Stable Homotopy Theory and Higher Algebra

An ∞-categorical introduction

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Change log

The following is an overview of the major updates of these notes:

- **November 2026:** Substantial reorganization of Part II. Expository improvements throughout the book.
- **August 2026:** Split the book into a publication version and an online version (this version); the latter uses color and contains additional material not in the published version due to space limitations. Revised Chapter 1. Added index and notation index.
- **July 2026:** Substantial revision. Improved exposition in Chapters 1–7, moved Chapter 3 on cohomology and Chapter 8 on ring spectra and modules, detailed revision of Part II with various reorganizations of the material.

  Some new material added: Section 10.3 on classical bordism groups; proof of universal property cartesian monoidal structures in Section 15.3; corrected proof of Proposition 20.2.3; Section 18.2 on mode theory; Section 17.4 on monads and monadicity; Chapter 23 on presentable ∞-categories.
- **March 2026:** Revision of Chapter 9, expanded Section 1.1 on the heuristics behind ∞-categories, added Section 21.1 on monoidal Dwyer-Kan localizations.
- **February 2026:** Revised Chapter 5, added discussion of descent in Section 24.7.
- **January 2026:** Added Chapter 6 on homological algebra, added Section 3.3 on ordinary and cellular homology. Restructured Chapter 17, removed chapter on deformation theory of ring spectra.
- **October 2025:** Substantial revision of Chapters 1-4, reorganized exercises.
- **August 2025:** Integrated remaining part of notes from *Introduction to Higher Algebra*, adding material on ring spectra. Rewrote Chapter 19 on operadic envelopes.
- **July 2025:** Expanded introduction. Ontological status changed from 'lecture notes' to 'book project' (to be published by Cambridge University Press in 2027).
- **June 2025:** Integrated Part II on the theory of ∞-operads. This material was written for the lecture course *Introduction to Higher Algebra*, taught at the University of Regensburg in the summer term of 2025.

- **May 2025:** Expanded Chapter 24 on cocartesian fibrations and the straightening/unstraightening correspondence.
- **April 2025:** Started a change log.
- **April 2025:** Substantial revision of the lecture notes:
  - Moved the introduction to ∞-categories to start, making it Chapter 1;
  - Added Chapter 2 relating ∞-categorical pushouts/pullbacks to classical homotopy pushouts/pullbacks;
  - Rewrote the chapter on Brown representability in ∞-categorical language;
  - Added a section on derived ∞-categories $D(\mathcal{A})$;
  - Reorganized the chapter on complex K-theory.
  - Expanded Appendix 25 on complete Segal animae; added model-independent discussion of homotopy coherent nerves;
  - Added an appendix with suggestions for further reading (which by now has been removed again).
- **February 2025:** Added two chapters on Thom spectra and Atiyah duality, respectively;
- **October 2024 - February 2025:** Weekly updates of the course material. We discussed, in this order, Brown representability, an introduction to ∞-categories, stable ∞-categories, (commutative) monoids and the recognition principle, complex K-theory, and Bousfield localizations.

Introduction

This book aims to provide an accessible introduction to the field of stable homotopy theory using the language of ∞-categories. Our primary goal is to develop the technical foundations of the subject, including stability and the ∞-category of spectra, the theory of ∞-operads, and some basic theory of ring spectra.

Historical overview

To put into perspective the modern viewpoint on stable homotopy theory, let us take a step back and consider some of the historical developments leading up to it.

In 19th-century topology, mathematicians like Bernhard Riemann, Enrico Betti and Henri Poincaré studied manifolds using numerical invariants such as Betti numbers and torsion coefficients. In 1925, Emmy Noether emphasized that the underlying homology groups should themselves be regarded as invariants. Their ranks and torsion subgroups recover the numerical data, but the groups provide a more natural algebraic framework. This helped shift attention from numerical invariants to homology groups. Samuel Eilenberg and Norman Steenrod subsequently developed an axiomatic approach to homology and cohomology, presented systematically in their 1952 book.

A central feature of reduced homology is its *stability*: the reduced homology of a space X agrees, up to a degree shift, with that of its suspension ΣX. Homotopy groups do not generally have this stability property, but the *Freudenthal suspension theorem*, originating in work of Hans Freudenthal in 1938 [Fre38], shows that they acquire it after sufficiently many suspensions. In its general form, it states that if X is an n-connected pointed CW complex, with $n \geq 0$, then the canonical map $X \to \Omega\Sigma X$ induces an isomorphism on π_k for $k \leq 2n$ and a surjection for $k = 2n + 1$. Applying this to successive suspensions ΣX, $\Sigma^2 X$, etcetera, one finds that the sequence of abelian groups[1]

$$\pi_k(X) \to \pi_k(\Omega\Sigma X) \to \pi_k(\Omega^2\Sigma^2 X) \to \ldots$$

eventually stabilizes for every $k \geq 0$. Its colimit is called the *k-th stable homotopy group* of X, denoted $\pi_k^{\mathrm{st}}(X)$. In contrast to their unstable analogues, the stable homotopy groups form a *(reduced) generalized homology theory*, satisfying the Eilenberg–Steenrod axioms except for the dimension axiom. Other important examples of generalized (co)homology theories include *cobordism theory*, arising from René Thom's work on manifolds with tangential structures modulo bordism [Tho54], and *topological K-theory*, developed by Atiyah and Hirzebruch from vector bundles over a space [AH59].

[1] For $k \leq 1$, the first few terms are not yet abelian groups.

The success of homology theory in topology inspired mathematicians to develop analogous theories in other fields, leading for example to the theories of group cohomology, Lie algebra cohomology, and associative algebra cohomology. A common framework for these developments was established by Henri Cartan and Samuel Eilenberg in their monumental 1956 book *Homological Algebra*, which gave the field its name. Its central notion is that of a *derived functor*, which in the case of the tensor product and the hom-functor yields Tor-groups and Ext-groups, respectively. Many previously ad hoc cohomology theories could now be described as instances of Ext-groups.

Extending the stabilization of homotopy groups to maps between spaces, Spanier and Whitehead introduced the set of *stable maps* between two pointed CW-complexes X and Y:

$$[X, Y]_*^{\mathrm{st}} \quad := \quad \operatorname{colim}_n [\Sigma^n X, \Sigma^n Y]_*.$$

Finite pointed CW-complexes and stable maps assemble into a category, which Spanier and Whitehead used to investigate duality phenomena of spaces [SW53]. Enlarging this category by formal desuspensions makes suspension invertible and yields what is now commonly called the *Spanier–Whitehead category*.

Building on Spanier and Whitehead's work, Spanier's student Lima [Lim59] introduced the first definition of a *spectrum*, and used it to extend their duality theory to a more general class of spaces. Several constructions followed, culminating in Boardman's construction of the stable homotopy category; a published treatment is given by Adams [Ada74]. One convenient presentation of spectra is the following:[2]

Definition. A *spectrum* X is a sequence of pointed spaces $\{X_n\}_{n\geq 0}$ equipped with pointed homotopy equivalences $X_n \simeq \Omega X_{n+1}$, where $\Omega(-)$ denotes the loop space; if in addition $\pi_k(X_n) = 0$ for $0 \leq k < n$, we say that X is *connective*.

The homotopy category of spectra contains the Spanier–Whitehead category as a full subcategory, represents all generalized cohomology theories on CW complexes via Brown's representability theorem, and admits an associative, commutative and unital smash product. It provides a common framework for computational tools such as the Adams spectral sequence [Ada58], Bousfield's localizations at homology theories [Bou79], and the methods of chromatic homotopy theory [Rav84; Rav92].

The years that followed saw a surge in the study of 'brave new algebra' [May77], i.e. the development of algebraic constructions and methods for spectra. Key technical ingredients included the theory of operads developed by May [May72] and Boardman–Vogt [BV73], and point-set models of spectra with suitably structured smash products [LMS86; EKMM97; HSS00]. Many familiar algebraic constructions on abelian groups have homotopical analogues for spectra:

- There is a *direct sum* $X \oplus Y$ of two spectra, which is simultaneously a product and a coproduct;
- There is a *tensor product* $X \otimes Y$ of two spectra. This leads to the notion of a *ring spectrum*: a spectrum R equipped with a multiplication operation $m\colon R \otimes R \to R$ that is 'coherently' associative and unital;

[2] These objects are classically called 'Ω-spectra'.

- There is a notion of a *module* over a ring spectrum. Familiar notions for modules over (commutative) rings, like the properties of being projective, perfect, or flat, admit generalizations to the setting of ring spectra;
- Given a spectrum X and a prime p, we may form its *p-localization* $X_{(p)}$ and its *p-completion* $X^{\wedge}_{p}$, which have similar behavior as their analogues for abelian groups;
- And so on and so forth...

When formulated intrinsically in the ∞-category of spectra, these constructions are already derived. For example, unlike the tensor product of abelian groups, the tensor product of spectra is exact in both variables. The exactness that homological algebra obtains by passing to derived functors is thus built into the setting. In homological algebra, this point of view goes back to the derived categories of Grothendieck and Verdier. This also explains why the techniques and concepts from stable homotopy theory have found significant applications in numerous neighboring fields of mathematics, like (derived) algebraic geometry, algebraic K-theory, geometric topology and representation theory.

Higher category theory

To fully appreciate the pervasive role of (stable) homotopy theory in modern mathematics, we must first embrace a new way of thinking about equality:

> **Fundamental Principle of Homotopy Theory:** When expressing that two entities are equal, we must always specify *how* they are equal by providing an identification between them.

This principle is already familiar to most mathematicians when applied to mathematical *objects*: for instance, when we want to express that two abelian groups A and B are 'the same', we usually mean that there is an isomorphism $A \cong B$ between them. In homotopy theory, we extend this principle to *morphisms* between objects; for example, two continuous maps $f, g\colon X \to Y$ are considered 'equal' when equipped with a homotopy $f \sim g$ between them. More broadly, in homotopy theory, the notion of 'sameness' is not a *property* (a binary yes/no question) but rather *structure* that must be explicitly provided. We use the word *identification* for this general notion, which takes the form of an isomorphism for abelian groups or a homotopy for continuous maps. This shift in perspective originated in algebraic topology, but has by now become fundamental in many neighboring fields of mathematics.

While this principle appears straightforward, it rapidly gives rise to intricate 'coherence problems'. Consider, for instance, the task of defining a homotopical version of an abelian group A. Focusing first on associativity of addition $+\colon A \times A \to A$, the Fundamental Principle requires providing an identification between the two maps $A \times A \times A \to A$ given by $(a, b, c) \mapsto a + (b + c)$ and $(a, b, c) \mapsto (a + b) + c$. Once such an identification is given, one can construct two identifications between the maps $(a, b, c, d) \mapsto a + (b + (c + d))$ and $(a, b, c, d) \mapsto ((a + b) + c) + d$. Identifying these requires an *identification between identifications*. Continuing with more inputs leads to higher associativity coherences, as studied by Stasheff [Sta63a]. Commutativity and its compatibility with associativity require further coherence data.

The modern solution to such coherence problems lies in the theory of ∞-*categories*. Conceptually, an ∞-category behaves much like a classical 1-category: we may speak of objects and morphisms that can be composed. But in accordance with the Fundamental Principle, we also keep track of identifications between parallel morphisms, identifications between those identifications, and so on. Homotopical versions of familiar categorical constructions are conveniently formulated within this framework: for example, we may define for any ∞-category C with finite products a new ∞-category $\mathrm{CGrp}(C)$ of *commutative groups* in C, where all the required higher coherences are built into the objects.

As we will see, there are ∞-categories $\mathcal{S}$ and Sp whose objects are spaces[3] and spectra, respectively, and whose identifications encode the familiar homotopical data. The following theorem establishes a crucial relation between these two ∞-categories:

Recognition Principle for Connective Spectra (Boardman–Vogt, May and Segal)**.** There is a fully faithful functor of ∞-categories

$$\mathrm{CGrp}(\mathcal{S}) \hookrightarrow \mathrm{Sp}$$

whose image consists of the connective spectra.[4]

In other words, we may think of a connective spectrum as a 'homotopy coherent' analogue of an abelian group. This result is part of a much broader phenomenon: many algebraic structures have natural 'homotopy coherent' analogues in the world of spectra (think for example of the ring spectra mentioned before).

While coherence problems in homotopy theory can be (and historically have been) handled without ∞-categories, the ∞-categorical framework builds the Fundamental Principle into its foundations, so that many homotopy-theoretic concepts admit native formulations. The methods from higher category theory are rapidly becoming a standard tool among researchers, and they are slowly making their way into more and more graduate courses at universities. It thus seems like the time is ripe for an accessible resource that develops the foundations of stable homotopy theory using the language of ∞-categories. Providing such a resource is the primary objective of this book.

From abelian categories to stable ∞-categories

With the concept of an ∞-category at hand, we can now return to the question of what makes stable homotopy theory such a suitable framework for doing homological algebra. The short answer is that it allows for a richer notion of *exact sequences*, refining the classical notion of short exact sequences in abelian categories. To see what this means, let us start from the classical situation.

Let us recall how short exact sequences work in an abelian category C. Such a category admits a *zero object* 0, meaning that for every object X there are unique maps $X \to 0$ and $0 \to X$, and every morphism admits a kernel and a cokernel. Moreover, given a diagram

$$X \overset{i}{\hookrightarrow} Y \overset{p}{\twoheadrightarrow} Z$$

[3] For reasons that will be explained, we will write An rather than $\mathcal{S}$ for this ∞-category, and refer to its objects as *animae*.

[4] The difference between spectra and connective spectra is analogous to the difference between arbitrary chain complexes and those whose homology vanishes in negative degrees.

with $pi = 0$, where i is a monomorphism and p is an epimorphism, the map i exhibits X as the kernel of p if and only if p exhibits Z as the cokernel of i. When these equivalent conditions hold, we call the diagram a *short exact sequence*.

The story changes a little if C is an ∞-category with a zero object. In this case, the equality $pi = 0$ from before should be replaced by a choice of nullhomotopy $pi \simeq 0$. Following the Fundamental Principle, we must take this choice as part of the data; whether the resulting diagram is exact can depend on it. Similarly, we replace the kernel of a map $g\colon Y \to Z$ by its *fiber*: the terminal example of an object X equipped with a map $f\colon X \to Y$ and a nullhomotopy $gf \simeq 0$; dually, we replace cokernels by *cofibers*. These considerations motivate the following central definition in stable homotopy theory:

Definition. An ∞-category C is called *stable* if it has a zero object, if it has fibers and cofibers, and if for every composable pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ equipped with a nullhomotopy $gf \simeq 0$, the map f and the given nullhomotopy exhibit X as the fiber of g if and only if g and the given nullhomotopy exhibit Z as the cofiber of f.

We refer to the diagrams satisfying these equivalent conditions as *exact sequences*. The agreement between fiber and cofiber sequences has powerful consequences. First, every map $f\colon X \to Y$ fits into an exact sequence

$$X \xrightarrow{f} Y \longrightarrow \operatorname{cofib}(f),$$

so exact sequences are no longer restricted to monomorphisms followed by epimorphisms, as they are in abelian categories. More importantly, a functor between stable ∞-categories is exact as soon as it preserves either finite limits or finite colimits. In particular, left and right adjoints between stable ∞-categories are exact. Tensoring with a fixed spectrum and taking internal hom from that spectrum form such an adjoint pair, so both functors are already exact.

An important example of a stable ∞-category is the ∞-category Sp of spectra; in fact, it is a *universal* example.[5] Furthermore, every abelian category $\mathcal{A}$ gives rise to a stable ∞-category by considering its *derived ∞-category* $\mathrm{D}(\mathcal{A})$, which comes equipped with a fully faithful inclusion $\mathcal{A} \hookrightarrow \mathrm{D}(\mathcal{A})$ that turns short exact sequences in $\mathcal{A}$ into exact sequences in $\mathrm{D}(\mathcal{A})$.

As we will explore during the course of this book, most of the usual constructions from homological algebra can be performed directly at the level of stable ∞-categories, without needing to first perform them at the level of abelian categories and then pass to derived functors. The classical derived functors are then obtained as the homotopy groups of these constructions. For example, every ordinary commutative ring R and every ordinary R-module M determine an *Eilenberg–MacLane ring spectrum* and a module spectrum concentrated in degree 0, which we denote by the same letters. For two R-modules M and N, their classical Tor- and Ext-groups are recovered as

$$\operatorname{Tor}_n^R(M,N) \cong \pi_n(M \otimes_R N) \qquad \text{and} \qquad \operatorname{Ext}_R^n(M,N) \cong \pi_{-n}(\hom_R(M,N)),$$

for $n \geq 0$, where $- \otimes_R -$ and $\hom_R(-,-)$ denote the tensor product and internal hom in the ∞-category of R-module spectra.

[5] For every stable ∞-category C with small colimits, a colimit-preserving functor $\mathrm{Sp} \to C$ is uniquely determined by its value on the sphere spectrum, and every object of C occurs this way. See Theorem 4.4.14.

The theory of ∞-operads

The higher coherences resulting from the Fundamental Principle do, however, make it considerably more subtle to define and manipulate algebraic structures in the setting of ∞-categories than in classical algebra. Writing out all these coherences individually quickly becomes unwieldy, so we need a formalism that organizes them systematically. One such formalism is the theory of ∞-*operads*, extending the classical theory of operads developed by May [May72] and Boardman and Vogt [BV73]. Our approach builds on Lurie's treatment of this theory [Lur17]. We devote a substantial part of this book to ∞-operads, with the aim of providing an accessible introduction to them.

Heuristically, an operad encodes operations and their composition laws. For example, the *commutative operad* has a single n-ary operation for every $n \geq 0$. Interpreted in abelian groups with their tensor product, it describes commutative rings: its operations act by the multiplication maps $R^{\otimes n} \to R$, with the nullary operation giving the unit map $\mathbb{Z} \to R$. More generally, an operad $\mathcal{O}$ specifies a set $\mathcal{O}(n)$ of n-ary operations for each $n \geq 0$, together with actions of permutations of the inputs and composition laws. For example, plugging an n-ary operation into one input of a k-ary operation yields an $(n + k - 1)$-ary operation. These composition laws must be associative, unital and compatible with permutations.

In the ∞-categorical setting, the composition laws and their higher coherences must be encoded together. Following Lurie's approach, we do this through an ∞-*category of operators*, which organizes operations and their composition into categorical data. We use an equivalent formulation over the $(2, 1)$-category Span(Fin) of *spans of finite sets*, in place of Lurie's category Fin_* of finite pointed sets. The comparison between these formulations is established in Section 19.4.

Content

The material is organized into a foundational chapter followed by three parts. Chapter 1 introduces ∞-categories and animae. Part I develops the foundations of stable homotopy theory: (co)homology theories, spectra and stable ∞-categories, the recognition principle, homological algebra, and localization and completion, followed by ring spectra, complex K-theory, Thom spectra and duality phenomena. Part II treats ∞-operads over Span(Fin) and the higher algebra built on them, including the tensor product of spectra and the theory of algebras and modules. Part III gathers the ∞-categorical background used throughout: adjunctions, presentable ∞-categories, cocartesian fibrations and straightening, and complete Segal animae. Each part opens with an introduction describing its chapters in more detail.

Prerequisites and how to read this book

The main prerequisites are 1-category theory and algebraic topology at the level of a first graduate course. In particular, the reader should be comfortable with limits and colimits, adjoint functors, CW complexes, homotopy groups, and singular homology and cohomology. Some familiarity with chain complexes and vector bundles is useful, but the relevant material is recalled when it first

becomes essential. No prior familiarity with ∞-categories, spectra, operads, model categories, or higher algebra is assumed.

This book is available in a published and an online version. The online version contains supplementary sections, identified by "(online only)" in their titles, and uses color in some explanatory diagrams. The supplementary sections are placed at the ends of chapters, so all material appearing in both versions has the same chapter, section, and theorem numbering. Page numbers, however, differ between the two versions.

The book is not intended to be read strictly from beginning to end. Chapter 1 provides the common foundation for Parts I and II, while Part III collects technical background that may be consulted when it is cited. Parts I and II have several mutual dependencies, which are summarized in Figure 1. For example, Chapter 8 on ring spectra requires the coherent tensor product of spectra, which is only constructed in Part II. So that readers can learn about ring spectra without first working through the higher algebra of Part II, the chapter begins by collecting the results it needs from Part II in a 'coherent algebra package' (Proposition 8.1.5). Part II in turn relies on the theory of spectra from Chapter 4 and, in its application to derived categories, on Chapter 6.

Here are several possible reading routes:

(1) For a first course in stable homotopy theory, read Chapter 1 followed by Part I in order, treating the forward references from Chapter 8 to Part II as black boxes. The final three chapters may be selected according to interest.

(2) For higher algebra, begin with Chapter 1 and then read Part II in order. The construction of the tensor product of spectra in Chapter 16 uses Chapter 4, and the derived-category application in Chapter 21 uses Chapter 6; these applications may be skipped if the reader is primarily interested in the abstract operadic theory. The relevant chapters of Part III may be consulted when they are first cited.

(3) For ring spectra and derived algebra, read Part I through Chapter 7, then Part II through Chapter 18, and then return to Chapter 8. For the derived-category constructions, also read Chapter 21.

(4) For the categorical foundations, read Chapter 1 followed by Part III.

Acknowledgments

This book has grown out of two graduate lecture courses on stable homotopy theory and higher algebra taught at the University of Regensburg in 2024/2025. In preparing the manuscript I have benefited from many existing sources on (stable) homotopy theory and higher algebra. I would like to acknowledge some of these influences more explicitly.

The ∞-categorical approach to stable homotopy theory and higher algebra adopted in this book is deeply indebted to Lurie's foundational text *Higher Algebra* [Lur17], which remains one of the central references in the field. Most of the theory of ∞-operads developed in Part II ultimately goes back to this work, even when the order of exposition or the proofs differ from Lurie's treatment. The exposition in Part I is strongly influenced by the lecture notes of Denis Nardin, *Introduction to stable homotopy theory* [Nar21], in particular in the choice and ordering of topics. For Part II, I have

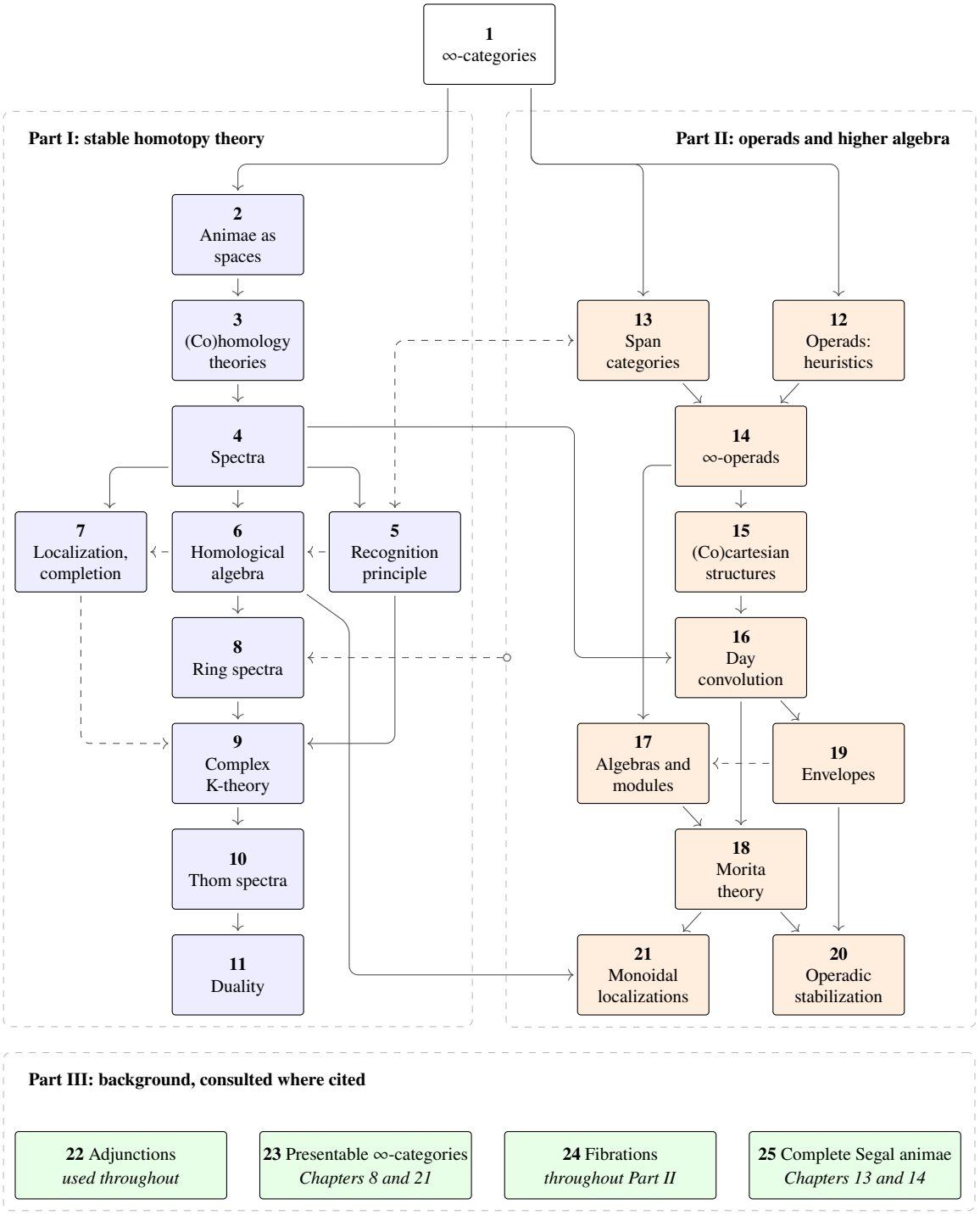


Figure 1: The dependency structure of the book. A solid arrow means that the source should substantially be read before the target. A dashed arrow marks a minor dependency, or material that may be black-boxed on a first reading; a double-headed dashed arrow marks a mutual dependency. The dashed arrow leaving the Part II panel indicates that Chapter 8 uses results from throughout Part II as black boxes. The chapters of Part III are labeled with where they are used.

benefited from Haugseng's introductory note on ∞-operads [Hau23a], and from Gepner's survey article on higher algebra [Gep20]. The brief historical sketch of stable homotopy theory in the introduction draws on the accounts of May and Weibel [May99b; Wei99]. Classical developments of the theory of spectra that have shaped my background, and from which I have borrowed a number of examples and perspectives, may be found in the textbooks of Switzer and Adams [Swi75; Ada74].

Although no single section of the present book follows them closely, Fabian Hebestreit's lecture course on algebraic and Hermitian K-theory [HW21] was my own first systematic introduction to ∞-categories and modern stable homotopy theory. It has had a lasting impact on how I think about the subject and, indirectly, on the way the material is presented here.

The model-agnostic approach to ∞-categories used throughout this book is based on joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde [CCNW26].

The pictorial illustrations for multimorphisms in ∞-operads were inspired by [BS22, Figure 1].

I am grateful to many colleagues and students for their help and encouragement. I would like to thank Denis-Charles Cisinski, Daniel Gratzer, Rune Haugseng, Sil Linskens, Maxime Ramzi, Stefan Schwede, Jan Steinebrunner, Tashi Walde, Ferdinand Wagner and Christoph Winges for numerous helpful conversations about the material in this book. I am especially grateful to Marc Hoyois for reading substantial parts of the manuscript and for pointing out several mathematical errors. The participants of the two Regensburg courses on which this book is based provided valuable feedback and many questions that led to clarifications and improvements in the exposition. I also thank Keima Aksaka, Karolis Dembickas, Alissa Doggwiler, Michael J. Glaeser, Tim Henke, Taiga Nakamura, Lorenzo Pascarella, Daniele Velati and Yiming Wang for sending corrections and further suggestions.

Use of generative AI

During the final preparation phase of this textbook[6], I have made substantial use of the AI coding assistants Claude Code and Codex. This includes:

- Editorial assistance, such as typo correction and LaTeX assistance;
- Mathematical review, such as finding errors and suggesting corrections;
- Substantive mathematical contribution, such as drafting proofs, explanations, and new material.

AI involvement was particularly substantial in the following chapters and sections: Chapter 6, Sections 5.5 and 8.1, Chapter 9, Section 10.3, Sections 15.1 and 15.3, Sections 16.3, 18.2, 19.5, 19.6 and 20.2, Chapter 21, and Chapter 23.

All AI-generated material incorporated into the text was reviewed and edited by me. This involved checking not only the mathematical arguments but also the compatibility with the book's overall structure and style. I remain solely responsible for the contents of the book and for any errors that remain.

[6]This disclosure was written in July 2026. It will likely be updated before final publication.

1 Introduction to ∞-categories

This chapter provides the ∞-categorical foundations that underlie the rest of the book. Our goal is to develop a usable language of ∞-categories that matches intuition from classical category theory and aligns with the way it is used in stable homotopy theory and higher algebra.

A characteristic feature of the theory of ∞-categories is that working with it requires a significant shift in perspective on several basic mathematical concepts, such as that of equality. The premise of this book is that the most effective way into the field is to become familiar with this perspective through basic mathematical examples and constructions, before discussing how it can be modeled within traditional set-based mathematics. We therefore take an axiomatic approach to ∞-categories, focusing on the basic properties that are needed in practice.

We discuss quasicategories as a concrete model for the axioms in Section 1.6, but the remainder of the book does not depend on this model. The specifically quasicategorical parts may therefore be skipped.

The exercises in this chapter vary considerably in difficulty. Some are routine checks, while others invite the reader to work through technical consequences of the axioms in detail. The latter may be treated as black boxes for the purpose of this book. A more detailed development of these foundations may be found in [CCNW26].

1.1 The heuristics behind ∞-categories

Before setting up the axiomatics of ∞-categories used in this book, we take a moment to explain the heuristics behind the theory, which will help motivate our choice of setup.

The fundamental principle of homotopy theory

The first concept we need to discuss is the notion of *equality*. In traditional mathematics, equality is typically viewed as a mere property: two mathematical objects are either equal or they are not. In homotopy theory and related fields, however, we encounter a notion of identification where it is not just important *that* two objects are identified, but precisely *how* they are identified. In particular, the identification between objects becomes a mathematical structure in its own right: think of a homotopy between two continuous maps, or of an isomorphism between two abelian groups. This perspective is so central to understanding ∞-categories that it deserves to be elevated to a guiding principle:

Fundamental Principle of Homotopy Theory: When expressing that two entities are equal, we must always specify *how* they are equal by providing an identification between them.

To get a feeling for this principle, let us look at three familiar settings in which it already appears:

Example 1.1.1 (Algebraic topology)**.** Algebraic topology provides the most natural example of the Fundamental Principle:

- Two continuous maps $f, g\colon X \to Y$ are considered 'equal up to homotopy' if we provide a homotopy $H\colon X \times [0,1] \to Y$ between f and g;
- Two spaces X and Y are then considered 'homotopically the same' when equipped with a *homotopy equivalence*: continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ together with chosen homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.
- When studying diagrams of spaces, we similarly replace strict commutativity by homotopy commutativity. For instance, given a square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{k} & W \end{array}$$

  we do not ask for a strict equality $h \circ f = k \circ g$, but instead require a specified homotopy $H\colon h \circ f \sim k \circ g$ witnessing how the two compositions are 'the same'. This perspective naturally leads to *homotopy coherent diagrams*, where the specified homotopies are accompanied by higher homotopies expressing their compatibility.

Example 1.1.2 (Homological algebra)**.** The phenomenon of 'equality as structure' also naturally appears in homological algebra, where it leads us to think about chain complexes not as rigid algebraic objects but as representatives of more flexible homotopical entities:

- Instead of requiring two chain maps $f, g\colon C_\bullet \to D_\bullet$ to be equal, we specify a *chain homotopy* between them: a family of maps $\varphi_n\colon C_n \to D_{n+1}$ satisfying $f_n - g_n = \partial^D \circ \varphi_n + \varphi_{n-1} \circ \partial^C$. A chain map admitting an inverse up to chain homotopy is called a *chain homotopy equivalence*.
- In practice, one often wants to go further and even identify two chain complexes whenever they are connected by a *quasi-isomorphism*, that is, a chain map inducing isomorphisms on all homology groups. Localizing at the quasi-isomorphisms gives the *derived category* of chain complexes.

Example 1.1.3 (Category theory)**.** A third example comes from category theory:

- For two objects X and Y in a 1-category C, the natural notion of equality between them is often not the strict set-theoretic equality, but rather a specified isomorphism $f\colon X \xrightarrow{\sim} Y$. Universal

constructions illustrate this: any two products of a fixed pair of objects are related by a unique isomorphism compatible with their projections.

- This pattern extends to functors: we wish to think of two functors $F, G\colon C \to D$ as 'the same' whenever we are given a natural isomorphism $\eta\colon F \xrightarrow{\sim} G$. If we think of the individual isomorphisms $\eta_X\colon F(X) \xrightarrow{\sim} G(X)$ as identifications, then naturality expresses their compatibility with morphisms.
- In monoidal categories, the associativity and unitality conditions are expressed not as strict equalities but as natural isomorphisms: for objects X, Y, and Z, we have an associator $\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ and unitors $\lambda_X\colon 1 \otimes X \xrightarrow{\sim} X$ and $\rho_X\colon X \otimes 1 \xrightarrow{\sim} X$. The coherence conditions (like the Mac Lane pentagon) then govern how these isomorphisms relate to each other.

The common pattern in all these examples is that it is not enough to know that two things are the same: one must also remember the identifications, and often also the compatibilities between different such identifications.

Animae

Once equality is no longer a mere property, the usual notion of a collection also has to change. In traditional set-based mathematics, the primitive notion of a collection is that of a *set*. The behavior of sets is governed by axioms which strongly reflect the traditional perspective on equality: given a set X, two of its elements x and y are either equal or not equal. Their equality carries no additional structure.

If we take the Fundamental Principle fully seriously, we are thus led to a different notion of collection, one in which equality between elements is a structure rather than a property. In recent years, the homotopy theory community has started to adopt the term *anima* for this concept. In terms of traditional mathematics, one may think of animae as weak homotopy types of topological spaces; we will discuss this viewpoint at length in Chapter 2. But the philosophy we adopt in this book is that animae are primitive objects in their own right, whose role is similar to that of sets in traditional mathematics. In particular, the familiar constructions on sets still have analogues for animae:

- Given animae X and Y, there is a *product* $X \times Y$, whose elements are pairs (x, y) consisting of an element x of X and an element y of Y;
- Given animae X and Y, there is a *disjoint union* $X \sqcup Y$, an element of which is either an element of X or an element of Y;
- There is an *empty anima* $\emptyset$, which does not have any elements;
- There is a *one-point anima* $*$, which has a unique element in the homotopical sense explained below.

The main difference between animae and sets lies in what it means for two elements x and y of an anima X to be *equal*. Rather than equality being a yes/no question, we should regard identifications between x and y as mathematical structure that can be manipulated. The collection of all such

identifications should then itself form an anima, which we denote by $(x =_X y)$, or simply by $x = y$ if the ambient anima is clear. Given two such identifications α and β, we may in turn form the anima $(\alpha =_{(x=_X y)} \beta)$ of identifications between them. This process may be repeated indefinitely, forming identifications between identifications at every level.

The following are some examples of animae:

Example 1.1.4. Every set X gives rise to an anima, which we will again denote by X. Given two elements $x, y \in X$, the identification anima $(x =_X y)$ is the one-point anima if x and y are equal as elements of the set X, and the empty anima otherwise.

Example 1.1.5. Every 1-category C gives rise to an anima $C^{\simeq}$, called its *groupoid core*. The elements of $C^{\simeq}$ are the objects of C. The identification anima $(X =_{C^{\simeq}} Y)$ between two objects X and Y is the set of isomorphisms between X and Y (regarded as an anima via the previous example).

Example 1.1.6. Every topological space X gives rise to an anima $\Pi_\infty(X)$, called its *underlying anima* (or *fundamental* ∞*-groupoid*). Every point in X defines an element of $\Pi_\infty(X)$. Given points $x, y \in X$, every path $p\colon [0,1] \to X$ from x to y defines an identification in $\Pi_\infty(X)$ between x and y. Given two such paths p and q, every homotopy between p and q relative to their endpoints defines an identification between them in $(x =_{\Pi_\infty(X)} y)$. Homotopies between these homotopies, relative to their boundaries, define identifications between identifications. This pattern repeats ad infinitum. We will study the assignment $X \mapsto \Pi_\infty(X)$ in more detail in Chapter 2.

Just as sets come with maps between them, animae come with maps between them. A map of animae $f\colon X \to Y$ assigns to every element x of X an element $f(x)$ of Y, to every identification between x and x' an identification between $f(x)$ and $f(x')$, to every identification between two such identifications an identification between their images, and so on, compatibly with identities and composition at every level. The role played by bijections of sets is now played by *equivalences of animae*: those maps which admit an inverse, together with the identifications witnessing that both composites are the identity.

Uniqueness

The homotopical perspective on equality also forces us to reconsider uniqueness. Classically, saying that a set X contains a unique element with some given property means that the subset of all valid choices has exactly one element, or equivalently that the map from this subset to the one-point set $*$ is a bijection. Replacing sets by animae and bijections by equivalences, the homotopical analogue reads: the *anima* of all valid choices is *contractible*, meaning that the map from it to the one-point anima $*$ is an equivalence. In this sense, contractibility is the homotopical incarnation of 'there is exactly one': it says that a choice exists, that any two choices are equal, that any two identifications between the same two choices are themselves identified, and so on.

For example, given elements x and y of the terminal anima $*$, we can express that there exists a unique identification between x and y by saying that the anima $(x =_* y)$ is contractible.

The homotopical interpretation of uniqueness appears under various names in the classical literature. In category theory, the phrase 'up to unique isomorphism' is often used for objects satisfying a universal property. It means that the anima (in this case: groupoid) of such objects, together with the isomorphisms compatible with their universal structures, is contractible. In homotopy theory, a

common phrase is 'up to contractible choice'. For example, to concatenate $n \geq 1$ loops in a pointed space, one may first choose a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval. The space of such subdivisions is convex, so its underlying anima is contractible. In this book, all uniqueness statements should be interpreted in the homotopical sense, and we simply say *unique*.

∞-categories

Having replaced sets by animae, we can now make the same replacement in the notion of a category. Instead of a set of objects and sets of morphisms between them, we now allow the objects to form an anima, and for each pair of objects the morphisms between them to form an anima as well. Composition is then accompanied by identifications expressing associativity and unitality, together with higher identifications expressing their coherence. This is the heuristic origin of the notion of an $(\infty, 1)$-*category*, or just an ∞-*category* for short.[7]

One should not take this heuristic too literally as a definition: replacing sets by animae suggests the right shape of the theory, but turning this into a workable notion of ∞-category requires encoding highly non-trivial coherence data for composition. This is one of the reasons why, later in this chapter, we will take ∞-categories as primitive and describe their behavior axiomatically rather than trying to define them directly from this heuristic.

Let us now go through some basic families of examples of ∞-categories:

Example 1.1.7. Every classical category C defines an ∞-category, whose anima of objects is the groupoid core $C^{\simeq}$ from Example 1.1.5, and whose hom animae are the hom sets of C.

Example 1.1.8. Every topologically enriched category also defines an ∞-category, whose hom animae are obtained by applying Π_∞ to its hom spaces. We construct this ∞-category in Section 25.3.

Example 1.1.9. Every anima X defines an ∞-category, again denoted X, whose objects are the elements of X and whose animae of morphisms from x to y are the animae $(x =_X y)$ of identifications. Identity morphisms and categorical composition arise from identity identifications and composition of identifications. Inversion of identifications makes every morphism invertible.

In our axiomatic setup, animae form a primitive class of ∞-categories. Once morphisms and their inverses have been introduced, we will impose the axiom that this class consists exactly of those ∞-categories in which every morphism is invertible. The animae $(x =_X y)$ from before are then identified with the hom animae $\mathrm{Hom}_X(x, y)$ of this ∞-category.

Many familiar constructions from classical category theory also make sense for ∞-categories, such as products, coproducts, pullbacks, pushouts, functor categories, groupoid cores, subcategories, and localizations.

1.2 The language of ∞-category theory

The previous section explained why one should expect ∞-categories to arise once equality is treated as structure rather than property. We now turn that heuristic picture into working syntax. We will do

[7] The terminology '∞-category' is standard but in some ways unfortunate: all morphisms in degrees greater than 1 are invertible, so the notion does not include arbitrary 2-categories, 3-categories, etcetera.

so in two steps. First we describe the ambient language of ∞-categories, functors, and identifications. Then we explain how the familiar internal notions of objects, morphisms, and commutative triangles can be recovered from this language by testing against a few small 1-categories.

1.2.1 ∞-categories, functors and identifications

We start with the basic objects of this language:

Axiom A.1 (∞-categories and identifications)**.** (1) We may speak of ∞-*categories*, denoted C, D, E, etcetera. Among them there is a distinguished class of *animae*. There is an anima $*$ called the *one-point anima*. For animae X and Y, there is an anima $X \times Y$ called their *product*.

(2) For ∞-categories C, D, we may speak of *functors* $f\colon C \to D$. We assume identity functors $\mathrm{id}_C\colon C \to C$, and composite functors $g \circ f\colon C \to E$ for $f\colon C \to D$ and $g\colon D \to E$. Products of animae have projection functors $\mathrm{pr}_X\colon X \times Y \to X$ and $\mathrm{pr}_Y\colon X \times Y \to Y$. For functors $f\colon T \to X$ and $g\colon T \to Y$, there is a *pairing* $(f, g)\colon T \to X \times Y$. Every ∞-category T has a functor $p_T\colon T \to *$.

(3) For parallel functors $f, g\colon C \to D$, there is an anima $(f = g)$, called their *anima of identifications*. An *identification* $\alpha\colon f = g$ is a functor $\alpha\colon * \to (f = g)$, also called a *point* of this anima.

For points $x, y\colon * \to X$ of an anima X, we also write $(x =_X y)$ for their identification anima, in agreement with the notation of Section 1.1. Given two identifications $\alpha, \beta\colon * \to (f = g)$, the same construction gives an anima $(\alpha = \beta)$, allowing us to speak of identifications between identifications. We may iterate this as often as needed. In practice, we frequently leave the chosen identification unnamed.

The universal properties of the one-point anima and products of animae will be stated in Section 1.3; we have included some of their accompanying structure in the foundational language as it is used to state various coherence relations below.

Axiom A.2. The composition of functors is unital and associative through specified identifications: for $f\colon C \to D$, $g\colon D \to E$, and $h\colon E \to F$, there are identifications

$$\lambda_f\colon \mathrm{id}_D \circ f = f, \qquad \rho_f\colon f \circ \mathrm{id}_C = f, \qquad \alpha_{f,g,h}\colon (h \circ g) \circ f = h \circ (g \circ f).$$

Every functor f has an identity identification $\mathrm{id}_f\colon f = f$. For parallel functors f, g, h, there are functors

$$- \circ -\colon (g = h) \times (f = g) \to (f = h), \qquad (-)^{-1}\colon (f = g) \to (g = f),$$

called *vertical composition* and *inversion*. Vertical composition is unital and associative through specified identifications, and inversion supplies two-sided inverses. These laws are identifications of functors on the relevant animae of inputs; for example, left unitality of vertical composition specifies an identification $\mathrm{id}_g \circ (-) = (-)$ of functors $(f = g) \to (f = g)$.

Identifications can also be composed with functors on either side.

Axiom A.3. For parallel functors $f, f' \colon C \to D$ and functors $h \colon D \to E$, $k \colon B \to C$, there are functors

$$h \star - \colon (f = f') \to (hf = hf'), \qquad - \star k \colon (f = f') \to (fk = f'k),$$

called *postwhiskering* and *prewhiskering*. Given also $g, g' \colon D \to E$, the two ways of pasting $\sigma \colon f = f'$ and $\tau \colon g = g'$ are identified:

$$(\tau \star f') \circ (g \star \sigma) = (g' \star \sigma) \circ (\tau \star f).$$

Either expression defines their *horizontal composite* $\tau * \sigma \colon gf = g'f'$.

Remark 1.2.1. Since the operations on identifications introduced above are functors between animae, they act on identifications between identifications, and so on at every level.

We also assume various coherence laws relating these operations. For example, whiskering a composite of identifications agrees with the composite of the whiskered identifications. Similarly, whiskering an identification between functors is compatible with the unitors and associator. The unitors and associator themselves satisfy the usual triangle and pentagon identities.

These laws are themselves specified identifications. When a law takes an identification as an input, we view it as an identification between the respective two functors between identification animae. When several identifications occur as inputs, the source of both functors is a product of their identification animae. For instance, the interchange law in Axiom A.3 is an identification between the two functors

$$(g = g') \times (f = f') \longrightarrow (gf = g'f')$$

obtained by composing projections, whiskering, and vertical composition according to the formulas

$$(\tau, \sigma) \longmapsto (\tau \star f') \circ (g \star \sigma), \qquad (\tau, \sigma) \longmapsto (g' \star \sigma) \circ (\tau \star f).$$

For the complete list of coherence laws, we refer to Section 1.2 of [CCNW26]. Throughout the book, we use these laws implicitly, usually leaving the parentheses and the necessary associativity and unit identifications unwritten.

Definition 1.2.2. A *commutative square of functors*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D' \end{array}$$

consists of four functors f, g, h and f' together with a specified identification $\alpha \colon h \circ f = f' \circ g$. A *commutative triangle of functors*

$$\begin{array}{ccccc} & & D & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ C & & \xrightarrow[h]{} & & E \end{array}$$

consists of three functors f, g and h together with a specified identification $\alpha \colon h = g \circ f$. We may similarly form commutative diagrams of identifications, and of identifications between identifications.

When applying a universal property, we must also account for the identifications in its input data. For example, an identification of two commutative squares must compare their chosen commutativity identifications, not just their four edges.

Definition 1.2.3. A functor $f\colon C \to D$ is said to be an *equivalence* if there exists another functor $g\colon D \to C$ and identifications $\mathrm{id}_C = gf$ and $\mathrm{id}_D = fg$. In this case we call g an *inverse* to f.

Exercise 1.2.4. Show that a functor f is an equivalence if and only if it admits both a section (i.e., a g such that $\mathrm{id}_D = fg$) and a retraction (i.e. an h such that $\mathrm{id}_C = hf$), and that in this case we have $g = h$.

Notation 1.2.5. It follows immediately from the previous exercise that if g and h are both inverses to f, then there is an identification $g = h$. When an inverse to f has been chosen, we denote it by $f^{-1}\colon D \to C$.

Exercise 1.2.6. Show that if $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.

Lemma 1.2.7. *Equivalences satisfy the 2-out-of-3 property: given functors $f\colon C \to D$ and $g\colon D \to E$, if two of the three functors f, g and gf are equivalences, then so is the third.*

Proof. If g and f are equivalences, then so is gf by Exercise 1.2.6. If f and gf are equivalences, then we may rewrite g as the composite $g = g \circ \mathrm{id}_D = g \circ (f \circ f^{-1}) = (gf) \circ f^{-1}$, so that g is the composite of two equivalences and hence itself an equivalence. A similar argument shows that f is an equivalence whenever g and gf are equivalences. □

Exercise 1.2.8. Show that equivalences of ∞-categories satisfy the 2-out-of-6 property: given functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.

Lemma 1.2.9. *Prewhiskering or postwhiskering by an equivalence induces equivalences of identification animae. In particular, if $f\colon A \to B$ is an equivalence and $x, y\colon * \to A$, then*

$$f_*\colon (x = y) \longrightarrow (fx = fy)$$

is an equivalence.

Proof. Let us merely prove the special case; the general proof is similar. Let $g\colon B \to A$ be an inverse to f, with identifications $\alpha\colon \mathrm{id}_A = gf$ and $\beta\colon \mathrm{id}_B = fg$. We may then consider the following maps of animae:

$$(x = y) \xrightarrow{f_*} (fx = fy) \xrightarrow{g_*} (gfx = gfy) \xrightarrow{f_*} (fgfx = fgfy).$$

By the 2-out-of-6 property, it suffices to show that the composites $g_*f_* = (gf)_*$ and $f_*g_* = (fg)_*$ are equivalances. We argue the first; the second is similar. Applying the interchange law to the identification $\alpha\colon \mathrm{id}_A = gf$, we see that $(gf)_*$ gets identified with the composite

$$(x = y) \xrightarrow{\alpha_y \star -} (x = gfy) \xrightarrow{- \star \alpha_x^{-1}} (gfx = gfy).$$

Since α_y and α_x^{-1} are both invertible, both of these functors are equivalences, hence so is their composite. □

1.2.2 Objects, morphisms and commutative triangles

We now explain how to describe objects, morphisms, and commutative triangles inside an ∞-category. For this, we first include classical 1-categories in our language.

Axiom B.1. Every classical 1-category C gives rise to an ∞-category, which we will abusively again denote by C. Given 1-categories C and D, every 1-functor $f\colon C \to D$ of 1-categories gives rise to a functor of ∞-categories, and every ∞-functor comes from a 1-functor. Every natural isomorphism $\alpha\colon f \xrightarrow{\cong} f'$ in the classical sense gives rise to an identification $\alpha\colon f = f'$ of functors of ∞-categories. These assignments preserve composition of functors and send composition of natural isomorphisms to composition of identifications.

Example 1.2.10. Every partially ordered set $(P, \leq)$ may be regarded as a category, hence as an ∞-category, by taking $\mathrm{Hom}_P(p, q)$ to be a singleton if $p \leq q$ and empty otherwise.

Notation 1.2.11. For $n \geq 0$, we write $[n]$ for the following linearly ordered set:

$$[n] \quad := \quad \{0 \leq 1 \leq \cdots \leq n\}.$$

For $n \geq 1$ and $0 \leq i \leq n$, we define the *face map* $d_i\colon [n-1] \to [n]$ as the unique injective map of posets whose image leaves out i. Similarly we define the *degeneracy map* $s_i\colon [n+1] \to [n]$ as the unique surjective map of posets that hits i twice.

Axiom B.2. The functor $p_{[0]}\colon [0] \to *$ is an equivalence.

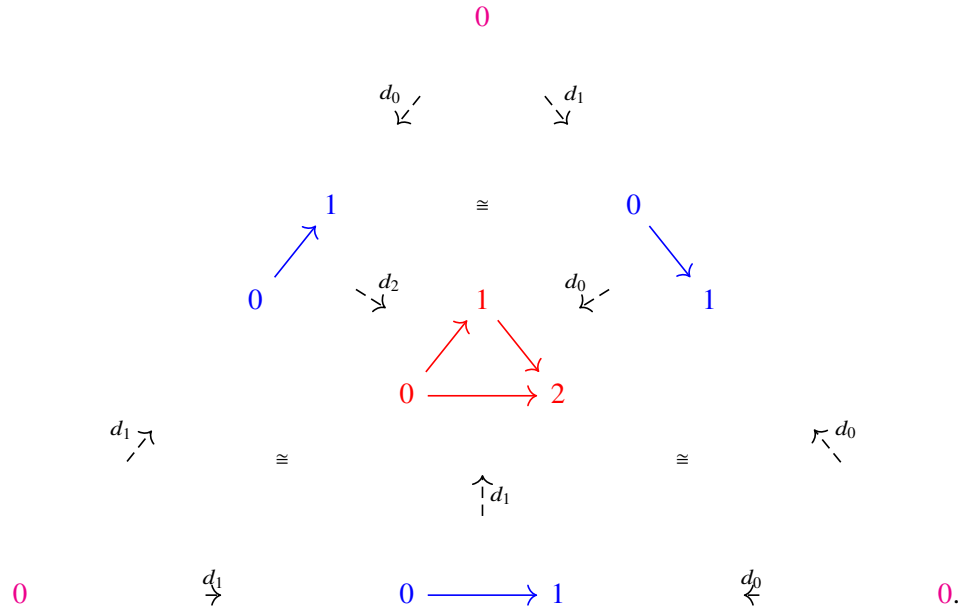
Thus a functor $[0] \to C$ may equally well be described as a functor $* \to C$.

For $0 \leq n \leq 2$, we denote the associated three ∞-categories again by $[0]$, $[1]$ and $[2]$. They may be displayed as follows:

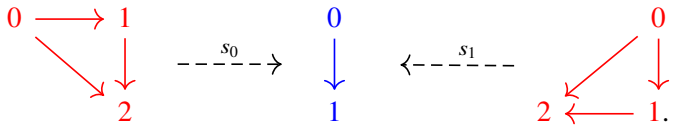
$$0, \qquad 0 \longrightarrow 1 \qquad \text{and} \qquad \begin{array}{ccc} & 1 & \\ \nearrow & & \searrow \\ 0 & \longrightarrow & 2. \end{array}$$

The five face maps $d_0, d_1\colon [0] \to [1]$ and $d_0, d_1, d_2\colon [1] \to [2]$ and the relations between them

may be displayed as follows:



The two degeneracy maps $s_0, s_1 \colon [2] \to [1]$ may be interpreted as the following 'projections':



Definition 1.2.12. Let C be an ∞-category.

(1) An *object of* C is a functor $x \colon [0] \to C$;

(2) A *morphism of* C is a functor $f \colon [1] \to C$. We write $f(0)$ for its *source/domain* and $f(1)$ for its *target/codomain*, which are defined as the following composites:

$$f(0) \colon [0] \xrightarrow{d_1} [1] \xrightarrow{f} C \qquad \text{and} \qquad f(1) \colon [0] \xrightarrow{d_0} [1] \xrightarrow{f} C.$$

We will also write $f \colon x \to y$ to indicate that f comes with identifications $x = f(0)$ and $y = f(1)$.

(3) A *commutative triangle in* C is a functor $\sigma \colon [2] \to C$. We will frequently denote commutative triangles by

$$\sigma \quad = \quad \begin{array}{ccccc} & & y & & \\ & f \nearrow & & \searrow g & \\ x & & \xrightarrow[h]{} & & z, \end{array}$$

where $f = \sigma \circ d_2$, $g = \sigma \circ d_0$ and $h = \sigma \circ d_1$. We will often leave commutative triangles unnamed, and only label their three edges.

Warning 1.2.13. While we may speak of *objects of* C, we may not speak of a *set of objects of* C! (We will introduce the *anima of objects* in Axiom F below.)

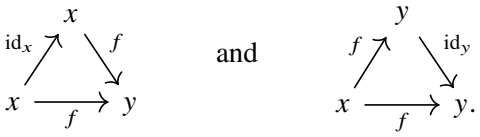
Remark 1.2.14. We want to think of a commutative triangle σ as saying that h is a candidate composite of f and g. Note that at this stage we do not yet say anything about the existence nor the uniqueness of composites; we get back to this in Section 1.4.2.

Example 1.2.15. Given an object x of C, we define the *identity morphism* $\mathrm{id}_x \colon x \to x$ as the composite $[1] \to [0] \xrightarrow{x} C$. Since the composites

$$[0] \xrightarrow{d_1} [1] \xrightarrow{s_0} [0] \qquad \text{and} \qquad [0] \xrightarrow{d_0} [1] \xrightarrow{s_0} [0]$$

are identified with the identity functor $\mathrm{id}_{[0]}$, we see that the source and target of id_x are indeed identified with x.

Exercise 1.2.16. Construct for every morphism $f \colon x \to y$ commutative triangles in C of the form



Once composition of morphisms is available, this exercise will give the expected relations $f = f \circ \mathrm{id}_x$ and $f = \mathrm{id}_y \circ f$, see Lemma 1.4.8.

1.3 Basic constructions of ∞-categories

The theory of ∞-categories admits most of the basic constructions familiar from classical category theory, like terminal and initial categories, products and coproducts of categories, pullbacks, functor categories, and pushouts. In classical category theory, one may give explicit constructions of these categories, and prove their universal properties 'by hand' by writing down what the relevant functors should do on objects and morphisms. When working with ∞-categories, we need to reverse this logic: since functors can no longer be prescribed on objects and morphisms, we must always construct them by invoking universal properties. We will now introduce these universal properties for the basic constructions of ∞-categories. They always come in two parts: first we prescribe how to construct functors from suitable data, and then we pin down the anima of identifications between two such functors.

1.3.1 Products and coproducts

We start with products and coproducts of ∞-categories. For animae, the products, projections, and pairings below are those already introduced in Axiom A.1. Likewise, the empty product is the one-point anima $*$, with its functors $p_T \colon T \to *$. The next axiom specifies their universal properties and extends products to arbitrary families of ∞-categories. It also introduces coproducts.

Axiom C.1 (Products and coproducts of ∞-categories)**.** Let I be a set and let C_i be an ∞-category for every $i \in I$.

(1) There is a *product* $\prod_{i \in I} C_i$, equipped with *projection functors* $\mathrm{pr}_i \colon \prod_{i \in I} C_i \to C_i$ for every i. Given another ∞-category T and functors $f_i \colon T \to C_i$ for every i, we obtain a functor $(f_i) \colon T \to$

$\prod_{i\in I} C_i$ together with identifications $\mathrm{pr}_i \circ (f_i) = f_i$. For two functors $g, h\colon T \to \prod_{i\in I} C_i$, the projection-induced functor

$$(g = h) \longrightarrow \prod_{i\in I} (\mathrm{pr}_i\, g = \mathrm{pr}_i\, h)$$

is an equivalence. If every C_i is an anima, their product is an anima.

(2) There is a *coproduct* or *disjoint union* $\bigsqcup_{i\in I} C_i$, equipped with *inclusion functors* $\mathrm{incl}_i\colon C_i \hookrightarrow \bigsqcup_{i\in I} C_i$ for every i. Given another ∞-category T and functors $f_i\colon C_i \to T$ for every i, we obtain a functor $\langle f_i\rangle\colon \bigsqcup_{i\in I} C_i \to T$ together with identifications $\langle f_i\rangle \circ \mathrm{incl}_i = f_i$. For two functors $g, h\colon \bigsqcup_{i\in I} C_i \to T$, restriction induces an equivalence

$$(g = h) \longrightarrow \prod_{i\in I} (g\,\mathrm{incl}_i = h\,\mathrm{incl}_i).$$

Notation 1.3.1. The empty product is the one-point anima $*$, which we also call the *terminal* ∞-*category*. We denote the empty coproduct by $\emptyset$ and call it the *empty* ∞-*category*. Thus every ∞-category C has canonical functors

$$C \to *, \qquad\qquad \emptyset \to C.$$

Remark 1.3.2. In the empty-index cases the displayed equivalences say that $(f = g)$ is contractible for any two functors $f, g\colon T \to *$, and likewise for functors $\emptyset \to T$. Thus any two such identifications are themselves identified.

Notation 1.3.3. When $I = \{1, 2\}$, we write $C_1 \times C_2$ and $C_1 \sqcup C_2$ for the resulting product and coproduct. Given functors $f_1\colon T \to C_1$ and $f_2\colon T \to C_2$, we write $(f_1, f_2)\colon T \to C_1 \times C_2$ for the induced functor. Given functors $g_1\colon C_1 \to T$ and $g_2\colon C_2 \to T$, we write $\langle g_1, g_2\rangle\colon C_1 \sqcup C_2 \to T$ for the induced functor.

Taking $T = *$, we see that objects of $C_1 \times C_2$ have the form (x, y), where x is an object of C_1 and y is an object of C_2.

Definition 1.3.4. Let C and D be ∞-categories and let x be an object of D, regarded as a functor $x\colon * \to D$. We define the *constant functor* $\mathrm{const}_x\colon C \to D$ as the following composite:

$$\mathrm{const}_x\colon C \xrightarrow{p_C} * \xrightarrow{x} D.$$

Definition 1.3.5 (Contractible ∞-category)**.** An ∞-category C is called *contractible* if the functor $p_C\colon C \to *$ is an equivalence.

Observe that an ∞-category C is contractible if and only if there exists an object $x\colon * \to C$ such that the composite $\mathrm{const}_x\colon C \to C$ is identified with the identity $\mathrm{id}_C\colon C \to C$. An identification $H\colon \mathrm{const}_x = \mathrm{id}_C$ is called a *contraction of* C *onto* x.

The familiar symmetries of products and coproducts follow from these universal properties just as in ordinary category theory, and we will use them freely.

