk, as in [KNP24], the colimit functor was denoted k.) The talk mostly worked with dualizable categories. Several of the results below hold more generally for compactly assembled categories, and we state them in this generality whenever the proof is the same; see [KNP24, Sections 2.2–2.4] for a systematic treatment of the non-stable case.

4.1 Weakly compact morphisms

Our definition of compact morphisms (Definition 3.3) asks for a factorization which is natural in the filtered diagram. In the talk, as in [Efi24, Definition 2.37], compact morphisms were instead defined by asking for such factorizations one map at a time. We now show that the two notions agree in compactly assembled categories, so that both definitions may be used interchangeably in these notes.

Definition 4.1 ([KNP24, Definition 2.2.3])**.** Let C be a category with filtered colimits. A morphism $f\colon x \to x'$ in C is called *weakly compact* if for every filtered diagram $(z_i)_{i\in I}$ in C and every morphism $g\colon x' \to \mathrm{colim}_i z_i$, the composite $gf\colon x \to \mathrm{colim}_i z_i$ factors through the canonical map $z_i \to \mathrm{colim}_i z_i$ for some $i \in I$.

Remark 4.2. Since $y(x)$ is compact in $\mathrm{Ind}(C)$, maps $y(x) \to$ "colim"$_i z_i$ are the same as maps $x \to z_i$ for some i, up to the usual identifications. Hence f is weakly compact if and only if for

every $Z \in \mathrm{Ind}(C)$, every diagram of solid arrows

$$\begin{array}{ccc} y(x) & \dashrightarrow & Z \\ {\scriptstyle y(f)}\downarrow & & \downarrow{\scriptstyle \eta_Z} \\ y(x') & \longrightarrow & y(\mathrm{colim}\, Z) \end{array}$$

admits a dashed arrow making the square commute. Here η_Z is the unit of the adjunction $\mathrm{colim} \dashv y$ of Construction A.21, and a map $y(x') \to y(\mathrm{colim}\, Z)$ is the same as a map $x' \to \mathrm{colim}\, Z$, since y is fully faithful.

A compact morphism in the sense of Definition 3.3 is weakly compact: the factorization of f^* through $\mathrm{Map}_{\mathrm{Ind}(C)}(y(x), -)$ provides, in particular, a lift of each point. In the terminology of [KNP24, Definition 2.2.4], compact morphisms in our sense are called *strongly compact*.

Example 4.3.

(1) If x is compact, then id_x is weakly compact. The converse holds for stable C, see Corollary 4.12 below.

(2) As in Remark 3.4, weakly compact morphisms form a two-sided ideal: if $f\colon x \to x'$ is weakly compact and $k\colon w \to x$, $h\colon x' \to x''$ are arbitrary, then for every $g\colon x'' \to \mathrm{colim}_i\, z_i$ the composite $(gh)f$ factors through some z_i, and hence so does $ghfk$.

(3) In particular, every morphism which factors through a compact object is weakly compact.

Proposition 4.4 ([Efi24, Proposition 2.39], [KNP24, Proposition 2.4.1])**.** *Let C be compactly assembled and let $f\colon x \to x'$ be a morphism in C. Then the following are equivalent:*

(1) *f is compact (in the sense of Definition 3.3);*

(2) *f is weakly compact;*

(3) *$y(f)\colon y(x) \to y(x')$ factors through $\varepsilon_{x'}\colon \hat{y}(x') \to y(x')$;*

(4) *$\hat{y}(f)\colon \hat{y}(x) \to \hat{y}(x')$ factors through $\varepsilon_x\colon \hat{y}(x) \to y(x)$;*

(5) *$\hat{y}(f)$ is a weakly compact morphism in* $\mathrm{Ind}(C)$.

Proof. We use repeatedly that maps $\hat{y}(w) \to Z$ in $\mathrm{Ind}(C)$ correspond, under the adjunction $\hat{y} \dashv \mathrm{colim}$, to maps $w \to \mathrm{colim}\, Z$; in particular, two maps $\hat{y}(w) \to Z$ are homotopic as soon as their images under colim are, where we identify $\mathrm{colim}\, \hat{y}(w) \simeq w$. Similarly, two maps $y(w) \to y(w')$ are homotopic as soon as their images under colim are, since y is fully faithful. Recall also that $\mathrm{colim}(\varepsilon_w) \simeq \mathrm{id}_w$.

(1) $\iff$ (3) is Lemma 3.5(1), and (1) $\implies$ (2) was observed in Remark 4.2.

(2) $\implies$ (3): Write $\hat{y}(x') \simeq$ "colim"$_i z_i$. Then $\mathrm{colim}_i\, z_i \simeq \mathrm{colim}\, \hat{y}(x') \simeq x'$. Applying weak compactness to the identity $x' \to \mathrm{colim}_i\, z_i$, we find a map $h\colon x \to z_i$ for some i such that the composite $x \to z_i \to \mathrm{colim}_i\, z_i \simeq x'$ is homotopic to f. Let $g\colon y(x) \to \hat{y}(x')$ be the composite of $y(h)$ with the canonical map $y(z_i) \to$ "colim"$_i z_i$. Then $\mathrm{colim}(\varepsilon_{x'} g) \simeq \mathrm{colim}(g) \simeq f$, and hence $\varepsilon_{x'} g \simeq y(f)$.

(3) $\implies$ (4): Let $g\colon y(x) \to \hat{y}(x')$ with $\varepsilon_{x'} g \simeq y(f)$. Then $g\varepsilon_x$ and $\hat{y}(f)$ are maps $\hat{y}(x) \to \hat{y}(x')$ with $\operatorname{colim}(g\varepsilon_x) \simeq \operatorname{colim}(g) \simeq f \simeq \operatorname{colim}(\hat{y}(f))$, so $\hat{y}(f) \simeq g\varepsilon_x$.

(4) $\implies$ (5): The object $y(x)$ is compact in $\operatorname{Ind}(C)$, so this follows from Example 4.3(3).

(5) $\implies$ (2): Let $g\colon x' \to \operatorname{colim}_i z_i$ be a map in C, and let $Z = \text{“colim”}_i z_i = \operatorname{colim}_i y(z_i)$. By adjunction, g corresponds to a map $\tilde{g}\colon \hat{y}(x') \to Z$ with $\operatorname{colim}(\tilde{g}) \simeq g$. By (5), the composite $\tilde{g}\hat{y}(f)\colon \hat{y}(x) \to \operatorname{colim}_i y(z_i)$ factors through some $y(z_i)$, say as $\hat{y}(x) \xrightarrow{u} y(z_i) \to Z$. Applying colim, we find that $gf \simeq \operatorname{colim}(\tilde{g}\hat{y}(f))$ factors as $x \xrightarrow{\operatorname{colim}(u)} z_i \to \operatorname{colim}_i z_i$. □

Remark 4.5. In (5), weak compactness in $\operatorname{Ind}(C)$ may be replaced by weak compactness in $\operatorname{Ind}(C^\kappa)$, for any κ such that $\hat{y}$ takes values in $\operatorname{Ind}(C^\kappa)$, by the same argument. In particular, if C is dualizable we may take $\kappa = \omega_1$ (Theorem 3.20), and then $\operatorname{Ind}(C^{\omega_1})$ is compactly generated.

From now on we simply speak of *compact morphisms* in compactly assembled categories, and use the characterizations of Proposition 4.4 freely. The following observation is fundamental; it says that compact morphisms can be “interpolated”.

Corollary 4.6 ([Efi24, Corollary 2.40], [KNP24, Proposition 2.4.4])**.** *Let C be compactly assembled. Then every compact morphism $f\colon x \to x'$ factors as a composite $x \xrightarrow{f_1} w \xrightarrow{f_2} x'$ of two compact morphisms.*

Proof. By Proposition 4.4, there is $g\colon y(x) \to \hat{y}(x')$ with $\varepsilon_{x'} g \simeq y(f)$. Write $\hat{y}(x') \simeq \text{“colim”}_i w_i$, so that $x' \simeq \operatorname{colim}_i w_i$. Since $\hat{y}$ preserves colimits, also $\hat{y}(x') \simeq \operatorname{colim}_i \hat{y}(w_i)$ in $\operatorname{Ind}(C)$. As $y(x)$ is compact in $\operatorname{Ind}(C)$, the map g factors as

$$y(x) \xrightarrow{h} \hat{y}(w_i) \xrightarrow{\hat{y}(f_2)} \hat{y}(x')$$

for some i, where $f_2\colon w_i \to x'$ is the canonical map. Put $w := w_i$ and $f_1 := \operatorname{colim}(h)\colon x \to w$. Then $\varepsilon_w h$ and $y(f_1)$ have the same image under colim, so $\varepsilon_w h \simeq y(f_1)$, and f_1 is compact by Proposition 4.4(3). The map $y(w) \to \text{“colim”}_j w_j = \hat{y}(x')$ is a lift of $y(f_2)$ along $\varepsilon_{x'}$, since both become f_2 after applying colim; so f_2 is compact as well. Finally, $f \simeq \operatorname{colim}(g) \simeq \operatorname{colim}(\hat{y}(f_2)) \operatorname{colim}(h) \simeq f_2 f_1$. □

By iterating Corollary 4.6, a single compact morphism can be “interpolated” by a dense family of compact morphisms. Since this is also needed for other classes of morphisms (for instance trace-class morphisms, see Proposition 6.19), we formulate it abstractly. We write $\mathbb{Q}_{[0,1]} = \mathbb{Q} \cap [0,1]$ and $\mathbb{Q}_{\geq 0} = \mathbb{Q} \cap [0,\infty)$, regarded as partially ordered sets.

Lemma 4.7. *Let D be a category, let S be a class of morphisms of D which is closed under homotopy and composition, and let P be a class of objects of D. Assume that every morphism in S is homotopic to a composite $x \to p \to x'$ of two morphisms in S with $p \in P$.*

(1) *For every morphism $s\colon x \to x'$ in S there is a functor $F\colon \mathbb{Q}_{[0,1]} \to D$ with $F(0) = x$, $F(1) = x'$ and $F(0 \leq 1) \simeq s$, such that $F(a \leq b) \in S$ for all $a < b$ and $F(a) \in P$ for all $0 < a < 1$.*

(2) *For every sequence $x_0 \to x_1 \to x_2 \to \cdots$ of morphisms in S there is a functor $F\colon \mathbb{Q}_{\geq 0} \to D$ whose restriction to $\mathbb{N} \subseteq \mathbb{Q}_{\geq 0}$ is equivalent to the given sequence, such that $F(a \leq b) \in S$ for*

all $a < b$ and $F(a) \in P$ for all $a \notin \mathbb{N}$. If D admits sequential colimits, then $\operatorname{colim}_{\mathbb{Q}_{\geq 0}} F \simeq \operatorname{colim}_n x_n$.

Proof. (1) The set T of dyadic rationals in $[0, 1]$ is a countable dense totally ordered set with a smallest and a largest element, so it is order-isomorphic to $\mathbb{Q}_{[0,1]}$ by Cantor's back-and-forth argument; we may therefore replace $\mathbb{Q}_{[0,1]}$ by T, with 0 and 1 preserved. Let $T_k = \{i/2^k \mid 0 \leq i \leq 2^k\}$, so that $T = \bigcup_k T_k$ and hence $N(T) = \bigcup_k N(T_k)$. Thus $\operatorname{Fun}(T, D) \simeq \lim_k \operatorname{Fun}(T_k, D)$, and on cores this is a limit of a tower of animae. So it suffices to construct functors $F_k \colon T_k \to D$, with $F_k(a \leq b) \in S$ for $a < b$ and $F_k(a) \in P$ for $0 < a < 1$, together with equivalences $F_{k+1}|_{T_k} \simeq F_k$. Indeed, a point of the limit of a tower of animae is specified by points at each stage and paths between their successive images; no separate higher compatibility choices are required. As in Remark A.10, a functor out of a finite totally ordered set amounts to a composable sequence of morphisms, and two such functors are equivalent as soon as they agree on objects and their consecutive morphisms are homotopic. Let F_0 be given by s. Given F_k, factor each morphism $F_k(a \leq a')$ between consecutive elements $a < a'$ of T_k as a composite of two morphisms in S through an object of P, and let F_{k+1} be the resulting functor on T_{k+1}. Then $F_{k+1}|_{T_k}$ and F_k agree on objects, and their consecutive morphisms are homotopic, so $F_{k+1}|_{T_k} \simeq F_k$. Since S is closed under composition, every $F_{k+1}(a \leq b)$ with $a < b$ is in S.

(2) Apply the construction of (1) simultaneously to all morphisms $x_n \to x_{n+1}$: on the dyadic rationals in $[0, \infty)$, which are the union of the subsets $2^{-k}\,\mathbb{N}$, we refine the given sequence on $\mathbb{N} = 2^0\,\mathbb{N}$ by factoring consecutive morphisms as before. The set of dyadic rationals in $[0, \infty)$ is order-isomorphic to $\mathbb{Q}_{\geq 0}$ by an isomorphism fixing $\mathbb{N}$ (apply the back-and-forth argument to each interval $[n, n + 1]$). Finally, $\mathbb{N} \subseteq \mathbb{Q}_{\geq 0}$ is cofinal, so the colimits agree by Lemma A.7. □

Proposition 4.8 ([KNP24, Propositions 2.4.4 and 2.4.6])**.** *Let C be compactly assembled.*

(1) *Every compact morphism $f \colon x \to x'$ factors as $x \to w \to x'$ with both morphisms compact and w an ω_1-compact object.*

(2) *Every compact morphism $f \colon x \to x'$ extends to a functor $F \colon \mathbb{Q}_{[0,1]} \to C$ with $F(0 \leq 1) \simeq f$, such that $F(a \leq b)$ is compact for all $a < b$ and $F(a)$ is ω_1-compact for $0 < a < 1$.*

(3) *An object of C is ω_1-compact if and only if it is the colimit of a functor $F \colon \mathbb{Q}_{\geq 0} \to C$, or equivalently of a functor $F \colon \mathbb{Q} \to C$, such that $F(a \leq b)$ is compact for all $a < b$. Moreover, F can be chosen to take values in ω_1-compact objects.*

Proof. (1) By Corollary 4.6, applied twice, f factors as $x \to x_0 \to x_0' \to x'$ with all three maps compact. By Theorem 3.20(1), x_0' is a filtered colimit of ω_1-compact objects, so the compact (hence weakly compact, by Proposition 4.4) morphism $x_0 \to x_0'$ factors as $x_0 \to w \to x_0'$ with w ω_1-compact. Since compact morphisms form an ideal, $x \to x_0 \to w$ and $w \to x_0' \to x'$ are compact.

(2) Apply Lemma 4.7(1) to the class S of compact morphisms and the class P of ω_1-compact objects, using (1).

(3) By Theorem 3.20(2), Remark 3.21 and Proposition 4.4, an object x is ω_1-compact if and only if $x \simeq \operatorname{colim}_n x_n$ for a sequence of compact morphisms $x_n \to x_{n+1}$. By (1), each $x_n \to x_{n+1}$ factors as $x_n \to w_n \to x_{n+1}$ through an ω_1-compact w_n, with both maps compact. The sequence

$w_0 \to w_1 \to \cdots$ of composites $w_n \to x_{n+1} \to w_{n+1}$ again consists of compact morphisms between ω_1-compact objects, and has colimit x by cofinality. Applying Lemma 4.7(2) to it (with P the ω_1-compact objects) gives a functor $F\colon \mathbb{Q}_{\geq 0} \to C$ with the required properties and colimit x. Since $\mathbb{Q}_{>0} \subseteq \mathbb{Q}_{\geq 0}$ is cofinal and $\mathbb{Q}_{>0}$ is order-isomorphic to $\mathbb{Q}$ (both are countable dense totally ordered sets without endpoints), we may equivalently use $\mathbb{Q}$. Conversely, the colimit of such a functor is the colimit of its restriction to the cofinal subset $\mathbb{N}$, which is ω_1-compact by Theorem 3.20(2). □

For stable categories there is a useful reformulation of compactness, which only involves ind-objects with vanishing colimit. This is the definition of compact morphisms used by Clausen and Mathew.

Proposition 4.9. *Let C be a stable category with filtered colimits (e.g. $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$), and let $f\colon x \to x'$ be a morphism in C. Then f is weakly compact if and only if for every $Z \in \mathrm{Ind}(C)$ with $\operatorname{colim} Z \simeq 0$, the map*

$$f^*\colon \pi_0 \operatorname{Map}_{\mathrm{Ind}(C)}(y(x'), Z) \to \pi_0 \operatorname{Map}_{\mathrm{Ind}(C)}(y(x), Z)$$

is zero. Explicitly, this means: for every filtered diagram $(z_i)_{i\in I}$ with $\operatorname{colim}_i z_i \simeq 0$ and every map $g\colon x' \to z_i$, there is $i' \geq i$ such that the composite $x \xrightarrow{f} x' \xrightarrow{g} z_i \to z_{i'}$ is nullhomotopic.

Proof. Note that $\mathrm{Ind}(C)$ is stable, and that filtered colimits in C as well as the colimit functor $\mathrm{Ind}(C) \to C$ are exact.

Suppose that f is weakly compact, and let $Z = \text{“colim”}_{i\in I} z_i$ with $\operatorname{colim}_i z_i \simeq 0$; by Theorem A.8 we may assume that I is a filtered partially ordered set. A map $y(x') \to Z$ is represented by some $g\colon x' \to z_{i_0}$. For $i \geq i_0$ write $g_i\colon x' \to z_i$ for the composite with the transition map, and $w_i := \mathrm{fib}(g_i)$. As $\{i \geq i_0\} \subseteq I$ is cofinal and filtered colimits are exact, the canonical map

$$\operatorname{colim}_{i\geq i_0} w_i \to \mathrm{fib}\left(x' \to \operatorname{colim}_{i\geq i_0} z_i\right) \simeq x'$$

is an equivalence. Applying weak compactness of f to the inverse of this equivalence, we find $i \geq i_0$ and a map $b\colon x \to w_i$ such that f is homotopic to the composite of b with $w_i \to x'$. Hence $g_i f$ factors through $g_i \circ (w_i \to x') \simeq 0$, so $g_i f \simeq 0$, and the image of our map under f^* vanishes.

Conversely, suppose the condition holds, and let $g\colon x' \to \operatorname{colim}_i z_i$. Put $Z = \text{“colim”}_i z_i$, let $\eta_Z\colon Z \to y(\operatorname{colim} Z)$ be the unit of $\operatorname{colim} \dashv y$, and let $W = \mathrm{cofib}(\eta_Z)$. Then $\operatorname{colim} W \simeq \mathrm{cofib}(\operatorname{colim}(\eta_Z)) \simeq 0$, since $\operatorname{colim}(\eta_Z)$ is an equivalence. In the exact sequence of abelian groups

$$\pi_0 \operatorname{Map}(y(x), Z) \xrightarrow{(\eta_Z)_*} \pi_0 \operatorname{Map}(y(x), y(\operatorname{colim} Z)) \to \pi_0 \operatorname{Map}(y(x), W),$$

the class of $y(gf) = y(g)y(f)$ maps to the image under f^* of the class of the composite $y(x') \xrightarrow{y(g)} y(\operatorname{colim} Z) \to W$, which vanishes by assumption. Hence $y(gf) \simeq \eta_Z \psi$ for some $\psi\colon y(x) \to Z$. Since $y(x)$ is compact, ψ is represented by a map $\psi_0\colon x \to z_i$ for some i, and $\eta_Z\psi$ is then y of the composite $x \xrightarrow{\psi_0} z_i \to \operatorname{colim}_i z_i$. As y is fully faithful, gf is homotopic to this composite. □

Remark 4.10. In the literature one also finds weaker, “elementwise” definitions of compact morphisms. For instance, Clausen and Mathew call a morphism $f\colon x \to x'$ in a presentable stable category compact if for every filtered diagram $(z_j)_j$ with $\operatorname{colim}_j z_j \simeq 0$, every map $x' \to z_j$ becomes nullhomotopic after precomposition with f and postcomposition with $z_j \to z_{j'}$ for some $j' \geq j$. In compactly assembled categories this agrees with Definition 3.3, by Propositions 4.4 and 4.9.

Remark 4.11. The vanishing-colimit test in Proposition 4.9 characterizes weak compactness in any stable category with filtered colimits. Compact assembly is used only to identify this with the natural factorization condition of Definition 3.3, via Proposition 4.4.

Corollary 4.12 (cf. [Efi24, Remark 2.38])**.** *Let C be a stable category with filtered colimits. Then $x \in C$ is compact if and only if* id_x *is weakly compact.*

Proof. Suppose that id_x is weakly compact. By Proposition 4.9, we have $\pi_0 \operatorname{Map}_{\mathrm{Ind}(C)}(y(x), W) = 0$ for every $W \in \mathrm{Ind}(C)$ with $\operatorname{colim} W \simeq 0$. Applying this to all shifts of W, we get $\hom_{\mathrm{Ind}(C)}(y(x), W) \simeq 0$. For $Z =$ "colim"$_i z_i$, applying this to the fiber W of the unit $\eta_Z \colon Z \to y(\operatorname{colim} Z)$ shows that

$$\begin{aligned}\operatorname{colim}_i \hom_C(x, z_i) &\simeq \hom_{\mathrm{Ind}(C)}(y(x), Z) \\ &\to \hom_{\mathrm{Ind}(C)}(y(x), y(\operatorname{colim} Z)) \simeq \hom_C(x, \operatorname{colim}_i z_i)\end{aligned}$$

is an equivalence, i.e. x is compact. The converse is Example 4.3(1). □

4.2 Compactly exhaustible objects

In Section 3.4 we saw that the ω_1-compact objects of a compactly assembled category are precisely the sequential colimits along compact morphisms (Theorem 3.20 and Remark 3.21). We now show that, for presentable stable categories, the existence of enough such objects characterizes dualizability.

Definition 4.13. Let C be a category with filtered colimits. A *compact exhaustion* of an object $x \in C$ is a sequence $x_0 \to x_1 \to x_2 \to \cdots$ of weakly compact morphisms together with an equivalence $\operatorname{colim}_n x_n \simeq x$. An object is called *compactly exhaustible* if it admits a compact exhaustion.

Proposition 4.14 ([KNP24, Theorem 2.2.15])**.** *Let $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Then C is dualizable if and only if C is generated under colimits by compactly exhaustible objects.*

Proof. If C is dualizable, then C is generated under colimits by its ω_1-compact objects, which are compactly exhaustible by Theorem 3.20 together with Proposition 4.4.

Conversely, assume that C is generated under colimits by compactly exhaustible objects. By Theorem 3.16, it suffices to show that $\operatorname{colim} \colon \mathrm{Ind}(C) \to C$ admits a left adjoint, i.e. that for every $x \in C$ the functor $\operatorname{Map}_C(x, \operatorname{colim}(-)) \colon \mathrm{Ind}(C) \to \mathrm{An}$ is corepresentable. The category $\mathrm{Ind}(C)$ is stable and admits small colimits (any small diagram lies in some $\mathrm{Ind}(C^\lambda)$, and the inclusions $\mathrm{Ind}(C^\lambda) \subseteq \mathrm{Ind}(C^\mu)$ are exact and preserve filtered colimits, see Construction A.21). Hence the class of x for which $\operatorname{Map}_C(x, \operatorname{colim}(-))$ is corepresentable is closed under small colimits: if $x \simeq \operatorname{colim}_a x_a$ and $\operatorname{Map}_C(x_a, \operatorname{colim}(-)) \simeq \operatorname{Map}_{\mathrm{Ind}(C)}(X_a, -)$ naturally, then

$$\begin{aligned}\operatorname{Map}_C(x, \operatorname{colim}(-)) &\simeq \lim_a \operatorname{Map}_C(x_a, \operatorname{colim}(-)) \\ &\simeq \lim_a \operatorname{Map}_{\mathrm{Ind}(C)}(X_a, -) \simeq \operatorname{Map}_{\mathrm{Ind}(C)}(\operatorname{colim}_a X_a, -).\end{aligned}$$

So it suffices to treat a compactly exhaustible object $x \simeq \operatorname{colim}_n x_n$, with $x_n \to x_{n+1}$ weakly compact. Put $X := \operatorname{colim}_n y(x_n) \in \mathrm{Ind}(C)$, the colimit being taken in $\mathrm{Ind}(C)$. Then $\operatorname{colim} X \simeq \operatorname{colim}_n x_n \simeq x$, as colim is a left adjoint and $\operatorname{colim} \circ y \simeq \mathrm{id}$. We claim that the map

$$\operatorname{Map}_{\mathrm{Ind}(C)}(X, Z) \to \operatorname{Map}_C(\operatorname{colim} X, \operatorname{colim} Z) \simeq \operatorname{Map}_C(x, \operatorname{colim} Z)$$

induced by colim is an equivalence, naturally in $Z \in \mathrm{Ind}(C)$. Under the adjunction $\mathrm{colim} \dashv y$, it is identified with the map $(\eta_Z)_* \colon \mathrm{Map}(X, Z) \to \mathrm{Map}(X, y(\mathrm{colim}\, Z))$. Since everything is stable, it suffices to show that $\hom_{\mathrm{Ind}(C)}(X, W) \simeq 0$ for $W = \mathrm{fib}(\eta_Z)$, which satisfies $\mathrm{colim}\, W \simeq 0$ (as $\mathrm{colim}(\eta_Z)$ is an equivalence).

Put $h_n := \hom_{\mathrm{Ind}(C)}(y(x_n), W)$. We have $\hom(X, W) \simeq \lim_n h_n$, so there are Milnor short exact sequences

$$0 \to \lim_n{}^1 \pi_{m+1} h_n \to \pi_m \hom(X, W) \to \lim_n \pi_m h_n \to 0.$$

Now $\pi_m h_n \simeq \pi_0 \mathrm{Map}(y(x_n), \Omega^m W)$, and $\mathrm{colim}\, \Omega^m W \simeq 0$. Since $x_n \to x_{n+1}$ is weakly compact, Proposition 4.9 shows that all transition maps in the tower $(\pi_m h_n)_n$ are zero, for every $m \in \mathbb{Z}$. A tower of abelian groups with zero transition maps has vanishing lim and $\lim^1$, so $\pi_m \hom(X, W) = 0$ for all m, as desired. □

Remark 4.15. In Proposition 4.14, stability is used to reduce to ind-objects with vanishing colimit. For general presentable categories, the analogous statement holds provided filtered colimits in C are exact, with "dualizable" replaced by "compactly assembled" [KNP24, Theorem 2.2.15].

4.3 A universal property of $\hat{y}$

For an object x of a compactly assembled category C, the functor $\mathrm{Map}_{\mathrm{Ind}(C)}(y(x), \hat{y}(-))$ is a "filtered-colimit-preserving approximation" of $\mathrm{Map}_C(x, -)$: if x is compact, the two agree, and in general the former is the best approximation of the latter by a functor that preserves filtered colimits. To make this precise, for categories C and D with filtered colimits we write $\mathrm{Fun}^{\mathrm{filt}}(C, D) \subseteq \mathrm{Fun}(C, D)$ for the full subcategory of functors preserving filtered colimits.

Proposition 4.16 ([KNP24, Proposition 2.4.8])**.** *Let C be compactly assembled, and let D be a category with filtered colimits. Then the inclusion $\mathrm{Fun}^{\mathrm{filt}}(C, D) \hookrightarrow \mathrm{Fun}(C, D)$ admits a right adjoint, given by*

$$F \mapsto \mathrm{colim}_D \circ \mathrm{Ind}(F) \circ \hat{y},$$

where $\mathrm{Ind}(F) \colon \mathrm{Ind}(C) \to \mathrm{Ind}(D)$ is the extension of F preserving filtered colimits.

Proof. Write $\mathrm{colim} = \mathrm{colim}_C$. The adjunction $\hat{y} \dashv \mathrm{colim}$ induces an adjunction

$$\mathrm{colim}^* \colon \mathrm{Fun}(C, D) \rightleftarrows \mathrm{Fun}(\mathrm{Ind}(C), D) : \hat{y}^*$$

given by precomposition: a transformation $G\,\mathrm{colim} \to H$ corresponds to the composite $G \to G\,\mathrm{colim}\,\hat{y} \to H\hat{y}$, using the unit $\mathrm{id} \to \mathrm{colim}\,\hat{y}$. Since colim and $\hat{y}$ preserve filtered colimits, this restricts to an adjunction between $\mathrm{Fun}^{\mathrm{filt}}(C, D)$ and $\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(C), D)$. By the universal property of $\mathrm{Ind}(C)$ (the analogue of Theorem A.15(2) for large C, see Construction A.21), restriction along y is an equivalence $\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(C), D) \simeq \mathrm{Fun}(C, D)$, with inverse $F \mapsto \mathrm{colim}_D \circ \mathrm{Ind}(F)$. Under this equivalence, the left adjoint colim^* becomes $G \mapsto G\,\mathrm{colim}\, y \simeq G$, i.e. the inclusion, and the right adjoint $\hat{y}^*$ becomes $F \mapsto \mathrm{colim}_D \circ \mathrm{Ind}(F) \circ \hat{y}$. □

Corollary 4.17. *Let C be compactly assembled and $x \in C$. Then the natural transformation*

$$\mathrm{Map}_{\mathrm{Ind}(C)}\big(y(x), \hat{y}(-)\big) \xrightarrow{(\varepsilon)_*} \mathrm{Map}_{\mathrm{Ind}(C)}\big(y(x), y(-)\big) \simeq \mathrm{Map}_C(x, -)$$

exhibits $\mathrm{Map}_{\mathrm{Ind}(C)}(y(x), \hat{y}(-))$ *as the terminal object of* $\mathrm{Fun}^{\mathrm{filt}}(C, \mathrm{An})_{/\mathrm{Map}_C(x,-)}$. *If C is moreover stable, the same holds for the mapping spectra* $\hom_{\mathrm{Ind}(C)}(y(x), \hat{y}(-)) \to \hom_C(x, -)$ *in* $\mathrm{Fun}^{\mathrm{filt}}(C, \mathrm{Sp})_{/\hom_C(x,-)}$.

Proof. For a right adjoint R of a fully faithful inclusion i, the counit $iR(F) \to F$ is a terminal object of the slice over F, since $\mathrm{Map}(G, R(F)) \simeq \mathrm{Map}(iG, F)$. Apply this to Proposition 4.16 with $D = \mathrm{An}$ and $F = \mathrm{Map}_C(x, -)$. For $Z = \text{“colim”}_j z_j \in \mathrm{Ind}(C)$ we have

$$\mathrm{colim}_{\mathrm{An}} \mathrm{Ind}(F)(Z) \simeq \mathrm{colim}_j \mathrm{Map}_C(x, z_j) \simeq \mathrm{Map}_{\mathrm{Ind}(C)}(y(x), Z),$$

as $y(x)$ is compact in $\mathrm{Ind}(C)$. Hence the right adjoint sends F to $\mathrm{Map}_{\mathrm{Ind}(C)}(y(x), \hat{y}(-))$. To identify the counit, write $\tilde{F} = \mathrm{colim}_{\mathrm{An}} \circ \mathrm{Ind}(F)$. By construction, the counit at F is obtained by restricting the counit $\tilde{F}\hat{y}\,\mathrm{colim} \to \tilde{F}$ of the precomposition adjunction along y, and the latter is $\tilde{F}$ applied to the counit $\hat{y}\,\mathrm{colim} \to \mathrm{id}$ of $\hat{y} \dashv \mathrm{colim}$. At $y(c)$, this counit is the map $\hat{y}(c) \simeq \hat{y}(\mathrm{colim}\, y(c)) \to y(c)$, which is ε_c by definition of ε. So the counit is $\tilde{F}(\varepsilon)$, i.e. the map induced by ε. The stable case is identical, with $D = \mathrm{Sp}$. □

4.4 The axiom (AB6)

We finally discuss the characterization of dualizable categories by the axiom (AB6), due to Clausen and Scholze. It expresses the fact that infinite products commute with filtered colimits, in the strongest possible sense.

Definition 4.18. Let C be a category with small products and filtered colimits. We say that C satisfies *(AB6)* if for every set S and every family of functors $F_s \colon J_s \to C$, $s \in S$, with each J_s filtered, the canonical map

$$\mathrm{colim}_{(j_s)_s \in \prod_s J_s} \prod_{s \in S} F_s(j_s) \to \prod_{s \in S} \mathrm{colim}_{j \in J_s} F_s(j)$$

is an equivalence. Note that $\prod_s J_s$ is again filtered.

Example 4.19. The category of sets and the category An of animae satisfy (AB6) [Efi24, Section 2.7].

Proof. For sets, surjectivity follows by choosing, for each s, a stage j_s representing the given element of $\mathrm{colim}_{J_s} F_s$. For injectivity, two representatives which agree in each colimit agree at a later stage in each coordinate. Choosing these stages gives equality in the colimit over $\prod_s J_s$. Here we use the axiom of choice.

For animae, we may replace each J_s by a cofinal directed poset, by Theorem A.8. The product of these cofinal functors is cofinal. Indeed, if $u \colon P \to J$ is such a functor and $j \in J$, the comma category $P \times_J J_{j/}$ is filtered: any finite diagram in it has an underlying cone in P, and the finitely many compatibility data in the animae $\mathrm{Map}_J(j, u(p))$ can be extended at a later stage, since their filtered colimit is contractible by cofinality. Here we use that filtered colimits of animae commute with finite limits. Thus the comma categories for the product functor are products of nonempty filtered categories, which are again nonempty and filtered (using choice), and hence contractible. The set calculation proves that the comparison map induces a bijection on π_0, since π_0 commutes with products and filtered colimits of animae.

Fix a point in the source, represented by points $x_s \in F_s(j_s)$. Restrict to the cofinal product of the tails $J_{s,\geq j_s}$, so that all diagrams are pointed by the images of x_s. For $n \geq 1$, homotopy groups commute with products and filtered colimits of pointed animae. Applying the set calculation to these homotopy groups therefore gives

$$\operatorname{colim}_{(k_s)_s \geq (j_s)_s} \prod_s \pi_n(F_s(k_s), x_s) \cong \prod_s \operatorname{colim}_{k_s \geq j_s} \pi_n(F_s(k_s), x_s).$$

Thus the comparison map induces isomorphisms on all homotopy groups at every basepoint, and is an isomorphism of animae. □

(AB6) in C is equivalent to the statement that the colimit functor $\operatorname{colim}\colon \operatorname{Ind}(C) \to C$ preserves products. This rests on the following description of products in $\operatorname{Ind}(C)$.

Lemma 4.20 ([Efi24, Proposition 2.50])**.** *Let C be a locally small category with small products. Then $\operatorname{Ind}(C)$ has small products, and for filtered diagrams $(x_j)_{j \in J_s}$ in C, $s \in S$, we have*

$$\prod_{s \in S} \text{“colim”}_{j \in J_s} x_j \simeq \text{“colim”}_{(j_s)_s \in \prod_s J_s} \prod_{s \in S} x_{j_s}.$$

Proof. We show that the right-hand side has the universal property of the product. Since every object of $\operatorname{Ind}(C)$ is a filtered colimit of representables, it suffices to test against $y(c)$, $c \in C$. As $y(c)$ is compact, and using (AB6) in An, we get natural equivalences

$$\begin{aligned} \operatorname{Map}\left(y(c), \text{“colim”}_{(j_s)_s} \prod_s x_{j_s}\right) &\simeq \operatorname{colim}_{(j_s)_s} \prod_s \operatorname{Map}_C(c, x_{j_s}) \\ &\simeq \prod_s \operatorname{colim}_{j \in J_s} \operatorname{Map}_C(c, x_j) \\ &\simeq \prod_s \operatorname{Map}\left(y(c), \text{“colim”}_{j \in J_s} x_j\right). \end{aligned}$$

□

Proposition 4.21 (Clausen–Scholze, see [Efi24, Proposition 2.51])**.** *Let $C \in \Pr^{\mathrm{L}}_{\mathrm{st}}$. Then C is dualizable if and only if C satisfies (AB6).*

Proof. By Lemma 4.20, and since colim sends $\text{“colim”}_{(j_s)_s} \prod_s x_{j_s}$ to $\operatorname{colim}_{(j_s)_s} \prod_s x_{j_s}$, the axiom (AB6) for C says exactly that $\operatorname{colim}\colon \operatorname{Ind}(C) \to C$ preserves products of families of ind-objects. Every family of objects of $\operatorname{Ind}(C)$ is of this form. So (AB6) holds if and only if colim preserves small products, which is equivalent to dualizability of C by Proposition 3.17. □

Remark 4.22. In the talk, the implication “(AB6) $\implies$ dualizable” was argued directly with $\operatorname{colim}\colon \operatorname{Ind}(C^\kappa) \to C$. Note that products in $\operatorname{Ind}(C^\kappa)$ are not computed by the formula of Lemma 4.20, since a product of κ-compact objects need not be κ-compact. This is why the argument passes through the large category $\operatorname{Ind}(C)$, as in the proof of (2) $\implies$ (3) in Proposition 3.17.

Remark 4.23. Since Sp is compactly generated, hence dualizable, it satisfies (AB6). More generally, by [Efi24, Remark 2.52] a presentable, not necessarily stable, category is compactly assembled if and only if it satisfies (AB6) and filtered colimits in it commute with finite limits.

4.5 Automatic dualizability and images

As an application of the two characterizations of dualizability in terms of compact exhaustions and (AB6), we show that dualizability propagates along strongly continuous functors whose images generate the target. As a consequence, a strongly continuous functor out of a dualizable category has a well-behaved image. These results will be used in Chapter 7 to study colimits and limits of dualizable categories. We begin with a criterion for strong continuity.

Lemma 4.24 ([Efi24, Proposition 1.8])**.** *Let $G\colon D \to E$ be a colimit-preserving functor between presentable stable categories, and let $F_i\colon C_i \to D$, $i \in I$, be a family of strongly continuous functors from presentable stable categories C_i. Assume that D is generated under colimits by the essential images of the F_i, and that each composite $G \circ F_i$ is strongly continuous. Then G is strongly continuous.*

Proof. The right adjoints F_i^R are jointly conservative: if $F_i^R(x) \simeq 0$ for all i, then the full subcategory of objects $d \in D$ with $\hom_D(d, x) \simeq 0$ is closed under colimits and contains the essential images of the F_i, so it is all of D, and in particular $x \simeq 0$. Now let $(e_a)_a$ be a diagram in E. Since the functors F_i^R and $F_i^R G^R \simeq (GF_i)^R$ preserve colimits, applying F_i^R to the comparison map $\operatorname{colim}_a G^R(e_a) \to G^R(\operatorname{colim}_a e_a)$ gives the comparison map for $(GF_i)^R$, which is an equivalence. Hence the comparison map is an equivalence, i.e. G^R preserves colimits. □

Proposition 4.25 ([Efi24, Proposition 2.53])**.** *Let D be a presentable stable category, and let $F_i\colon C_i \to D$, $i \in I$, be a family of strongly continuous functors from dualizable categories C_i. If D is generated under colimits by the essential images of the F_i, then D is dualizable.*

Proof. As in the proof of Lemma 4.24, the right adjoints $F_i^R\colon D \to C_i$ are jointly conservative. They preserve small products, being right adjoints, and filtered colimits, by strong continuity. Let S be a set and let $(x_{s,j})_{j \in J_s}$, $s \in S$, be filtered diagrams in D. Applying F_i^R to the comparison map

$$\operatorname{colim}_{(j_s)_s \in \prod_s J_s} \prod_{s \in S} x_{s,j_s} \to \prod_{s \in S} \operatorname{colim}_{j \in J_s} x_{s,j}$$

yields the corresponding comparison map for the diagrams $(F_i^R x_{s,j})_j$ in C_i, which is an equivalence by Proposition 4.21, as C_i is dualizable. Hence the comparison map in D is an equivalence. So D satisfies (AB6), and is dualizable by Proposition 4.21. □

Second proof. A strongly continuous functor $F\colon C \to D$ preserves weakly compact morphisms. Indeed, let $f\colon x \to x'$ be weakly compact in C, let $(z_j)_j$ be a filtered diagram in D, and let $g\colon F(x') \to \operatorname{colim}_j z_j$. The adjoint map $x' \to F^R(\operatorname{colim}_j z_j) \simeq \operatorname{colim}_j F^R(z_j)$ becomes, after precomposition with f, a map factoring through some $F^R(z_j)$. Passing back to adjoints, $gF(f)$ factors through z_j. Hence F also preserves compact exhaustions (Definition 4.13).

By Proposition 4.14, each C_i is generated under colimits by compactly exhaustible objects. Their images under F_i are compactly exhaustible, and they generate D under colimits, since the localizing subcategory they generate contains the essential images of the F_i. So D is dualizable by Proposition 4.14. □

Remark 4.26. In the talk, as in [Efi24], the proof of Proposition 4.25 was first reduced to the case of a single functor, using the induced functor $\prod_i C_i \to D$. Here we note that the product $\prod_i C_i$, computed in $\widehat{\mathrm{Cat}}$, is also the coproduct of the C_i in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, as colimits in Pr^{L} are computed as limits along the right adjoints (Section A.5). If each C_i is dualizable, then so is $\prod_i C_i$: products and filtered colimits in $\prod_i C_i$ are computed componentwise, so (AB6) holds in $\prod_i C_i$ since it holds in each C_i. Alternatively, if C_i is a retract of a compactly generated category $\widetilde{C}_i$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, then $\prod_i C_i$ is a retract of $\prod_i \widetilde{C}_i$, which is compactly generated by the objects with a single nonzero component, which is compact.

Definition 4.27 ([Efi24, Definition 2.54])**.** Let $F\colon C \to D$ be a colimit-preserving functor between presentable stable categories. The *image* $\mathrm{Im}(F) \subseteq D$ is the smallest full stable subcategory of D which contains the essential image of F and is closed under colimits, i.e. the localizing subcategory generated by the essential image of F.

If C is generated under colimits by a set S of objects, then $\mathrm{Im}(F)$ is generated under colimits by the set $F(S)$, and hence is presentable [Efi24, Section 1.3]. By Proposition 2.12(1), the cofiber of F in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is $D \to \mathrm{Im}(F)^{\perp}$, whose kernel is $\mathrm{Im}(F)$. So $\mathrm{Im}(F)$ is equivalently the fiber of the cofiber of F in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Corollary 4.28 ([Efi24, Corollary 2.55])**.** *Let $F\colon C \to D$ be a strongly continuous functor between presentable stable categories, and assume that C is dualizable. Then $\mathrm{Im}(F)$ is dualizable, and the inclusion $j\colon \mathrm{Im}(F) \hookrightarrow D$ is strongly continuous.*

Proof. Write $F = j \circ F'$ with $F'\colon C \to \mathrm{Im}(F)$. Since j is fully faithful, the right adjoint of F' is $F^R \circ j$, which preserves colimits. So F' is strongly continuous, and its essential image generates $\mathrm{Im}(F)$ under colimits. Hence $\mathrm{Im}(F)$ is dualizable by Proposition 4.25. Since $j \circ F' = F$ is strongly continuous, Lemma 4.24 shows that j is strongly continuous. □

In the talk, the strong continuity of the inclusion $\mathrm{Im}(F) \hookrightarrow D$ was stated without proof. Unwinding the proof of Lemma 4.24, the point is that the right adjoint $F'^R\colon \mathrm{Im}(F) \to C$ of F' is conservative, and its composite $F'^R j^R \simeq F^R$ with j^R preserves colimits.

The two criteria have different uses. Compact exhaustions describe objects and allow constructions by successive factorizations; (AB6) detects dualizability through jointly conservative right adjoints. The image and generation results above make the latter method available without choosing an exhaustion for each object.

5 Examples of dualizable categories

In this chapter we discuss several sources of dualizable (and, more generally, compactly assembled) categories:

- Kernels of homological epimorphisms of ring spectra, and in particular almost ring theory (Sections 5.1 and 5.2).
- Continuous posets, as a class of non-stable compactly assembled categories (Section 5.3).
- Gluing constructions and extensions, including a nondualizable extension of compactly generated categories (Section 5.4).

5.1 Kernels of homological epimorphisms

Let $A \to B$ be a map of $\mathbb{E}_1$-rings, and consider the base change functor $B \otimes_A -\colon \mathrm{D}(A) \to \mathrm{D}(B)$ between the categories of left modules, with right adjoint the restriction functor $\mathrm{res}\colon \mathrm{D}(B) \to \mathrm{D}(A)$. Recall from Definition 2.16 that $A \to B$ (or rather $\mathrm{Perf}(A) \to \mathrm{Perf}(B)$) is a *homological epimorphism* if $B \otimes_A -$ is a localization, i.e. if res is fully faithful. Since res preserves colimits, this is equivalent to the counit $B \otimes_A \mathrm{res}(N) \to N$ being an equivalence for all $N \in \mathrm{D}(B)$, and since both sides preserve colimits in N and $\mathrm{D}(B)$ is generated by B, this is in turn equivalent to the multiplication map

$$B \otimes_A B \to B$$

being an equivalence. In the language of Clausen, $A \to B$ is then called *idempotent*.

Proposition 5.1. *Let $A \to B$ be a homological epimorphism of $\mathbb{E}_1$-rings, and put*

$$\mathrm{D}(A \to B) := \ker\big(B \otimes_A -\colon \mathrm{D}(A) \to \mathrm{D}(B)\big).$$

Then $\mathrm{D}(A \to B)$ is dualizable, and the sequence $\mathrm{D}(A \to B) \hookrightarrow \mathrm{D}(A) \to \mathrm{D}(B)$ is a strongly continuous localization sequence. In particular,

$$K^{\mathrm{cont}}(\mathrm{D}(A \to B)) \simeq \mathrm{fib}\big(K(A) \to K(B)\big).$$

Proof. The functor $B \otimes_A -$ is a localization by assumption, and it is strongly continuous since its right adjoint res preserves colimits. Hence $\mathrm{D}(A \to B) \hookrightarrow \mathrm{D}(A) \to \mathrm{D}(B)$ is a strongly continuous localization sequence (Lemma 2.13), and $\mathrm{D}(A \to B)$ is dualizable by Proposition 3.19. The final statement follows from the definition of K^{cont}, see Section 1.4. □

Proposition 5.2. *Every dualizable category is of the form* $\mathrm{D}(E \to E')$ *for some homological epimorphism* $E \to E'$ *of* $\mathbb{E}_1$*-rings.*

Proof. Let C be dualizable. Choose a set S of representatives of the isomorphism classes of C^{ω_1}, a regular cardinal $\kappa > |S|$, and put $G := \bigoplus_{x \in S} x$. As a κ-small sum of κ-compact objects, G is κ-compact. Let $A \subseteq C^\kappa$ be the thick subcategory generated by G; it contains C^{ω_1}, since every object of C^{ω_1} is a retract of G. The functor $\hat{y}\colon C \to \mathrm{Ind}(C^\kappa)$ takes values in $\mathrm{Ind}(C^{\omega_1}) \subseteq \mathrm{Ind}(A)$ (Theorem 3.20, see also [Efi24, Corollary 2.14]), and hence is a fully faithful left adjoint of $\mathrm{colim}\colon \mathrm{Ind}(A) \to C$. Its essential image $\hat{y}(C)$ is closed under colimits, with right orthogonal $\ker(\mathrm{colim}\colon \mathrm{Ind}(A) \to C)$. Since the latter is closed under colimits in $\mathrm{Ind}(A)$, the localization

$$Q\colon \mathrm{Ind}(A) \to \mathrm{Ind}(A)/\hat{y}(C) \simeq \ker(\mathrm{colim})$$

has a colimit-preserving right adjoint and thus preserves compact objects. As Q is essentially surjective and $\mathrm{Ind}(A)$ is compactly generated by G, the category $\ker(\mathrm{colim})$ is compactly generated by the single object $Q(G)$. Hence $\ker(\mathrm{colim}) \simeq \mathrm{Ind}(B)$ for B the thick subcategory generated by $Q(G)$, and Q is Ind of a homological epimorphism $A \to B$ with kernel $\hat{y}(C) \simeq C$; this is the Calkin sequence of Theorem 3.16 with C^κ replaced by A. Finally, by Morita theory [Lur17, Theorem 7.1.2.1] we have $\mathrm{Ind}(A) \simeq \mathrm{D}(E)$ for $E := \mathrm{End}(G)^{\mathrm{op}}$ and $\mathrm{Ind}(B) \simeq \mathrm{D}(E')$ for $E' := \mathrm{End}(Q(G))^{\mathrm{op}}$, under which Q corresponds to $E' \otimes_E -$, where the ring map $E \to E'$ is induced by Q on endomorphisms. Indeed, $Q(G)$ has the resulting E-action, so the two colimit-preserving functors agree on the generator together with its endomorphism action, as required by Morita theory. So $C \simeq \mathrm{D}(E \to E')$. □

There is a convenient criterion for a map $A \to B$ with fiber I to be a homological epimorphism. Recall that a non-unital ring spectrum I is called *h-unital* if the multiplication $I \otimes_{\mathbb{S} \oplus I} I \to I$ is an isomorphism, where $\mathbb{S} \oplus I$ denotes the unitalization. Equivalently, the nonunital relative tensor product, defined by the semisimplicial bar construction with terms $I^{\otimes(n+2)}$, maps isomorphically to I; see [KNP24, Section 2.10 and Lemma 3.5.5].

Theorem 5.3 ([KNP24, Section 2.10, especially Corollary 2.10.6])**.** *For a non-unital ring spectrum* I*, the following are equivalent:*

(1) I *is h-unital;*

(2) *the map* $\mathbb{S} \oplus I \to \mathbb{S}$ *is a homological epimorphism;*

(3) *for every fiber sequence* $I \to A \to B$ *in which* $A \to B$ *is a map of unital ring spectra, the map* $A \to B$ *is a homological epimorphism.*

Conversely, if $A \to B$ is a homological epimorphism of connective ring spectra and $\pi_0(A) \to \pi_0(B)$ is surjective, then $\mathrm{fib}(A \to B)$ is h-unital [KNP24, Proposition 2.10.7]. The surjectivity hypothesis cannot be omitted: the fiber of $k[t] \to k[t, t^{-1}]$ is not h-unital [KNP24, Example 2.10.8].

Example 5.4. The following maps are homological epimorphisms, and hence give rise to dualizable categories $\mathrm{D}(A \to B)$:

(1) A localization $A \to A[S^{-1}]$ of a commutative ring at a multiplicative subset. More generally, if A is a commutative Noetherian ring, the kernel of every homological epimorphism out of A is compactly generated: the associated localization of $\mathrm{D}(A)$ is smashing, so this follows from [Nee92, Theorem 3.3 and Corollary 3.4].

(2) A surjection $A \twoheadrightarrow B$ of perfect $\mathbb{F}_p$-algebras: the higher Tor^A-groups between perfect A-algebras vanish by [BS17, Lemma 3.16], so $B \otimes^{\mathbf{L}}_A B \simeq B$. Similarly, if A is an integral perfectoid ring and B is a perfect $\mathbb{F}_p$-algebra, any surjection $A \twoheadrightarrow B$ is a homological epimorphism. Indeed, the derived p-completed tensor product of perfectoid rings is discrete by the proof of [BS22, Lemma 10.3]. Here $B \otimes^{\mathbf{L}}_A B$ is already derived p-complete, since it is a B-module, and its degree-zero homology is B.

(3) (Wodzicki) If A is a unital C^*-algebra and $I \subseteq A$ a closed two-sided $*$-ideal, then $A \to A/I$ is a homological epimorphism. For $A = B(\mathcal{H})$ and $I = \mathcal{K}(\mathcal{H})$, this gives the category $\mathrm{D}(\mathcal{K})$ from Example 1.14.

(4) (Almost ring theory) If A is a ring and $I \subseteq A$ is a two-sided ideal with $I^2 = I$ which is flat as a left or right A-module, then $A \to A/I$ is a homological epimorphism. We discuss this in detail in Section 5.2.

5.2 Almost ring theory

Almost ring theory (Faltings, Gabber–Ramero [GR03]) provides one of the most natural sources of dualizable categories which are not compactly generated. Throughout this section, let A be an (associative, unital) ring and let $I \subseteq A$ be a two-sided ideal such that

- $I^2 = I$, and
- I is flat as a right A-module.

We work with left A-modules, and write A-Mod for the abelian category of left A-modules and $\mathrm{D}(A)$ for its derived category.

The forgetful functor (A/I)-Mod $\hookrightarrow$ A-Mod is fully faithful, with essential image the A-modules M with $IM = 0$. Since $I^2 = I$, this subcategory is closed under extensions: if $0 \to M' \to M \to M'' \to 0$ is exact with $IM' = 0 = IM''$, then $IM = I^2M \subseteq IM' = 0$. As it is clearly closed under subobjects and quotients, it is a *Serre subcategory*.

Definition 5.5. The category of (A, I)-*almost modules* is the Serre quotient

$$(A, I)^a\text{-Mod} := A\text{-Mod}/(A/I)\text{-Mod}.$$

Example 5.6.

(1) If $I = A$, then (A/I)-Mod $= 0$ and $(A, A)^a$-Mod $= A$-Mod.

(2) The motivating examples come from non-discrete valuation rings: let A be a valuation ring of rank one whose value group is not discrete, e.g. $A = \mathcal{O}_{\mathbb{C}_p}$, and let $I = \mathfrak{m}$ be its maximal ideal. Then $\mathfrak{m}^2 = \mathfrak{m}$, since every element of $\mathfrak{m}$ is a product of two elements of $\mathfrak{m}$, and $\mathfrak{m}$ is

flat since it is torsion-free over a valuation ring. An A-module is called *almost zero* if it is killed by $\mathfrak{m}$; for instance, the residue field $A/\mathfrak{m}$ becomes zero in the almost category. This is the setting of Faltings' almost purity theorem in p-adic Hodge theory.

The derived picture

We have an adjoint triple

$$\mathrm{D}(A) \underset{R}{\overset{L}{\underset{\longleftarrow \mathrm{res} \longrightarrow}{\rightleftarrows}}} \mathrm{D}(A/I)$$

where res denotes restriction of scalars, $L = A/I \otimes^{\mathbf{L}}_A -$ and $R = \mathbf{R}\operatorname{Hom}_A(A/I, -)$.

Lemma 5.7. *The functor* $\mathrm{res}\colon \mathrm{D}(A/I) \to \mathrm{D}(A)$ *is fully faithful. In particular, L is a localization and $A \to A/I$ is a homological epimorphism.*

Proof. As explained at the beginning of Section 5.1, it suffices to show that the multiplication map $A/I \otimes^{\mathbf{L}}_A A/I \to A/I$ is an equivalence. Using the short exact sequence $0 \to I \to A \to A/I \to 0$ of right A-modules, we find

$$\begin{aligned}
&\operatorname{Tor}^A_0(A/I, A/I) = A/I,\\
&\operatorname{Tor}^A_1(A/I, A/I) \cong I/I^2 = 0,\\
&\operatorname{Tor}^A_k(A/I, A/I) \cong \operatorname{Tor}^A_{k-1}(I, A/I) = 0 \quad \text{for } k \geq 2,
\end{aligned}$$

where the last vanishing uses flatness of I as a right A-module. □

Lemma 5.8. *The essential image of* $\mathrm{res}\colon \mathrm{D}(A/I) \hookrightarrow \mathrm{D}(A)$ *is the full subcategory*

$$\mathrm{D}_{(A/I)\text{-Mod}}(A) := \{X \in \mathrm{D}(A) \mid IH_n(X) = 0 \text{ for all } n \in \mathbb{Z}\}$$

of complexes whose homology lies in the Serre subcategory (A/I)-Mod*. Moreover, the kernel of L consists of those $X \in \mathrm{D}(A)$ with $X \simeq I \otimes^{\mathbf{L}}_A X$, and $I \otimes^{\mathbf{L}}_A X \in \ker(L)$ for every $X \in \mathrm{D}(A)$.*

Proof. Since res is fully faithful with left adjoint L, its essential image consists of those X for which the unit $X \to \mathrm{res}\, L(X)$ is an equivalence. Tensoring the short exact sequence $0 \to I \to A \to A/I \to 0$ of A-bimodules with X gives a cofiber sequence

$$I \otimes^{\mathbf{L}}_A X \to X \to \mathrm{res}\, L(X),$$

so X lies in the essential image if and only if $I \otimes^{\mathbf{L}}_A X \simeq 0$. Since I is flat as a right module, $H_n(I \otimes^{\mathbf{L}}_A X) \cong I \otimes_A H_n(X)$. Finally, for a left A-module M, we have $I \otimes_A M = 0$ if and only if $IM = 0$: the multiplication map $I \otimes_A M \to IM$ is surjective, and conversely if $IM = 0$ then M is an A/I-module and $I \otimes_A M \cong (I \otimes_A A/I) \otimes_{A/I} M \cong I/I^2 \otimes_{A/I} M = 0$.

For the second statement, $L(I \otimes^{\mathbf{L}}_A X) \simeq (A/I \otimes^{\mathbf{L}}_A I) \otimes^{\mathbf{L}}_A X$, and $A/I \otimes^{\mathbf{L}}_A I \simeq \mathrm{fib}(A/I \to A/I \otimes^{\mathbf{L}}_A A/I) \simeq 0$ by Lemma 5.7. The characterization of $\ker(L)$ follows from the cofiber sequence above. □

Proposition 5.9. *There is an equivalence*

$$\mathrm{D}((A, I)^a\text{-Mod}) \simeq \ker\big(L\colon \mathrm{D}(A) \to \mathrm{D}(A/I)\big),$$

and $\ker(L) = \mathrm{D}(A \to A/I)$ *is dualizable. More precisely,* $\mathrm{D}((A,I)^a\text{-Mod}) \hookrightarrow \mathrm{D}(A) \xrightarrow{L} \mathrm{D}(A/I)$ *is a strongly continuous localization sequence, and* $\mathrm{D}((A,I)^a\text{-Mod})$ *is also the kernel of* L *computed in* $\mathrm{Pr}^{\mathrm{dual}}$.

Proof. Since (A/I)-Mod $\subseteq A$-Mod is a Serre subcategory of a Grothendieck abelian category which is closed under colimits, the derived category of the Serre quotient is the Verdier quotient

$$\mathrm{D}((A,I)^a\text{-Mod}) \simeq \mathrm{D}(A)/\mathrm{D}_{(A/I)\text{-Mod}}(A).$$

Indeed, the quotient functor A-Mod $\to (A,I)^a$-Mod is exact and has a fully faithful right adjoint, and for such functors this identification is proved in [Kra10, Section 7.7]; see also [Efi24, Example 2.26]. It thus remains to identify the Verdier quotient $\mathrm{D}(A)/\mathrm{D}_{(A/I)\text{-Mod}}(A)$ with $\ker(L)$. Consider the functor $q := I \otimes^{\mathbf{L}}_A - \colon \mathrm{D}(A) \to \ker(L)$, which lands in $\ker(L)$ by Lemma 5.8. For $K \in \ker(L)$ and $X \in \mathrm{D}(A)$, applying $\mathrm{Map}(K,-)$ to the cofiber sequence $I \otimes^{\mathbf{L}}_A X \to X \to \operatorname{res} L(X)$ and using $\mathrm{Map}(K, \operatorname{res} LX) \simeq \mathrm{Map}(LK, LX) \simeq *$ shows that $\mathrm{Map}(K, qX) \xrightarrow{\sim} \mathrm{Map}(K,X)$. So q is right adjoint to the fully faithful inclusion $\ker(L) \hookrightarrow \mathrm{D}(A)$, and hence q exhibits $\ker(L)$ as the localization of $\mathrm{D}(A)$ at the morphisms inverted by q [Lur09, Proposition 5.2.7.12]. Since q is exact, these are the morphisms whose cofiber lies in $\ker(q)$, which is $\mathrm{D}_{(A/I)\text{-Mod}}(A)$ by Lemma 5.8. Hence $\ker(L) \simeq \mathrm{D}(A)/\mathrm{D}_{(A/I)\text{-Mod}}(A)$.