Exercise 1.3.6. Show that (co)products of ∞-categories are associative, commutative and unital:

$$C \times D \xrightarrow{\sim} D \times C, \qquad (C \times D) \times E \xrightarrow{\sim} C \times (D \times E), \qquad C \times * \xrightarrow{\sim} C \xrightarrow{\sim} * \times C,$$

and

$$C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad (C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Construction 1.3.7. The formation of products and coproducts of ∞-categories is functorial: given two functors $f\colon C \to C'$ and $g\colon D \to D'$, we obtain new functors

$$\begin{aligned} f \times g &:= (f \circ \mathrm{pr}_C, g \circ \mathrm{pr}_D)\colon\ C \times D \to C' \times D' \\ f \sqcup g &:= \langle \mathrm{incl}_{C'} \circ f, \mathrm{incl}_{D'} \circ g \rangle\ : C \sqcup D \to C' \sqcup D'. \end{aligned}$$

We leave it to the reader to verify that there are identifications

$$\begin{aligned} \mathrm{id}_C \times \mathrm{id}_D &= \mathrm{id}_{C\times D} \quad \text{and} \quad (f' \times g') \circ (f \times g) = (f' \circ f) \times (g' \circ g), \\ \mathrm{id}_C \sqcup \mathrm{id}_D &= \mathrm{id}_{C\sqcup D} \quad \text{and} \quad (f' \sqcup g') \circ (f \sqcup g) = (f' \circ f) \sqcup (g' \circ g) \end{aligned}$$

for functors $f\colon C \to C'$, $f'\colon C' \to C''$, $g\colon D \to D'$ and $g'\colon D' \to D''$.

1.3.2 Pullbacks of ∞-categories

We will now axiomatize the *pullback* of two functors $f\colon C \to E$ and $g\colon D \to E$ of ∞-categories. This will be similar to the axiom for the product $C \times D$, but it is more involved due to the fact that we need to record compatibilities with the structure maps to E.

Axiom C.2 (Pullbacks of ∞-categories)**.** For functors $f\colon C \to E$ and $g\colon D \to E$, their *pullback*, or *fiber product over* E, is an ∞-category $C \times_E D$ with projections $\mathrm{pr}_1\colon C \times_E D \to C$ and $\mathrm{pr}_2\colon C \times_E D \to D$, and an identification $\tau\colon f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2$; in other words, a commutative square of the form

$$\begin{array}{ccc} C \times_E D & \xrightarrow{\mathrm{pr}_1} & C \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{g} & E. \end{array}$$

If C, D, E are animae, then $C \times_E D$ is an anima. Given $t_C\colon T \to C$, $t_D\colon T \to D$, and $\sigma\colon f \circ t_C = g \circ t_D$, there is a functor $t = (t_C, t_D)_\sigma\colon T \to C \times_E D$ with identifications $\alpha\colon \mathrm{pr}_1 \circ t = t_C$, $\beta\colon \mathrm{pr}_2 \circ t = t_D$, and

$$\sigma \circ (f \star \alpha) = (g \star \beta) \circ (\tau \star t).$$

For any $h, k\colon T \to C \times_E D$, the functor

$$(h = k) \longrightarrow (\mathrm{pr}_1 \circ h = \mathrm{pr}_1 \circ k) \times_{(f\circ \mathrm{pr}_1 \circ h = g \circ \mathrm{pr}_2 \circ k)} (\mathrm{pr}_2 \circ h = \mathrm{pr}_2 \circ k)$$

induced by whiskering with the two projections is an equivalence. The two functors defining the pullback on the right are

$$\alpha \longmapsto (\tau \star k) \circ (f \star \alpha), \qquad \beta \longmapsto (g \star \beta) \circ (\tau \star h),$$

and the identification needed to define the displayed functor is supplied by the interchange law.

Remark 1.3.8. More informally, the axiom says that functors into a pullback $C \times_E D$ may be constructed by giving functors into C and D which agree after mapping to E, and that an identification of two such functors $h, k\colon T \to C \times_E D$ consists of identifications $\alpha\colon \mathrm{pr}_1 \circ h = \mathrm{pr}_1 \circ k$ and $\beta\colon \mathrm{pr}_2 \circ h = \mathrm{pr}_2 \circ k$, together with an identification

$$H\colon (\tau \star k) \circ (f \star \alpha) = (g \star \beta) \circ (\tau \star h).$$

Exercise 1.3.9. Show that the fiber product is commutative, associative and unital: for functors $C \to E$, $D \to E$ and $B \to E$, there are equivalences

$$C \times_E D \xrightarrow{\sim} D \times_E C, \qquad B \times_E (C \times_E D) \xrightarrow{\sim} (B \times_E C) \times_E D, \qquad C \times_E E \xrightarrow{\sim} C.$$

Exercise 1.3.10. Given ∞-categories C and D, produce an equivalence $C \times D \xrightarrow{\sim} C \times_* D$.

Exercise 1.3.11. The construction of pullbacks of ∞-categories is functorial: given a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{f} & E & \xleftarrow{g} & D \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\psi} \\ C' & \xrightarrow[f']{} & E' & \xleftarrow[g']{} & D' \end{array}$$

there is an induced functor

$$\varphi \times_\chi \psi := (\varphi \circ \mathrm{pr}_C, \psi \circ \mathrm{pr}_D) \colon C \times_E D \to C' \times_{E'} D'.$$

Show that if each of the functors φ, ψ and χ is an equivalence, then so is $\varphi \times_\chi \psi$. Deduce that equivalences are stable under pullback: if $g \colon D \to E$ is an equivalence and $f \colon C \to E$ is an arbitrary functor, then the projection $C \times_E D \to C$ is an equivalence.

Pullbacks of ∞-categories lead to the notion of a *fiber* of a functor:

Definition 1.3.12. Let $f \colon C \to D$ be a functor. For every object x of D, we define the *fiber* of f over x as the pullback

$$\begin{array}{ccc} C_x & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ * & \xrightarrow{x} & D. \end{array}$$

The definition of pullbacks of ∞-categories naturally leads to the notion of a pullback square.

Definition 1.3.13 (Pullback square)**.** A commutative square

$$\begin{array}{ccc} T & \xrightarrow{s} & C \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow[g]{} & E \end{array}$$

is called a *pullback square* if the induced functor $(s, t) \colon T \to C \times_E D$ is an equivalence.

Given objects $x, y \colon * \to C$ and an ∞-category T, we expect identifications $\mathrm{const}_x = \mathrm{const}_y$ between constant functors $T \to C$ to correspond to functors from T to the identification anima $(x = y)$. The following axiom makes this precise.

Axiom C.3. For objects $x, y \colon * \to C$, there is an identification $\varepsilon \colon \mathrm{const}_x = \mathrm{const}_y$ of functors $(x = y) \to C$ exhibiting a pullback square

$$\begin{array}{ccc} (x = y) & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow{\scriptstyle y} \\ * & \xrightarrow{x} & C. \end{array}$$

For each identification $\alpha\colon x = y$, its restriction along $\alpha\colon * \to (x = y)$ is identified with α, using the identifications supplied by the terminal universal property. We write this as $\varepsilon(\alpha) = \alpha$.

Exercise 1.3.14 (Iterated pullbacks)**.** Consider a commutative diagram

$$\begin{array}{ccccc} C_1 & \xrightarrow{g_1} & C_2 & \xrightarrow{g_2} & C_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ D_1 & \xrightarrow[h_1]{} & D_2 & \xrightarrow[h_2]{} & D_3. \end{array}$$

Construct an equivalence

$$D_1 \times_{D_3} C_3 \xrightarrow{\sim} D_1 \times_{D_2} (D_2 \times_{D_3} C_3).$$

Hint: define one direction by $(\mathrm{pr}_{D_1}, (h_1 \circ \mathrm{pr}_{D_1}, \mathrm{pr}_{C_3}))$ and construct an inverse by projecting to D_1 and C_3.

Lemma 1.3.15 (Pasting lemma for pullback squares)**.** *Consider a commutative diagram*

$$\begin{array}{ccccc} C_1 & \xrightarrow{g_1} & C_2 & \xrightarrow{g_2} & C_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ D_1 & \xrightarrow[h_1]{} & D_2 & \xrightarrow[h_2]{} & D_3. \end{array}$$

If the right-hand square is a pullback square, then the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

Proof. By the previous exercise there is a preferred equivalence

$$D_1 \times_{D_3} C_3 \xrightarrow{\sim} D_1 \times_{D_2} (D_2 \times_{D_3} C_3).$$

If the right-hand square is a pullback square, the functor $C_2 \to D_2 \times_{D_3} C_3$ is an equivalence. Hence, by Exercise 1.3.11, the induced functor $D_1 \times_{D_2} C_2 \to D_1 \times_{D_2} (D_2 \times_{D_3} C_3)$ is an equivalence. The claim now follows from 2-out-of-3 applied to the square

$$\begin{array}{ccc} C_1 & \xrightarrow{(f_1, g_1)} & D_1 \times_{D_2} C_2 \\ {\scriptstyle (f_1, g_2 \circ g_1)}\downarrow & & \downarrow{\scriptstyle \simeq} \\ D_1 \times_{D_3} C_3 & \xrightarrow{\simeq} & D_1 \times_{D_2} (D_2 \times_{D_3} C_3). \end{array}$$

□

Exercise 1.3.16. Consider a commutative square

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ E & \xrightarrow{k} & F \end{array}$$

and assume that h is an equivalence. Show that g is an equivalence if and only if the square is a pullback square.

We record one further property of coproducts.

Axiom C.4 (Universality of coproducts)**.** Let I be a set, and let D_i be an ∞-category for every $i \in I$.

(1) Given functors $f_i \colon C_i \to D_i$ for all $i \in I$, the commutative square

$$\begin{array}{ccc} C_i & \xrightarrow{\mathrm{incl}_i} & \bigsqcup_{i\in I} C_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle \bigsqcup_{i\in I} f_i} \\ D_i & \xrightarrow{\mathrm{incl}_i} & \bigsqcup_{i\in I} D_i \end{array}$$

is a pullback square.

(2) For a functor $h \colon E \to \bigsqcup_{i\in I} D_i$, the functor

$$\langle \mathrm{pr}_E \rangle_i \colon \coprod_{i\in I} (E \times_{\bigsqcup_{i\in I} D_i} D_i) \to E$$

is an equivalence.

(3) The inclusions $\{i\} \hookrightarrow I$ for $i \in I$ induce an equivalence $\bigsqcup_{i\in I} \{i\} \xrightarrow{\sim} I$.

Remark 1.3.17. This is the categorical analogue of three basic facts about disjoint unions of sets: base change along an inclusion picks out the corresponding summand, every map into a disjoint union decomposes the source into the corresponding fibers, and every set is the disjoint union of its elements.

1.3.3 Functor categories

Functor categories let us package families of functors internally, and they will also allow us to define pushouts by duality in the next subsection.

Axiom C.5 (Functor category)**.** The *functor category* between two ∞-categories C and D is an ∞-category denoted by $\mathrm{Fun}(C, D)$. It comes equipped with a functor $\mathrm{ev} \colon \mathrm{Fun}(C, D) \times C \to D$, called the *evaluation functor*. For another ∞-category E and a functor $g \colon E \to \mathrm{Fun}(C, D)$, we may then define its *uncurrying* $g^u \colon E \times C \to D$ as the composite

$$E \times C \xrightarrow{g \times \mathrm{id}_C} \mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D.$$

We demand that every functor $f \colon E \times C \to D$ gives rise to a functor $f_c \colon E \to \mathrm{Fun}(C, D)$, called the *currying* of f, equipped with an identification $f = (f_c)^u$. Moreover, for two functors $g, h \colon E \to \mathrm{Fun}(C, D)$, we demand that uncurrying induces an equivalence

$$(g = h) \longrightarrow (g^u = h^u).$$

Thus a specified identification between the uncurried functors lifts to one between g and h. By Lemma 1.2.9, identifications between prescribed identifications lift as well. This allows us to translate back and forth between curried and uncurried functors.

In the case $E = *$, objects $* \to \mathrm{Fun}(C, D)$ of the functor category correspond to functors $C \simeq * \times C \to D$ from C to D, justifying the terminology for $\mathrm{Fun}(C, D)$.

Exercise 1.3.18. Show that, for $k\colon E' \to E$ and $g\colon E \to \mathrm{Fun}(C,D)$, there is a preferred identification $(gk)^u = g^u(k \times \mathrm{id}_C)$, compatible with identifications of g. Deduce the corresponding identification for currying. *Hint:* Expand uncurrying using evaluation and the identities expressing functoriality of products.

The functoriality of the product naturally provides functoriality for the functor categories:

Construction 1.3.19 (Functoriality of functor categories in C and D)**.** Given a functor $g\colon D \to E$, we define the functor $g \circ -\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,E)$ as the currying of the composite

$$\mathrm{Fun}(C,D) \times C \xrightarrow{\mathrm{ev}} D \xrightarrow{g} E.$$

Similarly, given a functor $f\colon C \to D$, we define the functor $- \circ f\colon \mathrm{Fun}(D,E) \to \mathrm{Fun}(C,E)$ as the currying of the composite

$$\mathrm{Fun}(D,E) \times C \xrightarrow{\mathrm{id}\times f} \mathrm{Fun}(D,E) \times D \xrightarrow{\mathrm{ev}} E.$$

In a completely analogous way, every identification $\beta\colon g = g'$ of functors $D \to E$ induces an identification $(\beta \circ -)\colon (g \circ -) = (g' \circ -)$ of functors $\mathrm{Fun}(C,D) \to \mathrm{Fun}(C,E)$, and similarly for the construction $- \circ f$.

Alternative notations for these functors that we will frequently use are g_* and f^*:

$$g_* := g \circ -\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,E), \qquad f^* := - \circ f\colon \mathrm{Fun}(D,E) \to \mathrm{Fun}(C,D).$$

Exercise 1.3.20. Formulate and prove that the assignments $g \mapsto (g \circ -)$ and $f \mapsto (- \circ f)$ are functorial, in the sense that they respect identity functors and composition of functors.

Exercise 1.3.21. Given ∞-categories C, D and E, construct a composition functor

$$- \circ -\colon \mathrm{Fun}(D,E) \times \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,E).$$

Show that it reduces to the functors $g \circ -$ and $- \circ f$ when fixing one of the two variables.

Exercise 1.3.22. Let $f, g\colon C \to D$ be functors, also regarded as objects of $\mathrm{Fun}(C,D)$ by currying. Show that their primitive identification anima $(f = g)$ fits into a pullback square

$$\begin{array}{ccc} (f=g) & \longrightarrow & \mathrm{Fun}(C,D) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ * & \xrightarrow{(f,g)} & \mathrm{Fun}(C,D) \times \mathrm{Fun}(C,D). \end{array}$$

Hint: First use currying at $*$ and Lemma 1.2.9 to compare the identification animae of the functors and their corresponding objects. Then apply Axiom C.3, and express the pullback of two objects as a fiber of the diagonal.

Proposition 1.3.23 (Internal universal properties)**.** *Let T be an ∞-category and let $\{C_i\}_{i\in I}$ be a set-indexed collection of ∞-categories. The canonical functors below are equivalences:*

$$\mathrm{Fun}(T,*) \xrightarrow{\sim} *, \qquad \mathrm{Fun}(\emptyset,T) \xrightarrow{\sim} *, \qquad \mathrm{Fun}(*,T) \xrightarrow{\sim} T,$$

$$\mathrm{Fun}\left(T, \prod_{i\in I} C_i\right) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Fun}(T,C_i), \qquad \mathrm{Fun}\left(\bigsqcup_{i\in I} C_i, T\right) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Fun}(C_i,T).$$

Moreover, for functors $f\colon C \to E$ and $g\colon D \to E$ there is an equivalence

$$\mathrm{Fun}(T, C \times_E D) \xrightarrow{\sim} \mathrm{Fun}(T, C) \times_{\mathrm{Fun}(T,E)} \mathrm{Fun}(T, D),$$

and for ∞-categories C and D there is an equivalence

$$\mathrm{Fun}(T, \mathrm{Fun}(C, D)) \xrightarrow{\sim} \mathrm{Fun}(T \times C, D).$$

Proof sketch. These properties are developed in Section 1.7.1 of [CCNW26]; the proof for binary (co)products treated there adapts to one for arbitrary (co)products.

The functor $\mathrm{Fun}(*, T) \to T$ is the evaluation map. In each other case, the displayed functor is constructed using the universal property of its target. For the inverse, we may map into the relevant functor category by first uncurrying and then using the universal property of the target. In the coproduct case, we use universality of coproducts to distribute the product with the parameter ∞-category over $\bigsqcup_i C_i$. In the pullback case, we must show that the projection identifications agree over the base; this may be done after uncurrying, where the equivalence of identification animae in Axiom C.5 allows us to use the pullback constructor. □

1.3.4 Pushouts of ∞-categories

Using functor categories, we may now dualize the notion of pullback square and define pushouts:

Definition 1.3.24 (Pushout square)**.** A commutative square of ∞-categories

$$\begin{array}{ccc} C' & \xrightarrow{u} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{v} & D \end{array}$$

is called a *pushout square* (or *cocartesian*) if, for every ∞-category E, the induced square

$$\begin{array}{ccc} \mathrm{Fun}(D, E) & \xrightarrow{v^*} & \mathrm{Fun}(D', E) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \mathrm{Fun}(C, E) & \xrightarrow{u^*} & \mathrm{Fun}(C', E) \end{array}$$

is a pullback square.

Axiom C.6 (Pushout of ∞-categories)**.** For functors $f\colon E \to C$ and $g\colon E \to D$, there exists a pushout square

$$\begin{array}{ccc} E & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{incl}_C} \\ D & \xrightarrow[\mathrm{incl}_D]{} & C \sqcup_E D. \end{array}$$

We refer to the ∞-category $C \sqcup_E D$ as the *pushout* of C and D along E.

Lemma 1.3.25 (Pasting lemma for pushout squares)**.** *Consider a commutative diagram*

$$\begin{array}{ccccc} C_1 & \xrightarrow{g_1} & C_2 & \xrightarrow{g_2} & C_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ D_1 & \xrightarrow[h_1]{} & D_2 & \xrightarrow[h_2]{} & D_3. \end{array}$$

If the left-hand square is a pushout square, then the right-hand square is a pushout square if and only if the outer rectangle is a pushout square.

Proof. This follows immediately from Lemma 1.3.15. □

1.4 Commutative diagrams, composition, and animae

After studying constructions that build new ∞-categories out of old ones, we now return to the internal structure of a single ∞-category. The first step is to enlarge the class of diagrams we can speak about and to formalize how triangles assemble into commutative squares. This leads to the Segal axiom, which makes composition of morphisms precise. Once composition is in place, we can define isomorphisms and characterize animae as the ∞-groupoids.

1.4.1 Commutative diagrams

Besides commutative triangles, several other simple diagram shapes occur constantly in practice: commutative squares, spans, cospans, and infinite strings of composable morphisms. The next axiom formalizes the idea that for these diagram shapes, the higher data is determined by the lower-dimensional data we choose to display.

Definition 1.4.1 (Diagram shapes)**.** A *commutative square* in an ∞-category C is a functor $[1] \times [1] \to C$. We will often display such a square as

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ z & \xrightarrow[k]{} & w. \end{array}$$

Throughout, we read the first coordinate of $[1] \times [1]$ as the row and the second as the column, so that the four objects $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ are displayed as x, y, z and w, respectively. In particular $\{0\} \times [1]$ and $\{1\} \times [1]$ are the top and bottom rows, while $[1] \times \{0\}$ and $[1] \times \{1\}$ are the left and right columns.

Let $\ulcorner \subseteq [1] \times [1]$ be the subposet spanned by the three objects $(0,0)$, $(0,1)$ and $(1,0)$, and let $\lrcorner \subseteq [1] \times [1]$ be the subposet spanned by the three objects $(1,0)$, $(0,1)$ and $(1,1)$. A *span* in C is a functor $\ulcorner \to C$, and a *cospan* in C is a functor $\lrcorner \to C$. We display them as

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \\ z. & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & y \\ & & \downarrow{\scriptstyle h} \\ z & \xrightarrow[k]{} & w. \end{array}$$

Every commutative square gives rise to both a span and a cospan via restriction along the two inclusions $\ulcorner \hookrightarrow [1] \times [1]$ and $\lrcorner \hookrightarrow [1] \times [1]$.

Finally, let $[\omega] := \{0 \leq 1 \leq 2 \leq 3 \leq \cdots\}$ be the poset of natural numbers. An *infinite sequence of morphisms in C* is a functor $x_\bullet \colon [\omega] \to C$, which we display as

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \dots,$$

where x_n is the evaluation of $x_\bullet$ at $n \in [\omega]$ and f_n is obtained from the inclusion $[1] \cong \{n \leq n+1\} \hookrightarrow [\omega]$.

These pictures suppress part of the underlying functorial data. The next axiom describes how composable strings determine their composites and how two commutative triangles with a common diagonal assemble into a commutative square. The four boundary edges of a square alone do not specify its commutativity data.

Axiom D (Commutative diagram axiom)**.** The following commutative squares are pushouts of ∞-categories:

$$\begin{array}{ccc} [0] & \xrightarrow{0} & [1] \\ {\scriptstyle n-1}\downarrow & & \downarrow{\scriptstyle n-1\to n} \\ [n-1] & \hookrightarrow & [n], \end{array} \qquad \begin{array}{ccc} [1] & \xrightarrow{d_1} & [2] \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle (s_1,s_0)} \\ [2] & \xrightarrow[(s_0,s_1)]{} & [1]\times[1], \end{array}$$

$$\begin{array}{ccc} [0] & \xrightarrow{d_1} & [1] \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle (0,0)\to(1,0)} \\ [1] & \xrightarrow[(0,0)\to(0,1)]{} & \ulcorner, \end{array} \qquad \begin{array}{ccc} [0] & \xrightarrow{d_0} & [1] \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle (1,0)\to(1,1)} \\ [1] & \xrightarrow[(0,1)\to(1,1)]{} & \lrcorner, \end{array}$$

$$\begin{array}{ccc} \bigsqcup_{l\in\mathbb{N}}\{l+1\} & \longrightarrow & \bigsqcup_{k\in\mathbb{N}}\{2k \leq 2k+1\} \\ \downarrow & & \downarrow \\ \bigsqcup_{k\in\mathbb{N}}\{2k+1 \leq 2k+2\} & \longrightarrow & [\omega]. \end{array}$$

Here the first square is required to be a pushout for every $n \geq 1$. In the last square, the right and bottom maps are the inclusions, while the top and left maps send the vertex $\{l+1\}$ to the corresponding vertex with the same label.

Exercise 1.4.2. Unwind the definitions, and show that these five pushout conditions express that finite and infinite strings of composable morphisms, commutative squares, spans, and cospans are determined by the lower-dimensional data displayed above.

We will use two maps that turn commutative triangles into commutative squares with an identity edge.

Construction 1.4.3. There are monotone maps $p_0, p_2 \colon [1] \times [1] \to [2]$, pictorially represented by

$$p_0 \quad = \quad \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \| & \searrow & \downarrow \\ 0 & \longrightarrow & 2, \end{array} \qquad\qquad p_2 \quad = \quad \begin{array}{ccc} 0 & \longrightarrow & 2 \\ \downarrow & \searrow & \| \\ 1 & \longrightarrow & 2. \end{array}$$

Writing $j_0 := (s_0, s_1)$ and $j_2 := (s_1, s_0)$ for the two functors $[2] \to [1] \times [1]$ appearing in Axiom D, there are identifications $p_0 \circ j_0 = \mathrm{id}_{[2]}$ and $p_2 \circ j_2 = \mathrm{id}_{[2]}$.

Precomposition with p_0 or p_2 displays a commutative triangle as a commutative square whose left or right column, respectively, is an identity. The following proposition makes this precise.

Proposition 1.4.4 (Triangles as squares with an identity edge)**.** *For every ∞-category C, the following two commutative squares are pullback squares:*

$$\begin{array}{ccc} \mathrm{Fun}([2], C) & \xrightarrow{p_0^*} & \mathrm{Fun}([1] \times [1], C) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ C & \xrightarrow{(p_{[1]})^*} & \mathrm{Fun}([1] \times \{0\}, C) \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \mathrm{Fun}([2], C) & \xrightarrow{p_2^*} & \mathrm{Fun}([1] \times [1], C) \\ {\scriptstyle \mathrm{ev}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ C & \xrightarrow{(p_{[1]})^*} & \mathrm{Fun}([1] \times \{1\}, C). \end{array}$$

Here the functors labeled 'res' are given by precomposition with the inclusions of the left and right columns into $[1] \times [1]$.

Proof sketch. Factor each displayed square into two squares. The commutative-square case and the case $n = 2$ of Axiom D identify the resulting squares as pullbacks, and Lemma 1.3.15 finishes the argument. For details, see Corollary 2.2.8 of [CCNW26]. □

1.4.2 Composition of morphisms

Given a commutative triangle $\sigma\colon [2] \to C$ in an ∞-category C, we may extract two morphisms $f := \sigma \circ d_2$ and $g := \sigma \circ d_0$:

$$\sigma \qquad = \qquad \begin{array}{ccccc} & & y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ x & & \longrightarrow & & z. \end{array}$$

By the case $n = 2$ of Axiom D, the pair (f, g) completely determines the commutative triangle. We refer to this special case as the *Segal Axiom*:

Proposition 1.4.5 (Segal Axiom)**.** *For an ∞-category C, the restriction functor*

$$(d_0^*, d_2^*)\colon \mathrm{Fun}([2], C) \to \mathrm{Fun}([1], C) \times_{\mathrm{ev}_0, C, \mathrm{ev}_1} \mathrm{Fun}([1], C), \qquad \sigma \mapsto (g, f)$$

is an equivalence. □

In particular, for morphisms $f\colon x \to y$ and $g\colon y \to z$ we obtain a unique commutative triangle $\sigma\colon [2] \to C$ satisfying $d_2^*(\sigma) = f$ and $d_0^*(\sigma) = g$. The resulting morphism $g \circ f := d_1^*(\sigma)\colon [1] \to C$ is called the *composite of f and g*.

Definition 1.4.6 (Arrow category)**.** Let C be an ∞-category. We write

$$\mathrm{Ar}(C) \quad := \quad \mathrm{Fun}([1], C)$$

and refer to it as the *arrow category* of C. Its objects are precisely the morphisms of C. Restriction along the two maps $[0] \to [1]$ defines the *source* and *target* functors

$$s := d_1^* = \mathrm{ev}_0\colon \mathrm{Ar}(C) \to C, \qquad t := d_0^* = \mathrm{ev}_1\colon \mathrm{Ar}(C) \to C.$$

The assignment $(g, f) \mapsto g \circ f$ is functorial:

Construction 1.4.7 (Composition functor)**.** Consider the zig-zag

$$\mathrm{Ar}(C) \times_{s,C,t} \mathrm{Ar}(C) \xleftarrow[\sim]{(d_0^*, d_2^*)} \mathrm{Fun}([2], C) \xrightarrow{d_1^*} \mathrm{Ar}(C).$$

The first map is an equivalence, and hence it admits an inverse, providing a composite functor

$$- \circ - \colon \mathrm{Ar}(C) \times_{s,C,t} \mathrm{Ar}(C) \xrightarrow{\sim} \mathrm{Fun}([2], C) \xrightarrow{d_1^*} \mathrm{Ar}(C).$$

We refer to this as the *composition functor*.

We will now show that composition in an ∞-category is unital and associative.

Lemma 1.4.8 (Unitality)**.** *For every morphism $f \colon x \to y$ in an ∞-category C, there are identifications*

$$\mathrm{id}_y \circ f = f = f \circ \mathrm{id}_x$$

in $\mathrm{Ar}(C)$*. Moreover, these identifications are natural in f, in the sense that they form identifications of functors* $\mathrm{Ar}(C) \to \mathrm{Ar}(C)$*.*

Proof. For a fixed morphism f, the identifications are an immediate consequence of Exercise 1.2.16, using the degenerate commutative triangles $s_1^*(f) := f \circ s_1$ and $s_0^*(f) := f \circ s_0$. For the naturality of the relation $f \circ \mathrm{id}_x = f$ we have to show that the composite

$$\mathrm{Ar}(C) \xrightarrow{\sim} \mathrm{Ar}(C) \times_{s,C,\mathrm{id}} C \xrightarrow{1 \times p_{[1]}^*} \mathrm{Ar}(C) \times_{s,C,t} \mathrm{Ar}(C) \xrightarrow{- \circ -} \mathrm{Ar}(C)$$

is identified with the identity functor. This follows from the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ar}(C) & & \xrightarrow{\quad s_0^* \quad} & & \mathrm{Fun}([2], C) \xrightarrow{\quad d_1^* \quad} \mathrm{Ar}(C) \\
\simeq \downarrow & & & & \downarrow \simeq \quad \| \\
\mathrm{Ar}(C) \times_{s,C,\mathrm{id}} C & \xrightarrow[1 \times p_{[1]}^*]{} & \mathrm{Ar}(C) \times_{s,C,t} \mathrm{Ar}(C) & \xrightarrow[- \circ -]{} & \mathrm{Ar}(C).
\end{array}$$

A similar discussion applies to the relation $f = \mathrm{id}_y \circ f$. □

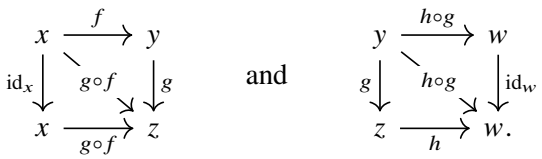
Proposition 1.4.9 (Associativity)**.** *For composable morphisms $f \colon x \to y$, $g \colon y \to z$ and $h \colon z \to w$ in an ∞-category C, there is an identification*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

in $\mathrm{Ar}(C)$*.*

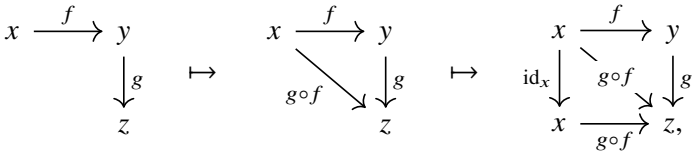
Proof. Throughout this proof we display a morphism in $\mathrm{Ar}(C)$ as a commutative square whose two *vertical* arrows are its source and its target, so that the horizontal direction records the morphism in $\mathrm{Ar}(C)$; the identification $\mathrm{Fun}([1], \mathrm{Ar}(C)) \simeq \mathrm{Fun}([1] \times [1], C)$ we use is the one that swaps the two

factors of $[1] \times [1]$ relative to the convention of Definition 1.4.1. With this reading, consider the following two morphisms in $\mathrm{Ar}(C)$:

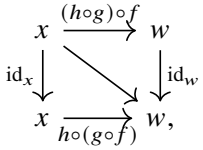


These morphisms are natural in f, g and h: we have functors

$$\mathrm{Ar}(C) \times_{s,C,t} \mathrm{Ar}(C) \quad \xrightarrow{\sim} \quad \mathrm{Fun}([2], C) \quad \xrightarrow{p_0^*} \quad \mathrm{Fun}([1] \times [1], C),$$



and similarly for the other diagram. Forming the composite of these two morphisms in $\mathrm{Ar}(C)$, we thus obtain a commutative square in C of the form

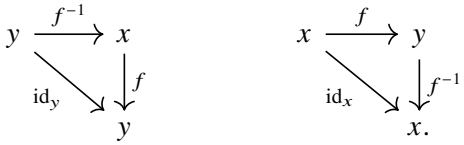


natural in f, g and h. Using Lemma 1.4.8, the diagonal of this square is then identified with both $(h \circ g) \circ f$ and $h \circ (g \circ f)$, finishing the proof. □

1.4.3 Isomorphisms, animae and groupoid cores

With composition available, we can now speak about invertibility of morphisms.

Definition 1.4.10 (Isomorphisms)**.** Consider a morphism $f\colon x \to y$ in an ∞-category C. We say that f is *invertible*, or that it is an *isomorphism*, if there exists a morphism $f^{-1}\colon y \to x$ together with commutative triangles in C of the form



Animae entered our language as a primitive class of ∞-categories. We can now characterize this class in terms of morphisms.

Axiom E (Recognition of animae)**.** An ∞-category is an anima if and only if every morphism in it is invertible.

An ∞-category satisfying this condition is also called an ∞-*groupoid*. This axiom connects our primitive language of collections with the internal structure of ∞-categories.

Recall from Example 1.1.5 that every 1-category C has an anima of objects $C^{\simeq}$, called its *groupoid core*: the subgroupoid of C which contains all objects but only the invertible morphisms. The groupoid $C^{\simeq}$ is uniquely characterized by the property that the inclusion $C^{\simeq} \hookrightarrow C$ is terminal among functors from a groupoid into C. We use this universal property to axiomatize the ∞-categorical version of the core:

Axiom F (Groupoid core axiom)**.** For every ∞-category C, there is an anima $C^{\simeq}$ called the *(groupoid) core of* C, with a functor $\gamma_C \colon C^{\simeq} \to C$. Every functor $F \colon X \to C$ from an anima X factors through γ_C with a specified identification of its composite with F. For $G, H \colon X \to C^{\simeq}$, postwhiskering induces an equivalence

$$(G = H) \longrightarrow (\gamma_C G = \gamma_C H).$$

Furthermore, the map $\langle 0, 1 \rangle \colon * \sqcup * \to [1]$ factors through an equivalence $* \sqcup * \xrightarrow{\sim} [1]^{\simeq}$.

Definition 1.4.11. For ∞-categories C and D, we write $\mathrm{Map}(C, D)$ for the anima $\mathrm{Fun}(C, D)^{\simeq}$, and refer to it as the *mapping anima* between C and D.

Exercise 1.4.12. Show that for an anima X and an ∞-category C the core functor $\gamma_C \colon C^{\simeq} \to C$ induces an equivalence

$$\mathrm{Map}(X, C^{\simeq}) \xrightarrow{\sim} \mathrm{Map}(X, C).$$

Hint: Apply the groupoid core axiom to the evaluation functor $\mathrm{Map}(X, C) \times X \to C$, and then curry and pass to the core once more.

Exercise 1.4.13. Show that the groupoid-core construction preserves pullbacks: for functors $C \to E \leftarrow D$, construct a preferred equivalence

$$(C \times_E D)^{\simeq} \simeq C^{\simeq} \times_{E^{\simeq}} D^{\simeq}.$$

Hint: Construct both directions from the groupoid-core and pullback universal properties, and compare the composites after projecting to the factors.

Exercise 1.4.14. Let $F \colon C \to D$ be a functor. Show that the following conditions are equivalent:

(1) The functor F is an equivalence;

(2) For every ∞-category E the functor $F \circ - \colon \mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$ is an equivalence;

(3) For every ∞-category E the functor $- \circ F \colon \mathrm{Map}(D, E) \to \mathrm{Map}(C, E)$ is an equivalence.

Exercise 1.4.15. Show that a morphism $f \colon X \to Y$ of animae is an equivalence if and only if the induced map $f^* \colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is an equivalence for every anima Z.

1.4.4 Hom animae

In Section 1.1, we heuristically described the morphisms between two objects of an ∞-category as forming an anima. We encode this object within our axiomatic setup by the following pullback.

Definition 1.4.16 (Hom anima)**.** Let C be any ∞-category. Given objects x and y of C, we define the *hom anima* $\mathrm{Hom}_C(x, y)$ via the following pullback square:

$$\begin{array}{ccc} \mathrm{Hom}_C(x,y) & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ * & \xrightarrow{(x,y)} & C \times C. \end{array}$$

Observe that the objects of $\mathrm{Hom}_C(x, y)$ are triples (f, α, β), where $f\colon x' \to y'$ is a morphism in C and $\alpha\colon x = x'$ and $\beta\colon y = y'$ are identifications of objects of C. We usually leave these identifications implicit.

Remark 1.4.17. At this point, the pullback only defines $\mathrm{Hom}_C(x, y)$ as an ∞-category. The terminology *hom anima* will be justified after the Rezk axiom: by Proposition 1.4.24, $\mathrm{Hom}_C(x, y)$ is indeed an anima.

Exercise 1.4.18. Let x, y and z be objects of an ∞-category C. Construct a composition functor

$$- \circ -\colon \mathrm{Hom}_C(y, z) \times \mathrm{Hom}_C(x, y) \to \mathrm{Hom}_C(x, z), \qquad (g, f) \mapsto g \circ f.$$

Hint: Restrict the composition functor $\mathrm{Ar}(C) \times_{s,C,t} \mathrm{Ar}(C) \to \mathrm{Ar}(C)$ along the pullbacks defining the three hom animae.

Definition 1.4.19 (Natural transformations)**.** If C and D are ∞-categories, we define a *natural transformation* of functors $C \to D$ to be a morphism in $\mathrm{Fun}(C, D)$. By (un)currying, this may equivalently be encoded as a functor $[1] \times C \to D$. We denote the hom anima in $\mathrm{Fun}(C, D)$ by $\mathrm{Nat}(f, g)$:

$$\mathrm{Nat}(f, g) := \mathrm{Hom}_{\mathrm{Fun}(C,D)}(f, g).$$

1.4.5 The Rezk axiom

In a 1-category, every invertible morphism is isomorphic in the arrow category to an identity morphism. Thus, if $\mathrm{Iso}(C) \subseteq \mathrm{Ar}(C)$ denotes the full subcategory spanned by the invertible morphisms, then the functor

$$C \to \mathrm{Iso}(C), \qquad x \mapsto \mathrm{id}_x,$$

is an equivalence. The Rezk axiom is the ∞-categorical analogue of this statement. Since we do not yet have subcategories at our disposal, we first construct $\mathrm{Iso}(C)$ as an explicit pullback:

Definition 1.4.20. Given an ∞-category C, we define the ∞-category $\mathrm{Iso}(C)$ via the following pullback diagram:

$$\begin{array}{ccc} \mathrm{Iso}(C) & \longrightarrow & \mathrm{Ar}(C) \times_{t,C,s} \mathrm{Ar}(C) \times_{t,C,s} \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (g,f,h)\mapsto(f\circ g, h\circ f)} \\ C \times C & \xrightarrow{(x,y)\mapsto(\mathrm{id}_y,\mathrm{id}_x)} & \mathrm{Ar}(C) \times \mathrm{Ar}(C). \end{array}$$

We define $\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(C) \to \mathrm{Ar}(C)$ to be the composite functor

$$\mathrm{Iso}(C) \to \mathrm{Ar}(C) \times_{t,C,s} \mathrm{Ar}(C) \times_{t,C,s} \mathrm{Ar}(C) \xrightarrow{\mathrm{pr}_2} \mathrm{Ar}(C).$$

An object of $\mathrm{Iso}(C)$ is a triple of morphisms $(g\colon y' \to x, f\colon x \to y, h\colon y \to x')$ together with identifications $f \circ g = \mathrm{id}_y$ and $h \circ f = \mathrm{id}_x$. Its image under π_{Iso} is the morphism f.

Exercise 1.4.21. Give a precise construction of the right vertical functor in the pullback defining $\mathrm{Iso}(C)$. *Hint:* Project to the composable pairs (f, g) and (h, f), apply the composition functor to each pair, and then take their product.

Remark 1.4.22. Note that $\mathrm{Iso}(C)$ encodes separate left and right inverses g and h of f, rather than a single two-sided inverse f^{-1}. The reason for this is to guarantee that the fibers of $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Ar}(C)$ are either empty or contractible, so that being an isomorphism is really a *property* of a morphism rather than additional structure. This is made precise by Proposition 1.5.6.

As one expects, the identity map $\mathrm{id}_x \colon x \to x$ is invertible in C for every object x of C:

Construction 1.4.23. The identity morphisms define a canonical functor

$$i \colon C \to \mathrm{Iso}(C).$$

Indeed, restriction along $p_{[1]} \colon [1] \to [0]$ gives a functor

$$p_{[1]}^* \colon C \to \mathrm{Ar}(C), \qquad x \mapsto \mathrm{id}_x,$$

whose composites with the source and target functors $s, t \colon \mathrm{Ar}(C) \to C$ are both identified with id_C. The triple $(p_{[1]}^*, p_{[1]}^*, p_{[1]}^*)$ therefore lifts to a functor

$$C \to \mathrm{Ar}(C) \times_{t,C,s} \mathrm{Ar}(C) \times_{t,C,s} \mathrm{Ar}(C), \qquad x \mapsto (\mathrm{id}_x, \mathrm{id}_x, \mathrm{id}_x).$$

By Lemma 1.4.8 there is an identification $\mathrm{id}_x \circ \mathrm{id}_x = \mathrm{id}_x$, so the composite of this functor with $(g, f, h) \mapsto (f \circ g, h \circ f)$ is identified with the composite of the diagonal $\Delta \colon C \to C \times C$ with $(x, y) \mapsto (\mathrm{id}_y, \mathrm{id}_x)$. The universal property of the pullback defining $\mathrm{Iso}(C)$ thus provides the desired functor i, and it satisfies $\pi_{\mathrm{Iso}} \circ i = p_{[1]}^*$.

Axiom G (Rezk axiom)**.** For every ∞-category C the functor $i \colon C \to \mathrm{Iso}(C)$ is an equivalence.

The Rezk axiom now implies that the hom animae constructed above really are animae.

Proposition 1.4.24 (Hom animae are animae)**.** *Let C be an ∞-category and let x and y be objects in C. Then the hom anima $\mathrm{Hom}_C(x, y)$ is indeed an anima.*

For a proof, see Corollary 4.3.5 of [CCNW26].

Remark 1.4.25. The condition that the functor $i \colon C \to \mathrm{Iso}(C)$ is an equivalence is known as the *completeness condition*. It was used by Rezk [Rez01] to provide a model of ∞-categories, called *complete Segal spaces*. A detailed discussion of the equivalence between ∞-categories and complete Segal animae will be given in Chapter 25.

For functors $f, g \colon C \to D$, write $(f \cong g)$ for the fiber over (f, g) of

$$(s, t) \colon \mathrm{Iso}(\mathrm{Fun}(C, D)) \to \mathrm{Fun}(C, D) \times \mathrm{Fun}(C, D).$$

Its objects are invertible natural transformations from f to g, together with inverse data. By Exercise 1.3.22, the primitive identification anima $(f = g)$ is the fiber over (f, g) of the diagonal of $\mathrm{Fun}(C, D)$. The identity-transformation functor $i \colon \mathrm{Fun}(C, D) \to \mathrm{Iso}(\mathrm{Fun}(C, D))$, whose composite with (s, t) is identified with this diagonal, therefore induces a functor

$$(f = g) \longrightarrow (f \cong g).$$

Lemma 1.4.26. *The functor $(f = g) \to (f \cong g)$ is an equivalence. In particular, $(f \cong g)$ is an anima.*

Proof. The Rezk axiom says that $i\colon \mathrm{Fun}(C, D) \to \mathrm{Iso}(\mathrm{Fun}(C, D))$ is an equivalence. Taking fibers over (f, g) gives the displayed functor, which is an equivalence because equivalences are stable under pullback. □

Thus an identification between functors is the same thing as a *natural isomorphism* between them, and we may henceforth write $\alpha\colon f \cong g$ for $\alpha\colon f = g$. Taking $C = *$, an identification between two objects is the same thing as an isomorphism between them.

1.5 Further constructions of ∞-categories

Opposite categories, subcategories, and localizations modify an ∞-category by reversing its morphisms, restricting its objects or morphisms, or formally inverting a chosen class of morphisms. This section develops these three constructions.

1.5.1 Opposite categories

Axiom H. For every ∞-category C there is an *opposite category* C^{op}. Similarly every functor $F\colon C \to D$ induces an opposite functor $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$, and this process respects composition of functors. It also induces functors $(F = G) \to (F^{\mathrm{op}} = G^{\mathrm{op}})$ on identification animae, compatibly with identities, composition, and whiskering of identifications. We have equivalences $(C^{\mathrm{op}})^{\mathrm{op}} \simeq C$, natural in functors and identifications; under these equivalences, $(F^{\mathrm{op}})^{\mathrm{op}}$ is identified with F.

If C and D come from classical 1-categories then C^{op} and F^{op} agree with the usual constructions of opposite categories.

If X is an anima then we have an equivalence $X^{\mathrm{op}} \simeq X$.

Note that functors $[1] \to C^{\mathrm{op}}$ correspond to functors $[1] \simeq [1]^{\mathrm{op}} \to (C^{\mathrm{op}})^{\mathrm{op}} \simeq C$, so that morphisms in C^{op} are just morphisms in C but with source and target flipped. Also since $[2] \simeq [2]^{\mathrm{op}}$, this correspondence reverses the order of composition: $(g \circ f)^{\mathrm{op}} = f^{\mathrm{op}} \circ g^{\mathrm{op}}$. In particular C is an anima if and only if C^{op} is an anima.

Proposition 1.5.1. *The construction $C \mapsto C^{\mathrm{op}}$ is compatible with all constructions of ∞-categories:*

(1) There are equivalences $*^{\mathrm{op}} \simeq *$ and $\emptyset^{\mathrm{op}} \simeq \emptyset$.

(2) For ∞-categories C and D there are equivalences

$$(C \times D)^{\mathrm{op}} \simeq C^{\mathrm{op}} \times D^{\mathrm{op}},$$
$$C^{\mathrm{op}} \sqcup D^{\mathrm{op}} \simeq (C \sqcup D)^{\mathrm{op}},$$
$$\mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}}) \simeq \mathrm{Fun}(C, D)^{\mathrm{op}}.$$

(3) For functors $F\colon C \to E$ and $G\colon D \to E$ there is an equivalence

$$(C \times_E D)^{\mathrm{op}} \simeq C^{\mathrm{op}} \times_{E^{\mathrm{op}}} D^{\mathrm{op}}.$$

(4) For an ∞-category C there is an equivalence $(C^{\mathrm{op}})^{\simeq} \simeq (C^{\simeq})^{\mathrm{op}}$.

Proof. All of these properties follow readily from the universal properties of these constructions. For illustration, let us merely sketch the proof that $(-)^{\mathrm{op}}$ preserves products: functors $T \to (C \times D)^{\mathrm{op}}$ correspond to functors $T^{\mathrm{op}} \to C \times D$, which in turn are pairs of functors $(T^{\mathrm{op}} \to C, T^{\mathrm{op}} \to D)$, which in turn translate back into pairs of functors $(T \to C^{\mathrm{op}}, T \to D^{\mathrm{op}})$, i.e. functors $T \to C^{\mathrm{op}} \times D^{\mathrm{op}}$. □

Exercise 1.5.2. Let x and y be objects of an ∞-category C. Show that there is an equivalence

$$\mathrm{Hom}_{C^{\mathrm{op}}}(x, y) \simeq \mathrm{Hom}_C(y, x).$$

Hint: apply $(-)^{\mathrm{op}}$ to the pullback square defining $\mathrm{Hom}_C(y, x)$ and use that $\mathrm{Hom}_C(y, x)$ is an anima.

1.5.2 Subcategories

Next, we would like to define the notion of *subcategories*. Just like in classical category theory, a subcategory of an ∞-category C is determined by choosing some objects of C and some morphisms between them, subject to closure under identities and composition. However, while the subcategory can be constructed 'by hand' in the classical situation, we must now make sure that all the higher coherence data from C is inherited by the subcategory. In order to formulate this, we start by isolating the homotopical analogue of a subcategory inclusion.

Definition 1.5.3. A functor $F\colon C \to D$ is called a *monomorphism* if the commutative square

$$\begin{array}{ccc} C & = & C \\ \| & & \downarrow{\scriptstyle F} \\ C & \xrightarrow[F]{} & D \end{array}$$

is a pullback square, i.e. the functor $\Delta_F := (\mathrm{id}_C, \mathrm{id}_C)\colon C \to C \times_D C$ is an equivalence of ∞-categories.

Remark 1.5.4. Note that this means that two functors $G, H\colon T \to C$ are naturally isomorphic whenever the functors $F \circ G$ and $F \circ H$ are naturally isomorphic, reflecting the usual categorical notion of monomorphism.

We may sometimes write $F\colon C \hookrightarrow D$ if F is a monomorphism. If y is an object of D, we say that y *is contained in* C, sometimes written as $y \in C$, if there exists some object x of C satisfying $F(x) \cong y$.

Exercise 1.5.5. Let $F\colon C \hookrightarrow D$ be a monomorphism, and let y be an object of D which is contained in C. Show that the fiber C_y of F at y is contractible.

Hint: Show that any functor $x\colon * \to C_y$ is an equivalence by exhibiting it as a pullback of the equivalence $\Delta_F\colon C \to C \times_D C$.

The projection from the ∞-category of isomorphisms introduced in Definition 1.4.20 is the principal example needed below.

Proposition 1.5.6 (Isomorphisms form a subcategory of arrows)**.** *For every ∞-category C, the functor*

$$\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Ar}(C)$$

is a monomorphism.

For a proof, see Lemma 2.3.3 of [CCNW26].

We postpone the remaining routine properties of monomorphisms to the exercise section at the end of the chapter.

Axiom I.1 (Monomorphisms define subcategories)**.** Let $i \colon A \to C$ be a monomorphism of ∞-categories. Then for every ∞-category T, the commutative square

$$\begin{array}{ccc} \mathrm{Map}(T, A) & \longrightarrow & \mathrm{Map}(T, C) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\mathrm{Map}([1], T), \mathrm{Map}([1], A)) & \hookrightarrow & \mathrm{Map}(\mathrm{Map}([1], T), \mathrm{Map}([1], C)) \end{array}$$

is a pullback square. The vertical functors record the action of a functor on morphisms.

Exercise 1.5.7. Construct the two vertical functors and the identification that makes the square in Axiom I.1 commute. *Hint:* For $D = A, C$, restrict the composition functor $\mathrm{Fun}(T, D) \times \mathrm{Fun}([1], T) \to \mathrm{Fun}([1], D)$ to the relevant cores, factor through the appropriate groupoid cores, and curry.

Informally, the axiom says that in order for a functor $F \colon T \to C$ to factor through A, it suffices to check that F maps morphisms of T to morphisms of C that lie in A. Accordingly, we will use the terms *monomorphism* and *subcategory inclusion* interchangeably.

We will now introduce a method for constructing subcategories of C by specifying a collection of morphisms closed under composition.

Definition 1.5.8. Let C be an ∞-category. A *collection of morphisms in C* is a monomorphism $M \hookrightarrow \mathrm{Map}([1], C)$. We say it is *closed under composition* if the following two conditions are satisfied:

- For a morphism $f \colon x \to y$ in C, if $f \in M$, then also $\mathrm{id}_x \in M$ and $\mathrm{id}_y \in M$;
- For morphisms $f \colon x \to y$ and $g \colon y \to z$ in C, if $g, f \in M$, then also $g \circ f \in M$.

Axiom I.2 (Subcategory axiom)**.** Consider an ∞-category C equipped with a collection of morphisms $m \colon M \hookrightarrow \mathrm{Map}([1], C)$ of C which is closed under composition. Then there exists a monomorphism $i_M \colon \langle M \rangle_C \to C$ such that the induced map on morphisms factors through an equivalence $\mathrm{Map}([1], \langle M \rangle_C) \xrightarrow{\sim} M$.

By Axiom I.1, the monomorphism i_M has the expected universal property of a subcategory. We refer to the ∞-category $\langle M \rangle_C$ as the *(non-full) subcategory spanned by the morphisms in M*.

Definition 1.5.9 (Full subcategory)**.** A functor $i\colon A \to C$ is called a *full subcategory inclusion* if $i^{\simeq}\colon A^{\simeq} \to C^{\simeq}$ is a monomorphism, and for every ∞-category T the commutative square

$$\begin{array}{ccc} \mathrm{Map}(T, A) & \longrightarrow & \mathrm{Map}(T, C) \\ \downarrow & & \downarrow \\ \mathrm{Map}(T^{\simeq}, A^{\simeq}) & \hookrightarrow & \mathrm{Map}(T^{\simeq}, C^{\simeq}) \end{array}$$

is a pullback square. The vertical functors are constructed by restricting evaluation to $\mathrm{Map}(T, A)\times T^{\simeq}$ and $\mathrm{Map}(T, C) \times T^{\simeq}$, factoring through the corresponding groupoid cores, and currying.

Definition 1.5.10. A *collection of objects* of an ∞-category C is a monomorphism $\Gamma \hookrightarrow C^{\simeq}$. The endpoint functors on $\mathrm{Ar}(C)$ restrict along the groupoid cores to functors $s, t\colon \mathrm{Map}([1], C) \to C^{\simeq}$, for which we use the same notation. Given a collection of objects Γ, we define M_Γ as the following pullback:

$$\begin{array}{ccc} M_\Gamma & \hookrightarrow & \mathrm{Map}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ \Gamma \times \Gamma & \hookrightarrow & C^{\simeq} \times C^{\simeq}. \end{array}$$

The collection M_Γ is closed under composition, since identities and composites have endpoints in Γ whenever the original morphisms do. Hence Axiom I.2 determines a monomorphism $\langle M_\Gamma\rangle_C \hookrightarrow C$. We refer to the subcategory $\langle M_\Gamma\rangle_C$ of C as the *full subcategory spanned by the objects in* Γ.

Proposition 1.5.11 (Full subcategories from collections of objects)**.** *The functor $\langle M_\Gamma\rangle_C \hookrightarrow C$ is a full subcategory inclusion, and there is an equivalence*

$$\langle M_\Gamma\rangle_C^{\simeq} \simeq \Gamma$$

over $C^{\simeq}$.

Proof sketch. Combine the pullback in Axiom I.1, first for $T = *$ and then for general T, with the pullback defining M_Γ. For $T = *$, pullback pasting identifies the core of $\langle M_\Gamma\rangle_C$ with $\Gamma \times_{C^{\simeq}} \Gamma$, which is equivalent to Γ because $\Gamma \to C^{\simeq}$ is a monomorphism. For general T, the same argument shows that a functor $T \to C$ factors through $\langle M_\Gamma\rangle_C$ precisely when its restriction to $T^{\simeq}$ factors through Γ. For details, see Lemma 3.2.6 and Proposition 3.2.7 of [CCNW26]. □

Recognizing equivalences

Recall that a functor between classical 1-categories is an equivalence if and only if it is both fully faithful and essentially surjective. The same statement is true for ∞-categories. Since this is such a fundamental result, we will take it as one of our axioms.

Definition 1.5.12. A functor $F\colon C \to D$ is called *fully faithful* if for every pair of objects x and y of C, the induced map of animae

$$F\colon \mathrm{Hom}_C(x, y) \to \mathrm{Hom}_D(Fx, Fy)$$

is an equivalence.

Definition 1.5.13. A functor $F\colon C \to D$ is called *essentially surjective* if for every object y of D there exists an object x of C together with an isomorphism $Fx \cong y$.

Axiom J (Joyal)**.** A functor $F\colon C \to D$ is an equivalence if and only if it is fully faithful and essentially surjective.

Exercise 1.5.14. Show that a functor $F\colon C \to D$ is an equivalence if and only if the induced functor

$$F_*\colon \operatorname{Map}([1], C) \to \operatorname{Map}([1], D)$$

is an equivalence.

Proposition 1.5.15 (Fully faithful functors and full subcategories)**.** *A functor $F\colon C \to D$ is fully faithful if and only if it is a full subcategory inclusion in the sense of Definition 1.5.9.*

Proof sketch. A full subcategory inclusion is fully faithful by applying its defining pullback square with $T = [1]$ and taking endpoint fibers. Conversely, let $F\colon C \to D$ be fully faithful. The formula for hom animae in a pullback (Exercise 1.4) shows that base change preserves full faithfulness. Thus $\mathrm{pr}_1\colon C \times_D C \to C$ is fully faithful; its diagonal section makes it essentially surjective. The Fundamental Theorem makes this projection an equivalence, so its diagonal section is an equivalence as well. Hence F, and therefore $F^{\simeq}$, is a monomorphism, since taking cores preserves pullbacks. Form the full subcategory of D spanned by $C^{\simeq}$. The induced functor from C to this subcategory is fully faithful and essentially surjective, hence an equivalence by the Fundamental Theorem. For details, see Proposition 6.4.6 of [CCNW26]. □

Another statement familiar from classical category theory is that natural isomorphisms can be checked pointwise. We will also assume this result as a black-box.

Axiom K (Joyal)**.** Let $F, G\colon C \to D$ be two functors and let $\alpha\colon F \Rightarrow G$ be a natural transformation. Then α is a natural isomorphism if and only if for every object x of C the morphism $\alpha(x)\colon F(x) \to G(x)$ is an isomorphism in D.

Remark 1.5.16. It is possible to formulate axioms for ∞-categories that would allow one to deduce Axioms J and K.

1.5.3 Localizations

Another important ∞-categorical construction is that of *localizations*, where we formally invert a collection of morphisms in an ∞-category.

Definition 1.5.17. Let C be an ∞-category, and let $W \hookrightarrow \operatorname{Map}([1], C)$ be a collection of morphisms in C. We say that a functor $F\colon C \to D$ *inverts the morphisms in W* if for every morphism $f\colon x \to y$ in C contained in $W \subseteq \operatorname{Map}([1], C)$, the image $F(f)\colon F(x) \to F(y)$ is invertible in D. We let $\operatorname{Fun}^W(C, D)$ be the full subcategory of $\operatorname{Fun}(C, D)$ spanned by those functors that invert the morphisms in W.

Exercise 1.5.18. Make the definition of $\operatorname{Fun}^W(C, D)$ precise by constructing the corresponding collection of objects of $\operatorname{Fun}(C, D)$. *Hint:* Pull back the collection of isomorphisms in D along the functor which records the action of a functor $C \to D$ on the morphisms in W.

Definition 1.5.19 (Localization)**.** Let C be an ∞-category, and let $W \hookrightarrow \mathrm{Map}([1], C)$ be a collection of morphisms in C. We say that a functor $l\colon C \to L$ is a *localization of C at W* if l inverts the morphisms in W, and if for every ∞-category D, the functor

$$l^*\colon \mathrm{Fun}(L, D) \to \mathrm{Fun}(C, D)$$

is a full subcategory inclusion which identifies $\mathrm{Fun}(L, D)$ with the full subcategory $\mathrm{Fun}^W(C, D)$.

Note that the universal property of localizations uniquely determines the localization. We will generally denote such a localization by $C[W^{-1}]$.

Axiom L (Existence of localizations)**.** Let C be an ∞-category, and let $W \hookrightarrow \mathrm{Map}([1], C)$ be a collection of morphisms in C. Then there exists a localization $l\colon C \to C[W^{-1}]$ of C at W.

Remark 1.5.20 (See Exercise 1.10)**.** One can show that the functor $p_{[1]}\colon [1] \to *$ exhibits $*$ as the localization of $[1]$ at the collection $W = \mathrm{Map}([1], [1])$ of all its morphisms.

Exercise 1.5.21. Let $l\colon C \to C[W^{-1}]$ be a localization of C at a collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$. Show that l can be computed by the following pushout square of ∞-categories:

$$\begin{array}{ccccc} [1] \times W & \longrightarrow & [1] \times \mathrm{Ar}(C) & \xrightarrow{\mathrm{ev}} & C \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & & & \downarrow{\scriptstyle l} \\ W & & \longrightarrow & & C[W^{-1}]. \end{array}$$

Hint: Show that if i is a full subcategory inclusion, then so is $\mathrm{Fun}(W, i)$, and then use Exercises 1.8 and 1.10.

Definition 1.5.22. We define the *geometric realization* $\|C\|$ of an ∞-category C as the localization $C[W^{-1}]$, where $W := \mathrm{Map}([1], C)$ is the collection of *all* morphisms in C.

Remark 1.5.23. The terminology 'geometric realization' will be justified in Lemma 25.2.5, where we will show that $\|C\|$ is equivalent to the geometric realization (i.e. colimit) of the nerve $N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ of C.

Exercise 1.5.24. Let C and D be ∞-categories with collections of morphisms W and W', respectively, and assume that both collections contain all identity morphisms. Use the universal property to show that

$$(C \times D)[(W \times W')^{-1}] \xrightarrow{\sim} C[W^{-1}] \times D[W'^{-1}].$$

Deduce that $\|C \times D\| \xrightarrow{\sim} \|C\| \times \|D\|$.

1.6 Simplicial objects and quasicategories

So far, we have been treating ∞-category theory very axiomatically, focusing on the essential features of the theory itself and ignoring how precisely one might model these objects within traditional set-based mathematics. In this section, we briefly discuss the most common traditional model for ∞-categories, namely *quasicategories*. Along the way, we will also introduce some simplicial constructions and notation that will be used later independently of quasicategories themselves.

Readers who are only interested in the axiomatic development may therefore treat the specifically quasicategorical parts of this section as optional.

Before we start, some words on history are in order. Quasicategories have played an important role in making ∞-category theory into a practical and usable tool for mathematics. Their defining condition already appeared in the work of Boardman and Vogt [BV73, Definition 4.8], who called it the 'restricted Kan condition'. Through deep insights and foundational work by André Joyal [Joy08] and subsequently Jacob Lurie [Lur09b; Lur17], the quasicategorical approach developed into a powerful framework for higher category theory and its applications.

We begin with some simplicial preliminaries that will be useful later in any case.

Definition 1.6.1. The *simplex category* $\mathbf{\Delta}$ is the 1-category of non-empty finite linearly ordered sets and order-preserving maps between them.

Observe that any object in $\mathbf{\Delta}$ is isomorphic to the linear order $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for some n, and that the isomorphism is unique if it exists. We will therefore frequently denote generic objects of $\mathbf{\Delta}$ simply by $[n]$.