By Proposition 5.1 and Lemma 5.7, $\ker(L)$ is dualizable and its inclusion into $\mathrm{D}(A)$ is strongly continuous; explicitly, its right adjoint is $X \mapsto I \otimes^{\mathbf{L}}_A X$ by Lemma 5.8. To see that $\ker(L)$ is also the kernel in $\mathrm{Pr}^{\mathrm{dual}}$, let E be dualizable and $F\colon E \to \mathrm{D}(A)$ strongly continuous with $LF \simeq 0$. Then F factors uniquely through a colimit-preserving functor $F'\colon E \to \ker(L)$, and F' is strongly continuous since its right adjoint is the composite of the colimit-preserving functors $\ker(L) \hookrightarrow \mathrm{D}(A) \xrightarrow{F^R} E$. □

Remark 5.10. In the talk, the last part of Proposition 5.9 was deduced from the general fact that the inclusion $\mathrm{Pr}^{\mathrm{dual}} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ preserves pullbacks along localizations (Proposition 7.21(2)). Efimov [Efi24, Example 2.26] shows that $\mathrm{D}((A,I)^a\text{-Mod})$ is dualizable even without any flatness assumption on I: in general, $\mathrm{D}(A) \to \mathrm{D}_{(A/I)\text{-Mod}}(A)$ is still a strongly continuous localization, identified with $B \otimes_A -$ for a connective DG ring B with $H_0(B) = A/I$, and $B = A/I$ if I is flat as a right or left A-module.

The following is a special case of Keller's example (Example 2.20), and shows that almost module categories are typically very far from being compactly generated.

Proposition 5.11. *If* $I \subseteq \mathrm{Jac}(A)$ *is contained in the Jacobson radical of* A*, then* $\mathrm{D}((A,I)^a\text{-Mod})$ *has no nonzero compact objects.*

Proof. By Proposition 5.9, the inclusion $\ker(L) \hookrightarrow \mathrm{D}(A)$ is strongly continuous and hence preserves compact objects. So it suffices to show: if $P \in \mathrm{Perf}(A)$ satisfies $A/I \otimes^{\mathbf{L}}_A P \simeq 0$, then $P \simeq 0$.

Suppose $P \neq 0$ and let n be minimal with $H_n(P) \neq 0$. We may represent P by a bounded complex of finitely generated projective modules concentrated in homological degrees $\geq n$: starting from any bounded complex $P_\bullet$ of finitely generated projectives, if $a < n$ is the lowest degree with $P_a \neq 0$, then $P_{a+1} \to P_a$ is surjective as $H_a(P) = 0$, hence split, and we may replace P_{a+1} by its kernel (again

finitely generated projective) and remove P_a. In this representation, $H_n(P) = \operatorname{coker}(P_{n+1} \to P_n)$ is a finitely generated A-module, and by right exactness of the tensor product

$$0 = H_n(A/I \otimes^{\mathbf{L}}_A P) \cong A/I \otimes_A H_n(P) \cong H_n(P)/IH_n(P).$$

So $H_n(P) = IH_n(P) \subseteq \operatorname{Jac}(A)H_n(P)$, and Nakayama's lemma implies $H_n(P) = 0$, a contradiction. □

Example 5.12. In the situation of Example 5.6(2), $A = O_{\mathbb{C}_p}$ is local with maximal ideal $\mathfrak{m} = \operatorname{Jac}(A)$. Hence $\mathrm{D}((O_{\mathbb{C}_p}, \mathfrak{m})^a\text{-Mod})$ is a dualizable category without nonzero compact objects.

Hilbert-space models for operator ideals

Proposition 5.13. *Let $A = B(\mathcal{H})$ for a separable infinite-dimensional complex Hilbert space, and let J be either $\mathcal{K}(\mathcal{H})$ or $S_{0+} = \bigcap_{p>0} S_p(\mathcal{H})$. Let E be the additive category of separable complex Hilbert spaces and bounded linear maps, with the split exact structure, and put $D = \operatorname{Ind}(\mathrm{D}^b(E))$. Then $A \to A/J$ is a homological epimorphism. Under Morita theory, its kernel identifies with the full stable subcategory $C_J \subseteq D$ generated under colimits by sequences of Hilbert spaces whose transition maps belong to J. Here membership in J means compactness, respectively membership in every Schatten class, also for operators between different Hilbert spaces.*

The completed Hilbert tensor product restricts to a symmetric monoidal structure on C_J whose unit and multiplication functors are strongly continuous. In particular, $C_J \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{dual}})$.

The proof separates the algebraic and categorical steps. Positive operators provide cofinal principal left ideals and root factorizations; these establish flatness and idempotence and identify the kernel by telescopes. The last step checks strong continuity of the restricted tensor product through its right adjoint.

Proof. We use right A-modules in this proof. Every separable Hilbert space is a retract of $\mathcal{H}$, and every bounded idempotent has closed image. Hence $V \mapsto \operatorname{Hom}_E(\mathcal{H}, V)$ identifies E with the finitely generated projective right A-modules, and D with their unbounded derived category. To recover our left-module convention, choose a conjugation c on $\mathcal{H}$: the complex-linear transpose $a \mapsto ca^*c$ identifies A^{op} with A and preserves both ideals. The intrinsic category C_J itself does not require this choice.

We first record a factorization fact: given countably many $a_n \in J$, there is a positive injective $d \in J$ with $a_n \in dA$ for every n. Include a fixed positive injective operator in J among the a_n, and choose $\lambda_n > 0$ sufficiently small that

$$s = \sum_n \lambda_n a_n a_n^*$$

converges in operator norm and, when $J = S_{0+}$, in every Schatten quasi-norm $\|-\|_{1/m}$, $m \geq 1$. Such a choice is possible by imposing the first n quasi-norm bounds on the nth summand, with summable bounds in the corresponding quasi-norm metrics. Then s is positive and injective and belongs to J, as does $d = s^{1/2}$. Since $a_n a_n^* \leq \lambda_n^{-1} s$, the map $dx \mapsto a_n^* x$ is bounded on the dense image of d. Extending it to $\mathcal{H}$ and taking adjoints gives $a_n = db_n$ with $b_n \in A$.

Consequently, J is the directed union of the right ideals dA with $d \in J$ positive and injective. Each dA is a free right module of rank one, so J is flat on the right. Also $J^2 = J$: polar decomposition

writes $a = u|a|^{1/2}|a|^{1/2}$ with both factors in J. Both ideals are closed under positive square roots; for S_{0+} this follows directly from the definition by singular values. The Tor calculation in Lemma 5.7 now shows that $A \to A/J$ is a homological epimorphism.

For a positive injective $d \in J$, put

$$T_d = \bigcup_{n \geq 0} d^{1/2^n} A \subseteq J.$$

Under the identifications $d^{1/2^n} A \cong A$, this is the sequential colimit of free modules with transition maps given by left multiplication by $d^{1/2^{n+1}} \in J$. The family of submodules T_d is directed: apply the factorization fact to all the roots of two given operators to obtain e with both submodules contained in $eA \subseteq T_e$. It covers J, so J is a filtered colimit of such telescopes.

Let $L = - \otimes^{\mathbf{L}}_A A/J$, and let $C = \ker(L)$. Its inclusion $i\colon C \to D$ has the colimit-preserving right adjoint $r(M) = M \otimes^{\mathbf{L}}_A J$. This follows by tensoring the fiber sequence $J \to A \to A/J$ with M and using that L is a localization. Since D is generated as a stable category under colimits by A, the category C is generated in the same sense by $r(A) = J$. Every J-transition telescope lies in C, because applying L gives a sequence with zero transition maps, whose colimit is zero. Conversely, the preceding expression for J as a filtered colimit of the T_d puts J in the stable subcategory generated by these telescopes. Thus $C = C_J$.

The completed Hilbert tensor product extends to a colimit-preserving multiplication $\mu_D\colon D \otimes D \to D$. It preserves compact objects: it sends pairs of Hilbert spaces to Hilbert spaces, and compact objects are obtained from these by finite stable operations and retracts. Thus μ_D^R preserves colimits. Tensor products of two J-transition telescopes again have J-transitions, using the cofinal diagonal in $\mathbb{N} \times \mathbb{N}$. For S_{0+}, the relevant assertion follows from

$$\sum_{i,j} (s_i(a) s_j(b))^p = \left(\sum_i s_i(a)^p \right) \left(\sum_j s_j(b)^p \right) < \infty$$

for every $p > 0$; for compact operators it follows by finite-rank approximation. Hence the tensor product restricts to C_J. The unit $\mathbb{C}$ belongs to C_J, since its identity has finite rank, and is compact there because it is compact in D and the inclusion preserves colimits.

Finally, $i \otimes i$ has right adjoint $r \otimes r$: the adjunction $i \dashv r$ is an adjunction between colimit-preserving functors, so it can be tensored. The restricted multiplication has right adjoint

$$\mu^R_{C_J} = (r \otimes r) \mu^R_D i,$$

as follows by applying these adjunctions to mapping spectra. All three functors preserve colimits. The unit functor is strongly continuous by compactness of $\mathbb{C}$, and C_J is dualizable by Proposition 5.1. This proves the last assertion. □

5.3 Continuous posets

We now study a class of non-stable compactly assembled categories in detail, which was already briefly discussed in Example 3.7. They will reappear in Chapter 9, where we study sheaves on continuous posets. See [Gie+03] for a comprehensive treatment.

Throughout, let P be a partially ordered set, regarded as a category.

Lemma 5.14. *The category* $\mathrm{Ind}(P)$ *is (equivalent to) a partially ordered set. Explicitly, every object of* $\mathrm{Ind}(P)$ *is of the form* "colim"$_{s\in S} s$ *for a directed subset* $S \subseteq P$*, and for directed subsets* $S, T \subseteq P$ *we have*

$$\mathrm{Map}_{\mathrm{Ind}(P)}\left(\text{"colim"}_{s\in S} s, \text{"colim"}_{t\in T} t\right) = \lim_{s\in S^{\mathrm{op}}} \mathrm{colim}_{t\in T} \mathrm{Map}_P(s,t)$$
$$\simeq \begin{cases} * & \text{if for all } s \in S \text{ there is } t \in T \text{ with } s \leq t, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence $\mathrm{Ind}(P)$ *is equivalent to the poset of* ideals *of* P*, i.e. non-empty, directed, downward closed subsets* $J \subseteq P$*, ordered by inclusion, via* $S \mapsto \downarrow S := \{p \in P \mid p \leq s \text{ for some } s \in S\}$.

Proof. By Theorem A.8, every object of $\mathrm{Ind}(P)$ is a colimit of representables indexed by a filtered poset I, i.e. of the form "colim"$_{i\in I} p_i$ for a functor $p\colon I \to P$. The image $S = p(I)$ is a directed subset, and the map $I \to S$ is cofinal by Lemma A.7 (for $s \in S$, the set $\{i \mid s \leq p_i\}$ is non-empty and upward closed in the filtered poset I, hence filtered). This proves the first claim. The formula for mapping spaces follows from Construction A.21: each $\mathrm{Map}_P(s,t)$ is either empty or contractible, filtered colimits of such spaces are again empty or contractible (and contractible if and only if one of the terms is), and a limit of such spaces is contractible if and only if all terms are. In particular, all mapping spaces in $\mathrm{Ind}(P)$ are empty or contractible, so $\mathrm{Ind}(P)$ is a poset. Finally, the condition in the formula only depends on $\downarrow S$ and $\downarrow T$: it says precisely that $\downarrow S \subseteq \downarrow T$. □

Suppose now that P admits directed suprema. Then the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(P) \to P$ sends a directed subset S to $\sup S$, and the Yoneda embedding sends b to $\downarrow b$.

Lemma 5.15. *Let* P *be a small partially ordered set admitting directed suprema. Then* P *is accessible.*

Proof. Let κ be a regular cardinal with $\kappa > |P|$. Let $D \subseteq P$ be a κ-directed subset. Since $|D| < \kappa$, the set D itself has an upper bound in D (Lemma A.5), i.e. D has a largest element. It follows that every κ-filtered diagram in P has a cofinal constant subdiagram, so that κ-filtered colimits in P are "trivial": every object of P is κ-compact, and the Yoneda embedding $P \to \mathrm{Ind}_\kappa(P)$ is an equivalence, as every κ-filtered colimit of representable presheaves on P is representable. Hence $P \simeq \mathrm{Ind}_\kappa(P^\kappa)$ is κ-accessible. □

Definition 5.16. A partially ordered set P is called *continuous* if it admits directed suprema and the functor $\mathrm{colim} = \sup\colon \mathrm{Ind}(P) \to P$ admits a left adjoint $\hat{y}\colon P \to \mathrm{Ind}(P)$.

By Lemma 5.15, a small poset is continuous if and only if it is compactly assembled. Unwinding the definitions using Lemma 5.14, the existence of the left adjoint $\hat{y}$ means: for every $b \in P$ there is a directed subset $\hat{y}(b) \subseteq P$ such that for every directed subset $T \subseteq P$,

$$b \leq \sup T \iff \text{for all } s \in \hat{y}(b) \text{ there exists } t \in T \text{ with } s \leq t. \tag{5.1}$$

Definition 5.17. Let P be a poset admitting directed suprema, and let $a, b \in P$. We say that a is *way below* b, written $a \ll b$, if for every directed subset $T \subseteq P$ with $b \leq \sup T$ there exists $t \in T$ with $a \leq t$. We write

$$P_{\ll b} := \{a \in P \mid a \ll b\}.$$

Remark 5.18.

(1) If $a \ll b$, then $a \leq b$: apply the definition to $T = \{b\}$.

(2) If $a' \leq a \ll b \leq b'$, then $a' \ll b'$.

(3) An element b is compact in P (i.e. $\mathrm{Map}_P(b, -)$ preserves filtered colimits) if and only if $b \ll b$.

(4) If P is continuous, then $\hat{y}(b) \subseteq P_{\ll b}$ for every b. Indeed, let $s \in \hat{y}(b)$ and let T be directed with $b \leq \sup T$. By (5.1), there exists $t \in T$ with $s \leq t$, so $s \ll b$.

Proposition 5.19. *Let P be a poset admitting directed suprema. Then P is continuous if and only if for every $b \in P$ the subset $P_{\ll b}$ is directed and* $\sup P_{\ll b} = b$. *In this case, $\hat{y}(b) = P_{\ll b}$, viewed as an object of* $\mathrm{Ind}(P)$.

Proof. Suppose that each $P_{\ll b}$ is directed with supremum b. We claim that $b \mapsto P_{\ll b}$ satisfies (5.1), and hence defines a left adjoint to sup (a monotone map, by Remark 5.18(2)). If $b \leq \sup T$, then by definition every $a \ll b$ satisfies $a \leq t$ for some $t \in T$. Conversely, if every $a \in P_{\ll b}$ is bounded by some element of T, then $b = \sup P_{\ll b} \leq \sup T$.

Conversely, suppose that P is continuous. We claim that

$$P_{\ll b} = \{a \in P \mid a \leq s \text{ for some } s \in \hat{y}(b)\} = {\downarrow}\hat{y}(b).$$

For "$\supseteq$", note that $\hat{y}(b) \subseteq P_{\ll b}$ by Remark 5.18(4), and $P_{\ll b}$ is downward closed by Remark 5.18(2). For "$\subseteq$", note that $\hat{y}$ is fully faithful (Lemma A.26), so the counit $\sup \hat{y}(b) \leq b$ is an equality, i.e. $b = \sup \hat{y}(b)$. Applying the definition of $a \ll b$ to the directed set $T = \hat{y}(b)$ gives $s \in \hat{y}(b)$ with $a \leq s$. Since $\hat{y}(b)$ is directed, so is ${\downarrow}\hat{y}(b) = P_{\ll b}$, and $\sup P_{\ll b} = \sup \hat{y}(b) = b$. Finally, $P_{\ll b}$ and $\hat{y}(b)$ define the same object of $\mathrm{Ind}(P)$ by Lemma 5.14. □

Example 5.20. The poset $P = \mathbb{R} \cup \{\infty\}$ (with its usual order) admits directed suprema: the supremum of a non-empty subset is its usual supremum if the subset is bounded, and ∞ otherwise. We claim that $a \ll b$ if and only if $a < b$. If $a < b$ and T is directed with $b \leq \sup T$, then $\sup T > a$, so some $t \in T$ satisfies $t > a$. Conversely, suppose $a \ll b$; then $a \leq b$ by Remark 5.18(1). If $a = b$, consider the directed set $T = \{b - 1/n \mid n \geq 1\}$ (for $b \in \mathbb{R}$) resp. $T = \mathbb{N}$ (for $b = \infty$): it has supremum b, but no element $t \in T$ satisfies $b \leq t$. Hence $a < b$. It follows that $P_{\ll b} = \{a \mid a < b\}$ is directed with supremum b, so P is continuous by Proposition 5.19, with $\hat{y}(b) = \text{"colim"}_{a<b} a$. Note that P has no compact elements.

Example 5.21. Let X be a locally compact Hausdorff space and let $\mathrm{Open}(X)$ be the poset of open subsets of X, ordered by inclusion. Directed suprema in $\mathrm{Open}(X)$ are unions. We claim that

$$U \ll V \iff \overline{U} \text{ is compact and } \overline{U} \subseteq V,$$

in which case we write $U \Subset V$. Suppose first that $\overline{U}$ is compact and $\overline{U} \subseteq V$, and let $(W_i)_{i \in I}$ be a directed family of open subsets with $V \subseteq \bigcup_i W_i$. Then the W_i cover the compact set $\overline{U}$, so finitely many of them do, and since the family is directed, a single W_i contains $\overline{U} \supseteq U$. Conversely, let $U \ll V$. Since X is locally compact Hausdorff, every point $x \in V$ has an open neighbourhood W with $\overline{W}$ compact and $\overline{W} \subseteq V$. Hence V is the union of the family $\{W \in \mathrm{Open}(X) \mid W \Subset V\}$, which

is directed since $W_1 \cup W_2 \Subset V$ if $W_1, W_2 \Subset V$. By assumption, $U \subseteq W$ for some $W \Subset V$, and then $\overline{U} \subseteq \overline{W}$ is a closed subset of a compact set, hence compact, and $\overline{U} \subseteq \overline{W} \subseteq V$.

The argument shows moreover that $\mathrm{Open}(X)_{\ll V}$ is directed with union V, so $\mathrm{Open}(X)$ is a continuous poset by Proposition 5.19, with $\hat{y}(V) =$ "colim"$_{U \Subset V} U$. Its compact elements are the open subsets U with $U \Subset U$, i.e. the compact open subsets of X.

Remark 5.22. If X is a locally compact Hausdorff space, then the morphism $h_U \to h_V$ of representable sheaves on X is a compact morphism in $\mathrm{Shv}(X; \mathrm{Sp})$ whenever $U \Subset V$ (Mathew, Lecture 3). Since every open V is the union of a sequence $U_0 \Subset U_1 \Subset \cdots$ if X is second countable, this suggests (and, suitably generalized, proves) that $\mathrm{Shv}(X; \mathrm{Sp})$ is generated by sequential colimits along compact morphisms and hence dualizable, cf. Remark 3.21. We will return to this in Chapter 9.

5.4 Extensions and semiorthogonal gluing

We first show that semiorthogonal decompositions and gluings behave well with respect to dualizability. We then explain why a general extension behaves differently. The Tate-object classification at the end of this section is a more specialized application and may be omitted on a first reading.

Proposition 5.23 ([Efi24, Proposition 2.76])**.** *Let $E = \langle C, D \rangle$ be a semiorthogonal decomposition of presentable stable categories in the sense of Definition 2.21. Then E is dualizable if and only if C and D are dualizable. In this case $E^\omega = \langle C^\omega, D^\omega \rangle$.*

Proof. The projections i_C^L and i_D^R preserve colimits, so C and D are retracts of E in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Thus dualizability of E implies that of C and D.

Conversely, if C and D are dualizable, then E is dualizable by Proposition 4.25, since the strongly continuous inclusions i_C and i_D jointly generate E.

The inclusions preserve compact objects. The projection i_C^L does too, since its right adjoint i_C preserves colimits. For compact $e \in E$, the cofiber sequence $i_D i_D^R e \to e \to i_C i_C^L e$ therefore shows that $i_D i_D^R e$ is compact. Since i_D is fully faithful and preserves filtered colimits, $i_D^R e$ is compact in D. This restricts the semiorthogonal decomposition to compact objects. □

Construction 5.24 (Gluing along a functor)**.** Let $F\colon D \to C$ be an accessible exact functor between presentable stable categories. Let E be the category of triples (x, y, u) with $x \in C$, $y \in D$, and $u\colon x \to F(y)$. The same comma category occurs in the construction of a recollement in *Lectures on Topos Theory*, Construction 5.48. This is a presentable stable category [Efi24, Proposition 1.2]; colimits are computed on x and y, using the natural maps $\mathrm{colim}\, F(y_j) \to F(\mathrm{colim}\, y_j)$.

The fully faithful colimit-preserving functors

$$i_C(c) = (c, 0, 0), \qquad i_D(d) = (0, d, 0)$$

have right adjoints $i_C^R(x, y, u) = \mathrm{fib}(u)$ and $i_D^R(x, y, u) = y$. Moreover, $i_C^L(x, y, u) = x$ is left adjoint to i_C, and every object fits into a cofiber sequence

$$i_D(y) \to (x, y, u) \to i_C(x).$$

We have $\hom_E(i_D(d), i_C(c)) \simeq 0$. The functor i_D is always strongly continuous, since its right adjoint $(x, y, u) \mapsto y$ preserves colimits.

In the talk, as in [Efi24, Definition 1.1], the inclusion of C was instead defined as $c \mapsto (c[-1], 0, 0)$. Its right adjoint is then $(x, y, u) \mapsto \mathrm{cofib}(u)$, and its composite with i_D is F itself, while in our convention $i_C^R i_D \simeq \Omega F$. This shift plays no role in what follows.

Proposition 5.25 ([Efi24, Proposition 2.77])**.** *Let $F\colon D \to C$ be an accessible exact functor between dualizable categories, and let E be the gluing of Construction 5.24. Then the following are equivalent:*

(1) *F preserves colimits;*

(2) *E is dualizable, and i_C and i_D are strongly continuous.*

In this case, i_C and i_D exhibit a semiorthogonal decomposition $E = \langle C, D\rangle$.

Proof. (2) $\implies$ (1): If i_C is strongly continuous, then i_C^R preserves colimits, and hence so does $i_C^R i_D \simeq \Omega F$. So F preserves colimits. This does not use the dualizability of E.
(1) $\implies$ (2): If F preserves colimits, then $i_C^R(x, y, u) = \mathrm{fib}(u\colon x \to F(y))$ preserves colimits, so i_C is strongly continuous. The functor i_D is always strongly continuous. The functors i_C and i_D are fully faithful, we have $\hom_E(i_D(d), i_C(c)) \simeq 0$, and every object of E is an extension of an object of $i_C(C)$ by one of $i_D(D)$. So we obtain a semiorthogonal decomposition $E = \langle C, D\rangle$ in the sense of Definition 2.21, and E is dualizable by Proposition 5.23. □

The strong continuity condition in Proposition 5.25 is essential: a general strongly continuous localization sequence with dualizable outer terms need not have dualizable middle term (Example 5.30).

Semiorthogonal decompositions are inherited by categories of strongly continuous functors. For presentable stable categories T and E, we write $\mathrm{Fun}^{\mathrm{LL}}(T, E) \subseteq \mathrm{Fun}^{\mathrm{colim}}(T, E)$ for the full subcategory of strongly continuous functors. It is a stable subcategory: shifts of strongly continuous functors are strongly continuous, and the right adjoint of the cofiber of a map $G \to G'$ of colimit-preserving functors is the fiber of the induced map $G'^R \to G^R$ of right adjoints, which preserves colimits if G^R and G'^R do. In the following, we use the notion of a semiorthogonal decomposition of a (not necessarily presentable) stable category from Remark 2.24.

Proposition 5.26 ([Efi24, Proposition 2.78])**.** *Let $E = \langle i_1(C), i_2(D)\rangle$ be a semiorthogonal decomposition of a presentable stable category, let $\pi_1 := i_1^L\colon E \to C$ and $\pi_2 := i_2^R\colon E \to D$ be the projections, and let T be a presentable stable category.*

(1) *A colimit-preserving functor $G\colon T \to E$ is strongly continuous if and only if $\pi_1 G$ and $\pi_2 G$ are strongly continuous. Postcomposition with i_1 and i_2 gives a semiorthogonal decomposition*

$$\mathrm{Fun}^{\mathrm{LL}}(T, E) = \langle \mathrm{Fun}^{\mathrm{LL}}(T, C), \mathrm{Fun}^{\mathrm{LL}}(T, D)\rangle.$$

(2) *A colimit-preserving functor $G\colon E \to T$ is strongly continuous if and only if Gi_1 and Gi_2 are strongly continuous. Precomposition with π_1 and π_2 gives a semiorthogonal decomposition*

$$\mathrm{Fun}^{\mathrm{LL}}(E, T) = \langle \mathrm{Fun}^{\mathrm{LL}}(C, T), \mathrm{Fun}^{\mathrm{LL}}(D, T)\rangle.$$

Proof. Recall from Proposition 2.23 that i_1, i_2, π_1 and π_2 are strongly continuous, that $\pi_1 i_1 \simeq \mathrm{id}$, $\pi_2 i_2 \simeq \mathrm{id}$ and $\pi_1 i_2 \simeq 0$, and that there is a cofiber sequence $i_2\pi_2 \to \mathrm{id}_E \to i_1\pi_1$. In particular, composition with any of these four functors preserves strong continuity.

(1) If G is strongly continuous, then so are $\pi_1 G$ and $\pi_2 G$. Conversely, if $\pi_1 G$ and $\pi_2 G$ are strongly continuous, then so are $i_1\pi_1 G$ and $i_2\pi_2 G$, and hence so is G, by the cofiber sequence $i_2\pi_2 G \to G \to i_1\pi_1 G$ and the stability of $\mathrm{Fun}^{\mathrm{LL}}(T, E)$. Let $j_1 := i_1 \circ -$ and $j_2 := i_2 \circ -$. The pointwise adjunctions $\pi_1 \dashv i_1$ and $i_2 \dashv \pi_2$ show that $\pi_1 \circ -$ is left adjoint to j_1 and that $\pi_2 \circ -$ is right adjoint to j_2. Hence j_1 and j_2 are fully faithful, since $\pi_1 i_1 \simeq \mathrm{id}$ and $\pi_2 i_2 \simeq \mathrm{id}$, and

$$\mathrm{Map}(j_2(H'), j_1(H)) \simeq \mathrm{Map}(\pi_1 i_2 H', H) \simeq *$$

for $H \in \mathrm{Fun}^{\mathrm{LL}}(T, C)$ and $H' \in \mathrm{Fun}^{\mathrm{LL}}(T, D)$. Finally, every G is an extension $i_2\pi_2 G \to G \to i_1\pi_1 G$ of an object in the image of j_1 by one in the image of j_2.

(2) The criterion for strong continuity follows from Lemma 4.24, applied to the functors i_1 and i_2, whose essential images generate E; the converse holds since i_1 and i_2 are strongly continuous. Let $j_1' := - \circ \pi_1$ and $j_2' := - \circ \pi_2$. For an adjunction $L \dashv R$, precomposition with R is left adjoint to precomposition with L. Hence j_1' has the left adjoint $- \circ i_1$, and j_2' has the right adjoint $- \circ i_2$. Again j_1' and j_2' are fully faithful, as $\pi_1 i_1 \simeq \mathrm{id}$ and $\pi_2 i_2 \simeq \mathrm{id}$. Moreover,

$$\mathrm{Map}(j_2'(H'), j_1'(H)) \simeq \mathrm{Map}(H', H\pi_1 i_2) \simeq *$$

for $H \in \mathrm{Fun}^{\mathrm{LL}}(C, T)$ and $H' \in \mathrm{Fun}^{\mathrm{LL}}(D, T)$. Finally, precomposing G with the cofiber sequence $i_2\pi_2 \to \mathrm{id}_E \to i_1\pi_1$ gives a cofiber sequence $j_2'(Gi_2) \to G \to j_1'(Gi_1)$. □

In the talk, the decomposition in (2) was written with the two factors in the opposite order. The order above, which agrees with [Efi24, Proposition 2.78], is forced by the vanishing of maps from the image of j_2' to the image of j_1'.

Warning 5.27. The converse of Proposition 3.19 fails: if K and E are dualizable, then D need not be dualizable, even if K and E are compactly generated. An explicit counterexample is given in Example 5.30 below. However, dualizable categories are closed under pullbacks along strongly continuous localizations, see Proposition 7.21(2).

Let us take a closer look at extensions of compactly generated categories. Consider a strongly continuous localization sequence

$$\mathrm{Ind}(A) \xrightarrow{F} D \xrightarrow{G} \mathrm{Ind}(B)$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, with $A, B \in \mathrm{Cat}_{\mathrm{perf}}$. Since $\hom_D(Fa, G^R b) \simeq \hom_{\mathrm{Ind}(B)}(GFa, b) \simeq 0$, the category D admits a semiorthogonal decomposition

$$D = \langle G^R(\mathrm{Ind}(B)), F(\mathrm{Ind}(A))\rangle$$

in the ordinary sense of semiorthogonality and generation. It need not satisfy Definition 2.21: although G^R preserves colimits, its right adjoint need not do so. Thus Proposition 5.23 does not apply to an arbitrary extension. Since G is strongly continuous, G^R admits a right adjoint G^{RR}, and the extension D is encoded by the accessible functor

$$G^{RR} \circ F \colon \mathrm{Ind}(A) \to \mathrm{Ind}(B).$$

An accessible exact functor $\Phi \colon \mathrm{Ind}(A) \to \mathrm{Ind}(B)$ is in turn determined by the functor $\Psi \colon B \to \mathrm{Pro}(\mathrm{Ind}(A))$ sending $b \in B$ to the pro-object corepresenting the accessible functor $x \mapsto \hom_{\mathrm{Ind}(B)}(b, \Phi(x))$.

Following Hennion [Hen17], we denote by $\mathrm{Tate}^{\mathrm{el}}(A) \subseteq \mathrm{Pro}(\mathrm{Ind}(A))$ the stable subcategory generated by $\mathrm{Ind}(A)$ and $\mathrm{Pro}(A)$ (the *elementary Tate objects*), and by $\mathrm{Tate}(A) \subseteq \mathrm{Pro}(\mathrm{Ind}(A))$ its idempotent completion. By [Hen17, Theorem 3], an object of $\mathrm{Pro}(\mathrm{Ind}(A))$ lies in $\mathrm{Tate}^{\mathrm{el}}(A)$ if and only if it is an extension $u \to t \to v$ with $u \in \mathrm{Pro}(A)$ and $v \in \mathrm{Ind}(A)$.

Proposition 5.28 (Efimov [Efi24, Proposition 6.3])**.** *In the above situation, let* $\Psi\colon B \to \mathrm{Pro}(\mathrm{Ind}(A))$ *be the functor corresponding to* $\Phi = G^{RR} \circ F$*. Then the following are equivalent:*

(1) *D is dualizable;*

(2) *D is compactly generated;*

(3) *The essential image of* Ψ *is contained in* $\mathrm{Tate}(A)$.

Proof. We first show that D is compactly generated if and only if for every $b \in B$ there is a compact object $d \in D^\omega$ with $G(d) \simeq b \oplus \Sigma b$. If D is compactly generated, this follows from Theorem 2.17(2) and Remark 2.9. Conversely, suppose such lifts d_b exist. Since F is strongly continuous, the objects $F(a)$, $a \in A$, are compact, and we claim that together with the d_b they generate D. Indeed, if $x \in D$ is right orthogonal to all of them, then $F^R(x) = 0$, so $x \simeq G^R G(x)$, and $0 \simeq \hom_D(d_b, G^R G(x)) \simeq \hom_{\mathrm{Ind}(B)}(b \oplus \Sigma b, G(x))$ for all b, so $G(x) = 0$ and hence $x = 0$.

(1) $\Rightarrow$ (2): If D is dualizable, then the required lifts exist by [Efi24, Proposition 2.10] (which reduces to the compactly generated case by embedding D strongly continuously into a compactly generated category). (2) $\Rightarrow$ (1) is clear.

(2) $\Leftrightarrow$ (3): Let $d \in D$, and put $z := G(d)$ and $w := F^R(d)$, so that we have a fiber sequence $F(w) \to d \to G^R(z)$. Applying $\hom_D(-, F(w'))$ for $w' \in \mathrm{Ind}(A)$ and using $\hom_D(G^R(z), F(w')) \simeq \hom_{\mathrm{Ind}(B)}(z, \Phi(w'))$, we obtain a fiber sequence of functors $\mathrm{Ind}(A) \to \mathrm{Sp}$

$$\hom_{\mathrm{Ind}(B)}(z, \Phi(-)) \to \hom_D(d, F(-)) \to \hom_{\mathrm{Ind}(A)}(w, -).$$

Every filtered colimit in D is an extension of a filtered colimit in $F(\mathrm{Ind}(A))$ by one in $G^R(\mathrm{Ind}(B))$, as F^R and G preserve colimits. Since $\hom_D(d, G^R(-)) \simeq \hom_{\mathrm{Ind}(B)}(z, -)$, it follows that d is compact if and only if z is compact and $\hom_D(d, F(-))$ preserves filtered colimits. Now identify exact accessible functors $\mathrm{Ind}(A) \to \mathrm{Sp}$ with $\mathrm{Pro}(\mathrm{Ind}(A))^{\mathrm{op}}$; under this identification, the functors preserving filtered colimits correspond to $\mathrm{Pro}(A)$. The first term of the fiber sequence corresponds to $\Psi(z)$ by definition, and the last one to $w \in \mathrm{Ind}(A)$. So d is compact if and only if $z \in B$ and there is a cofiber sequence $u \to \Psi(z) \to \Sigma w$ in $\mathrm{Pro}(\mathrm{Ind}(A))$ with $u \in \mathrm{Pro}(A)$. Conversely, given $z \in B$ and a cofiber sequence $u \to \Psi(z) \to v$ with $u \in \mathrm{Pro}(A)$ and $v \in \mathrm{Ind}(A)$, the map $\Psi(z) \to v$ corresponds to a map $z \to \Phi(v)$, i.e. to a map $G^R(z) \to F(v)$, and its fiber d is a compact object with $G(d) \simeq z$ by the same computation. Using Hennion's theorem, we conclude that $z \in B$ lifts to D^ω if and only if $\Psi(z) \in \mathrm{Tate}^{\mathrm{el}}(A)$.

Combining this with the first paragraph, D is compactly generated if and only if $\Psi(b) \oplus \Sigma\Psi(b) \in \mathrm{Tate}^{\mathrm{el}}(A)$ for all $b \in B$. This implies $\Psi(b) \in \mathrm{Tate}(A)$. Conversely, if $\Psi(b) \in \mathrm{Tate}(A)$, then $\Psi(b) \oplus \Sigma\Psi(b) \in \mathrm{Tate}^{\mathrm{el}}(A)$ by Thomason's classification of dense subcategories, as in Remark 2.9. □

Remark 5.29. Applied to compactly generated D, the proof shows that short exact sequences $A \to D_0 \to B$ in $\mathrm{Cat}_{\mathrm{perf}}$ correspond to exact functors $B \to \mathrm{Tate}(A)$; this classification first appeared in [Cal+20, Theorem A.5.1 and Corollary A.5.10], see also [Efi24, Theorem 6.4].

Example 5.30 (Efimov)**.** Let k be a field, and consider the category

$$D := \left\{ (V, W, \varphi) \;\middle|\; V, W \in \mathrm{D}(k),\ \varphi\colon \bigoplus_{\mathbb{N}} V \to \bigoplus_{\mathbb{N}} W \right\}.$$

The functors $W \mapsto (0, W, 0)$ and $(V, W, \varphi) \mapsto V$ exhibit D as an extension of $\mathrm{D}(k)$ by $\mathrm{D}(k)$, both of which are compactly generated. Nevertheless, D is not dualizable.

Proof (following [Efi24, Proposition 6.8]). Here $\Phi(W) = \prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} W$, since $\hom(\bigoplus_{\mathbb{N}} V, \bigoplus_{\mathbb{N}} W) \simeq \hom(V, \prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} W)$. By Proposition 5.28, it suffices to show that $\Psi(k) \notin \mathrm{Tate}(\mathrm{Perf}(k))$, where $\Psi(k)$ is the pro-object corepresenting $W \mapsto \hom(k, \Phi(W)) = \prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} W$. Using (AB6) in $\mathrm{D}(k)$, we may rewrite this functor as

$$\prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} W \simeq \mathrm{colim}_{f\colon \mathbb{N}\to\mathbb{N}} \prod_{n\in\mathbb{N}} \bigoplus_{i=0}^{f(n)-1} W \simeq \mathrm{colim}_{f\colon \mathbb{N}\to\mathbb{N}} \hom(X_f, W), \qquad X_f := \bigoplus_{n\in\mathbb{N}} k^{f(n)},$$

where $\mathbb{N}^{\mathbb{N}}$ is ordered pointwise, and for $f \le g$ the transition map is induced by the projection $X_g \to X_f$. Hence $\Psi(k) \simeq$ "$\lim_f X_f$, a pro-system in which all transition maps are split epimorphisms.

Let $q\colon \mathrm{D}(k) \to \mathrm{Calk}(\mathrm{Perf}(k))$ be the projection to $\mathrm{Calk}(\mathrm{Perf}(k)) := \bigcup_\kappa \mathrm{Calk}_\kappa(\mathrm{Perf}(k))$ (the transition functors are fully faithful by Remark 2.27); this is an idempotent complete stable category, and $q(x) = 0$ if and only if $x \in \mathrm{Perf}(k)$. Suppose that $\Psi(k) \in \mathrm{Tate}(\mathrm{Perf}(k))$. Then $\Psi(k) \oplus \Sigma\Psi(k) \in \mathrm{Tate}^{\mathrm{el}}(\mathrm{Perf}(k))$, so there is a cofiber sequence $u \to \Psi(k) \oplus \Sigma\Psi(k) \to v$ with $u \in \mathrm{Pro}(\mathrm{Perf}(k))$ and $v \in \mathrm{D}(k)$. Applying $\mathrm{Pro}(q)$ kills u, so $\mathrm{Pro}(q)(\Psi(k)) =$ "$\lim_f q(X_f)$ is a retract of the constant pro-object $q(v)$, and hence is itself isomorphic to a constant pro-object c.

We now use the following elementary observation: if a pro-system $(Y_f)_f$ indexed by a directed poset, with split epimorphisms as transition maps $t_{gf}\colon Y_g \to Y_f$, is isomorphic to a constant pro-object c, then there is f_0 such that t_{gf} is an isomorphism for all $f_0 \le f \le g$. Indeed, the isomorphism is given by maps $p_f\colon Y_f \to c$ for $f \ge f_0$ (compatible with the t_{gf}) and a compatible family $s_f\colon c \to Y_f$, such that $p_f s_f = \mathrm{id}_c$ for $f \ge f_0$, and for every f there is $g \ge f, f_0$ with $s_f p_g = t_{gf}$. If σ is a section of t_{gf}, then $s_f(p_g\sigma) = \mathrm{id}$. So for $f \ge f_0$, the map s_f is both a split mono and a split epi, hence an isomorphism, and then $t_{gf} = s_f s_g^{-1}$ is an isomorphism for $f_0 \le f \le g$.

Applied to $Y_f = q(X_f)$, this gives a contradiction: for every f, the fiber of $X_{f+1} \to X_f$ is $\bigoplus_{\mathbb{N}} k \notin \mathrm{Perf}(k)$, so $q(X_{f+1}) \to q(X_f)$ is not an isomorphism.

6 Rigidity and nuclear objects

This chapter develops the symmetric monoidal counterpart of compact assembly. Trace-class morphisms take the role of compact morphisms, and nuclear objects are built from compact objects along trace-class maps. We first discuss rigidity and self-duality, then construct nuclear categories and examine the analytic examples. The interpolation results of Chapter 4 will be used in the rigidification argument.

6.1 Rigid symmetric monoidal categories

A rich source of dualizable categories are *rigid* symmetric monoidal categories, introduced by Gaitsgory–Rozenblyum [GR17, Chapter 1, Section 9]. We have already met them in Definition 1.4; we now give a more detailed treatment.

Definition 6.1 ([GR17, Chapter 1, Definition 9.1.2])**.** Let $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a presentably symmetric monoidal stable category, with unit $\mathbb{1}_C$ and multiplication $\mu\colon C \otimes C \to C$. Then C is called *rigid* if

(1) the unit $\mathbb{1}_C$ is compact in C, or equivalently, the unit functor $u\colon \mathrm{Sp} \to C$, $\mathbb{S} \mapsto \mathbb{1}_C$, is strongly continuous;

(2) the right adjoint $\mu^R\colon C \to C \otimes C$ preserves colimits, and its canonical lax $C \otimes C$-linear structure is strict; equivalently, μ admits a right adjoint in $\mathrm{Mod}_{C\otimes C}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$.

Remark 6.2. The equivalence of the two formulations in (2) is a general fact about adjunctions internal to module categories: a right adjoint of a $C \otimes C$-linear functor inherits a lax $C \otimes C$-linear structure, and it is a right adjoint in $\mathrm{Mod}_{C\otimes C}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ if and only if it preserves colimits and this lax structure is strict. Definition 6.1 is the definition of Gaitsgory–Rozenblyum, and it agrees with Ramzi's definition [Ram24b, Definitions 4.5 and 4.36] (Ramzi additionally requires C to be dualizable, which is automatic by Proposition 6.3). It is also equivalent to Definition 1.4, which asks that for every finite set I the multiplication $\mu_I\colon C^{\otimes I} \to C$ admits a right adjoint in $\mathrm{Mod}_{C^{\otimes I}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$. Indeed, for $I = \emptyset$ the map $\mu_\emptyset$ is the unit functor $u\colon \mathrm{Sp} \to C$, and $\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}) = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, so this case is condition (1); the case $|I| = 1$ is vacuous, and the case $|I| = 2$ is condition (2). Conversely, if (2) holds, then by induction $\mu_{I\sqcup\{*\}} = \mu \circ (\mu_I \otimes \mathrm{id}_C)$ is a composite of two functors admitting right adjoints in $\mathrm{Mod}_{C^{\otimes I}\otimes C}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$: for $\mu_I \otimes \mathrm{id}_C$ this follows by tensoring the adjunction for μ_I with C, and for μ by restricting the $C \otimes C$-linear adjunction along the algebra map $\mu_I \otimes \mathrm{id}_C\colon C^{\otimes I} \otimes C \to C \otimes C$.

Proposition 6.3 ([GR17, Chapter 1, Section 9.2.1])**.** *Let C be rigid. Then C is self-dual in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, with evaluation and coevaluation given by*

$$\mathrm{ev}\colon C \otimes C \xrightarrow{\mu} C \xrightarrow{\Gamma := \hom_C(\mathbb{1}_C, -)} \mathrm{Sp}, \qquad \mathrm{coev}\colon \mathrm{Sp} \xrightarrow{u} C \xrightarrow{\mu^R} C \otimes C.$$

In particular, C is dualizable.

Proof. Note first that both functors preserve colimits: Γ preserves colimits since $\mathbb{1}_C$ is compact (and C is stable), and μ^R preserves colimits by assumption. Moreover, Γ is right adjoint to u. Write $\eta := \mu^R(\mathbb{1}_C) \in C \otimes C$, so that coev is the colimit-preserving functor $\mathrm{Sp} \to C \otimes C$ sending $\mathbb{S}$ to η. We verify the first triangle identity, i.e. that the composite

$$C \simeq \mathrm{Sp} \otimes C \xrightarrow{\mathrm{coev} \otimes C} C \otimes C \otimes C \xrightarrow{C \otimes \mathrm{ev}} C \otimes \mathrm{Sp} \simeq C$$

is equivalent to the identity; the second one follows by symmetry. Since $C \otimes \mathrm{ev} = (C \otimes \Gamma) \circ (C \otimes \mu)$, the composite sends $c \in C$ to

$$(C \otimes \Gamma)\big((C \otimes \mu)(\eta \boxtimes c)\big) \simeq (C \otimes \Gamma)\big(\eta \cdot (\mathbb{1}_C \boxtimes c)\big),$$

where $\cdot$ denotes the multiplication in $C \otimes C$. Now we use the linearity of μ^R: for $a \in C \otimes C$ and $x \in C$ we have $\mu^R(\mu(a) \otimes x) \simeq a \cdot \mu^R(x)$. Taking $a = \mathbb{1}_C \boxtimes c$ and $x = \mathbb{1}_C$ gives

$$\eta \cdot (\mathbb{1}_C \boxtimes c) = (\mathbb{1}_C \boxtimes c) \cdot \mu^R(\mathbb{1}_C) \simeq \mu^R(c).$$

Hence the composite is $(C \otimes \Gamma) \circ \mu^R\colon C \to C$, naturally in c (as all functors involved preserve colimits, it suffices to identify the functors, which the above does naturally in c). Since $C \otimes -$ is a functor of $(\infty, 2)$-categories, $C \otimes \Gamma$ is right adjoint to $C \otimes u\colon C \to C \otimes C$, $x \mapsto x \boxtimes \mathbb{1}_C$. Hence $(C \otimes \Gamma) \circ \mu^R$ is right adjoint to $\mu \circ (C \otimes u) \simeq \mathrm{id}_C$, and is therefore equivalent to the identity. □

The linearity of the right adjoint to multiplication is what makes the triangle identity work. Trace-class morphisms express the corresponding duality data for individual morphisms.

Next, we introduce trace-class morphisms, following Clausen–Scholze [Sch26].

Definition 6.4. Let C be a symmetric monoidal category. A morphism $f\colon x \to y$ in C is called *trace-class* if there exist an object $z \in C$ and maps $\alpha\colon \mathbb{1}_C \to y \otimes z$ and $\beta\colon z \otimes x \to \mathbb{1}_C$ such that f is homotopic to the composite

$$x \simeq \mathbb{1}_C \otimes x \xrightarrow{\alpha \otimes x} y \otimes z \otimes x \xrightarrow{y \otimes \beta} y \otimes \mathbb{1}_C \simeq y.$$

Lemma 6.5. *Let C be a symmetric monoidal category whose homotopy category is idempotent complete; for instance, C stable and idempotent complete [Lur17, Lemma 1.2.4.6]. Then an object $x \in C$ is dualizable if and only if id_x is trace-class.*

Proof. If x is dualizable with dual $x^\vee$, then the coevaluation and evaluation are a witness for id_x. Conversely, let (z, α, β) be a witness for id_x, so that $\alpha\colon \mathbb{1} \to x \otimes z$ and $\beta\colon z \otimes x \to \mathbb{1}$ satisfy the first triangle identity $(x \otimes \beta)(\alpha \otimes x) \simeq \mathrm{id}_x$, but not necessarily the second one. Consider

$$e := (\beta \otimes z)(z \otimes \alpha)\colon z \to z \otimes x \otimes z \to z.$$

Inserting the first triangle identity in the middle of $e \circ e$ shows that $e \circ e \simeq e$, and similarly $\beta(e \otimes x) \simeq \beta$ and $(x \otimes e)\alpha \simeq \alpha$. As the homotopy category is idempotent complete, e splits: $e \simeq sr$ with $r\colon z \to z'$, $s\colon z' \to z$ and $rs \simeq \mathrm{id}_{z'}$. Put $\alpha' := (x \otimes r)\alpha$ and $\beta' := \beta(s \otimes x)$. Then

$$(x \otimes \beta')(\alpha' \otimes x) \simeq (x \otimes \beta)(x \otimes e \otimes x)(\alpha \otimes x) \simeq (x \otimes \beta)(\alpha \otimes x) \simeq \mathrm{id}_x$$

and $(\beta' \otimes z')(z' \otimes \alpha') \simeq res \simeq rsrs \simeq \mathrm{id}_{z'}$. So (α', β') exhibits z' as a dual of x; since dualizability can be checked in the homotopy category, x is dualizable. □

Remark 6.6.

(1) If C is closed, write $x^\vee := \underline{\mathrm{Hom}}(x, \mathbb{1}_C)$ for the *predual* of x. Then f is trace-class if and only if the map $\mathbb{1}_C \to \underline{\mathrm{Hom}}(x, y)$ classifying f factors through the canonical map $y \otimes x^\vee \to \underline{\mathrm{Hom}}(x, y)$: given (z, α, β) as above, β is adjoint to a map $z \to x^\vee$, and composing α with it gives the desired factorization; conversely, given a factorization $\mathbb{1}_C \to y \otimes x^\vee$, take $z = x^\vee$ and β the evaluation map.

(2) Trace-class morphisms form a two-sided ideal: if f is trace-class with witness (z, α, β), then hfk is trace-class with witness $(z, (h \otimes z)\alpha, \beta(z \otimes k))$.

(3) If C is stable and idempotent complete (for instance presentable and stable), then the identity of x is trace-class if and only if x is dualizable, see Lemma 6.5.

(4) A symmetric monoidal functor preserves trace-class morphisms, since it preserves the data (z, α, β).

Lemma 6.7. *Let $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be such that $\mathbb{1}_C$ is compact. Then every trace-class morphism in C is compact in the sense of Definition 3.3.*

Proof. Let $f\colon x \to y$ be trace-class with witness (z, α, β). We construct a factorization of $f^*\colon \mathrm{Map}_C(y, \mathrm{colim}(-)) \to \mathrm{Map}_C(x, \mathrm{colim}(-))$ through $\mathrm{Map}_{\mathrm{Ind}(C)}(y(x), -)$, naturally in $W = \text{“colim”}_j w_j \in \mathrm{Ind}(C)$. Given $g\colon y \to \mathrm{colim}_j w_j$, consider

$$(g \otimes z) \circ \alpha\colon \mathbb{1}_C \to (\mathrm{colim}_j w_j) \otimes z \simeq \mathrm{colim}_j (w_j \otimes z).$$

Since $\mathbb{1}_C$ is compact, this defines a point of $\mathrm{colim}_j \mathrm{Map}_C(\mathbb{1}_C, w_j \otimes z)$, and postcomposing with the natural map

$$\mathrm{colim}_j \mathrm{Map}_C(\mathbb{1}_C, w_j \otimes z) \to \mathrm{colim}_j \mathrm{Map}_C(x, w_j), \qquad \gamma \mapsto (w_j \otimes \beta) \circ (\gamma \otimes x),$$

we obtain a point of $\mathrm{colim}_j \mathrm{Map}_C(x, w_j) = \mathrm{Map}_{\mathrm{Ind}(C)}(y(x), W)$. All these maps are natural in W, and the image of this point in $\mathrm{Map}_C(x, \mathrm{colim}_j w_j)$ is $(\mathrm{colim}_j w_j \otimes \beta) \circ ((g \otimes z)\alpha \otimes x) \simeq g \circ (y \otimes \beta) \circ (\alpha \otimes x) \simeq gf$. □

The converse holds for rigid categories; this is the main structural result about rigid categories. It was formulated in Clausen's lectures as the *definition* of rigidity for dualizable symmetric monoidal categories.

Theorem 6.8 (Ramzi [Ram24b, Corollaries 2.6, 4.55 and 4.60, Example 2.9]; see also [Efi25, Propositions 1.6 and 1.7])**.** *Let $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$. Then the following are equivalent:*

(1) *C is rigid;*

(2) *$\mathbb{1}_C$ is compact, and C is generated under colimits by objects of the form* $\operatorname{colim}(x_0 \to x_1 \to x_2 \to \cdots)$ *in which each map $x_n \to x_{n+1}$ is trace-class;*

(3) *C is dualizable, and a morphism in C is compact if and only if it is trace-class.*

We will not prove this theorem. Note however that some implications are formal given what we know: (1) $\implies$ the first half of (3) is Proposition 6.3, and (3) $\implies$ (2) follows from Lemma 3.5(2), since $\mathrm{id}_{\mathbb{1}_C}$ is trace-class and hence compact, and Remark 3.21, which shows that a dualizable category is generated by sequential colimits along compact morphisms. The hard part is that in a rigid category every compact morphism is trace-class [Ram24b, Corollary 4.55, Example 2.9 and Corollary 2.6], and that (2) implies (1) [Ram24b, Corollary 4.60]. Note that the dualizability hypothesis in (3) was not stated in the talk.

Corollary 6.9. *Let $F\colon C \to D$ be a colimit-preserving symmetric monoidal functor between rigid categories. Then F is strongly continuous. Consequently, the category* Rig *of rigid symmetric monoidal categories and colimit-preserving symmetric monoidal functors is a full subcategory of* $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{dual}})$.

Proof. By Theorem 6.8, the compact morphisms in C are trace-class. Since F is symmetric monoidal, it preserves trace-class morphisms (Remark 6.6(4)), and trace-class morphisms in D are compact by Lemma 6.7. So F preserves compact morphisms, and hence is strongly continuous by Proposition 3.22. □

Remark 6.10. The argument shows more generally that a colimit-preserving symmetric monoidal functor $F\colon C \to D$ with C rigid and D dualizable with compact unit is strongly continuous. In fact, the dualizability hypothesis on D can be removed [Efi25, Proposition 1.9].

Example 6.11. Let $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be compactly generated. Then C is rigid if and only if $\mathbb{1}_C$ is compact and every compact object of C is dualizable, if and only if the compact objects of C are precisely the dualizable objects [GR17, Chapter 1, Lemma 9.1.5 and Corollary 9.1.7]. Indeed, if $\mathbb{1}_C$ is compact, then dualizable objects are compact (as $\mathrm{Map}(x, -) \simeq \mathrm{Map}(\mathbb{1}_C, x^\vee \otimes -)$), and conversely if all compact objects are dualizable, then $C = \mathrm{Ind}(C^\omega)$ is generated by objects whose identity is trace-class, so (2) of Theorem 6.8 holds. Examples include $\mathrm{D}(R)$ for a commutative ring R, more generally $\mathrm{D}_{\mathrm{qc}}(X)$ for a qcqs scheme X, and Sp itself.

Example 6.12. Let X be a compact Hausdorff space. Then $\mathrm{Shv}(X; \mathrm{Sp})$ is rigid; this can be seen as a manifestation of Verdier duality, see [Efi25, Example 1.4]. For instance, the unit (the constant sheaf) is compact since global sections $\Gamma(X; -)$ commute with filtered colimits for compact X [Lur09, Corollary 7.3.4.12]. In particular, $\mathrm{Shv}(X; \mathrm{Sp})$ is dualizable and self-dual. For every presentable stable category D we have $\mathrm{Shv}(X; D) \simeq \mathrm{Shv}(X; \mathrm{Sp}) \otimes D$ [KNP24, Proposition 2.8.7], so $\mathrm{Shv}(X; D)$ is dualizable whenever D is. We will study these categories in detail in Chapter 9.

Example 6.13. If X is merely locally compact Hausdorff and not compact, then $\mathrm{Shv}(X; \mathrm{Sp})$ is not rigid, since the unit is not compact. It is still dualizable, and even *locally rigid* in the sense of Ramzi [Ram24b, Definition 4.5 and Example 4.10] (see also [Efi25, Definition 1.36 and Example 1.38] and [KNP24, Example 4.2.9, Corollary 4.4.13]): it satisfies condition (2) of Definition 6.1. Indeed, for $j\colon X \hookrightarrow X^+$ the one-point compactification, $j_!$ identifies $\mathrm{Shv}(X; \mathrm{Sp})$ with a smashing ideal in the rigid category $\mathrm{Shv}(X^+; \mathrm{Sp})$.

6.2 Nuclear modules and condensed mathematics

The most important examples of dualizable categories which are not compactly generated arise in condensed mathematics, as categories of *nuclear modules*. These are extracted from large compactly generated categories of "complete" modules, which are not rigid, via the following general construction due to Clausen and Scholze [Sch26].

Throughout this section, let $C \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be compactly generated with compact unit. We write $x^\vee = \underline{\mathrm{Hom}}(x, \mathbb{1}_C)$ for the predual of x.

Definition 6.14 ([Efi25, Definition 1.25]; cf. [CS26, Definition 8.5] and [Sch26, Definitions 13.10 and 13.12])**.** An object $M \in C$ is called *nuclear* if every morphism $P \to M$ from a compact object $P \in C^\omega$ is trace-class. We denote by $\mathrm{Nuc}(C) \subseteq C$ the full subcategory of nuclear objects. An object M is called *basic nuclear* if $M \simeq \mathrm{colim}(P_0 \to P_1 \to P_2 \to \cdots)$ with each P_n compact and each map $P_n \to P_{n+1}$ trace-class.

Note that a compact object P is nuclear if and only if id_P is trace-class, i.e. if and only if P is dualizable (Remark 6.6(3)). So the nuclear objects should be thought of as the objects which are "locally dualizable".

The definition only asks that every morphism $P \to M$ from a compact object be trace-class, i.e. that a certain map is surjective on π_0. We will see that this map is then automatically an equivalence. The key input is the following elementary observation, which says that trace-class morphisms can be "pulled back" through the canonical map $y \otimes P^\vee \to \underline{\mathrm{Hom}}(P, y)$. For $x \in C$ and objects $P, Q \in C$ we write

$$\lambda_x \colon \mathrm{Map}_C(Q, x \otimes P^\vee) \to \mathrm{Map}_C(Q \otimes P, x), \qquad s \mapsto (x \otimes \mathrm{ev}_P)(s \otimes P),$$

where $\mathrm{ev}_P \colon P^\vee \otimes P \to \mathbb{1}_C$ is the evaluation. Under the adjunction $\mathrm{Map}_C(Q \otimes P, x) \simeq \mathrm{Map}_C(Q, \underline{\mathrm{Hom}}(P, x))$, this is the map induced by the canonical map $x \otimes P^\vee \to \underline{\mathrm{Hom}}(P, x)$. It is natural in x. For $Q = \mathbb{1}_C$, a morphism $P \to x$ is trace-class if and only if its homotopy class lies in the image of $\pi_0 \lambda_x$, by Remark 6.6(1).

Lemma 6.15. *Let $f \colon x \to y$ be a trace-class morphism in C, and let $P, Q \in C$. Then there is a map of sets*

$$\psi_f \colon \pi_0 \,\mathrm{Map}_C(Q \otimes P, x) \to \pi_0 \,\mathrm{Map}_C(Q, y \otimes P^\vee)$$

such that $\lambda_y \circ \psi_f = f \circ (-)$ and $\psi_f \circ \lambda_x = (f \otimes P^\vee) \circ (-)$ on homotopy classes.