Observation 1.6.2. Every morphism in $\mathbf{\Delta}$ can be written as a finite composite of face maps $d_i \colon [n-1] \to [n]$ and degeneracy maps $s_i \colon [n+1] \to [n]$ (see Notation 1.2.11). The relations these maps satisfy are known as the *simplicial identities*.

Definition 1.6.3. Let C be an ∞-category. A *simplicial object in* C is defined to be a functor $\mathbf{\Delta}^{\mathrm{op}} \to C$. We will denote the ∞-category of simplicial objects by

$$\mathrm{s}C := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C).$$

In light of Observation 1.6.2, we may informally think of a simplicial object in C as a collection of objects X_n of C for every natural number n together with face maps $d_i \colon X_n \to X_{n-1}$ and degeneracy maps $s_i \colon X_n \to X_{n+1}$ satisfying the simplicial identities. We frequently draw simplicial objects as follows:

$$\cdots \;\; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow} \;\; X_2 \;\; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} \;\; X_1 \;\; \substack{\longrightarrow\\ \longleftarrow} \;\; X_0.$$

Since this looks somewhat chaotic, it is common to leave the degeneracy maps implicit and only draw the face maps:

$$\cdots \;\; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \;\; X_2 \;\; \substack{\longrightarrow\\ \longrightarrow} \;\; X_1 \;\; \longrightarrow \;\; X_0.$$

We will now specialize to $C = \mathrm{Set}$. For a simplicial set $X \in \mathrm{sSet}$, we refer to the set X_n as its *set of n-simplices*.

Definition 1.6.4 (Simplices)**.** For every natural number n, we define the *n-simplex* Δ^n as the simplicial set represented by $[n]$:

$$\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]) \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}.$$

By the Yoneda lemma this defines a fully faithful functor

$$\Delta^{\bullet} \colon \mathbf{\Delta} \hookrightarrow \mathrm{sSet}.$$

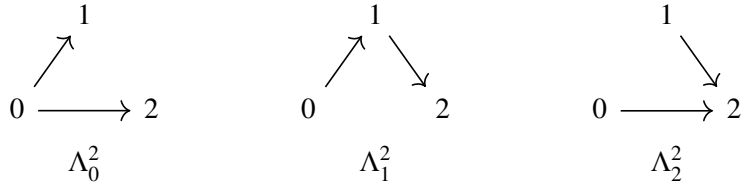
Again by the Yoneda lemma there is a natural bijection $X_n \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X)$, so that we may think of an n-simplex in X as a morphism $\sigma \colon \Delta^n \to X$ of simplicial sets.

Definition 1.6.5 (Horns)**.** For $n \geq 1$ and $0 \leq k \leq n$, we define the *k-horn* Λ^n_k as the subsimplicial set

$$\Lambda^n_k \subseteq \Delta^n$$

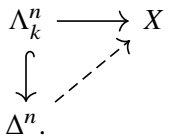
whose m-simplices are those maps $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ such that there is some element $i \in [n] \setminus \{k\}$ which is not in the image of φ. Equivalently, it is the union inside Δ^n of all of its faces except for the k-th face.

Example 1.6.6. For $n = 2$ the three horns Λ^2_0, Λ^2_1 and Λ^2_2 may be displayed as follows:



Definition 1.6.7. Let X be a simplicial set.

(1) We say that X is a *Kan complex* if for every $n \geq 1$ and $0 \leq k \leq n$, every morphism of simplicial sets $\Lambda^n_k \to X$ extends to a morphism $\Delta^n \to X$:



(2) We say that X is a *quasicategory* if the above condition holds for all $n \geq 1$ but only for $0 < k < n$.

We denote by

$$\text{Kan} \quad \subseteq \quad \text{qCat} \quad \subseteq \quad \text{sSet}$$

the full subcategories on the Kan complexes and the quasicategories, respectively.

We now discuss three basic sources of quasicategories, of which the first two will be most relevant for us:

Example 1.6.8 (Singular simplicial complex)**.** Let X be a topological space. We will construct a simplicial set $\text{Sing}(X) \in \text{sSet}$ called the *singular simplicial complex of X*.

- For every $n \geq 0$, let $|\Delta^n|$ denote the topological n-simplex:

$$|\Delta^n| := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1,\ x_i \geq 0 \text{ for } i = 0, \ldots, n\}.$$

- For every order-preserving map $\varphi\colon [n] \to [m]$ we get a continuous map

$$|\Delta^\varphi|\colon |\Delta^n| \to |\Delta^m|, \quad (x_0, \ldots, x_n) \mapsto (y_0, \ldots, y_m), \qquad y_i := \sum_{j \in \varphi^{-1}(i)} x_j.$$

  This defines a functor

$$|\Delta^\bullet|\colon \mathbf{\Delta} \to \text{Top}, \qquad [n] \mapsto |\Delta^n|, \qquad \varphi \mapsto |\Delta^\varphi|.$$

- We now define the simplicial set $\mathrm{Sing}(X)$ as

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X),$$

  i.e. it is the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{|\Delta^\bullet|} \mathrm{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Top}}(-,X)} \mathrm{Set}$.

The simplicial set $\mathrm{Sing}(X)$ is a familiar object from algebraic topology, and is used for example in the definition of the singular (co)homology of X. It is not difficult to see that $\mathrm{Sing}(X)$ is functorial in X, so that it defines a functor

$$\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}.$$

Definition 1.6.9 (Geometric realization)**.** Let $K\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. Its *geometric realization* is the quotient space

$$|K| := \left(\bigsqcup_{n \geq 0} K_n \times |\Delta^n| \right) / \sim,$$

where each set K_n carries the discrete topology and the equivalence relation is generated by

$$(\varphi^*(x), y) \sim (x, |\Delta^\varphi|(y)),$$

for all maps $\varphi\colon [n] \to [m]$ in $\mathbf{\Delta}$, elements $x \in K_m$, and points $y \in |\Delta^n|$. This construction defines a functor

$$|-|\colon \mathrm{sSet} \to \mathrm{Top}$$

which is left adjoint to the singular complex functor $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$.

Exercise 1.6.10. Show that the singular simplicial complex $\mathrm{Sing}(X)$ of a topological space X is a Kan complex.

Example 1.6.11 (Nerve of a 1-category)**.** Let C be a (small, classical) 1-category. We define the *nerve of* C as the simplicial set $N(C) \in \mathrm{sSet}$ given by

$$N(C)_n := \mathrm{Hom}_{\mathrm{Cat}^{(1)}}([n], C),$$

i.e. it is the composite $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathrm{Cat}_1^{(1)})^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}^{(1)}}(-,C)} \mathrm{Set}$. Here $\mathrm{Cat}_1^{(1)}$ denotes the 1-category of small 1-categories.[8] Again this construction is functorial in C, resulting in a functor

$$N\colon \mathrm{Cat}_1^{(1)} \to \mathrm{sSet}.$$

Exercise 1.6.12. Show that the nerve $N(C)$ of a classical 1-category C is a quasicategory. Provide an example for which it is not a Kan complex. Give a complete characterization of those 1-categories C for which $N(C)$ is a Kan complex.

Exercise 1.6.13. Show that the functor $N\colon \mathrm{Cat}_1^{(1)} \to \mathrm{sSet}$ is fully faithful, and that its essential image consists precisely of the simplicial sets in which every inner horn has a unique filler.

[8] We use this notation to distinguish it from the (2,1)-category Cat_1 considered in Example 1.8.15(2).

Example 1.6.14 (Homotopy coherent nerve)**.** A further family of quasicategories is given by the homotopy coherent nerve $N^{\mathrm{top}}(D)$ of a topologically enriched category D (see Definition 25.3.1). Since this construction will not be used in the present book, we do not discuss it here in detail; see [Lur09b, Definition 1.1.5.5]. In Section 25.3, we will instead construct the homotopy coherent nerve directly as an ∞-category, without passing through quasicategories.

We now relate the theory of quasicategories to our axioms of ∞-category theory:

Proposition 1.6.15. *Quasicategories provide a model of the axioms for ∞-categories stated above.*

Proof sketch. We describe the interpretation of our axioms in the theory of quasicategories. Functors are maps of simplicial sets, whose composition is strictly unital and associative, and animae are modeled by Kan complexes. For functors $f, g\colon C \to D$, their identification anima is modeled by the mapping anima from f to g in the maximal Kan complex of $\mathrm{Fun}(C, D)$. Composition and whiskering of natural isomorphisms supply the operations on identifications and their coherence laws. The nerve from Exercise 1.6.12 supplies the classical 1-categories.

The categorical constructions used in our axioms exist in this model and satisfy the stated universal properties, including those on identification animae. In particular, our pullbacks are modeled by homotopy pullbacks of quasicategories. The representability axiom identifies the anima of identifications between two objects with their homotopy pullback, while the Rezk axiom expresses the equivalence between a quasicategory and its ∞-category of invertible morphisms.

The remaining recognition statements are non-trivial results of quasicategory theory. A quasicategory is a Kan complex if and only if all its morphisms are invertible; see [Lan21, Corollary 2.1.12]. This result, as well as the criteria for equivalences and invertible natural transformations in Axioms J and K, is due to Joyal [Joy08]; for the latter two, see also [Lan21, Theorems 2.2.1 and 2.3.20]. We refer to these sources for the model-specific results; the rest of the book uses the axioms directly. □

Warning 1.6.16. Quasicategories provide a concrete model in which to establish the foundations. However, their simplicial presentation can obscure the conceptual content of ∞-category theory. Statements that refer to the underlying simplicial set, such as equality of vertices or horn fillers, concern the model rather than the abstract ∞-category. In the rest of the book we therefore continue to formulate everything model-independently.

1.7 Limits and colimits

Limits and colimits are among the most important universal constructions in category theory. In the ∞-categorical setting, their definitions look formally almost identical to the classical ones, provided one replaces sets of morphisms by hom animae.

Definition 1.7.1 (Limit/colimit)**.** Let I and C be ∞-categories and let $F\colon I \to C$ be a functor.

(1) A *cone on F* is an object X in C together with a natural transformation $\varepsilon\colon \mathrm{const}_X \to F$ of functors $I \to C$.

(2) A cone (X, ε) is called a *limit cone* if for every other object Y of C the composite

$$\mathrm{Hom}_C(Y, X) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_Y, \mathrm{const}_X) \xrightarrow{\varepsilon\circ -} \mathrm{Nat}(\mathrm{const}_Y, F)$$

is an equivalence.

In this case, the object X is called the *limit* of F. It is unique, and will be denoted either by $\lim_I F$ or $\lim_{i \in I} F(i)$.

Dually, one defines a *cocone* to be a natural transformation $\eta \colon F \to \mathrm{const}_X$. It is a *colimit cocone* if for every Y the composite

$$\mathrm{Hom}_C(X, Y) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_X, \mathrm{const}_Y) \xrightarrow{-\circ\eta} \mathrm{Nat}(F, \mathrm{const}_Y)$$

is an equivalence. The object X is then called the *colimit* of F.

Observation 1.7.2. Note that a functor $F \colon I \to C$ admits a limit in C if and only if the opposite functor $F^{\mathrm{op}} \colon I^{\mathrm{op}} \to C^{\mathrm{op}}$ admits a colimit in C^{op}.

Remark 1.7.3. If I and C are 1-categories, then so is $\mathrm{Fun}(I, C)$, and hence all animae occurring in Definition 1.7.1 are sets. The condition on a cone (X, ε) then says that for every object Y the map of sets

$$\mathrm{Hom}_C(Y, X) \to \mathrm{Nat}(\mathrm{const}_Y, F), \qquad u \mapsto \varepsilon \circ \mathrm{const}_u,$$

is a bijection, that is, that every cone on F with tip Y factors uniquely through (X, ε). So Definition 1.7.1 recovers the classical notion of a limit, and dually of a colimit, in a 1-category.

Definition 1.7.4. We say an ∞-category C *admits all I-indexed (co)limits* if for every functor $F \colon I \to C$ there exists a (co)limit (co)cone for F. If $G \colon C \to D$ is a functor between ∞-categories that admit I-indexed (co)limits, then we say that G *preserves I-indexed (co)limits* if for every functor $F \colon I \to C$, the functor G sends (co)limit (co)cones of F in C to (co)limit (co)cones of GF in D.

For later diagrammatic arguments, it is convenient to encode cones and cocones as functors out of auxiliary indexing ∞-categories:

Definition 1.7.5 (Cocone ∞-categories)**.** For an ∞-category I, we define the ∞-category $I^{\triangleright}$, called the *cocone of I*, via the following pushout square of ∞-categories:

$$\begin{array}{ccc} I \times \{1\} & \hookrightarrow & I \times [1] \\ \downarrow & & \downarrow \\ * & \longrightarrow & I^{\triangleright}. \end{array}$$

We will often denote the object classified by the bottom morphism as ∞. Note that a functor $\overline{F} \colon I^{\triangleright} \to C$ consists of a functor $I \times [1] \to C$ whose restriction to $I \times \{1\}$ is constant. Equivalently, it consists of a functor $F := \overline{F}|_{I \times \{0\}} \colon I \to C$ and an object $W := \overline{F}(\infty)$ in C together with a cocone $F \to \mathrm{const}_W$.

We say that a functor $\overline{F} \colon I^{\triangleright} \to C$ is a *colimit diagram* if the associated cocone $F \to \mathrm{const}_W$ is a colimit cocone.

We dually define $I^{\triangleleft}$, called the *cone of I*, as the following pushout of ∞-categories:

$$\begin{array}{ccc} I \times \{0\} & \hookrightarrow & I \times [1] \\ \downarrow & & \downarrow \\ * & \longrightarrow & I^{\triangleleft}. \end{array}$$

A functor $\overline{F}\colon I^{\triangleleft} \to C$ consists of a functor $F\colon I \to C$ together with a cone $\mathrm{const}_W \to F$. We say that $\overline{F}$ is a *limit diagram* if this cone is a limit cone.

The notions of limits and colimits are closely related to that of *adjunctions*, introduced in Chapter 22. We discuss the relations between (co)limits and adjunctions in Section 22.2.

Let us discuss some special cases of limits/colimits that will be relevant.

Products and coproducts

Definition 1.7.6. Let I be a set, regarded as a discrete ∞-category. Given a collection of objects x_i of C, we refer to a colimit of the corresponding functor $I \to C$ as a *coproduct*, and denote it by $\coprod_{i\in I} x_i$ if it exists. We refer to a limit of $I \to C$ as a *product* and denote it by $\prod_{i\in I} x_i$.

Definition 1.7.7 (Initial/terminal objects)**.** An object x of an ∞-category C is called *terminal* if for every other object y the hom anima $\mathrm{Hom}_C(y, x)$ is contractible. We say that x is *initial* if for every other y the hom anima $\mathrm{Hom}_C(x, y)$ is contractible.

Exercise 1.7.8. Let I be a set, and let $\{x_i\}$ be a collection of objects of an ∞-category C.

(1) Show that an object x in C is a product of the objects x_i if and only if it comes equipped with morphisms $\mathrm{pr}_i\colon x \to x_i$ such that for any other object y of C the map

$$(\mathrm{pr}_i \circ -)_i\colon \mathrm{Hom}_C(y, x) \to \prod_{i\in I} \mathrm{Hom}_C(y, x_i)$$

is an equivalence of animae.

(2) Formulate and prove the dual statement for coproducts.

Hint: show that there is an equivalence $\mathrm{Fun}(I, C) \simeq \prod_{i\in I} C$.

Exercise 1.7.9. Show that the terminal objects of C are precisely the limits of the unique functor $F\colon \emptyset \to C$, and that the initial objects are the colimits of $\emptyset \to C$.

Example 1.7.10. The empty anima $\emptyset$ is initial in An and the one-point anima $*$ is terminal in An. Indeed, by Axiom M below the hom anima $\mathrm{Hom}_{\mathrm{An}}(X, Y)$ may be identified with the mapping anima $\mathrm{Map}(X, Y) = \mathrm{Fun}(X, Y)^{\simeq}$, and both $\mathrm{Fun}(\emptyset, Y)$ and $\mathrm{Fun}(X, *)$ are contractible.

Pullbacks and pushouts

Recall from Definition 1.4.1 the posets $\ulcorner$ and $\lrcorner$:

$$\ulcorner \quad = \quad \begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ \downarrow & & \\ (1,0) & & \end{array} \qquad \text{and} \qquad \lrcorner \quad = \quad \begin{array}{ccc} & & (0,1) \\ & & \downarrow \\ (1,0) & \longrightarrow & (1,1). \end{array}$$

By Axiom D, restricting along the two edges induces equivalences

$$\mathrm{Fun}(\ulcorner, C) \xrightarrow{\sim} \mathrm{Fun}([1], C) \times_{d_1^*, C, d_1^*} \mathrm{Fun}([1], C),$$
$$\mathrm{Fun}(\lrcorner, C) \xrightarrow{\sim} \mathrm{Fun}([1], C) \times_{d_0^*, C, d_0^*} \mathrm{Fun}([1], C)$$

for every ∞-category C. In particular, a span $\ulcorner \to C$ is equivalently encoded by a pair of morphisms $(f\colon x \to y, g\colon x \to z)$ with the same source, while a cospan $\lrcorner \to C$ is equivalently encoded by a pair of morphisms $(h\colon y \to w, k\colon z \to w)$ with the same target.

Definition 1.7.11. Let C be an ∞-category, and let $f\colon x \to y$ and $g\colon x \to z$ be two morphisms in C, corresponding to a functor $\ulcorner \to C$. We refer to a colimit of this functor as a *pushout of y and z along x*. We will denote a pushout by $y \sqcup_x z$, if it exists.

Dually, given morphisms $h\colon y \to w$ and $k\colon z \to w$, we refer to a limit of the corresponding functor $\lrcorner \to C$ as a *pullback of y and z over w*, and denote it by $y \times_w z$.

Remark 1.7.12. For $C = \mathrm{An}$, these notions recover the classical homotopy pushouts and homotopy pullbacks of algebraic topology. We will show in Section 2.3 that the functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ sends homotopy pullbacks of spaces to pullbacks of animae, and homotopy pushouts of diagrams of spaces having the homotopy type of cell complexes to pushouts of animae, see Propositions 2.3.10 and 2.3.22. This is one of the main ways in which the abstract theory of this chapter connects to classical homotopy theory.

Recall that cocones on $\ulcorner$-indexed diagrams are encoded by the cocone category $\ulcorner^\triangleright$ from Definition 1.7.5. We will now see that this ∞-category is equivalent to the indexing diagram encoding commutative squares.

Lemma 1.7.13. *The functor $\ulcorner \times [1] \to [1] \times [1]$ described informally by the diagram*

$$\begin{array}{ccccc} (1,0) & \longleftarrow & (0,0) & \longrightarrow & (0,1) \\ \downarrow & & \downarrow & & \downarrow \\ (1,1) & = & (1,1) & = & (1,1) \end{array}$$

induces an equivalence $\ulcorner^\triangleright \xrightarrow{\sim} [1] \times [1]$.

Proof. Since $\ulcorner \times [1]$ is a pushout of two copies of $[1] \times [1]$ and $[1] \times [1]$ is a pushout of two copies of $[2]$, we may construct such a functor by gluing two copies of the functor $p_2\colon [1] \times [1] \to [2]$ from Construction 1.4.3. By construction, the restriction of this functor to $\ulcorner \times \{1\}$ is the identity on $(1,1)$, hence it factors through a map $\ulcorner^\triangleright \to [1] \times [1]$. To show that it is an equivalence, we must show that for any ∞-category C the induced commutative square

$$\begin{array}{ccc} \mathrm{Fun}([1] \times [1], C) & \longrightarrow & \mathrm{Fun}(\ulcorner \times [1], C) \\ {\scriptstyle \mathrm{ev}_{(1,1)}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ C & \longrightarrow & \mathrm{Fun}(\ulcorner \times \{1\}, C) \end{array}$$

is a pullback square. Again using the pushout descriptions of $\ulcorner \times [1]$, $[1] \times [1]$ and $\ulcorner \times \{1\}$, this square may be identified with

$$\begin{array}{ccc} \mathrm{Fun}([2], C) \times_{d_1^*, \mathrm{Fun}([1], C), d_1^*} \mathrm{Fun}([2], C) & \longrightarrow & \mathrm{Fun}([1] \times [1], C) \times_{\mathrm{res}, \mathrm{Fun}([1], C), \mathrm{res}} \mathrm{Fun}([1] \times [1], C) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ C \times_C C & \longrightarrow & \mathrm{Fun}([1], C) \times_{d_1^*, C, d_1^*} \mathrm{Fun}([1], C). \end{array}$$

The claim now follows from the right pullback square in Proposition 1.4.4. □

Exercise 1.7.14. Construct an analogous equivalence $\lrcorner^{\triangleleft} \xrightarrow{\sim} [1] \times [1]$.

Convention 1.7.15. From now on, given a diagram $\ulcorner \to C$ encoded by two morphisms $f : x \to y$ and $g : x \to z$, we will always use the lemma to identify cocones over this diagram with commutative squares in C of the form

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ z & \xrightarrow{k} & w. \end{array} \tag{1.1}$$

Similarly, given morphisms $h : y \to w$ and $k : z \to w$, we will identify cones over the corresponding diagram $\lrcorner \to C$ with commutative squares of the above form.

Definition 1.7.16. We say a commutative square in C of the form (1.1) is a *pushout square* if the corresponding cocone in C is a colimit cocone, in which case we have $w \simeq y \sqcup_x z$. Dually, we say the square is a *pullback square* if the corresponding cone in C is a limit cone, in which case we have $x \simeq y \times_w z$.

Exercise 1.7.17 (Pasting lemma for pullback squares)**.** Consider a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g_1} & Y & \xrightarrow{g_2} & Z \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ X' & \xrightarrow[h_1]{} & Y' & \xrightarrow[h_2]{} & Z'. \end{array}$$

If the right-hand square is a pullback square, then the left-hand square is a pullback square if and only if the outer rectangle is a pullback square. *Hint: compare with Lemma 1.3.15.*

Exercise 1.7.18. Formulate and prove the dual pasting lemma for pushout squares.

Geometric realizations and totalizations

Definition 1.7.19. Let C be an ∞-category, and let $X : \mathbf{\Delta}^{\mathrm{op}} \to C$ be a simplicial object. The colimit of X, if it exists, is called the *geometric realization of* X, and is denoted by

$$|X| \quad := \quad \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} X_n.$$

If C admits all $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits, we denote the resulting colimit functor by $|-| : \mathrm{s}C \to C$.

Remark 1.7.20. The terminology is inspired by the classical geometric realization functor from Definition 1.6.9, which glues topological simplices according to the structure maps of a simplicial set.

Definition 1.7.21. Let C be an ∞-category and let $X\colon \mathbf{\Delta} \to C$ be a cosimplicial object. The limit of X, if it exists, is called the *totalization of* X.

1.8 Foundational results in ∞-category theory

Having established the basic language and constructions of ∞-categories, we now collect a small amount of foundational infrastructure that will be used repeatedly later without further proof. Since our ultimate interest in this book is stable homotopy theory and higher algebra, we treat the results below as black boxes. Table 1.1 records additional background material developed later in the book.

Background material	Discussed in
Adjunctions and Kan extensions	Chapter 22
Presheaves and free cocompletions	Chapter 23
The Hom functor	Section 24.4

Table 1.1: Background material on ∞-category theory that is used as a black box here and developed later in the book.

The ∞-categories Cat_∞ and An

To formulate Yoneda-style constructions and presheaf categories, we need ambient ∞-categories of small animae and small ∞-categories. The following axiom isolates their key features that are used throughout this book.

Axiom M. There exist ∞-categories An and Cat_∞, together with a fully faithful functor $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$. They satisfy the following properties:

- There is a functor $\pi_{\mathrm{univ}}\colon (\mathrm{Cat}_\infty)_\bullet \to \mathrm{Cat}_\infty$ called the *universal cocartesian fibration*.[9] Every object $c \in \mathrm{Cat}_\infty$ defines an ∞-category C via the following pullback square:

$$\begin{array}{ccc} C & \longrightarrow & (\mathrm{Cat}_\infty)_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{c} & \mathrm{Cat}_\infty. \end{array}$$

  An ∞-category C arising this way is called a *small ∞-category*. If c is contained in the full subcategory An, then C is an anima; animae arising this way are called *small animae*.

- For objects $c, d \in \mathrm{Cat}_\infty$ corresponding to ∞-categories C and D, there is an equivalence of animae

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(c, d) \xrightarrow{\sim} \mathrm{Map}(C, D),$$

  and in particular morphisms in Cat_∞ correspond to functors of ∞-categories. Composition in Cat_∞ corresponds to composition of functors.

[9] This terminology will be explained in Chapter 24.

- The ∞-categories An and Cat_∞ admit all small limits and colimits; here the adjective 'small' means that the indexing ∞-category is required to be small. The inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$ preserves small limits and colimits.

- The initial and terminal ∞-categories are small. Small ∞-categories are closed under products, coproducts, pullbacks, pushouts, functor categories, subcategories, and localizations.

- There is a fully faithful functor
$$\mathbf{\Delta} \hookrightarrow \mathrm{Cat}_\infty$$
given on objects by sending a linearly ordered set P to its associated ∞-category. In particular, the ∞-categories $[n]$ are small for all $n \geq 0$.

- There is a fully faithful functor
$$\mathrm{Set} \hookrightarrow \mathrm{An}$$
given on objects by sending a set to its associated anima.

Remark 1.8.1 (Large ∞-categories)**.** The ∞-categories An and Cat_∞ are sometimes referred to as the *universe of animae* and the *universe of ∞-categories*. It will sometimes be convenient to assume there are two more such universes $\widehat{\mathrm{An}}$ and $\widehat{\mathrm{Cat}}_\infty$ that satisfy all the same conditions as An and Cat_∞, with the following two further conditions:

- There is a fully faithful inclusion $\mathrm{Cat}_\infty \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ which on small animae restricts to a functor $\mathrm{An} \hookrightarrow \widehat{\mathrm{An}}$;

- The ∞-categories An and Cat_∞ are small with respect to $\widehat{\mathrm{Cat}}_\infty$.

These inclusions preserve small limits and colimits. We will refer to ∞-categories that are small with respect to $\widehat{\mathrm{Cat}}_\infty$ as *large ∞-categories*. The two points may thus be summarized by saying that every small ∞-category is also large, and that the ∞-categories An and Cat_∞ are large.

All categorical constructions are understood relative to fixed ambient universes, which we silently enlarge when necessary.

Remark 1.8.2 (Two meanings of 'pullback square')**.** The ambient ∞-category Cat_∞ lets us reconcile two pieces of terminology that we have so far used in parallel. In Section 1.3 we defined products, coproducts, pullbacks and pushouts *of* ∞-categories, together with the corresponding notions of pullback and pushout square, purely in terms of functors and identifications. In Section 1.7 we defined limits and colimits *inside* an ∞-category, and hence in particular inside Cat_∞. For small ∞-categories the two notions agree:

- A commutative square of functors in the sense of Definition 1.2.2 between small ∞-categories is the same thing as a commutative square in Cat_∞ in the sense of Definition 1.4.1. Indeed, by Axiom M the hom animae of Cat_∞ are the mapping animae $\mathrm{Map}(C, D)$, so filling a square of functors amounts to giving an identification between the two composites.

- Such a square is a pullback square of ∞-categories in the sense of Section 1.3 if and only if it is a pullback square in Cat_∞ in the sense of Section 1.7, and similarly for pushouts, products and coproducts. For pullbacks and products, the first condition can be detected by bijectivity on

π_0 of the mapping animae out of every ∞-category T, whereas the second is characterized by equivalences of these mapping animae. The former nevertheless suffices to conclude that the canonical functor to the pullback or product is an equivalence. The statements for pushouts and coproducts are dual.

The one point to keep in mind is a matter of universes: the definitions in Section 1.3 quantify over *all* ∞-categories T, whereas a (co)limit in Cat_∞ only tests against small ones. By Remark 1.8.1 this is harmless, since the constructions of Section 1.3 are compatible with enlarging the ambient universe. From now on we use the two readings interchangeably.

Remark 1.8.3. In the quasicategorical model for ∞-categories, the statements in Axiom M are proved by [Lur09b]. The quasicategory modeling Cat_∞ is taken to be the homotopy coherent nerve of the Kan-enriched category of quasicategories, while the quasicategory modeling An is the homotopy coherent nerve of the Kan-enriched category of Kan complexes.

Presheaves and free cocompletions

With these ambient ∞-categories in place, we can record the presheaf formalism that will be used repeatedly later.

Let C be an ∞-category. Then the hom animae $\mathrm{Hom}_C(x, y)$ for $x, y \in C$ assemble into a functor

$$\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}$$

called the *Hom functor*. Formally constructing this functor turns out to be surprisingly involved: it relies on the advanced concept of straightening/unstraightening from Chapter 24. We refer to Section 24.4 for a construction of the Hom functor.

Remark 1.8.4 (Functoriality of mapping animae)**.** Applying the Hom functor construction to the large ∞-category Cat_∞ gives a functor

$$\mathrm{Map}(-,-) := \mathrm{Hom}_{\mathrm{Cat}_\infty}(-,-) \colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty \to \mathrm{An}.$$

By Axiom M, its value at (C, D) is $\mathrm{Map}(C, D) = \mathrm{Fun}(C, D)^{\simeq}$, which is small when C and D are small. Its action on morphisms is given by precomposition and postcomposition. Thus the Hom functor supplies the higher coherences needed to make these operations functorial in both variables. We use this functor, for example, when constructing the nerve in Definition 25.1.1. The same construction applies to large ∞-categories after enlarging the universe.

Definition 1.8.5. A *presheaf* on an ∞-category C is a functor $F \colon C^{\mathrm{op}} \to \mathrm{An}$. We define the *presheaf category of* C as

$$\mathrm{PSh}(C) \quad := \quad \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

By currying, the Hom functor defines a functor $Y \colon C \to \mathrm{PSh}(C)$ called the *Yoneda embedding*; for $x \in C$ we refer to the presheaf $Y(x) := \mathrm{Hom}_C(-, x)$ as the *presheaf represented by* x. A presheaf is called *representable* if it is isomorphic to $Y(x)$ for some x.

The ∞-categorical analogue of the Yoneda lemma is even more important than its classical analogue:

Theorem 1.8.6 (Yoneda lemma, [Lan21, Propositions 4.2.10 and 4.2.11])**.** *For a presheaf F on C and an object $x \in C$, evaluation at $\mathrm{id}_x \in \mathrm{Hom}_C(x,x) = Y(x)(x)$ induces an equivalence*

$$\mathrm{Nat}(Y(x), F) \xrightarrow{\sim} F(x), \qquad (\varphi \colon Y(x) \to F) \mapsto \varphi(\mathrm{id}_x).$$

In particular the Yoneda embedding $Y \colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ is fully faithful.

The following two results will be proved in Corollary 24.6.8 and Corollary 22.2.9, respectively.

Proposition 1.8.7. *The functor $\mathrm{Hom}_C(-,-) \colon C^{\mathrm{op}} \times C \to \mathrm{An}$ preserves limits in both variables separately.*

Proposition 1.8.8. *The presheaf category $\mathrm{PSh}(C)$ admits all small limits and colimits. The evaluation functors $\mathrm{ev}_x \colon \mathrm{PSh}(C) \to \mathrm{An}$ preserve all small limits and colimits.*

Exercise 1.8.9. Let C be an ∞-category. Deduce that the Yoneda embedding $Y \colon C \hookrightarrow \mathrm{PSh}(C)$ preserves all limits that exist in C.

While the inclusion of C into $\mathrm{PSh}(C)$ preserves all limits, it does not in general preserve colimits. Instead, it *freely adds colimits* to C, in the following sense:

Theorem 1.8.10 (Lurie [Lur09b, Theorem 5.1.5.6])**.** *Let C be a small ∞-category and let D be a large ∞-category admitting all small colimits. Then restriction along the Yoneda embedding $Y \colon C \hookrightarrow \mathrm{PSh}(C)$ induces an equivalence*

$$Y^* \colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D),$$

where the left-hand side denotes the full subcategory of colimit-preserving functors $\mathrm{PSh}(C) \to D$.

More informally, this says that every functor $u \colon C \to D$ admits a unique colimit-preserving extension $Y_!(u) \colon \mathrm{PSh}(C) \to D$. This functor can be described directly in terms of the concept of *left Kan extension*, developed systematically in Section 22.4. Informally, a left Kan extension $i_!(u) \colon E \to D$ of a functor $u \colon C \to D$ along a functor $i \colon C \to E$ is the best approximation to an extension of u to E: its value at an object e of E is assembled as a colimit of the values $u(c)$ over all maps $i(c) \to e$. In the situation $i = Y$ of the theorem, the full faithfulness of the Yoneda embedding guarantees that for $e = Y(c)$, this Kan extension recovers $u(c)$.

The special case $C = *$ is the form that we will use most often in Part I. Since $\mathrm{PSh}(*) \simeq \mathrm{An}$, it gives the following universal property.

Corollary 1.8.11 (Universal property of animae)**.** *Let D be a large ∞-category admitting all small colimits. Evaluation at $* \in \mathrm{An}$ induces an equivalence*

$$\mathrm{ev}_* \colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{An}, D) \xrightarrow{\sim} D.$$

Given an object $d \in D$, the inverse equivalence sends d to a functor of the form

$$\mathrm{An} \to D, \qquad X \mapsto \mathrm{colim}_X d,$$

where the right-hand side denotes the colimit of the constant diagram with value d, indexed by the anima X.

Returning now to the simplicial language from Section 1.6, we may use this theorem to assign an ∞-category to every simplicial anima:

Construction 1.8.12 (Associated ∞-category)**.** Every simplicial anima $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ has an associated ∞-category $\mathrm{ac}(X) \in \mathrm{Cat}_\infty$. To construct it, consider the inclusion $\mathbf{\Delta} \hookrightarrow \mathrm{Cat}_\infty$ from Axiom M. By Theorem 1.8.10, this uniquely extends to a colimit-preserving functor

$$\mathrm{ac}\colon \mathrm{sAn} = \mathrm{PSh}(\mathbf{\Delta}) \to \mathrm{Cat}_\infty.$$

In particular, every simplicial set X (hence every quasicategory) defines an ∞-category $\mathrm{ac}(X)$ by regarding it as a simplicial anima via the inclusion $\mathrm{sSet} \hookrightarrow \mathrm{sAn}$.

Remark 1.8.13. Note that the composite

$$\mathrm{Cat}_1^{(1)} \xrightarrow{N} \mathrm{sSet} \hookrightarrow \mathrm{sAn} \xrightarrow{\mathrm{ac}} \mathrm{Cat}_\infty$$

provides a way of associating an ∞-category to a classical 1-category. From this point forward, we stop working with Axiom B.1 and use this refined assignment. The two assignments agree on the indexing categories used so far in this chapter: on the linear orders $[n]$ this is the fully faithful functor $\mathbf{\Delta} \hookrightarrow \mathrm{Cat}_\infty$ from Axiom M. For $[1]\times[1]$, $\ulcorner$, and $\lrcorner$, this follows by expressing their nerves as pushouts of representables and applying ac. For $[\omega]$, it follows by writing its nerve as the sequential colimit of the nerves of its finite initial segments and using the finite- and infinite-string clauses of Axiom D.

Adjunctions

The last piece of black-box infrastructure we record here is the notion of an *adjunction*, which interacts closely with the limit and colimit constructions introduced above:

Definition 1.8.14. Let $F\colon C \to D$ and $G\colon D \to C$ be two functors. An *adjunction* between C and D consists of a natural isomorphism

$$\mathrm{Hom}_D(F(-), -) \cong \mathrm{Hom}_C(-, G(-))$$

of functors $C^{\mathrm{op}} \times D \to \mathrm{An}$. We often write $F \dashv G$ to indicate that such an isomorphism has been given.

As in classical category theory, adjunctions may equivalently be characterized in terms of a unit $\eta\colon \mathrm{id}_C \to GF$ and a counit $\varepsilon\colon FG \to \mathrm{id}_D$ satisfying the triangle identities $(\varepsilon F)(F\eta) \cong \mathrm{id}_F$ and $(G\varepsilon)(\eta G) \cong \mathrm{id}_G$. An important feature of adjunctions is that they can be characterized *pointwise*: given a functor $F\colon C \to D$, if one can provide for every object Y of D a suitable object $G(Y)$ of C equipped with a map $\varepsilon_Y\colon FG(Y) \to Y$ satisfying a suitable adjointness condition, then these objects automatically assemble into a right adjoint $G\colon D \to C$, and the maps ε_Y assemble into a counit ε for the adjunction.

The theory of adjunctions is very rich and there is a lot to say about it. Details on the pointwise characterization of adjunctions, along with related concepts such as Kan extensions, (co)final functors, and Bousfield localizations, can be found in Chapter 22.

Example 1.8.15. Let us state as black boxes two fundamental examples of adjunctions we will use:

(1) (Path components) The fully faithful inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ admits a left adjoint $\pi_0 \colon \mathrm{An} \to \mathrm{Set}$ by Corollary 12.3.24 of [CCNW26]. This functor preserves arbitrary products.

We call $\pi_0(X)$ the *set of path components* of the anima X. We often abuse terminology and refer to an anima as a *set* if it is in the essential image of the inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ (cf. Example 1.1.4).

(2) (Homotopy category) We say that an ∞-category C is a *1-category* if the hom anima $\mathrm{Hom}_C(X, Y)$ is a set for all $X, Y \in C$. We denote by

$$\mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}_\infty$$

the full subcategory spanned by the 1-categories. Then this inclusion admits a left adjoint (see [Lur09b, Section 1.2.3])

$$\mathrm{Ho} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_1 .$$

Given an ∞-category C, we refer to the 1-category $\mathrm{Ho}(C)$ as its *homotopy category*; it has the same set of isomorphism classes of objects as C, but has hom sets given by the set of path components of the hom anima of C:

$$\mathrm{Hom}_{\mathrm{Ho}(C)}(X, Y) \cong \pi_0(\mathrm{Hom}_C(X, Y)).$$

Remark 1.8.16. The functor $\mathrm{Cat}_1^{(1)} \to \mathrm{Cat}_\infty$ from Remark 1.8.13 lands in the full subcategory Cat_1 of 1-categories, by Theorem 15.5.1 of [CCNW26]. In particular, every classical 1-category defines a 1-category in the above sense. Conversely, every 1-category arises this way. We will use the phrase 'classical 1-category' if we wish to refer to the classical definition of categories in terms of sets of objects and morphisms.

Exercises

The opening exercises develop fluency with coproducts, pullbacks, hom animae, monomorphisms, and full subcategories.

Exercise 1.1. Show that coproducts of ∞-categories are *disjoint*: given ∞-categories C and D, the commutative square

$$\begin{array}{ccc} \emptyset & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle \mathrm{incl}_C} \\ D & \xrightarrow[\mathrm{incl}_D]{} & C \sqcup D \end{array}$$

is a pullback square. In a similar spirit, show that for every functor $f \colon C \to D$ the commutative square

$$\begin{array}{ccc} \emptyset & \xrightarrow{\mathrm{id}_\emptyset} & \emptyset \\ \downarrow & & \downarrow \\ C & \xrightarrow[f]{} & D \end{array}$$

is a pullback square. In both cases, explain also which commutative square is meant, that is, which identification between the composites makes it commute. *Hint:* use Axiom C.4.

Exercise 1.2. Consider two functors $f\colon C \to E$ and $g\colon D \to E$. Show that the commutative square

$$\begin{array}{ccc} C \times_E D & \xrightarrow{f \circ \mathrm{pr}_C} & E \\ {\scriptstyle (\mathrm{pr}_C, \mathrm{pr}_D)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_E, \mathrm{id}_E)} \\ C \times D & \xrightarrow{f \times g} & E \times E \end{array}$$

is a pullback square. In other words, construct an equivalence

$$C \times_E D \simeq (C \times D) \times_{E \times E} E.$$

Hint: Consider the commutative diagram

$$\begin{array}{ccccc} C \times_E D & \xrightarrow{\mathrm{pr}_D} & D & \xrightarrow{g} & E \\ {\scriptstyle (\mathrm{pr}_C, \mathrm{pr}_D)}\downarrow & & \downarrow{\scriptstyle (g, \mathrm{id}_D)} & & \downarrow{\scriptstyle (\mathrm{id}_E, \mathrm{id}_E)} \\ C \times D & \xrightarrow{f \times \mathrm{id}_D} & E \times D & \xrightarrow{\mathrm{id}_E \times g} & E \times E \\ {\scriptstyle \mathrm{pr}_C}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_E} & & \\ C & \xrightarrow{f} & E. & & \end{array}$$

The vertical composite of the two left-hand squares is the square defining $C \times_E D$. Use Exercises 1.3.10 and 1.3.16 and Lemma 1.3.15 to show that the bottom-left and top-right squares are pullback squares, and then apply pullback pasting twice.

Exercise 1.3. Let C and D be ∞-categories and let $x, x' \in C$ and $y, y' \in D$ be objects. Produce an equivalence

$$\mathrm{Hom}_{C \times D}((x, y), (x', y')) \simeq \mathrm{Hom}_C(x, x') \times \mathrm{Hom}_D(y, y').$$

Show that composition in $C \times D$ corresponds to composition in C and D under these equivalences.

Exercise 1.4. Let $f\colon C \to E$ and $g\colon D \to E$ be functors, and let $x, x' \in C \times_E D$ be objects. Show that there is a pullback square

$$\begin{array}{ccc} \mathrm{Hom}_{C \times_E D}(x, x') & \longrightarrow & \mathrm{Hom}_C(\mathrm{pr}_C(x), \mathrm{pr}_C(x')) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Hom}_D(\mathrm{pr}_D(x), \mathrm{pr}_D(x')) & \longrightarrow & \mathrm{Hom}_E(f\,\mathrm{pr}_C(x), f\,\mathrm{pr}_C(x')). \end{array}$$

Here the lower horizontal functor is induced by g, followed by transport along the structural isomorphisms $g\,\mathrm{pr}_D(x) \cong f\,\mathrm{pr}_C(x)$ and $g\,\mathrm{pr}_D(x') \cong f\,\mathrm{pr}_C(x')$ carried by the two objects of the pullback.

Exercise 1.5. Show that a functor $F\colon C \to D$ is a monomorphism if and only if the projection $\mathrm{pr}_1\colon C \times_D C \to C$ is an equivalence.

Exercise 1.6. Let $F\colon C \to D$ and $G\colon C' \to D'$ be functors, and suppose there is a diagram

$$\begin{array}{ccccc} C & \xrightarrow{u} & C' & \xrightarrow{r} & C \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle F} \\ D & \xrightarrow[v]{} & D' & \xrightarrow[s]{} & D \end{array}$$

in which both squares commute and whose outer rectangle is isomorphic, as a commutative square, to the identity square on F. In this situation, we say that F is a *retract* of G.

Show that if G is a monomorphism, then so is F. Conclude that if $i\colon A \to C$ induces a monomorphism $\mathrm{Map}([1], A) \hookrightarrow \mathrm{Map}([1], C)$, then it also induces a monomorphism $A^{\simeq} \hookrightarrow C^{\simeq}$.

Exercise 1.7. Let $i\colon A \to C$ be a monomorphism. Show that $i^{\simeq}\colon A^{\simeq} \to C^{\simeq}$ is a monomorphism. Show that for every ∞-category T, the functor $i_*\colon \mathrm{Fun}(T, A) \to \mathrm{Fun}(T, C)$ is a monomorphism.

Exercise 1.8. Consider a pullback square of ∞-categories

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle i'}\downarrow & \lrcorner & \downarrow{\scriptstyle i} \\ C' & \longrightarrow & C. \end{array}$$

Show that if i is a monomorphism, then so is i'. Show moreover that if i is a full subcategory inclusion, then so is i'.

The next exercise gives the concrete calculation needed for the subsequent localization arguments.

Exercise 1.9. The goal of this exercise is to show that the anima $\mathrm{Map}([1], [1])$ is equivalent to a disjoint union of three points.

(1) Construct a retraction r of the evaluation functor

$$e := (\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Map}([1], [1]) \to [1]^{\simeq} \times [1]^{\simeq},$$

that is, a map $r\colon [1]^{\simeq} \times [1]^{\simeq} \to \mathrm{Map}([1], [1])$ together with an equivalence $r \circ e \simeq \mathrm{id}$.

Hint: consider the functor obtained from the ordinary minimum and maximum maps on $[1]$:

$$m\colon [1] \times [1] \times [1] \to [1], \qquad (x, y, z) \mapsto \min\big(\max(x, y), \min(\max(x, z), \max(y, z))\big).$$

(2) Deduce from (1) that $\mathrm{Map}([1], [1])$ is a set. Under the equivalence $[1]^{\simeq} \times [1]^{\simeq} \simeq \{(0,0), (0,1), (1,0), (1,1)\}$, show that the fiber of e over $(1, 0)$ is empty, while each of the other three fibers is inhabited.

(3) Conclude that the canonical map

$$* \sqcup * \sqcup * \to \mathrm{Map}([1], [1])$$

selecting the three functors const_0, $\mathrm{id}_{[1]}$, and const_1 is an equivalence.

The remaining exercises connect the walking morphism, localization, groupoid cores, and geometric realization. The final exercise uses the adjunction theory of Part III and may be postponed.

Exercise 1.10. Show that the functor $p_{[1]}\colon [1] \to *$ exhibits $*$ as the localization of $[1]$ at the collection $W = \mathrm{Map}([1], [1])$ of all morphisms.

(1) Show that $p_{[1]}$ inverts all morphisms in $\mathrm{Map}([1], [1])$.

(2) Let D be an ∞-category. Use Exercise 1.9 to rewrite the pullback square defining $\mathrm{Map}^W([1], D)$, and deduce that $\mathrm{Map}^W([1], D)$ is equivalent to $\mathrm{Iso}(D)^{\simeq}$. Use the Rezk axiom to conclude that the functor

$$p^*_{[1]} \colon D^{\simeq} \to \mathrm{Map}^W([1], D)$$

is an equivalence.

(3) By Axiom L, let $l \colon [1] \to L$ be a localization of $[1]$ at W. Use the universal property of l together with (2) to show that for every ∞-category D, the unique functor $L \to *$ induces an equivalence

$$\mathrm{Map}(*, D) \to \mathrm{Map}(L, D),$$

and thus is an equivalence.

Exercise 1.11. Let $l \colon C \to L$ be a functor, and let $W \hookrightarrow \mathrm{Map}([1], C)$ be the collection defined by the pullback square

$$\begin{array}{ccc} W & \longrightarrow & \mathrm{Map}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle l_*} \\ \mathrm{Iso}(L)^{\simeq} & \xhookrightarrow{\pi_{\mathrm{Iso}}} & \mathrm{Map}([1], L). \end{array}$$

Show that l is a localization of C at *some* collection of morphisms if and only if it is a localization of C at this class W.

Exercise 1.12. Show that any localization $l \colon C \to L$ is essentially surjective: for any object $x \in L$, there exists some $y \in C$ with $x \cong l(y)$. You may use without proof that every map of animae admits an image factorization. *Hint:* Apply this fact to $C^{\simeq} \to L^{\simeq}$, let $L' \subseteq L$ be the full subcategory on the resulting image collection, and apply the universal property of l to the map $C \to L'$.

Exercise 1.13. Show that localizations are closed under pushouts: for a pushout square of ∞-categories

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle l}\downarrow & \ulcorner & \downarrow{\scriptstyle l'} \\ L & \longrightarrow & L', \end{array}$$

if l is a localization, then so is l'.

Exercise 1.14 (cf. Proposition 1.4.4)**.** Use the previous exercise to show that the two functors

$$p_0 \colon [1] \times [1] \to [2] \qquad \text{and} \qquad p_2 \colon [1] \times [1] \to [2]$$

are localizations of $[1] \times [1]$ at the morphism $(0, 0) \to (1, 0)$ and $(0, 1) \to (1, 1)$, respectively; that is, at the left and the right column of the square. Here p_0 and p_2 are the functors constructed in Construction 1.4.3, pictorially represented by the following two diagrams:

$$p_0 \quad = \quad \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \| & & \downarrow \\ 0 & \longrightarrow & 2, \end{array} \qquad\qquad p_2 \quad = \quad \begin{array}{ccc} 0 & \longrightarrow & 2 \\ \downarrow & & \| \\ 1 & \longrightarrow & 2. \end{array}$$

Exercise 1.15. Let C be an ∞-category. Show that the following statements are equivalent:

(1) The ∞-category C is an anima;

(2) The functor $p^*_{[1]} \colon C \to \mathrm{Ar}(C)$ is an equivalence;

(3) The functor $p^*_{[1]} \colon C^{\simeq} \to \mathrm{Map}([1], C)$ is an equivalence;

(4) The functor $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Ar}(C)$ is an equivalence.

Hint: Use Exercise 1.10 and the Rezk axiom to show that $p^*_{[1]} \colon C \to \mathrm{Ar}(C)$ identifies C with $\mathrm{Iso}(C)$.

Exercise 1.16. Let C be an ∞-category. The goal of this exercise is to identify the groupoid core $C^{\simeq}$ of C with the subcategory $\langle M \rangle_C$ of C spanned by the isomorphisms $M := \mathrm{Iso}(C)^{\simeq} \hookrightarrow \mathrm{Map}([1], C)$.

(1) Show that M is closed under composition, so that the subcategory $\langle M \rangle_C$ is defined.

(2) Show that the map $p^*_{[1]} \colon \langle M \rangle_C^{\simeq} \hookrightarrow \mathrm{Map}([1], \langle M \rangle_C)$ is an equivalence, and deduce from Exercise 1.15 that $\langle M \rangle_C$ is an anima.

(3) Show that the core functor $\gamma_C \colon C^{\simeq} \to C$ sends all morphisms to isomorphisms, hence induces a functor $C^{\simeq} \to \langle M \rangle_C$.

(4) Show that this functor induces an equivalence on groupoid cores. Use (2) to conclude that it is an equivalence.

Exercise 1.17. Let C be an ∞-category. The goal of this exercise is to show that its geometric realization $\|C\|$ is an anima.

(1) Use Exercise 1.16 to show that the localization functor $C \to \|C\|$ factors through a functor $C \to \|C\|^{\simeq}$.

(2) Argue that this functor inverts all morphisms, hence induces a functor $\|C\| \to \|C\|^{\simeq}$.

(3) Show that this provides an inverse to the inclusion $\|C\|^{\simeq} \to \|C\|$ and conclude that $\|C\|$ is an anima.

Exercise 1.18. This exercise uses the theory developed in Chapter 22. Use the pointwise criterion for adjunctions (Lemma 22.1.4) to show that the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$ admits both a left adjoint $\| - \| \colon \mathrm{Cat}_\infty \to \mathrm{An}$ as well as a right adjoint $(-)^{\simeq} \colon \mathrm{Cat}_\infty \to \mathrm{An}$.

Part I

Stable homotopy theory

Introduction to Part I

This first part develops the foundations of stable homotopy theory in the language of ∞-categories. Its central object is the ∞-category Sp of spectra. Every generalized cohomology theory arises from a spectrum, and since suspension is an equivalence on Sp, the suspension isomorphism of a cohomology theory holds automatically. Moreover, Sp is a stable ∞-category, so the constructions of homological algebra, such as cofibers and tensor products, can be carried out directly, without first passing to derived functors.

Throughout we build on Chapter 1, which precedes both mathematical parts and supplies the model-independent language of ∞-categories and of *animae*, the ∞-categories in which every morphism is invertible. We begin here with the topological preliminaries. Chapter 2 identifies animae with spaces up to weak homotopy equivalence, and transports the classical homotopy pushouts and pullbacks, suspensions and loop spaces into this setting. Chapter 3 treats (co)homology theories axiomatically and introduces spectra together with Brown's representability theorem; the online version also develops cellular homology from the axioms and proves Brown representability.

Spectra are then studied in their own right. Chapter 4 develops fiber and cofiber sequences, the notion of a stable ∞-category, and the stabilization $\mathrm{Sp}(C)$. Chapter 5 proves the recognition principle, identifying connective spectra with commutative groups in An, so that a connective spectrum becomes a coherent analogue of an abelian group. Chapter 6 develops homological algebra from this viewpoint: derived ∞-categories, t-structures and their hearts, Ext- and Tor-groups, and derived functors. Chapter 7 treats localization and completion of spectra at a prime.

The remaining chapters treat the principal examples. Chapter 8 develops ring spectra and their modules, taking the coherent algebra of Part II as a black box. Chapter 9 constructs complex K-theory, Chapter 10 the Thom spectra and their bordism theories, and Chapter 11 treats Spanier–Whitehead, Atiyah and Poincaré duality.

2 Animae as spaces

This chapter relates the abstract notion of an anima to classical homotopy theory. Recall from Chapter 1 that an anima is the homotopical analogue of a set, characterized as an ∞-category in which all morphisms are invertible. The homotopy hypothesis makes the relationship precise: animae may be thought of as *spaces up to (weak) homotopy equivalence*. In particular, we will see that the classical constructions of cones, suspensions, loop spaces, homotopy pushouts, homotopy pullbacks and wedges have analogues phrased purely in terms of the ∞-category An.

The connection between spaces and animae is the content of Grothendieck's *homotopy hypothesis*. To make it precise, we first need a way to pass from spaces to animae, that is, a functor $\mathrm{Top} \to \mathrm{An}$. A topological space already behaves informally like an ∞-groupoid: its points are objects, paths are morphisms, homotopies between paths are higher morphisms, and this hierarchy continues indefinitely. This structure is recorded by the singular complex of X. Recall from Example 1.6.8 that a topological space X has a simplicial set

$$\mathrm{Sing}(X)_n \;:=\; \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X).$$

Its 0-simplices are the points of X, its 1-simplices are paths, and its higher simplices encode homotopies and their higher coherences.

We may regard $\mathrm{Sing}(X)$ as a simplicial anima via the inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ from Axiom M. We then define the *underlying anima*, or *fundamental ∞-groupoid*, of X by

$$\Pi_\infty(X) \quad := \quad \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Sing}(X)_n \quad \in \quad \mathrm{An}.$$

In other words, $\Pi_\infty(X)$ glues together the singular simplices of X. Although each individual topological simplex is contractible, the colimit still captures the homotopy type of X by recording the way these simplices meet along their faces. Here it is crucial that the colimit is taken in An: a colimit in Set would only identify points, whereas the colimit in An glues by adding identifications between points, identifications between these identifications, and so on. The construction is functorial in X, and thus defines a functor $\Pi_\infty \colon \mathrm{Top} \to \mathrm{An}$.

How faithfully does Π_∞ reflect the homotopy theory of spaces? As a first hint, one can show with a little effort that it sends every homotopy equivalence of topological spaces to an isomorphism of animae; in short, this holds because it preserves finite products and sends the interval to a contractible anima. In particular, the underlying anima of a topological space is a homotopy invariant of the space.

With some more effort, one can show that $\Pi_\infty(X)$ is in fact a *weak homotopy invariant* of the space X. Recall that algebraic topology studies spaces by assigning algebraic invariants to them. Among these invariants are the *homotopy groups*: generalizing the definition of the fundamental group as

homotopy classes of loops in a space, one defines for each $n \geq 1$ the n-th homotopy group of a space X at a basepoint x as the set

$$\pi_n(X, x) \quad := \quad [S^n, X]_*$$

of pointed homotopy classes of pointed continuous maps from the topological n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ into X, with group structure defined via some form of concatenation. A continuous map $f\colon X \to Y$ is called a *weak homotopy equivalence* if it induces a bijection on sets of path components and isomorphisms $\pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x))$ on homotopy groups for all $n \geq 1$ and all choices of basepoint $x \in X$. An algebraic invariant of topological spaces is called a *weak homotopy invariant* if it sends every weak homotopy equivalence to an isomorphism in its target category; examples are the singular homology and cohomology of a space. A large part of algebraic topology is devoted to studying spaces up to weak homotopy equivalence.

If we allow our invariants of spaces to take values in arbitrary ∞-categories, then the functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ constructed before turns out to be a weak homotopy invariant. Proving this requires classical results in simplicial homotopy theory that go beyond the scope of this book, and we refer to [Lur26, Tag 00U2 and Tag 013P] for an exposition of this theory. The key fact relevant here is that a continuous map $X \to Y$ of topological spaces is a weak homotopy equivalence if and only if the induced map of Kan complexes $\mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a simplicial homotopy equivalence. Since geometric realization preserves finite products (Lemma 22.6.12) and $|\Delta^1|$ is contractible, it sends simplicial homotopies to identifications and therefore inverts simplicial homotopy equivalences. We conclude that Π_∞ is a weak homotopy invariant.

A modern formulation of Grothendieck's homotopy hypothesis says that Π_∞ is the *universal* weak homotopy invariant, in the sense of the following axiom:

Axiom N (Grothendieck's homotopy hypothesis)**.** The morphisms inverted by the functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ are precisely the weak homotopy equivalences. Moreover, Π_∞ exhibits An as a localization of Top at the weak homotopy equivalences, that is, it induces an equivalence of ∞-categories

$$\mathrm{Top}[\{\text{weak homotopy equivalences}\}^{-1}] \xrightarrow{\sim} \mathrm{An}.$$

In light of the homotopy hypothesis, we may think of An as the universal home for studying topological spaces up to weak homotopy equivalence. The rest of the chapter adds substance to this claim by transporting familiar constructions from algebraic topology into the world of animae. The main focus will be the classical theory of *homotopy pushouts and pullbacks*, which we recall in Section 2.1. In Section 2.2 we introduce a general framework for computing pushouts and pullbacks in localizations due to Cisinski [Cis19, Chapter 7], which we apply in Section 2.3 to show that Π_∞ sends homotopy pushouts and pullbacks of spaces to ∞-categorical pushouts and pullbacks of animae. This yields the basic constructions on pointed animae, such as fibers, cofibers, suspensions, and loop spaces, developed in Section 2.4, which will become crucial in the remainder of the book.

Remark. As justification for Axiom N, we note that it is satisfied in the quasicategorical model. Applied to the Kan–Quillen model structure, [Lur17, Theorem 1.3.4.20] identifies the localization of simplicial sets at weak homotopy equivalences with the quasicategory $\mathcal{S}$ of spaces. The unit and counit of the classical realization–singular adjunction are weak homotopy equivalences [Lur26, Tag 0142 and Tag 0143], so this adjunction induces an equivalence with the localization of Top. Under this equivalence, the localization functor agrees with our construction of Π_∞ by [Lur26, Tag 04QS].

2.1 Homotopy pushouts and pullbacks

Ordinary pushouts and pullbacks are among the most basic constructions one performs on spaces: think of gluing two spaces along a common subspace, or of the fiber of a fiber bundle over a point. For homotopy theory, however, they have a fundamental defect: they are not automatically homotopy invariant, in that two diagrams of spaces related by homotopy equivalences may have pushouts, or pullbacks, that are not homotopy equivalent.

Example 2.1.1. Given two maps $f\colon Z \to X$ and $g\colon Z \to Y$ of topological spaces, recall that the pushout in Top is given by the quotient space

$$X \sqcup_Z Y := (X \sqcup Y)/\!\sim, \qquad \text{where} \qquad f(z) \sim g(z) \text{ for all } z \in Z.$$

For $n \geq 1$, let D^n denote the n-dimensional disk, and let $S^{n-1} \hookrightarrow D^n$ denote its boundary. We have the following two pushout squares in Top:

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & S^n \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & *. \end{array}$$

Collapsing D^n to a point gives a map between the two spans which is a homotopy equivalence at every vertex, yet their pushouts S^n and $*$ are not homotopy equivalent.

Example 2.1.2. Given two continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$, their pullback $X \times_Z Y$ is given by the subspace

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subseteq X \times Y.$$

To see that pullbacks are not homotopy invariant, suppose that Z is a topological space and $p\colon [0,1] \to Z$ a path whose endpoints $z_0 := p(0)$ and $z_1 := p(1)$ differ, and consider the following two pullback squares in Top:

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow{\scriptstyle z_0} \\ * & \xrightarrow{z_0} & Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \emptyset & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow{\scriptstyle z_0} \\ * & \xrightarrow{z_1} & Z. \end{array}$$

The two cospans differ only by replacing $z_0\colon * \to Z$ with the homotopic map z_1, yet their pullbacks $*$ and $\emptyset$ are not homotopy equivalent (there is not even a map $* \to \emptyset$).

We may get around the lack of homotopy invariance of pushouts and pullbacks by replacing them by their homotopy invariant analogues, called *homotopy pushouts* and *homotopy pullbacks*. For the convenience of the reader we recall their definitions.