Proof. By Remark 6.6(1), there is a map $t \colon \mathbb{1}_C \to y \otimes x^\vee$ such that $f \simeq (y \otimes \mathrm{ev}_x)(t \otimes x)$. For $b \colon Q \otimes P \to x$, let $\tilde{b} \colon x^\vee \otimes Q \to P^\vee$ be the map adjoint to $\mathrm{ev}_x(x^\vee \otimes b) \colon x^\vee \otimes Q \otimes P \to \mathbb{1}_C$, so that $\mathrm{ev}_P(\tilde{b} \otimes P) \simeq \mathrm{ev}_x(x^\vee \otimes b)$, and put

$$\psi_f(b) := (y \otimes \tilde{b})(t \otimes Q) \colon Q \to y \otimes x^\vee \otimes Q \to y \otimes P^\vee.$$

Here and below we suppress the unit and symmetry isomorphisms. Then

$$\lambda_y(\psi_f(b)) = (y \otimes \mathrm{ev}_P)(y \otimes \tilde{b} \otimes P)(t \otimes Q \otimes P) \simeq (y \otimes \mathrm{ev}_x)(y \otimes x^\vee \otimes b)(t \otimes Q \otimes P) \simeq (y \otimes \mathrm{ev}_x)(t \otimes x)\, b \simeq fb.$$

Conversely, let $s\colon Q \to x \otimes P^\vee$ and $b = \lambda_x(s) = (x \otimes \mathrm{ev}_P)(s \otimes P)$. Then $\mathrm{ev}_x(x^\vee \otimes b) \simeq (\mathrm{ev}_x \otimes \mathrm{ev}_P)(x^\vee \otimes s \otimes P)$, which is also adjoint to $(\mathrm{ev}_x \otimes P^\vee)(x^\vee \otimes s)\colon x^\vee \otimes Q \to P^\vee$; hence $\tilde{b} \simeq (\mathrm{ev}_x \otimes P^\vee)(x^\vee \otimes s)$. Therefore

$$\psi_f(\lambda_x(s)) \simeq (y \otimes \mathrm{ev}_x \otimes P^\vee)(y \otimes x^\vee \otimes s)(t \otimes Q) \simeq (y \otimes \mathrm{ev}_x \otimes P^\vee)(t \otimes s) \simeq (y \otimes \mathrm{ev}_x \otimes P^\vee)(t \otimes x \otimes P^\vee)\, s \simeq (f \otimes P^\vee)\, s.$$

□

Lemma 6.16. *Let $M \simeq \mathrm{colim}_{i \in I} M_i$ for a directed poset I such that for every i there is $j \geq i$ with $M_i \to M_j$ trace-class. Let $P, Q \in C$ be such that Q and $Q \otimes P$ are compact. Then the map $\hom_C(Q, M \otimes P^\vee) \to \hom_C(Q \otimes P, M)$ induced by $M \otimes P^\vee \to \underline{\mathrm{Hom}}(P, M)$ is an equivalence of spectra.*

Proof. Since $\pi_n \hom_C(Q, -) = \pi_0 \mathrm{Map}_C(\Sigma^n Q, -)$ for $n \in \mathbb{Z}$, and $\Sigma^n Q$ satisfies the same hypotheses as Q, it suffices to show that $\pi_0 \lambda_M : \pi_0 \mathrm{Map}_C(Q, M \otimes P^\vee) \to \pi_0 \mathrm{Map}_C(Q \otimes P, M)$ is a bijection. As Q and $Q \otimes P$ are compact and $- \otimes P^\vee$ preserves colimits, and as π_0 commutes with filtered colimits, $\pi_0 \lambda_M$ is the colimit over $i \in I$ of the maps $\pi_0 \lambda_{M_i}$. Write $t_{ij} : M_i \to M_j$ for the transition maps.

Surjectivity: an element of the target is represented by some $b\colon Q \otimes P \to M_i$. Choose $j \geq i$ such that t_{ij} is trace-class. By Lemma 6.15, $\lambda_{M_j}(\psi_{t_{ij}}(b)) = t_{ij} b$, which represents the same element.

Injectivity: let $s, s'\colon Q \to M_i \otimes P^\vee$ with the same image in the colimit. Then there is $i' \geq i$ with $t_{ii'} \lambda_{M_i}(s) = t_{ii'} \lambda_{M_i}(s')$, i.e. $\lambda_{M_{i'}}(\sigma) = \lambda_{M_{i'}}(\sigma')$ for $\sigma = (t_{ii'} \otimes P^\vee)s$ and $\sigma' = (t_{ii'} \otimes P^\vee)s'$. Choose $j \geq i'$ such that $f = t_{i'j}$ is trace-class. By Lemma 6.15, $(f \otimes P^\vee)\sigma = \psi_f(\lambda_{M_{i'}}(\sigma)) = \psi_f(\lambda_{M_{i'}}(\sigma')) = (f \otimes P^\vee)\sigma'$, so s and s' have the same image in $\pi_0 \mathrm{Map}_C(Q, M_j \otimes P^\vee)$. □

Proposition 6.17 (Clausen–Scholze; see [CS26, Theorem 8.6] and [Efi25, Propositions 1.27 and 1.28])**.**

(1) *For $M \in C$, the following are equivalent:*

   (a) M is nuclear;

   (b) for every compact P, the map $\hom_C(\mathbb{1}_C, P^\vee \otimes M) \to \hom_C(P, M)$ is an equivalence of spectra (equivalently, since we may replace P by its shifts, the corresponding map of mapping spaces is an equivalence for every compact P);

   (d) $M \simeq \mathrm{colim}_{i \in I} M_i$ for a directed poset I and compact objects M_i, such that for every i there is $j \geq i$ with $M_i \to M_j$ trace-class.

   Moreover, these conditions are implied by

   (c) for every compact P, the map $P^\vee \otimes M \to \underline{\mathrm{Hom}}(P, M)$ is an equivalence,

   and if the tensor product of any two compact objects of C is compact, then (c) is equivalent to (a), (b) and (d).

(2) *The subcategory $\mathrm{Nuc}(C) \subseteq C$ is closed under colimits. It is ω_1-presentable, and its ω_1-compact objects are precisely the basic nuclear objects.*

The main step is to turn the pointwise trace-class condition into a presentation whose transition maps are eventually trace-class. The interleaving lemma then upgrades surjectivity on π_0 to the isomorphism of mapping spectra in (b). For (2), we decompose these presentations into countable pieces, as in the proof of ω_1-presentability for compactly assembled categories.

Proof. We first prove (1). (b) $\Rightarrow$ (a): Taking π_0 of the mapping spaces, every morphism $P \to M$ lies in the image of $\pi_0\lambda_M$ (with $Q = \mathbb{1}_C$), i.e. is trace-class.

(a) $\Rightarrow$ (d): Since C is compactly generated, M is a filtered colimit of compact objects, and by Theorem A.8 we may write $M \simeq \operatorname{colim}_{i\in I} M_i$ with I a directed poset and M_i compact; denote by t_{ij} the transition maps and by $\iota_i \colon M_i \to M$ the canonical maps. Fix i. Since ι_i is trace-class, it is $\lambda_M(w)$ for some $w \in \pi_0 \operatorname{Map}_C(\mathbb{1}_C, M \otimes M_i^\vee)$. As $\mathbb{1}_C$ is compact and $M \otimes M_i^\vee \simeq \operatorname{colim}_j M_j \otimes M_i^\vee$, the class w is the image of some $w' \in \pi_0 \operatorname{Map}_C(\mathbb{1}_C, M_j \otimes M_i^\vee)$ with $j \geq i$. Then $g := \lambda_{M_j}(w') \colon M_i \to M_j$ is trace-class, and $\iota_j g \simeq \lambda_M(w) \simeq \iota_i$ by naturality of λ. As M_i is compact, $\pi_0 \operatorname{Map}_C(M_i, M) \simeq \operatorname{colim}_k \pi_0 \operatorname{Map}_C(M_i, M_k)$, so there is $k \geq j$ with $t_{jk}g \simeq t_{ik}$. Since trace-class morphisms form an ideal (Remark 6.6(2)), t_{ik} is trace-class.

(d) $\Rightarrow$ (b): This is Lemma 6.16 with $Q = \mathbb{1}_C$.

(c) $\Rightarrow$ (b): Apply $\hom_C(\mathbb{1}_C, -)$ and use $\hom_C(\mathbb{1}_C, \underline{\mathrm{Hom}}(P, M)) \simeq \hom_C(P, M)$.

(d) $\Rightarrow$ (c), if compact objects are closed under tensor products: For compact Q, the map $\hom_C(Q, P^\vee \otimes M) \to \hom_C(Q, \underline{\mathrm{Hom}}(P, M)) \simeq \hom_C(Q \otimes P, M)$ is an equivalence by Lemma 6.16. Since C is compactly generated, the compact objects detect equivalences, so $P^\vee \otimes M \to \underline{\mathrm{Hom}}(P, M)$ is an equivalence.

We now prove (2). By (1), M is nuclear if and only if the maps in (b) are equivalences. Since $\mathbb{1}_C$ and P are compact, the functors $\hom_C(\mathbb{1}_C, P^\vee \otimes -)$ and $\hom_C(P, -)$ preserve filtered colimits, and being exact they preserve all small colimits. Hence $\mathrm{Nuc}(C)$ is closed under colimits in C; in particular, colimits in $\mathrm{Nuc}(C)$ are computed in C.

Basic nuclear objects are nuclear by (d) (with $I = \mathbb{N}$). They are countable colimits of compact objects, hence ω_1-compact in C (Corollary A.4), and hence also in $\mathrm{Nuc}(C)$.

Next, every nuclear M is an ω_1-filtered colimit of basic nuclear objects. Write $M \simeq \operatorname{colim}_{i\in I} M_i$ as in (d), and choose for every i an element $\sigma(i) \geq i$ such that $M_i \to M_{\sigma(i)}$ is trace-class. Let Σ_σ be the set of countable directed subsets $J \subseteq I$ with $\sigma(J) \subseteq J$, ordered by inclusion. The proof of Proposition A.9 applies verbatim to Σ_σ instead of Σ (adding in the inductive construction also $\sigma(u)$ for $u \in U_k$), and shows that Σ_σ is ω_1-filtered and that $M \simeq \operatorname{colim}_{J\in\Sigma_\sigma} M_J$ with $M_J = \operatorname{colim}_{i\in J} M_i$. Each M_J is basic nuclear: enumerate $J = \{q_0, q_1, \dots\}$, and choose inductively $j_0 = q_0$ and $j_{n+1} \in J$ with $j_{n+1} \geq \sigma(j_n), q_{n+1}$. Then $n \mapsto j_n$ is cofinal (Lemma A.7), so $M_J \simeq \operatorname{colim}_n M_{j_n}$, and each $M_{j_n} \to M_{j_{n+1}}$ factors through the trace-class map $M_{j_n} \to M_{\sigma(j_n)}$, hence is trace-class.

The basic nuclear objects form an essentially small collection, since C^ω is essentially small. Thus $\mathrm{Nuc}(C)$ admits small colimits and is generated under ω_1-filtered colimits by an essentially small collection of ω_1-compact objects, so it is ω_1-accessible by [Lur09, Proposition 5.4.2.2], and hence ω_1-presentable.

Finally, let $X \in \mathrm{Nuc}(C)$ be ω_1-compact. Writing $X \simeq \operatorname{colim}_{J\in\Sigma_\sigma} X_J$ as above, the identity of X factors through some X_J, so X is a retract of a basic nuclear object $Y = \operatorname{colim}_n a_n$, where the a_n are compact and the transition maps $a_n \to a_{n+1}$ are trace-class. Now we argue as in the proof of Proposition A.13, applied in $C \simeq \mathrm{Ind}(C^\omega)$: by Lemma A.11 and Lemma A.12, we find a strictly increasing $f \colon \mathbb{N} \to \mathbb{N}$ with $f(n) \geq n$ and a natural transformation $\eta \colon a_\bullet \to a_{f(\bullet)}$ such that $X \simeq \operatorname{colim}_n \Phi(n, n)$, where $\Phi(k, n) = a_{f^k(n)}$. The transition map $\Phi(n, n) \to \Phi(n+1, n+1)$ is the composite of the transition map $a_{f^n(n)} \to a_{f^n(n+1)}$ of $a_\bullet$, which is trace-class as $f^n(n+1) > f^n(n)$, with $\eta_{f^n(n+1)}$. So it is trace-class, and X is basic nuclear. □

Remark 6.18. Proposition 6.17 shows that the condition in Definition 6.14, which only asks that the maps $\pi_0 \operatorname{Map}_C(\mathbb{1}_C, P^\vee \otimes M) \to \pi_0 \operatorname{Map}_C(P, M)$ be surjective, already implies that $\hom_C(\mathbb{1}_C, P^\vee \otimes M) \to \hom_C(P, M)$ is an equivalence. Clausen–Scholze [CS26, Definition 8.5] instead define nuclearity by the latter condition (and [Sch26, Definition 13.10] by an analogous condition for analytic rings), whereas Definition 6.14 follows [Efi25, Definition 1.25]. The two definitions agree; the injectivity comes from the fact that, by (a) $\Rightarrow$ (d), M is a filtered colimit of compact objects along trace-class maps, for which the two sides become interleaved (Lemma 6.16). The hypothesis on compact objects in (1)(c) holds for instance for the derived categories of solid and liquid modules, which are generated by compact objects closed under tensor products.

In general, $\operatorname{Nuc}(C)$ need not be rigid, or even dualizable. The following criterion is the one that is used in practice.

Proposition 6.19 ([Efi25, Propositions 1.23 and 1.29], [Ram24b, Construction 4.79 and Theorem 4.81])**.** *Suppose that every trace-class morphism between compact objects of C is a composite of two trace-class morphisms between compact objects. Then* $\operatorname{Nuc}(C)$ *is a rigid symmetric monoidal category. More precisely,* $\operatorname{Nuc}(C)$ *is equivalent to the* rigidification C^{rig} *of C, the value at C of the right adjoint to the inclusion of rigid categories into* $\operatorname{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\kappa})$*, for an uncountable regular cardinal κ for which C belongs to this category. Here the morphisms preserve κ-compact objects, and the tensor product preserves them separately. This value is independent of the sufficiently large choice of κ [Efi25, Proposition 1.23].*

Sketch. By [Efi25, Proposition 1.23 and Remark 1.24(i)], based on [Ram24b, Theorem 4.81 and Corollary 4.75], the rigidification $C^{\mathrm{rig}} \subseteq C$ is the full subcategory generated under colimits by colimits $\operatorname{colim}_{a\in\mathbb{Q}} F(a)$ of diagrams $F\colon (\mathbb{Q}, \leq) \to C^\omega$ in which every map $F(a) \to F(b)$, $a < b$, is trace-class. Such colimits are nuclear by Proposition 6.17(1)(d), so $C^{\mathrm{rig}} \subseteq \operatorname{Nuc}(C)$. Conversely, let $P_0 \to P_1 \to \cdots$ be a sequence of trace-class maps between compact objects. By the assumption and Lemma 4.7(2), applied to $D = C^\omega$ and the class of trace-class morphisms (which is closed under composition by Remark 6.6(2)), it extends to a functor $F\colon \mathbb{Q}_{\geq 0} \to C^\omega$ with trace-class transition maps and the same colimit. Restricting F to the cofinal subset $\mathbb{Q}_{>0} \cong \mathbb{Q}$, we see that the basic nuclear objects lie in C^{rig}. By Proposition 6.17(2), these generate $\operatorname{Nuc}(C)$. □

Remark 6.20. Compare this with the construction of $(C, S)^{\mathrm{dual}}$ in Clausen's lectures: given a class S of morphisms forming an ideal with $S = S^2$ (and some further closure properties), one obtains a dualizable category generated by colimits along morphisms in S. For S the trace-class morphisms of C, this gives C^{rig}. The condition $S = S^2$ in Proposition 6.19 is what allows one to work with sequential colimits instead of $\mathbb{Q}$-indexed ones (Lemma 4.7; for compact morphisms in compactly assembled categories, see Proposition 4.8), cf. also the construction of C_S in Lecture 7 of the Bonn workshop.

Examples from condensed mathematics

We now describe the main examples, without proofs. We refer to [Sch26] for the solid and liquid theory, to [CS26] for the gaseous theory, and to [And23] and [Efi25, Section 7.2] for the comparison with Efimov's theory.

Example 6.21 (Solid modules and nuclear modules over formal schemes)**.** *Ambient category.* Let $\mathrm{D}(\mathrm{Solid}_{\mathbb{Z}})$ be the derived category of light solid abelian groups, as in Efimov's convention in [Efi25, Section 7.2]. This is a compactly generated symmetric monoidal category, with compact projective generator $\prod_{\mathbb{N}} \mathbb{Z}$ and compact unit $\mathbb{Z}$. It is far from rigid: the compact object $\prod_{\mathbb{N}} \mathbb{Z}$ is not dualizable (its predual is $\bigoplus_{\mathbb{N}} \mathbb{Z}$). In fact, there is a symmetric monoidal equivalence

$$\mathrm{D}(\mathrm{Solid}_{\mathbb{Z}}) \simeq \mathrm{Ind}\left(\mathrm{D}^b(\mathrm{Proj}^{\omega_1\text{-}}\mathbb{Z})^{\mathrm{op}}\right),$$

where $\mathrm{D}^b(\mathrm{Proj}^{\omega_1\text{-}}\mathbb{Z}) \subseteq \mathrm{D}(\mathbb{Z})$ is the stable subcategory generated by $\bigoplus_{\mathbb{N}} \mathbb{Z}$, under which $\prod_{\mathbb{N}} \mathbb{Z}$ corresponds to $(\bigoplus_{\mathbb{N}} \mathbb{Z})^{\mathrm{op}}$ [Efi25, Section 7.2].

Completion and nuclear objects. More generally, let R be a commutative ring and $I = (a_1, \dots, a_m) \subseteq R$ a finitely generated ideal, and let $R^\wedge_I$ denote its derived I-adic completion. We use the induced solid analytic structure, with the notation of [Efi25, Section 7.2]. All completions and tensor products in this example are derived; for Noetherian R, the derived completion of R agrees with its ordinary completion. Its category of modules $\mathrm{D}(\mathrm{Solid}_{(R^\wedge_I,\mathbb{Z})}) = \mathrm{Mod}_{R^\wedge_I}(\mathrm{D}(\mathrm{Solid}_{\mathbb{Z}}))$ is compactly generated by $R^\wedge_I \otimes^{\square} \prod_{\mathbb{N}} \mathbb{Z}$, with compact unit. The category of *nuclear solid modules* is

$$\mathrm{Nuc}^{CS}(R^\wedge_I) := \mathrm{Nuc}\left(\mathrm{D}(\mathrm{Solid}_{(R^\wedge_I,\mathbb{Z})})\right),$$

with respect to this specified solid structure. (For $R = \mathbb{Z}$ and $I = 0$ one recovers $\mathrm{D}(\mathbb{Z})$: an object of $\mathrm{D}(\mathrm{Solid}_{\mathbb{Z}})$ is nuclear if and only if it is discrete [CS26, Lecture VIII, Exercise 1].) One has $\mathrm{Nuc}^{CS}(R^\wedge_I) \simeq \mathrm{D}(\mathrm{Solid}_{(R^\wedge_I,\mathbb{Z})})^{\mathrm{rig}}$.

Rigidity and invariants. The key technical input is that the preduals of compact objects are nuclear, which goes back to Andreychev [And23, Lemma 3.8 and Korollar 3.14]; see [Efi25, Propositions 7.4 and 7.5] (Efimov deduces the rigidity from this via [Efi25, Propositions 1.33 and 1.34] rather than via Proposition 6.19). The tensor products here are those of the complete solid module category; the cited propositions supply the nuclearity and rigidity statements. Hence:

- $\mathrm{Nuc}^{CS}(R^\wedge_I)$ is rigid, hence dualizable, but in general not compactly generated.
- Its compact objects are the perfect complexes over $R^\wedge_I$.
- Its continuous K-theory is $K^{\mathrm{cont}}(\mathrm{Nuc}^{CS}(R^\wedge_I)) \simeq \lim_n K(R/\!\!/(a_1^n, \dots, a_m^n))$, which is $\lim_n K(R/I^n)$ if R is Noetherian (Efimov [Efi25, Corollary 7.10], as conjectured by Clausen–Scholze), cf. Theorem 1.15.

Failure of compact generation. To see that compact generation fails already for $\mathrm{Nuc}^{CS}(\mathbb{Z}_p)$, suppose otherwise. Its compact objects are $\mathrm{Perf}(\mathbb{Z}_p)$, so the canonical $\mathbb{Z}$-linear functor $\mathrm{D}(\mathbb{Z}_p) \to \mathrm{Nuc}^{CS}(\mathbb{Z}_p)$ would be an equivalence. But Efimov computes

$$\mathrm{HH}^{\mathrm{cont}}(\mathrm{Nuc}^{CS}(\mathbb{Z}_p)/\mathbb{Z}) \simeq \mathbb{Z}_p$$

[Efi25, Corollary 7.20(i)], whereas

$$\mathrm{HH}_1(\mathbb{Z}_p/\mathbb{Z}) \cong \Omega^1_{\mathbb{Z}_p/\mathbb{Z}} \cong \Omega^1_{\mathbb{Q}_p/\mathbb{Q}} \neq 0.$$

Here $\mathrm{HH}^{\mathrm{cont}}$ denotes the continuous extension of relative Hochschild homology, which agrees with ordinary Hochschild homology on compactly generated categories. This gives the required contradiction; see also the discussion after [Efi25, Remark 7.21].

Operator-algebra presentation. There is also an operator-algebra presentation, analogous to the compact-operator example of Example 1.14. Let R be a commutative ring, $I = (f_1, \dots, f_m) \subseteq R$ a finitely generated ideal. In this paragraph all completions and tensor products are derived, following [Efi25, Section 7.2]. Let $P := (\bigoplus_{\mathbb{N}} R)^\wedge_I$ be the I-completed countable direct sum, which plays the role of a separable Hilbert space. In place of $B(\mathcal{H})$ we take the endomorphism ring $\mathrm{End}_R(P)$, and in place of the Calkin algebra $B(\mathcal{H})/\mathcal{K}$ the I-adically completed endomorphism ring

$$\mathrm{Calk}^{\mathrm{top}}_R(P) := \mathrm{End}_{\mathrm{Calk}(R)}(P)^\wedge_I$$

of P in the Calkin category of $\mathrm{Perf}(R)$. The fiber of $\mathrm{End}_R(P) \to \mathrm{Calk}^{\mathrm{top}}_R(P)$ is $\big(\prod_{\mathbb{N}} R \otimes_R \bigoplus_{\mathbb{N}} R\big)^\wedge_I$, which one may think of as the I-adic completion of the endomorphisms of finite rank, just as $\mathcal{K}$ is the norm closure of the operators of finite rank. Efimov shows that $\mathrm{End}_R(P) \to \mathrm{Calk}^{\mathrm{top}}_R(P)$ is a homological epimorphism, whose kernel is the category of *nuclear* modules over $R^\wedge_I$ in the sense of Clausen–Scholze [Efi25, Corollary 7.7]:

$$\mathrm{Nuc}^{CS}(R^\wedge_I) \simeq \mathrm{D}\big(\mathrm{End}_R(P) \to \mathrm{Calk}^{\mathrm{top}}_R(P)\big).$$

Efimov's variant. Efimov also introduced a variant $\mathrm{Nuc}(R^\wedge_I)$ [Efi25, Definition 3.28], which can be described as $\lim^{\mathrm{dual}}_n \mathrm{D}(R/\!\!/(a_1^n, \dots, a_m^n))$, the limit computed in $\mathrm{Pr}^{\mathrm{dual}}$ [Efi25, Proposition 5.23]. It receives a fully faithful strongly continuous functor from $\mathrm{Nuc}^{CS}(R^\wedge_I)$ [Efi25, Corollary 7.6], which induces an equivalence on compact objects, both sides having compact objects $\mathrm{Perf}(R^\wedge_I)$ [Efi25, Proposition 3.29]. It can equivalently be described as the rigidification $\mathrm{D}_{I\text{-compl}}(R)^{\mathrm{rig}}$ [Efi25, Proposition 4.3] of the category of derived I-complete R-modules; the latter need not itself be rigid. For example, for $R = \mathbb{Z}$ and $I = (p)$ its unit $\mathbb{Z}_p$ is not compact.

Example 6.22 (Banach spaces as nuclear modules)**.** Heuristically, nuclear modules are "Banach-type" objects, as opposed to the compact objects such as $\prod_{\mathbb{N}} \mathbb{Z}_p$, which are of "profinite" or "compact" type. For instance, for every profinite set S the module $C(S, \mathbb{Z}_p)$ of continuous functions is nuclear over $\mathbb{Z}_p^\square$ [And23, Lemma 3.8]. Taking $S = \mathbb{N} \cup \{\infty\}$, it follows that the p-adic Banach space $c_0(\mathbb{N}, \mathbb{Z}_p)$ of null sequences, a direct summand of $C(S, \mathbb{Z}_p)$, is nuclear, and inverting p gives nuclear solid $\mathbb{Q}_p$-vector spaces such as $c_0(\mathbb{N}, \mathbb{Q}_p)$. In contrast, the compact object $\prod_{\mathbb{N}} \mathbb{Z}_p$ of $\mathrm{D}(\mathrm{Solid}_{\mathbb{Z}_p})$ is not nuclear: a compact nuclear object is dualizable, and the dualizable objects of $\mathrm{D}(\mathrm{Solid}_{\mathbb{Z}_p})$ are just the perfect complexes of $\mathbb{Z}_p$-modules [Sch26, Theorem 14.10].

Example 6.23 (Liquid and gaseous modules)**.** *Liquid modules.* Over the real and complex numbers, the solid theory is replaced by the *liquid* theory: for $0 < p \leq 1$ there is a compactly generated symmetric monoidal category $\mathrm{D}(\mathrm{Liq}_p(\mathbb{R}))$ of p-liquid $\mathbb{R}$-vector spaces [Sch26], and one can form $\mathrm{Nuc}(\mathbb{R}^{p\text{-liq}})$. The nuclear objects of $\mathrm{D}(\mathrm{Liq}_p)$ do not depend on p [CS26, Corollary 8.18].

The lecture model. According to Clausen [Cla23b], there is an explicit model for the rigidification of the category of liquid $\mathbb{C}$-vector spaces in terms of operator algebras:

$$\mathrm{D}^{\mathrm{liq}}(\mathbb{C})^{\mathrm{rig}} \simeq \mathrm{D}\big(B(\mathcal{H}) \to B(\mathcal{H})/S_{0+}\big),$$

Here $S_{0+} = \bigcap_{p>0} S_p$ is the ideal of operators which are p-summable for all $p > 0$. The kernel on the right and its Hilbert-space model are constructed in Proposition 5.13.

Gaseous modules. The gaseous theory of Clausen–Scholze [CS26] provides yet another analytic ring structure on $\mathbb{C}$, with a category of nuclear modules $\mathrm{Nuc}(\mathbb{C}^{\mathrm{gas}})$. Rigidification produces rigid, hence dualizable, categories whose continuous K-theory is a candidate for analytic K-theory, cf. Section 1.5. Failure of compact generation is a separate assertion; we do not deduce it from rigidity.

Remark 6.24. The reason for working with Nuc rather than with all complete (solid) modules is that the former has much better formal properties: for instance, it is rigid, hence dualizable with a good continuous K-theory, and $\mathrm{Spa}(A, A^+) \mapsto \mathrm{Nuc}(A)$ satisfies descent for the analytic topology on affinoid adic spaces (Andreychev), see Section 1.5.

7 Calkin categories, colimits and limits of dualizable categories

In this chapter we study the category $\mathrm{Pr}^{\mathrm{dual}}$ of dualizable categories and strongly continuous functors as a category in its own right. Colimits in $\mathrm{Pr}^{\mathrm{dual}}$ turn out to be computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and filtered colimits in $\mathrm{Pr}^{\mathrm{dual}}$ are much better behaved than in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$: they preserve fully faithful functors and short exact sequences. Limits in $\mathrm{Pr}^{\mathrm{dual}}$ are more subtle and are in general *not* computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Kernels and pullbacks exist, but they are given by the largest dualizable subcategories of the corresponding kernels and pullbacks in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ on which the relevant functors are strongly continuous.

Along the way we introduce the continuous Calkin category, which is the key ingredient in the definition of localizing invariants of dualizable categories (Chapter 8). The main result about it is that it sends short exact sequences of dualizable categories to short exact sequences of small categories (Proposition 7.9). Its functoriality relies on the description of colimits, which we therefore treat first.

We follow [Efi24, Sections 2.9–2.12]. Recall from Section 2.3 that $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ denotes the subcategory of presentable stable categories and strongly continuous functors. By Remark 3.14 and Remark 3.13, we have full subcategories

$$\mathrm{Pr}^{\mathrm{L}}_{\omega,\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{dual}} \subseteq \mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}.$$

7.1 Colimits of dualizable categories

Colimits in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ are computed as limits along the right adjoints (Section A.5). If all functors in the diagram are strongly continuous, then the right adjoints preserve colimits, and this allows us to show that colimits in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ and in $\mathrm{Pr}^{\mathrm{dual}}$ are computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Proposition 7.1 ([Efi24, Proposition 1.7])**.** *Let $I \to \mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$, $i \mapsto C_i$, be a small diagram, let $C := \mathrm{colim}_i C_i$ be its colimit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and let $F_i \colon C_i \to C$ be the structure functors.*

(1) *The functors F_i are strongly continuous.*

(2) *A colimit-preserving functor $G \colon C \to D$ to a presentable stable category D is strongly continuous if and only if each composite $G \circ F_i$ is strongly continuous.*

In particular, $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ is cocomplete, and the inclusion $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ preserves colimits.

Proof. We identify C with the limit $\lim_i C_i$ in $\widehat{\mathrm{Cat}}$, taken along the right adjoints F_{ij}^R of the transition functors F_{ij}. Under this identification, $F_i^R \colon C \to C_i$ is the projection. Since the transition functors F_{ij}^R preserve colimits, colimits in $\lim_i C_i$ exist and are computed pointwise [Lur09, Remark 5.4.7.13]. In particular, the projections F_i^R preserve colimits, which proves (1). Similarly, the right adjoint $G^R \colon D \to C \simeq \lim_i C_i$ corresponds to the compatible family of functors $F_i^R G^R \simeq (GF_i)^R$. As colimits in the limit are computed pointwise, G^R preserves colimits if and only if each $(GF_i)^R$ does, which proves (2).

For the final assertion, note that $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}}(C, D) \subseteq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(C, D)$ is a union of path components, and similarly for the C_i. By (2), the equivalence $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(C, D) \simeq \lim_i \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(C_i, D)$ restricts to an equivalence $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}}(C, D) \simeq \lim_i \mathrm{Map}_{\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}}(C_i, D)$, so that C, together with the cocone of (1), is also a colimit in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$. □

Proposition 7.2 ([Efi24, Proposition 2.63])**.** *The full subcategory* $\mathrm{Pr}^{\mathrm{dual}} \subseteq \mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ *is closed under small colimits. In particular,* $\mathrm{Pr}^{\mathrm{dual}}$ *is cocomplete, and the inclusion* $\mathrm{Pr}^{\mathrm{dual}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ *preserves colimits.*

Proof. Let $I \to \mathrm{Pr}^{\mathrm{dual}}$, $i \mapsto C_i$, be a small diagram, and let $C = \mathrm{colim}_i C_i$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. By Proposition 7.1, it suffices to show that C is dualizable. The structure functors $F_i \colon C_i \to C$ are strongly continuous by Proposition 7.1(1). Their right adjoints are the projections $\lim_i C_i \to C_i$, which are jointly conservative. Hence the localizing subcategory $L \subseteq C$ generated by the essential images of the F_i, which is generated by a set of objects and hence presentable, satisfies $L^\perp = 0$. By Proposition 2.12(1), we get $L = C$. So C is dualizable by Proposition 4.25. □

As a first consequence, we can describe the epimorphisms in these categories.

Proposition 7.3 ([Efi24, Propositions 1.13 and 1.14 and Corollary 2.64])**.** *A morphism* $F \colon C \to D$ *in* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ *or* $\mathrm{Pr}^{\mathrm{dual}}$ *is an epimorphism if and only if it is a localization, i.e.* F^R *is fully faithful.*

Proof. A morphism $F \colon C \to D$ in a category with pushouts is an epimorphism if and only if the codiagonal $\nabla \colon D \sqcup_C D \to D$ is an equivalence. By Propositions 7.1 and 7.2, pushouts in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ and $\mathrm{Pr}^{\mathrm{dual}}$ are computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. So it suffices to treat $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Computing the pushout as a pullback along the right adjoints, we have $D \sqcup_C D \simeq D \times_C D$, where both functors $D \to C$ are F^R, and the right adjoint of ∇ is the diagonal $\delta \colon D \to D \times_C D$. So F is an epimorphism if and only if δ is an equivalence. If F^R is fully faithful, then F^R is a monomorphism in $\widehat{\mathrm{Cat}}$, so δ is an equivalence. Conversely, suppose that δ is an equivalence, and let $x, y \in D$. Then the map

$$\begin{aligned}\hom_D(x, y) \to \hom_{D \times_C D}(\delta x, \delta y) &\simeq \hom_D(x, y) \times_{\hom_C(F^R x, F^R y)} \hom_D(x, y) \\ &\simeq \hom_D(x, y) \oplus \mathrm{fib}\left(\hom_D(x, y) \to \hom_C(F^R x, F^R y)\right)\end{aligned}$$

is the diagonal, and it is an equivalence. Here the splitting of the pullback is induced by the first projection and its section given by the diagonal, so the diagonal corresponds to the inclusion of the first summand. We find that $\hom_D(x, y) \to \hom_C(F^R x, F^R y)$ is an equivalence. So F^R is fully faithful. □

Remark 7.4. Dually, the monomorphisms in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ and in $\mathrm{Pr}^{\mathrm{dual}}$ are precisely the fully faithful functors [Efi24, Propositions 1.13 and 1.14 and Corollary 2.87]. For $\mathrm{Pr}^{\mathrm{dual}}$, this requires the description of pullbacks in $\mathrm{Pr}^{\mathrm{dual}}$, see Corollary 7.22 below.

7.2 Continuous Calkin categories

For a small idempotent complete stable category A, the Calkin category $\mathrm{Calk}_\kappa(A)$ of Section 2.5 fits into a short exact sequence $A \to \mathrm{Ind}(A)^\kappa \to \mathrm{Calk}_\kappa(A)$. For K-theory, the middle term is invisible: it admits countable coproducts, so its K-theory vanishes by the Eilenberg swindle, and $K(\mathrm{Calk}_\kappa(A)) \simeq \Sigma K(A)$. We now extend this construction to dualizable categories, which may have no nonzero compact objects at all. Throughout this section, κ denotes an uncountable regular cardinal.

Let C be dualizable. By Theorem 3.20(1), C is generated under colimits by ω_1-compact objects, hence by a set of κ-compact objects, so C is κ-compactly generated [Ram24a, Lemma A.21 and Addendum A.22]. By Proposition 3.17, the functor $\mathrm{colim}\colon \mathrm{Ind}(C^\kappa) \to C$ thus admits a fully faithful, strongly continuous left adjoint $\hat{y}\colon C \to \mathrm{Ind}(C^\kappa)$. As in the proof of Theorem 3.16, its cofiber in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is compactly generated.

Definition 7.5 ([Efi24, Definition 2.58])**.** Let C be a dualizable category. The *continuous κ-Calkin category* $\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)$ is the small idempotent complete stable category of compact objects of the quotient

$$\mathrm{Ind}(C^\kappa)/\hat{y}(C) \simeq \ker\big(\mathrm{colim}\colon \mathrm{Ind}(C^\kappa) \to C\big),$$

so that we have a strongly continuous localization sequence

$$C \xrightarrow{\hat{y}} \mathrm{Ind}(C^\kappa) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)).$$

Here the identification of the quotient with the kernel comes from Proposition 2.12: the right orthogonal of $\hat{y}(C)$ consists of those X with $\mathrm{Map}(x, \mathrm{colim}\, X) \simeq \mathrm{Map}(\hat{y}x, X) \simeq *$ for all $x \in C$.

Remark 7.6. If $C = \mathrm{Ind}(A)$ is compactly generated, the continuous Calkin category $\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)$ of Definition 7.5 agrees with $\mathrm{Calk}_\kappa(A)$ [Efi24, Proposition 2.59]. Indeed, by Example 3.6, the functor $\hat{y}\colon C \to \mathrm{Ind}(C^\kappa)$ is $\mathrm{Ind}(j)$ for the inclusion $j\colon A \hookrightarrow C^\kappa = \mathrm{Ind}(A)^\kappa$, so by Theorem 2.17(1) its cofiber is $\mathrm{Ind}\big((\mathrm{Ind}(A)^\kappa/A)^{\mathrm{idem}}\big) = \mathrm{Ind}(\mathrm{Calk}_\kappa(A))$.

Construction 7.7. Let $F\colon C \to D$ be a strongly continuous functor between dualizable categories. Then F preserves κ-compact objects, since F^R preserves colimits, so it restricts to a functor $F^\kappa\colon C^\kappa \to D^\kappa$. The left adjoint of $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ is the composite of $\hat{y}\colon C \to \mathrm{Ind}(C^\kappa)$ with the inclusion $\mathrm{Ind}(C^\kappa) \subseteq \mathrm{Ind}(C)$: indeed, by the proof of Proposition 3.17, the colimit functor of $\mathrm{Ind}(C)$ factors as $\mathrm{colim}_\kappa \circ r$, where r is the right adjoint of the inclusion $\mathrm{Ind}(C^\kappa) \subseteq \mathrm{Ind}(C)$. Since $\mathrm{Ind}(F)$ restricts to $\mathrm{Ind}(F^\kappa)$ on $\mathrm{Ind}(C^\kappa)$, Proposition A.27 gives a commutative square

$$\begin{array}{ccc} C & \xrightarrow{\hat{y}_C} & \mathrm{Ind}(C^\kappa) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathrm{Ind}(F^\kappa)} \\ D & \xrightarrow[\hat{y}_D]{} & \mathrm{Ind}(D^\kappa) \end{array}$$

of strongly continuous functors. Passing to cofibers in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$, which are computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ by Proposition 7.1, we obtain a strongly continuous functor $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(D))$. It preserves compact objects, and hence restricts to an exact functor

$$\mathrm{Calk}^{\mathrm{cont}}_\kappa(F)\colon \mathrm{Calk}^{\mathrm{cont}}_\kappa(C) \to \mathrm{Calk}^{\mathrm{cont}}_\kappa(D).$$

The squares above are coherent in F: the colimit functors form a natural transformation $\mathrm{Ind}((-)^\kappa) \to \mathrm{id}$, and taking mates along $\hat{y}_C \dashv \mathrm{colim}_C$ gives the natural maps $\hat{y}_D F \to \mathrm{Ind}(F^\kappa)\hat{y}_C$ used above. These are isomorphisms for strongly continuous F, by Proposition A.27. The compatibility of mates with composition supplies a natural transformation $\mathrm{id} \to \mathrm{Ind}((-)^\kappa)$ in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$. Applying the cofiber functor and then taking compact objects therefore defines a functor $\mathrm{Calk}^{\mathrm{cont}}_\kappa \colon \mathrm{Pr}^{\mathrm{dual}} \to \mathrm{Cat}_{\mathrm{perf}}$, including its higher coherences.

Under the identifications of the quotients with kernels, the functor $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)) \to \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(D))$ is the restriction of $\mathrm{Ind}(F^\kappa)$. Indeed, the functors $\mathrm{colim}_D \circ \mathrm{Ind}(F^\kappa)$ and $F \circ \mathrm{colim}_C$ preserve colimits and agree on C^κ, so $\mathrm{Ind}(F^\kappa)$ maps $\ker(\mathrm{colim}_C)$ into $\ker(\mathrm{colim}_D)$. The left adjoint q_C of the inclusion $\ker(\mathrm{colim}_C) \subseteq \mathrm{Ind}(C^\kappa)$ is given by $q_C(X) = \mathrm{cofib}(\hat{y}_C \,\mathrm{colim}_C\, X \to X)$. Using the square above, we get $\mathrm{Ind}(F^\kappa)(q_C X) \simeq q_D(\mathrm{Ind}(F^\kappa)X)$, which identifies the functor induced on the quotients with the restriction of $\mathrm{Ind}(F^\kappa)$.

The main property of the continuous Calkin construction is that it preserves short exact sequences. We first fix terminology.

Definition 7.8. A *short exact sequence* of dualizable categories is a strongly continuous localization sequence $C \to D \to E$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (Definition 2.11) in which C, D and E are dualizable. Equivalently, it is a sequence in $\mathrm{Pr}^{\mathrm{dual}}$ which is both a fiber sequence and a cofiber sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (Proposition 2.12).

By Proposition 3.19, it suffices to require that D is dualizable. A short exact sequence of dualizable categories is also both a fiber sequence and a cofiber sequence in $\mathrm{Pr}^{\mathrm{dual}}$, see Remark 7.23 below.

Proposition 7.9 ([Efi24, Proposition 2.61])**.** *The functor* $\mathrm{Calk}^{\mathrm{cont}}_\kappa \colon \mathrm{Pr}^{\mathrm{dual}} \to \mathrm{Cat}_{\mathrm{perf}}$ *sends short exact sequences of dualizable categories to short exact sequences.*

Proof. Let $C \xrightarrow{F} D \xrightarrow{G} E$ be a short exact sequence of dualizable categories. As shown above, C and D are κ-compactly generated. Since G preserves κ-compact objects, [Ram24a, Proposition A.26] shows that $C^\kappa \to D^\kappa \to E^\kappa$ is a short exact sequence. By Theorem 2.17(1), the sequence $\mathrm{Ind}(C^\kappa) \to \mathrm{Ind}(D^\kappa) \to \mathrm{Ind}(E^\kappa)$ is a strongly continuous localization sequence. By Construction 7.7, we obtain a commutative diagram

$$\begin{array}{ccccc}
C & \xrightarrow{\hat{y}} & \mathrm{Ind}(C^\kappa) & \longrightarrow & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(C)) \\
{\scriptstyle F}\downarrow & & \downarrow & & \downarrow{\scriptstyle \Phi} \\
D & \xrightarrow{\hat{y}} & \mathrm{Ind}(D^\kappa) & \longrightarrow & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(D)) \\
{\scriptstyle G}\downarrow & & \downarrow & & \downarrow \\
E & \xrightarrow[\hat{y}]{} & \mathrm{Ind}(E^\kappa) & \longrightarrow & \mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(E))
\end{array}$$

in which the rows and the first two columns are cofiber sequences in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Since colimits commute with colimits, the third column is a cofiber sequence in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ as well.

We claim that Φ is fully faithful. By Construction 7.7, Φ is identified with the restriction of $\mathrm{Ind}(F^\kappa)$ to the kernels of the colimit functors. Since F is fully faithful, so are F^κ and $\mathrm{Ind}(F^\kappa)$, and hence so is Φ.

Since Φ is fully faithful and colimit-preserving, its essential image is closed under colimits, and the cofiber sequence of the third column is a localization sequence by Proposition 2.12(1). All functors

involved preserve compact objects, so it is strongly continuous. By Theorem 2.17(2), the sequence $\mathrm{Calk}^{\mathrm{cont}}_\kappa(C) \to \mathrm{Calk}^{\mathrm{cont}}_\kappa(D) \to \mathrm{Calk}^{\mathrm{cont}}_\kappa(E)$ of compact objects is short exact. □

In the talk, the fully faithfulness of Φ was explained by passing to the right adjoints of the horizontal functors: the colimit functors are compatible with the vertical functors, and the kernel of a map between fully faithful functors is fully faithful.

7.3 Filtered colimits of dualizable categories

Filtered colimits in $\mathrm{Cat}_{\mathrm{perf}}$ are exact: they preserve fully faithful functors and short exact sequences. The analogous statement fails in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (Remark 7.13). We now show that it does hold in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ and in $\mathrm{Pr}^{\mathrm{dual}}$, whose colimits are computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ by Section 7.1. As a consequence, the functor $C \mapsto C^\omega$ on $\mathrm{Pr}^{\mathrm{dual}}$ preserves filtered colimits. By Theorem A.8, every filtered colimit can be computed as a colimit over a directed poset, so we may restrict attention to directed posets.

Lemma 7.10 ([Efi24, Proposition 2.65])**.** *Let I be a directed poset, let $I \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, $i \mapsto C_i$, be a diagram with transition functors $F_{ij}\colon C_i \to C_j$, let $C := \mathrm{colim}_i C_i$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and let $F_i\colon C_i \to C$ be the structure functors.*

(1) *The counits induce an equivalence* $\mathrm{colim}_i F_i F_i^R \simeq \mathrm{id}_C$.

(2) *If the functors F_{ij} are strongly continuous, then for all $i, j \in I$ there is a natural equivalence*

$$\mathrm{colim}_{k\geq i,j} F^R_{jk} F_{ik} \simeq F^R_j F_i.$$

Proof. We identify C with $\lim_i C_i$, taken along the functors F^R_{ij}, so that F^R_i becomes the projection. Mapping spectra in this limit are the limits of the mapping spectra.

(1) For $i \leq j$, the counit of $F_{ij} \dashv F^R_{ij}$ induces a map $F_i F_i^R \simeq F_j F_{ij} F^R_{ij} F^R_j \to F_j F^R_j$, so the functors $F_i F_i^R$ form an I-indexed diagram with a cocone to id_C, given by the counits. For $x, y \in C$ we get natural equivalences

$$\hom_C\left(\mathrm{colim}_i F_i F_i^R x, y\right) \simeq \lim_i \hom_C(F_i F_i^R x, y) \simeq \lim_i \hom_{C_i}(F_i^R x, F_i^R y) \simeq \hom_C(x, y),$$

whose composite is induced by the cocone. By the Yoneda lemma, the cocone exhibits id_C as the colimit.

(2) By Proposition 7.1(1), the functors F_i are strongly continuous, so the projections F^R_j preserve colimits, and the transition functors $F^R_{jj'}$ of the limit preserve colimits by assumption. Fix i, and for $x \in C_i$ consider the family of objects

$$F'_i(x)_j := \mathrm{colim}_{k\geq i,j} F^R_{jk} F_{ik}(x) \in C_j, \qquad j \in I.$$

For $k \leq \ell$, the transition map in this colimit is

$$F^R_{jk} F_{ik} \longrightarrow F^R_{jk} F^R_{k\ell} F_{k\ell} F_{ik} \simeq F^R_{j\ell} F_{i\ell},$$

induced by the unit of $F_{k\ell} \dashv F^R_{k\ell}$. The units for composites give these diagrams coherently. For $j \leq j'$, we have $F^R_{jj'} F'_i(x)_{j'} \simeq \mathrm{colim}_{k\geq i,j'} F^R_{jk} F_{ik}(x) \simeq F'_i(x)_j$, using that $F^R_{jj'}$ preserves colimits

and that $\{k \geq i, j'\} \subseteq \{k \geq i, j\}$ is cofinal. These identifications are induced by restriction to common cofinal tails and the composition isomorphisms for right adjoints. Their compatibility for chains of indices follows from that of the units above and the universal property of colimits. Thus they define a functor $F_i' \colon C_i \to \lim_j C_j \simeq C$, and the claim is equivalent to $F_i' \simeq F_i$. We show that F_i' is left adjoint to F_i^R. For $x \in C_i$ and $y = (y_j)_j \in C$, we have

$$\begin{aligned}\hom_C(F_i' x, y) &\simeq \lim_j \hom_{C_j}\Big(\operatorname{colim}_{k\geq i,j} F_{jk}^R F_{ik} x,\ y_j\Big)\\ &\simeq \lim_{(j,k)\in P^{\mathrm{op}}} \hom_{C_j}(F_{jk}^R F_{ik} x, y_j)\\ &\simeq \lim_{j\geq i} \hom_{C_j}(F_{ij} x, y_j) \simeq \lim_{j\geq i} \hom_{C_i}(x, F_{ij}^R y_j)\\ &\simeq \hom_{C_i}(x, y_i) = \hom_{C_i}(x, F_i^R y).\end{aligned}$$

Here $P = \{(j,k) \mid i \leq j \leq k\}$, with the product order. In the second step we first restrict the limit over j to the cofinal subset $I_{\geq i}$. The functor $(j,k) \mapsto \hom_{C_j}(F_{jk}^R F_{ik} x, y_j)$ is contravariant in (j,k). In the third step we restrict along the diagonal $I_{\geq i} \to P$, $j \mapsto (j,j)$, which is cofinal: for $(j,k) \in P$, the poset of $m \geq i$ with $(j,k) \leq (m,m)$ is $I_{\geq k}$, which is directed and hence weakly contractible. Thus the opposite of the diagonal is initial, as required for this limit over P^{op}. On the diagonal, $F_{jj}^R = \mathrm{id}$. The fourth step is the adjunction $F_{ij} \dashv F_{ij}^R$, and the remaining diagram is constant with value $\hom_{C_i}(x, y_i)$, as $F_{ij}^R y_j \simeq y_i$. The unit of the resulting adjunction $F_i' \dashv F_i^R$ is the map $x \to F_i'(x)_i$ from the summand $k = i$. To identify the resulting isomorphism with the canonical map, use the cocones

$$F_{jk}^R F_{ik} \longrightarrow F_{jk}^R F_k^R F_k F_{ik} \simeq F_j^R F_i,$$

induced by the unit of $F_k \dashv F_k^R$. They define $F_i' \to F_i$, and at $j = k = i$ carry the unit just described to the unit of $F_i \dashv F_i^R$. By the adjunction, this map is the isomorphism between the two left adjoints. Evaluating at the j-th component proves the claim with its canonical cocone. □

Proposition 7.11 ([Efi24, Proposition 2.67])**.**

(1) *In* $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ *and in* $\mathrm{Pr}^{\mathrm{dual}}$*, the class of fully faithful functors is closed under filtered colimits.*

(2) *A filtered colimit of strongly continuous localization sequences, formed in* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$*, is again a strongly continuous localization sequence. In particular, filtered colimits of short exact sequences of dualizable categories are short exact.*

Proof. (1) By Proposition 7.2, it suffices to treat $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$. Let I be a directed poset, let (C_i) and (D_i) be I-indexed diagrams in $\mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ with transition functors F_{ij} and G_{ij}, and let $\Phi_i \colon C_i \to D_i$ be a natural transformation whose components are fully faithful and strongly continuous. Let $C = \operatorname{colim}_i C_i$ and $D = \operatorname{colim}_i D_i$, with structure functors F_i and G_i, and let $\Phi \colon C \to D$ be the induced functor, which is strongly continuous by Proposition 7.1. We show that the unit $\mathrm{id}_C \to \Phi^R\Phi$ is an equivalence. Both functors preserve colimits, the essential images of the F_i generate C under colimits, and the functors F_j^R are jointly conservative (see the proof of Proposition 7.2). So it suffices to show that $F_j^R F_i \to F_j^R \Phi^R \Phi F_i$ is an equivalence for all $i, j \in I$. Using $\Phi F_i \simeq G_i \Phi_i$, Lemma 7.10(2), the strong continuity of Φ_j and the fully faithfulness of Φ_k, we compute

$$\begin{aligned}F_j^R \Phi^R \Phi F_i &\simeq \Phi_j^R G_j^R G_i \Phi_i \simeq \Phi_j^R \Big(\operatorname{colim}_{k\geq i,j} G_{jk}^R G_{ik}\Big)\Phi_i \simeq \operatorname{colim}_{k\geq i,j} \Phi_j^R G_{jk}^R G_{ik} \Phi_i\\ &\simeq \operatorname{colim}_{k\geq i,j} F_{jk}^R \Phi_k^R \Phi_k F_{ik} \simeq \operatorname{colim}_{k\geq i,j} F_{jk}^R F_{ik} \simeq F_j^R F_i.\end{aligned}$$

More precisely, under the canonical cocones in Lemma 7.10(2), the map $F_j^R F_i \to F_j^R \Phi^R \Phi F_i$ is the colimit of the maps

$$F_{jk}^R F_{ik} \longrightarrow F_{jk}^R \Phi_k^R \Phi_k F_{ik}$$

induced by the units of $\Phi_k \dashv \Phi_k^R$. This follows by pasting the adjunction units for the squares $\Phi_k F_{ik} \simeq G_{ik} \Phi_i$ and for the colimit cocones. Each of these maps is an isomorphism because Φ_k is fully faithful, so their colimit is an isomorphism as required.

(2) Let $C_i \to D_i \to E_i$ be a filtered diagram of strongly continuous localization sequences, and let $C \to D \to E$ be its colimit in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. By Proposition 7.1, it is a sequence of strongly continuous functors. Since colimits commute with colimits, $D \to E$ is the cofiber of $C \to D$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. By (1), $C \to D$ is fully faithful, and it preserves colimits. Hence $C \to D \to E$ is a localization sequence by Proposition 2.12(1). If the D_i are dualizable, then so is D by Proposition 7.2. □

Corollary 7.12 ([Efi24, Corollary 2.68])**.** *Let C be a presentable stable category, and let $(C_i)_{i \in I}$ be a directed family of presentable stable full subcategories $C_i \subseteq C$ whose inclusion functors are strongly continuous. If C is generated under colimits by the C_i, then the functor $\mathrm{colim}_i C_i \to C$ is an equivalence, where the colimit is formed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.*

Proof. For $i \leq j$, the right adjoint of $C_i \hookrightarrow C_j$ is the restriction of the right adjoint of $C_i \hookrightarrow C$, so the transition functors are strongly continuous. The colimit of the constant diagram with value C over the weakly contractible poset I is C. Applying Proposition 7.11(1) to the inclusions $C_i \hookrightarrow C$ shows that $\mathrm{colim}_i C_i \to C$ is fully faithful. Its essential image is closed under colimits and contains all C_i, so it is all of C. □

Remark 7.13. Proposition 7.11 and Corollary 7.12 fail in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, i.e. without the strong continuity assumptions. Efimov [Efi24, Remark 2.69] considers $C = \mathrm{D}(\mathbb{Z})$ and $C_n = \mathrm{D}(\mathbb{Z}[p_k^{-1} \mid k \geq n]) \subseteq C$, where p_k is the k-th prime number. The subcategories $C_1 \subseteq C_2 \subseteq \cdots$ generate C under colimits, but $\mathrm{colim}_n C_n \to C$ is not an equivalence, as one shows using Lemma 7.10(1).

To compute compact objects in filtered colimits, we use the following observation. Let $C \to \mathrm{Ind}(A) \to \mathrm{Ind}(B)$ be a short exact sequence of dualizable categories, where $A, B \in \mathrm{Cat}_{\mathrm{perf}}$ and the second functor is $\mathrm{Ind}(p)$ for a functor $p \colon A \to B$. Then

$$C^\omega \simeq \ker(p \colon A \to B). \tag{7.1}$$

Indeed, a fully faithful strongly continuous functor preserves compact objects, and it detects them since it preserves filtered colimits and is fully faithful. So C^ω consists of the compact objects of $\mathrm{Ind}(A)$, i.e. the objects of A, which lie in $C = \ker(\mathrm{Ind}(p))$.

We recall the categorical facts used below. Limits and filtered colimits of stable categories and exact functors are computed in Cat [Lur17, Theorem 1.1.4.4 and Proposition 1.1.4.6]. The same holds for $\mathrm{Cat}_{\mathrm{perf}}$: idempotent completeness is closed under limits; in a filtered colimit, an idempotent in the homotopy category is represented by an endomorphism at some stage, and its idempotence holds at a later stage, where it splits. Here we use the characterization of idempotent completeness of a stable category by idempotent completeness of its homotopy category.

Filtered colimits in Cat commute with finite limits, and taking the core preserves filtered colimits. One can check these two facts in the quasicategory model: filtered colimits are computed on underlying simplicial sets; an edge is invertible precisely when it has a two-sided inverse in the

homotopy category, a condition witnessed by finite data. Thus taking the maximal Kan complex commutes with filtered colimits. Pullbacks of categories can be computed by triples (x, y, u) with u an isomorphism between the two images; this construction, as well as finite products, commutes with filtered colimits. Consequently filtered colimits in $\mathrm{Cat}_{\mathrm{perf}}$ commute with finite limits as well.

Proposition 7.14 ([Efi24, Proposition 2.70])**.** *The functor* $(-)^{\omega}\colon \mathrm{Pr}^{\mathrm{dual}} \to \mathrm{Cat}_{\mathrm{perf}}$ *preserves filtered colimits. In particular,* Sp *is a compact object of* $\mathrm{Pr}^{\mathrm{dual}}$.

Proof. Let $(C_i)_{i\in I}$ be a filtered diagram in $\mathrm{Pr}^{\mathrm{dual}}$, and put $C := \mathrm{colim}_i\, C_i$, $A := \mathrm{colim}_i\, C_i^{\omega_1}$ and $B := \mathrm{colim}_i\, \mathrm{Calk}^{\mathrm{cont}}_{\omega_1}(C_i)$, the latter two colimits taken in $\mathrm{Cat}_{\mathrm{perf}}$. The functor $\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ preserves colimits: for $T \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, we have $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(A'), T) \simeq \mathrm{Fun}^{\mathrm{ex}}(A', T)$ by Theorem A.15(3), and $\mathrm{Fun}^{\mathrm{ex}}(-, T)$ sends colimits in $\mathrm{Cat}_{\mathrm{perf}}$ to limits, since T is idempotent complete. By Propositions 7.9 and 7.11(2), applied to the Calkin sequences of the C_i for $\kappa = \omega_1$, we therefore obtain a short exact sequence

$$C \to \mathrm{Ind}(A) \to \mathrm{Ind}(B).$$

By (7.1), applied to this sequence and to the Calkin sequences of the C_i, we get

$$C^{\omega} \simeq \ker(A \to B) \simeq \mathrm{colim}_i \ker\left(C_i^{\omega_1} \to \mathrm{Calk}^{\mathrm{cont}}_{\omega_1}(C_i)\right) \simeq \mathrm{colim}_i\, C_i^{\omega},$$

where the middle equivalence uses the commutation of filtered colimits with finite limits established above.

Finally, a colimit-preserving functor $\mathrm{Sp} \to C$ is determined by the image x of $\mathbb{S}$, and it is strongly continuous if and only if its right adjoint $\hom_C(x, -)$ preserves colimits, i.e. if and only if x is compact. So $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{dual}}}(\mathrm{Sp}, C) \simeq (C^{\omega})^{\simeq}$. Since $(-)^{\simeq}\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{An}$ preserves filtered colimits, Sp is compact in $\mathrm{Pr}^{\mathrm{dual}}$. □

To see why the core functor is conservative on stable categories and exact functors, observe that $\mathrm{Map}_C(x, y)$ is a retract of the anima of automorphisms of $x \oplus y$: send f to the shear $\left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right)$, and send an automorphism u to $p_y u i_x$. These maps are natural under exact functors. A functor inducing an isomorphism on cores therefore induces isomorphisms on these automorphism animae and hence on their retracts $\mathrm{Map}_C(x, y)$; it is also essentially surjective, so it is an equivalence.

Conversely, compactness of Sp implies that $(-)^{\omega}$ preserves filtered colimits, since $(-)^{\simeq}\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{An}$ is conservative [Efi24, Proposition 2.70].

7.4 Finite limits of dualizable categories

Limits in $\mathrm{Pr}^{\mathrm{dual}}$ are much more complicated than colimits. In general, the kernel in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ of a strongly continuous functor between dualizable categories is not dualizable, or its inclusion is not strongly continuous (Example 7.20). Instead, the kernel in $\mathrm{Pr}^{\mathrm{dual}}$ is the largest dualizable subcategory of the kernel whose inclusion is strongly continuous. The existence of such a largest subcategory relies on the following refinement of the description of ω_1-compact objects in terms of compact morphisms.

Proposition 7.15 ([Efi24, Proposition 2.81])**.** *Let C be a dualizable category and let $x \in C^{\omega_1}$. Then there exists a functor $\Phi\colon (\mathbb{Q}, \leq) \to C$ such that*

(1) $\operatorname{colim}_{a\in\mathbb{Q}} \Phi(a) \simeq x$;

(2) *for every* $a \in \mathbb{Q}$, *the canonical map* $\operatorname{colim}_{b<a} \Phi(b) \to \Phi(a)$ *is an equivalence;*

(3) *for all* $a < b$ *in* $\mathbb{Q}$, *the morphism* $\Phi(a) \to \Phi(b)$ *is compact.*

Proof. By Proposition 4.8(3), there is a functor $F\colon \mathbb{Q} \to C$ with $\operatorname{colim}_{\mathbb{Q}} F \simeq x$ such that $F(a) \to F(b)$ is compact for all $a < b$. Define $\Phi(a) := \operatorname{colim}_{b<a} F(b)$; formally, Φ is the left Kan extension of $F \circ \mathrm{pr}_1$ along $\mathrm{pr}_2\colon \{(b,a) \in \mathbb{Q}^2 \mid b < a\} \to \mathbb{Q}$. For $a < a'$, choose c with $a < c < a'$. Then $\Phi(a) \to \Phi(a')$ factors as

$$\Phi(a) \to F(a) \to F(c) \to \Phi(a'),$$

so it is compact, since $F(a) \to F(c)$ is compact and compact morphisms form an ideal (Remark 3.4). Since $\mathbb{Q}$ is dense, we have $\operatorname{colim}_{b<a} \Phi(b) \simeq \operatorname{colim}_{b<a} \operatorname{colim}_{c<b} F(c) \simeq \operatorname{colim}_{c<a} F(c) = \Phi(a)$, and similarly $\operatorname{colim}_{\mathbb{Q}} \Phi \simeq \operatorname{colim}_{\mathbb{Q}} F \simeq x$. □

In the talk, the functor F was constructed directly: one extends a compact exhaustion $x_1 \to x_2 \to \cdots$ of x by zero to a functor on $\mathbb{Z}$ and then refines it to the dyadic rationals by repeatedly factoring the transition maps, as in the proof of Lemma 4.7.