Definition 2.1.3. Consider maps $f\colon Z \to X$ and $g\colon Z \to Y$ of topological spaces. Their *homotopy pushout* $X \sqcup^h_Z Y$ (or *double mapping cylinder*) is defined as

$$X \sqcup^h_Z Y \quad := \quad X \sqcup_Z (Z \times [0,1]) \sqcup_Z Y,$$

where the right-hand side is the quotient of $X \sqcup (Z \times [0,1]) \sqcup Y$ by the equivalence relation generated by $(z,0) \sim f(z)$ and $(z,1) \sim g(z)$ for all $z \in Z$. We denote by $\iota_X \colon X \hookrightarrow X \sqcup^h_Z Y$ and $\iota_Y \colon Y \hookrightarrow X \sqcup^h_Z Y$ the canonical inclusions. There is also a canonical homotopy

$$H_{\mathrm{can}} \colon \iota_X \circ f \sim \iota_Y \circ g,$$

represented by the map $Z \times [0,1] \to X \sqcup^h_Z Y$ sending (z,t) to $[(z,t)]$.

This construction follows the Fundamental Principle: where the strict pushout imposes the equality $f(z) = g(z)$ for all $z \in Z$, the homotopy pushout instead introduces a path $f(z) \rightsquigarrow g(z)$ witnessing their identification. As a result, $X \sqcup^h_Z Y$ classifies *homotopy commutative squares*: continuous maps from $X \sqcup^h_Z Y$ into some other space T are given by two maps $t \colon X \to T$ and $s \colon Y \to T$ together with a homotopy $H \colon t \circ f \sim s \circ g$:

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & H & \downarrow{\scriptstyle s} \\ X & \xrightarrow[t]{} & T. \end{array}$$

The homotopy pushout $X \sqcup^h_Z Y$ is illustrated in Figure 2.1.

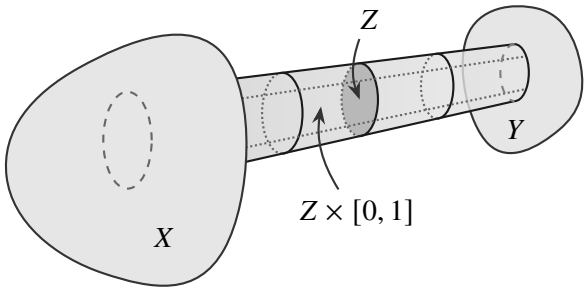


Figure 2.1: The homotopy pushout $X \sqcup^h_Z Y$, obtained by gluing the cylinder $Z \times [0,1]$ to X at one end and to Y at the other.

Exercise 2.1.4. Show that the homotopy type of $X \sqcup^h_Z Y$ depends only on the homotopy types of X, Y, Z and the homotopy classes of f, g. In more detail, consider a homotopy commutative diagram of topological spaces and continuous maps of the form

$$\begin{array}{ccccc} X & \xleftarrow{f} & Z & \xrightarrow{g} & Y \\ {\scriptstyle \alpha}\downarrow & H & \downarrow{\scriptstyle \gamma} & K & \downarrow{\scriptstyle \beta} \\ X' & \xleftarrow[f']{} & Z' & \xrightarrow[g']{} & Y', \end{array}$$

where $H \colon \alpha \circ f \sim f' \circ \gamma$ and $K \colon \beta \circ g \sim g' \circ \gamma$ are homotopies. Construct a map $\alpha \sqcup^h_\gamma \beta \colon X \sqcup^h_Z Y \to X' \sqcup^h_{Z'} Y'$, and show it is a homotopy equivalence whenever α, β and γ are homotopy equivalences.

Let us mention the two most important examples of homotopy pushouts, whose ∞-categorical incarnations will recur throughout the entire book:

- For a continuous map $f \colon X \to Y$, we define its (unreduced) *homotopy cofiber* $C(f)$ as the homotopy pushout of f with $X \to *$. As a special case, we have the *cone* $C(X) := C(\mathrm{id}_X)$ of a topological space X.

- For a topological space X, we define its (unreduced) *suspension* SX as the homotopy pushout of $X \to *$ with $X \to *$.

Both are illustrated in Figure 2.2.

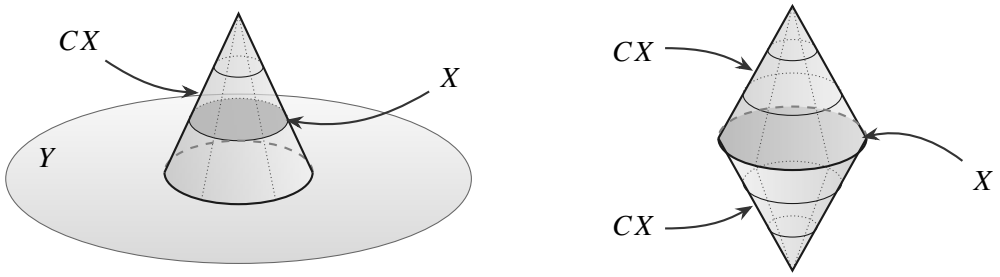


Figure 2.2: The homotopy cofiber $C(f) = Y \sqcup^h_X *$ of a map $f\colon X \to Y$ (left), and the suspension $SX = * \sqcup^h_X *$ of a space X (right).

Remark 2.1.5. In practice, we will mostly be interested in the *reduced* variants of each of these constructions: if X, Y and Z are pointed topological spaces and f and g preserve the basepoints, then the reduced homotopy pushout is obtained from $X \sqcup^h_Z Y$ by collapsing the basepoint interval $\{z_0\} \times [0,1]$ to a point, where $z_0 \in Z$ is the basepoint of Z. The reduced form of the homotopy cofiber and suspension are denoted by $\widetilde{C}(f)$ and ΣX, respectively.

The definition of homotopy pullbacks is dual.

Definition 2.1.6. Consider maps $f\colon X \to Z$ and $g\colon Y \to Z$ of topological spaces. We define their *homotopy pullback* $X \times^h_Z Y$ as the subspace of the product $X \times Z^{[0,1]} \times Y$ given as

$$X \times^h_Z Y \quad := \quad \{(x, p, y) \in X \times Z^{[0,1]} \times Y \mid p(0) = f(x), p(1) = g(y)\}.$$

Here $Z^{[0,1]} = \mathrm{Map}([0,1], Z)$ is the space of paths in Z with the compact-open topology, which in this case is completely determined by the requirement that a map $T \to Z^{[0,1]}$ is continuous if and only if the curried map $T \times [0,1] \to Z$ is continuous.

Again, note how the homotopy pullback adheres to the spirit of the Fundamental Principle: for a point in the homotopy pullback, we do not require $f(x)$ and $g(y)$ to be strictly equal; instead, we require the additional data of a path p from $f(x)$ to $g(y)$ that expresses that these points are 'equal up to homotopy'. As a result, a map $T \to X \times^h_Z Y$ into the homotopy pullback corresponds to a pair of maps $T \to X$ and $T \to Y$ together with a homotopy between the two composites $T \to Z$:

$$\begin{array}{ccc} T & \xrightarrow{t} & Y \\ {\scriptstyle s}\downarrow & H & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z. \end{array}$$

Exercise 2.1.7. Formulate and prove the homotopy invariance of homotopy pullbacks, dual to Exercise 2.1.4.

Two important examples of homotopy pullbacks:

- Given a map $f\colon X \to Y$, the *homotopy fiber* of f at a point $y \in Y$ is the homotopy pullback of f along the map $y\colon * \to Y$. Its elements consist of a point $x \in X$ equipped with a path $p\colon f(x) \rightsquigarrow y$.
- The *loop space* ΩX of a pointed space (X, x) is the homotopy pullback of the map $x\colon * \to X$ along itself. Its elements are x-centered loops $p\colon x \rightsquigarrow x$.

While the homotopy invariance of homotopy pushouts and pullbacks ensures that the problems illustrated by Example 2.1.1 and Example 2.1.2 do not occur, it is not immediately clear that the localization functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ turns homotopy pushouts and pullbacks into ∞-categorical pushouts and pullbacks. In the next section, we provide the tools to tackle this problem.

2.2 Computing pushouts and pullbacks in localizations

The problem of computing pushouts and pullbacks of animae in terms of pushouts and pullbacks of topological spaces is a special case of a problem that appears in other situations as well: given an ∞-category C equipped with a collection of morphisms W designated as weak equivalences, how can pushouts and pullbacks in the localization $C[W^{-1}]$ be computed in terms of pushouts and pullbacks in the original ∞-category C? The goal of this section is to introduce a general framework for addressing this question, due to Cisinski [Cis19, Chapter 7] and closely related to Quillen's framework of *model categories* [Qui67]. In the case of pullbacks, the idea is to find a suitable class of morphisms in C called *fibrations* such that the localization functor preserves pullbacks along fibrations between fibrant objects, provided that the other object in the cospan is also fibrant. Such strict pullbacks are already the 'homotopically correct' pullbacks. Dually, pushouts in $C[W^{-1}]$ may be computed using strict pushouts of cofibrant objects along *cofibrations*. For a synthetic treatment of localizations and the calculus of fractions, see also [CCNW26, Chapter 14].

Remark 2.2.1. Model category theory has played an important role in the development of homotopy theory, being one of the main theoretical frameworks researchers used for studying homotopy theories before the widespread adoption of ∞-category theory. A model category is a 1-category equipped with weak equivalences, fibrations, and cofibrations satisfying various axioms that ensure these classes interact appropriately. From an ∞-categorical perspective, we may think of a model category C as providing a 'model' for its associated ∞-categorical localization $C[W^{-1}]$. In particular, the fibrations and cofibrations in a model category do not affect the 'underlying homotopy theory' presented by the model category, and should rather be thought of as tools for gaining control over categorical constructions like pushouts and pullbacks in this localization.

Remark 2.2.2. This section can be skipped on a first reading. The only results we will use are the specific consequences for the localization functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ stated in Theorems 2.3.8 and 2.3.18; readers willing to accept those as black boxes may safely move to Section 2.3.

Definition 2.2.3 (Class of fibrations)**.** Let C be an ∞-category with a terminal object $*$. A collection of morphisms $F \subseteq \mathrm{Map}([1], C)$ is called a *class of fibrations* if the following conditions are satisfied:

(1) It contains all the isomorphisms;

(2) It is closed under composition;

(3) It is closed under base change: for all morphisms $f\colon X \to Y$ in F and $v\colon Y' \to Y$ in C, the pullback $Y' \times_Y X$ exists, and the projection $Y' \times_Y X \to Y'$ is again in F.

A morphism in F is called a *fibration*. An object X in C is called *fibrant* if the unique map $X \to *$ is a fibration. Dually, a collection of morphisms I is called a *class of cofibrations* if the corresponding class I^{op} in C^{op} defines a class of fibrations in C^{op}. A morphism in I is a *cofibration*, and an object X is *cofibrant* if $\emptyset \to X$ is a cofibration.

Definition 2.2.4 (∞-category with weak equivalences and fibrations)**.** An *∞-category with weak equivalences and fibrations* is a triple (C, W, F), where C is an ∞-category with a terminal object, and W and F are collections of morphisms satisfying the following conditions:

(1) (2-out-of-3) For composable morphisms f and g, if two of f, g, gf are in W, then so is the third.

(2) (Fibrations) The class F is a class of fibrations.

(3) (Pullback axiom) For any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{v} & Y \end{array}$$

in which p is a fibration between fibrant objects, and Y' is fibrant, if p belongs to W then so does p'.

(4) (Factorization axiom) For any morphism $f\colon X \to Y$ in C with fibrant codomain, there exists a morphism $w\colon X \to X'$ in W and a fibration $p\colon X' \to Y$ such that $f = p \circ w$.

The morphisms in W are called *weak equivalences*. Morphisms in $W \cap F$ are *trivial fibrations*. We similarly define an *∞-category with weak equivalences and cofibrations* by dualizing (replacing F with a class of cofibrations I and appropriately modifying axioms (3) and (4) to involve pushouts and cofibrant objects).

Example 2.2.5. Let C be an ∞-category that admits a terminal object and admits all pullbacks. Then we may turn C into an ∞-category with weak equivalences and fibrations by taking W to consist only of the isomorphisms and F to consist of all morphisms in C.

Example 2.2.6 (Structures on Top)**.** As will be discussed below, the category Top can be equipped with such structures relevant to $\mathrm{An} \simeq \mathrm{Top}[W^{-1}]$ where W is the class of weak homotopy equivalences:

- $(\mathrm{Top}, W, I = \text{relative cell complexes})$ is an ∞-category with weak equivalences and cofibrations (see Theorem 2.3.8). The cofibrant objects are the cell complexes.
- $(\mathrm{Top}, W, F = \text{Serre fibrations})$ is an ∞-category with weak equivalences and fibrations (see Theorem 2.3.18). All objects in Top are fibrant in this structure.

The following definition captures those functors that preserve the relevant structure:

Definition 2.2.7 (Left exact functor)**.** Let (C, W_C, F_C) and (D, W_D, F_D) be ∞-categories with weak equivalences and fibrations. A functor $F\colon C \to D$ is called *left exact* if:

(1) F preserves terminal objects.

(2) F sends fibrations between fibrant objects in C to fibrations in D, and trivial fibrations between fibrant objects in C to trivial fibrations in D.

(3) For any pullback square in C

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{v} & Y \end{array}$$

in which $p \in F_C$ and Y, Y' are fibrant in C, the induced square

$$\begin{array}{ccc} F(X') & \xrightarrow{F(u)} & F(X) \\ {\scriptstyle F(p')}\downarrow & \lrcorner & \downarrow{\scriptstyle F(p)} \\ F(Y') & \xrightarrow{F(v)} & F(Y) \end{array}$$

is a pullback square in D.

If C, D have weak equivalences and *co*fibrations, we say that $F\colon C \to D$ is *right exact* if $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is left exact (relative to the dual fibration structures). In particular, F preserves initial objects and pushouts along cofibrations between cofibrant objects, provided that the other object in the span is also cofibrant.

We now come to the main result of this section, which we will state as a black box:

Theorem 2.2.8 (Cisinski [Cis19, Proposition 7.5.6, Theorem 7.5.18])**.** *Let (C, W, F) be an ∞-category with weak equivalences and fibrations, let $C_f \subseteq C$ be the full subcategory of fibrant objects, and write W_f for the weak equivalences between fibrant objects.*

(1) The inclusion $\iota\colon C_f \hookrightarrow C$ induces an equivalence

$$\bar{\iota}\colon C_f[W_f^{-1}] \xrightarrow{\sim} C[W^{-1}].$$

(2) The localization $C[W^{-1}]$ has finite limits (i.e., a terminal object and pullbacks), and the localization functor $\gamma\colon C \to C[W^{-1}]$ is left exact (when $C[W^{-1}]$ is given the trivial structure as in Example 2.2.5).

(3) For any ∞-category D with finite limits, composition with γ induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(C[W^{-1}], D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}_W(C, D)$$

between left exact functors $C[W^{-1}] \to D$ and left exact functors $C \to D$ that invert the morphisms in W. In particular, every left exact functor $C \to D$ which inverts the morphisms W induces a left exact functor $C[W^{-1}] \to D$.

Corollary 2.2.9. *Let (C, W, I) be an ∞-category with weak equivalences and cofibrations, let $C_c \subseteq C$ be the full subcategory of cofibrant objects, and write W_c for the weak equivalences between cofibrant objects.*

(1) The inclusion $\iota\colon C_c \hookrightarrow C$ induces an equivalence

$$\bar{\iota}\colon C_c[W_c^{-1}] \xrightarrow{\sim} C[W^{-1}].$$

(2) The localization $C[W^{-1}]$ has finite colimits (i.e., an initial object and pushouts), and the localization functor $\gamma\colon C \to C[W^{-1}]$ is right exact.

(3) For any ∞-category D with finite colimits, composition with γ induces an equivalence

$$\mathrm{Fun}^{\mathrm{rex}}(C[W^{-1}], D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{rex}}_W(C, D).$$

Proof. Apply Theorem 2.2.8 to the opposite category C^{op}. □

Thus the localization of an ∞-category with weak equivalences and fibrations (respectively, cofibrations) may be computed using only its fibrant (respectively, cofibrant) objects, without choosing replacements functorially.

The exactness results above cover the interaction of localizations with pushouts and pullbacks. We will now extend this discussion to arbitrary limits and colimits, following Cisinski [Cis19, Section 7.7]

Definition 2.2.10. Let C be an ∞-category with weak equivalences and fibrations. We say that C is *homotopy complete* if the following two conditions are satisfied:

(1) The product $\prod_{i\in I} X_i$ of any small collection of fibrant objects $(X_i)_{i\in I}$ exists in C and is again fibrant;

(2) Given a small collection $(f_i\colon X_i \to Y_i)$ of morphisms between fibrant objects, if each f_i is a fibration (resp. trivial fibration), then also $\prod_i f_i$ is a fibration (resp. trivial fibration).

A functor between homotopy complete ∞-categories with weak equivalences and fibrations is called *homotopy continuous* if it is left exact and preserves small products of fibrant objects.

Dually, we say an ∞-category with weak equivalences and cofibrations C is *homotopy cocomplete* if C^{op} is homotopy complete; we obtain the analogous notion of a *homotopy cocontinuous* functor.

Theorem 2.2.11 (Cisinski [Cis19, Proposition 7.7.4, Theorem 7.7.6])**.** *Let C be a homotopy complete ∞-category with weak equivalences and fibrations, and write $L(C) := C[W^{-1}]$ for its localization.*

(1) The localization $L(C)$ has small limits, and the localization functor $\gamma\colon C \to L(C)$ is homotopy continuous;

(2) If all objects of C are fibrant, then $L(C)$ is in fact universal *with this property: for any ∞-category E with small limits, the functor γ induces an equivalence of ∞-categories*

$$\gamma^*\colon \mathrm{Fun}^{\lim}(L(C), E) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{hcont}}(C, E),$$

where the right-hand side denotes the full subcategory spanned by the homotopy continuous functors.

(3) Given a homotopy continuous functor $F\colon C \to D$, the functor[10]

$$L(C) \simeq C_f[W_f^{-1}] \longrightarrow L(D)$$

induced by the restriction of F to fibrant objects preserves small limits.

The dual result holds for homotopy cocomplete ∞-categories with weak equivalences and cofibrations.

2.3 Relative cell complexes and Serre fibrations

Having established a general toolkit for understanding limits and colimits in localizations, we return our attention to the localization functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$. In order to apply the theorems from the previous section, we need to equip Top with suitable notions of fibrations and cofibrations. The standard choice, originating from Quillen's work [Qui67], uses Serre fibrations and relative cell complexes. We will recall the relevant definitions and statements.

2.3.1 Relative cell complexes

We start with the cofibrations.

Definition 2.3.1. A continuous map $i\colon A \to X$ is called a *relative cell complex* if it is isomorphic under A to a map obtained by iteratively attaching cells: there exists an ordinal λ, a sequence $(X_\alpha)_{\alpha\leq\lambda}$ and a homeomorphism $X_\lambda \cong X$ under A such that

- $X_0 = A$;
- For $\alpha < \lambda$, $X_{\alpha+1}$ is formed from X_α via a pushout square

$$\begin{array}{ccc} \bigsqcup_{j\in I_\alpha} S^{n_j-1} & \longrightarrow & X_\alpha \\ \downarrow & \ulcorner & \downarrow \\ \bigsqcup_{j\in I_\alpha} D^{n_j} & \longrightarrow & X_{\alpha+1}, \end{array}$$

  for some index set I_α and dimensions $n_j \geq 0$.
- For limit ordinals $\beta \leq \lambda$, $X_\beta = \mathrm{colim}_{\alpha<\beta} X_\alpha$.

A topological space X is a *cell complex* if the map $\emptyset \to X$ is a relative cell complex. We denote by $\mathrm{Cell} \subseteq \mathrm{Top}$ the full subcategory spanned by the cell complexes.

Remark 2.3.2. While the definition involves ordinals to handle potentially transfinite constructions, for many common examples the process involves only a finite or countable number of steps. For example, every cell complex is homotopy equivalent to a *CW-complex* [MP12, Section 17.4, p. 358], which is a cell complex where $\lambda = \omega$ is the set of natural numbers, and where the dimensions of the cells attached at step $n \in \omega$ is n.

[10] This functor is often denoted $\mathbf{R}F$ and called the *right derived functor* of F; see Section 6.6.

Cell complexes play an important role in homotopy theory, due to the following two important theorems:

Theorem 2.3.3 (Whitehead's theorem, [Hat02, Theorem 4.5])**.** *Let $f\colon X \to Y$ be a weak homotopy equivalence between spaces having the homotopy type of cell complexes. Then f is a homotopy equivalence.*

Proof. Choose homotopy equivalences $u\colon X' \to X$ and $v\colon Y' \to Y$ with X' and Y' CW-complexes, and let $w\colon Y \to Y'$ be a homotopy inverse to v. The composite $wfu\colon X' \to Y'$ is a weak homotopy equivalence, hence a homotopy equivalence by the usual Whitehead theorem. Since u and w are homotopy equivalences, so is f. □

Theorem 2.3.4 (CW-approximation, [Hir15, Theorem 2.1])**.** *For every topological space X, there exists a cell complex Z equipped with a weak homotopy equivalence $q\colon Z \xrightarrow{\sim} X$. Moreover, Z and q can be taken to be functorial in X.*

Remark 2.3.5. An explicit construction of such a functorial CW-approximation may be given as follows. Recall from Definition 1.6.9 that the singular complex functor $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$ admits the geometric realization functor $|-|\colon \mathrm{sSet} \to \mathrm{Top}$ as a left adjoint. One can prove that the geometric realization $|K|$ is a CW-complex for every simplicial set K. Furthermore, for any topological space X the counit $|\mathrm{Sing}(X)| \to X$ of the adjunction is a weak homotopy equivalence.

As a consequence of these two theorems, animae are cell complexes up to homotopy equivalence:

Proposition 2.3.6. *The functor $\Pi_\infty\colon \mathrm{Cell} \to \mathrm{An}$ induces an equivalence*

$$\mathrm{Cell}[\{\text{homotopy equivalences}\}^{-1}] \xrightarrow{\sim} \mathrm{An}.$$

Proof. In light of the homotopy hypothesis, Axiom N, it suffices to show that the functor

$$\mathrm{Cell}[\{\text{homotopy equivalences}\}^{-1}] \to \mathrm{Top}[\{\text{weak homotopy equivalences}\}^{-1}]$$

induced by the inclusion is an equivalence of ∞-categories. By Theorem 2.3.4 there exists a functor $Z\colon \mathrm{Top} \to \mathrm{Cell}$ equipped with a natural transformation $q\colon Z \Rightarrow \mathrm{id}_{\mathrm{Top}}$ such that the map $q_X\colon Z(X) \to X$ is a weak homotopy equivalence for all X. Given a weak homotopy equivalence $f\colon X \to Y$, the 2-out-of-3 property applied to the diagram

$$\begin{array}{ccc} Z(X) & \xrightarrow{q_X} & X \\ {\scriptstyle Z(f)}\downarrow & & \downarrow{\scriptstyle f} \\ Z(Y) & \xrightarrow{q_Y} & Y \end{array}$$

shows that the map $Z(f)$ is again a weak homotopy equivalence, and by Whitehead's theorem it is in fact a homotopy equivalence. It follows that Z induces a functor $\mathrm{Top}[\{\text{weak homotopy equivalences}\}^{-1}] \to \mathrm{Cell}[\{\text{homotopy equivalences}\}^{-1}]$. The natural transformation q becomes invertible in the localization of Top; its restriction to Cell also becomes invertible, since its components are homotopy equivalences by Whitehead's theorem. Thus the two induced functors are inverse equivalences. □

Corollary 2.3.7. *The homotopy category* $\mathrm{Ho}(\mathrm{An})$ *is equivalent to the category* hCell *whose objects are the cell complexes and whose morphism set between X and Y is the set $[X, Y]$ of homotopy classes of continuous maps $X \to Y$.*

Proof. It suffices to observe that a functor $F\colon \mathrm{Cell} \to C$ into a 1-category inverts homotopy equivalences if and only if it identifies homotopic maps. One implication follows by applying F to homotopy inverses. Conversely, since $X \times [0,1]$ is again a cell complex, a homotopy $H\colon f \sim g$ gives

$$F(f) = F(H)F(i_0) = F(H)F(\mathrm{pr})^{-1} = F(H)F(i_1) = F(g),$$

where $i_0, i_1\colon X \to X \times [0,1]$ are the endpoint inclusions and $\mathrm{pr}\colon X \times [0,1] \to X$ is the projection. Thus hCell has the same universal property as the homotopy category of the localization in Proposition 2.3.6. □

We are now ready to establish the claimed structure on Top:

Theorem 2.3.8 (cf. Hovey [Hov99, Theorem 2.4.19])**.** *Let W_{whe} be the collection of weak homotopy equivalences in* Top*, and let I_{cell} be the collection of relative cell complexes. Then* $(\mathrm{Top}, W_{whe}, I_{cell})$ *is an ∞-category with weak equivalences and cofibrations. Furthermore, it admits functorial factorizations and is homotopy cocomplete.*

Proof sketch. We need to verify the axioms from Definition 2.2.4 (dualized for cofibrations):

(1) *(2-out-of-3):* It is clear from the definition that the class W_{whe} satisfies the 2-out-of-3 property.

(2) *(Class of cofibrations):* The class I_{cell} contains all isomorphisms (by attaching no cells) and is closed under composition (attaching cells iteratively). It is also closed under pushouts: if $i\colon A \to X$ is a relative cell complex and $f\colon A \to A'$ is any map, consider the pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle i'} \\ X & \xrightarrow[f']{} & X'. \end{array}$$

Then i' is again a relative cell complex, with filtration $X'_\alpha = A' \sqcup_A X_\alpha$.

(3) *(Pushout axiom):* Every relative cell complex is a cofibration in the standard model structure on Top. Thus, if $i\colon A \to X$ is also a weak homotopy equivalence, it is a trivial cofibration in that model structure. Its pushout is again a trivial cofibration, hence a weak homotopy equivalence. See Hovey [Hov99, Section 2.4].

(4) *(Factorization axiom):* The small object argument functorially factors every map $f\colon X \to Y$ as $X \xrightarrow{i} Z \xrightarrow{p} Y$, where i is a relative cell complex and p is a trivial Serre fibration, hence in particular a weak homotopy equivalence. See Hovey [Hov99, Sections 2.1 and 2.4].

Since all axioms are satisfied, $(\mathrm{Top}, W_{\mathrm{whe}}, I_{\mathrm{cell}})$ is an ∞-category with weak equivalences and cofibrations (with functorial factorizations). It is straightforward to see that a small coproduct of (trivial) cofibrations is again a (trivial) cofibration, so that Top is in fact homotopy cocomplete. □

Proposition 2.3.9. *The functor* $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ *sends every pushout square in* Top *whose vertices are cell complexes and one of whose maps is a relative cell complex to a pushout square in* An*.*

Proof. This follows directly from Theorem 2.3.8 and the right exactness of localization functors stated in Corollary 2.2.9. □

We now arrive at the main result of this section.

Proposition 2.3.10. *Let $f\colon Z \to X$ and $g\colon Z \to Y$ be continuous maps, where all three spaces have the homotopy type of a cell complex. The canonical homotopy H_{can} from Definition 2.1.3 determines a commutative square of animae*

$$\begin{array}{ccc} \Pi_\infty(Z) & \xrightarrow{\Pi_\infty(g)} & \Pi_\infty(Y) \\ {\scriptstyle \Pi_\infty(f)}\downarrow & & \downarrow{\scriptstyle \Pi_\infty(\iota_Y)} \\ \Pi_\infty(X) & \xrightarrow{\Pi_\infty(\iota_X)} & \Pi_\infty(X \sqcup^h_Z Y). \end{array}$$

and this square is a pushout.

Proof. Consider the *mapping cylinder* of the map f:

$$M(f) \quad := \quad (X \sqcup (Z \times [0,1]))/(z,0) \sim f(z).$$

The map f factors as $Z \xrightarrow{i} M(f) \xrightarrow{p} X$, where $i(z) := [(z,1)]$, where $p|_X = \mathrm{id}_X$ and where $p|_{Z\times[0,1]}(z,t) := f(z)$. Observe that the map $p\colon M(f) \to X$ is a homotopy equivalence, with homotopy inverse sending x to $[x]$. In particular, it induces an equivalence $\Pi_\infty(M(f)) \xrightarrow{\sim} \Pi_\infty(X)$ of animae. Further observe that the homotopy pushout $X \sqcup^h_Z Y$ is the strict pushout of $i\colon Z \to M(f)$ and $g\colon Z \to Y$:

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow \\ M(f) & \longrightarrow & X \sqcup^h_Z Y. \end{array} \tag{2.1}$$

We will show that the functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ preserves this pushout square.

Case 1: Assume first that the spaces X, Y and Z are cell complexes and that the map f is a relative cell complex. In this case, the map $i\colon Z \to M(f)$ is also a relative cell complex: it may be written as the composite of $Z \times \{1\} \hookrightarrow Z \times [0,1]$, which is a relative cell complex by the product cell filtration, and the map $Z \times [0,1] \to M(f)$, which is a pushout of the map $Z \to X$ and hence also a relative cell complex. The fact that $\Pi_\infty(-)$ preserves the pushout square in (2.1) is now an instance of Proposition 2.3.9. Under the homotopy equivalence $M(f) \to X$, the displayed square corresponds to the square in the statement, with its specified homotopy H_{can}.

Case 2: We will now prove the general case. By Theorem 2.3.4, there exists a cell complex Z' and a weak homotopy equivalence $Z' \xrightarrow{\sim} Z$. By factorizing the composite maps $Z' \to X$ and $Z' \to Y$ into a relative cell complex followed by a weak equivalence, we obtain a commutative diagram

$$\begin{array}{ccccc} X' & \xleftarrow{f'} & Z' & \xrightarrow{g'} & Y' \\ {\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ X & \xleftarrow[f]{} & Z & \xrightarrow[g]{} & Y, \end{array}$$

where X', Y' and Z' are cell complexes, the vertical maps are weak homotopy equivalences, and f' and g' are relative cell complexes. Since all six spaces have the homotopy type of a cell complex, it follows from Whitehead's theorem that the vertical maps are even homotopy equivalences. By

homotopy invariance of homotopy pushouts, Exercise 2.1.4, the induced map $X' \sqcup^h_{Z'} Y' \to X \sqcup^h_Z Y$ is again a homotopy equivalence, and in particular it becomes an equivalence after applying $\Pi_\infty(-)$. This means that we have reduced the problem to Case 1, thus finishing the proof. □

We next record when an actual quotient computes a homotopy cofiber. Recall that a pair (Y, A) consisting of a closed subspace $A \subseteq Y$ is called a *neighborhood deformation retract pair*, or *NDR-pair*, if there are a continuous map $u\colon Y \to [0,1]$ and a homotopy $h\colon Y \times [0,1] \to Y$ such that

$$u^{-1}(0) = A, \qquad h(y,0) = y, \qquad h(a,t) = a,$$

and $h(y,1) \in A$ whenever $u(y) < 1$. The NDR condition is equivalent to the homotopy extension property; see May [May99a, Chapter 6, Section 4].

Corollary 2.3.11. *Let (Y, A) be an NDR-pair with A nonempty, and assume that A and Y have the homotopy type of cell complexes. Then the quotient map $Y \to Y/A$ induces an isomorphism*

$$\Pi_\infty(Y) \sqcup_{\Pi_\infty(A)} * \cong \Pi_\infty(Y/A)$$

in An_*.

Proof. The NDR condition implies that the canonical map from the homotopy cofiber of $A \hookrightarrow Y$ to the actual quotient Y/A is a homotopy equivalence; see May [May99a, Chapter 8, Section 4]. The claim now follows by applying Proposition 2.3.10 to the diagram $Y \leftarrow A \to *$. □

2.3.2 Serre fibrations

We now turn to the fibration structure on Top.

Definition 2.3.12. Let $p\colon E \to B$ be a continuous map of topological spaces. We say that p is a *Serre fibration* if it satisfies the homotopy lifting property with respect to all disks: for every $n \geq 0$ and every solid commutative diagram of the form

$$\begin{array}{ccc} D^n \times \{0\} & \xrightarrow{f} & E \\ {\scriptstyle i}\big\downarrow & \overset{\overline{H}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ D^n \times [0,1] & \xrightarrow{H} & B, \end{array}$$

there exists a dashed morphism making the diagram commute.

We state without proof some standard facts about Serre fibrations:

Theorem 2.3.13 (Locality for Serre fibrations, tom Dieck [tDie08, Theorem 6.3.3])**.** *Let $p\colon E \to B$ be a continuous map, and assume that there exists an open cover $\{U_i\}_{i \in I}$ such that the restriction $p|_{U_i}\colon p^{-1}(U_i) \to U_i$ is a Serre fibration for each $i \in I$. Then p is itself a Serre fibration.*

In particular, locally trivial fiber bundles are Serre fibrations.

Theorem 2.3.14 (Long exact sequence of homotopy groups, Hatcher [Hat02, Theorem 4.41])**.** *Let $p\colon E \to B$ be a Serre fibration and consider a point $e \in E$ with image $b := p(e)$. Define the fiber F_b as the following (strict) pullback of topological spaces:*

$$\begin{array}{ccc} F_b & \xhookrightarrow{i} & E \\ \downarrow & \lrcorner & \downarrow p \\ \mathrm{pt} & \xhookrightarrow{b} & B. \end{array}$$

Then there is a long exact sequence of homotopy groups of the form

$$\cdots \to \pi_{n+1}(B,b) \to \pi_n(F_b,e) \xrightarrow{i_*} \pi_n(E,e) \xrightarrow{p_*} \pi_n(B,b) \to \pi_{n-1}(F_b,e) \to \ldots$$

Corollary 2.3.15. *A Serre fibration $p\colon E \to B$ is a weak homotopy equivalence if and only if the fiber F_b is weakly contractible for every $b \in B$.*

Proof. By working separately over each path component of B, we may assume that B is path-connected. Picking a preferred basepoint b_0 of B, it then suffices to show that p is a weak homotopy equivalence if and only if the fiber $F := F_{b_0}$ is weakly contractible. Path lifting handles nonemptiness of the fibers and the comparison of path components; the remaining assertions follow from the long exact sequence from Theorem 2.3.14, including its low-degree terms. □

A general source of Serre fibrations is the path space construction.

Construction 2.3.16. We define the *mapping path space* $P(f)$ of a continuous map $f\colon E \to B$ as

$$P(f) := \{(e,\gamma) \in E \times B^{[0,1]} \mid f(e) = \gamma(0)\}.$$

Lemma 2.3.17. *The projection $p\colon P(f) \to B$ given by $(e,\gamma) \mapsto \gamma(1)$ is a Serre fibration.*

Proof. Consider a solid commutative diagram of the form

$$\begin{array}{ccc} D^n \times \{0\} & \xrightarrow{g} & P(f) \\ {\scriptstyle i}\downarrow & \overset{\overline{H}}{\nearrow} & \downarrow p \\ D^n \times [0,1] & \xrightarrow{H} & B. \end{array}$$

We may define the dashed map $\overline{H}$ as $\overline{H}(x,s) := (e(x), K(x,s,-))$, where

- The map $e\colon D^n \to E$ is the first component of g;
- The map $K\colon D^n \times [0,1] \times [0,1] \to B$ is a continuous map satisfying the following three constraints:
  - To guarantee that $\overline{H}(x,s) \in P(f)$ we need $K(x,s,0) = f(e(x))$;
  - To guarantee commutativity of the top triangle, we need $K(x,0,t) = \gamma(x)(t)$, where $\gamma(x)$ is the second component of $g(x)$;
  - To guarantee commutativity of the bottom triangle, we need $K(x,s,1) = H(x,s)$.

Such a continuous map exists, since the inclusion $\big([0,1] \times \{0,1\}\big) \cup \big(\{0\} \times [0,1]\big) \hookrightarrow [0,1] \times [0,1]$ has a continuous retraction. □

Theorem 2.3.18 (cf. Hovey [Hov99, Theorem 2.4.19])**.** *Consider the category* Top *of topological spaces, let F_{Serre} denote the collection of Serre fibrations and let W_{whe} denote the collection of weak homotopy equivalences. Then the triple* $(\mathrm{Top}, W_{whe}, F_{Serre})$ *is an ∞-category with weak equivalences and fibrations. It has functorial factorizations, and is homotopy complete.*

Proof. It is clear that the weak homotopy equivalences satisfy 2-out-of-3, and it is also straightforward to check that Serre fibrations are closed under pullbacks and composition and contain all homeomorphisms. This gives conditions (1) and (2) from Definition 2.2.4.

To see condition (3), consider a pullback square

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{g} & B \end{array}$$

of topological spaces, where p is both a Serre fibration and a weak homotopy equivalence. We need to show that also p' is a weak homotopy equivalence. By Corollary 2.3.15, it suffices to show that for every point $b' \in B'$ the fiber $F'_{b'} := {p'}^{-1}(b')$ is weakly contractible. But since p' is a pullback of p, this fiber is homeomorphic to the fiber $F_b = p^{-1}(b)$, which in turn is weakly contractible by Corollary 2.3.15.

Finally, for condition (4), note that a continuous map $f\colon E \to B$ may be factored as

$$E \xrightarrow{i} P(f) \xrightarrow{p} B,$$

where p is the path space fibration from Lemma 2.3.17, and where i maps e to $(e, \mathrm{const}_{f(e)})$. The first map is a homotopy equivalence, hence in particular a weak homotopy equivalence, and the second map is a Serre fibration by Lemma 2.3.17. Since the definition of $P(f)$ is functorial in f, this finishes the proof that $(\mathrm{Top}, W_{\mathrm{whe}}, F_{\mathrm{Serre}})$ is an ∞-category with weak equivalences and fibrations, with functorial factorizations. Finally, all topological spaces are fibrant and small products exist in Top. Products of Serre fibrations are again Serre fibrations, and products of trivial Serre fibrations are trivial by Corollary 2.3.15, since products of weakly contractible spaces are weakly contractible. □

Proposition 2.3.19. *The localization functor* $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ *preserves all pullbacks along Serre fibrations.*

Proof. In light of Theorem 2.3.18 this is an instance of Theorem 2.2.8. □

Corollary 2.3.20. *The functor* $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ *preserves finite products:*

$$\Pi_\infty(X \times Y) \xrightarrow{\sim} \Pi_\infty(X) \times \Pi_\infty(Y).$$ □

Exercise 2.3.21. Prove the statement of Corollary 2.3.20 directly from the definition of Π_∞. *Hint:* Use Lemma 22.6.12.

We are now ready to prove our main application concerning homotopy pullbacks of topological spaces:

Proposition 2.3.22. *The localization functor* $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ *sends homotopy pullbacks of topological spaces to pullbacks of animae.*

Proof. Consider continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$, and consider the pullback square

$$\begin{array}{ccc} X \times_Z^h Y & \longrightarrow & P(f) \\ \downarrow & \lrcorner & \downarrow p \\ Y & \xrightarrow{g} & Z \end{array}$$

defining $X \times_Z^h Y$. The path fibration p is a Serre fibration by Lemma 2.3.17, hence this square is sent to a pullback square in An by Proposition 2.3.19. The canonical inclusion $i\colon X \hookrightarrow P(f)$ is a homotopy equivalence, giving an isomorphism $\Pi_\infty(P(f)) \simeq \Pi_\infty(X)$. Under this identification, the displayed square corresponds to the homotopy-commutative square defining $X \times_Z^h Y$, since the path coordinate supplies precisely its canonical homotopy. This finishes the proof. □

2.4 Pointed animae, suspensions and loops

The previous sections show that homotopy pushouts and pullbacks of spaces are carried by Π_∞ to pushouts and pullbacks in An. We can therefore define suspensions, cofibers, loop spaces, and fibers directly in the ∞-category An_* by their universal properties, and compare them with the classical constructions from algebraic topology. We write $\mathrm{Top}_* := \mathrm{Top}_{*/}$ for the category of pointed topological spaces and basepoint-preserving continuous maps.

Definition 2.4.1 (Pointed animae)**.** A *pointed anima* is a pair (X, x) consisting of an anima X and a distinguished point $x \in X$. A morphism of pointed animae $f\colon (X, x) \to (Y, y)$ consists of a map $f\colon X \to Y$ in An together with an identification $f(x) =_Y y$.

We denote by An_* the ∞-category of pointed animae and basepoint-preserving maps, defined as the slice category $\mathrm{An}_{*/}$ from Definition 22.3.1.

Remark 2.4.2. The forgetful functor $\mathrm{An}_* \to \mathrm{An}$ preserves pullbacks and pushouts.

Notation 2.4.3. The one-point anima $*$ is canonically pointed; we denote the corresponding pointed anima by $0 \in \mathrm{An}_*$ and call it the *zero object*. One can show it is both initial and terminal in An_*: given pointed animae $X, Y \in \mathrm{An}_*$, there are unique morphisms $X \to 0$ and $0 \to Y$. (See Lemma 4.1.9 for more details.) We denote their composite by

$$0\colon X \to Y$$

and call it the *null map*.

Definition 2.4.4 (Fiber and cofiber sequences in An_*)**.** A *nullsequence* in An_* is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

equipped with a specified identification $g \circ f = 0$, called a *nullhomotopy*. Equivalently, it is a commutative square in An_* of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z. \end{array}$$

Such a nullsequence is called:

- a *fiber sequence* if this square is a pullback square in An_* (or, equivalently, in An). In this case we call X the *fiber of* g and write $X \simeq \mathrm{fib}(g)$;
- a *cofiber sequence* if the same square is a pushout square in An_* (or, equivalently, in An). In this case we call Z the *cofiber of* f and write $Z \simeq \mathrm{cofib}(f)$.

For pointed animae $U, V \in \mathrm{An}_*$, we write

$$[U, V]_* := \pi_0 \mathrm{Hom}_{\mathrm{An}_*}(U, V)$$

for the pointed set of homotopy classes of pointed maps $U \to V$, whose basepoint is the null map. Recall that a sequence $A \to B \to C$ of pointed sets is *exact at* B if the image of $A \to B$ is the preimage of the basepoint under $B \to C$.

Lemma 2.4.5. *Let $W \in \mathrm{An}_*$ be a pointed anima.*

(1) Every cofiber sequence $X \to Y \to Z$ induces a fiber sequence of pointed mapping animae

$$\mathrm{Hom}_{\mathrm{An}_*}(Z, W) \longrightarrow \mathrm{Hom}_{\mathrm{An}_*}(Y, W) \longrightarrow \mathrm{Hom}_{\mathrm{An}_*}(X, W),$$

and hence an exact sequence of pointed sets

$$[Z, W]_* \longrightarrow [Y, W]_* \longrightarrow [X, W]_*.$$

(2) Every fiber sequence $X \to Y \to Z$ induces a fiber sequence of pointed mapping animae

$$\mathrm{Hom}_{\mathrm{An}_*}(W, X) \longrightarrow \mathrm{Hom}_{\mathrm{An}_*}(W, Y) \longrightarrow \mathrm{Hom}_{\mathrm{An}_*}(W, Z),$$

and hence an exact sequence of pointed sets

$$[W, X]_* \longrightarrow [W, Y]_* \longrightarrow [W, Z]_*.$$

Proof. The functor $\mathrm{Hom}_{\mathrm{An}_*}(-, W) \colon \mathrm{An}_*^{\mathrm{op}} \to \mathrm{An}$ sends pushout squares to pullback squares. Since $\mathrm{Hom}_{\mathrm{An}_*}(0, W)$ is terminal, applying this functor to the pushout square defining a cofiber sequence shows that the first sequence of mapping animae is a fiber sequence. Dually, the functor $\mathrm{Hom}_{\mathrm{An}_*}(W, -) \colon \mathrm{An}_* \to \mathrm{An}$ preserves pullbacks and sends 0 to a terminal anima, which proves the corresponding claim for fiber sequences.

It remains to pass to pointed homotopy classes. For any fiber sequence $F \to E \to B$ of pointed animae, a component of E lies in the image of $\pi_0(F) \to \pi_0(E)$ precisely when its image in B lies in the component of the basepoint. Thus $\pi_0(F) \to \pi_0(E) \to \pi_0(B)$ is exact as a sequence of pointed sets, proving both claims. □

Using fiber and cofiber sequences, we can define suspension and loop functors on An_* in a purely ∞-categorical way.

Definition 2.4.6 (Loop and suspension in An_*)**.** Consider a pointed anima $X \in \mathrm{An}_*$. Its *loop anima* ΩX is defined as the fiber of the unique map $0 \to X$:

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & X. \end{array}$$

Its *suspension* ΣX is defined as the cofiber of the unique map $X \to 0$:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma X. \end{array}$$

Exercise 2.4.7. Show that the assignments $X \mapsto \Omega X$ and $X \mapsto \Sigma X$ define endofunctors

$$\Omega, \Sigma \colon \mathrm{An}_* \to \mathrm{An}_*.$$

Hint: Use the pushout description of the diagram $\lrcorner$ from Axiom D to construct a functor $\mathrm{An}_* \to \mathrm{Fun}(\lrcorner, \mathrm{An}_*)$ given on objects by $X \mapsto (0 \to X \leftarrow 0)$.

Lemma 2.4.8. *The functor* $\Sigma \colon \mathrm{An}_* \to \mathrm{An}_*$ *is left adjoint to* $\Omega \colon \mathrm{An}_* \to \mathrm{An}_*$*: for all* $X, Y \in \mathrm{An}_*$ *there is a natural equivalence*

$$\mathrm{Hom}_{\mathrm{An}_*}(\Sigma X, Y) \simeq \mathrm{Hom}_{\mathrm{An}_*}(X, \Omega Y).$$

Proof. Applying the contravariant functor $\mathrm{Hom}_{\mathrm{An}_*}(-, Y)$ to the pushout defining ΣX, and the covariant functor $\mathrm{Hom}_{\mathrm{An}_*}(X, -)$ to the pullback defining ΩY, gives pullback squares

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{An}_*}(\Sigma X, Y) & \longrightarrow & \mathrm{Hom}_{\mathrm{An}_*}(0, Y) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Hom}_{\mathrm{An}_*}(0, Y) & \longrightarrow & \mathrm{Hom}_{\mathrm{An}_*}(X, Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Hom}_{\mathrm{An}_*}(X, \Omega Y) & \longrightarrow & \mathrm{Hom}_{\mathrm{An}_*}(X, 0) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Hom}_{\mathrm{An}_*}(X, 0) & \longrightarrow & \mathrm{Hom}_{\mathrm{An}_*}(X, Y). \end{array}$$

Since 0 is both initial and terminal, the copies of $\mathrm{Hom}_{\mathrm{An}_*}(0, Y)$ and $\mathrm{Hom}_{\mathrm{An}_*}(X, 0)$ are contractible. The maps from them to $\mathrm{Hom}_{\mathrm{An}_*}(X, Y)$ select the zero morphism $X \to 0 \to Y$, and it follows that

$$\mathrm{Hom}_{\mathrm{An}_*}(\Sigma X, Y) \simeq * \times_{\mathrm{Hom}_{\mathrm{An}_*}(X,Y)} * \simeq \mathrm{Hom}_{\mathrm{An}_*}(X, \Omega Y).$$

Both of these equivalences are natural contravariantly in X and covariantly in Y, proving the adjunction. □

Definition 2.4.9. Let $(X_i)_{i \in I}$ be a collection of pointed animae. Their coproduct in An_* is called the *wedge* and is denoted by $\bigvee_{i \in I} X_i$. It may be computed by the following pushout square in An:

$$\begin{array}{ccc} \bigsqcup_{i \in I} * & \longrightarrow & \bigsqcup_{i \in I} X_i \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \bigvee_{i \in I} X_i. \end{array}$$

Definition 2.4.10 (Smash product)**.** Let X and Y be pointed animae. Their *smash product* is the pointed anima

$$X \wedge Y := \mathrm{cofib}(X \vee Y \longrightarrow X \times Y),$$

where the map is induced by $(\mathrm{id}_X, 0)\colon X \to X \times Y$ and $(0, \mathrm{id}_Y)\colon Y \to X \times Y$. Interchanging the two factors induces a natural *swap map*

$$\sigma_{X,Y}\colon X \wedge Y \xrightarrow{\cong} Y \wedge X.$$

We now relate these ∞-categorical constructions in An_* to the classical constructions for pointed topological spaces. Applying Π_∞ to the basepoint defines a functor $\Pi_\infty\colon \mathrm{Top}_* \to \mathrm{An}_*$. It follows from Propositions 2.3.10 and 2.3.22 that this functor sends homotopy pullbacks to pullbacks and suitable homotopy pushouts to pushouts.

We will also use the following direct consequence of the construction of Π_∞.

Lemma 2.4.11. *The functor* $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ *preserves small coproducts.*

Proof. Every continuous map from a topological simplex into a coproduct of spaces lands in a unique summand, since the simplex is connected. Hence the singular complex functor preserves coproducts. The claim follows because $\Pi_\infty(X)$ is the colimit in An of the simplicial set $\mathrm{Sing}(X)$ and colimits commute with colimits. □

In particular:

Proposition 2.4.12. *The functor* $\Pi_\infty\colon \mathrm{Top}_* \to \mathrm{An}_*$ *has the following properties:*

(1) *Given a pointed map* $f\colon X \to Y$ *between spaces having the homotopy type of cell complexes, the unreduced homotopy cofiber* $C(f)$*, pointed by its cone point, satisfies* $\Pi_\infty(C(f)) \cong \mathrm{cofib}(\Pi_\infty(f))$.

(2) *If* X *has the homotopy type of a cell complex, the unreduced suspension* SX*, pointed by either cone point, satisfies* $\Pi_\infty(SX) \cong \Sigma\Pi_\infty(X)$.

(3) *For a collection* $(X_i)_{i\in I}$ *of pointed spaces having the homotopy type of cell complexes, there is an isomorphism* $\Pi_\infty(\bigvee^h_{i\in I} X_i) \cong \bigvee_{i\in I} \Pi_\infty(X_i)$*, where the homotopy wedge* $\bigvee^h_i X_i$ *is the homotopy pushout of the span* $\bigsqcup_i X_i \leftarrow \bigsqcup_i * \to *$.

(4) *Given a pointed map* $f\colon X \to Y$ *with homotopy fiber* F_f *over the basepoint, there is an isomorphism* $\Pi_\infty(F_f) \cong \mathrm{fib}(\Pi_\infty(f))$.

(5) *The loop space of a pointed space* X *satisfies* $\Pi_\infty(\Omega X) \cong \Omega\Pi_\infty(X)$.

Proof. Parts (1) and (2) are direct consequences of Proposition 2.3.10, while parts (4) and (5) follow from Proposition 2.3.22. For part (3), combine Proposition 2.3.10 with Lemma 2.4.11 and the pushout description of wedges in Definition 2.4.9. □

Remark 2.4.13. A pointed topological space (X, x_0) is called *well-pointed* if $(X, \{x_0\})$ is an NDR-pair. Under this hypothesis, the familiar reduced models compute the constructions in the preceding proposition. More precisely, for every pointed map $f\colon X \to Y$ with X well-pointed, the collapse

maps $C(f) \to \widetilde{C}(f)$ and $SX \to \Sigma X$ are homotopy equivalences. Likewise, if every X_i is well-pointed, then the strict wedge $\bigvee_i X_i$ computes the homotopy wedge $\bigvee_i^h X_i$. Thus the conclusions of the proposition apply in particular to reduced cofibers, reduced suspensions and strict wedges of pointed cell complexes whose basepoints are 0-cells.

This result motivates the following homotopical definition of spheres in An_*:

Definition 2.4.14. We define the *homotopical n-sphere* $S^n \in \mathrm{An}_*$ inductively by setting $S^0 := * \sqcup *$ (with the first point as basepoint, say), and taking $S^{n+1} := \Sigma(S^n)$.

Corollary 2.4.15. *The functor* $\Pi_\infty(-) \colon \mathrm{Top}_* \to \mathrm{An}_*$ *sends the topological n-sphere* $S^n \subseteq \mathbb{R}^{n+1}$ *to the homotopical n-sphere* S^n.

Proof. Since Π_∞ preserves coproducts, we have $\Pi_\infty(S^0) \simeq S^0$. The claim now follows inductively from part (2) of Proposition 2.4.12. □

Lemma 2.4.16 (Basic properties of smash products)**.** *For a pointed anima X and animae Y, Z, write* $Y_+ := Y \sqcup *$ *and* $Z_+ := Z \sqcup *$ *for the result of adjoining a disjoint basepoint. There are natural isomorphisms*

$$S^0 \wedge X \cong X, \qquad Y_+ \wedge Z_+ \cong (Y \times Z)_+, \qquad S^1 \wedge X \cong \Sigma X.$$

Moreover, the functor $(-) \wedge X \colon \mathrm{An}_* \to \mathrm{An}_*$ *preserves pushouts, and there are natural isomorphisms* $S^m \wedge S^n \cong S^{m+n}$ *for all* $m, n \geq 0$.

Proof. The required calculations from the pushout descriptions of wedges and cofibers are left to Exercise 2.9. □

Homotopy groups of animae

We now introduce homotopy groups of animae and prove some basic facts about them, like a version of the Whitehead theorem for animae and the long exact sequence of homotopy groups for a fiber sequence.

Definition 2.4.17 (Path components of an anima)**.** Recall from Example 1.8.15(1) that the inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ admits a left adjoint $\pi_0 \colon \mathrm{An} \to \mathrm{Set}$, sending an anima X to its *set of path components* $\pi_0(X)$. The unit of the adjunction gives a map of animae $X \to \pi_0(X)$, and for an element $[x] \in \pi_0(X)$ we define the corresponding path component of X as the fiber of this map over $[x]$. We say that X is *connected* if $\pi_0(X)$ is a singleton.

Definition 2.4.18 (Homotopy groups of an anima)**.** Let $X \in \mathrm{An}_*$ be a pointed anima and $n \geq 0$. The *n-th homotopy group* of X is defined as

$$\pi_n(X) := [S^n, X]_*.$$

We also write $\pi_n(X, x)$ if we wish to make explicit the basepoint x of X.

Since $S^n = \Sigma^n(S^0)$, the adjunction Lemma 2.4.8 implies that

$$\pi_n(X) \cong \pi_0(\Omega^n X),$$

where the functor Ω^n is defined inductively as the n-fold iteration of Ω.

Remark 2.4.19. Let (Y, y_0) be any pointed topological space, and let $X := \Pi_\infty(Y, y_0) \in \mathrm{An}_*$. Then for every $n \geq 0$ there is a natural isomorphism

$$\pi_n(X) \;=\; [S^n, X]_* \;\cong\; [S^n, Y]_*$$

with the classical n-th homotopy group of (Y, y_0). Indeed, Proposition 2.4.12 gives natural isomorphisms

$$\Omega^n X \cong \Pi_\infty(\Omega^n Y).$$

The path components of the underlying anima of a topological space agree with its usual path components: after choosing a cell-complex approximation, this follows from Corollary 2.3.7 by mapping out of a point. It follows that

$$\pi_n(X) \cong \pi_0\Pi_\infty(\Omega^n Y) \cong \pi_0(\Omega^n Y) = [S^n, Y]_*.$$

Remark 2.4.20 (Degree)**.** For every $n \geq 1$, the degree of a self-map of the n-sphere induces an isomorphism

$$\deg\colon \pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}\,.$$

Under these isomorphisms, suspension preserves degree; see Hatcher [Hat02, Corollary 4.25 and Proposition 2.33]. In particular, a pointed self-map of S^n can be identified with the identity as a pointed map if and only if it has degree 1.

Remark 2.4.21 (The suspension as a cogroup)**.** For every pointed anima X, the suspension ΣX is a *cogroup object* in the homotopy category $\mathrm{Ho}(\mathrm{An}_*)$: there are pointed maps

$$\Sigma X \to *, \qquad \mathrm{pinch}\colon \Sigma X \to \Sigma X \vee \Sigma X, \qquad i\colon \Sigma X \to \Sigma X,$$

satisfying counitality, coassociativity, and the coinverse axiom in $\mathrm{Ho}(\mathrm{An}_*)$. To see this, use the homotopy hypothesis to choose a topological space Y whose underlying anima is isomorphic to the underlying anima of X, and choose a point of Y in the component corresponding to the basepoint of X. Applying the factorization from Theorem 2.3.8 to this basepoint map, we may further assume that Y is a cell complex whose basepoint is a 0-cell. Then $\Sigma X \simeq \Pi_\infty(\Sigma Y)$ by Proposition 2.4.12 and Remark 2.4.13, and the maps above are the images under Π_∞ of the classical pinch comultiplication on the topological suspension ΣY. The classical pointed homotopies expressing the cogroup axioms are carried by Π_∞ to identifications between maps in An_*. The double suspension $\Sigma^2 X$ carries comultiplications in two a priori distinct ways, but by the Eckmann–Hilton argument they agree and the resulting comultiplication is cocommutative.

Consequently, for every pointed anima Y the representable functor $[-, Y]_*$ sends wedges to products, so the cogroup structure on ΣX makes $[\Sigma X, Y]_*$ into a group; this group is abelian when the source is a double suspension, as for $[\Sigma^2 X, Y]_*$.

Proposition 2.4.22 (Whitehead theorem for animae)**.** *A morphism $f\colon X \to Y$ of animae is an isomorphism if and only if it induces a bijection $\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$ on path components and for every $x \in X$ it induces an isomorphism $\pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x))$ on higher homotopy groups for all $n \geq 1$.*

Proof. By Corollary 2.3.7, the map f is the image under $\Pi_\infty\colon \mathrm{Cell} \to \mathrm{An}$ of a continuous map between cell complexes. In light of Remark 2.4.19, the claim is thus an instance of the usual Whitehead theorem for cell complexes, recorded in Theorem 2.3.3. □

Corollary 2.4.23. *Let $f\colon X \to Y$ be a map of pointed animae with X and Y connected. Then f is an isomorphism if and only if the induced map*

$$\Omega f\colon \Omega X \to \Omega Y$$

is an isomorphism in An.

Proof. If f is an isomorphism, then clearly so is Ωf. Conversely, if Ωf is an isomorphism, then it in particular induces isomorphisms on all homotopy groups, and using the identification $\pi_n(X) \cong \pi_{n-1}(\Omega X)$ for $n \geq 1$ it follows that the map $\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n \geq 1$. By connectedness, transport along paths gives the same conclusion at every basepoint. Since X and Y are connected, $\pi_0(X) \to \pi_0(Y)$ is automatically a bijection, so f is an isomorphism by Proposition 2.4.22. □

We now briefly recall that the homotopy groups carry natural group structures.

Construction 2.4.24 (Loop anima as a group object)**.** Given a pointed anima $X \in \mathrm{An}_*$, we may consider the maps

$$\begin{aligned} e\colon * &\xrightarrow{(\mathrm{id},\mathrm{id})} * \times_X * = \Omega X, \\ M\colon \Omega X \times \Omega X \cong * \times_X * \times_X * &\xrightarrow{(\mathrm{pr}_1,\mathrm{pr}_3)} * \times_X * = \Omega X, \\ i\colon \Omega X = * \times_X * &\xrightarrow{(\mathrm{pr}_2,\mathrm{pr}_1)} * \times_X * = \Omega X. \end{aligned}$$

The usual group relations hold for these maps in $\mathrm{Ho}(\mathrm{An}_*)$, so that ΩX acquires the structure of a group object in $\mathrm{Ho}(\mathrm{An}_*)$. We leave the details to the reader.

Corollary 2.4.25. *For every pointed anima X and $n \geq 1$, the set $\pi_n(X)$ carries a natural group structure, and for $n \geq 2$ this group is abelian.*

Proof. The group structure on $\pi_1(X) \cong \pi_0(\Omega X)$ comes from the group object structure of ΩX in Construction 2.4.24. For $n \geq 2$, $\pi_n(X) \cong \pi_{n-2}(\Omega^2 X)$ acquires two group structures which commute with each other, and it follows from the Eckmann–Hilton argument that they agree and are abelian. □

By Lemma 2.4.5, applying pointed homotopy classes to a fiber sequence produces an exact sequence. We now extend such a sequence indefinitely by repeatedly passing to fibers.

Recall that the loop object $\Omega X = 0 \times_X 0$ admits a canonical *inversion map*

$$\iota_X\colon \Omega X \to \Omega X,$$

induced by interchanging the two copies of 0 in the pullback. Given a pointed map $f\colon Y \to \Omega X$, we write

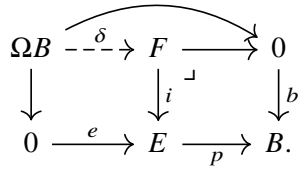
$$-f := \iota_X \circ f\colon Y \to \Omega X.$$

Proposition 2.4.26 (Puppe sequence)**.** *Given a fiber sequence $F \xrightarrow{i} E \xrightarrow{p} B$ in An_*, there is a sequence*

$$\cdots \xrightarrow{\Omega^2 p} \Omega^2 B \xrightarrow{-\Omega\delta} \Omega F \xrightarrow{-\Omega i} \Omega E \xrightarrow{-\Omega p} \Omega B \xrightarrow{\delta} F \xrightarrow{i} E \xrightarrow{p} B,$$

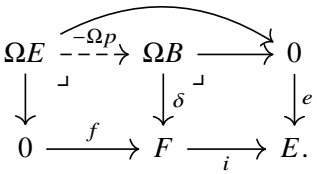
where each consecutive triple forms a fiber sequence.

Proof. To see that ΩB is indeed the fiber of i, consider the following commutative diagram:



Here the outer square is the one defining ΩB, and the right square is obtained by flipping the given fiber sequence. The map δ is induced by the universal property of the pullback F, and it follows from the pasting law of pullback squares that also the left square is a pullback square, thus exhibiting ΩB as the fiber of i.

Next, we identify the fiber of δ. In the pullback description, an element of F consists of $e \in E$ and an identification $\beta\colon p(e) \simeq b_0$, and δ sends a loop λ to (e_0, λ). An identification from (e_0, λ) to the basepoint (e_0, id) consists of a loop α in E together with an identification $\lambda \simeq p(\alpha)$. Using α^{-1} as the coordinate on this fiber therefore identifies its map to ΩB with $-\Omega p$. This gives the following pullback diagram:



Looping these three fiber squares and using the natural inversion maps gives all remaining triples with the displayed signs. □

Corollary 2.4.27 (Long exact sequence on homotopy groups)**.** *A fiber sequence $F \xrightarrow{i} E \xrightarrow{p} B$ of pointed animae induces a long exact sequence*

$$\cdots \to \pi_2(B) \xrightarrow{\delta_*} \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B) \xrightarrow{\delta_*} \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

Proof. Since each consecutive triple in the Puppe sequence is a fiber sequence, Lemma 2.4.5, applied with $W = S^0$, shows that applying $\pi_0(-) = [S^0, -]_*$ produces an exact sequence

$$\cdots \to \pi_0(\Omega^2 B) \to \pi_0(\Omega F) \to \pi_0(\Omega E) \to \pi_0(\Omega B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

In light of the natural isomorphisms $\pi_0(\Omega^n X) \cong \pi_n(X)$ we are done. (The minus signs do not affect kernels or images; the maps at the pointed-set end already have the displayed signs.) □

Dually, interchanging the two copies of 0 in $\Sigma X = 0 \sqcup_X 0$ defines the suspension reversal $r_X\colon \Sigma X \to \Sigma X$. For a pointed map $f\colon X \to Y$, we write

$$-\Sigma f := r_Y \circ \Sigma f = \Sigma f \circ r_X,$$

where the last identification follows from naturality.

Lemma 2.4.28 (coPuppe sequence)**.** *Given a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in An_*, there is a sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma g} \Sigma Z \xrightarrow{-\Sigma \delta} \Sigma^2 X \to \dots$$

in which every consecutive triple forms a cofiber sequence.

Proof. This is dual to the construction and proof of the Puppe sequence in Proposition 2.4.26. □

Lemma 2.4.29 (Sequential colimits as pushouts)**.** *Let C be an ∞-category with countable coproducts and pushouts, and let $Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} \cdots$ be a sequence in C. Its colimit exists and may be computed as the pushout*

$$\begin{array}{ccc} \coprod_{n\geq 0} Z_n & \xrightarrow{a} & \coprod_{k\geq 0} Z_{2k} \\ {\scriptstyle b}\downarrow & & \downarrow \\ \coprod_{k\geq 0} Z_{2k+1} & \longrightarrow & \operatorname{colim}_n Z_n, \end{array}$$

where on the summand Z_n the maps a and b are defined as follows. If $n = 2k$ is even, then a is the inclusion $\mathrm{id}\colon Z_{2k} \to Z_{2k}$ *of a coproduct summand and b is the map $f_{2k}\colon Z_{2k} \to Z_{2k+1}$; if $n = 2k+1$ is odd, then a is the map $f_{2k+1}\colon Z_{2k+1} \to Z_{2k+2}$ and b is the inclusion* $\mathrm{id}\colon Z_{2k+1} \to Z_{2k+1}$.

Proof. Let P denote the displayed pushout. For every object $T \in C$, mapping from P to T gives the pullback of the animae of maps from the coproducts of the even and odd terms over the anima of maps from all terms. By the pushout description of $[\omega]$ in Axiom D, this is identified with the anima of cocones from the sequence $(Z_n)_n$ to T. Thus P has the universal property of $\operatorname{colim}_n Z_n$. □

Exercises

Exercise 2.1 (Basic underlying animae)**.** Compute the underlying anima $\Pi_\infty(X)$ when X is a discrete set and when $X = [0, 1]$. Deduce the answers for

$$X = \emptyset, \qquad X = *, \qquad X = S^0, \qquad X = \coprod_{i=1}^{n} *.$$

Check directly how Π_∞ behaves with respect to finite products and finite coproducts of discrete spaces.

Exercise 2.2 (The pushout counterexample, repaired)**.** For $n \geq 1$, recall the two strict pushout squares from Example 2.1.1:

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^n \end{array} \qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & *. \end{array}$$

Show that the corresponding homotopy pushouts are homotopy equivalent. Identify both of them with S^n, and translate the calculation into An_* using Proposition 2.4.12.

Exercise 2.3 (Maps out of a homotopy pushout)**.** Let $f\colon Z \to X$ and $g\colon Z \to Y$ be continuous maps, and let T be another topological space. Show that continuous maps

$$X \sqcup^h_Z Y \to T$$

are naturally the same as triples (t_X, t_Y, H), where $t_X \colon X \to T$ and $t_Y \colon Y \to T$ are continuous maps and $H \colon Z \times [0,1] \to T$ is a homotopy

$$H \colon t_X \circ f \sim t_Y \circ g.$$

In other words, maps out of the homotopy pushout are homotopy coherent cocones under the span $X \leftarrow Z \rightarrow Y$.

Exercise 2.4 (Maps into a homotopy pullback)**.** Let $f \colon X \to Z$ and $g \colon Y \to Z$ be continuous maps, and let T be another topological space. Show that continuous maps

$$T \to X \times^h_Z Y$$

are naturally the same as triples (t_X, t_Y, H), where $t_X \colon T \to X$ and $t_Y \colon T \to Y$ are continuous maps and $H \colon T \times [0,1] \to Z$ is a homotopy

$$H \colon f \circ t_X \sim g \circ t_Y.$$

Exercise 2.5 (Homotopy fibers in examples)**.** Compute the homotopy fiber over the basepoint in each of the following cases:

(1) the inclusion of the basepoint $* \to S^1$;

(2) the covering map $\mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$;

(3) for $n \geq 1$, the degree-n map $S^1 \to S^1$, $z \mapsto z^n$.

For the first example, show that the homotopy fiber is homotopy equivalent to a discrete copy of $\mathbb{Z}$. For the last two examples, compare the homotopy fiber with the ordinary fiber.

Exercise 2.6 (Loop spaces as homotopy fibers)**.** Let (X, x) be a pointed topological space. Use the definition of homotopy pullback to show that the homotopy pullback of

$$* \xrightarrow{x} X \xleftarrow{x} *$$

is the usual based loop space of X at x.

Exercise 2.7 (Mapping path factorization)**.** Let $f \colon E \to B$ be a continuous map, and let

$$P(f) := \{(e, \gamma) \in E \times B^{[0,1]} \mid f(e) = \gamma(0)\}$$

be its mapping path space. Let $i \colon E \to P(f)$ be the map $e \mapsto (e, \mathrm{const}_{f(e)})$, and let $p \colon P(f) \to B$ be the map $(e, \gamma) \mapsto \gamma(1)$.

(1) Write down an explicit homotopy inverse to i and show that i is a homotopy equivalence.

(2) Identify $P(f)$ in the special cases $f = \mathrm{id}_B$ and $f \colon E \to *$.

(3) For $b \in B$, identify the ordinary fiber $p^{-1}(b)$ with the homotopy fiber of f at b.

Exercise 2.8 (Suspension examples)**.** Use the intrinsic definition of suspension in An_*.

(1) Show that $\bigvee_{i=1}^{r} S^0$ is the pointed discrete anima with r non-basepoint elements.

(2) Show that $\Sigma S^n \cong S^{n+1}$ for all $n \geq 0$, and compute $\Sigma(*)$.

(3) Show that $\Sigma(X \vee Y) \cong \Sigma X \vee \Sigma Y$ for pointed animae X and Y, and compute $\Sigma(\bigvee_{i=1}^{r} S^0)$.

Exercise 2.9 (Smash products)**.** Prove Lemma 2.4.16 directly from the pushout descriptions of wedges and cofibers. In particular, establish the natural isomorphisms $S^0 \wedge X \cong X$ and $X_+ \wedge Y_+ \cong (X \times Y)_+$, show that $(-) \wedge X$ preserves pushouts, and deduce that

$$S^1 \wedge X \cong \Sigma X \qquad \text{and} \qquad S^m \wedge S^n \cong S^{m+n}$$

for all $m, n \geq 0$.

Exercise 2.10. Show by counterexample that the conclusion of Corollary 2.4.23 fails if X and Y are not assumed connected.

Exercise 2.11 (The coPuppe sequence for a cell attachment)**.** For $n \geq 1$, apply the coPuppe sequence of Lemma 2.4.28 to the cofiber sequence

$$S^{n-1} \to D^n \to S^n$$

in pointed animae.

(1) Write out the first six terms of the resulting sequence.

(2) Use the contractibility of D^n to simplify the sequence.

(3) Identify the connecting map $S^n \to \Sigma S^{n-1}$ up to isomorphism.

Exercise 2.12 (Whitehead's theorem through the homotopy hypothesis)**.** Let $f\colon X \to Y$ be a map between spaces having the homotopy type of cell complexes. Show that $\Pi_\infty(f)$ is an isomorphism if and only if f is a homotopy equivalence. Explain why the hypothesis on X and Y cannot simply be omitted from the conclusion about homotopy equivalences.

Exercise 2.13 (When a strict pushout is homotopically correct)**.** Let $i\colon A \to X$ be a relative cell complex between cell complexes, and let $f\colon A \to Y$ be a map to a cell complex. Compare the strict pushout $X \sqcup_A Y$ with the homotopy pushout $X \sqcup_A^h Y$.

(1) Show that applying Π_∞ to either construction gives a pushout of the span $\Pi_\infty(X) \leftarrow \Pi_\infty(A) \to \Pi_\infty(Y)$.

(2) Deduce that the canonical map $X \sqcup_A^h Y \to X \sqcup_A Y$ is a weak homotopy equivalence.

(3) Show that both spaces have the homotopy type of cell complexes, and conclude that the canonical map is a homotopy equivalence.

3 Homology and cohomology

A central theme in algebraic topology is the assignment of algebraic invariants to topological spaces. Among the most important are the *homology groups* $H_n(X;\mathbb{Z})$ and *cohomology groups* $H^n(X;\mathbb{Z})$. They are computable, satisfy convenient formal properties, and distinguish many spaces that are otherwise hard to tell apart.

More generally, a *generalized (co)homology theory* assigns graded abelian groups to spaces in a weakly homotopy invariant way and interacts predictably with wedges, cofiber sequences, and suspensions. We formulate these axioms directly for pointed animae, where homotopy invariance is automatic. We then explain the close relation with *spectra*, viewed for now as sequences of pointed animae $E_0, E_1, E_2, \ldots$ equipped with isomorphisms $E_n \simeq \Omega E_{n+1}$. Every spectrum represents a cohomology theory, and Brown representability gives the converse.

Ordinary homology and cohomology are singled out among generalized theories by the *dimension axiom*. We take their existence for granted here and construct them intrinsically in Chapter 6. The online version also develops their description in terms of cellular homology and gives a constructive proof of Brown representability. Readers already comfortable with (co)homology theories may begin with Section 3.2.

3.1 Homology and cohomology theories

This section formulates reduced homology and cohomology theories directly on pointed animae, and explains how they determine unreduced and relative theories.

Convention 3.1.1 (Graded abelian groups)**.** A *graded abelian group* is a sequence $A_* = (A_n)_{n\in\mathbb{Z}}$ of abelian groups indexed by the integers. These form a 1-category $\mathrm{Ab}^{\mathbb{Z}}$ whose morphisms $f_*\colon A_* \to B_*$ consist of homomorphisms $f_n\colon A_n \to B_n$ for each $n \in \mathbb{Z}$.

Definition 3.1.2 (Homology and cohomology theories)**.** A *reduced homology theory* is a pair $(\widetilde{E}_*, \sigma)$ consisting of a functor

$$\widetilde{E}_*\colon \mathrm{An}_* \to \mathrm{Ab}^{\mathbb{Z}}$$

and a natural isomorphism

$$\sigma\colon \widetilde{E}_*(-) \xrightarrow{\sim} \widetilde{E}_{*+1}(\Sigma(-)),$$

called the *suspension isomorphism*, satisfying:

(1) *(Wedge axiom)* For every collection $(X_i)_{i\in I}$ of pointed animae, the canonical map

$$\bigoplus_{i\in I} \widetilde{E}_*(X_i) \to \widetilde{E}_*\Big(\bigvee_{i\in I} X_i\Big)$$

is an isomorphism.

(2) *(Exactness)* For every cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in An_*, the sequence of abelian groups

$$\widetilde{E}_*(X) \xrightarrow{f_*} \widetilde{E}_*(Y) \xrightarrow{g_*} \widetilde{E}_*(Z)$$

is exact in each degree.

Dually, a *reduced cohomology theory* is a pair $(\widetilde{E}^*, \sigma)$ consisting of a functor $\widetilde{E}^*\colon \mathrm{An}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$ and a natural isomorphism $\sigma\colon \widetilde{E}^*(-) \xrightarrow{\sim} \widetilde{E}^{*+1}(\Sigma(-))$ satisfying:

(1) *(Wedge axiom)* For every collection $(X_i)_{i\in I}$ of pointed animae, the canonical map $\widetilde{E}^*\Big(\bigvee_{i\in I} X_i\Big) \to \prod_{i\in I} \widetilde{E}^*(X_i)$ is an isomorphism.

(2) *(Exactness)* For every cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in An_*, the sequence $\widetilde{E}^*(Z) \xrightarrow{g^*} \widetilde{E}^*(Y) \xrightarrow{f^*} \widetilde{E}^*(X)$ is exact in each degree.

The suspension isomorphism extends exactness to the familiar long exact sequences. Indeed, applying exactness to the coPuppe sequence of a cofiber sequence $X \to Y \to Z$ produces boundary maps

$$\partial\colon \widetilde{E}_n(Z) \to \widetilde{E}_{n-1}(X) \qquad \text{and} \qquad \partial\colon \widetilde{E}^n(X) \to \widetilde{E}^{n+1}(Z).$$

We leave the resulting long exact sequences as an exercise because their construction is a useful way to become comfortable with cofiber sequences.

Exercise 3.1.3 (Long exact sequences of homology theories)**.** Use the coPuppe sequence of Lemma 2.4.28 to show that every homology theory gives rise to a long exact sequence

$$\cdots \to \widetilde{E}_{n+1}(Z) \xrightarrow{\partial} \widetilde{E}_n(X) \xrightarrow{f_*} \widetilde{E}_n(Y) \xrightarrow{g_*} \widetilde{E}_n(Z) \xrightarrow{\partial} \widetilde{E}_{n-1}(X) \to \cdots.$$

Formulate and prove the analogous statement for cohomology theories.

Reduced theories give unreduced theories by adding a disjoint basepoint. For an unpointed anima X, write $X_+ := X \sqcup \mathrm{pt}$, pointed by the added point. We then set

$$E_n(X) := \widetilde{E}_n(X_+) \qquad \text{and} \qquad E^n(X) := \widetilde{E}^n(X_+).$$

For a map $A \to X$ of unpointed animae, the relative groups are defined by

$$E_n(X, A) := \widetilde{E}_n(X/A) \qquad \text{and} \qquad E^n(X, A) := \widetilde{E}^n(X/A),$$

where X/A denotes the cofiber of $A_+ \to X_+$.

Exercise 3.1.4 (Long exact sequence of a pair)**.** Let $i\colon A \to X$ be a map of animae, and let $j\colon X_+ \to X/A$ denote the cofiber map. Show that every homology theory gives rise to a natural long exact sequence

$$\cdots \to E_n(A) \xrightarrow{i_*} E_n(X) \xrightarrow{j_*} E_n(X, A) \xrightarrow{\partial} E_{n-1}(A) \xrightarrow{i_*} E_{n-1}(X) \to \cdots.$$

Formulate and prove the analogous long exact sequence for cohomology:

$$\cdots \to E^n(X, A) \xrightarrow{j^*} E^n(X) \xrightarrow{i^*} E^n(A) \xrightarrow{\partial} E^{n+1}(X, A) \xrightarrow{j^*} E^{n+1}(X) \to \cdots.$$

Exercise 3.1.5. Show that precomposition with $\Pi_\infty \colon \mathrm{Top}_* \to \mathrm{An}_*$ establishes a bijection between reduced cohomology theories in the sense of Definition 3.1.2 and pairs (E^*, σ) consisting of a functor $E^* \colon \mathrm{Top}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$ and isomorphisms $\sigma_X \colon E^*(X) \xrightarrow{\sim} E^{*+1}(\Sigma X)$ defined for pointed CW-complexes X whose basepoints are 0-cells, natural in pointed maps between such complexes, satisfying:

(1) Weak homotopy equivalences induce isomorphisms on E^*.

(2) The wedge axiom holds for wedges of pointed CW-complexes whose basepoints are 0-cells.

(3) For every relative CW-pair (X, A), the sequence $E^*(X/A) \to E^*(X) \to E^*(A)$ is exact.

Formulate the analogous characterization of homology theories.

Hint: Use Theorem 2.3.4 to reduce to CW-complexes, and Proposition 2.4.12 to transport the relevant constructions.

3.2 Cohomology theories and spectra

We now relate cohomology theories to spectra. We first explain how every spectrum determines a reduced cohomology theory. Brown representability supplies the converse, showing that every reduced cohomology theory arises in this way.

Definition 3.2.1 (Spectrum)**.** A *spectrum* E is a sequence of pointed animae $(E_m)_{m \geq 0}$ together with isomorphisms

$$\sigma_m \colon E_m \xrightarrow{\sim} \Omega E_{m+1}$$

for all $m \geq 0$. We call the maps σ_m the *structure maps* of E.

For convenience, we extend the sequence to all integer indices by setting

$$E_k := \Omega^{-k} E_0 \qquad (k < 0).$$

With this convention, the structure maps extend to all $k \in \mathbb{Z}$: for $k < 0$, we take σ_k to be the identity map under the identification

$$E_k = \Omega^{-k} E_0 = \Omega(\Omega^{-k-1} E_0) = \Omega E_{k+1}.$$

Definition 3.2.2 (Morphism of spectra)**.** Let $E = (E_m, \sigma_m)$ and $F = (F_m, \tau_m)$ be spectra. A *morphism of spectra* $f \colon E \to F$ consists of pointed maps

$$f_m \colon E_m \to F_m \qquad (m \geq 0)$$

together with specified identifications making the following squares commute for every $m \geq 0$:

$$\begin{array}{ccc} E_m & \xrightarrow[\sim]{\sigma_m} & \Omega E_{m+1} \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle \Omega f_{m+1}} \\ F_m & \xrightarrow[\sim]{\tau_m} & \Omega F_{m+1}. \end{array}$$

For $k < 0$, we extend f to the negative levels by setting $f_k := \Omega^{-k} f_0 \colon E_k \to F_k$.

Definition 3.2.3 (E-cohomology)**.** Let E be a spectrum and let $X \in \mathrm{An}_*$ be a pointed anima. For $k \in \mathbb{Z}$, we define the *reduced E-cohomology* of X by

$$\widetilde{E}^k(X) := [X, E_k]_*.$$

The isomorphism $E_k \xrightarrow{\sim} \Omega^2 E_{k+2}$ identifies this set with the second homotopy group of the pointed mapping anima $\mathrm{Hom}_{\mathrm{An}_*}(X, E_{k+2})$, and therefore equips it with a natural abelian group structure. The structure isomorphisms of E induce natural isomorphisms

$$\widetilde{E}^k(X) = [X, E_k]_* \xrightarrow{\cong} [X, \Omega E_{k+1}]_* \xrightarrow{\cong} [\Sigma X, E_{k+1}]_* = \widetilde{E}^{k+1}(\Sigma X),$$

which define the suspension isomorphism for $\widetilde{E}^*$. Under the iterated structure isomorphisms, this composite is identified with the suspension–loop adjunction isomorphism $\pi_3 \mathrm{Hom}_{\mathrm{An}_*}(X, E_{k+3}) \cong \pi_2 \mathrm{Hom}_{\mathrm{An}_*}(\Sigma X, E_{k+3})$. Hence it is an isomorphism of abelian groups.

A morphism of spectra $f\colon E \to F$ induces a natural transformation $\widetilde{E}^* \to \widetilde{F}^*$ by postcomposition with the maps $f_k\colon E_k \to F_k$. These maps are homomorphisms for the abelian group structures above, and the specified identifications in Definition 3.2.2 ensure that the transformation commutes with the suspension isomorphisms.

Proposition 3.2.4. *For every spectrum E, the pair $(\widetilde{E}^*, \sigma)$ is a reduced cohomology theory.*

Proof. The suspension isomorphism was constructed above. For a collection $(X_i)_{i\in I}$ of pointed animae, the universal property of the wedge gives an isomorphism

$$\widetilde{E}^k\left(\bigvee_{i\in I} X_i\right) = \left[\bigvee_{i\in I} X_i, E_k\right]_* \cong \prod_{i\in I} [X_i, E_k]_* = \prod_{i\in I} \widetilde{E}^k(X_i),$$

which is the wedge axiom. Finally, Lemma 2.4.5 shows that every cofiber sequence $X \to Y \to Z$ induces an exact sequence

$$[Z, E_k]_* \longrightarrow [Y, E_k]_* \longrightarrow [X, E_k]_*$$

of pointed sets. Since these maps are homomorphisms for the abelian group structures constructed above, this proves the exactness axiom. □

Remark 3.2.5. After constructing the tensor product of spectra, we will associate to E a reduced homology theory by the formula

$$\widetilde{E}_k(X) := \pi_k(\Sigma^\infty X \otimes E);$$

we refer to Section 4.4.5 for details. The construction of generalized homology theories from spectra was developed by Whitehead [Whi62].

Theorem 3.2.6 (Brown representability, Brown [Bro62], Switzer [Swi75, Theorems 9.27–9.28])**.** *For every reduced cohomology theory (h^*, σ), there is a spectrum E and a natural isomorphism $h^* \xrightarrow{\sim} \widetilde{E}^*$ compatible with the suspension isomorphisms. Moreover, every natural transformation $\widetilde{E}^* \to \widetilde{F}^*$ of reduced cohomology theories that commutes with the suspension isomorphisms is induced by a morphism of spectra $E \to F$.*

The first assertion is a standard consequence of Brown's representability theorem. After translating Brown's result to the language of animae, apply it degreewise to the restriction of h^k to connected pointed animae, and let Y_k be a connected representing anima. Since ΣX is connected for every pointed anima X, the suspension isomorphism gives natural isomorphisms

$$h^k(X) \cong h^{k+1}(\Sigma X) \cong [\Sigma X, Y_{k+1}]_* \cong [X, \Omega Y_{k+1}]_*.$$

Thus we may take $E_k := \Omega Y_{k+1}$. The suspension isomorphisms induce maps $Y_{k+1} \to \Omega Y_{k+2}$ that are isomorphisms on all positive homotopy groups; looping them gives the required isomorphisms $E_k \xrightarrow{\sim} \Omega E_{k+1}$. The final assertion follows degreewise from the Yoneda lemma, as in Exercise 3.1. A complete proof is given in Section 3.4.

3.3 Ordinary homology and cohomology

Ordinary homology and cohomology are the generalized theories singled out by the dimension axiom; see Hatcher [Hat02, Theorem 4.59]. In this chapter we use them axiomatically, postponing their intrinsic construction to Section 6.2. The online version continues with the cellular computation of ordinary homology from a CW-structure.

Theorem 3.3.1 (Existence of ordinary homology and cohomology)**.** *For every abelian group A, there exist reduced homology and cohomology theories*

$$\widetilde{H}_*(-;A)\colon \mathrm{An}_* \to \mathrm{Ab}^{\mathbb{Z}} \qquad \textit{and} \qquad \widetilde{H}^*(-;A)\colon \mathrm{An}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$$

satisfying the dimension axiom

$$\widetilde{H}_n(S^0;A) \cong \begin{cases} A & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases} \qquad \textit{and} \qquad \widetilde{H}^n(S^0;A) \cong \begin{cases} A & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The associated unreduced theories are denoted $H_*(-;A)$ and $H^*(-;A)$. When $A = \mathbb{Z}$, we usually omit it from the notation.

Remark 3.3.2 (Singular (co)homology)**.** Given a topological space X, one can write down an explicit chain complex of abelian groups $C_*^{\mathrm{sing}}(X;\mathbb{Z})$ called the *singular chain complex of X*. The n-th term of this chain complex is given by

$$C_n^{\mathrm{sing}}(X;\mathbb{Z}) := \mathbb{Z}[\mathrm{Sing}_n(X)],$$

the free abelian group generated by the set $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$ of continuous maps from the geometric n-simplex to X. The boundary map $d_n\colon C_n^{\mathrm{sing}}(X;\mathbb{Z}) \to C_{n-1}^{\mathrm{sing}}(X;\mathbb{Z})$ sends a generating n-simplex σ to the alternating sum $\sum_{i=0}^{n}(-1)^i d_i(\sigma)$ of its faces.

For an abelian group A, tensoring with A and taking hom into A produces chain and cochain complexes

$$C_*^{\mathrm{sing}}(X;A) := C_*^{\mathrm{sing}}(X;\mathbb{Z}) \otimes A \qquad \text{and} \qquad C^*_{\mathrm{sing}}(X;A) := \mathrm{Hom}_{\mathrm{Ab}}(C_*^{\mathrm{sing}}(X;\mathbb{Z}), A).$$

Their homology groups are the classical singular homology and cohomology groups of X with coefficients in A. This is the traditional construction of ordinary (co)homology. The fact that this satisfies the axioms is non-trivial.

In this book, we will not take singular chains as the definition, because it depends on choosing a topological presentation of an anima. Instead, in Section 6.2 we will construct ordinary homology and cohomology intrinsically using the derived category $\mathrm{D}(\mathbb{Z})$. There we show that, for a topological space X, the intrinsic ordinary (co)homology of the underlying anima $\Pi_\infty(X)$ agrees with the singular (co)homology of X.

Remark 3.3.3. The dimension axiom implies the usual values on spheres. Namely, since $S^n \simeq \Sigma^n S^0$ as pointed animae, the suspension isomorphism gives

$$\widetilde{H}_k(S^n; A) \cong \begin{cases} A & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases}$$

and similarly for cohomology.

The next chapter organizes spectra and their morphisms into an ∞-category. Before turning to it, the online version continues with an optional cellular computation of ordinary homology, followed by a proof of Brown representability. Neither is needed later.

3.3.1 Cellular homology (online only)

Convention 3.3.4. Throughout this section, we write D^n for the n-disk and S^{n-1} for its boundary sphere, viewed as animae. While contractibility of the disk implies that $D^n \simeq \mathrm{pt}$, this notation emphasizes the analogy to cell attachments in CW-complexes. Given a map $f\colon A \to X$ of animae, we write X/A for the cofiber of $A_+ \to X_+$.

Definition 3.3.5 (CW-structure)**.** A *CW-structure* on an anima X consists of a sequence of animae

$$\emptyset = X^{-2} = X^{-1} \to X^0 \to X^1 \to X^2 \to \cdots$$

called the *skeleta* of X, with X^0 a set, together with an equivalence $X \simeq \mathrm{colim}_n X^n$ and, for each $n \geq 1$, a set I_n of *n-cells* and a pushout square of animae

$$\begin{array}{ccc} \coprod_{\alpha \in I_n} S^{n-1} & \xrightarrow{\varphi} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\alpha \in I_n} D^n & \longrightarrow & X^n. \end{array}$$

The map φ is called the *attaching map* for the n-cells. We say that X is *finite-dimensional* if there are no n-cells for sufficiently large n, and *finite* if it has finitely many cells in total.

Remark 3.3.6 (Cellular approximation)**.** We will freely use the *cellular approximation theorem*: every pointed map between CW-complexes with basepoints that are 0-cells is pointed homotopic to a cellular map; see Hatcher [Hat02, Theorem 4.8]. Together with the comparison of classical and intrinsic homotopy groups in Remark 2.4.19, it has the following consequence for an anima X with a CW-structure, which is all we will need. The inclusion of the n-skeleton induces a map

$$[S^k, X^n]_* \to [S^k, X]_*$$

that is a bijection for $k < n$ and a surjection for $k = n$; the same holds for the inclusion $X^n \to X^m$ of skeleta with $m > n$. More generally, if W is obtained from an anima Z by attaching cells of dimension $\geq m$, then $[S^k, Z]_* \to [S^k, W]_*$ is a bijection for $k < m - 1$ and a surjection for $k = m - 1$. Equivalently, every pointed map $S^k \to W$ can be identified with one factoring through Z once $k \leq m - 1$, and two such maps that can be identified in W can already be identified after attaching the cells of dimension m.

Example 3.3.7. The results from Chapter 2 show: for a topological CW-complex X with skeleta X^n, its underlying anima $\Pi_\infty(X)$ obtains a CW-structure with skeleta $\Pi_\infty(X^n)$.

Example 3.3.8. For $n \geq 1$, the n-sphere S^n admits a CW-structure with one 0-cell and one n-cell: we have a pushout square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathrm{pt} \\ \downarrow & \ulcorner & \downarrow \\ D^n & \longrightarrow & S^n. \end{array}$$

The key to cellular homology is understanding how homology changes as we pass from one skeleton to the next. Recall from Section 3.1 that the relative groups are defined by $H_k(X, A) := \widetilde{H}_k(X/A)$ and fit into the long exact sequence of Exercise 3.1.4.

Lemma 3.3.9. *Let X be an anima with a CW-structure. For all $k, n \in \mathbb{Z}$ with $n \geq 0$:*

(a) The relative homology $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with one generator for each n-cell of X.

(b) $H_k(X^n) = 0$ for $k > n$.

(c) The map $H_k(X^n) \to H_k(X)$ induced by the structure map $X^n \to X$ is an isomorphism for $k < n$ and surjective for $k = n$.

Proof. For part (a), the case $n = 0$ follows because X^0 is a disjoint union of points. For $n \geq 1$, the pushout square defining X^n from X^{n-1} shows that

$$X^n/X^{n-1} \simeq \bigvee_{\alpha \in I_n} D^n/S^{n-1} \simeq \bigvee_{\alpha \in I_n} S^n,$$

a wedge of n-spheres indexed by the n-cells. Therefore

$$H_k(X^n, X^{n-1}) = \widetilde{H}_k\Big(\bigvee_{\alpha \in I_n} S^n\Big) \cong \bigoplus_{\alpha \in I_n} \widetilde{H}_k(S^n) \cong \begin{cases} \bigoplus_{\alpha \in I_n} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

For part (b), the claim is clear for $n = 0$: we have $H_k(X^0) = 0$ for $k > 0$ as X^0 is a disjoint union of points, which have trivial homology. Consider now the long exact sequence of the pair (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}).$$

By part (a), the first term vanishes when $k \neq n - 1$ and the last term vanishes when $k \neq n$. In particular, for $k > n$ the map $H_k(X^{n-1}) \to H_k(X^n)$ is an isomorphism and the claim follows

by induction. For part (c), we may similarly use the long exact sequence of the pair (X^{n+1}, X^n) to conclude that the map $H_k(X^n) \to H_k(X^{n+1})$ is surjective for $k \leq n$ and injective (hence an isomorphism) for $k < n$.

It remains to pass from the skeleta to their colimit. Adding a disjoint basepoint commutes with colimits. Apply the pushout presentation of Lemma 2.4.29 to the sequence $(X^r)_+$. Since the resulting square is a pushout, the cofiber of its top horizontal map is isomorphic to the cofiber of its bottom horizontal map. Comparing the corresponding long exact sequences gives the Mayer–Vietoris sequence of the pushout. By the wedge axiom, and after multiplying the odd source summands by -1, the relevant portion is

$$\bigoplus_{r\geq 0} H_k(X^r) \xrightarrow{\delta_k} \bigoplus_{r\geq 0} H_k(X^r) \longrightarrow H_k(X) \longrightarrow \bigoplus_{r\geq 0} H_{k-1}(X^r) \xrightarrow{\delta_{k-1}} \bigoplus_{r\geq 0} H_{k-1}(X^r).$$

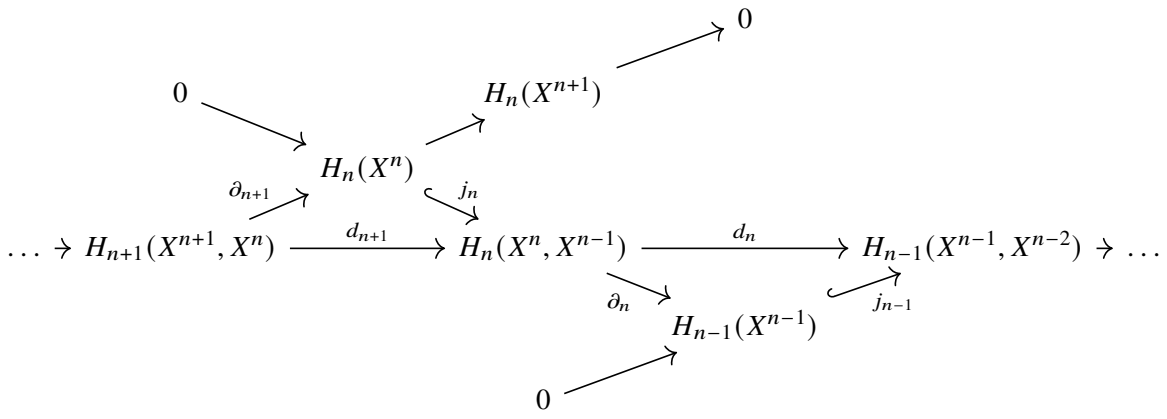
Writing ι_r for the inclusion of the r-th summand, the first map is given by

$$\delta_k(\iota_r(x)) = \iota_r(x) - \iota_{r+1}((X^r \to X^{r+1})_*(x)).$$

The map δ_k is injective: if an element of the direct sum lies in its kernel, its components vanish successively, starting with the zeroth component. Its cokernel is the algebraic colimit $\operatorname{colim}_r H_k(X^r)$. The same injectivity statement for δ_{k-1} therefore identifies the middle map with an isomorphism $\operatorname{colim}_r H_k(X^r) \xrightarrow{\sim} H_k(X)$. The claim now follows from the stabilization properties established above. □

We now construct the cellular chain complex.

Construction 3.3.10 (Cellular chain complex)**.** Let X be an anima with a CW-structure. Using Lemma 3.3.9, portions of the long exact sequences for the pairs (X^{n+1}, X^n), (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}) fit into the following commutative diagram:



Here the maps j_n and ∂_n come from the long exact sequence of the pair (X^n, X^{n-1}). Since $H_n(X^{n-1}) = 0$ by Lemma 3.3.9(b), the map j_n is injective for every n. We define the *cellular boundary maps* as the compositions

$$d_n \;:=\; j_{n-1} \circ \partial_n \colon H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2}).$$

Since $\partial_n \circ j_n = 0$ (as these are consecutive maps in an exact sequence), we have $d_n \circ d_{n+1} = 0$. Thus the groups

$$C_n^{\mathrm{CW}}(X) := H_n(X^n, X^{n-1})$$

with differentials d_n form a chain complex, called the *cellular chain complex* of X.

By Lemma 3.3.9(a), the cellular chain group $C_n^{\mathrm{CW}}(X)$ is a free abelian group with one generator for each n-cell of X. Choose orientations of the characteristic disks and their boundary spheres compatibly. We write $e_\alpha^n \in C_n^{\mathrm{CW}}(X)$ for the resulting generator corresponding to the n-cell indexed by $\alpha \in I_n$.

Definition 3.3.11 (Cellular homology)**.** The *cellular homology groups* of an anima X with CW-structure are the homology groups of the cellular chain complex:

$$H_n^{\mathrm{CW}}(X) \;:=\; H_n(C_*^{\mathrm{CW}}(X)) \;=\; \ker(d_n)/\mathrm{im}(d_{n+1}).$$

The main theorem of this section is that cellular homology agrees with ordinary homology:

Theorem 3.3.12. *For any anima X with a CW-structure and any $n \geq 0$, there is an isomorphism*

$$H_n^{\mathrm{CW}}(X) \;\cong\; H_n(X).$$

Proof. By Lemma 3.3.9(c), the map $H_n(X^{n+1}) \to H_n(X)$ is an isomorphism. Combining this with the previous commutative diagram, we conclude that $H_n(X)$ is isomorphic to the cokernel of the map $\partial_{n+1}\colon H_{n+1}(X^{n+1}, X^n) \to H_n(X^n)$. By injectivity of j_n, the target $H_n(X^n)$ of this map may be identified with the kernel of $\partial_n\colon H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$, which in turn may be identified with $\ker(d_n)$ because of the injectivity of j_{n-1}. Under the resulting isomorphism $j_n\colon H_n(X^n) \xrightarrow{\cong} \ker(d_n)$, the image of ∂_{n+1} gets mapped to the image of $d_{n+1} = j_n\partial_{n+1}$, and we conclude that

$$H_n(X) \;\cong\; \ker(d_n)/\mathrm{im}(d_{n+1}) \;=\; H_n^{\mathrm{CW}}(X). \qquad \square$$

We leave the following easy consequences to the reader:

Exercise 3.3.13. Let X be an anima with a CW-structure.

(1) If X has no n-cells, then $H_n(X) = 0$.

(2) If X has k cells in dimension n, then $H_n(X)$ is generated by at most k elements.

(3) If X has no two cells in adjacent dimensions, then $H_n(X)$ is free abelian with rank equal to the number of n-cells.

Example 3.3.14. Complex projective space $\mathbb{C}\mathrm{P}^n$ has a filtration whose successive strata are the affine cells $\mathbb{C}^0, \mathbb{C}^1, \ldots, \mathbb{C}^n$, and hence a CW-structure with one cell in each even dimension $0, 2, 4, \ldots, 2n$. Since no two cells are in adjacent dimensions, Exercise 3.3.13(3) gives

$$H_k(\mathbb{C}\mathrm{P}^n) \;\cong\; \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.3.15. For $n, m \geq 1$ with $n \neq m$, the product $S^n \times S^m$ has a CW-structure with one 0-cell, one n-cell, one m-cell, and one $(n+m)$-cell. If $n, m \geq 2$ and $|n - m| > 1$, then no two cells are in adjacent dimensions, so

$$H_k(S^n \times S^m) \;\cong\; \begin{cases} \mathbb{Z} & \text{if } k = 0, n, m, \text{ or } n+m, \\ 0 & \text{otherwise.} \end{cases}$$

In the remaining cases, some cells lie in adjacent dimensions, but the corresponding cellular boundary maps are zero, giving the same answer.

To perform explicit computations, we need a formula for the cellular boundary maps in terms of the attaching maps of the cells. The formula involves the degree of maps between spheres.

Remark 3.3.16 (Degree)**.** For every $n \geq 1$, a map $f\colon S^n \to S^n$ induces multiplication by a unique integer on $H_n(S^n) \cong \mathbb{Z}$. This integer is called the *degree* of f. For pointed maps, this agrees with the degree recorded in Remark 2.4.20.

Proposition 3.3.17 (Cellular boundary formula)**.** *Let X be an anima with a CW-structure, and let $n \geq 2$. Then the cellular boundary map is given at the generator e^n_α by*

$$d_n(e^n_\alpha) \;=\; \sum_{\beta \in I_{n-1}} d_{\alpha\beta} \cdot e^{n-1}_\beta,$$

where the sum is over the set of $(n-1)$-cells, and $d_{\alpha\beta}$ is the degree of the composite map

$$\Delta_{\alpha\beta}\colon S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \simeq \bigvee_{\gamma \in I_{n-1}} S^{n-1}_\gamma \xrightarrow{p_\beta} S^{n-1}_\beta.$$

Here q is the quotient map and p_β is the projection onto the β-th wedge summand.

Only finitely many coefficients $d_{\alpha\beta}$ are nonzero, since $d_n(e^n_\alpha)$ belongs to the direct sum $C^{\mathrm{CW}}_{n-1}(X)$.

Proof. Recall that d_n is the composite

$$\widetilde{H}_n(X^n/X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}),$$

where ∂_n is the connecting homomorphism and $j_{n-1} = q_*$ is induced by the quotient map q. Let

$$\iota_\alpha\colon S^n \;\hookrightarrow\; \bigvee_{\gamma \in I_n} S^n_\gamma \;\cong\; X^n/X^{n-1}$$

be the inclusion of the α-th wedge summand. The generator $e^n_\alpha \in \widetilde{H}_n(X^n/X^{n-1})$ is the image of the generator $[\iota_n] \in \widetilde{H}_n(S^n)$ under the map on homology induced by ι_α. The map ι_α may alternatively be obtained from the attaching map $\varphi_\alpha\colon S^{n-1} \to X^{n-1}$ by passing to cofibers in the following diagram:

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & D^n & \longrightarrow & S^n \\ {\scriptstyle\varphi_\alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle\iota_\alpha} \\ X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^n/X^{n-1}. \end{array}$$

By the naturality of the boundary map, it follows that the top square in the following diagram commutes:

$$\begin{array}{ccccc} \widetilde{H}_n(S^n) & \xrightarrow[\cong]{\partial_n} & \widetilde{H}_{n-1}(S^{n-1}) & & \\ {\scriptstyle(\iota_\alpha)_*}\downarrow & & \downarrow{\scriptstyle(\varphi_\alpha)_*} & \searrow^{(\Delta_{\alpha\beta})_*} & \\ \widetilde{H}_n(X^n/X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}) & & \\ & {\scriptstyle d_n}\searrow & \downarrow{\scriptstyle q_*} & & \\ & & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) & \xrightarrow{(p_\beta)_*} & \widetilde{H}_{n-1}(S^{n-1}_\beta). \end{array}$$

It follows that the image under d_n of the generator e^n_α is the same as the image of $[\iota_{n-1}] \in \widetilde{H}_{n-1}(S^{n-1})$ of the middle vertical map. Under the isomorphism $\widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \cong \bigoplus_{\gamma \in I_{n-1}} \widetilde{H}_{n-1}(S^{n-1}_\gamma)$, the β-th component is obtained by applying the projection $(p_\beta)_*$. We conclude that $d_{\alpha\beta}$ is the coefficient of the map $(p_\beta \circ q \circ \varphi_\alpha)_* = (\Delta_{\alpha\beta})_* \colon H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$. By Remark 3.3.16, this is $\deg(\Delta_{\alpha\beta})$. □

For $n = 1$, the boundary of a 1-cell is the difference of the classes of its two endpoints. This follows directly from the connecting homomorphism for the pair (X^1, X^0). In particular, this boundary vanishes when X^0 consists of a single point.

Example 3.3.18 (Orientable surfaces)**.** Let M_g be the closed orientable surface of genus g. It admits a CW-structure with one 0-cell, $2g$ one-cells $a_1, b_1, \ldots, a_g, b_g$, and one 2-cell attached along the word $[a_1, b_1] \cdots [a_g, b_g] = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. The cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0.$$

The boundary d_1 is zero because each one-cell starts and ends at the unique zero-cell. For d_2, each generator a_i appears in the attaching word with total exponent $1 + (-1) = 0$, and similarly for b_i. Thus each map $\Delta_{\alpha\beta}$ has degree zero, giving $d_2 = 0$. Since both boundary maps vanish, we obtain $H_0(M_g) \cong \mathbb{Z}$, $H_1(M_g) \cong \mathbb{Z}^{2g}$, and $H_2(M_g) \cong \mathbb{Z}$.

Example 3.3.19 (Non-orientable surfaces)**.** For $g \geq 1$, let N_g be the closed non-orientable surface of genus g (the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$). It admits a CW-structure with one 0-cell, g one-cells $a_1, \ldots, a_g$, and one 2-cell attached along the word $a_1^2 a_2^2 \cdots a_g^2$. The cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{g} \xrightarrow{d_1} \mathbb{Z} \to 0.$$

Again $d_1 = 0$, since each one-cell starts and ends at the unique zero-cell. For d_2, each a_i appears with total exponent 2, so the degree of each $\Delta_{\alpha\beta}$ equals 2. Thus $d_2(1) = (2, 2, \ldots, 2) \in \mathbb{Z}^g$. Since d_2 is injective, $H_2(N_g) = 0$. To compute $H_1(N_g) = \ker(d_1)/\mathrm{im}(d_2) = \mathbb{Z}^g/\langle(2, \ldots, 2)\rangle$, we change basis: replacing the last standard basis vector $(0, \ldots, 0, 1)$ by $(1, \ldots, 1)$, we see that $H_1(N_g) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\,\mathbb{Z}$.

Example 3.3.20 (Real projective space)**.** Real projective space $\mathbb{R}\mathrm{P}^n$ has a CW-structure with one cell in each dimension $0, 1, \ldots, n$. The attaching map $\varphi_k \colon S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$ for the k-cell is the quotient map identifying antipodal points. For $k \geq 2$, one may compute using geometric methods that the composite $\Delta \colon S^{k-1} \xrightarrow{\varphi_k} \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} \simeq S^{k-1}$ has degree $1 + (-1)^k$: the two preimages of a generic point contribute with the same sign when k is even and opposite signs when k is odd. For $k = 1$, both endpoints attach to the unique 0-cell, so $d_1 = 0$. Thus, for $k \geq 1$, the cellular boundary maps are

$$d_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

This gives

$$H_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}/2\,\mathbb{Z} & \text{if } 0 < k < n \text{ and } k \text{ is odd,} \\ \mathbb{Z} & \text{if } k = n \text{ and } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

3.4 Brown representability (online only)

We now prove the Brown representability theorem stated in Theorem 3.2.6. The proof has two parts. We first prove Brown's general criterion for a set-valued functor on connected pointed animae. We then apply it degreewise to a reduced cohomology theory and assemble the representing animae into a spectrum.

3.4.1 The representability theorem

Write $\mathrm{An}_{*,\geq 1}$ for the full subcategory of An_* spanned by the connected pointed animae. Since π_0 preserves colimits, this subcategory is closed under wedges and pushouts.

Definition 3.4.1 (Brown's conditions)**.** Let

$$F\colon \mathrm{Ho}(\mathrm{An}_{*,\geq 1})^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

be a functor. We say that:

(1) The functor F is *representable* if it is naturally isomorphic to $[-, Z]_*$ for some $Z \in \mathrm{An}_{*,\geq 1}$;

(2) The functor F satisfies the *wedge axiom* if, for every small collection $(X_i)_{i\in I}$ of connected pointed animae, the canonical map

$$F\left(\bigvee_{i\in I} X_i\right) \longrightarrow \prod_{i\in I} F(X_i)$$

is a bijection;

(3) The functor F satisfies the *Mayer–Vietoris property* if every pushout square

$$\begin{array}{ccc} C & \xrightarrow{k} & A \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[j]{} & X \end{array}$$

in $\mathrm{An}_{*,\geq 1}$ induces a surjection

$$(i^*, j^*)\colon F(X) \longrightarrow F(A) \times_{F(C)} F(B).$$

The wedge axiom for the empty collection implies that $F(\mathrm{pt})$ is a singleton. It also gives the values of F on suspensions an algebraic structure which will be used in the proof.

Remark 3.4.2. Suppose that F satisfies the wedge axiom. The cogroup structure on ΣX from Remark 2.4.21 makes $F(\Sigma X)$ a group: the pinch map induces its multiplication, while the collapse and reflection maps induce its unit and inverses. The two cogroup structures on $\Sigma^2 X$ agree by the Eckmann–Hilton argument, so $F(\Sigma^2 X)$ is an abelian group. A natural transformation between two functors satisfying the wedge axiom preserves these group structures.

Theorem 3.4.3 (Brown representability, Brown [Bro62])**.** *A functor* $F\colon \mathrm{Ho}(\mathrm{An}_{*,\geq 1})^{\mathrm{op}} \to \mathrm{Set}$ *is representable if and only if it satisfies the wedge axiom and the Mayer–Vietoris property.*

For a connected pointed anima Z and an element $\xi \in F(Z)$, the Yoneda lemma associates to ξ a natural transformation

$$T_\xi \colon [X, Z]_* \longrightarrow F(X), \qquad [f] \longmapsto f^*(\xi).$$

Brown's construction produces a representing object by successively improving the behavior of this transformation on spheres.

Definition 3.4.4 (Universal element)**.** Let $n \geq 1$. An element $\xi \in F(Z)$ is *n-universal* if

$$T_\xi \colon [S^k, Z]_* \longrightarrow F(S^k)$$

is a bijection for $1 \leq k < n$ and a surjection for $k = n$. It is *universal* if this map is a bijection for every $k \geq 1$.

Proposition 3.4.5 (Universal approximation)**.** *Let F satisfy the wedge axiom and the Mayer–Vietoris property. For every connected pointed anima X and every $\eta \in F(X)$, there are a connected pointed anima Z_X, a pointed map $f_X \colon X \to Z_X$, and a universal element $\xi_X \in F(Z_X)$ such that $f_X^*(\xi_X) = \eta$.*

Proof. Step 1: Constructing increasingly universal elements. We construct a sequence

$$X =: Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} Z_2 \longrightarrow \cdots$$

and elements $\xi_n \in F(Z_n)$ such that $f_n^*(\xi_{n+1}) = \xi_n$, the element ξ_n is n-universal for $n \geq 1$, and T_{ξ_n} is surjective on S^k for every $n \geq 1$ and $k \geq 1$.

Set $\xi_0 := \eta$ and define

$$Z_1 := X \vee \bigvee_{m \geq 1} \bigvee_{\gamma \in F(S^m)} S^m.$$

By the wedge axiom, there is a unique element $\xi_1 \in F(Z_1)$ whose restriction to X is η and whose restriction to the sphere indexed by $\gamma \in F(S^m)$ is γ. The inclusion of that sphere shows that T_{ξ_1} is surjective on S^m for every $m \geq 1$. In particular, ξ_1 is 1-universal.

Suppose now that $n \geq 1$ and that (Z_n, ξ_n) has been constructed. By Remark 3.4.2, the map

$$T_{\xi_n} \colon [S^n, Z_n]_* \longrightarrow F(S^n)$$

is a group homomorphism. Let K_n denote its kernel. Choose a representative $g \colon S^n \to Z_n$ of every class $[g] \in K_n$, and form the pushout

$$\begin{array}{ccc} \displaystyle\bigvee_{[g] \in K_n} S^n & \xrightarrow{\vee g} & Z_n \\ \downarrow & & \downarrow{\scriptstyle f_n} \\ \mathrm{pt} & \longrightarrow & Z_{n+1}. \end{array}$$

The restrictions of ξ_n to all the attaching spheres are zero. The Mayer–Vietoris property therefore supplies an element $\xi_{n+1} \in F(Z_{n+1})$ satisfying $f_n^*(\xi_{n+1}) = \xi_n$.

The map $T_{\xi_{n+1}}$ remains surjective on every sphere, since T_{ξ_n} is surjective and $\xi_n = f_n^* \xi_{n+1}$. Attaching $(n+1)$-cells induces bijections

$$[S^k, Z_n]_* \xrightarrow{\cong} [S^k, Z_{n+1}]_*$$

for $k < n$, so ξ_{n+1} remains n-universal. Finally, every map $S^n \to Z_{n+1}$ can be identified with a composite through Z_n. If its class lies in the kernel of $T_{\xi_{n+1}}$, a lift to Z_n lies in K_n and hence becomes null-homotopic after the corresponding cell is attached. Thus $T_{\xi_{n+1}}$ is injective on S^n, proving that ξ_{n+1} is $(n+1)$-universal.

Step 2: Passing to the limit. Set

$$Z_X := \operatorname{colim}_n Z_n.$$

The mapping-telescope description from Lemma 2.4.29 gives a pushout square

$$\begin{array}{ccc} \bigvee_{n\geq 0} Z_n & \longrightarrow & \bigvee_{r\geq 0} Z_{2r} \\ \downarrow & & \downarrow \\ \bigvee_{r\geq 0} Z_{2r+1} & \longrightarrow & Z_X. \end{array}$$

On the summand Z_n, the two maps are the identity and the transition map to the adjacent stage, assigned to the even and odd target wedges according to the parity of n. The relations $f_n^*(\xi_{n+1}) = \xi_n$, together with the wedge axiom and the Mayer–Vietoris property, therefore produce an element $\xi_X \in F(Z_X)$ restricting to ξ_n on every stage. In particular, the composite $f_X \colon X = Z_0 \to Z_X$ satisfies $f_X^*(\xi_X) = \eta$.

For a fixed $k \geq 1$, the map $Z_n \to Z_X$ is obtained by attaching cells of dimension at least $n+1$. Hence Remark 3.3.6 shows that

$$[S^k, Z_n]_* \longrightarrow [S^k, Z_X]_*$$

is a bijection once $n > k$. Since ξ_n is n-universal, T_{ξ_X} is consequently a bijection on S^k. Thus ξ_X is universal. □

Proof of Theorem 3.4.3. If $F \cong [-, Z]_*$ is representable, the wedge axiom follows from the universal property of coproducts. Mapping a pushout into Z gives a pullback of mapping animae. A pair of homotopy classes whose restrictions to the intersection agree can be represented together with an identification between those restrictions, and therefore lifts to a point of this pullback. This proves the Mayer–Vietoris property.

Conversely, suppose that F satisfies Brown's conditions. Apply Proposition 3.4.5 to $X = \mathrm{pt}$ and the unique element of $F(\mathrm{pt})$. We obtain a connected pointed anima Z and a universal element $\xi \in F(Z)$. We claim that

$$T_\xi \colon [X, Z]_* \longrightarrow F(X)$$

is a bijection for every connected pointed anima X.

For surjectivity, let $\eta \in F(X)$ and let $\widetilde{\eta} \in F(X \vee Z)$ correspond under the wedge axiom to (η, ξ). Applying Proposition 3.4.5 to $(X \vee Z, \widetilde{\eta})$ produces a universal element $\widetilde{\xi} \in F(\widetilde{Z})$ and a map $X \vee Z \to \widetilde{Z}$ carrying $\widetilde{\xi}$ to $\widetilde{\eta}$. Its restriction $g \colon Z \to \widetilde{Z}$ induces isomorphisms on all positive homotopy groups because both ξ and $\widetilde{\xi}$ are universal. Since both animae are connected, g is an isomorphism by Proposition 2.4.22. Composing the restriction $X \to \widetilde{Z}$ with an inverse of g gives a class in $[X, Z]_*$ which maps to η.

For injectivity, suppose that $f, g\colon X \to Z$ satisfy $f^*\xi = g^*\xi =: \eta$. Form the pushout

$$\begin{array}{ccc} X \vee X & \xrightarrow{(f,g)} & Z \\ {\scriptstyle\nabla}\downarrow & & \downarrow \\ X & \longrightarrow & \widetilde{X}, \end{array}$$

where ∇ is the fold map. The Mayer–Vietoris property gives an element $\widetilde{\eta} \in F(\widetilde{X})$ restricting to ξ on Z and to η on X. Apply Proposition 3.4.5 to $(\widetilde{X}, \widetilde{\eta})$. As in the surjectivity argument, the resulting composite $Z \to \widetilde{X} \to \widetilde{Z}$ is an isomorphism because it carries one universal element to another. The pushout square then shows, after composing with its inverse, that both f and g are identified with the same map $X \to Z$. Hence $[f] = [g]$. □

3.4.2 From representing animae to spectra

We now apply the abstract theorem to reduced cohomology theories. The only condition which does not follow immediately from the axioms is the Mayer–Vietoris property.

Lemma 3.4.6. *Let (h^*, σ) be a reduced cohomology theory. For each $n \in \mathbb{Z}$, the restriction*

$$h^n\colon \mathrm{Ho}(\mathrm{An}_{*,\geq 1})^{\mathrm{op}} \longrightarrow \mathrm{Ab} \longrightarrow \mathrm{Set}$$

satisfies the Mayer–Vietoris property.

Proof. Consider a pushout square of connected pointed animae

$$\begin{array}{ccc} C & \xrightarrow{k} & A \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[j]{} & X. \end{array}$$

The pushout property gives an isomorphism $\mathrm{cofib}(k) \xrightarrow{\sim} \mathrm{cofib}(j)$. The long exact cohomology sequences of Exercise 3.1.3 therefore fit into a commutative diagram

$$\begin{array}{ccccccccc} h^n(\mathrm{cofib}(j)) & \longrightarrow & h^n(X) & \xrightarrow{j^*} & h^n(B) & \longrightarrow & h^{n+1}(\mathrm{cofib}(j)) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle l^*} & & \downarrow{\scriptstyle\cong} \\ h^n(\mathrm{cofib}(k)) & \longrightarrow & h^n(A) & \xrightarrow{k^*} & h^n(C) & \longrightarrow & h^{n+1}(\mathrm{cofib}(k)). \end{array}$$

Suppose that $\alpha \in h^n(A)$ and $\beta \in h^n(B)$ satisfy $k^*\alpha = l^*\beta$. The boundary of β vanishes, since its image under the right-hand isomorphism is the boundary of $l^*\beta = k^*\alpha$. Hence β lifts to some $\gamma_0 \in h^n(X)$. The element $\alpha - i^*\gamma_0$ lies in the kernel of k^*, so it comes from $h^n(\mathrm{cofib}(k))$. Transporting a preimage across the left-hand isomorphism and then mapping it to $h^n(X)$ gives an element γ_1 whose restriction to B vanishes and whose restriction to A is $\alpha - i^*\gamma_0$. Thus $\gamma_0 + \gamma_1$ restricts to (α, β), proving the required surjectivity. □

Proof of Theorem 3.2.6. Let (h^*, σ) be a reduced cohomology theory. For each $m \geq 0$, the restriction of h^m to connected pointed animae satisfies the wedge axiom by definition and the Mayer–Vietoris property by Lemma 3.4.6. The abstract Brown representability theorem, Theorem 3.4.3, gives a connected pointed anima Y_m and a natural isomorphism

$$h^m(X) \cong [X, Y_m]_*$$

on connected pointed animae.

Set

$$E_m := \Omega Y_{m+1} \qquad (m \geq 0).$$

Since ΣX is connected for every pointed anima X, the suspension isomorphism gives natural isomorphisms

$$h^m(X) \cong h^{m+1}(\Sigma X) \cong [\Sigma X, Y_{m+1}]_* \cong [X, E_m]_*$$

for every pointed anima X. Thus E_m represents h^m on all of An_*. The same calculation shows that ΩE_{m+1} also represents h^m. The Yoneda lemma therefore supplies an isomorphism

$$E_m \xrightarrow{\sim} \Omega E_{m+1}$$

compatible with the given suspension isomorphism. These isomorphisms make $(E_m)_{m\geq 0}$ into a spectrum E. The representing isomorphisms respect the abelian group structures: after double suspension, both group laws are induced by the pinch map. They identify h^k with $\widetilde{E}^k$ for $k \geq 0$, and for $k < 0$ we obtain

$$\widetilde{E}^k(X) = [X, \Omega^{-k} E_0]_* \cong h^0(\Sigma^{-k} X) \cong h^k(X).$$

This proves the first assertion.

Now let $\alpha \colon \widetilde{E}^* \to \widetilde{F}^*$ be a natural transformation commuting with suspension. For each $m \geq 0$, the degree-m component is a natural transformation

$$[-, E_m]_* \longrightarrow [-, F_m]_*,$$

so Exercise 3.1 identifies it with postcomposition by a unique homotopy class of pointed maps $f_m \colon E_m \to F_m$. Compatibility with suspension says precisely that the squares

$$\begin{array}{ccc} E_m & \xrightarrow{\sim} & \Omega E_{m+1} \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle \Omega f_{m+1}} \\ F_m & \xrightarrow{\sim} & \Omega F_{m+1} \end{array}$$

commute in $\mathrm{Ho}(\mathrm{An}_*)$. Choosing representatives and the corresponding identifications gives a morphism of spectra $f \colon E \to F$ in the sense of Definition 3.2.2. Its induced natural transformation is α in nonnegative degrees, and hence in every degree by compatibility with suspension. □

Exercise 3.4.7 (Mayer–Vietoris sequence)**.** Let $F \colon \mathrm{Ho}(\mathrm{An}_{*,\geq 1})^{\mathrm{op}} \to \mathrm{Ab}$ satisfy the wedge axiom and the Mayer–Vietoris property, and consider a pushout square of connected pointed animae

$$\begin{array}{ccc} C & \xrightarrow{k} & A \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[j]{} & X. \end{array}$$

Show that the cofiber of $(i, j)\colon A \vee B \to X$ is isomorphic to ΣC, and use the coPuppe sequence to construct a long exact Mayer–Vietoris sequence

$$\cdots \to F(\Sigma X) \to F(\Sigma A) \times F(\Sigma B) \to F(\Sigma C) \to F(X) \to F(A) \times F(B) \to F(C).$$

Determine the maps and signs explicitly.