Lemma 7.16 ([Efi24, Lemma 2.82])**.** *Let C be a dualizable category, and let $D \subseteq C$ be a full stable subcategory closed under colimits. Then there is a largest full subcategory $E \subseteq D$ such that E is dualizable and the inclusion $E \hookrightarrow C$ is strongly continuous. Moreover, E is generated under colimits by the objects* $\operatorname{colim}_{\mathbb{Q}} F$, *where $F\colon \mathbb{Q} \to D$ is a functor such that $F(a) \to F(b)$ is compact in C for all $a < b$.*

Proof. Let $E' \subseteq D$ be the full stable subcategory generated under colimits by the objects $\operatorname{colim}_{\mathbb{Q}} F$ as in the statement. Each such object is the colimit of the sequence $F(0) \to F(1) \to \cdots$ of compact morphisms, hence is ω_1-compact in C by Theorem 3.20(2) and Remark 3.21. As C^{ω_1} is essentially small, E' is generated under colimits by a set of objects, and hence is presentable.

Let $F\colon \mathbb{Q} \to D$ be as in the statement. Replacing F by $a \mapsto \operatorname{colim}_{b<a} F(b)$ as in the proof of Proposition 7.15, which does not change $\operatorname{colim}_{\mathbb{Q}} F$ or leave D, we may assume that $\operatorname{colim}_{b<a} F(b) \simeq F(a)$ for all a. Then each $F(a)$ lies in E', being the colimit of the restriction of F to $\mathbb{Q}_{<a}$, which is order-isomorphic to $\mathbb{Q}$. As in the proof of Proposition 3.22, we have

$$\hat{y}_C\big(\operatorname{colim}_{\mathbb{Q}} F\big) \simeq \text{“colim”}\big(F(0) \to F(1) \to F(2) \to \cdots\big),$$

since the maps $F(n) \to F(n+1)$ are compact. This lies in the essential image of the fully faithful functor $\mathrm{Ind}(E') \to \mathrm{Ind}(C)$. This essential image is closed under colimits, since $\mathrm{Ind}(E') \to \mathrm{Ind}(C)$ is exact and preserves filtered colimits. Since $\hat{y}_C$ preserves colimits, it follows that $\hat{y}_C$ maps E' into $\mathrm{Ind}(E')$. For $e \in E'$ and $Z \in \mathrm{Ind}(E')$, we then have

$$\mathrm{Map}_{\mathrm{Ind}(E')}(\hat{y}_C e, Z) \simeq \mathrm{Map}_{\mathrm{Ind}(C)}(\hat{y}_C e, Z) \simeq \mathrm{Map}_C(e, \operatorname{colim}_C Z) \simeq \mathrm{Map}_{E'}(e, \operatorname{colim}_{E'} Z),$$

as the inclusion $E' \hookrightarrow C$ preserves colimits. So $\operatorname{colim}_{E'}\colon \mathrm{Ind}(E') \to E'$ admits the left adjoint $\hat{y}_C|_{E'}$, i.e. E' is compactly assembled, hence dualizable by Theorem 3.16. Moreover, the inclusion $E' \hookrightarrow C$ commutes with $\hat{y}$ by construction, so it is strongly continuous (Definition 3.9 and Remark 3.10).

Now let $E'' \subseteq D$ be any full subcategory such that E'' is dualizable and the inclusion $E'' \hookrightarrow C$ is strongly continuous. Let $x \in E''$ be ω_1-compact. By Proposition 7.15, there is a functor $F\colon \mathbb{Q} \to E''$ with $\operatorname{colim}_{\mathbb{Q}} F \simeq x$ such that $F(a) \to F(b)$ is compact in E'' for $a < b$. These maps are compact in C by Proposition 3.22, and $\operatorname{colim}_{\mathbb{Q}} F \simeq x$ in C since the inclusion preserves colimits. Hence $x \in E'$. As E'' is generated under colimits by its ω_1-compact objects (Theorem 3.20(1)), we get $E'' \subseteq E'$. So $E := E'$ is the largest such subcategory. □

In the talk, the strong continuity of the inclusion $E \hookrightarrow C$ was asked about: it is precisely the compatibility $\hat{y}_C|_E \simeq \hat{y}_E$ established in the proof.

Remark 7.17. Efimov also shows that the inclusion $\mathrm{Pr}^{\mathrm{dual}} \hookrightarrow \mathrm{Pr}^{\mathrm{LL}}_{\mathrm{st}}$ admits a right adjoint [Efi24, Remark 2.57]: for a presentable stable category C, the strictly full dualizable subcategories of C with strongly continuous inclusion form a small complete lattice, whose join is given by images of products in the sense of Definition 4.27 (using Corollary 4.28 and Remark 4.26). Its largest element is the value of the right adjoint at C.

Definition 7.18. Let $F\colon C \to D$ be a strongly continuous functor between dualizable categories, and let $\ker(F) \subseteq C$ be its kernel in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. The *dualizable kernel* $\ker^{\mathrm{dual}}(F) \subseteq \ker(F)$ is the largest full subcategory of $\ker(F)$ which is dualizable and whose inclusion into C is strongly continuous; it exists by Lemma 7.16.

Proposition 7.19 ([Efi24, Proposition 2.83])**.** *Let $F\colon C \to D$ be a strongly continuous functor between dualizable categories. Then $\ker^{\mathrm{dual}}(F) \to C \xrightarrow{F} D$ is a fiber sequence in $\mathrm{Pr}^{\mathrm{dual}}$.*

Proof. Write $j\colon \ker^{\mathrm{dual}}(F) \hookrightarrow C$ for the inclusion. Let T be dualizable and let $G\colon T \to C$ be strongly continuous with $FG \simeq 0$. Then $\mathrm{Im}(G) \subseteq \ker(F)$, and by Corollary 4.28, $\mathrm{Im}(G)$ is dualizable and its inclusion into C is strongly continuous. By maximality, $\mathrm{Im}(G) \subseteq \ker^{\mathrm{dual}}(F)$, so $G \simeq jG'$ for a colimit-preserving functor $G'\colon T \to \ker^{\mathrm{dual}}(F)$. Its right adjoint $G^R j$ preserves colimits, so G' is strongly continuous. Since j is fully faithful, $j\circ -$ identifies $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{dual}}}(T, \ker^{\mathrm{dual}}(F))$ with the union of those path components of $\mathrm{Map}_{\mathrm{Pr}^{\mathrm{dual}}}(T, C)$ which consist of functors G with $FG \simeq 0$. Since a nullhomotopy of FG is unique if it exists (Remark 2.3), the latter is the fiber of $F \circ -\colon \mathrm{Map}_{\mathrm{Pr}^{\mathrm{dual}}}(T, C) \to \mathrm{Map}_{\mathrm{Pr}^{\mathrm{dual}}}(T, D)$ over the zero functor. □

Example 7.20 ([Efi24, Example 2.84])**.** Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K, and consider the base change functor $F\colon \mathrm{D}(R) \to \mathrm{D}(R/\mathfrak{m})$, which is strongly continuous. Its kernel in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ consists of the R-modules on which a uniformizer acts invertibly, i.e. $\ker(F) \simeq \mathrm{D}(K)$. However, the inclusion $\mathrm{D}(K) \hookrightarrow \mathrm{D}(R)$ is not strongly continuous, since its right adjoint $\hom_R(K, -)$ does not preserve colimits, K not being a compact R-module. The only localizing subcategories of $\mathrm{D}(K)$ are 0 and $\mathrm{D}(K)$, so $\ker^{\mathrm{dual}}(F) = 0$.

Efimov uses a variant of this example to show that sequential colimits in $\mathrm{Pr}^{\mathrm{dual}}$ do not commute with kernels, so that $\mathrm{Pr}^{\mathrm{dual}}$ is not compactly assembled [Efi24, Corollary 2.85].

We now describe pullbacks in $\mathrm{Pr}^{\mathrm{dual}}$. Let $F\colon A \to C$ and $G\colon B \to C$ be strongly continuous functors between dualizable categories. Their *lax pullback* L is the category of triples (a, b, f) with $a \in A$, $b \in B$ and $f\colon F(a) \to G(b)$. By adjunction, it is equivalent to the gluing of Construction 5.24 along the colimit-preserving functor $F^R G\colon B \to A$. By Proposition 5.25, L is dualizable, and we have

a semiorthogonal decomposition $L = \langle i_A(A), i_B(B)\rangle$ with $i_A(a) = (a, 0, 0)$ and $i_B(b) = (0, b, 0)$, and with projections $\pi_A(a, b, f) = a$ and $\pi_B(a, b, f) = b$. Consider the colimit-preserving functor

$$\Phi\colon L \to C, \qquad (a, b, f) \mapsto \operatorname{cofib}\big(f\colon F(a) \to G(b)\big).$$

Since $\Phi i_A \simeq \Sigma F$ and $\Phi i_B \simeq G$ are strongly continuous, Φ is strongly continuous by Proposition 5.26(2). Its kernel $\ker(\Phi)$ is the full subcategory of triples in which f is an equivalence, i.e. $\ker(\Phi) \simeq A \times_C B$ is the pullback in $\Pr^{\mathrm{L}}_{\mathrm{st}}$, which is computed in $\widehat{\mathrm{Cat}}$.

Proposition 7.21 ([Efi24, Proposition 2.86])**.** *Let $F\colon A \to C$ and $G\colon B \to C$ be strongly continuous functors between dualizable categories, and let $A \times^{\mathrm{dual}}_C B := \ker^{\mathrm{dual}}(\Phi)$, with Φ as above.*

(1) *The category $A \times^{\mathrm{dual}}_C B$ is a pullback of F and G in $\Pr^{\mathrm{dual}}$. The functor $A \times^{\mathrm{dual}}_C B \to A \times_C B$ is fully faithful and strongly continuous, and its essential image is the largest full subcategory $E \subseteq A \times_C B$ such that E is dualizable and the functors $E \to A$ and $E \to B$ are strongly continuous.*

(2) *If G is a localization, then $A \times^{\mathrm{dual}}_C B \simeq A \times_C B$, and the projection $A \times_C B \to A$ is a localization.*

Proof. (1) Let T be dualizable. By Proposition 7.19, $\mathrm{Map}_{\Pr^{\mathrm{dual}}}(T, A \times^{\mathrm{dual}}_C B)$ is the union of the path components of $\mathrm{Map}_{\Pr^{\mathrm{dual}}}(T, L)$ consisting of functors H with $\Phi H \simeq 0$, i.e. of strongly continuous functors $T \to L$ landing in $\ker(\Phi) = A \times_C B$. By Proposition 5.26(1), a colimit-preserving functor $H\colon T \to L$ is strongly continuous if and only if $\pi_A H$ and $\pi_B H$ are. So $\mathrm{Map}_{\Pr^{\mathrm{dual}}}(T, A \times^{\mathrm{dual}}_C B)$ is the union of the path components of $\mathrm{Map}_{\Pr^{\mathrm{L}}_{\mathrm{st}}}(T, A \times_C B)$ consisting of functors whose two components are strongly continuous. As $A \times_C B$ is a pullback in $\Pr^{\mathrm{L}}_{\mathrm{st}}$, this is

$$\mathrm{Map}_{\Pr^{\mathrm{dual}}}(T, A) \times_{\mathrm{Map}_{\Pr^{\mathrm{dual}}}(T, C)} \mathrm{Map}_{\Pr^{\mathrm{dual}}}(T, B),$$

naturally in T. This proves that $A \times^{\mathrm{dual}}_C B$ is a pullback in $\Pr^{\mathrm{dual}}$.

Write $k\colon A \times^{\mathrm{dual}}_C B \hookrightarrow A \times_C B$ and $j\colon A \times_C B \hookrightarrow L$ for the inclusions. Then jk is strongly continuous and j preserves colimits, so the right adjoint $k^R \simeq (jk)^R j$ preserves colimits, i.e. k is strongly continuous. Finally, colimits in L are computed on the components a and b, so a functor $E \to L$ preserves colimits if and only if its composites with π_A and π_B do. Together with Proposition 5.26(1), this shows that for a full subcategory $E \subseteq A \times_C B$, the functors $E \to A$ and $E \to B$ are strongly continuous if and only if the inclusion $E \to L$ is. So the final assertion follows from the definition of $\ker^{\mathrm{dual}}(\Phi)$.

(2) Suppose that G^R is fully faithful. For $c \in C$ and $(a, b, f) \in L$, we have

$$\mathrm{Map}_L\big((a, b, f), (0, G^R c, 0)\big) \simeq \mathrm{fib}\big(\mathrm{Map}_B(b, G^R c) \to \mathrm{Map}_C(F(a), GG^R c)\big) \simeq \mathrm{Map}_C\big(\mathrm{cofib}(f), c\big),$$

using $GG^R \simeq \mathrm{id}$. So Φ has the right adjoint $\Phi^R(c) = (0, G^R c, 0)$, which preserves colimits and satisfies $\Phi\Phi^R \simeq GG^R \simeq \mathrm{id}$. Hence Φ is a strongly continuous localization. By Lemma 2.13 and Proposition 3.19, its kernel $\ker(\Phi) = A \times_C B$ is dualizable and its inclusion into L is strongly continuous. Hence $\ker^{\mathrm{dual}}(\Phi) = \ker(\Phi)$.

The functor $R\colon A \to A \times_C B$, $a \mapsto (a, G^R F(a), \theta)$, where $\theta\colon F(a) \simeq GG^R F(a)$ is the inverse of the counit, is right adjoint to the projection: for $(a', b', \theta') \in A \times_C B$, the anima of maps $(a', b', \theta') \to R(a)$ is

$$\mathrm{Map}_A(a', a) \times_{\mathrm{Map}_C(F(a'), F(a))} \mathrm{Map}_B(b', G^R F(a)) \simeq \mathrm{Map}_A(a', a),$$

since $\mathrm{Map}_B(b', G^R F(a)) \simeq \mathrm{Map}_C(G(b'), F(a)) \simeq \mathrm{Map}_C(F(a'), F(a))$ via θ'. The composite of R with the projection is the identity, so R is fully faithful and the projection is a localization. □

In the talk, the functor $A \times_C^{\mathrm{dual}} B \to L$ was shown to be strongly continuous using the strong continuity of the projections π_A and π_B; part (2) was not covered.

Corollary 7.22 ([Efi24, Corollary 2.87])**.** *A strongly continuous functor $F\colon C \to D$ between dualizable categories is a monomorphism in* $\mathrm{Pr}^{\mathrm{dual}}$ *if and only if it is fully faithful.*

Proof. If F is fully faithful, then $F \circ -$ is fully faithful on functor categories, so it induces monomorphisms on mapping spaces. Conversely, if F is a monomorphism, then the diagonal $C \to C \times_D^{\mathrm{dual}} C$ is an equivalence. Composing with the fully faithful functor $C \times_D^{\mathrm{dual}} C \to C \times_D C$ of Proposition 7.21(1), we find that the diagonal $C \to C \times_D C$ is fully faithful. As in the proof of Proposition 7.3, it follows that $\hom_C(x, y) \to \hom_D(Fx, Fy)$ is an equivalence for all $x, y \in C$. □

Remark 7.23. A short exact sequence $C \xrightarrow{i} D \xrightarrow{p} E$ of dualizable categories (Definition 7.8) is both a fiber sequence and a cofiber sequence in $\mathrm{Pr}^{\mathrm{dual}}$. It is a cofiber sequence since colimits in $\mathrm{Pr}^{\mathrm{dual}}$ are computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ (Proposition 7.2). It is a fiber sequence since $\ker^{\mathrm{dual}}(p) = \ker(p) = C$, as the inclusion $C \hookrightarrow D$ is strongly continuous; alternatively, apply Proposition 7.21(2) to the functors $0 \to E$ and p.

8 Localizing invariants of dualizable categories

This chapter is planned. No proofs are included yet.

9 Sheaves on continuous posets and locally compact Hausdorff spaces

This chapter is planned. No proofs are included yet.

10 Dualizability and flatness

This chapter is planned. No proofs are included yet.

11 $\aleph_1$-presentability of $\mathbf{Pr}^{\mathrm{dual}}$

This chapter is planned. No proofs are included yet.

A Categorical background

In this appendix we collect background material on accessible and presentable categories that is used throughout the notes. Most of this material is standard, see [Lur09; Lur17], and much of it is also treated in [Cno; CCW26]. We give complete proofs of the statements that are less easy to find in the literature or that are used in a specific form in the main text, and we refer to the literature for the foundational results on accessible categories.

Throughout, *category* means ∞-category, and An denotes the category of animae. We fix a regular cardinal κ whenever it appears. For a small category C we write $\mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ and $y\colon C \hookrightarrow \mathrm{PSh}(C)$ for the Yoneda embedding.

A.1 Filtered and κ-filtered categories

Filtered colimits are the basic tool throughout these notes: compactness, Ind-categories and accessibility are all defined in terms of them. We recall the relevant definitions, and prove some technical results on κ-filtered partially ordered sets and sequential diagrams, which are used in the study of ω_1-presentability in Section 3.4.

Definition A.1. We use the quasicategory model to specify the size of diagram shapes: a simplicial set is *κ-small* if it has fewer than κ nondegenerate simplices. A category I is *κ-filtered* if every diagram $K \to I$ indexed by a κ-small simplicial set extends to a cocone $K^{\triangleright} \to I$. This property is independent of the chosen model of I. The phrases *κ-small limit* and *κ-small colimit* likewise refer to these diagram shapes. We say *filtered* for ω-filtered. A colimit indexed by a small κ-filtered category is called a *κ-filtered colimit*.

Definition A.2. Let C be a category which admits κ-filtered colimits. An object $x \in C$ is called *κ-compact* if the functor $\mathrm{Map}_C(x, -)\colon C \to \mathrm{An}$ preserves κ-filtered colimits. We write $C^{\kappa} \subseteq C$ for the full subcategory of κ-compact objects, and say *compact* for ω-compact.

We will freely use the following fundamental fact.

Theorem A.3 ([Lur09, Proposition 5.3.3.3])**.** *In* An*, κ-filtered colimits commute with κ-small limits.*

Corollary A.4. *Let C be a category with κ-filtered colimits.*

(1) *If $\kappa \leq \lambda$, then every κ-compact object of C is λ-compact.*

(2) *If C admits κ-small colimits, then $C^{\kappa} \subseteq C$ is closed under κ-small colimits.*

(3) *Let* $x \in C$ *and let* $u\colon \mathrm{Map}_C(x,-) \to \mathrm{Map}_C(x',-)$ *exhibit* $\mathrm{Map}_C(x,-)$ *as a retract of* $\mathrm{Map}_C(x',-)$ *in* $\mathrm{Fun}(C,\mathrm{An})$. *If* x' *is* κ*-compact, then so is* x. *In particular,* C^κ *is closed under retracts.*

Proof. (1) A λ-filtered category is κ-filtered.

(2) If $x = \mathrm{colim}_{k\in K} x_k$ with K κ-small and each x_k κ-compact, then $\mathrm{Map}_C(x,-) = \lim_{k\in K^{\mathrm{op}}} \mathrm{Map}_C(x_k,-)$ is a κ-small limit of functors preserving κ-filtered colimits, hence preserves κ-filtered colimits by Theorem A.3.

(3) A retract of a functor preserving κ-filtered colimits again preserves them, since a retract of an equivalence in An (namely the comparison map from the colimit) is an equivalence. □

For partially ordered sets, κ-filteredness admits the expected elementary description.

Lemma A.5. *A partially ordered set* P *is* κ*-filtered if and only if every subset* $S \subseteq P$ *of cardinality* $< \kappa$ *admits an upper bound in* P. *In particular,* P *is filtered if and only if it is non-empty and every two elements admit a common upper bound.*

Proof. Suppose P is κ-filtered and let $S \subseteq P$ have cardinality $< \kappa$. Regarding S as a discrete category, the inclusion $S \to P$ is a κ-small diagram, and a cocone on it is precisely an upper bound of S.

Conversely, let $F\colon K \to P$ be a functor with K κ-small. Then K has fewer than κ objects up to equivalence, so the image S of F on objects has cardinality $< \kappa$. Choose an upper bound u of S. Since $\mathrm{Fun}(K,P)$ is again a partially ordered set, in which $F \le G$ if and only if $F(k) \le G(k)$ for all k, there is a (unique) natural transformation from F to the constant functor with value u, i.e. a cocone on F.

For the last statement, note that every finite subset of P has an upper bound if and only if P is non-empty and every two elements have an upper bound, by induction on the size of the subset. □

Lemma A.6. *A filtered partially ordered set* P *is* ω_1*-filtered if and only if every increasing sequence* $p_0 \le p_1 \le p_2 \le \dots$ *in* P *admits an upper bound.*

Proof. The condition is clearly necessary. Conversely, let $S = \{s_0, s_1, s_2, \dots\} \subseteq P$ be a countable subset (a finite subset has an upper bound since P is filtered). Using that P is filtered, choose inductively $p_0 \ge s_0$ and $p_n \ge p_{n-1}, s_n$ for $n \ge 1$. Then an upper bound for the increasing sequence $(p_n)_n$ is an upper bound for S. We conclude by Lemma A.5. □

We will often use the following criterion for cofinality, see [Lur09, Theorem 4.1.3.1]: a functor $f\colon A \to B$ is cofinal if and only if for every $b \in B$ the category $A \times_B B_{b/}$ is weakly contractible. If A and B are partially ordered sets, then $A \times_B B_{b/}$ is the partially ordered set $\{a \in A \mid b \le f(a)\}$. Since filtered categories are weakly contractible [Lur09, Lemma 5.3.1.20], we obtain:

Lemma A.7. *Let* $f\colon A \to B$ *be a map of partially ordered sets such that for every* $b \in B$ *the subset* $\{a \in A \mid b \le f(a)\} \subseteq A$ *is non-empty and filtered. Then* f *is cofinal. In particular:*

(1) *A map* $f\colon \mathbb{N} \to B$ *into a partially ordered set is cofinal if for every* $b \in B$ *there exists* n *with* $b \le f(n)$.

(2) *Every countable filtered partially ordered set Q admits a cofinal map $\mathbb{N} \to Q$.*

(3) *The diagonal $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is cofinal, and so is every map $f\colon \mathbb{N} \to \mathbb{N}$ with $f(n) \geq n$ for all n.*

Proof. The first statement was explained above. For (1), the subset $\{n \mid b \leq f(n)\}$ is non-empty and upwards closed in $\mathbb{N}$ (as f is order preserving), hence of the form $\{n \geq n_0\}$, which is filtered. For (2), if Q is finite it has a largest element (Lemma A.5) and we may take the constant map. Otherwise enumerate $Q = \{q_0, q_1, \dots\}$ and choose inductively $f(0) = q_0$ and $f(n) \geq f(n-1), q_n$; then f is order preserving and cofinal by (1). Statement (3) follows directly from (1). □

Theorem A.8 ([Lur09, Proposition 5.3.1.18])**.** *For every κ-filtered category I there exist a κ-filtered partially ordered set P and a cofinal functor $P \to I$.*

Proposition A.9. *Let P be a filtered partially ordered set, and let Σ be the set of countable filtered subsets $Q \subseteq P$, ordered by inclusion. Then Σ is ω_1-filtered and $P = \bigcup_{Q \in \Sigma} Q$. Moreover, for every functor $F\colon P \to D$ into a category with filtered colimits, there is a natural equivalence*

$$\operatorname{colim}_{p \in P} F(p) \simeq \operatorname{colim}_{Q \in \Sigma} \operatorname{colim}_{q \in Q} F(q).$$

Proof. Every $p \in P$ lies in the countable filtered subset $\{p\}$. Let $Q_0, Q_1, \dots \in \Sigma$ be countably many elements. Put $U_0 = \bigcup_n Q_n$ (and add one element of P if the family is empty), and inductively let $U_{k+1} \supseteq U_k$ be obtained by adding a chosen common upper bound in P for every pair of elements of U_k. Then $U = \bigcup_k U_k$ is countable, and any two elements of U have an upper bound in U; hence $U \in \Sigma$ by Lemma A.5, and U is an upper bound of the Q_n. So Σ is ω_1-filtered by Lemma A.5.

Consider the partially ordered set $E = \{(Q, q) \mid Q \in \Sigma, q \in Q\}$, with $(Q, q) \leq (Q', q')$ if $Q \subseteq Q'$ and $q \leq q'$, together with the projections $\pi\colon E \to \Sigma$ and $\rho\colon E \to P$. The map ρ is cofinal by Lemma A.7: for $p \in P$, the subset $\{(Q, q) \mid p \leq q\}$ is non-empty, and it is filtered since two elements $(Q, q), (Q', q')$ are bounded by (Q'', q'') where q'' is an upper bound of q, q' and $Q'' \in \Sigma$ contains $Q \cup Q' \cup \{q''\}$. Hence $\operatorname{colim}_P F \simeq \operatorname{colim}_E F\rho \simeq \operatorname{colim}_\Sigma \pi_!(F\rho)$, where $\pi_!$ denotes left Kan extension along π. All indexing categories in this calculation are filtered, so the assumed filtered colimits suffice. For $Q \in \Sigma$ we have $\pi_!(F\rho)(Q) = \operatorname{colim}_{(Q', q') \in E,\, Q' \subseteq Q} F(q')$, and the map $Q \to \{(Q', q') \in E \mid Q' \subseteq Q\}$, $q \mapsto (Q, q)$, is cofinal by Lemma A.7: for (Q', q') with $Q' \subseteq Q$, the subset $\{q \in Q \mid q' \leq q\}$ is non-empty and filtered. Hence $\pi_!(F\rho)(Q) \simeq \operatorname{colim}_{q \in Q} F(q)$, which proves the claim. □

Remark A.10. Since the nerve of $\mathbb{N}$ contains the simplicial subset $\bigcup_n \Delta^{\{n, n+1\}}$ via an inner anodyne map, a functor $\mathbb{N} \to D$ amounts to a sequence of objects and morphisms $x_0 \to x_1 \to x_2 \to \dots$ in D, without further coherence data. Similarly, since $\Delta^1 \times \bigcup_n \Delta^{\{n, n+1\}} \subseteq \Delta^1 \times \mathbb{N}$ is inner anodyne [Lur09, Corollary 2.3.2.4], a natural transformation $\eta\colon x_\bullet \to x'_\bullet$ of sequences amounts to morphisms $\eta_n\colon x_n \to x'_n$ together with homotopies witnessing the commutativity of each square

$$\begin{array}{ccc} x_n & \longrightarrow & x_{n+1} \\ {\scriptstyle \eta_n}\downarrow & & \downarrow{\scriptstyle \eta_{n+1}} \\ x'_n & \longrightarrow & x'_{n+1}. \end{array}$$

We will use this freely.

Lemma A.11 (Telescopes)**.** *Let D be a category with sequential colimits, and let $X \xrightarrow{s} Y \xrightarrow{r} X$ be maps with $rs \simeq \mathrm{id}_X$. Then $X \simeq \mathrm{colim}(Y \xrightarrow{e} Y \xrightarrow{e} Y \to \dots)$, where $e = sr$.*

Proof. Consider the sequence $Y \xrightarrow{r} X \xrightarrow{s} Y \xrightarrow{r} X \xrightarrow{s} \dots$. The maps $\mathbb{N} \to \mathbb{N}$ given by $n \mapsto 2n$ and $n \mapsto 2n+1$ are cofinal by Lemma A.7. Restricting along the first gives the sequence $Y \xrightarrow{e} Y \xrightarrow{e} \dots$, and restricting along the second gives a sequence $X \to X \to \dots$ all of whose maps are equivalences $rs \simeq \mathrm{id}_X$; the colimit of the latter is X. Hence both colimits agree with the colimit of the interleaved sequence. □

We now describe the ω_1-compact objects of $\mathrm{Ind}(C)$. Recall that for a small category C, the category $\mathrm{Ind}(C) \subseteq \mathrm{PSh}(C)$ is the full subcategory spanned by the filtered colimits of representable presheaves. It admits filtered colimits, computed in $\mathrm{PSh}(C)$, and the representable presheaves are compact in $\mathrm{Ind}(C)$ [Lur09, Section 5.3.5]. We usually suppress the Yoneda embedding y when no confusion can arise, and write "colim"$_i c_i$ for the colimit in $\mathrm{Ind}(C)$ of a diagram of representables when we want to stress that it is a formal colimit.

Lemma A.12 (Strictification of maps between sequential ind-objects)**.** *Let C be a small category, let $a_\bullet, b_\bullet \colon \mathbb{N} \to C$ be sequences, and put $Y = \mathrm{colim}_n\, y(a_n)$ and $Z = \mathrm{colim}_n\, y(b_n)$ in $\mathrm{Ind}(C)$. For every map $g \colon Y \to Z$ there exist a strictly increasing map $f \colon \mathbb{N} \to \mathbb{N}$ with $f(n) \geq n$ and a natural transformation $\eta \colon a_\bullet \to b_{f(\bullet)}$ of functors $\mathbb{N} \to C$ such that the induced map*

$$Y = \mathrm{colim}_n\, y(a_n) \xrightarrow{\mathrm{colim}\,\eta} \mathrm{colim}_n\, y(b_{f(n)}) \simeq Z$$

is homotopic to g. Here the last equivalence comes from cofinality of f (Lemma A.7).