Exercises

Exercise 3.1. Use the Yoneda lemma to show that for two pointed animae X and Y, any natural transformation $[-, X]_* \to [-, Y]_*$ of functors $\mathrm{Ho}(\mathrm{An}_*)^{\mathrm{op}} \to \mathrm{Set}$ is given by postcomposition with a unique morphism $f\colon X \to Y$ in $\mathrm{Ho}(\mathrm{An}_*)$.

Exercise 3.2 (Cohomology of spheres and Moore animae)**.** Let $\widetilde{E}^*$ be a reduced cohomology theory, and write $A^q := \widetilde{E}^q(S^0)$ for its coefficient groups. For an abelian group A, write $A[m] := \ker(m\colon A \to A)$.

(1) Show that $\widetilde{E}^q(S^n) \cong A^{q-n}$ for every $q \in \mathbb{Z}$ and $n \geq 0$.

(2) For $m \geq 2$ and $n \geq 1$, let $M(\mathbb{Z}/m, n)$ be the cofiber of a degree-m map $S^n \to S^n$. Deduce natural short exact sequences

$$0 \to A^{q-n-1}/m \to \widetilde{E}^q(M(\mathbb{Z}/m, n)) \to A^{q-n}[m] \to 0.$$

Exercise 3.3 (A Moore anima)**.** Fix integers $m \geq 2$ and $n \geq 1$, and let $M(\mathbb{Z}/m, n)$ be the cofiber in An_* of a degree-m map $S^n \to S^n$. Show that

$$\widetilde{H}_k(M(\mathbb{Z}/m, n); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m, & k = n, \\ 0, & k \neq n, \end{cases}$$

Use Exercise 3.2 to show that

$$\widetilde{H}^k(M(\mathbb{Z}/m, n); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m, & k = n+1, \\ 0, & k \neq n+1. \end{cases}$$

Explain where the shift between the two answers comes from.

Exercise 3.4 (Uniqueness in Brown representability)**.** Suppose that spectra E and F both represent a reduced cohomology theory h^*, compatibly with the suspension isomorphisms. Show that $E \cong F$.

4 Spectra and stable ∞-categories

This chapter studies the ∞-category of spectra in its own right and develops the general notion of a *stable ∞-category* that it exemplifies. In the previous chapter, we encountered spectra as objects representing generalized cohomology theories. We now shift perspective to their formal properties, which are reminiscent of homological algebra.

As we will see, there are two equivalent perspectives on stable ∞-categories. The first emphasizes the relation to the stability property of cohomology theories by considering the suspension and loop functors and asking them to be invertible. The second emphasizes the analogy with exact sequences in an abelian category by considering fibers and cofibers, the homotopical analogues of kernels and cokernels, and demanding that every nullsequence

$$X \to Y \to Z$$

is a fiber sequence if and only if it is a cofiber sequence. It is this second perspective that connects stable ∞-categories to homological algebra.

The ∞-category of spectra admits a universal property: it is the terminal stable ∞-category equipped with a left exact functor to An. This is a special case of the *stabilization* $\mathrm{Sp}(C)$ of an ∞-category C with finite limits. We may think of $\mathrm{Sp}(C)$ as obtained from C by formally making the loops operation invertible.

4.1 Pointed ∞-categories

In Section 2.4 we introduced the basic constructions of fibers, cofibers, suspensions and loops in the ∞-category An_* of pointed animae. In this section we develop the corresponding formalism for an arbitrary pointed ∞-category C. This will provide the language needed to formulate the definition of a stable ∞-category in Section 4.2, and will also be used repeatedly in later chapters.

Definition 4.1.1 (Pointed ∞-category)**.** We call an ∞-category C *pointed* if it has a *zero object*, i.e. an object 0 that is both initial and terminal (Definition 1.7.7). Given objects $X, Y \in C$, we define the *null map* $0\colon X \to Y$ as the composite of the unique maps $X \to 0$ and $0 \to Y$.

A functor $F\colon C \to D$ between pointed ∞-categories is called *pointed* if it preserves zero objects.

Example 4.1.2. The category Ab of abelian groups is pointed: the zero object is the trivial group 0. More generally, any additive ∞-category is pointed.

Example 4.1.3. The category Top_* of pointed topological spaces is pointed, with the one-point space functioning as zero object.

Example 4.1.4. The ∞-category An_* of pointed animae (see Section 2.4) is pointed, with the one-point anima as zero object.

Definition 4.1.5 (Fiber and cofiber sequences)**.** Let C be a pointed ∞-category. A *nullsequence* in C is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

equipped with a specified identification $g \circ f = 0$, called a *nullhomotopy*. Equivalently, a nullsequence is a commutative square in C of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z. \end{array}$$

If this square is a pullback square in C, we say it is a *fiber sequence*, and refer to X as the *fiber of* g, written $X \simeq \mathrm{fib}(g)$. If the square is a pushout square, we say it is a *cofiber sequence*, and refer to Z as the *cofiber of* f, written $Z \simeq \mathrm{cofib}(f)$.

We say that C *admits fibers* if such a fiber sequence exists for every morphism $g\colon Y \to Z$. Dually, we say that C *admits cofibers* if such a cofiber sequence exists for every morphism $f\colon X \to Y$.

Definition 4.1.6 (Loop and suspension)**.** Let C be a pointed ∞-category. If C admits fibers, we define the *loop object* of $X \in C$ by

$$\Omega X := \mathrm{fib}(0 \to X).$$

If C admits cofibers, we define its *suspension* by

$$\Sigma X := \mathrm{cofib}(X \to 0).$$

Just as in Exercise 2.4.7, the loop and suspension constructions define endofunctors $\Omega, \Sigma\colon C \to C$.

Lemma 4.1.7. *Let C be a pointed ∞-category that admits fibers and cofibers. Then the functor $\Sigma\colon C \to C$ is left adjoint to $\Omega\colon C \to C$: for all $X, Y \in C$ there is a natural equivalence*

$$\mathrm{Hom}_C(\Sigma X, Y) \simeq \mathrm{Hom}_C(X, \Omega Y).$$

Proof. The proof is identical to that of Lemma 2.4.8. □

Pointed objects

The pointed ∞-categories Top_* and An_* are a special case of a general construction for turning an ∞-category C with terminal object into a pointed ∞-category.

Definition 4.1.8 (Pointed objects)**.** If C is an ∞-category with a terminal object $*$, we define its *∞-category of pointed objects* C_* as the slice category of C under $*$, i.e. as the following pullback square:

$$\begin{array}{ccc} C_* & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ * & \xrightarrow{*} & C. \end{array}$$

Objects of C_* are objects X of C equipped with a map $x\colon * \to X$ from the terminal object, thought of as a chosen 'basepoint' of X. We will frequently denote such objects by (X, x), or sometimes abusively as X.

A simple computation shows that the hom anima between pointed objects (X, x) and (Y, y) in C is computed as the following fiber:

$$\mathrm{Hom}_{C_*}((X,x),(Y,y)) \;\simeq\; \mathrm{fib}_y\left(\mathrm{Hom}_C(X,Y) \xrightarrow{-\circ x} \mathrm{Hom}_C(*,Y)\right).$$

(See Lemma 22.3.2 for a proof of the more general computation of hom animae in slice categories.) In other words, a morphism in C_* from (X, x) to (Y, y) consists of a morphism $f\colon X \to Y$ in C together with an isomorphism $f(x) \cong y$ of maps $* \to Y$.

Lemma 4.1.9. *Let C be an ∞-category with terminal object $*$. Then:*

(1) *The ∞-category C_* is pointed, with zero object given by $0 := (*, \mathrm{id}_*\colon * \to *)$.*

(2) *The ∞-category C is pointed if and only if the forgetful functor*

$$\mathrm{fgt}\colon C_* = C_{*/} \longrightarrow C$$

is an equivalence.

Proof. (1) For initiality of 0, note that the map $\mathrm{Hom}_C(*, Y) \to \mathrm{Hom}_C(*, Y)$ given by precomposing with the identity is an equivalence, hence its fiber $\mathrm{Hom}_{C_*}(0, Y)$ is contractible for every $Y \in C_*$. For terminality, we observe that

$$\mathrm{Hom}_{C_*}(X, 0) \simeq \mathrm{fib}\left(\mathrm{Hom}_C(X,*) \to \mathrm{Hom}_C(*,*)\right)$$

is the fiber of a map between two contractible animae, which thus is itself contractible.

(2) The 'if' part is clear from (1). Conversely, assume that C is pointed. Then the object $*$ is also an initial object in C, and hence the target functor $C_* = C_{*/} \to C$ is an equivalence by Lemma 22.3.4. □

Exercise 4.1.10. Show that C_* admits pullbacks whenever C admits pullbacks, and that the functor $C_* \to C$ preserves pullbacks in this case. Similarly, show that C_* admits pushouts whenever C admits pushouts, and that $C_* \to C$ preserves pushouts.

The pointed ∞-category C_* is in a precise sense the 'universal approximation' of C by a pointed ∞-category. We will end this section by recording a precise formulation of this universal property, which will be used later.

Notation 4.1.11. Let $\mathrm{Cat}_\infty^{\mathrm{term}} \subseteq \mathrm{Cat}_\infty$ denote the subcategory of ∞-categories with a terminal object, and functors preserving the terminal object. Let $\mathrm{Cat}_\infty^{\mathrm{pt}} \subseteq \mathrm{Cat}_\infty^{\mathrm{term}}$ denote the full subcategory of pointed ∞-categories. Given ∞-categories C and D with terminal objects, we denote by

$$\mathrm{Fun}_*(C, D) \quad \subseteq \quad \mathrm{Fun}(C, D)$$

the full subcategory spanned by functors preserving the terminal object.

The constructions of C_* and the forgetful functor $\mathrm{fgt}\colon C_* \to C$ are functorial in C, giving rise to a functor

$$(-)_*\colon \mathrm{Cat}_\infty^{\mathrm{term}} \to \mathrm{Cat}_\infty^{\mathrm{pt}}$$

and a natural transformation $\mathrm{fgt}\colon (-)_* \to \mathrm{id}$.

Corollary 4.1.12. *Let D be a pointed ∞-category and let C be an ∞-category with a terminal object. Then the ∞-category $\mathrm{Fun}_*(D, C)$ is pointed.*

Proof. The constant functor $\mathrm{const}_*\colon D \to C$ is terminal in $\mathrm{Fun}(D, C)$, hence also in the full subcategory $\mathrm{Fun}_*(D, C)$. We claim that it is also initial. Let $0 \in D$ denote a zero object, and let $F\colon D \to C$ be a functor preserving terminal objects. Since 0 is also initial in D, limits over D are given by evaluation at 0 by Lemma 22.2.4. We therefore get

$$\mathrm{Hom}_{\mathrm{Fun}_*(D,C)}(\mathrm{const}_*, F) \simeq \mathrm{Hom}_C(*, \lim_{d\in D} F(d)) \simeq \mathrm{Hom}_C(*, F(0)).$$

Since F preserves terminal objects and 0 is terminal in D, the object $F(0)$ is terminal in C. The last hom anima is therefore contractible, proving that const_* is initial. □

Lemma 4.1.13. *Let D be pointed and let C have a terminal object. Then the forgetful functor $\mathrm{fgt}\colon C_* \to C$ induces an equivalence*

$$\mathrm{fgt}_*\colon \mathrm{Fun}_*(D, C_*) \xrightarrow{\sim} \mathrm{Fun}_*(D, C).$$

In particular, the functor $(-)_*\colon \mathrm{Cat}_\infty^{\mathrm{term}} \to \mathrm{Cat}_\infty^{\mathrm{pt}}$ is right adjoint to the inclusion $\mathrm{Cat}_\infty^{\mathrm{pt}} \hookrightarrow \mathrm{Cat}_\infty^{\mathrm{term}}$.

Proof. Since $C_* = C_{*/}$, there is an equivalence

$$\mathrm{Fun}_*(D, C_*) \simeq \mathrm{Fun}_*(D, C)_*.$$

Indeed, a natural transformation $\mathrm{const}_* \to F$ is necessarily an isomorphism at the zero object as soon as F preserves terminal objects. By Corollary 4.1.12 and Lemma 4.1.9, the forgetful functor from the right-hand side to $\mathrm{Fun}_*(D, C)$ is an equivalence. Passing to underlying animae exhibits $(-)_*$ as a right adjoint to the inclusion. □

4.2 Stable ∞-categories

We now come to the central definition of stable homotopy theory: that of a stable ∞-category. To motivate the definition, let us take a look at some of the features of the category Ab of abelian groups which make it suitable for doing homotopical algebra:

(1) The category Ab is *additive*: the direct sum $A \oplus B$ of two abelian groups is both a product and a coproduct in Ab, and the hom sets $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ inherit the structure of abelian groups;

(2) The category Ab admits *kernels* and *cokernels*;

(3) There is a notion of *short exact sequence*: given a sequence of abelian groups

$$0 \to A \overset{i}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C \to 0$$

satisfying $pi = 0$ with i a monomorphism (i.e. an injective map) and p an epimorphism (i.e. a surjective map), the map i exhibits A as a kernel of p if and only if the map p exhibits C as a cokernel of i.

Categories with properties (1)-(3) are called *abelian categories*; we will study them more closely in Section 6.1.1. The definition of stable ∞-categories closely mirrors that of abelian categories:

Definition 4.2.1 (Stable ∞-category)**.** An ∞-category C is called *stable* if:

(1) The ∞-category C is pointed;

(2) The ∞-category C admits fibers and cofibers;

(3) A nullsequence $X \overset{f}{\to} Y \overset{g}{\to} Z$ in C is a fiber sequence if and only if it is a cofiber sequence.

We refer to the nullsequences in (3) as *exact sequences*.[11] A functor $F\colon C \to D$ between two stable ∞-categories is called *exact* if it preserves the zero object and sends exact sequences to exact sequences. Given two stable ∞-categories C and D, we denote by

$$\mathrm{Fun}^{\mathrm{ex}}(C, D) \quad \subseteq \quad \mathrm{Fun}(C, D)$$

the full subcategory spanned by the exact functors.

Note that this definition differs in two crucial ways from the definition of an abelian category. First, in condition (3) there is no requirement for f and g to be monomorphisms or epimorphisms; they are allowed to be arbitrary morphisms in C. Second, in condition (1) we are merely assuming that C is pointed, and are not a priori requiring any additive structure. In some sense, condition (3) has absorbed the requirement of additivity: as we will show in Lemma 4.2.17, stable ∞-categories are automatically additive.

While the definition of stable ∞-categories we gave is the one that most closely resembles the definition of abelian categories, there are various equivalent characterizations of stable ∞-categories that we will frequently use.

Theorem 4.2.2. *For a pointed ∞-category C, the following conditions are equivalent:*

(1) The ∞-category C admits pushouts and pullbacks, and a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{k} & Z \end{array}$$

in C is a pushout square if and only if it is a pullback square.

[11] Alternative terminology for 'exact sequence' includes *bifiber sequence*, *exact triangle* and *distinguished triangle*.

(2) The ∞-category C is stable;

(3) The ∞-category C admits fibers and the loop functor $\Omega\colon C \to C$ is an equivalence;

(4) The ∞-category C admits cofibers and the suspension functor $\Sigma\colon C \to C$ is an equivalence.

Proof. It is clear that (1) implies (2) by taking $W = 0$. If we assume (2), then the fiber sequence

$$\begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

is by assumption also a pushout square and thus the counit map $\Sigma\Omega X \to X$ is an equivalence. Similarly the unit map $X \to \Omega\Sigma X$ is an equivalence. It follows that Σ and Ω are inverse equivalences, so that (3) and (4) hold.

We now show that (3) implies (1); applying the argument to C^{op} will then show that also (4) implies (1), finishing the argument. We will first show that C admits pushouts and that pushout squares agree with pullback squares. To this end, consider the full subcategory

$$P \subseteq \mathrm{Fun}([1] \times [1], C)$$

spanned by the pullback squares. We then have the following claim:

Claim: The restriction functor $P \to \mathrm{Fun}(\ulcorner\ , C)$ is an equivalence.

Proof of claim: To construct an inverse, consider a diagram $z \leftarrow x \to y$ in C, and consider the following pullback diagram (ignoring the dashed morphisms):

$$\begin{array}{ccccccc} \Omega x & \longrightarrow & \Omega z & \longrightarrow & 0 & & \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \\ \Omega y & \longrightarrow & a & \longrightarrow & b & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & c & \longrightarrow & x & \longrightarrow & y \\ & & \downarrow & \lrcorner & \downarrow & & \vdots \\ & & 0 & \longrightarrow & z & \dashrightarrow & \Omega^{-1}a. \end{array}$$

Note that each of the pullbacks that appear here may be obtained as a fiber, hence is assumed to exist in C. More explicitly, starting from the span $z \leftarrow x \to y$, set

$$b := \mathrm{fib}(x \to y), \qquad c := \mathrm{fib}(x \to z).$$

Then define a as the fiber of the composite $b \to x \to z$. The nullhomotopy of $a \to b \to x \to z$ induces a map $a \to c$, and the square

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & x \end{array}$$

is a pullback by the pasting law. The other displayed pullback squares are obtained in the same way from the fiber sequences $b \to x \to y$, $c \to x \to z$, and $a \to b \to z$. The construction is functorial in the input span because each step is obtained by applying the fiber construction to a functorially specified morphism, together with the universal maps in its defining pullback square. Thus a morphism of spans induces compatible morphisms of all the objects in the displayed diagram. We now claim that an inverse to the restriction functor is given by

$$\begin{pmatrix} x & \longrightarrow & y \\ \downarrow & & \\ z & & \end{pmatrix} \quad \mapsto \quad \Omega^{-1}\begin{pmatrix} \Omega x & \longrightarrow & \Omega y \\ \downarrow & \lrcorner & \downarrow \\ \Omega z & \longrightarrow & a \end{pmatrix}.$$

Note that this functor does indeed land in P since the equivalence $\Omega^{-1}\colon C \xrightarrow{\sim} C$ preserves pullback squares. Furthermore, it provides an extension of the original diagram, hence defines a right inverse. Finally, if we start with any pullback square, we may rewrite it in the form

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \lrcorner & \downarrow \\ z & \longrightarrow & \Omega^{-1}a, \end{array}$$

and we see that the above procedure recovers the original pullback square, showing that it is also a left inverse. This finishes the proof of the claim. □

We will now use the claim to produce pushouts in C. Consider again a diagram $z \leftarrow x \to y$ in C, and use the claim to extend it uniquely to a pullback diagram:

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & \lrcorner & \downarrow \\ z & \longrightarrow & w. \end{array}$$

We claim that this square is also a pushout square in C. Indeed, for every other object a in C we have natural equivalences

$$\mathrm{Nat}\left(\begin{pmatrix} x & \longrightarrow & y \\ \downarrow & & \\ z & & \end{pmatrix}, \begin{pmatrix} a & = & a \\ \| & & \\ a & & \end{pmatrix}\right) \simeq \mathrm{Nat}\left(\begin{pmatrix} x & \longrightarrow & y \\ \downarrow & \lrcorner & \downarrow \\ z & \longrightarrow & w \end{pmatrix}, \begin{pmatrix} a & = & a \\ \| & \lrcorner & \| \\ a & = & a \end{pmatrix}\right) \simeq \mathrm{Hom}_C(w, a),$$

where the first equivalence is induced from the equivalence $P \simeq \mathrm{Fun}(\ulcorner, C)$, whereas the second equivalence holds because $[1]\times[1]$ has a terminal object $(1,1)$, so that the functor $\mathrm{ev}_{(1,1)}\colon \mathrm{Fun}([1]\times[1], C) \to C$ is left adjoint to the constant diagram functor $C \to \mathrm{Fun}([1]\times[1], C)$. In particular, we see that C admits pushouts, and that every pullback square is a pushout square.

Note that in particular C admits cofibers and that $\Sigma\colon C \to C$ is an equivalence, so applying the above logic to C^{op} we see that C also admits all pullbacks, and that every pushout square is a pullback square. We conclude that condition (1) is satisfied, finishing the proof. □

Terminology 4.2.3. Let C be a stable ∞-category. We say that a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{k} & Z \end{array}$$

is an *exact square* if it is a pullback square, or equivalently a pushout square.

Warning 4.2.4. Despite the similarity in their definitions, abelian categories are *never* stable ∞-categories: the loop functor $\Omega\colon \mathcal{A} \to \mathcal{A}$ in a pointed 1-category $\mathcal{A}$ is necessarily the zero-functor, hence is only an equivalence if $\mathcal{A}$ is the trivial category.

Nevertheless, every abelian category $\mathcal{A}$ embeds fully faithfully into a stable ∞-category $\mathrm{D}(\mathcal{A})$, called its *derived ∞-category*, to be introduced in Section 6.1 below. The inclusion $\mathcal{A} \hookrightarrow \mathrm{D}(\mathcal{A})$ sends short exact sequences in $\mathcal{A}$ to exact sequences in $\mathrm{D}(\mathcal{A})$.

Proposition 4.2.5. *Let $F\colon C \to D$ be a functor between stable ∞-categories. Then the following are equivalent:*

(1) The functor F is exact;

(2) The functor F is right exact, i.e. preserves initial objects and pushouts;

(3) The functor F is left exact, i.e. preserves terminal objects and pullbacks.

Proof. Since C and D are pointed, it is clear that F is pointed in each of the three cases. The argument from Theorem 4.2.2 shows that a pointed functor preserves exact sequences if and only if it preserves pushout squares if and only if it preserves pullback squares. □

Remark 4.2.6. Conditions (1)-(3) on F are also equivalent to F preserving loop objects, or equivalently suspensions; see Corollary 16.5.6.

Since exact functors are clearly closed under composition, the following definition makes sense:

Definition 4.2.7. We define the *∞-category of small stable ∞-categories* as the (non-full) subcategory

$$\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty$$

whose objects are the stable ∞-categories and whose morphisms are the exact functors between them.

Lemma 4.2.8. *The ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$ admits small limits, and the inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \hookrightarrow \mathrm{Cat}_\infty$ preserves small limits.*

Proof. Let $C_\bullet\colon K \to \mathrm{Cat}_\infty^{\mathrm{st}}$ be a diagram of stable ∞-categories and exact functors. By Corollary 24.5.6, the limit $\lim_{k\in K} C_k$ admits an initial and terminal object, and admits pushouts and pullbacks. Furthermore, each of the forgetful functors $\lim_{k\in K} C_k \to C_k$ preserve these (co)limits. Stability of $\lim_{k\in K} C_k$ now follows directly from the stability of the ∞-categories C_k for all $k \in K$. □

Exercise 4.2.9. Let I and C be ∞-categories and assume C is stable. Show that also $\mathrm{Fun}(I, C)$ is stable. *Hint:* Use Lemma 22.2.8.

4.2.1 Shift functors

Let C be a stable ∞-category. Since $\Sigma\colon C \to C$ and $\Omega\colon C \to C$ are inverse equivalences, we can define shift functors for any integer n.

Definition 4.2.10 (Shift functors)**.** For $n \in \mathbb{Z}$, we define the *n-th shift functor* $[n]\colon C \to C$ by

$$[n] := \begin{cases} \Sigma^n & \text{if } n \geq 0 \\ \Omega^{-n} & \text{if } n \leq 0 \end{cases}.$$

The two formulas agree for $n = 0$.

Warning 4.2.11. While the shift notation is convenient, it should be used with care to avoid sign issues.

For integers n and m, one may identify the composite $[n] \circ [m]$ with $[n+m]$ by explicitly checking each of the four cases. In particular, this shows that $[n] \circ [m]$ is equivalent to $[m] \circ [n]$: both are shifting by the sum $m + n$.

The subtlety is that there are several inequivalent ways to exhibit this identification. For example, since $[n]\colon C \to C$ is exact, it commutes with $[m]$, giving another identification $[n]\circ[m] \simeq [m]\circ[n]$. To see that this identification differs from the previous one, let us consider the case $n = m = 1$, so that both shift functors are simply the suspension functor. The first identification is then nothing but the identity on $\Sigma^2\colon C \to C$. However, the second identification $\Sigma^2 \simeq \Sigma^2$ is the one coming from commutation of Σ with itself. To see that these are not the same, consider the case $C = \mathrm{An}_*$, for which the commutation map is obtained by forming smash products with the swap map $S^1 \wedge S^1 \xrightarrow{\sim} S^1 \wedge S^1$ from Definition 2.4.10. Since this map has degree -1, it cannot be identified with the identity.

We will now discuss various basic facts about stable ∞-categories that illustrate their general behavior.

Lemma 4.2.12. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence in a stable ∞-category C, then for any integer n, the sequence*

$$X[n] \xrightarrow{f[n]} Y[n] \xrightarrow{g[n]} Z[n]$$

is also exact. Furthermore, the sequences

$$Y \xrightarrow{g} Z \to X[1] \qquad \textit{and} \qquad Z[-1] \to X \xrightarrow{f} Y$$

are exact sequences.

Proof. Since $[n]$ is an equivalence, it preserves finite limits and colimits, hence exact sequences. The second statement follows from the following pushout/pullback diagrams:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & X[1] \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} Z[-1] & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ 0 & \longrightarrow & Y & \xrightarrow{g} & Z. \end{array}$$

Here we used the pasting law for pushouts/pullbacks to identify the objects $X[1]$ and $Z[-1]$. □

Lemma 4.2.13. *Let $f\colon X \to Y$ be a morphism in a stable ∞-category. Then there is an isomorphism* $\mathrm{cofib}(f) \cong \mathrm{fib}(f)[1]$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} \mathrm{fib}(f) & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} & \ulcorner & \downarrow \\ 0 & \longrightarrow & Y & \longrightarrow & \mathrm{cofib}(f). \end{array}$$

The left square is a pullback square, and hence by stability also a pushout square. The right square is by definition a pushout square. Hence the outer rectangle is a pushout square, giving the desired isomorphism $\mathrm{cofib}(f) \simeq \Sigma(\mathrm{fib}(f)) = \mathrm{fib}(f)[1]$. □

Lemma 4.2.14. *Let $f\colon X \to Y$ be a morphism in a stable ∞-category C. The following are equivalent:*

(1) The morphism f is an isomorphism;

(2) We have $\mathrm{cofib}(f) \simeq 0$*;*

(3) We have $\mathrm{fib}(f) \simeq 0$*.*

Proof. We show that (1) is equivalent to (2); the equivalence between (1) and (3) is dual. Consider the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f). \end{array}$$

Since this square is both a pushout square as well as a pullback square and isomorphisms are closed under pushouts and pullbacks, we see that f is an isomorphism if and only if the map $0 \to \mathrm{cofib}(f)$ is an isomorphism, proving the claim. □

4.2.2 The additive structure of stable ∞-categories

Stable ∞-categories possess a strong additive structure.

Definition 4.2.15 ((Semi)additive ∞-category)**.** An ∞-category C is *semiadditive* if it has a zero object, finite products, finite coproducts, and the canonical map

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix}\colon X \sqcup Y \to X \times Y$$

is an isomorphism. We write $X \oplus Y$ for the resulting biproducts. We say that C is *additive* if, in addition, the shear map

$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix}\colon X \oplus X \to X \oplus X$$

is an equivalence.

Semiadditivity is sometimes also called *preadditivity*.

Remark 4.2.16. For objects X, Y in a semiadditive ∞-category C, we may equip $\pi_0 \operatorname{Hom}_C(X,Y)$ with the structure of an abelian monoid: given morphisms $f, g\colon X \to Y$, we define their sum $f+g$ as the following composite:

$$X \xrightarrow{(\mathrm{id},\mathrm{id})} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\binom{\mathrm{id}}{\mathrm{id}}} Y.$$

We leave it to the reader to check that this is indeed unital, associative and commutative. Then C is *additive* if and only if $\pi_0 \operatorname{Hom}_C(X,Y)$ is in fact a *group*. In particular, for every morphism $f\colon X \to Y$ we may form its negative $-f\colon X \to Y$ which satisfies $f + (-f) = 0$.

Lemma 4.2.17. *Every stable ∞-category C is additive.*

Proof. Stable ∞-categories have zero objects and pullbacks/pushouts, and therefore finite products ($X \times Y = X \times_0 Y$) and coproducts ($X \sqcup Y = X \sqcup_0 Y$). Additivity of C then amounts to showing that for all objects X and Y in C, the two commutative squares

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ Y & \xrightarrow{(0,\mathrm{id})} & X \times Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},\mathrm{id})} \\ X & \xrightarrow{(0,\mathrm{id})} & X \times X \end{array}$$

are exact. To see this, we consider the following two diagrams:

$$\begin{array}{ccccc} 0 & \longrightarrow & X & & \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \\ Y & \xrightarrow{(0,\mathrm{id})} & X \times Y & \xrightarrow{\mathrm{pr}_2} & Y \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0 \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} 0 & \longrightarrow & X & & \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},\mathrm{id})} & & \\ X & \xrightarrow{(0,\mathrm{id})} & X \times X & \xrightarrow{\mathrm{pr}_2} & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

In both cases, the right bottom square, the bottom rectangle and the left rectangle are pullback squares, and hence by the pasting law for pullback diagrams also the upper left square is exact. This finishes the proof. □

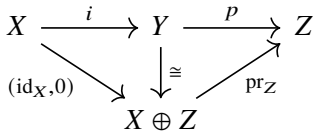
Recall that the *splitting lemma* provides equivalent conditions for a short exact sequence in an abelian category to be a *split short exact sequence*. The following is an analogue of the splitting lemma for stable ∞-categories:

Exercise 4.2.18 (Splitting lemma)**.** Let $X \xrightarrow{i} Y \xrightarrow{p} Z$ be an exact sequence in a stable ∞-category C. The following are equivalent:

(1) The map i admits a retraction $r\colon Y \to X$ (i.e., $ri \simeq \mathrm{id}_X$);

(2) The map p admits a section $s\colon Z \to Y$ (i.e., $ps \simeq \mathrm{id}_Z$);

(3) There is an isomorphism $Y \cong X \oplus Z$ making the following diagram commute:



Hint. Given a retraction of i, paste the two resulting pushout squares to identify Y with $X \oplus Z$. The argument starting from a section of p is dual.

Exercise 4.2.19 (Recognizing exact squares)**.** Let C be a stable ∞-category and consider a commutative square in C:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Z & \xrightarrow[g]{} & W. \end{array}$$

The following conditions are equivalent:

(1) The square is exact;

(2) The sequence $X \xrightarrow{(f,u)} Y \oplus Z \xrightarrow{\binom{v}{-g}} W$ is an exact sequence.

Hint. Show that the square formed by the diagonal $W \to W \oplus W$, the difference map $W \oplus W \to W$ and the zero object is a pullback, and paste it with the square exhibiting $Y \times_W Z$ as the pullback of $Y \oplus Z \to W \oplus W$ along the diagonal.

4.2.3 Finite limits and colimits

From the fact that stable ∞-categories admit zero objects, pushouts and pullbacks, we may deduce that they in fact admit *all* finite limits and colimits.

Definition 4.2.20 (Finite ∞-categories)**.** We denote by

$$\mathrm{Cat}_\infty^{\mathrm{fin}} \subseteq \mathrm{Cat}_\infty$$

the smallest full subcategory which contains the ∞-categories $\emptyset$, $*$ and $[1]$, and which is closed under pushouts of ∞-categories. A small ∞-category I is called *finite* if it is contained in $\mathrm{Cat}_\infty^{\mathrm{fin}}$.

Remark 4.2.21. It is proved by [Vol25, Proposition 2.5] that this model-independent definition of finite ∞-categories is compatible with the definition used in quasicategory theory: an ∞-category is finite if and only if it is the image of a finite simplicial set under the associated ∞-category functor $\mathrm{ac}\colon \mathrm{sSet} \to \mathrm{Cat}_\infty$ from Construction 1.8.12.

Example 4.2.22. The following ∞-categories are finite:

- The ∞-categories $\emptyset$, $*$ and $[1]$;
- The ∞-category $* \sqcup *$;

- The ∞-categories $\lrcorner$ and $\ulcorner$;
- The ∞-categories $[n]$.

Remark 4.2.23. There is a functor $(-)^{\mathrm{op}} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ implementing the assignment $C \mapsto C^{\mathrm{op}}$. This functor is an equivalence, hence in particular preserves pushouts and full subcategories. Since each of the ∞-categories $\emptyset$, $*$ and $[1]$ are equivalent to their opposites, it follows that an ∞-category C is finite if and only if its opposite C^{op} is finite.

Definition 4.2.24 (Left exact ∞-categories)**.** An ∞-category C is called *left exact*, or said to *admit finite limits*, if it admits all I-indexed limits for all finite ∞-categories I. Similarly, a functor $F \colon C \to D$ between left exact ∞-categories is called *left exact* if it preserves I-indexed limits for all $I \in \mathrm{Cat}_\infty^{\mathrm{fin}}$. We denote by

$$\mathrm{Cat}_\infty^{\mathrm{lex}} \quad \subseteq \quad \mathrm{Cat}_\infty$$

the subcategory consisting of the left exact ∞-categories and left exact functors. Given two left exact ∞-categories C and D, we denote by

$$\mathrm{Fun}^{\mathrm{lex}}(C, D) \quad \subseteq \quad \mathrm{Fun}(C, D)$$

the full subcategory spanned by the left exact functors.

One dually obtains a notion of *right exactness* in terms of finite *co*limits.

Theorem 4.2.25 (Lurie [Lur09b, Corollaries 4.4.2.4 and 4.4.2.5])**.** *(1) An ∞-category C admits finite limits if and only if it has a terminal object and admits pullbacks.*

(2) A functor $F \colon C \to D$ between ∞-categories with finite limits is left exact if and only if it preserves terminal objects and pullbacks.

Dually C has finite colimits if and only if it has an initial object and admits pushouts, and a functor between ∞-categories with finite colimits preserves finite colimits if and only if it preserves initial objects and pushouts.

Proof sketch. One direction is clear. The other direction needs the non-trivial fact that when the ∞-category I can be written as a colimit $I = \mathrm{colim}_j I_j$, we get an alternative description of $\lim_{i \in I} F(i)$ as $\lim_{j \in J^{\mathrm{op}}} \lim_{i \in I_j} F(i)$. (See Theorem 22.2.11 for a more precise formulation). □

Corollary 4.2.26. *Every stable ∞-category admits finite limits and colimits.* □

Corollary 4.2.27. *A functor $F \colon C \to D$ between stable ∞-categories is exact if and only if it is left exact if and only if it is right exact: we have equalities*

$$\mathrm{Fun}^{\mathrm{ex}}(C, D) \quad = \quad \mathrm{Fun}^{\mathrm{lex}}(C, D) \quad = \quad \mathrm{Fun}^{\mathrm{rex}}(C, D)$$

as full subcategories of $\mathrm{Fun}(C, D)$.

We record an alternative characterization of stable ∞-categories, due to Moritz Groth [Gro16]:

Proposition 4.2.28. *An ∞-category C is stable if and only if it has all finite limits and colimits and finite limits commute with finite colimits.*

Proof. Let C be a stable ∞-category. If I is a finite ∞-category, then C has I-indexed colimits by the previous corollary, hence there is a functor $\mathrm{colim}\colon \mathrm{Fun}(I, C) \to C$. Its domain is also stable and colim preserves finite colimits, because colimits commute with colimits. So the functor is exact and hence preserves finite limits as well.

Conversely, assume that C is finitely complete and cocomplete and that finite limits commute with finite colimits in C. Considering the empty category, the fact that empty limits commute with empty colimits expresses that C is pointed. To show that C is stable, consider for every $X \in C$ the following diagram:

$$\begin{array}{ccccc} X & \longleftarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & X & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \longleftarrow & X & \longrightarrow & X. \end{array}$$

Taking pushouts of the rows we get the diagram $0 \to \Sigma X \leftarrow 0$ whose pullback is $\Omega\Sigma X$. If instead we take pullbacks of the columns, we get $X \leftarrow X \to X$, whose pushout is X. Pullbacks commuting with pushouts tell us then the canonical map $X \to \Omega\Sigma X$ is an equivalence. A similar argument shows that $\Sigma\Omega X \to X$ is an equivalence, so that C is stable by Theorem 4.2.2. □

4.3 Stabilization

If C is an ∞-category with finite limits, there is a stable ∞-category $\mathrm{Sp}(C)$ that universally 'approximates C from the left' by a stable ∞-category, known as the *stabilization of* C. The stabilization comes equipped with a left exact functor $\mathrm{Sp}(C) \to C$ satisfying the property that any other left exact functor $D \to C$ from a stable ∞-category D uniquely factors through some left exact functor $D \to \mathrm{Sp}(C)$. The following definition captures this more precisely:

Definition 4.3.1. A *stabilization* of an ∞-category C with finite limits is a stable ∞-category $\mathrm{Sp}(C)$ equipped with a left exact functor $\Omega^\infty\colon \mathrm{Sp}(C) \to C$ such that for every stable ∞-category D the composition functor

$$\Omega^\infty \circ -\colon \mathrm{Fun}^{\mathrm{ex}}(D, \mathrm{Sp}(C)) = \mathrm{Fun}^{\mathrm{lex}}(D, \mathrm{Sp}(C)) \to \mathrm{Fun}^{\mathrm{lex}}(D, C)$$

is an equivalence, where the first equality holds by Corollary 4.2.27.

Note that this universal property uniquely determines the stabilization whenever it exists; in fact, it determines a right adjoint $\mathrm{Sp}\colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ to the inclusion functor, cf. Corollary 4.3.17 below.

Example 4.3.2. The prototypical example is $C = \mathrm{An}$, whose stabilization $\mathrm{Sp}(\mathrm{An})$ is the ∞-category of spectra Sp that we will study in Section 4.4.

We prove the existence of $\mathrm{Sp}(C)$ by providing an explicit construction: first we define prespectra as structured sequences in C_*, and then we define spectra as those prespectra whose structure maps are isomorphisms. We will prove the following properties of $\mathrm{Sp}(C)$ along the way:

- If C admits limits, then so does $\mathrm{Sp}(C)$ (Lemma 4.3.18);

- If C admits colimits and the loop functor $\Omega\colon C_* \to C_*$ preserves sequential colimits, then $\mathrm{Sp}(C)$ admits colimits (Lemma 4.3.20);
- In the latter case, the functor $\Omega^\infty\colon \mathrm{Sp}(C) \to C$ admits a left adjoint $\Sigma^\infty\colon C \to \mathrm{Sp}(C)$ (Lemma 4.3.22 and Corollary 4.3.23).

4.3.1 Prespectra and spectrum objects

The construction of (pre)spectra we use closely matches the classical definition of a spectrum as a sequence of pointed spaces $\{X_n\}$ equipped with maps $X_n \to \Omega X_{n+1}$. We first allow these maps to be arbitrary; the stabilization is obtained by imposing that they be isomorphisms.

Definition 4.3.3 (Prespectra)**.** Let C be an ∞-category admitting finite limits. Write

$$C_*^{\mathbb{N}} := \prod_{n\geq 0} C_*$$

for the ∞-category of sequences of pointed objects of C. We define the shift functor and the levelwise loop functor by

$$\mathrm{sh}\colon C_*^{\mathbb{N}} \to C_*^{\mathbb{N}}, \qquad (X_n)_{n\geq 0} \mapsto (X_{n+1})_{n\geq 0},$$

and

$$\Omega\colon C_*^{\mathbb{N}} \to C_*^{\mathbb{N}}, \qquad (X_n)_{n\geq 0} \mapsto (\Omega X_n)_{n\geq 0}.$$

The ∞-category of *prespectra in C* is defined as

$$\mathrm{PSp}(C) := C_*^{\mathbb{N}} \times_{C_*^{\mathbb{N}}\times C_*^{\mathbb{N}}} \mathrm{Ar}(C_*^{\mathbb{N}}),$$

where the map $C_*^{\mathbb{N}} \to C_*^{\mathbb{N}} \times C_*^{\mathbb{N}}$ is $(\mathrm{id}, \Omega\,\mathrm{sh})$ and the map $\mathrm{Ar}(C_*^{\mathbb{N}}) \to C_*^{\mathbb{N}} \times C_*^{\mathbb{N}}$ is the source-target functor (s,t).

Unwinding definitions, a prespectrum X is a sequence $(X_n)_{n\geq 0}$ of pointed objects of C, together with structure maps

$$\sigma_n^X\colon X_n \to \Omega X_{n+1}.$$

A morphism $f\colon X \to Y$ of prespectra is a sequence of pointed maps $f_n\colon X_n \to Y_n$ together with identifications making the squares commute

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle \sigma_n^X}\downarrow & & \downarrow{\scriptstyle \sigma_n^Y} \\ \Omega X_{n+1} & \xrightarrow{\Omega f_{n+1}} & \Omega Y_{n+1}. \end{array}$$

Definition 4.3.4 (Spectrum objects)**.** Let C be an ∞-category admitting finite limits. We define its ∞-*category of spectrum objects*, or *stabilization*, as the full subcategory

$$\mathrm{Sp}(C) \subseteq \mathrm{PSp}(C)$$

spanned by those prespectra X for which every structure map $\sigma_n^X\colon X_n \to \Omega X_{n+1}$ is an isomorphism. We denote the evaluation functors by

$$(-)_n\colon \mathrm{Sp}(C) \to C_*.$$

The functor $(-)_0\colon \mathrm{Sp}(C) \to C_*$ will also be denoted by Ω^∞; when the target is written as C, we implicitly compose with the forgetful functor $C_* \to C$.

Remark 4.3.5. Assume C is a small ∞-category. Then the structure maps $\sigma_n \colon (-)_n \xrightarrow{\sim} \Omega(-)_{n+1}$ turn the functors $(-)_n \colon \mathrm{Sp}(C) \to C_*$ into a cone in Cat_∞, resulting in a comparison functor

$$\mathrm{Sp}(C) \to \lim(\cdots \to C_* \xrightarrow{\Omega} C_* \xrightarrow{\Omega} C_*).$$

This functor is an equivalence of ∞-categories.

Lemma 4.3.6. *The functors $(-)_n \colon \mathrm{PSp}(C) \to C_*$ are jointly conservative. In particular, a morphism $f \colon X \to Y$ of spectra in C is an isomorphism if and only if each induced map $f_n \colon X_n \to Y_n$ is an isomorphism.*

Proof. The projection $\mathrm{PSp}(C) \to C_*^{\mathbb{N}}$ is conservative because a morphism in the arrow category is an isomorphism precisely when its source and target maps are isomorphisms. The projections $C_*^{\mathbb{N}} \to C_*$ are jointly conservative by definition of the product. □

Lemma 4.3.7. *Let X and Y be prespectra in C. Then the hom anima $\mathrm{Hom}_{\mathrm{PSp}(C)}(X, Y)$ is the equalizer of the two maps*

$$\mathrm{Hom}_{C_*^{\mathbb{N}}}(X, Y) \rightrightarrows \mathrm{Hom}_{C_*^{\mathbb{N}}}(X, \Omega \,\mathrm{sh}\, Y).$$

The first map sends $f \colon X \to Y$ to the composite

$$X \xrightarrow{f} Y \xrightarrow{\sigma^Y} \Omega \,\mathrm{sh}\, Y,$$

and the second sends it to the composite

$$X \xrightarrow{\sigma^X} \Omega \,\mathrm{sh}\, X \xrightarrow{\Omega \,\mathrm{sh}(f)} \Omega \,\mathrm{sh}\, Y.$$

Proof. This is just the definition of $\mathrm{PSp}(C)$ as the lax equalizer of id and $\Omega \,\mathrm{sh}$. Indeed, a morphism $X \to Y$ in $\mathrm{PSp}(C)$ is a map of the underlying sequences together with an identification making the square commute

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \sigma^X}\downarrow & & \downarrow{\scriptstyle \sigma^Y} \\ \Omega \,\mathrm{sh}\, X & \xrightarrow{\Omega \,\mathrm{sh}(f)} & \Omega \,\mathrm{sh}\, Y. \end{array}$$

Equivalently, it is a point of the stated equalizer. □

Lemma 4.3.8. *Let I be an ∞-category.*

(1) If C admits I-indexed limits, then $\mathrm{PSp}(C)$ admits I-indexed limits and the projection $\mathrm{PSp}(C) \to C_*^{\mathbb{N}}$ creates them.

(2) If C_* admits I-indexed colimits, then $\mathrm{PSp}(C)$ admits I-indexed colimits and the projection $\mathrm{PSp}(C) \to C_*^{\mathbb{N}}$ creates them.

Proof. For limits, let $X \colon I \to \mathrm{PSp}(C)$ be a diagram and write $\overline{X} \colon I \to C_*^{\mathbb{N}}$ for the underlying diagram of sequences. Since C_* has I-indexed limits and Ω preserves limits, the endofunctor $\Omega \,\mathrm{sh}$ preserves I-indexed limits. Hence the limit $L := \lim_i \overline{X}_i$ in $C_*^{\mathbb{N}}$ inherits a structure map

$$L \simeq \lim_i \overline{X}_i \to \lim_i \Omega \,\mathrm{sh}(\overline{X}_i) \simeq \Omega \,\mathrm{sh}(L),$$

which makes it the limit in $\mathrm{PSp}(C)$.

For colimits, let $D := \mathrm{colim}_i \overline{X}_i$ in $C_*^{\mathbb{N}}$. For every i, the composite

$$\overline{X}_i \xrightarrow{\sigma^{X_i}} \Omega\,\mathrm{sh}(\overline{X}_i) \to \Omega\,\mathrm{sh}(D)$$

is compatible with the diagram, and therefore induces a structure map $D \to \Omega\,\mathrm{sh}(D)$. We claim that the resulting prespectrum is the colimit. Indeed, for any prespectrum Y the hom anima $\mathrm{Hom}_{\mathrm{PSp}(C)}(D, Y)$ is the equalizer of the two maps

$$\mathrm{Hom}_{C_*^{\mathbb{N}}}(D, Y) \rightrightarrows \mathrm{Hom}_{C_*^{\mathbb{N}}}(D, \Omega\,\mathrm{sh}\,Y)$$

given by postcomposition with σ^Y and precomposition with the structure map of D, by Lemma 4.3.7. Since D is the colimit of the $\overline{X}_i$ in $C_*^{\mathbb{N}}$, and since limits commute with limits in An, this equalizer identifies with

$$\lim_i \mathrm{Hom}_{\mathrm{PSp}(C)}(X_i, Y),$$

which is the desired universal property. □

We now turn to the main point: the loop functor on $\mathrm{Sp}(C)$ is an equivalence. The naive argument is to shift levels, since $X_n \simeq \Omega X_{n+1}$ by definition. The subtlety is that comparing the structure maps of X with those of $\Omega(\mathrm{shift}\,X)$ requires identifying $\Omega^2\Omega$ with $\Omega\Omega^2$, and there are two ways to do so. The next results handle this coherence issue by passing first to an even-level model and then showing that the resulting automorphism of Ω^3 is trivial.

Lemma 4.3.9. *Let* $\mathrm{PSp}^{(2)}(C)$ *be the lax equalizer*

$$\mathrm{PSp}^{(2)}(C) := C_*^{\mathbb{N}} \times_{C_*^{\mathbb{N}} \times C_*^{\mathbb{N}}} \mathrm{Ar}(C_*^{\mathbb{N}}),$$

where the map $C_*^{\mathbb{N}} \to C_*^{\mathbb{N}} \times C_*^{\mathbb{N}}$ *is* $(\mathrm{id}, \Omega^2\,\mathrm{sh})$ *and the map from the arrow category is the source-target functor. Let* $\mathrm{Sp}^{(2)}(C) \subseteq \mathrm{PSp}^{(2)}(C)$ *be the full subcategory spanned by those objects whose structure maps are isomorphisms. Then the even-level functor*

$$E\colon \mathrm{Sp}(C) \to \mathrm{Sp}^{(2)}(C), \qquad E(X)_k := X_{2k},$$

with structure maps

$$\widetilde{\sigma}_k^X\colon X_{2k} \xrightarrow{\sigma_{2k}^X} \Omega X_{2k+1} \xrightarrow{\Omega(\sigma_{2k+1}^X)} \Omega^2 X_{2k+2},$$

is an equivalence.

Proof. We construct an inverse. Given $Y \in \mathrm{Sp}^{(2)}(C)$ with structure maps $\tau_k\colon Y_k \xrightarrow{\sim} \Omega^2 Y_{k+1}$, define a spectrum QY by

$$(QY)_{2k} := Y_k \qquad \text{and} \qquad (QY)_{2k+1} := \Omega Y_{k+1}.$$

The structure maps of QY are $\tau_k\colon Y_k \to \Omega^2 Y_{k+1} = \Omega(QY)_{2k+1}$ in even degrees and the identity map $\Omega Y_{k+1} \to \Omega Y_{k+1} = \Omega(QY)_{2k+2}$ in odd degrees.

It is immediate that $EQ = \mathrm{id}$. Conversely, for $X \in \mathrm{Sp}(C)$ there is a natural isomorphism $X \to QEX$ which is the identity in even degrees and the structure map $\sigma_{2k+1}^X\colon X_{2k+1} \to \Omega X_{2k+2}$ in odd degrees. The compatibility with the structure maps follows directly from the definition of $\widetilde{\sigma}_k^X$. □

Remark 4.3.10. When C is small, the limit description of $\mathrm{Sp}(C)$ from Remark 4.3.5 identifies $\mathrm{Sp}^{(2)}(C)$ with the analogous limit over the subdiagram indexed by the even natural numbers. Since these are cofinal, the result also follows formally from Theorem 22.5.1.

Next up, we need to understand the interaction of the double loop functor Ω^2 with the loop functor Ω. If D is a pointed ∞-category with finite limits, the functor $\Omega^2 \colon D \to D$ commutes with limits, and in particular commutes with the formation of loop objects. This provides a natural isomorphism

$$\Omega^2(\Omega X) \xrightarrow{\cong} \Omega(\Omega^2 X).$$

The following proposition records the specific case we need.

Proposition 4.3.11 (Universal property of finite animae)**.** *Let* $\mathrm{An}^{\mathrm{fin}}$ *denote the smallest full subcategory of* An *which contains the point and is closed under finite colimits. For every* ∞*-category* D *with finite limits, evaluation at the point induces an equivalence*

$$\mathrm{ev}_* \colon \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}}, D) \xrightarrow{\sim} D.$$

Proof. This is the opposite of the universal property of the free finite cocompletion of the point, which is a special case of Proposition 23.1.2. □

Lemma 4.3.12 (Universal property of finite pointed animae)**.** *Let* $\mathrm{An}_*^{\mathrm{fin}}$ *be the small full subcategory of* An_* *generated under finite colimits by* S^0*. For every pointed* ∞*-category* D *with finite limits, evaluation at* S^0 *induces an equivalence*

$$\mathrm{ev}_{S^0} \colon \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, D) \xrightarrow{\sim} D.$$

Proof. Precomposition with $(-)_+ \colon \mathrm{An}^{\mathrm{fin}} \to \mathrm{An}_*^{\mathrm{fin}}$ defines a functor

$$\mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, D) \longrightarrow \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}}, D).$$

An inverse sends F to the functor

$$(Y, y) \longmapsto \mathrm{fib}(F(Y) \to F(*)).$$

This is pointed and left exact. The two inverse identities follow from $Y_+ = Y \sqcup *$ and from the pushout $Y_+ \sqcup_{S^0} * \simeq (Y, y)$. The result now follows from Proposition 4.3.11. □

Proposition 4.3.13. *For any pointed* ∞*-category* D *with finite limits, the composite map*

$$\Omega^3 X \simeq \Omega^2(\Omega X) \xrightarrow{\cong} \Omega(\Omega^2 X) \simeq \Omega^3 X$$

admits an identification with the identity map on $\Omega^3 X$*, natural in* $X \in D$*.*

Proof. Given $X \in D$, Lemma 4.3.12 gives a pointed left exact functor

$$X^{(-)} \colon (\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}} \to D$$

satisfying $X^{S^0} \simeq X$. Since $X^{(-)}$ sends pushouts in $\mathrm{An}_*^{\mathrm{fin}}$ to pullbacks in D, we have

$$X^{S^{n+1}} \simeq X^{\Sigma S^n} \simeq \Omega X^{S^n}$$

for every $n \geq 0$. Thus the resulting automorphism of $\Omega^3 X$ is induced by the analogous automorphism of S^3 in $\mathrm{An}_*^{\mathrm{fin}}$.

It remains to prove the claim in $\mathrm{An}_*^{\mathrm{fin}}$ for the generator $X = S^0$. We must show that the composite

$$\tau \colon S^3 \simeq \Sigma(\Sigma^2 S^0) \xrightarrow{\cong} \Sigma^2(\Sigma S^0) \simeq S^3$$

can be identified with the identity as a pointed map. By Remark 2.4.20, it suffices to show that τ has degree 1. Using the swap map $\sigma_{X,Y}$ from Definition 2.4.10, we may identify τ with σ_{S^1,S^2}. The isomorphism $S^2 \cong S^1 \wedge S^1$ from Lemma 2.4.16 then gives a decomposition

$$S^3 \simeq S^1 \wedge S^1 \wedge S^1 \xrightarrow{\mathrm{id} \wedge \sigma_{S^1,S^1}} S^1 \wedge S^1 \wedge S^1 \xrightarrow{\sigma_{S^1,S^1} \wedge \mathrm{id}} S^1 \wedge S^1 \wedge S^1 \simeq S^3.$$

Both maps in this decomposition are suspensions of σ_{S^1,S^1}, up to the natural identifications of smash products with suspensions. Their degrees are therefore equal to that of σ_{S^1,S^1} by Remark 2.4.20. Since this swap map is an isomorphism, its degree is ± 1. We conclude that

$$\deg(\tau) = \deg(\mathrm{id} \wedge \sigma_{S^1,S^1}) \cdot \deg(\sigma_{S^1,S^1} \wedge \mathrm{id}) = \deg(\sigma_{S^1,S^1})^2 = 1,$$

as desired. The construction is natural in X, since the map is induced by the automorphism τ of S^3 under the equivalence of Lemma 4.3.12. □

Theorem 4.3.14. *For every ∞-category C with finite limits, the ∞-category $\mathrm{Sp}(C)$ is stable.*

Proof. By Lemma 4.3.8, finite limits in $\mathrm{PSp}(C)$ are computed pointwise. Since Ω preserves finite limits, the full subcategory $\mathrm{Sp}(C) \subseteq \mathrm{PSp}(C)$ is closed under finite limits. It is also pointed: the constant zero prespectrum is both initial and terminal, since the zero object of C_* is both initial and terminal.

It remains, by Theorem 4.2.2, to show that the loop functor on $\mathrm{Sp}(C)$ is an equivalence. Define the shift functor

$$\mathrm{shift} \colon \mathrm{Sp}(C) \to \mathrm{Sp}(C), \qquad \mathrm{shift}(X)_n := X_{n+1},$$

with structure maps inherited from X. This functor is an equivalence: an inverse sends X to the spectrum $\mathrm{shift}^{-1} X$ with

$$(\mathrm{shift}^{-1} X)_0 := \Omega X_0 \qquad \text{and} \qquad (\mathrm{shift}^{-1} X)_n := X_{n-1} \quad (n \geq 1),$$

whose structure map in degree 0 is the identity of ΩX_0, and whose structure map in degree $n \geq 1$ is σ^X_{n-1}.

We claim that there is a natural isomorphism

$$\alpha \colon X \xrightarrow{\cong} \Omega(\mathrm{shift}\, X).$$

By Lemma 4.3.9, it suffices to construct this map after passing to the even-level model. There we set

$$\alpha_{2k} := \sigma^X_{2k} \colon X_{2k} \xrightarrow{\cong} \Omega X_{2k+1} = \Omega(\mathrm{shift}\, X)_{2k}.$$

We must check compatibility with the two-step structure maps. For X, the relevant structure map is

$$\widetilde{\sigma}^X_k = \left(X_{2k} \xrightarrow{\sigma^X_{2k}} \Omega X_{2k+1} \xrightarrow{\Omega(\sigma^X_{2k+1})} \Omega^2 X_{2k+2} \right).$$

For $\Omega(\text{shift}\, X)$, finite limits are computed pointwise, but its two-step structure map is not obtained by merely writing down the same composite with an extra Ω in front. After precomposing with $\alpha_{2k} = \sigma^X_{2k}$, the first composite in the compatibility square is

$$X_{2k} \xrightarrow{\sigma^X_{2k}} \Omega X_{2k+1} \xrightarrow{\Omega(\sigma^X_{2k+1})} \Omega^2 X_{2k+2} \xrightarrow{\Omega(\Omega(\sigma^X_{2k+2}))} \Omega(\Omega^2 X_{2k+3}) \xrightarrow{\cong} \Omega^2(\Omega X_{2k+3}),$$

where the final arrow is the canonical isomorphism coming from the fact that Ω^2 preserves loop objects. The other composite in the compatibility square is

$$X_{2k} \xrightarrow{\sigma^X_{2k}} \Omega X_{2k+1} \xrightarrow{\Omega(\sigma^X_{2k+1})} \Omega^2 X_{2k+2} \xrightarrow{\Omega^2(\sigma^X_{2k+2})} \Omega^2(\Omega X_{2k+3}).$$

Thus the two composites whose equality is required differ only by the resulting automorphism of $\Omega^3 X_{2k+3}$. This automorphism is the identity by Proposition 4.3.13. Hence α is an isomorphism of spectra.

Consequently $\mathrm{id}_{\mathrm{Sp}(C)} \simeq \Omega \circ \text{shift}$. Since shift is an equivalence, Ω is an equivalence as well. □

Corollary 4.3.15. *The suspension functor on* $\mathrm{Sp}(C)$ *is naturally isomorphic to the shift functor. In particular, for every spectrum* X *and all* $m, n \geq 0$*, there is a natural isomorphism*

$$(X[n])_m \cong X_{m+n}.$$

Proof. The proof of Theorem 4.3.14 gives a natural isomorphism $\mathrm{id}_{\mathrm{Sp}(C)} \cong \Omega \circ \text{shift}$. Since Σ is inverse to Ω, this identifies Σ with shift. □

Proposition 4.3.16. *For an* ∞*-category* C *with finite limits, the functor* $\Omega^\infty \colon \mathrm{Sp}(C) \to C$ *exhibits* $\mathrm{Sp}(C)$ *as a stabilization of* C*.*

Proof. Let D be a stable ∞-category. Since finite limits of spectrum objects are computed levelwise, the definition of spectrum objects gives an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(D, \mathrm{Sp}(C)) \simeq \lim\big(\cdots \xrightarrow{\Omega_C \circ -} \mathrm{Fun}^{\mathrm{lex}}(D, C_*) \xrightarrow{\Omega_C \circ -} \mathrm{Fun}^{\mathrm{lex}}(D, C_*)\big).$$

For every left exact $F \colon D \to C_*$, preservation of the pullback defining loops gives a natural isomorphism

$$\Omega_C \circ F \simeq F \circ \Omega_D.$$

Since D is stable, Ω_D is an equivalence. Thus all transition functors in this tower are equivalences, and evaluation at level zero identifies its limit with $\mathrm{Fun}^{\mathrm{lex}}(D, C_*)$.

Finally, forgetting the pointing gives an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(D, C_*) \simeq \mathrm{Fun}^{\mathrm{lex}}(D, C),$$

whose inverse equips $F(d)$ with the pointing $* \simeq F(0) \to F(d)$. The composite equivalence is postcomposition with Ω^∞, as required. □

Corollary 4.3.17. *The fully faithful inclusion functor* $\mathrm{Cat}^{\mathrm{st}}_\infty \hookrightarrow \mathrm{Cat}^{\mathrm{lex}}_\infty$ *admits a right adjoint*

$$\mathrm{Sp} \colon \mathrm{Cat}^{\mathrm{lex}}_\infty \to \mathrm{Cat}^{\mathrm{st}}_\infty$$

given on objects by sending C *to its stabilization* $\mathrm{Sp}(C)$*. The counit* $\mathrm{Sp}(C) \to C$ *of the adjunction is given by* Ω^∞*.*

Proof. By the pointwise criterion for adjunctions from Lemma 22.1.4, it suffices to show that for every stable ∞-category D, composition with $\Omega^\infty\colon \mathrm{Sp}(C) \to C$ induces an equivalence

$$\Omega^\infty \circ -\colon \mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(D, \mathrm{Sp}(C)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{lex}}}(D, C).$$

But since a functor $D \to \mathrm{Sp}(C)$ is exact if and only if it is left exact, this map is obtained from the defining equivalence $\mathrm{Fun}^{\mathrm{lex}}(D, \mathrm{Sp}(C)) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(D, C)$ by passing to groupoid cores. □

4.3.2 Spectrification and suspension spectra

We now explain how to compute colimits of spectra. The point is that colimits are easy in prespectra, and a prespectrum can be turned into a spectrum by repeatedly applying the shift-loop operation.