Proof. Denote by $\alpha_n \colon a_n \to a_{n+1}$ and $\beta_{m,m'} \colon b_m \to b_{m'}$ the transition maps, and by $\iota_m \colon b_m \to Z$ the canonical maps. Since $\mathrm{Map}(Y, Z) \simeq \lim_n \mathrm{Map}(a_n, Z)$, the map g amounts to maps $g_n \colon a_n \to Z$ together with paths $\gamma_n \colon g_{n+1}\alpha_n \simeq g_n$ in $\mathrm{Map}(a_n, Z)$. Since a_n is compact in $\mathrm{Ind}(C)$, we have $\mathrm{Map}(a_n, Z) \simeq \mathrm{colim}_m \mathrm{Map}(a_n, b_m)$. By Theorem A.3, path spaces in this filtered colimit are computed as filtered colimits of path spaces; in particular every point and every path in $\mathrm{Map}(a_n, Z)$ between images of points of $\mathrm{Map}(a_n, b_m)$ lifts, up to homotopy, to some stage $m' \geq m$.

By Remark A.10, the data of f and η amount to maps $\eta_n \colon a_n \to b_{f(n)}$ and homotopies $\theta_n \colon \eta_{n+1}\alpha_n \simeq \beta_{f(n),f(n+1)}\eta_n$. The induced map $\mathrm{colim}\,\eta$ corresponds to the point $(\iota_{f(n)}\eta_n)_n$ of $\lim_n \mathrm{Map}(a_n, Z)$, with the paths $\iota_{f(n+1)}\eta_{n+1}\alpha_n \simeq \iota_{f(n+1)}\beta_{f(n),f(n+1)}\eta_n \simeq \iota_{f(n)}\eta_n$ induced by θ_n. To exhibit a homotopy $\mathrm{colim}\,\eta \simeq g$, it therefore suffices to choose paths $\sigma_n \colon \iota_{f(n)}\eta_n \simeq g_n$ in $\mathrm{Map}(a_n, Z)$ such that for each n the two resulting paths from $\iota_{f(n+1)}\eta_{n+1}\alpha_n$ to g_n, namely $\sigma_n \circ \iota\theta_n$ and $\gamma_n \circ \sigma_{n+1}\alpha_n$, are homotopic.

We construct these data inductively. First choose $f(0)$, $\eta_0 \colon a_0 \to b_{f(0)}$ and $\sigma_0 \colon \iota_{f(0)}\eta_0 \simeq g_0$. Suppose $f(0), \dots, f(n)$, $\eta_0, \dots, \eta_n$, $\theta_0, \dots, \theta_{n-1}$ and $\sigma_0, \dots, \sigma_n$ have been constructed. Choose $m > f(n)$, a map $\eta' \colon a_{n+1} \to b_m$ and a path $\sigma' \colon \iota_m\eta' \simeq g_{n+1}$. The maps $\eta'\alpha_n$ and $\beta_{f(n),m}\eta_n$ in $\mathrm{Map}(a_n, b_m)$ have images in $\mathrm{Map}(a_n, Z)$ connected by the path

$$\iota_m\eta'\alpha_n \overset{\sigma'\alpha_n}{\simeq} g_{n+1}\alpha_n \overset{\gamma_n}{\simeq} g_n \overset{\sigma_n^{-1}}{\simeq} \iota_{f(n)}\eta_n.$$

Hence there exist $m' \geq m$ and a path θ from $\beta_{m,m'}\eta'\alpha_n$ to $\beta_{f(n),m'}\eta_n$ in $\mathrm{Map}(a_n, b_{m'})$ whose image in $\mathrm{Map}(a_n, Z)$ is homotopic to this path. Put $f(n+1) = \max(m', n+1)$, $\eta_{n+1} = \beta_{m,f(n+1)}\eta'$, let

θ_n be the image of θ, and let σ_{n+1} be the path $\iota_{f(n+1)}\eta_{n+1} \simeq \iota_m\eta' \overset{\sigma'}{\simeq} g_{n+1}$. By construction, the compatibility condition above holds. This completes the induction. □

Proposition A.13. *Let C be a small category and let $X \in \mathrm{Ind}(C)$. Then X is ω_1-compact if and only if $X \simeq \mathrm{colim}_n\, y(c_n)$ for some sequence $c_0 \to c_1 \to c_2 \to \dots$ in C.*

Proof. If X is a sequential colimit of representables, then X is a countable colimit of compact objects, hence ω_1-compact by Corollary A.4.

Conversely, let X be ω_1-compact. By definition of $\mathrm{Ind}(C)$ and Theorem A.8, we may write $X = \mathrm{colim}_{p\in P}\, y(c_p)$ for a filtered partially ordered set P. By Proposition A.9, we have $X \simeq \mathrm{colim}_{Q\in\Sigma} X_Q$ with $X_Q = \mathrm{colim}_{q\in Q}\, y(c_q)$ and Σ ω_1-filtered. As X is ω_1-compact, $\mathrm{Map}(X, X) \simeq \mathrm{colim}_Q \mathrm{Map}(X, X_Q)$, so the identity of X factors as $X \overset{s}{\to} X_Q \overset{r}{\to} X$ for some $Q \in \Sigma$. By Lemma A.7(2), X_Q is a sequential colimit $Y = \mathrm{colim}_n\, y(a_n)$ of representables. By Lemma A.11, $X \simeq \mathrm{colim}(Y \overset{e}{\to} Y \overset{e}{\to} \dots)$ with $e = sr$.

By Lemma A.12, there exist $f\colon \mathbb{N} \to \mathbb{N}$ with $f(n) \geq n$ and a natural transformation $\eta\colon a_\bullet \to a_{f(\bullet)}$ inducing e on colimits. Define a functor $\Phi\colon \mathbb{N}\times\mathbb{N} \to C$ by $\Phi(k, n) = a_{f^k(n)}$, with horizontal maps the transition maps of $a_\bullet$ and vertical maps $\eta_{f^k(n)}\colon a_{f^k(n)} \to a_{f^{k+1}(n)}$; by Remark A.10 this is the functor $\mathbb{N} \to \mathrm{Fun}(\mathbb{N}, C)$ given by the sequence $a_\bullet \to a_{f(\bullet)} \to a_{f^2(\bullet)} \to \dots$ of whiskerings of η. Since each f^k is cofinal, $\mathrm{colim}_n\, y\Phi(k, n) \simeq Y$, and under these identifications the induced map in the k-direction is $\mathrm{colim}\,\eta \simeq e$. Hence

$$X \simeq \mathrm{colim}_k\, \mathrm{colim}_n\, y\Phi(k, n) \simeq \mathrm{colim}_{(k,n)\in\mathbb{N}\times\mathbb{N}}\, y\Phi(k, n) \simeq \mathrm{colim}_n\, y\Phi(n, n),$$

where the last equivalence uses cofinality of the diagonal (Lemma A.7(3)). So X is a sequential colimit of representables. □

A.2 Accessible and presentable categories

We recall the basic notions; see [Lur09, Chapter 5] for proofs of the results quoted in this section.

Definition A.14. For a small category C_0, we let $\mathrm{Ind}_\kappa(C_0) \subseteq \mathrm{PSh}(C_0)$ denote the full subcategory spanned by the κ-filtered colimits of representable presheaves. We write $\mathrm{Ind} = \mathrm{Ind}_\omega$.

Theorem A.15 ([Lur09, Section 5.3.5])**.** *Let C_0 be a small category.*

(1) *The category $\mathrm{Ind}_\kappa(C_0)$ admits κ-filtered colimits, computed in $\mathrm{PSh}(C_0)$, and the Yoneda embedding $y\colon C_0 \hookrightarrow \mathrm{Ind}_\kappa(C_0)$ lands in κ-compact objects.*

(2) *For every category D admitting κ-filtered colimits, restriction along y induces an equivalence between the category of functors $\mathrm{Ind}_\kappa(C_0) \to D$ preserving κ-filtered colimits and $\mathrm{Fun}(C_0, D)$. The inverse is given by left Kan extension along y.*

(3) *If C_0 admits κ-small colimits, then $\mathrm{Ind}_\kappa(C_0)$ admits all small colimits, y preserves κ-small colimits, and for every category D admitting small colimits, restriction along y induces an equivalence between colimit-preserving functors $\mathrm{Ind}_\kappa(C_0) \to D$ and functors $C_0 \to D$ preserving κ-small colimits.*

Definition A.16. A category C is *κ-accessible* if it is locally small, admits κ-filtered colimits, the full subcategory C^κ is essentially small, and every object of C is a κ-filtered colimit of κ-compact objects. It is *accessible* if it is κ-accessible for some κ. A functor between accessible categories is *accessible* if it preserves κ-filtered colimits for some κ. A category is *presentable* if it is accessible and admits all small colimits.

Theorem A.17 ([Lur09, Section 5.4.2])**.**

(1) *A category C is κ-accessible if and only if $C \simeq \mathrm{Ind}_\kappa(C_0)$ for some small category C_0. In this case, the inclusion $C^\kappa \subseteq C$ induces an equivalence $\mathrm{Ind}_\kappa(C^\kappa) \simeq C$, and C^κ is the idempotent completion of C_0.*

(2) *If C is κ-accessible, then C^λ is essentially small for every regular cardinal λ, and C is λ-accessible for arbitrarily large λ.*

Theorem A.18 ([Lur09, Proposition 5.4.7.3])**.** *The subcategory $\mathrm{Acc} \subseteq \widehat{\mathrm{Cat}}$ of accessible categories and accessible functors admits small limits, and the inclusion $\mathrm{Acc} \subseteq \widehat{\mathrm{Cat}}$ preserves them.*

Theorem A.19 (Adjoint functor theorem, [Lur09, Corollary 5.5.2.9])**.** *Let $F\colon C \to D$ be a functor between presentable categories. Then F admits a right adjoint if and only if it preserves small colimits, and F admits a left adjoint if and only if it is accessible and preserves small limits.*

Lemma A.20. *Let C be an accessible category. Then every object of C is λ-compact for some regular cardinal λ. Consequently, $C = \bigcup_\lambda C^\lambda$, where each C^λ is essentially small.*

Proof. Let C be κ-accessible and $x \in C$. Write $x = \mathrm{colim}_{i\in I} x_i$ with I small and κ-filtered and each x_i κ-compact. Choose a regular cardinal $\lambda \geq \kappa$ such that I is λ-small. Each x_i is λ-compact by Corollary A.4(1). Moreover $\mathrm{Map}(x, -) \simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Map}(x_i, -)$ is a λ-small limit of functors preserving λ-filtered colimits, hence preserves them by Theorem A.3. The last statement follows from Theorem A.17(2). □

Construction A.21. Let C be a locally small, possibly large, category. We let $\mathrm{Ind}(C) \subseteq \mathrm{Fun}(C^{\mathrm{op}}, \widehat{\mathrm{An}})$ denote the full subcategory spanned by the small filtered colimits of representable functors, where $\widehat{\mathrm{An}}$ denotes the category of large animae. If $C' \subseteq C$ is a small full subcategory, then left Kan extension along $C' \subseteq C$ restricts to a fully faithful functor $\mathrm{Ind}(C') \hookrightarrow \mathrm{Ind}(C)$ which preserves filtered colimits: indeed, for filtered diagrams (c_i) and (d_j) in C' both categories compute

$$\mathrm{Map}(\text{“colim”}_i c_i, \text{“colim”}_j d_j) \simeq \lim_i \mathrm{colim}_j \mathrm{Map}_C(c_i, d_j).$$

If C is accessible, then Lemma A.20 shows that

$$\mathrm{Ind}(C) = \bigcup_\lambda \mathrm{Ind}(C^\lambda),$$

since a small filtered diagram in C takes values in C^λ for λ large enough.

If C admits filtered colimits, then the functor $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ sending “colim”$_i c_i$ to $\mathrm{colim}_i c_i$ (the unique filtered-colimit-preserving extension of id_C, by the analogue of Theorem A.15(2)) is left adjoint to the Yoneda embedding $y\colon C \hookrightarrow \mathrm{Ind}(C)$, since

$$\mathrm{Map}_C(\mathrm{colim}_i c_i, c) \simeq \lim_i \mathrm{Map}_C(c_i, c) \simeq \mathrm{Map}_{\mathrm{Ind}(C)}(\text{“colim”}_i c_i, y(c)).$$

A.3 Idempotent completeness

Recall from [Lur09, Section 4.4.5] that a category C is *idempotent complete* if every idempotent in C splits. We will use the following characterization, see [Lur09, Section 5.1.4]: a locally small category C is idempotent complete if and only if the essential image of the Yoneda embedding $C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \widehat{\mathrm{An}})$ is closed under retracts, if and only if the essential image of $C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \widehat{\mathrm{An}})$ is closed under retracts. Every category admitting sequential colimits is idempotent complete [Lur09, Section 4.4.5] (compare Lemma A.11); this applies in particular to presentable categories and to all categories admitting filtered colimits. Moreover, every accessible category is idempotent complete [Lur09, Chapter 5], and every small category C_0 admits an idempotent completion $C_0 \hookrightarrow C_0^{\mathrm{idem}}$, which may be computed as the full subcategory of $\mathrm{PSh}(C_0)$ spanned by retracts of representables.

Lemma A.22. *Let $g\colon C \to D$ be a retract of $g'\colon C' \to D'$ in $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}})$, where C, C', D, D' are locally small. Suppose that g' has a left adjoint f' and that C is idempotent complete. Then g has a left adjoint.*

Proof. Consider a retract diagram

$$\begin{array}{ccccc} C & \xrightarrow{u} & C' & \xrightarrow{v} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ D & \xrightarrow{\hat{u}} & D' & \xrightarrow{\hat{v}} & D, \end{array}$$

with $vu \simeq \mathrm{id}_C$ and $\hat{v}\hat{u} \simeq \mathrm{id}_D$, compatibly with the squares. By [Lur09, Proposition 5.2.4.2], it suffices to show that for every $d \in D$ the functor $F_d = \mathrm{Map}_D(d, g(-))\colon C \to \mathrm{An}$ is corepresentable. Since C is idempotent complete, it suffices to exhibit F_d as a retract of the corepresentable functor $\mathrm{Map}_C(x, -)$, where $x = vf'\hat{u}(d)$. Denote by η the unit of the adjunction $f' \dashv g'$ and by $\varphi \mapsto \varphi^\flat$ the adjunction bijection $\mathrm{Map}_{D'}(e, g'(c')) \simeq \mathrm{Map}_{C'}(f'(e), c')$. We define natural transformations

$$F_d \xrightarrow{i} \mathrm{Map}_C(x, -) \xrightarrow{p} F_d$$

as follows. For $c \in C$ and $\varphi\colon d \to g(c)$, let $i(\varphi)$ be the composite

$$x = vf'\hat{u}(d) \xrightarrow{v((\hat{u}\varphi)^\flat)} vu(c) \simeq c,$$

where $\hat{u}\varphi\colon \hat{u}(d) \to \hat{u}g(c) \simeq g'u(c)$. For $\chi\colon x \to c$, let $p(\chi)$ be the composite

$$d \simeq \hat{v}\hat{u}(d) \xrightarrow{\hat{v}(\eta_{\hat{u}d})} \hat{v}g'f'\hat{u}(d) \simeq gvf'\hat{u}(d) = g(x) \xrightarrow{g(\chi)} g(c).$$

Both constructions are natural in c. Using $gv \simeq \hat{v}g'$ and the triangle identity $g'((\hat{u}\varphi)^\flat) \circ \eta_{\hat{u}d} = \hat{u}\varphi$, we compute

$$p(i(\varphi)) = gv((\hat{u}\varphi)^\flat) \circ \hat{v}(\eta_{\hat{u}d}) \simeq \hat{v}\big(g'((\hat{u}\varphi)^\flat) \circ \eta_{\hat{u}d}\big) = \hat{v}\hat{u}(\varphi) \simeq \varphi,$$

so $pi \simeq \mathrm{id}$. Hence F_d is a retract of $\mathrm{Map}_C(x, -)$, as desired. □

A.4 Mates

We freely use the calculus of mates in the homotopy 2-category of categories, see [KS74] and [RV22]. Since invertibility of a natural transformation can be checked objectwise in homotopy categories, all statements in this section only concern natural transformations up to homotopy.

Construction A.23. Consider a square of categories

$$\begin{array}{ccc} A & \xrightarrow{G} & B \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ A' & \xrightarrow[G']{} & B' \end{array}$$

commuting via a natural isomorphism $\alpha\colon VG \simeq G'U$, and suppose that G and G' admit left adjoints F and F', with units η, η' and counits $\varepsilon, \varepsilon'$. The *mate* of α is the natural transformation $\beta\colon F'V \to UF$ given by the composite

$$F'V \xrightarrow{F'V\eta} F'VGF \overset{\alpha}{\cong} F'G'UF \xrightarrow{\varepsilon' UF} UF.$$

If instead U and V admit right adjoints U^R and V^R, the *vertical mate* of α is the natural transformation $\gamma\colon GU^R \to V^R G'$ given by

$$GU^R \to V^R VGU^R \overset{\alpha}{\cong} V^R G'UU^R \to V^R G'.$$

Lemma A.24 (Conjugate transformations)**.** *Let $F_1 \dashv G_1$ and $F_2 \dashv G_2$ be adjunctions between categories A and B, with $F_i\colon A \to B$. For every natural transformation $\varphi\colon F_1 \to F_2$ there is a unique natural transformation $\varphi^\vee\colon G_2 \to G_1$, its* conjugate*, such that the square*

$$\begin{array}{ccc} \mathrm{Map}_B(F_2 a, b) & \xrightarrow{\varphi^*} & \mathrm{Map}_B(F_1 a, b) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ \mathrm{Map}_A(a, G_2 b) & \xrightarrow{(\varphi^\vee)_*} & \mathrm{Map}_A(a, G_1 b) \end{array}$$

commutes naturally in a and b. Moreover, φ is an isomorphism if and only if $\varphi^\vee$ is.

Proof. The composite of the top map with the vertical isomorphisms is a natural transformation $\mathrm{Map}_A(-, G_2(-)) \to \mathrm{Map}_A(-, G_1(-))$, which by the Yoneda lemma is induced by a unique natural transformation $\varphi^\vee\colon G_2 \to G_1$. Again by the Yoneda lemma, φ is an isomorphism if and only if φ^* is an isomorphism for all a, b, if and only if $(\varphi^\vee)_*$ is an isomorphism for all a, b, if and only if $\varphi^\vee$ is an isomorphism. □

Lemma A.25 ([KS74, Section 2.2, Propositions 2.1 and 2.2])**.** *In the situation of Construction A.23, assume that G, G' admit left adjoints F, F' and that U, V admit right adjoints U^R, V^R. Then the mate $\beta\colon F'V \to UF$ (a transformation between left adjoints of $V^R G'$ and GU^R) is conjugate to the vertical mate $\gamma\colon GU^R \to V^R G'$. In particular, β is an isomorphism if and only if γ is.*

Lemma A.26 (Adjoint triples)**.** *Let $F \dashv G \dashv H$ be an adjoint triple, with $F, H\colon A \to B$ and $G\colon B \to A$. Then F is fully faithful if and only if H is fully faithful.*

Proof. We have adjunctions $GF \dashv GH$ and $\mathrm{id}_A \dashv \mathrm{id}_A$. The unit $\eta\colon \mathrm{id}_A \to GF$ of $F \dashv G$ is a transformation between the left adjoints, and its conjugate is the counit $\varepsilon\colon GH \to \mathrm{id}_A$ of $G \dashv H$: indeed, under the identifications $\mathrm{Map}_A(GFa, a') \simeq \mathrm{Map}_B(Fa, Ha') \simeq \mathrm{Map}_A(a, GHa')$, precomposition with η_a corresponds to postcomposition with $\varepsilon_{a'}$, by the triangle identities. Now F is fully faithful if and only if η is an isomorphism, and H is fully faithful if and only if ε is an isomorphism, so the claim follows from Lemma A.24. □

The following was used in the definition of strongly continuous functors between compactly assembled categories, see Definition 3.9 and Remark 3.10.

Proposition A.27. *Let C and D be compactly assembled categories, with left adjoints $\hat{y}_C$ and $\hat{y}_D$ of the colimit functors, and let $f\colon C \to D$ be a functor with a right adjoint g. Then the mate*

$$\beta\colon \hat{y}_D f \to \mathrm{Ind}(f)\hat{y}_C$$

of the commutative square $\mathrm{colim}_D \circ \mathrm{Ind}(f) \simeq f \circ \mathrm{colim}_C$ is an isomorphism if and only if g preserves filtered colimits.

Proof. Applying Ind to the unit and counit of $f \dashv g$ shows that $\mathrm{Ind}(f) \dashv \mathrm{Ind}(g)$. By Lemma A.25, β is an isomorphism if and only if the vertical mate

$$\gamma\colon \mathrm{colim}_C \mathrm{Ind}(g) \to gf\,\mathrm{colim}_C \mathrm{Ind}(g) \simeq g\,\mathrm{colim}_D \mathrm{Ind}(f)\,\mathrm{Ind}(g) \to g\,\mathrm{colim}_D$$

is an isomorphism. Evaluating at an object "colim"$_i d_i \in \mathrm{Ind}(D)$, the map γ becomes

$$\mathrm{colim}_i\, g(d_i) \to gf\big(\mathrm{colim}_i\, g(d_i)\big) \simeq g\big(\mathrm{colim}_i\, fg(d_i)\big) \to g\big(\mathrm{colim}_i\, d_i\big),$$

whose restriction to each $g(d_j)$ is $g(d_j) \to gfg(d_j) \to g(d_j)$ followed by the canonical map $g(d_j) \to g(\mathrm{colim}_i\, d_i)$. By the triangle identity, γ is therefore the canonical comparison map $\mathrm{colim}_i\, g(d_i) \to g(\mathrm{colim}_i\, d_i)$. Hence γ is an isomorphism if and only if g preserves filtered colimits. □

A.5 Tensor products of presentable categories

We denote by $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathrm{Cat}}$ the subcategory of presentable categories and colimit-preserving functors, and by $\mathrm{Pr}^{\mathrm{R}} \subseteq \widehat{\mathrm{Cat}}$ the subcategory of presentable categories and accessible limit-preserving functors. By Theorem A.19, passing to adjoints gives an equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$ [Lur09, Corollary 5.5.3.4]. Both Pr^{L} and Pr^{R} admit small limits, computed in $\widehat{\mathrm{Cat}}$ [Lur09, Proposition 5.5.3.13 and Theorem 5.5.3.18]; in particular, colimits in Pr^{L} may be computed as limits in $\widehat{\mathrm{Cat}}$ along the right adjoints.

Theorem A.28 ([Lur17, Section 4.8.1])**.** *The category Pr^{L} admits a symmetric monoidal structure $- \otimes -$ with unit An, characterized by the property that colimit-preserving functors $C \otimes D \to E$ correspond to functors $C \times D \to E$ preserving colimits in each variable separately:*

$$\mathrm{Fun}^{\mathrm{colim}}(C \otimes D, E) \simeq \mathrm{Fun}^{\mathrm{colim}}(C, \mathrm{Fun}^{\mathrm{colim}}(D, E)).$$

In particular, Pr^{L} *is closed, with internal hom given by* $\mathrm{Fun}^{\mathrm{colim}}(-,-)$*. Moreover, there is a natural equivalence*

$$C \otimes D \simeq \mathrm{Fun}^{\mathrm{lim}}(C^{\mathrm{op}}, D),$$

where the right-hand side denotes the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, D)$ *spanned by the functors preserving small limits.*

Remark A.29. The second description arises as follows: one first shows that $\mathrm{Fun}^{\mathrm{lim}}(C^{\mathrm{op}}, D)$ is presentable, and then one computes that

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Fun}^{\mathrm{lim}}(C^{\mathrm{op}}, D), E) \simeq \mathrm{Fun}^{\mathrm{colim}}(C, \mathrm{Fun}^{\mathrm{colim}}(D, E))$$

for every presentable E.

The full subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{L}}$ of stable presentable categories is closed under the tensor product, and becomes a symmetric monoidal category with unit Sp; moreover, a presentable category C is stable if and only if the map $C \to \mathrm{Sp} \otimes C$ is an equivalence [Lur17, Section 4.8.2].

Proposition A.30. *Let* C_0 *be a small category with finite colimits and let* D *be a presentable category. There is an equivalence, natural in* D*,*

$$\mathrm{Ind}(C_0) \otimes D \simeq \mathrm{Fun}^{\mathrm{lex}}(C_0^{\mathrm{op}}, D),$$

where the right-hand side denotes the full subcategory of $\mathrm{Fun}(C_0^{\mathrm{op}}, D)$ *spanned by the functors preserving finite limits. If* C_0 *and* D *are stable, this gives natural equivalences*

$$\mathrm{Ind}(C_0) \otimes D \simeq \mathrm{Fun}^{\mathrm{ex}}(C_0^{\mathrm{op}}, D) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(C_0^{\mathrm{op}}), D).$$

Proof. By Theorem A.28, we have

$$\mathrm{Ind}(C_0) \otimes D \simeq \mathrm{Fun}^{\mathrm{lim}}(\mathrm{Ind}(C_0)^{\mathrm{op}}, D) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(C_0), D^{\mathrm{op}})^{\mathrm{op}}.$$

Since D admits small limits, D^{op} admits small colimits, so by Theorem A.15(3) restriction along y gives an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(C_0), D^{\mathrm{op}}) \simeq \mathrm{Fun}^{\mathrm{rex}}(C_0, D^{\mathrm{op}}) = \mathrm{Fun}^{\mathrm{lex}}(C_0^{\mathrm{op}}, D)^{\mathrm{op}}.$$

This proves the first claim. In the stable case, a functor between stable categories preserves finite limits if and only if it is exact, and applying Theorem A.15(3) to C_0^{op} (which again has finite colimits) and the cocomplete category D gives $\mathrm{Fun}^{\mathrm{ex}}(C_0^{\mathrm{op}}, D) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(C_0^{\mathrm{op}}), D)$. □

Corollary A.31. *For a small stable category* A*, the restricted Yoneda functor induces an equivalence* $\mathrm{Ind}(A) \simeq \mathrm{Fun}^{\mathrm{ex}}(A^{\mathrm{op}}, \mathrm{Sp})$.

Proof. Apply Proposition A.30 with $D = \mathrm{Sp}$, using that $\mathrm{Ind}(A) \otimes \mathrm{Sp} \simeq \mathrm{Ind}(A)$ since $\mathrm{Ind}(A)$ is stable. □

Corollary A.32. *Let* C_0 *be a small stable category. Then* $C = \mathrm{Ind}(C_0)$ *is dualizable in* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$*, with dual*

$$C^{\vee} = \mathrm{Ind}(C_0^{\mathrm{op}}).$$

The evaluation map $\mathrm{Ind}(C_0^{\mathrm{op}}) \otimes \mathrm{Ind}(C_0) \to \mathrm{Sp}$ *is the colimit-preserving functor which restricts to the mapping spectrum functor*

$$\hom_{C_0} \colon C_0^{\mathrm{op}} \times C_0 \to \mathrm{Sp},$$

and the coevaluation $\mathrm{Sp} \to \mathrm{Ind}(C_0) \otimes \mathrm{Ind}(C_0^{\mathrm{op}}) \simeq \mathrm{Fun}^{\mathrm{ex}}(C_0, \mathrm{Ind}(C_0))$ *sends the sphere spectrum to the Yoneda embedding* $C_0 \hookrightarrow \mathrm{Ind}(C_0)$.

Proof. We use the criterion [Cno, Section "Symmetric monoidal duality"]: an object X of a closed symmetric monoidal category is dualizable if and only if for every Z the canonical map $Z \otimes \underline{\mathrm{Hom}}(X, \mathbb{1}) \to \underline{\mathrm{Hom}}(X, Z)$, adjoint to $Z \otimes \underline{\mathrm{Hom}}(X, \mathbb{1}) \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} Z$, is an equivalence. For $X = \mathrm{Ind}(C_0)$ we have $\underline{\mathrm{Hom}}(X, Z) = \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(C_0), Z) \simeq \mathrm{Fun}^{\mathrm{ex}}(C_0, Z)$ by Theorem A.15(3), and in particular $\underline{\mathrm{Hom}}(X, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{ex}}(C_0, \mathrm{Sp}) \simeq \mathrm{Ind}(C_0^{\mathrm{op}})$ by Corollary A.31. Under these identifications, $\mathrm{ev} \colon \mathrm{Ind}(C_0^{\mathrm{op}}) \otimes \mathrm{Ind}(C_0) \to \mathrm{Sp}$ is the colimit-preserving functor restricting on $C_0^{\mathrm{op}} \times C_0$ to $(a, c) \mapsto \hom_{C_0}(a, c)$, and the canonical map $Z \otimes \mathrm{Ind}(C_0^{\mathrm{op}}) \to \mathrm{Fun}^{\mathrm{ex}}(C_0, Z)$ agrees with the equivalence of Proposition A.30. Indeed, both are colimit-preserving functors out of $Z \otimes \mathrm{Ind}(C_0^{\mathrm{op}})$, hence determined by their restrictions to $Z \times C_0^{\mathrm{op}}$ by Theorem A.28 and Theorem A.15(3); the canonical map sends (z, a) to $z \otimes \mathrm{ev}(a, -) = z \otimes \hom_{C_0}(a, -)$, and so does the equivalence of Proposition A.30, since the equivalence $C \otimes D \simeq \mathrm{Fun}^{\mathrm{lim}}(C^{\mathrm{op}}, D)$ of Theorem A.28 sends an elementary tensor $c \otimes d$ to the functor $c' \mapsto \mathrm{Map}_C(c', c) \otimes d$ [Lur17, Proof of Proposition 4.8.1.17]. Hence X is dualizable with dual $\mathrm{Ind}(C_0^{\mathrm{op}})$ and evaluation as stated. The coevaluation corresponds, under $\underline{\mathrm{Hom}}(X, X) \simeq X \otimes X^\vee$, to the identity of X, which under $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Ind}(C_0), \mathrm{Ind}(C_0)) \simeq \mathrm{Fun}^{\mathrm{ex}}(C_0, \mathrm{Ind}(C_0))$ corresponds to the Yoneda embedding. □

Remark A.33. In other words, tensoring with $\mathrm{Ind}(C_0)$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is equivalent to forming the internal hom out of $\mathrm{Ind}(C_0^{\mathrm{op}})$.

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