Lemma 4.3.18 (Limits of spectra)**.** *Let C be an ∞-category with finite limits. If C admits I-indexed limits, then $\mathrm{Sp}(C)$ admits I-indexed limits, and the evaluation functors $(-)_n\colon \mathrm{Sp}(C) \to C_*$ preserve them.*

Proof. By Lemma 4.3.8, the limit of a diagram of spectra can first be computed in $\mathrm{PSp}(C)$. Since Ω preserves limits, the structure maps of this pointwise limit are again equivalences. Thus the limit prespectrum is a spectrum. □

Proposition 4.3.19 (Spectrification)**.** *Let C be an ∞-category with finite limits and sequential colimits. Assume that $\Omega\colon C_* \to C_*$ preserves sequential colimits. Then the inclusion $\mathrm{Sp}(C) \hookrightarrow \mathrm{PSp}(C)$ admits a left adjoint*

$$(-)^{\mathrm{sp}}\colon \mathrm{PSp}(C) \to \mathrm{Sp}(C), \qquad X \mapsto X^{\mathrm{sp}},$$

called the spectrification functor.

Proof. The endofunctor $\Omega\,\mathrm{sh}$ of $C_*^{\mathbb{N}}$ induces an endofunctor, still denoted $\Omega\,\mathrm{sh}$, of $\mathrm{PSp}(C)$ by

$$(\Omega\,\mathrm{sh}\, X)_n := \Omega X_{n+1},$$

with structure maps $\Omega(\sigma^X_{n+1})\colon \Omega X_{n+1} \to \Omega^2 X_{n+2}$. The structure maps of X define a natural map $\eta_X\colon X \to \Omega\,\mathrm{sh}\, X$. Since sh preserves colimits and Ω preserves sequential colimits, the endofunctor $\Omega\,\mathrm{sh}$ preserves sequential colimits of prespectra.

For a prespectrum X, define

$$X^{\mathrm{sp}} := \mathrm{colim}\left(X \xrightarrow{\eta_X} \Omega\,\mathrm{sh}\, X \xrightarrow{\Omega\,\mathrm{sh}(\eta_X)} (\Omega\,\mathrm{sh})^2 X \to \cdots\right)$$

in $\mathrm{PSp}(C)$. Then

$$\Omega\,\mathrm{sh}(X^{\mathrm{sp}}) \simeq \mathrm{colim}_{k\geq 0}(\Omega\,\mathrm{sh})^{k+1} X \simeq \mathrm{colim}_{k\geq 0}(\Omega\,\mathrm{sh})^k X = X^{\mathrm{sp}},$$

where the second isomorphism uses that the successor map $k \mapsto k+1$ is cofinal in $\mathbb{N}$. Under this isomorphism, the structure map of X^{sp} is an isomorphism, so X^{sp} is a spectrum.

It remains to prove the universal property. Let Y be a spectrum. Since $\eta_Y\colon Y \to \Omega\,\mathrm{sh}\, Y$ is an isomorphism, composition with η_X induces an equivalence

$$\mathrm{Hom}_{\mathrm{PSp}(C)}(\Omega\,\mathrm{sh}\, X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSp}(C)}(X, Y);$$

an inverse sends a map $f\colon X \to Y$ to the composite $\Omega\,\mathrm{sh}\,X \xrightarrow{\Omega\,\mathrm{sh}(f)} \Omega\,\mathrm{sh}\,Y \xrightarrow{\eta_Y^{-1}} Y$. Therefore

$$\mathrm{Hom}_{\mathrm{Sp}(C)}(X^{\mathrm{sp}}, Y) \simeq \lim_k \mathrm{Hom}_{\mathrm{PSp}(C)}((\Omega\,\mathrm{sh})^k X, Y) \simeq \mathrm{Hom}_{\mathrm{PSp}(C)}(X, Y),$$

which is the desired adjunction. □

Lemma 4.3.20 (Colimits of spectra)**.** *Let C be an ∞-category with finite limits and sequential colimits, and assume that $\Omega\colon C_* \to C_*$ preserves sequential colimits. If C_* admits I-indexed colimits for some ∞-category I, then $\mathrm{Sp}(C)$ admits I-indexed colimits.*

Proof. Let $X\colon I \to \mathrm{Sp}(C)$ be a diagram. Compute its colimit in $\mathrm{PSp}(C)$, where colimits are pointwise by Lemma 4.3.8, and then apply spectrification. Since spectrification is left adjoint to the inclusion $\mathrm{Sp}(C) \hookrightarrow \mathrm{PSp}(C)$, the resulting spectrum has the universal property of the colimit in $\mathrm{Sp}(C)$. □

Lemma 4.3.21. *Filtered colimits in* Sp *are computed levelwise. In particular, the functor $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$ preserves filtered colimits.*

Proof. Filtered colimits in An_* are created in An, and the loop functor $\Omega\colon \mathrm{An}_* \to \mathrm{An}_*$ preserves them. Hence the levelwise colimit in $\mathrm{PSp}(\mathrm{An})$ of a filtered diagram of spectra again has isomorphisms as its structure maps, so it is already a spectrum. The final claim follows by evaluating at level zero. □

Lemma 4.3.22 (Suspension spectra)**.** *Let C be an ∞-category admitting finite limits, finite colimits, and sequential colimits, and assume that $\Omega\colon C_* \to C_*$ preserves sequential colimits. Then the functor $\Omega^\infty\colon \mathrm{Sp}(C) \to C_*$ admits a left adjoint*

$$\Sigma^\infty\colon C_* \to \mathrm{Sp}(C).$$

Proof. First define the free prespectrum functor

$$\Sigma^{\infty,\mathrm{pre}}\colon C_* \to \mathrm{PSp}(C)$$

by

$$\Sigma^{\infty,\mathrm{pre}}(X)_n := \Sigma^n X,$$

with structure maps $\Sigma^n X \to \Omega\Sigma^{n+1}X$ given by the unit of the adjunction $\Sigma \dashv \Omega$ in C_*. We claim that $\Sigma^{\infty,\mathrm{pre}}$ is left adjoint to evaluation at level zero $\mathrm{ev}_0\colon \mathrm{PSp}(C) \to C_*$. Indeed, a map $\Sigma^{\infty,\mathrm{pre}}(X) \to Y$ is determined by its level-zero component $X \to Y_0$: once $f_n\colon \Sigma^n X \to Y_n$ is known, compatibility with the structure maps forces $f_{n+1}\colon \Sigma^{n+1}X \to Y_{n+1}$ to be adjoint to the composite

$$\Sigma^n X \xrightarrow{f_n} Y_n \xrightarrow{\sigma_n^Y} \Omega Y_{n+1}.$$

Conversely, this recursive construction produces a compatible map of prespectra. The construction is natural in X and Y and applies equally to parametrized families of maps, so it gives an isomorphism of hom animae

$$\mathrm{Hom}_{\mathrm{PSp}(C)}(\Sigma^{\infty,\mathrm{pre}}X, Y) \simeq \mathrm{Hom}_{C_*}(X, Y_0).$$

Now set

$$\Sigma^\infty(X) := (\Sigma^{\infty,\mathrm{pre}}X)^{\mathrm{sp}}.$$

For $Y \in \mathrm{Sp}(C)$, spectrification gives natural equivalences

$$\mathrm{Hom}_{\mathrm{Sp}(C)}(\Sigma^\infty X, Y) \simeq \mathrm{Hom}_{\mathrm{PSp}(C)}(\Sigma^{\infty,\mathrm{pre}} X, Y) \simeq \mathrm{Hom}_{C_*}(X, \Omega^\infty Y),$$

which proves the adjunction. □

Corollary 4.3.23. *In the situation of Lemma 4.3.22, the functor $\Omega^\infty \colon \mathrm{Sp}(C) \to C$ admits a left adjoint*

$$\Sigma^\infty_+ \colon C \to \mathrm{Sp}(C)$$

given by $\Sigma^\infty_+(X) := \Sigma^\infty(X_+)$, where $X_+ := X \sqcup *$.

Proof. Since adjunctions compose, this is immediate from the previous lemma and the fact that a left adjoint to the forgetful functor $C_* \to C$ is given by $(-)_+ \colon C \to C_*$, $X \mapsto X_+ := X \sqcup *$. □

4.3.3 Standard presentations

Spectrification also gives a canonical way to build a spectrum from its levels.

Definition 4.3.24. Let C be an ∞-category with finite limits. For $n \geq 0$, define

$$\Omega^{\infty-n} \colon \mathrm{Sp}(C) \to C_*, \qquad \Omega^{\infty-n}(X) := \Omega^\infty(X[n]),$$

where $X[n]$ denotes the n-fold suspension of X in the stable ∞-category $\mathrm{Sp}(C)$. By Corollary 4.3.15, this functor evaluates at the n-th level in the prespectrum model: $\Omega^{\infty-n}(X) \simeq X_n$.

Lemma 4.3.25. *Let C be an ∞-category with finite limits. For $n \geq 0$ there are natural isomorphisms*

$$\Omega^n \Omega^{\infty-n} \cong \Omega^\infty \qquad \text{and} \qquad \Omega\Omega^{\infty-(n+1)} \cong \Omega^{\infty-n}.$$

Proof. This follows from stability of $\mathrm{Sp}(C)$ and the fact that $\Omega^\infty \colon \mathrm{Sp}(C) \to C_*$ preserves finite limits. □

Lemma 4.3.26. *In the situation of Lemma 4.3.22, the functor $\Omega^{\infty-n} \colon \mathrm{Sp}(C) \to C_*$ admits a left adjoint*

$$\Sigma^{\infty-n} \colon C_* \to \mathrm{Sp}(C)$$

given by $\Sigma^{\infty-n}(X) := (\Sigma^\infty(X))[-n]$. It satisfies the relations

$$\Sigma^{\infty-n}\Sigma^n \cong \Sigma^\infty \qquad \text{and} \qquad \Sigma^{\infty-(n+1)}\Sigma \simeq \Sigma^{\infty-n}.$$

Proof. Since adjunctions compose, the functor $(\Sigma^\infty(-))[-n]$ is left adjoint to $\Omega^\infty((-)[n]) = \Omega^{\infty-n}$. The stated relations follow by passing to left adjoints in the previous lemma. □

Remark 4.3.27. In the situation of Lemma 4.3.22, the suspension spectrum $\Sigma^\infty(X)$ of a pointed object $X \in C_*$ is the spectrification of the suspension prespectrum

$$(X, \Sigma X, \Sigma^2 X, \dots).$$

Equivalently, its n-th level is given by

$$\Sigma^\infty(X)_n \simeq \mathrm{colim}_{k \geq 0} \Omega^k \Sigma^{k+n} X.$$

The structure maps are the inverses of the natural isomorphisms

$$\Omega\Sigma^\infty(X)_{n+1} \simeq \Omega(\mathrm{colim}_{k\geq 0}\, \Omega^k\Sigma^{k+n+1}X) \simeq \mathrm{colim}_{k\geq 0}\, \Omega^{k+1}\Sigma^{k+n+1}X \simeq \Sigma^\infty(X)_n,$$

where we use that $\Omega\colon C_* \to C_*$ preserves sequential colimits.

Exercise 4.3.28. In the situation of Lemma 4.3.22, use the prespectrum model to verify directly that for any spectrum $Y \in \mathrm{Sp}(C)$ and any pointed object $X \in C_*$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sp}(C)}(\Sigma^\infty X, Y) \xrightarrow{\sim} \mathrm{Hom}_{C_*}(X, \Omega^\infty Y).$$

Proposition 4.3.29 (Associated spectrum)**.** *In the situation of Lemma 4.3.22, let X be a prespectrum in C. Its associated spectrum X^{sp} is naturally isomorphic to the following colimit in* $\mathrm{Sp}(C)$*:*

$$X^{\mathrm{sp}} \quad \simeq \quad \mathrm{colim}(\, \Sigma^\infty X_0 \to \Sigma^{\infty-1}X_1 \to \Sigma^{\infty-2}X_2 \to \Sigma^{\infty-3}X_3 \to \cdots).$$

Here the n-th transition map is given by the composite

$$\Sigma^{\infty-n}X_n \xrightarrow{\cong} \Sigma^{\infty-(n+1)}\Sigma X_n \xrightarrow{\Sigma\sigma_n^X} \Sigma^{\infty-(n+1)}\Sigma\Omega X_{n+1} \xrightarrow{\varepsilon} \Sigma^{\infty-(n+1)}X_{n+1},$$

where the first isomorphism is from Lemma 4.3.26, and the last map is induced by the counit $\varepsilon\colon \Sigma\Omega \to \mathrm{id}$ of the adjunction $\Sigma \dashv \Omega$ on C_*.

Proof. Let Y be a spectrum in $\mathrm{Sp}(C)$. Mapping out of the displayed colimit gives

$$\lim_n \mathrm{Hom}_{\mathrm{Sp}(C)}(\Sigma^{\infty-n}X_n, Y) \simeq \lim_n \mathrm{Hom}_{C_*}(X_n, \Omega^{\infty-n}Y).$$

The transition maps in this limit encode the compatibilities for a map of prespectra $X \to Y$. Hence the right-hand side identifies with $\mathrm{Hom}_{\mathrm{PSp}(C)}(X, Y)$. By the spectrification adjunction from Proposition 4.3.19, this is naturally equivalent to $\mathrm{Hom}_{\mathrm{Sp}(C)}(X^{\mathrm{sp}}, Y)$. The claim follows from Yoneda. □

Corollary 4.3.30 (Standard presentation)**.** *In the situation of Lemma 4.3.22, every spectrum $X \in \mathrm{Sp}(C)$ is naturally isomorphic to the spectrum associated to its underlying prespectrum:*

$$\mathrm{colim}_{n\geq 0}\, \Sigma^{\infty-n}X_n \quad \xrightarrow{\cong} \quad X,$$

where $X_n = \Omega^{\infty-n}X$.

Proof. Apply the previous proposition to the underlying prespectrum of X. Since X is already a spectrum, spectrification fixes it. □

4.4 The ∞-category of spectra

We now specialize the preceding construction to animae. After recording the basic properties of the resulting ∞-category of spectra, we construct hom spectra and tensoring by spectra, and use these to describe colimit-preserving functors out of Sp.

Definition 4.4.1. The ∞-*category of spectra* Sp is defined as the stabilization of the ∞-category of animae,

$$\mathrm{Sp} \quad := \quad \mathrm{Sp}(\mathrm{An}).$$

Objects of Sp are called *spectra*. Concretely, a spectrum is a sequence of pointed animae X_n equipped with isomorphisms

$$\sigma_n \colon X_n \xrightarrow{\sim} \Omega(X_{n+1}).$$

By Lemma 4.3.7, a morphism in $\mathrm{Sp} = \mathrm{Sp}(\mathrm{An})$ is precisely a sequence of levelwise pointed maps together with identifications making the structure squares commute. Thus these are exactly the morphisms introduced in Definition 3.2.2. The following are the central properties governing the behavior of Sp:

(1) The ∞-category Sp comes equipped with a left exact functor $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$. Given a spectrum X, we refer to $\Omega^\infty(X) = X_0$ as its *underlying anima*.

(2) Given a stable ∞-category D, any left exact functor $D \to \mathrm{An}$ uniquely lifts to an exact functor $D \to \mathrm{Sp}$: composition with Ω^∞ induces an equivalence:

$$\Omega^\infty \circ - \colon \mathrm{Fun}^{\mathrm{ex}}(D, \mathrm{Sp}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(D, \mathrm{An}).$$

(3) The ∞-category Sp admits small limits (Lemma 4.3.18) and colimits (Lemma 4.3.20).

(4) The functors $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$ and $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}_*$ admit left adjoints

$$\Sigma^\infty \colon \mathrm{An}_* \to \mathrm{Sp} \qquad \text{and} \qquad \Sigma^\infty_+ \colon \mathrm{An} \to \mathrm{Sp}$$

satisfying $\Sigma^\infty_+(X) \cong \Sigma^\infty(X_+)$ (Lemma 4.3.22 and Corollary 4.3.23).

Given a pointed anima X, we refer to the spectrum $\Sigma^\infty(X)$ as the *reduced suspension spectrum* of X. For unpointed X, we call $\Sigma^\infty_+(X)$ the *unreduced suspension spectrum* of X.

Definition 4.4.2 (Sphere spectrum)**.** We define the *sphere spectrum* as $\mathbb{S} := \Sigma^\infty(S^0) \simeq \Sigma^\infty_+(*) \in \mathrm{Sp}$. For an anima X, we will frequently use the alternative notation

$$\mathbb{S}[X] \quad := \quad \Sigma^\infty_+(X)$$

for the unreduced suspension spectrum. We wish to think of the spectrum $\mathbb{S}[X]$ as the 'free spectrum generated by X'. This notation mimics the notation for the free abelian group $\mathbb{Z}[S]$ generated by a set S, characterized by the property that group homomorphisms $\mathbb{Z}[S] \to A$ correspond to maps of sets $S \to A$.

Table 4.1 collects the resulting analogies between ordinary algebra and 'higher algebra'.

Ordinary algebra	Higher algebra
Set	Anima
Abelian group	Spectrum
Integers $\mathbb{Z}$	Sphere spectrum $\mathbb{S}$
Underlying set	Underlying anima $\Omega^\infty(X)$
Free abelian group $\mathbb{Z}[S]$	Unreduced suspension spectrum $\mathbb{S}[X]$

Table 4.1: The dictionary between ordinary algebra and 'higher algebra'.

Remark 4.4.3. Given a pointed anima $(X, x) \in \mathrm{An}_*$, note that we may write X as the cofiber of the basepoint inclusion $*_+ \to X_+$. Since $\Sigma^\infty(-)$ preserves cofibers, we may write the reduced suspension spectrum of X as

$$\Sigma^\infty(X, x) \quad \simeq \quad \mathrm{cofib}(\mathbb{S} \xrightarrow{x} \mathbb{S}[X]).$$

4.4.1 Hom spectra

Given an abelian category $\mathcal{A}$, the hom sets $\mathrm{Hom}_{\mathcal{A}}(A, B)$ can be equipped with a canonical abelian group structure, where the sum $f + g$ of two morphisms $f, g \colon A \to B$ is given by the composite

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\nabla} B.$$

In an analogous way, the hom animae $\mathrm{Hom}_C(X, Y)$ in a stable ∞-category C can be canonically lifted to spectra:

Proposition 4.4.4. *For a stable ∞-category C, there exists a unique functor*

$$\mathrm{hom}_C(-,-) \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}$$

that is exact in both variables and fits in a commutative triangle as follows:

$$\begin{array}{ccc} & & \mathrm{Sp} \\ & \overset{\mathrm{hom}_C}{\nearrow} & \big\downarrow{\scriptstyle \Omega^\infty} \\ C^{\mathrm{op}} \times C & \xrightarrow[\mathrm{Hom}_C]{} & \mathrm{An}. \end{array}$$

Proof. Consider the Yoneda embedding

$$C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}), \qquad Y \mapsto \mathrm{Hom}_C(-, Y).$$

This lands in left exact functors, so by the equivalence $\mathrm{Fun}^{\mathrm{ex}}(C^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(C^{\mathrm{op}}, \mathrm{An})$ we may uniquely lift it to a (fully faithful) functor

$$C \hookrightarrow \mathrm{Fun}^{\mathrm{ex}}(C^{\mathrm{op}}, \mathrm{Sp}), \qquad Y \mapsto \mathrm{hom}_C(-, Y).$$

The resulting functor $\mathrm{hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}$ is exact in the first variable by construction. It remains to show it is also exact in the second variable: for fixed X, we need to show that the functor $\mathrm{hom}_C(X, -) \colon C \to \mathrm{Sp}$ is exact. It suffices to check this after composing with the functors $(-)_n \colon \mathrm{Sp} \to \mathrm{An}$, which are left exact and jointly conservative. Since we have $\mathrm{hom}_C(X, Y)_n \simeq \mathrm{hom}_C(X[-n], Y)_0 \simeq \mathrm{Hom}_C(X[-n], Y)$, this follows from the left exactness of $\mathrm{Hom}_C(X[-n], -) \colon C \to \mathrm{An}$. □

Terminology 4.4.5. We refer to the spectra $\mathrm{hom}_C(X, Y)$ as the *hom spectra* or *mapping spectra* of C.

We explicitly record the following observation made in the proof of the proposition:

Corollary 4.4.6 (Stable Yoneda lemma)**.** *For a stable ∞-category C, the functor*

$$C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Sp}), \qquad Y \mapsto \hom_C(-, Y)$$

is fully faithful. □

Exercise 4.4.7. Show that for $C = \mathrm{Sp}$, the Hom spectrum $\hom(X, Y) := \hom_{\mathrm{Sp}}(X, Y)$ may be explicitly described at level n by

$$\hom(X, Y)_n \quad \simeq \quad \mathrm{Hom}_{\mathrm{Sp}}(X, Y[n]).$$

Lemma 4.4.8. *The functor* $\hom(\mathbb{S}, -) \colon \mathrm{Sp} \to \mathrm{Sp}$ *is equivalent to the identity.*

Proof. It suffices to show that these two functors agree after applying Ω^∞, and indeed we have

$$\Omega^\infty \hom(\mathbb{S}, -) \cong \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, -) \cong \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}[*], -) \cong \mathrm{Hom}_{\mathrm{An}}(*, \Omega^\infty(-)) \cong \Omega^\infty(-).$$

Here we use that $\mathrm{Hom}_{\mathrm{An}}(*, -) \colon \mathrm{An} \to \mathrm{An}$ is equivalent to the identity, since its left adjoint $- \times * \colon \mathrm{An} \to \mathrm{An}$ is equivalent to the identity. □

We will use the standard presentation from Corollary 4.3.30 in the following form: if X is a spectrum with levels $X_n = \Omega^{\infty - n} X$, then

$$X \simeq \mathrm{colim}_{n \geq 0} \Sigma^{\infty - n} X_n.$$

Together with Remark 4.4.3, this says that every spectrum can be built from unreduced suspension spectra using shifts, cofibers and colimits.

4.4.2 The universal property of spectra

The ∞-category of spectra is defined via a universal property for left exact functors into it. We will now prove another universal property in terms of functors *out of it*: colimit-preserving functors $\mathrm{Sp} \to C$ into a cocomplete stable ∞-category C are uniquely determined by their value on the sphere spectrum.

Proposition 4.4.9 (Tensoring by spectra)**.** *Let C be a stable ∞-category with small colimits. Then C is tensored over* Sp*: there exists a unique functor*

$$- \otimes - \colon \mathrm{Sp} \times C \to C$$

equipped with a natural equivalence of spectra

$$\hom_C(X \otimes Y, Z) \quad \cong \quad \hom_{\mathrm{Sp}}(X, \hom_C(Y, Z))$$

for $X \in \mathrm{Sp}$ and $Y, Z \in C$.

Proof. The uniqueness of $X \otimes Y$ is clear from the stable Yoneda lemma, Corollary 4.4.6. For existence, we need to show that for every $Y \in C$ the functor

$$\Phi \colon \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Fun}(C, \mathrm{Sp}), \qquad \Phi(X) := \hom_{\mathrm{Sp}}(X, \hom_C(Y, -))$$

lands in the full subcategory $C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{Sp})$ of corepresentable functors, i.e. functors of the form $\hom_C(W, -)$ for some $W \in C$.

Step 1: We first show that $\Phi(X)$ is corepresentable whenever $X = \mathbb{S}[X']$ is an unreduced suspension spectrum for some anima $X' \in \mathrm{An}$. To this end, recall from Theorem 1.8.10 that evaluation at $*$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{An}, C) \quad \xrightarrow{\sim} \quad C.$$

We let $- \otimes Y \colon \mathrm{An} \to C$ denote the unique colimit-preserving functor satisfying $* \otimes Y \cong Y$. We then claim that there exists a natural isomorphism

$$\Phi(\mathbb{S}[X']) \quad \cong \quad \hom_C(X' \otimes Y, -)$$

of functors $C \to \mathrm{Sp}$ for every $X' \in \mathrm{An}$. For this, note that both sides define colimit-preserving functors $\mathrm{An} \to \mathrm{Fun}(C, \mathrm{Sp})^{\mathrm{op}}$ by varying X'. Indeed, on either side a colimit in X' becomes a limit in $\mathrm{Fun}(C, \mathrm{Sp})$ (that is, a colimit in $\mathrm{Fun}(C, \mathrm{Sp})^{\mathrm{op}}$), because the relevant mapping-spectrum functor is contravariant in the variable determined by X'. It therefore suffices to produce an isomorphism for $X' = *$. And indeed, in that case both sides evaluate to $\hom_C(Y, -)$ by Lemma 4.4.8 and the given isomorphism $* \otimes Y \cong Y$.

Step 2: We now show the claim for an arbitrary spectrum X. Observe that the functor Φ is exact and sends colimits of spectra to limits in the functor category. Also observe that the corepresentable functors in $\mathrm{Fun}(C, \mathrm{Sp})$ are closed under limits and shifts. It thus remains to show that every spectrum X can be built from unreduced suspension spectra using shifts, cofibers and colimits. By Corollary 4.3.30, every spectrum is a colimit of shifts of reduced suspension spectra. Moreover, Remark 4.4.3 writes every reduced suspension spectrum $\Sigma^\infty(Z, z)$ as the cofiber of a map $\mathbb{S} \to \mathbb{S}[Z]$ between unreduced suspension spectra. □

Corollary 4.4.10. *Let C be a stable ∞-category with small colimits. For every object $Y \in C$, the functor* $\hom_C(Y, -) \colon C \to \mathrm{Sp}$ *admits a left adjoint*

$$- \otimes Y \colon \mathrm{Sp} \to C.$$

Proof. Applying Ω^∞ to the natural isomorphism from Proposition 4.4.9 provides the adjunction equivalence

$$\mathrm{Hom}_C(X \otimes Y, Z) \quad \cong \quad \mathrm{Hom}_{\mathrm{Sp}}(X, \hom_C(Y, Z))$$

for $X \in \mathrm{Sp}$ and $Z \in C$. □

Fixing some stable ∞-category C with small colimits, let us now establish various basic properties of tensoring on C.

Lemma 4.4.11. *The functor $- \otimes - \colon \mathrm{Sp} \times C \to C$ preserves colimits in both variables.*

Proof. By the Yoneda lemma, it suffices to show that $\mathrm{Hom}_C(- \otimes -, Z) \colon \mathrm{Sp}^{\mathrm{op}} \times C^{\mathrm{op}} \to \mathrm{An}$ preserves limits in both variables, for all $Z \in C$. This functor is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Sp}}(-, \hom_C(-, Z))$. This clearly preserves limits in the first variable. For the second variable, this follows from the fact that $\Omega^\infty \hom_C(-, Z) = \mathrm{Hom}_C(-, Z) \colon C^{\mathrm{op}} \to \mathrm{An}$ preserves limits. □

Lemma 4.4.12. *The functor $\mathbb{S} \otimes - \colon C \to C$ is equivalent to the identity.*

Proof. For objects $Y, Z \in C$, there are natural isomorphisms

$$\hom_C(Y,Z) \overset{4.4.8}{\cong} \hom_{\mathrm{Sp}}(\mathbb{S}, \hom_C(Y,Z)) \cong \hom_C(\mathbb{S} \otimes Y, Z),$$

hence by the stable Yoneda lemma we obtain a natural isomorphism $\mathbb{S} \otimes Y \cong Y$. □

Lemma 4.4.13. *For a colimit-preserving functor $F\colon \mathrm{Sp} \to C$, there exists a natural isomorphism*

$$F(-) \;\cong\; - \otimes F(\mathbb{S}).$$

Moreover, this natural isomorphism can be chosen naturally in $F \in \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, C)$.

Proof. We first produce a natural isomorphism $F(\mathbb{S}[-]) \cong \mathbb{S}[-] \otimes F(\mathbb{S})$ of functors $\mathrm{An} \to C$. In light of the equivalence $\mathrm{ev}_*\colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{An}, C) \xrightarrow{\sim} C$, it suffices to produce an isomorphism $F(\mathbb{S}) \cong \mathbb{S} \otimes F(\mathbb{S})$, for which we take the one from Lemma 4.4.12.

Now, for computing F on an arbitrary spectrum X, we use the standard presentation from Corollary 4.3.30. Using that F preserves colimits and shifts, we then get

$$F(X) \simeq F(\mathrm{colim}_n \Sigma^{\infty-n} X_n) \simeq \mathrm{colim}_n \Omega^n F(\Sigma^\infty X_n) \simeq \mathrm{colim}_n \Omega^n \,\mathrm{cofib}(F(\mathbb{S}) \to F(\mathbb{S}[X_n])),$$

where the last equivalence uses $\Sigma^\infty X_n \simeq \mathrm{cofib}(\mathbb{S} \to \mathbb{S}[X_n])$. Applying the same reasoning to $F' = - \otimes F(\mathbb{S})$, we similarly obtain

$$X \otimes F(\mathbb{S}) \simeq \mathrm{colim}_n \Omega^n \,\mathrm{cofib}(F(\mathbb{S}) \to \mathbb{S}[X_n] \otimes F(\mathbb{S})).$$

The claim now follows from the identification $F(\mathbb{S}[X_n]) \cong \mathbb{S}[X_n] \otimes F(\mathbb{S})$ established before. These equivalences are natural in both F and X: the standard presentation is natural in X, while the initial identification of colimit-preserving functors $\mathrm{An} \to C$ is natural in F. We therefore obtain the claimed natural isomorphism in $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, C)$. □

Theorem 4.4.14 (Universality of Sp)**.** *Let C be a stable ∞-category with small colimits. Then evaluation at the sphere spectrum induces an equivalence*

$$\mathrm{ev}_{\mathbb{S}}\colon\; \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, C) \quad\xrightarrow{\sim}\quad C.$$

Proof. We claim that an inverse is given by the tensor product functor

$$C \;\to\; \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, C), \qquad\qquad X \;\mapsto\; - \otimes X.$$

A natural isomorphism $\mathbb{S} \otimes X \;\cong\; X$ is provided by Lemma 4.4.12. A natural isomorphism $F \cong - \otimes F(\mathbb{S})$ is provided by Lemma 4.4.13. □

We may summarize this result by saying that Sp is the *free cocomplete stable ∞-category generated by $\mathbb{S}$*.

Corollary 4.4.15. *Let C be a stable ∞-category with small colimits. Then every colimit-preserving functor $F\colon \mathrm{Sp} \to C$ admits a right adjoint $F^R\colon C \to \mathrm{Sp}$.*

Proof. By Theorem 4.4.14, F is of the form $- \otimes X$ for some $X \in C$, hence has right adjoint given by the mapping spectrum functor $\hom_C(X, -)\colon C \to \mathrm{Sp}$. □

4.4.3 Tensor products of spectra

Given two abelian groups A and B, we may form their *tensor product* $A \otimes B$, which is characterized by the property that group homomorphisms $A \otimes B \to C$ correspond to bilinear maps $A \times B \to C$, or equivalently to morphisms of abelian groups $A \to \mathrm{Hom}_{\mathrm{Ab}}(B, C)$. We will now construct a tensor product of spectra with analogous properties.

Proposition 4.4.16. *There exists a unique tensor product functor*

$$- \otimes - \colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$$

which preserves colimits in both variables and satisfies $\mathbb{S} \otimes \mathbb{S} \cong \mathbb{S}$.

Proof. Currying and two applications of Theorem 4.4.14 give equivalences

$$\mathrm{Fun}^{\mathrm{colim},\mathrm{colim}}(\mathrm{Sp} \times \mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, \mathrm{Sp})) \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Sp},$$

where the intermediate functor category is stable and admits small colimits computed pointwise. Under this equivalence, a functor is sent to its value on $(\mathbb{S}, \mathbb{S})$. The sphere spectrum therefore uniquely determines the required functor. □

Let us now establish various basic properties of the tensor product.

Lemma 4.4.17. *For spectra* $X, Y, Z \in \mathrm{Sp}$*, there are natural isomorphisms of spectra*

$$\mathbb{S} \otimes X \cong X \cong X \otimes \mathbb{S}, \qquad X \otimes Y \cong Y \otimes X \qquad (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$$

and

$$\hom(Y, \hom(X, Z)) \cong \hom(X \otimes Y, Z) \cong \hom(X, \hom(Y, Z)).$$

Proof. For the first row, each of these expressions define functors that are colimit-preserving in each of their arguments. By Theorem 4.4.14, it thus suffices to produce such isomorphisms after evaluation at the sphere spectrum, where they are immediate consequences of the isomorphism $\mathbb{S} \otimes \mathbb{S} \cong \mathbb{S}$.

For the second row, we may fix Z and regard these expressions as functors $\mathrm{Sp}^{\mathrm{op}} \times \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Sp}$ that preserve limits in both variables, or equivalently as functors $\mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{op}}$ preserving colimits in both variables. It then again suffices to prove they are isomorphic after setting $X = Y = \mathbb{S}$, which follows from the natural isomorphism $\hom(\mathbb{S}, Z) \cong Z$ from Lemma 4.4.8. □

Lemma 4.4.18. *There exists a natural isomorphism*

$$\mathbb{S}[X] \otimes \mathbb{S}[Y] \quad \cong \quad \mathbb{S}[X \times Y]$$

for all $X, Y \in \mathrm{An}$.

Proof. Both of the functors $\mathbb{S}[-] \otimes \mathbb{S}[-], \mathbb{S}[- \times -] \colon \mathrm{An} \times \mathrm{An} \to \mathrm{Sp}$ preserve colimits in both variables, hence they are naturally isomorphic if and only if they agree on $(*, *)$, see Corollary 1.8.11. But on $(*, *)$ both sides evaluate to $\mathbb{S} \otimes \mathbb{S} \simeq \mathbb{S}$. □

Lemma 4.4.19. *There exists a natural isomorphism*

$$\Sigma^\infty(X) \otimes \Sigma^\infty(Y) \quad \cong \quad \Sigma^\infty(X \wedge Y)$$

for all $X, Y \in \mathrm{An}_*$.

Proof. Since $- \otimes -$ preserves cofibers in each variable, this follows by combining the previous lemma with the expression $\Sigma^\infty(Z) \simeq \mathrm{cofib}(\mathbb{S} \to \mathbb{S}[Z])$ from Remark 4.4.3 and the isomorphism $X_+ \wedge Y_+ \cong (X \times Y)_+$ from Lemma 2.4.16. □

Remark 4.4.20. Let X and Y be spectra. The standard presentations of X and Y provide the following description of their tensor product:

$$X \otimes Y \quad \cong \quad \mathrm{colim}_n \mathrm{colim}_m (\Sigma^{\infty-n} X_n \otimes \Sigma^{\infty-m} Y_m) \quad \simeq \quad \mathrm{colim}_n \mathrm{colim}_m \Sigma^{\infty-n-m}(X_n \wedge Y_m),$$

where the last equivalence holds by the previous lemma. However, this formula should be used with care: it makes implicit use of the additivity of the shift functors $X[-n][-m] \cong X[-(n+m)]$ from Warning 4.2.11. To avoid introducing sign errors, we will not work with this formula for the tensor product.

4.4.4 Homotopy groups

Important invariants of spectra are their *homotopy groups*:

Definition 4.4.21. For a spectrum X and an integer k, we define its *k-th homotopy group* $\pi_k(X) \in \mathrm{Ab}$ as

$$\pi_k(X) := [\mathbb{S}[k], X] := \pi_0 \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}[k], X).$$

The assignment $X \mapsto \pi_k(X)$ defines a functor $\pi_* \colon \mathrm{Sp} \to \mathrm{Ab}$.

Exercise 4.4.22. Show that we have $\pi_k(X[n]) \simeq \pi_{k-n}(X)$ for all $k, n \in \mathbb{Z}$.

Exercise 4.4.23. Show that for $n \geq 0$ and $k + n \geq 0$ we have $\pi_k(X) \cong \pi_{n+k}(X_n)$. In particular we have $\pi_k(X) = \pi_k(\Omega^\infty X)$ for $k \geq 0$.

Example 4.4.24. For a pointed anima X, the negative homotopy groups of its suspension spectrum $\Sigma^\infty X$ vanish. For $k \geq 0$, Remark 4.3.27 gives

$$\pi_k(\Sigma^\infty(X)) \;\cong\; \pi_k(\Omega^\infty\Sigma^\infty(X)) \;\cong\; \mathrm{colim}_{n\geq 0}\, \pi_k(\Omega^n\Sigma^n X) = \mathrm{colim}_n\, \pi_{n+k}(\Sigma^n X).$$

These groups are known as the *stable homotopy groups* of X. By the classical Freudenthal suspension theorem [Fre38] (see also [Swi75, Theorem 6.26]), the maps $\pi_{n+k}(\Sigma^n X) \to \pi_{n+k+1}(\Sigma^{n+1} X)$ are isomorphisms for $n \geq k + 2$.

Example 4.4.25. By Remark 2.4.20, there is an isomorphism $\deg \colon \pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ for every $n \geq 1$, and the suspension maps correspond to the identity of $\mathbb{Z}$. It follows that there is an isomorphism $\pi_0(\mathbb{S}) \cong \mathrm{colim}_n\, \pi_n(S^n) \cong \mathbb{Z}$.

Example 4.4.26 (The stable Hopf map)**.** The first positive stable stem already contains information that ordinary homology does not detect. Let

$$h\colon S^3 \longrightarrow S^2$$

be the Hopf fibration. Concretely, it is obtained by applying Π_∞ to the quotient map

$$\left\{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\right\} \longrightarrow \mathbb{C}\mathrm{P}^1, \qquad (x,y) \longmapsto [x:y],$$

and identifying its source and target with S^3 and S^2, respectively. Applying the suspension-spectrum functor and shifting by -2 gives the *stable Hopf map*

$$\eta\colon \mathbb{S}[1] \longrightarrow \mathbb{S}\,.$$

The classical calculation $\pi_4(S^3) \cong \mathbb{Z}/2$ identifies the suspension of h with its nonzero element. The Freudenthal suspension theorem then shows that all subsequent suspensions are isomorphisms, and hence

$$\pi_1(\mathbb{S}) \cong \mathbb{Z}/2,$$

generated by η; see, for example, [Swi75, p. 87].

If we equip $\mathbb{C}\mathrm{P}^2$ with its usual cell structure $\mathbb{C}\mathrm{P}^2 \cong S^2 \cup_h e^4$ and use its 0-cell as basepoint, we obtain an exact sequence of spectra

$$\mathbb{S}[3] \xrightarrow{\Sigma^\infty h} \mathbb{S}[2] \longrightarrow \Sigma^\infty \Pi_\infty(\mathbb{C}\mathrm{P}^2).$$

The attaching map is the nonzero class $\eta[2]$. Ordinary reduced integral homology, however, sees only a copy of $\mathbb{Z}$ in degrees 2 and 4 and does not detect this attaching map. Thus the stable cell structure of $\mathbb{C}\mathrm{P}^2$ remembers information invisible to ordinary integral homology.

Corollary 4.4.27. *The functors $\pi_k\colon \mathrm{Sp} \to \mathrm{Ab}$ for $k \in \mathbb{Z}$ are jointly conservative: a morphism of spectra $f\colon X \to Y$ is an isomorphism if and only if the induced map $\pi_k(f)\colon \pi_k(X) \to \pi_k(Y)$ is an isomorphism.*

Proof. By Lemma 4.3.6, f is an isomorphism if and only if each map $f_n\colon X_n \to Y_n$ of animae is an isomorphism. Translation between components of the loop animae X_n and Y_n reduces this to homotopy groups at the basepoint. By the Whitehead theorem for animae, Proposition 2.4.22, it therefore suffices that f_n induces isomorphisms on these homotopy groups, including π_0. By Exercise 4.4.23 this condition is equivalent to the condition that f induces isomorphisms on all homotopy groups. □

Lemma 4.4.28. *For every $k \in \mathbb{Z}$, the functor $\pi_k\colon \mathrm{Sp} \to \mathrm{Ab}$ preserves filtered colimits.*

Proof. By Lemma 4.3.21, a filtered colimit of spectra is computed levelwise. The claim then follows from Exercise 4.4.23 together with the fact that the homotopy groups of pointed animae preserve filtered colimits. □

Lemma 4.4.29. *For every $k \in \mathbb{Z}$, the functor $\pi_k\colon \mathrm{Sp} \to \mathrm{Ab}$ preserves arbitrary coproducts.*

Proof. Finite coproducts in Sp agree with finite products, and π_k preserves both of these. Since an arbitrary coproduct is the filtered colimit of its finite subcoproducts, the claim follows from Lemma 4.4.28. □

Proposition 4.4.30 (Long exact sequence of homotopy groups)**.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be an exact sequence of spectra. Then there is a long exact sequence of homotopy groups of the form*

$$\cdots \to \pi_1(Y) \xrightarrow{g_*} \pi_1(Z) \to \pi_0(X) \xrightarrow{f_*} \pi_0(Y) \xrightarrow{g_*} \pi_0(Z) \to \pi_{-1}(X) \xrightarrow{f_*} \pi_{-1}(Y) \to \dots.$$

Proof. Since every exact sequence of spectra $X \xrightarrow{f} Y \xrightarrow{g} Z$ leads to two new exact sequences

$$Z[-1] \to X \xrightarrow{f} Y \qquad \text{and} \qquad Y \xrightarrow{g} Z \to X[1],$$

it inductively suffices to prove that the sequence $\pi_0(X) \xrightarrow{f} \pi_0(Y) \xrightarrow{g} \pi_0(Z)$ is exact. It is clear that the composite is the zero map. Conversely, consider a class $[\sigma] \in \pi_0(Y)$, represented by a morphism of spectra $\sigma\colon \mathbb{S} \to Y$, and assume that the class $[g \circ \sigma] \in \pi_0(Z)$ is zero. Then $\sigma\colon \mathbb{S} \to Y$ factors through the fiber X of $g\colon Y \to Z$, and hence is of the form $f \circ \tau$ for some $\tau\colon \mathbb{S} \to X$. It follows that $[\sigma]$ is in the image of $f\colon \pi_0(X) \to \pi_0(Y)$, as desired. □

4.4.5 Homology theories represented by spectra

In Chapter 3, a spectrum E directly gave a cohomology theory by mapping into its individual pointed animae E_k. We now have the additional structure needed to associate a homology theory to E as well.

Definition 4.4.31 (E-homology)**.** Let E be a spectrum and let $X \in \mathrm{An}_*$ be a pointed anima. For $k \in \mathbb{Z}$, we define the *reduced E-homology* of X by

$$\widetilde{E}_k(X) := \pi_k(\Sigma^\infty X \otimes E).$$

Proposition 4.4.32. *For every spectrum E, the functors $\widetilde{E}_*$ admit natural suspension isomorphisms that make them into a reduced homology theory in the sense of Definition 3.1.2.*

Proof. The natural isomorphism $\Sigma^\infty(\Sigma X) \cong (\Sigma^\infty X)[1]$ gives suspension isomorphisms

$$\widetilde{E}_{k+1}(\Sigma X) = \pi_{k+1}((\Sigma^\infty X \otimes E)[1]) \cong \pi_k(\Sigma^\infty X \otimes E) = \widetilde{E}_k(X).$$

Both Σ^∞ and $- \otimes E$ preserve cofiber sequences. Exactness therefore follows from the long exact sequence of homotopy groups in Proposition 4.4.30.

It remains to verify the wedge axiom. The functor $\Sigma^\infty(-) \otimes E$ preserves coproducts, and each $\pi_k\colon \mathrm{Sp} \to \mathrm{Ab}$ preserves coproducts by Lemma 4.4.29. Thus

$$\bigoplus_{i \in I} \widetilde{E}_k(X_i) \xrightarrow{\cong} \widetilde{E}_k\left(\bigvee_{i \in I} X_i\right),$$

as required. □

Remark 4.4.33 (Comparison with the earlier definition)**.** The cohomology theory associated to E in Definition 3.2.3 can now be expressed in terms of the mapping spectrum. Indeed, the adjunction

$\Sigma^\infty \dashv \Omega^\infty$ gives

$$\begin{aligned}\pi_{-k}\hom(\Sigma^\infty X, E) &\cong [\Sigma^\infty X[-k], E] \\ &\cong [\Sigma^\infty X, E[k]] \\ &\cong [X, \Omega^\infty(E[k])]_* \\ &\cong [X, E_k]_* = \widetilde{E}^k(X).\end{aligned}$$

Here the last isomorphism uses the spectrum structure maps for $k \geq 0$ and the convention $E_k = \Omega^{-k}E_0$ for $k < 0$. All displayed isomorphisms are isomorphisms of abelian groups: under $\Omega^\infty(E[k]) \simeq E_k$, the canonical infinite-loop group structure agrees with the double-loop group structure used in Definition 3.2.3.

4.4.6 Mapping telescopes and the Milnor sequence

We record a general formula for sequential colimits in stable ∞-categories, an abstract form of the classical *mapping telescope* which will let us later resolve a spectrum by direct sums of spheres. By duality this provides a description of an inverse limit, giving us the Milnor sequence.

Lemma 4.4.34 (Mapping telescope)**.** *Let C be a stable ∞-category with countable coproducts, and let $Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} Y_2 \to \dots$ be a sequential diagram in C. Write $\iota_k \colon Y_k \to \bigoplus_{j\geq 0} Y_j$ for the inclusion of the k-th summand. Then there is an exact sequence*

$$\bigoplus_{k\geq 0} Y_k \xrightarrow{\mathrm{id}-f} \bigoplus_{k\geq 0} Y_k \longrightarrow \operatorname{colim}_k Y_k,$$

where $\mathrm{id}-f$ *restricts on the k-th summand to the difference $\iota_k - \iota_{k+1}\circ f_k$.*

Proof. By Lemma 2.4.29, the colimit $\operatorname{colim}_k Y_k$ is the pushout of the two maps from $\bigoplus_k Y_k$ to the coproducts of the even and odd terms. In a stable ∞-category, the pushout of a span $B \xleftarrow{a} A \xrightarrow{b} C$ is the cofiber of

$$(a, -b) \colon A \longrightarrow B \oplus C.$$

After identifying the coproduct of the even and odd terms with $\bigoplus_k Y_k$ and changing the signs on the odd source summands, this map becomes $\mathrm{id}-f$. □

Corollary 4.4.35 (Dual mapping telescope)**.** *Let C be a stable ∞-category with countable products. For every tower*

$$\dots \xrightarrow{f_2} Y_2 \xrightarrow{f_1} Y_1 \xrightarrow{f_0} Y_0$$

in C, there is an exact sequence

$$\lim_k Y_k \longrightarrow \prod_{k\geq 0} Y_k \xrightarrow{\mathrm{id}-f} \prod_{k\geq 0} Y_k,$$

where the k-th component of $(\mathrm{id}-f)(y)$ *is $y_k - f_k(y_{k+1})$.*

Proof. Apply Lemma 4.4.34 in the opposite stable ∞-category C^{op}. □

To state the resulting short exact sequence, we first introduce the one derived-limit construction that we need.

Definition 4.4.36 (First derived limit)**.** Let $\cdots \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_0} A_0$ be a tower of abelian groups. Define a homomorphism

$$d\colon \prod_{k\geq 0} A_k \longrightarrow \prod_{k\geq 0} A_k, \qquad d(a)_k := a_k - f_k(a_{k+1}).$$

The *first derived limit* of the tower is the abelian group

$$\lim_k^1 A_k := \operatorname{coker}(d).$$

Notice that the ordinary inverse limit $\lim_k A_k$ is the kernel of d.

Corollary 4.4.37 (Milnor sequence)**.** *For every tower $\cdots \to Y_2 \to Y_1 \to Y_0$ of spectra and every $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \longrightarrow \lim_k^1 \pi_{n+1}(Y_k) \longrightarrow \pi_n\big(\lim_k Y_k\big) \longrightarrow \lim_k \pi_n(Y_k) \longrightarrow 0.$$

Proof. Homotopy groups commute with products of spectra: indeed, the mapping anima functor $\mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}[n], -)$ preserves limits, and π_0 preserves products of animae. Consequently, applying the long exact sequence of Proposition 4.4.30 to the fiber sequence of Corollary 4.4.35 gives an exact sequence

$$\prod_k \pi_{n+1}(Y_k) \xrightarrow{d} \prod_k \pi_{n+1}(Y_k) \longrightarrow \pi_n\big(\lim_k Y_k\big) \longrightarrow \prod_k \pi_n(Y_k) \xrightarrow{d} \prod_k \pi_n(Y_k).$$

The claim follows by identifying the cokernel of the first map and the kernel of the last map using Definition 4.4.36. □

Exercises

Exercise 4.1. Let C and D be ∞-categories with finite limits. Show that there is an equivalence

$$\mathrm{Sp}(C \times D) \xrightarrow{\sim} \mathrm{Sp}(C) \times \mathrm{Sp}(D).$$

Exercise 4.2. Let C be a semiadditive ∞-category.

(1) Show that for all objects X and Y in C the set $[X, Y] := \pi_0 \mathrm{Hom}_C(X, Y)$ of homotopy classes of morphisms from X to Y admits a canonical structure of an abelian monoid, where addition of $f, g \in [X, Y]$ is given by the composite

$$f + g\colon \quad X \xrightarrow{\Delta} X \oplus X \xrightarrow{f\oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

(2) Show that for all morphisms $f\colon X' \to X$ and $g\colon Y \to Y'$ in C the composition maps

$$- \circ f\colon [X, Y] \to [X', Y] \qquad \text{and} \qquad g \circ -\colon [X, Y] \to [X, Y']$$

are homomorphisms of abelian monoids.

(3) Let M be an abelian monoid. Show that M is an abelian group if and only if the map

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : M \oplus M \to M \oplus M, \qquad (x, y) \mapsto (x + y, y)$$

is a bijection of sets.

(4) Conclude that the semiadditive ∞-category C is in fact *additive* if and only if the abelian monoid $[X, Y]$ is an abelian group for all X and Y.

Exercise 4.3. Let C be a pointed ∞-category with finite limits.

(1) Show that for every object X, the loop object ΩX admits the structure of a group object in the homotopy category $\mathrm{Ho}(C)$: there are maps

$$e : * \to \Omega X, \qquad m : \Omega X \times \Omega X \to \Omega X, \qquad \text{and} \qquad i : \Omega X \to \Omega X$$

whose classes satisfy the usual group relations in $\mathrm{Ho}(C)$.

(2) Show that for objects X and Y in C, the set $[X, \Omega Y]$ is canonically a group. Show that $[X, \Omega^2 Y]$ is an abelian group.

(3) Deduce that when C is stable, the Hom-set $[X, Y]$ in $\mathrm{Ho}(C)$ is an abelian group for all X and Y, and that composition in C defines group homomorphisms.

(4) Show that the resulting abelian group structure on $[X, Y]$ agrees with the one defined in Exercise 4.2.

Remark 4.4.38. Dually, if C is pointed and admits finite colimits, then for every X the suspension ΣX is a *cogroup object* in $\mathrm{Ho}(C)$, and hence that $[\Sigma X, Y]$ admits a group structure for all Y.

Remark 4.4.39. In Chapter 5, we will upgrade the group structure of ΩX in $\mathrm{Ho}(C)$ to a group structure in C.

Exercise 4.4. Let X be a pointed anima. Consider the prespectrum $\Sigma^{\infty,\mathrm{pre}}(X) := (\Sigma^n X, \sigma_n)$, where the structure maps are given by the unit maps

$$\Sigma^n X \to \Omega\Sigma(\Sigma^n X) = \Omega\Sigma^{n+1} X$$

of the adjunction $\Sigma \dashv \Omega$. Show that the spectrum associated to the prespectrum $\Sigma^{\infty,\mathrm{pre}}(X)$ is $\Sigma^\infty(X)$.

Hint: show that all the maps

$$\Sigma^{\infty-n}(\Sigma^n X) \to \Sigma^{\infty-(n+1)}(\Sigma^{n+1} X)$$

appearing in the definition of the associated spectrum are isomorphisms.

Exercise 4.5. Let X be a finite cell complex (Definition 2.3.1). Show that $\Pi_\infty(X)$ is a finite anima.

Exercise 4.6. Show that for every spectrum X, the functor $\hom(-, X) : \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Sp}$ admits a left adjoint, which is given by $\hom(-, X)^{\mathrm{op}} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{op}}$.

Exercise 4.7 (Shifts of hom spectra)**.** Let C be a stable ∞-category and let $X, Y \in C$. Prove that

$$\hom_C(X[k], Y) \cong \hom_C(X, Y)[-k] \qquad \text{and} \qquad \hom_C(X, Y[k]) \cong \hom_C(X, Y)[k]$$

Conclude that $\hom(\mathbb{S}[m], \mathbb{S}[n]) \cong \mathbb{S}[n-m]$ for all $m, n \in \mathbb{Z}$.

Exercise 4.8 (Represented theories of spheres and Moore spectra)**.** For $m \geq 2$, let $\mathbb{S}/m$ be the cofiber of multiplication by m on the sphere spectrum. Let E be a spectrum.

(1) Prove that $\widetilde{E}_k(S^n) \cong \pi_{k-n}(E)$ for every $n \geq 0$.

(2) Deduce natural short exact sequences

$$0 \to \pi_k(E)/m \to \pi_k(E \otimes \mathbb{S}/m) \to \pi_{k-1}(E)[m] \to 0.$$

(3) Deduce natural short exact sequences

$$0 \to \pi_{k+1}(E)/m \to \pi_k \hom(\mathbb{S}/m, E) \to \pi_k(E)[m] \to 0.$$

(4) Express the reduced E-homology of the Moore anima from Exercise 3.3 in terms of the groups in part (2).

5 The recognition principle

The recognition principle makes precise the analogy between spectra and abelian groups. In this chapter, we define an ∞-category $\mathrm{CGrp}(\mathrm{An})$ of commutative groups in An, also known as E_∞-*groups*, and construct a fully faithful functor

$$\mathbf{B}^\infty \colon \mathrm{CGrp}(\mathrm{An}) \hookrightarrow \mathrm{Sp}$$

whose image is the full subcategory $\mathrm{Sp}_{\geq 0}$ of connective spectra. This result is classically known as the *recognition principle for infinite loop spaces*.

The ∞-categorical definition of commutative groups is more subtle than the one from classical algebra. Homotopical analogues of familiar algebraic structures come with infinite hierarchies of coherence conditions, making it impossible to define them by simply writing down a list of relations. Following ideas of Segal, we take a direct approach[12]: we define (commutative) monoids and groups as functors out of suitable indexing ∞-categories, namely the opposite simplex category $\mathbf{\Delta}^{\mathrm{op}}$ and the span category of finite sets $\mathrm{Span}(\mathrm{Fin})$, which encode all required coherences at once.

The equivalence between commutative groups in An and connective spectra is a consequence of another result called the *recognition principle for loop spaces*. For every pointed anima X the loop space ΩX inherits the structure of a group object in An_*. The recognition principle says that, conversely, every group object G arises this way: there exists a pointed anima X with $G \simeq \Omega X$, and if we require X to be connected then it is unique. This X is called the *delooping* of G and is denoted $\mathbf{B}G$.

If G is in fact a *commutative* group object, the delooping $\mathbf{B}G$ is again a commutative group, so the construction can be iterated: $\mathbf{B}^2 G := \mathbf{B}(\mathbf{B}G)$, and so on. The resulting sequence of deloopings assembles into a connective spectrum $\mathbf{B}^\infty G$. Conversely, the underlying anima of a spectrum comes with a preferred commutative group structure, which for $\mathbf{B}^\infty G$ recovers G.

5.1 Monoids and groups

Recall that a monoid is a set M equipped with a unit element $1 \in M$ and a product $x \cdot y \in M$ for all $x, y \in M$, satisfying associativity and unitality. Given such a monoid, we obtain for every $n \geq 0$ an n-fold multiplication map $M^n \to M$ sending $(x_1, \ldots, x_n)$ to $x_1 \cdots x_n$. In the ∞-categorical setting, we make all these n-fold multiplication maps part of the data, together with all the coherences between them. These coherences are encoded by the simplex category $\mathbf{\Delta}$.

[12] A more general approach to homotopy coherent algebraic structures is given in Part II via the theory of ∞-operads.

Definition 5.1.1 (Monoid)**.** Let C be an ∞-category that admits finite products. We define a *monoid object in C* (or *E_1-monoid*) to be a simplicial object

$$M\colon \mathbf{\Delta}^{\mathrm{op}} \to C, \qquad [n] \mapsto M_n := M([n]),$$

satisfying the *Segal condition*: for every $[n] \in \mathbf{\Delta}$ the *Segal map*

$$(e_i^*)_{i=1}^n\colon M_n \to \prod_{i=1}^n M_1$$

induced by the n inclusion maps $e_i\colon [1] \cong \{i-1 \leq i\} \hookrightarrow [n]$ is an isomorphism. In particular, the condition for $n = 0$ requires M_0 to be a terminal object of C. We write

$$\mathrm{Mon}(C) \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$$

for the full subcategory spanned by the monoids in C. We refer to M_1 as the *underlying object* of M.

Convention 5.1.2. Given a monoid M, we will often denote its underlying object in C again by M. We will also often identify M_n with the n-fold product $M^n := \prod_{i=1}^n M$ via the Segal map.

The unique map $s^0\colon [1] \to [0]$ and the inclusion $d^1\colon [1] \cong \{0 \leq 2\} \hookrightarrow [2]$ in $\mathbf{\Delta}$ induce maps $M_0 \to M_1$ and $M_2 \to M_1$, which under our convention take the form

$$e\colon * \to M \qquad \text{and} \qquad m\colon M \times M \to M.$$

We refer to e as the *unit* or *neutral element* of M, and refer to m as the *multiplication* of M. The unitality and associativity of multiplication are encoded by the following commutative diagrams in $\mathbf{\Delta}$, where we denote an order-preserving map $[m] \to [n]$ by listing the images $(a_0, \ldots, a_m)$ of the elements $0, \ldots, m$:

$$\begin{array}{ccccc} & & [1] & & \\ & \swarrow\!\!\swarrow & \downarrow{\scriptstyle (0,2)} & \searrow\!\!\searrow & \\ [1] & \xleftarrow[(0,0,1)]{} & [2] & \xrightarrow[(0,1,1)]{} & [1] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} [1] & \xrightarrow{(0,2)} & [2] \\ {\scriptstyle (0,2)}\downarrow & & \downarrow{\scriptstyle (0,2,3)} \\ [2] & \xrightarrow{(0,1,3)} & [3]. \end{array}$$

Similarly, all the expected 'higher coherences' relating the various ways of multiplying four, five, or more elements in M follow from suitable higher dimensional diagrams in $\mathbf{\Delta}$. More explicitly, for a map $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ we should think of the resulting map $\varphi^*\colon M^n \to M^m$ as follows:

$$\varphi^*(x_1, \ldots, x_n) := (y_1, \ldots, y_m), \qquad y_i := \prod_{\varphi(i-1) < j \leq \varphi(i)} x_j.$$

This formula becomes transparent once we think of the elements $x_1, \ldots, x_n$ as labels on the n *inequalities* of $[n]$, and likewise of $y_1, \ldots, y_m$ as labels on the m inequalities of $[m]$. Drawing the map φ then cuts the strip between $[m]$ and $[n]$ into m blocks, one for each inequality of $[m]$, and the label of an inequality of $[m]$ is the product of the labels it sees in its block. For $\varphi\colon [4] \to [5]$ with $\varphi(0) = 0$, $\varphi(1) = \varphi(2) = 2$, $\varphi(3) = 4$ and $\varphi(4) = 5$ this looks as follows:

$$\begin{array}{ccccccccccccccccc} [4] = \{ & 0 & & \leq & & 1 & & \leq & & 2 & \leq & 3 & \leq & 4 & \} \\ & & & & y_1 = x_1x_2 & & & y_2 = e & & & y_3 = x_3x_4 & & y_4 = x_5 & & \\ \varphi\downarrow & & & & & & & & & & & & & & \\ & & x_1 & & & & x_2 & & x_3 & & & x_4 & & x_5 & \\ [5] = \{ & 0 & \leq & 1 & \leq & 2 & \leq & 3 & \leq & 4 & \leq & 5 & & & \} \end{array} \tag{5.1}$$

The i-th block is bounded by the arrows starting at $i-1$ and at i, so the inequalities of [5] it contains are exactly those indexed by $\varphi(i-1) < j \leq \varphi(i)$; multiplying their labels gives y_i. Note that the strip is read against the direction of φ: the labels travel from [5] up to [4], which reflects the contravariance of M. Since $\varphi(1) = \varphi(2)$, the two arrows bounding the second block meet at 2, so this block contains no inequality of [5] at all; its label is therefore the empty product, that is, the unit e. In this example we thus obtain

$$\varphi^*(x_1, \ldots, x_5) = (x_1 x_2,\ e,\ x_3 x_4,\ x_5).$$

Exercise 5.1.3. Let $F\colon C \to D$ be a functor preserving finite products. Show that F induces a functor $\mathrm{Mon}(C) \to \mathrm{Mon}(D)$.

Example 5.1.4. By Exercise 5.1, a topological monoid is a monoid object in Top. Combining the preceding exercise with the fact that $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ preserves finite products (see Corollary 2.3.20), we see that every topological monoid M defines a monoid object $\Pi_\infty(M)$ in An.

Definition 5.1.5. We say that a monoid $M \in \mathrm{Mon}(C)$ is *grouplike*, or a *group in* C, if the so-called *shear map*

$$(\mathrm{pr}_1, m)\colon M \times M \to M \times M, \qquad (x, y) \mapsto (x, xy)$$

is an isomorphism in C. We denote by $\mathrm{Grp}(C) \subseteq \mathrm{Mon}(C)$ the full subcategory spanned by the groups in C.

Exercise 5.1.6. Show that a monoid $M \in \mathrm{Mon}(\mathrm{Set})$ is a group in the sense of Definition 5.1.5 if and only if every element x admits an inverse x^{-1}.

Exercise 5.1.7. For a group object $G \in \mathrm{Grp}(C)$, construct an *inverse map* $(-)^{-1}\colon G \to G$ in C.

5.1.1 Group structures on loop spaces

We will now introduce a general class of examples of group objects in ∞-categories: the loop objects ΩX on pointed objects X. If C is an ∞-category with finite limits, then we may apply Exercise 4.3 in the pointed ∞-category C_* to obtain a group structure on ΩX in $\mathrm{Ho}(C_*)$, and hence in $\mathrm{Ho}(C)$. We will now explain how to lift this structure to a group structure in C.

Let us first explain informally where the multiplication on ΩX comes from. The loop object is defined as the pullback $\Omega X = * \times_X *$; for $C = \mathrm{An}$, its points are the loops in X at the basepoint x. Multiplication should be given by composing loops, and it is the universal property of pullbacks that produces this map for us: consider the iterated pullback $* \times_X * \times_X *$, which comes with three natural projections to $* \times_X *$. Projecting away the third or the first factor yields two maps which exhibit the iterated pullback as the product $\Omega X \times \Omega X$, as verified in Lemma 5.1.13. Projecting away the *middle* factor instead yields a map

$$m\colon \Omega X \times \Omega X \to \Omega X,$$

and this is composition of loops. Similarly, the universal property of the pullback $\Omega X = * \times_X *$ turns the identity identification of the basepoint into a unit map $e\colon * \to \Omega X$, the constant loop.

The higher coherences arise from the same principle. The iterated pullback $* \times_X \cdots \times_X *$ with $n+1$ factors is the object of n-tuples of composable loops, and projecting onto pairs of consecutive factors exhibits it as the product $(\Omega X)^n$, as verified in Lemma 5.1.13. The remaining maps between these

iterated pullbacks encode the structure: projecting away an inner factor composes two consecutive loops, and repeating a factor inserts a constant loop. Coherences arise because different composites of such maps are identified by the universal property. For example, forgetting the two inner factors of $* \times_X * \times_X * \times_X *$ one at a time, in either order, gives the projection onto the two outer factors, and this identification is the associativity of m. Instead of listing such coherences one by one, we record the entire family of iterated pullbacks together with all of these maps. This is a simplicial object, and the identification of its n-th term with $(\Omega X)^n$ via these projections is precisely the Segal condition.

It remains to construct this simplicial object, with all its coherences, functorially from the basepoint map $x\colon * \to X$. This will be achieved by a right Kan extension, indexed by the following enlargement of the simplex category.

Definition 5.1.8. We define the *augmented simplex category* as $\mathbf{\Delta}_+ := \mathbf{\Delta}^{\triangleleft}$, the cone of the simplex category. We denote the cone point by $[-1]$. A functor ${\mathbf{\Delta}_+}^{\mathrm{op}} \to C$ is called an *augmented simplicial object*. It corresponds to a simplicial object $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ equipped with a cocone $X_\bullet \Rightarrow \mathrm{const}_{X_{-1}}$.

Note that we may identify $\mathbf{\Delta}_+$ with the full subcategory of the category Poset of posets spanned by the linearly ordered sets $[n] = \{0 \leq \cdots \leq n\}$, where we set $[-1] = \emptyset$. There are fully faithful inclusions

$$i\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_+ \qquad \text{and} \qquad j\colon [1] \hookrightarrow \mathbf{\Delta}_+, \quad j(0) = [-1], \quad j(1) = [0].$$

Definition 5.1.9. Let C be an ∞-category with pullbacks. We define the *Čech nerve functor* $\check{C}\colon \mathrm{Ar}(C) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ as the composite

$$\check{C}_\bullet\colon \mathrm{Ar}(C) \simeq \mathrm{Fun}([1]^{\mathrm{op}}, C) \xrightarrow{j_*} \mathrm{Fun}({\mathbf{\Delta}_+}^{\mathrm{op}}, C) \xrightarrow{i^*} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C),$$

where j_* is the right Kan extension functor along the inclusion $j^{\mathrm{op}}\colon [1]^{\mathrm{op}} \hookrightarrow {\mathbf{\Delta}_+}^{\mathrm{op}}$, and where i^* is restriction along the inclusion i^{op}. Given a morphism $f\colon X \to Y$ in C, we refer to the simplicial object $\check{C}_\bullet(f)$ as the *Čech nerve of* f.

Here the equivalence $\mathrm{Ar}(C) \simeq \mathrm{Fun}([1]^{\mathrm{op}}, C)$ uses the order-reversing isomorphism $[1] \cong [1]^{\mathrm{op}}$: a morphism $f\colon X \to Y$ corresponds to the functor with value X at 1 and Y at 0. In the pointwise calculation below, we use the same order reversal to write it in the usual form $f\colon [1] \to C$, with $f(0) = X$ and $f(1) = Y$.

For a discussion of the theory of Kan extensions, we refer the reader to Section 22.4.

Discussion 5.1.10. The existence of the right adjoint j_* to j^* follows from the pointwise formula for Kan extensions from Theorem 22.4.3. Given a morphism in C of the form $f\colon [1] \to C$, the pointwise formula says that the value of $j_*(f)\colon \mathbf{\Delta}_+^{\mathrm{op}} \to C$ at $[n] \in \mathbf{\Delta}_+$ is given by the limit $\lim_{I_n} f_n$, where I_n is the full subcategory of the slice $((\mathbf{\Delta}_+)_{/[n]})^{\mathrm{op}}$ spanned by all objects of the form $[-1] \to [n]$ and $[0] \to [n]$, and where f_n is the composite $I_n \to [1] \xrightarrow{f} C$. Since there is a unique map $\infty\colon [-1] \to [n]$ and there is one map $i\colon [0] \to [n]$ for every $0 \leq i \leq n$, the category I_n may be

identified with the following poset:

$$I_n \quad := \quad \begin{array}{ccc} 0 & & \\ 1 & & \\ \vdots & & \infty. \\ n-1 & & \\ n & & \end{array}$$

The forgetful functor $I_n \to [1]$ sends ∞ to 1 and sends each $0 \leq i \leq n$ to 0. In particular, the functor $f_n \colon I_n \to C$ takes the form

$$f_n \quad = \quad \begin{array}{ccc} X & & \\ X & \overset{f}{} & \\ \vdots & & Y. \\ X & \underset{f}{} & \\ X & & \end{array}$$

Since C admits pullbacks, the limit of this diagram exists in C and is given by the $(n+1)$-fold fiber product of X with itself over Y. All in all, we see that the Čech nerve of a morphism $f \colon X \to Y$ in C is well-defined and is given in degree n by

$$\check{C}_n(f) \quad \simeq \quad X^{\times_Y^{n+1}} \quad = \quad X \times_Y X \times_Y \cdots \times_Y X.$$

Definition 5.1.11 (Loop group)**.** Let C be an ∞-category with finite limits. We define the *loop group* $\mathbf{\Omega}(X,x) \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ of a pointed object (X,x) in C as the Čech nerve of the morphism $x \colon * \to X$. This results in a functor $\mathbf{\Omega} \colon C_* \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$.

Remark 5.1.12. Since the loop object $\Omega(X,x)$ is defined as the pullback $* \times_X *$ in C, there is a natural isomorphism

$$\mathbf{\Omega}(X,x)_1 \cong \Omega(X,x),$$

justifying the notation.

Lemma 5.1.13. *For every pointed object X, the loop group $\mathbf{\Omega}X$ is a group object in C.*

Proof. To show that $\mathbf{\Omega}X$ is a monoid, we will show by induction that for every n the map

$$(e_i^*)_{i=1}^n \colon (\mathbf{\Omega}X)_n \to \prod_{i=1}^n \Omega X$$

is an isomorphism in C. For $n = 0$, observe that I_0 is isomorphic to the walking morphism $[1]$, consisting of objects 0 and ∞, and the functor $f_0 \colon I_0 \to C$ is the diagram $* \to X$. Since $[1]$ has 0 as

an initial object, this limit is simply $*$; see Lemma 22.2.4. The case $n = 1$ is the identification from the preceding remark. For $n \geq 1$, note that we may write I_{n+1} as a pushout of I_n and I_1 along I_0:

$$\begin{array}{ccc} I_0 & \longrightarrow & I_n \\ \downarrow & \ulcorner & \downarrow \\ I_1 & \xrightarrow[I_{e_{n+1}}]{} & I_{n+1} \end{array}$$

By applying Theorem 22.2.11(2) with $J = \ulcorner$, we thus obtain a pullback square of the form

$$\begin{array}{ccc} \lim(f_{n+1}\colon I_{n+1} \to C) & \longrightarrow & \lim(f_n\colon I_n \to C) \\ \downarrow & \lrcorner & \downarrow \\ \lim(f_1\colon I_1 \to C) & \longrightarrow & \lim(f_0\colon I_0 \to C), \end{array}$$

where we take f to be the basepoint map $x\colon * \to X$. By induction, this pullback square is isomorphic to the commutative square

$$\begin{array}{ccc} (\mathbf{\Omega}X)_{n+1} & \xrightarrow{(e_i^*)_{i=1}^n} & \prod_{i=1}^n \Omega X \\ {\scriptstyle e_{n+1}^*}\downarrow & & \downarrow \\ \Omega X & \longrightarrow & *, \end{array}$$

and so this being a pullback square means that the map $(\mathbf{\Omega}X)_{n+1} \xrightarrow{(e_i^*)_{i=1}^{n+1}} \prod_{i=1}^{n+1} \Omega X$ is an isomorphism, finishing the induction step. This shows that $\mathbf{\Omega}X$ is a monoid in C.

To see that $\mathbf{\Omega}X$ is grouplike, write $p_{ij}\colon * \times_X * \times_{X^*} \to * \times_X *$ for the projection onto the i-th and j-th factors. Under the Segal isomorphism

$$(p_{01}, p_{12})\colon * \times_X * \times_{X^*} \xrightarrow{\sim} \Omega X \times \Omega X,$$

the shear map is identified with (p_{01}, p_{02}). But associativity of pullbacks gives another isomorphism

$$(p_{01}, p_{02})\colon * \times_X * \times_{X^*} \xrightarrow{\sim} (* \times_X *) \times_* (* \times_X *) \cong \Omega X \times \Omega X.$$

Thus the shear map is an isomorphism. □

5.1.2 Classifying animae

The loop group functor admits a left adjoint:

Construction 5.1.14 (Classifying anima of a monoid)**.** Let C be an ∞-category admitting finite products and geometric realizations. Given a monoid M, we define its *classifying anima* $\mathbf{B}M$ as

$$\mathbf{B}M \quad := \quad |M| \quad = \quad \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} M_n \quad \in \quad C.$$

We may turn this into a pointed object in C via the map $* = M_0 \to \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} M_n$. This construction is functorial in M, resulting in a functor $\mathbf{B}\colon \mathrm{Mon}(C) \to C_*$.

Proposition 5.1.15. *Assume that C admits both finite limits and geometric realizations. Then the functors $\mathbf{B}$ and $\mathbf{\Omega}$ define adjunctions*

$$\mathbf{B}\colon \mathrm{Mon}(C) \rightleftarrows C_* : \mathbf{\Omega} \qquad \textit{and} \qquad \mathbf{B}\colon \mathrm{Grp}(C) \rightleftarrows C_* : \mathbf{\Omega}.$$

Proof. Consider the following two adjunctions:

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \underset{i^*}{\overset{i_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Fun}((\mathbf{\Delta}_+)^{\mathrm{op}}, C) \underset{j_*}{\overset{j^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Fun}([1]^{\mathrm{op}}, C).$$

The bottom composite $i^* j_*$ is the Čech nerve functor $\check{C}_\bullet$. The functor $i_!$ is given by left Kan extension along i. From the pointwise formula for left Kan extensions, we see that the top composite $j^* i_!$ sends a simplicial object X in C to the map $X_0 \to |X| = \mathrm{colim}_n X_n$.

Now, note that the top composite $j^* i_!$ sends the full subcategory $\mathrm{Mon}(C) \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ into the full subcategory $C_* \subseteq \mathrm{Ar}(C)$, and that the resulting functor $\mathrm{Mon}(C) \to C_*$ is the bar construction $\mathbf{B}$. Similarly, the bottom composite $i^* j_*$ restricts to $\mathbf{\Omega} \colon C_* \to \mathrm{Mon}(C)$. It follows that the adjunction $j^* i_! \dashv i^* j_*$ restricts to an adjunction $\mathbf{B} \dashv \mathbf{\Omega}$ between $\mathrm{Mon}(C)$ and C_*. Since $\mathbf{\Omega}$ takes values in $\mathrm{Grp}(C) \subseteq \mathrm{Mon}(C)$, this adjunction further restricts to an adjunction between $\mathrm{Grp}(C)$ and C_*. □

For later use, we record the following observation:

Lemma 5.1.16. *The classifying anima functor* $\mathbf{B} \colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{An}$ *preserves finite products.*

Proof. This functor factors as

$$\mathrm{Mon}(\mathrm{An}) \hookrightarrow \mathrm{sAn} \xrightarrow{|-|} \mathrm{An}.$$

The geometric realization functor $|-|$ preserves finite products by Lemma 22.6.12, and the inclusion preserves finite products since monoids are closed under products in sAn. □

5.2 The recognition principle for loop spaces

Group objects in animae are precisely the loop animae of connected pointed animae. This is the content of the following recognition principle:

Theorem 5.2.1 (Recognition principle for loop spaces, cf. Stasheff [Sta63b], May [May72], Boardman and Vogt [BV73], Lurie [Lur09b, Lemma 7.2.2.11])**.** *Let M be a monoid in* An *and let X be a pointed anima.*

(1) The unit map $M \to \mathbf{\Omega}\mathbf{B}M$ is an isomorphism if and only if M is a group in An*;*

(2) The pointed anima $\mathbf{B}M$ is connected;

(3) The counit $\mathbf{B}\mathbf{\Omega}X \to X$ is an inclusion of the connected component of the basepoint of X.

The classical formulation of this result is in terms of the operadic language of A_∞-spaces or E_1-spaces: it says that any grouplike E_1-space is equivalent to a loop space. The ∞-categorical formulation in terms of the adjunction $\mathbf{B} \dashv \mathbf{\Omega}$ has the advantage that it immediately provides us with an equivalence of ∞-categories:

Corollary 5.2.2. *The adjunction $\mathbf{B} \dashv \mathbf{\Omega}$ restricts to an equivalence of ∞-categories*

$$\mathbf{B} \colon \mathrm{Grp}(\mathrm{An}) \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathrm{An}_{*,\geq 1} : \mathbf{\Omega}$$

between group objects in An *and connected pointed animae.*

Proof. The adjunction from Proposition 5.1.15 restricts to groups and pointed connected animae since $\mathbf{\Omega}(X)$ is always a group and $\mathbf{B}M$ is always connected. The unit $M \to \mathbf{\Omega}\mathbf{B}M$ is always an isomorphism by part (1) of the theorem, while the counit $\mathbf{B}\mathbf{\Omega}X \to X$ is always an isomorphism by part (3) of the theorem. This shows that the two functors are in fact inverse to one another. □

5.2.1 Proof of the recognition principle

The proof of Theorem 5.2.1 will occupy the remainder of this section; a reader willing to take the theorem on faith may safely continue with Section 5.3. Let us first explain the two main ingredients of the proof. To identify a grouplike monoid M with $\mathbf{\Omega}\mathbf{B}M$, we must compute the pullback square defining the loop anima of $\mathbf{B}M$ and identify it with the underlying anima of M. We will obtain this from a descent property of the ∞-category An, by comparing the simplicial object M with its *décalage* PM, obtained by shifting every simplicial degree up by one. The shifted object admits extra degeneracies, which force its geometric realization to be contractible, while grouplikeness of M ensures that the map $PM \to M$ is cartesian. Descent then turns this levelwise statement into the desired pullback square after geometric realization.

Extra degeneracies

We record a useful technique for computing geometric realizations.

Definition 5.2.3 (Extra degeneracies)**.** We define the subcategory $\mathbf{\Delta}^{\mathrm{deg}} \subseteq \mathbf{\Delta}$ as the subcategory consisting of all objects $[n]$, but with only those morphisms $\varphi\colon [n] \to [m]$ satisfying $\varphi(n) = m$. An *augmented simplicial object with extra degeneracies* in an ∞-category C is a functor $Y\colon (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \to C$.

There exists a functor $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ given on objects by $[n] \mapsto [n+1]$, and on morphisms by sending $\varphi\colon [n] \to [m]$ to the map

$$[n+1] \to [m+1], \qquad i \mapsto \begin{cases} \varphi(i) & 0 \leq i \leq n \\ m+1 & i = n+1. \end{cases}$$

Given an augmented simplicial object with extra degeneracies $Y\colon (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \to C$, precomposition with $(+1)$ yields the *underlying augmented simplicial object* of Y. Conversely, we say that an augmented simplicial object $X\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to C$ *admits extra degeneracies* if there exists an augmented simplicial object with extra degeneracies Y such that X is isomorphic to the underlying augmented simplicial object of Y.

To understand the structure of $\mathbf{\Delta}^{\mathrm{deg}}$, note that all the face maps that are allowed in $\mathbf{\Delta}^{\mathrm{deg}}$ are actually already in the image of the functor $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$. Among the degeneracy maps, the map $s_n\colon [n+1] \to [n]$ in $\mathbf{\Delta}^{\mathrm{deg}}$ is the only one which does not lie in this image. Hence an augmented simplicial object with extra degeneracies can be pictured as follows, with non-dashed arrows encoding the underlying augmented simplicial object and with dashed arrows encoding the 'extra degeneracies':

$$\cdots \;\; \overleftarrow{\overleftarrow{\overleftarrow{\dashleftarrow}}} \; Y_2 \;\; \overleftarrow{\overleftarrow{\dashleftarrow}} \; Y_1 \;\; \overleftarrow{\dashleftarrow} \; Y_0.$$

The utility of extra degeneracies lies in the following easy computation of geometric realizations:

Lemma 5.2.4 (Extra degeneracy argument)**.** *Let $X\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to C$ be an augmented simplicial object that admits extra degeneracies. Then X is a colimit diagram, in the sense that the corresponding cocone $X|_{\mathbf{\Delta}^{\mathrm{op}}} \Rightarrow \mathrm{const}_{X_{-1}}$ is a colimit cocone:*

$$\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n \xrightarrow{\sim} X_{-1}.$$

Proof. By assumption, there exists an augmented simplicial object with extra degeneracies $Y\colon (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \to C$ such that $X_n = Y_{n+1}$. We need to show that the preferred map from $\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n$ to X_{-1} induced by the cocone is an isomorphism.

Observe that the inclusion functor $(+1)\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ admits a right adjoint given by the inclusion $\mathbf{\Delta}^{\mathrm{deg}} \hookrightarrow \mathbf{\Delta}$: for natural numbers $n, m \geq 0$, providing a map $\varphi\colon [n] \to [m]$ of partially ordered sets is the same as providing a map $\overline{\varphi}\colon [n+1] \to [m]$ of posets satisfying the condition that $\overline{\varphi}(n+1) = m$. It follows from Corollary 22.5.7 that the functor $(+1)\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ is an initial functor, and hence its opposite $(+1)^{\mathrm{op}}\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}$ is a final functor. By Theorem 22.5.1, this means that the geometric realization of X may be computed as

$$\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n = \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} Y_{n+1} \simeq \mathrm{colim}_{[m]\in(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}} Y_m.$$

But since $\mathbf{\Delta}^{\mathrm{deg}}$ has an initial object given by $[0]$, $(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}$ has a terminal object, and so it follows from Lemma 22.2.4 that $\mathrm{colim}_{[m]\in(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}} Y_m$ is isomorphic to $Y_0 = X_{-1}$ as desired. □

Descent

The ∞-category An has a special feature that distinguishes it from other ∞-categories: it satisfies *descent*. In the following statement, we use the notation $I^{\triangleright} := (I\times[1]) \sqcup_{I\times\{1\}} *$ from Definition 1.7.5.

Theorem 5.2.5 (Descent for colimits)**.** *Let I be a small ∞-category, let $\overline{F}, \overline{G}\colon I^{\triangleright} \to \mathrm{An}$ be functors and let $\overline{\alpha}\colon \overline{F} \Rightarrow \overline{G}$ be a natural transformation such that the restriction $\alpha := \overline{\alpha}|_I\colon F \Rightarrow G$ is a cartesian transformation, in the sense that for every morphism $i \to j$ in I the commutative square*

$$\begin{array}{ccc} F(i) & \longrightarrow & F(j) \\ {\scriptstyle \alpha(i)}\downarrow & & \downarrow{\scriptstyle \alpha(j)} \\ G(i) & \longrightarrow & G(j) \end{array} \tag{5.2}$$

is a pullback square. Assume that $\overline{G}$ is a colimit diagram. Then $\overline{F}$ is a colimit diagram if and only if $\overline{\alpha}$ is a cartesian transformation, i.e. also all the squares

$$\begin{array}{ccc} F(i) & \longrightarrow & \overline{F}(\infty) \\ {\scriptstyle \alpha(i)}\downarrow & & \downarrow{\scriptstyle \overline{\alpha}(\infty)} \\ G(i) & \longrightarrow & \overline{G}(\infty) = \mathrm{colim}_I\, G. \end{array}$$

are pullback squares.

Proof. See Section 24.7; see also [Lur09b, Theorem 6.1.3.9]. □

The theorem has two complementary uses. If both $\overline{F}$ and $\overline{G}$ are colimit diagrams and α is cartesian, then every $F(i)$ can be recovered by pulling back the map $\operatorname{colim}_I F \to \operatorname{colim}_I G$ along $G(i) \to \operatorname{colim}_I G$. Conversely, if only $\overline{G}$ is known to be a colimit diagram, cartesianness of the extended transformation $\overline{\alpha}$ forces $\overline{F}$ to be a colimit diagram as well. See Section 24.7 for the conceptual interpretation in terms of slice categories.

Example 5.2.6 (Universality of coproducts)**.** Consider $I = * \sqcup *$, so that $I^{\triangleright} \simeq \sqsqcup$. The transformation $\overline{\alpha}$ will thus be a diagram of the form

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_2 & \longleftarrow & X_1 \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} \\ Y_0 & \longrightarrow & Y_0 \sqcup Y_1 & \longleftarrow & Y_1, \end{array}$$

where in the bottom we get a coproduct, reflecting the condition that $\overline{G}$ is a colimit diagram. The theorem then expresses that the following two conditions are equivalent:

- The two squares in the diagram are both pullback squares;
- The functor $X_0 \sqcup X_1 \to X_2$ is an isomorphism.

The equivalence of these two conditions is an instance of the universality of coproducts from Axiom C.4, with the two parts of the axiom corresponding to the two implications.

The décalage argument

We are now ready to prove the recognition principle.

Proof of Theorem 5.2.1. We start with (2): the fact that $\mathbf{B}M$ is connected for any monoid object M in An. By definition, $\mathbf{B}M$ is connected if and only if its set of path components $\pi_0(\mathbf{B}M)$ is a single point. Since $\pi_0 \colon \mathrm{An} \to \mathrm{Set}$ is a left adjoint, it preserves colimits (see Lemma 22.2.6), and thus $\pi_0(\mathbf{B}M)$ is a colimit in sets of the following simplicial diagram:

$$\cdots \;\; \pi_0(M_2) \;\; \pi_0(M_1) \;\; \pi_0(M_0).$$

The colimit of a simplicial object in sets is the coequalizer of its two face maps in degrees 1 and 0. Hence the canonical map $\pi_0(M_0) \to \operatorname{colim}_{[n]} \pi_0(M_n)$ is surjective. Since $M_0 \cong *$, it follows that $\pi_0(\mathbf{B}M)$ has a single element.

Let us assume (1) for a moment and use it to deduce (3), the fact that the counit $\varepsilon_X \colon \mathbf{B}\mathbf{\Omega}X \to X$ is the inclusion of the path component X_0 of the basepoint of X. Since $\mathbf{B}\mathbf{\Omega}X$ is connected, the counit map certainly lands in X_0 and we need to show that the resulting map $\mathbf{B}\mathbf{\Omega}X \to X_0$ is an isomorphism of animae. Since both sides are connected pointed animae, it suffices by Corollary 2.4.23 to show that the map $\mathbf{\Omega}(\varepsilon_X) \colon \mathbf{\Omega}(\mathbf{B}\mathbf{\Omega}X) \to \mathbf{\Omega}(X_0) \simeq \mathbf{\Omega}(X)$ is an isomorphism. By the triangle identities for the adjunction, there is a commutative triangle of the form

$$\begin{array}{ccc} & \mathbf{\Omega}(\mathbf{B}\mathbf{\Omega}X) & \\ {\scriptstyle \eta_{\mathbf{\Omega}X}} \nearrow & & \searrow {\scriptstyle \mathbf{\Omega}(\varepsilon_X)} \\ \mathbf{\Omega}X & = & \mathbf{\Omega}X, \end{array}$$

and so $\mathbf{\Omega}(\varepsilon_X)$ is an isomorphism if and only if $\eta_{\mathbf{\Omega}}\colon \mathbf{\Omega}X \to \mathbf{\Omega}(\mathbf{B}\mathbf{\Omega}X)$ is an isomorphism. But since $\mathbf{\Omega}X$ is a group object, this is an instance of part (1).

It remains to prove part (1), i.e. that M is a group if and only if the map $\eta_M\colon M \to \mathbf{\Omega}\mathbf{B}M$ is an isomorphism of monoids. One direction is clear: $\mathbf{\Omega}\mathbf{B}M$ is a group, so if this map is an isomorphism then also M is a group. For the other direction, assume that M is a group. To show that $\eta_M\colon M \to \mathbf{\Omega}\mathbf{B}M$ is an isomorphism, it suffices to show that the map $(\eta_M)_1\colon M_1 \to (\mathbf{\Omega}\mathbf{B}M)_1 = \Omega\mathbf{B}M$ on underlying animae is an isomorphism: since $M_n \simeq M_1^{\times n}$ and $(\mathbf{\Omega}\mathbf{B}M)_n \simeq ((\mathbf{\Omega}\mathbf{B}M)_1)^{\times n}$ we then get that it is an isomorphism at every level. The map $M_1 \to \Omega\mathbf{B}M$ is induced by the commutative square

$$\begin{array}{ccc} M_1 & \xrightarrow{d_0} & M_0 \simeq * \\ {\scriptstyle d_1}\downarrow & & \downarrow \\ M_0 \simeq * & \longrightarrow & \mathbf{B}M \end{array}$$

and so is an isomorphism if and only if this is a pullback square of animae. We will deduce this as an instance of Theorem 5.2.5. To this end, consider the diagram $\overline{M}\colon (\mathbf{\Delta}_+)^{\mathrm{op}} \to \mathrm{An}$ that extends M with its colimit $\overline{M}_{-1} := \mathbf{B}M = \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} M_n$. Also consider the functor

$$\overline{PM}\colon (\mathbf{\Delta}_+)^{\mathrm{op}} \to \mathrm{An}, \qquad [n] \mapsto \overline{M}_{n+1},$$

and consider the natural transformation $\overline{\alpha}\colon \overline{PM} \to \overline{M}$ whose components are the maps $d_{n+1}\colon M_{n+1} \to M_n$. Let us spell out why these maps assemble functorially. The successor functor $P\colon \mathbf{\Delta}_+ \to \mathbf{\Delta}_+$, obtained by adjoining a new final element, admits a natural transformation $\mathrm{id} \to P$ whose component at $[n]$ is the final coface $d^{n+1}\colon [n] \to [n+1]$. Precomposition with $\overline{M}\colon (\mathbf{\Delta}_+)^{\mathrm{op}} \to \mathrm{An}$ therefore gives $\overline{\alpha}\colon P^*\overline{M} = \overline{PM} \to \overline{M}$; at $[-1]$ its component is the augmentation $M_0 \to \mathbf{B}M$. This is the usual décalage construction. The successor functor is the composite $\mathbf{\Delta}_+ \xrightarrow{(+1)} \mathbf{\Delta}^{\mathrm{deg}} \hookrightarrow \mathbf{\Delta}_+$ from Definition 5.2.3. Thus $\overline{PM}$ is the underlying augmented simplicial object of $\overline{M}|_{(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}}$ and admits extra degeneracies. It is therefore a colimit diagram by Lemma 5.2.4, so that $\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \overline{PM}_n \simeq \overline{PM}_{-1} = M_0$. The pullback square we wish to prove now takes the form

$$\begin{array}{ccc} PM_0 & \longrightarrow & \overline{PM}_{-1} \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \overline{\alpha}_{-1}} \\ M_0 & \longrightarrow & \overline{M}_{-1}. \end{array}$$

By Theorem 5.2.5, it thus remains to show that α is a cartesian natural transformation. In other words, we need to show that for every morphism $\varphi\colon [n] \to [m]$ the square

$$\begin{array}{ccc} PM_m & \xrightarrow{\varphi^*} & PM_n \\ {\scriptstyle \alpha_m}\downarrow & & \downarrow{\scriptstyle \alpha_n} \\ M_m & \xrightarrow{\varphi^*} & M_n \end{array} \tag{5.3}$$

is a pullback square. We will prove this claim in increasingly general cases:

Step 1: We start with the case of the map $\varphi = d^1 \colon [0] \to [1]$. In this case, the square (5.3) reduces to the following:

$$\begin{array}{ccc} M_1^2 & \xrightarrow{m} & M_1 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\ M_1 & \longrightarrow & *. \end{array} \tag{5.4}$$

This square is a pullback square if and only if the shear map $(\mathrm{pr}_1, m) \colon M_1 \times M_1 \to M_1 \times M_1$ is an isomorphism, which holds by assumption since M is a group.

Step 2: We will now prove the claim for $\varphi = d^i \colon [n] \to [n+1]$, where $n \geq 0$ and $0 \leq i \leq n+1$. We use the identifications $M_k \cong M_1^k$ coming from the Segal condition. For $i = 0$, the square (5.3) is the square which drops the first coordinate horizontally and the last coordinate vertically. It is a pullback because a pair of $(n+1)$-tuples with the same middle n coordinates glues uniquely to an $(n+2)$-tuple. For $0 < i < n+1$, the horizontal maps multiply the i-th and $(i+1)$-st coordinates, while the vertical maps drop the last coordinate. Since the multiplication does not involve the last coordinate, this square is the product with M_1 of a degenerate pullback square, and is therefore again a pullback.

It remains to consider the final coface $i = n+1$. Here the multiplication involves the coordinate which is forgotten by the vertical maps, and the square takes the form

$$\begin{array}{ccc} M_1^{n+2} & \xrightarrow{(m_1,\dots,m_{n+2}) \mapsto (m_1,\dots,m_n, m_{n+1}m_{n+2})} & M_1^{n+1} \\ {\scriptstyle (m_1,\dots,m_{n+2}) \mapsto (m_1,\dots,m_{n+1})}\downarrow & & \downarrow{\scriptstyle (x_1,\dots,x_{n+1}) \mapsto (x_1,\dots,x_n)} \\ M_1^{n+1} & \xrightarrow[(m_1,\dots,m_{n+1}) \mapsto (m_1,\dots,m_n)]{} & M_1^n. \end{array}$$

None of the maps affect the first n coordinates, and on the last two coordinates it is the square (5.4). Hence it is the product of a degenerate pullback square with the pullback square from Step 1. This proves the claim for all coface maps d^i.

Step 3: Next, assume that φ is a codegeneracy map of the form $s^i \colon [n+1] \to [n]$. Since s^i has a section of the form $d^i \colon [n] \to [n+1]$, it follows from the pasting law of pullback squares that the square (5.3) is a pullback square:

$$\begin{array}{ccccc} PM_n & \xrightarrow{s_i^*} & PM_{n+1} & \xrightarrow{d_i^*} & PM_n \\ {\scriptstyle \alpha_n}\downarrow & & {\scriptstyle \alpha_{n+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha_n} \\ M_n & \xrightarrow{s_i^*} & M_{n+1} & \xrightarrow{d_i^*} & M_n. \end{array}$$

Step 4: We have now shown the claim when φ is either a coface map d^i or a codegeneracy map s^i. Since every map $\varphi \colon [n] \to [m]$ is an iterated composite of coface maps and codegeneracy maps, we then get the claim for general φ by using the pasting law of pullback squares again. This finishes the proof of the theorem. □

5.3 Commutative monoids and commutative groups

We now move to *commutative* monoids in an ∞-category. Just as for monoids, these will be defined as functors out of a suitable indexing ∞-category that encodes all the operations we have in a

commutative monoid.

For a commutative monoid M, given a tuple of elements $(x_s)_{s\in S}$ indexed by a finite set S, we can form the sum $\sum_{s\in S} x_s$. Commutativity says precisely that this sum is independent of any chosen ordering of S. Writing $M^S := \prod_{s\in S} M$ for the S-fold product, we thus have addition maps $M^S \to M$ for every finite set S.

How do these addition maps behave with respect to maps of finite sets? Let Fin denote the category of finite sets and all maps between them. For a map $g\colon S \to T$ of finite sets, there is an induced map $g_\oplus\colon M^S \to M^T$ given by summing over fibers: the t-th component of $g_\oplus((x_s)_{s\in S})$ is $\sum_{s\in g^{-1}(\{t\})} x_s$. This makes the assignment $S \mapsto M^S$ into a functor Fin $\to$ Set.

Just as for monoids, we should not look at arbitrary functors Fin $\to C$, but only at those satisfying a Segal-type condition of the form $M(S) \cong M(*)^S$. To phrase this condition precisely, we need to specify projections $M(S) \to M(*)$ for each $s \in S$, that is, we need an *a priori* way to compare $M(S)$ with the product $\prod_{s\in S} M(*)$. Observe that the S-fold product M^S is not only covariantly functorial in S via addition, but also *contravariantly* functorial via restriction: a map $f\colon S \to T$ induces a restriction map $f^*\colon M^T \to M^S$ given by $f^*(x_t)_{t\in T} := (x_{f(s)})_{s\in S}$.

To encode commutative monoids, we need both functorialities, addition and restriction, packaged into a single ∞-category. The objects of this ∞-category are finite sets. What should the morphisms from S to T be? These should encode the operation of 'first restricting along f, then adding along g': given maps $f\colon U \to S$ and $g\colon U \to T$, we obtain a composite

$$M^S \xrightarrow{f^*} M^U \xrightarrow{g_\oplus} M^T,$$

where the component of the output at $t \in T$ is given by summing $x_{f(u)}$ over all $u \in g^{-1}(\{t\})$. Thus, a morphism from S to T should be a pair of maps $S \leftarrow U \to T$, that is, a *span* from S to T.

What about composition? Suppose we first apply $S \xleftarrow{f} U \xrightarrow{g} T$ and then $T \xleftarrow{h} V \xrightarrow{k} R$. The first operation sends $(x_s)_{s\in S}$ to $(y_t)_{t\in T}$ where $y_t = \sum_{u\in g^{-1}(\{t\})} x_{f(u)}$. Applying the second operation, the component at $r \in R$ is

$$\sum_{v\in k^{-1}(\{r\})} y_{h(v)} \quad = \quad \sum_{v\in k^{-1}(\{r\})} \sum_{u\in g^{-1}(\{h(v)\})} x_{f(u)}.$$

This double sum ranges over pairs (u, v) with $g(u) = h(v)$ and $k(v) = r$. Setting $P := \{(u, v) \in U \times V \mid g(u) = h(v)\} = U \times_T V$, with its projections $h'\colon P \to U$ and $g'\colon P \to V$, we see that:

- the fiber $g^{-1}(\{h(v)\})$ is in bijection with the fiber $(g')^{-1}(\{v\})$, and
- the element $f(u)$ can be written as $(f \circ h')(u, v)$.

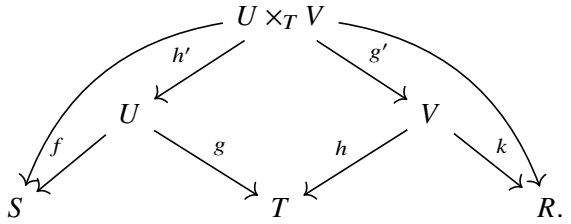
Hence the double sum equals $\sum_{p\in(k\circ g')^{-1}(\{r\})} x_{(f\circ h')(p)}$, which is exactly the result of applying the span $S \xleftarrow{f\circ h'} P \xrightarrow{k\circ g'} R$. In other words, composing 'first add along g, then restrict along h' gives the same result as 'first restrict along h', then add along g''. Spans thus compose by taking pullbacks, leading to the *span category of finite sets*.

Construction 5.3.1 (Span category)**.** There is an ∞-category Span(Fin), called the *span category of finite sets*. Its objects are finite sets, and for finite sets S and T its hom animae are given by the

groupoids

$$\mathrm{Hom}_{\mathrm{Span(Fin)}}(S,T) \qquad := \qquad \left\{ \begin{array}{ccccc} & & U & & \\ & \overset{f}{\swarrow} & \downarrow \cong & \overset{g}{\searrow} & \\ S & & & & T \\ & \underset{f'}{\nwarrow} & & \underset{g'}{\nearrow} & \end{array} \right\}$$

of *spans from S to T*; so the objects of this groupoid are spans $S \leftarrow U \rightarrow T$ and the morphisms are isomorphisms of spans. The identity span is given by $S \xleftarrow{\mathrm{id}_S} S \xrightarrow{\mathrm{id}_S} S$, and composition of spans is defined by taking pullbacks:



Remark 5.3.2. Since the hom animae of Span(Fin) are groupoids rather than sets, Span(Fin) is a $(2,1)$-category rather than a 1-category. The rigorous construction of this ∞-category is deferred to Chapter 13; for now, it suffices to know that Span(Fin) exists and behaves as described above.

Lemma 5.3.3. *The ∞-category* Span(Fin) *is semiadditive, with biproducts given by disjoint unions of finite sets.*

Proof. Consider finite sets S_i for $i = 1, \ldots, n$. We have spans of the form

$$\coprod_{i=1}^{n} S_i \hookleftarrow S_i \xrightarrow{\mathrm{id}_{S_i}} S_i$$

We claim that these morphisms exhibit $\bigsqcup_{i=1}^n S_i$ as a product of the sets S_i in Span(Fin). Indeed, note that for every map $U \rightarrow \bigsqcup_{i=1}^n S_i$ of finite sets we may write $U = \bigsqcup_{i=1}^n U_i$, where U_i is the preimage of S_i. This gives the desired equivalence of groupoids

$$\mathrm{Hom}_{\mathrm{Span(Fin)}}(T, \coprod_{i=1}^{n} S_i) = \left\{ \begin{array}{ccccc} & & \bigsqcup_{i=1}^n U_i & & \\ & \overset{f}{\swarrow} & \downarrow \cong & \overset{\bigsqcup_{i=1}^n g_i}{\searrow} & \\ T & & & & \bigsqcup_{i=1}^n S_i \\ & \underset{f'}{\nwarrow} & & \underset{\bigsqcup_{i=1}^n g_i'}{\nearrow} & \\ & & \bigsqcup_{i=1}^n U_i' & & \end{array} \right\} \simeq \prod_{i=1}^{n} \mathrm{Hom}_{\mathrm{Span(Fin)}}(T, S_i).$$

By the dual argument (reversing a span gives a bijection between spans from S to T and spans from T to S) we see that similarly the spans

$$S_i \xleftarrow{\mathrm{id}_{S_i}} S_i \hookrightarrow \coprod_{i=1}^{n} S_i.$$

exhibit $\bigsqcup_{i=1}^n S_i$ also as the *coproduct* of the S_i in Span(Fin). The composite from S_i into this coproduct and then to the j-th product factor is represented by the pullback $S_i \times_{\bigsqcup_k S_k} S_j$. It is the identity span when $i = j$ and the zero span otherwise. Hence the canonical map from the coproduct to the product is an isomorphism. For $n = 0$, the same argument says that the empty set is a zero object. Thus Span(Fin) is semiadditive. □

Definition 5.3.4 (Commutative monoid)**.** Let C be an ∞-category with finite products. We define a *commutative monoid* in C to be a product-preserving functor

$$M\colon \mathrm{Span}(\mathrm{Fin}) \to C.$$

We denote by

$$\mathrm{CMon}(C) \quad \subseteq \quad \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$$

the full subcategory spanned by the commutative monoids. We refer to the evaluation $M(*)$ at the one-point set as the *underlying object* of M. We will sometimes abuse notation and simply refer to $M(*)$ as M.

Remark 5.3.5 (Restriction and addition maps)**.** Let $M\colon \mathrm{Span}(\mathrm{Fin}) \to C$ be a commutative monoid. Every map $f\colon S \to T$ of finite sets determines two 'half-degenerate' spans

$$f^* \quad = \quad (T \xleftarrow{f} S \xrightarrow{\mathrm{id}_S} S) \qquad \text{and} \qquad f_\oplus \quad = \quad (S \xleftarrow{\mathrm{id}_S} S \xrightarrow{f} T),$$

called the *restriction* and the *addition* spans, respectively. Since products in $\mathrm{Span}(\mathrm{Fin})$ are computed as disjoint unions by Lemma 5.3.3, the condition that M preserves finite products means that for every finite set S the map

$$(e_s^*)_{s\in S}\colon M(S) \to M^S := \prod_{s\in S} M(*)$$

is an isomorphism, where $e_s\colon \{s\} \hookrightarrow S$ is the inclusion. We will henceforth identify $M(S)$ with M^S. It follows that every map $f\colon S \to T$ defines both a restriction map $f^*\colon M^T \to M^S$ as well as an addition map $f_\oplus\colon M^S \to M^T$, functorially in f. As a special case, the maps $\emptyset \to *$ and $* \sqcup * \to *$ in Fin then yield the *unit* and *addition*:

$$0\colon * \to M \qquad \text{and} \qquad +\colon M \times M \to M.$$

Definition 5.3.6 (Commutative group)**.** A commutative monoid M is called a *commutative group* if the *shear map* $(\mathrm{pr}_1, +)\colon M \times M \to M \times M$ is an isomorphism. We denote by

$$\mathrm{CGrp}(C) \quad \subseteq \quad \mathrm{CMon}(C)$$

the full subcategory spanned by the commutative groups in C.

5.3.1 Alternative definition

We now give an alternative characterization of commutative monoids that is commonly used in the literature and goes back to Segal [Seg74]. This characterization uses the category Fin_* of finite pointed sets in place of the span category $\mathrm{Span}(\mathrm{Fin})$.

Definition 5.3.7 (Segal condition)**.** Let C be an ∞-category with finite products. Denote by Fin_* the category of *finite pointed sets*, whose objects are pairs $(S, *)$ consisting of a finite set S and a distinguished basepoint $* \in S$, and whose morphisms are basepoint-preserving maps. For a finite set S, we write $S_+ := S \sqcup \{*\}$ for S with an added disjoint basepoint.

A functor $M\colon \mathrm{Fin}_* \to C$ is said to satisfy the *Segal condition* if for every finite set S the canonical map

$$M(S_+) \to \prod_{s\in S} M(\{s\}_+)$$

induced by the pointed maps $S_+ \to \{s\}_+$ is an isomorphism, where the pointed map sends everything except for $s \in S$ to the base point of $\{s\}_+$. We denote by

$$\mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Fin}_*, C) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, C)$$

the full subcategory spanned by functors satisfying the Segal condition.

Lemma 5.3.8. *Consider the wide subcategory* $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ *of* $\mathrm{Span}(\mathrm{Fin})$ *whose morphisms are those spans* $S \hookleftarrow U \to T$ *for which the left-pointing map is an injection. This is a 1-category, and there is an equivalence of 1-categories* $\mathrm{Fin}_* \simeq \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$.

Proof. It is clear that $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ is a 1-category: given two injections $U \hookrightarrow S$ and $U' \hookrightarrow S$, if there exists a bijection $U \xrightarrow{\sim} U'$ over S, then it is unique.

We now construct mutually inverse functors to Fin_*. Define a functor

$$\Phi\colon\ \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to \mathrm{Fin}_*$$

as follows: on objects, Φ sends a finite set S to $S_+ = S \sqcup \{*\}$. On morphisms, Φ sends a span $S \hookleftarrow U \to T$ (where the left-pointing map is an injection) to the pointed map $S_+ \to T_+$ that agrees with $U \to T$ on U and sends the complement $S \setminus U$ (together with the basepoint of S_+) to the basepoint of T_+.

Conversely, define a functor

$$\Psi\colon \mathrm{Fin}_* \to \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$$

as follows: on objects, Ψ sends a finite pointed set $(S, *)$ to $S \setminus \{*\}$. On morphisms, Ψ sends a pointed map $f\colon (S, *) \to (T, *)$ to the span

$$S \setminus \{*\} \hookleftarrow f^{-1}(T \setminus \{*\}) \xrightarrow{f} T \setminus \{*\},$$

where the left-pointing map is the inclusion. One readily verifies that Φ and Ψ are inverse equivalences. □

Proposition 5.3.9. *Let C be an ∞-category with finite products. Restriction along the inclusion* $i\colon \mathrm{Fin}_* \simeq \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ *induces an equivalence of ∞-categories*

$$i^*\colon \mathrm{CMon}(C) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Fin}_*, C).$$

The inverse is given by right Kan extension along i.

Proof. It is clear from the definitions that restriction along i sends product-preserving functors to functors satisfying the Segal condition.

The functor i^* is conservative: if $\alpha\colon M \to N$ is a natural transformation between commutative monoids such that $i^*\alpha$ is an isomorphism, then in particular $\alpha_*\colon M(*) \to N(*)$ is an isomorphism, and since both M and N preserve products we have that the map $\alpha_S\colon M(S) \cong M(*)^S \to N(*)^S \cong N(S)$ is an isomorphism for all finite sets S.

It thus suffices to show that for every functor $M\colon \mathrm{Fin}_* \to C$ satisfying the Segal condition, the right Kan extension $i_*(M)$ exists and the counit map $i^*i_*(M) \to M$ is an isomorphism. By the pointwise formula for right Kan extensions, we must show that for every finite set X the canonical map

$$\lim_{(X \to S) \in i_{X/}} M(S) \longrightarrow M(X)$$

is an isomorphism, where the limit is taken over the relative slice category $i_{X/} = \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Span}(\mathrm{Fin})_{X/}$. Consider the embedding

$$\varphi\colon (\mathrm{Fin}^{\mathrm{op}})_{X/} \times_{\mathrm{Fin}^{\mathrm{op}}} \mathrm{Fin}^{\mathrm{op}}_{\mathrm{inj}} \hookrightarrow \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Span}(\mathrm{Fin})_{X/}$$

that sends a map $f\colon S \to X$ to the span $X \xleftarrow{f} S \xrightarrow{\mathrm{id}} S$. This functor φ has a right adjoint, given by sending a span $X \leftarrow U \to S$ to the left-pointing map $U \to X$. By Corollary 22.5.7, the functor φ is therefore initial, so the limit may be computed over the left-hand category.

On this category, the restriction of M sends a map $f\colon S \to X$ to $M(S) \cong M(*)^S = \lim_{s \in S} M(*)$ (using the Segal condition). By the formula for iterated limits from Theorem 22.2.11, we may thus write the left-hand side as the limit of $M(*)$ over the set $\mathrm{colim}_{(X \to S)} S$. It thus suffices to show that X is the colimit of the forgetful functor

$$\mathrm{Fin}_{/X} \times_{\mathrm{Fin}} \mathrm{Fin}_{\mathrm{inj}} \longrightarrow \mathrm{Fin}.$$

This functor is left Kan extended from the full subcategory spanned by the inclusions $\{x\} \hookrightarrow X$ for $x \in X$, and the colimit over this subcategory is X. Moreover, the product projections in Span(Fin) have injective left legs and therefore belong to the wide subcategory on which i is defined. For $i_* M$, the canonical maps from the value on a disjoint union to the product of the values consequently agree, under the counit, with the Segal maps of M, and are therefore isomorphisms. Thus $i_* M$ preserves finite products. □

Remark 5.3.10. The ∞-category $\mathrm{CMon}(C)$ is often defined as $\mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Fin}_*, C)$ in the literature; this was also the original approach taken by Segal [Seg74] after which the Segal condition is named. Proposition 5.3.9 shows that our definition agrees with this classical one.

5.3.2 The underlying monoid of a commutative monoid

As the name suggests, every commutative monoid has an underlying monoid.

Construction 5.3.11. We will construct a functor $\mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \subseteq \mathrm{Span}(\mathrm{Fin})$:

- On objects, it sends $[n]$ to the set $\langle n \rangle := \{1, \dots, n\}$.
- On morphisms, it sends $\varphi\colon [m] \to [n]$ to the span

$$\langle n \rangle \longhookleftarrow \{\varphi(0)+1, \varphi(0)+2, \dots, \varphi(m)-1, \varphi(m)\} \longrightarrow \langle m \rangle,$$

  where the left-pointing map is the inclusion and the right-pointing map sends i to the unique $1 \leq j \leq m$ such that $\varphi(j-1) < i \leq \varphi(j)$.

More conceptually, we may think of the set $\mathrm{Cut}(P)$ for a poset P as the set of 'Dedekind cuts' of P, defined as pairs (P_0, P_1) of non-empty subposets $P_0, P_1 \subseteq P$ satisfying $P = P_0 \cup P_1$ and $p_0 < p_1$ for all $p_0 \in P_0$ and $p_1 \in P_1$. We want to think of the element $k \in \{1, \dots, n\} = \mathrm{Cut}([n])$ as the following partition of $[n]$:

$$k \quad \leftrightsquigarrow \quad (\{0, \dots, k-1\}, \{k, \dots, n\}).$$

On morphisms, $\mathrm{Cut}(\varphi)$ is then simply given by taking preimages of the two subsets P_0 and P_1, which is only defined if both of these preimages are non-empty.

Alternatively, we may think of the set $\mathrm{Cut}([n])$ as labeling the n inequalities in the partially ordered set $[n]$ that we can use to cut $[n]$ into two pieces:

$$[n] \quad = \quad \{0 \overset{1}{\leq} 1 \overset{2}{\leq} 2 \overset{3}{\leq} \dots \overset{n}{\leq} n\}.$$

This is the bookkeeping already used in (5.1): the middle term of the span $\mathrm{Cut}(\varphi)$ consists of those inequalities of $[n]$ which lie in one of the blocks cut out by φ, the right-pointing map records for each of them the inequality of $[m]$ whose block contains it, and the left-pointing inclusion discards the inequalities of $[n]$ lying in no block at all.

Exercise 5.3.12. Let $M\colon \mathrm{Span}(\mathrm{Fin}) \to C$ be a commutative monoid. Show that $M \circ \mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ defines a monoid in C, whose multiplication agrees with the addition map of M. Deduce that $M \circ \mathrm{Cut}$ is a group if and only if M is a commutative group, and that there is a pullback square

$$\begin{array}{ccc} \mathrm{CGrp}(C) & \longrightarrow & \mathrm{CMon}(C) \\ \downarrow & & \downarrow \\ \mathrm{Grp}(C) & \longrightarrow & \mathrm{Mon}(C). \end{array}$$

In particular, we obtain forgetful functors

$$U := \mathrm{Cut}^*\colon \mathrm{CMon}(C) \to \mathrm{Mon}(C) \qquad \text{and} \qquad U := \mathrm{Cut}^*\colon \mathrm{CGrp}(C) \to \mathrm{Grp}(C).$$

Warning 5.3.13. For a classical 1-category C, the forgetful functor $\mathrm{CMon}(C) \to \mathrm{Mon}(C)$ is fully faithful; see Exercise 5.6. For a general ∞-category, this is *not* true: commutativity is no longer a *property* of a monoid, but additional *structure* one needs to provide.

5.3.3 Monoids and commutative monoids in semiadditive ∞-categories

For a semiadditive ∞-category C, it turns out that every object admits a unique structure of a (commutative) monoid. The idea is simple: in a semiadditive ∞-category, every object M comes equipped with a canonical addition map

$$+\colon M \times M \xrightarrow{\sim} M \oplus M \xrightarrow{\nabla} M,$$

given by the fold map of the biproduct, and a zero map $0\colon * \to M$, the unique map from the initial object. It readily follows from the universal property of coproducts that this operation is unital, associative and commutative, resulting in a commutative monoid object in the homotopy category $\mathrm{Ho}(C)$. To promote M to a genuine commutative monoid object in C, we need these identities to hold coherently. While this is again a consequence of the universal property of coproducts, the required bookkeeping is more difficult as we need to handle all coherences simultaneously. The goal of this subsection is to carry out this bookkeeping. Only the statements of the results below will be used later, and the proofs all follow the same pattern of restriction and Kan extension along suitable subcategories, so they may be skimmed on a first reading. Let us start with the non-commutative case.

Proposition 5.3.14. *Let C be a semiadditive ∞-category. Then the forgetful functor $\mathrm{Mon}(C) \to C$ is an equivalence. If C is additive, then also the forgetful functor $\mathrm{Grp}(C) \to C$ is an equivalence.*

Proof. Let $\mathbf{\Delta}_{\mathrm{int}}^{\leq 1} \subseteq \mathbf{\Delta}$ be the subcategory with objects $[0]$ and $[1]$ and whose only non-identity morphisms are the two face maps $d^0, d^1 \colon [0] \to [1]$. We will show that restriction along the inclusions

$$j \colon \{[1]\} \hookrightarrow (\mathbf{\Delta}_{\mathrm{int}}^{\leq 1})^{\mathrm{op}} \qquad \text{and} \qquad i \colon (\mathbf{\Delta}_{\mathrm{int}}^{\leq 1})^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$$

define equivalences

$$\mathrm{Mon}(C) \xrightarrow{\sim} \mathrm{Fun}^{*}\left((\mathbf{\Delta}_{\mathrm{int}}^{\leq 1})^{\mathrm{op}}, C\right) \xrightarrow{\sim} C,$$

where Fun^{*} denotes the full subcategory of those functors whose value at $[0]$ is terminal.

For the first equivalence, we use left Kan extension along i. We apply the pointwise formula (Theorem 22.4.3) to compute the left Kan extension. For this, we need to understand the relative slice $i_{/[n]}$ for each $[n] \in \mathbf{\Delta}$. Objects of this category are pairs (x, φ) where $x \in \{[0], [1]\}$ and $\varphi \colon x \to [n]$ is a morphism in $\mathbf{\Delta}^{\mathrm{op}}$, i.e. a morphism $[n] \to x$ in $\mathbf{\Delta}$. There are $n+2$ order-preserving maps $[n] \to [1]$, which we may label by the sequence of values they take: there are the two *constant* maps $\underline{0}, \underline{1} \colon [n] \to [1]$, and for each $1 \leq k \leq n$ a map φ_k sending $0, 1, \ldots, k-1$ to 0 and $k, \ldots, n$ to 1. There is also a unique map $\star \colon [n] \to [0]$. For morphisms in $i_{/[n]}$, we observe that the only non-identity morphisms in $(\mathbf{\Delta}_{\mathrm{int}}^{\leq 1})^{\mathrm{op}}$ are the maps $[1] \to [0]$ induced by the face maps $d^0, d^1 \colon [0] \to [1]$. By checking the commutative triangles, one sees that the only non-identity morphisms in $i_{/[n]}$ are the maps $\underline{0} \to \star$ and $\underline{1} \to \star$. It follows that the category $i_{/[n]}$ has precisely $n+1$ connected components: there is a 'span' component $\underline{0} \to \star \leftarrow \underline{1}$, and there are n isolated objects $\varphi_1, \ldots, \varphi_n$.

Now let $F \colon (\mathbf{\Delta}_{\mathrm{int}}^{\leq 1})^{\mathrm{op}} \to C$ be a functor with $F([0]) = *$, and write $X := F([1])$. The left Kan extension $i_! F$ evaluated at $[n]$ is the colimit over $i_{/[n]}$ of the composite $i_{/[n]} \to (\mathbf{\Delta}_{\mathrm{int}}^{\leq 1})^{\mathrm{op}} \xrightarrow{F} C$. This colimit decomposes as a coproduct over the connected components. The span component contributes the colimit $\mathrm{colim}(X \to * \leftarrow X) = *$, the terminal object, which is also initial in the pointed ∞-category C. Each isolated object φ_k contributes a copy of X. Hence $(i_! F)([n]) \simeq X^{\sqcup n}$, the n-fold coproduct. Under these identifications, the Segal map $(i_! F)([n]) \to (i_! F)([1])^n$ is the canonical map from the n-fold coproduct of X to its n-fold product, hence is an isomorphism by semiadditivity. Thus $i_! F$ is a monoid.

Conversely, let $M \colon \mathbf{\Delta}^{\mathrm{op}} \to C$ be a monoid. The counit gives a canonical map from the left Kan extension of the restriction, $i_! i^* M \to M$. To show this is an isomorphism, it suffices to check this at the underlying object $[1]$. There it is clear: as we just computed, the value at $[1]$ of both sides is simply $M([1])$.

For the second equivalence, we use right Kan extension along j. Since j is fully faithful, right Kan extension j_* is fully faithful whenever it exists. The relative slice $j_{x/}$ over $x \in \{[0], [1]\}$ is as follows: for $x = [1]$, the identity $\mathrm{id}_{[1]}$ is an initial object, so the limit evaluates to the original value. For $x = [0]$, the relative slice is empty, so the limit is the terminal object. This shows that right Kan extension along j always exists and produces a functor in $\mathrm{Fun}^{*}((\mathbf{\Delta}_{\mathrm{int}}^{\leq 1})^{\mathrm{op}}, C)$. Conversely, any functor with value $*$ at $[0]$ is right Kan extended from $\{[1]\}$.

For the claim about groups, note that if C is additive and M is a monoid in C with underlying object X, then the shear map is given by $\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix} \colon X^2 \to X^2$. Its inverse is $\begin{pmatrix} \mathrm{id}_X & -\mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix}$, so every monoid is automatically a group. □

Remark 5.3.15. Once we know that $\mathrm{CMon}(C)$ is semiadditive (which will be proved in general in Proposition 5.3.20), the present proposition applied to $\mathrm{CMon}(C)$ gives an equivalence $\mathrm{Mon}(\mathrm{CMon}(C)) \simeq \mathrm{CMon}(C)$, under which the canonical monoid structure on an object $M \in \mathrm{CMon}(C)$ has multiplication given by the fold map $M \oplus M \to M$, which agrees with the algebraic addition $+\colon M \times M \to M$ of the commutative monoid structure. As a consequence, the monoid shear map (pr_1, m) from Definition 5.1.5 coincides with the algebraic shear map $(\mathrm{pr}_1, +)$ from Definition 5.3.6, and there are canonical identifications

$$\mathrm{CMon}(\mathrm{Grp}(C)) \quad \simeq \quad \mathrm{Grp}(\mathrm{CMon}(C)) \quad \simeq \quad \mathrm{CGrp}(C).$$

The first equivalence is an instance of currying functors $\mathrm{Span}(\mathrm{Fin}) \times \mathbf{\Delta}^{\mathrm{op}} \to C$: being a commutative monoid in the first coordinate commutes with being a group in the second. The second equivalence uses that both sides are the full subcategory of $\mathrm{Mon}(\mathrm{CMon}(C)) \simeq \mathrm{CMon}(C)$ spanned by those objects whose shear map is invertible.

Lemma 5.3.16 (A slice adjunction for spans)**.** *For every finite set S, the inclusion of forward maps induces a functor*

$$\mathrm{Fin}_{/S} \longrightarrow \mathrm{Span}(\mathrm{Fin})_{/S}$$

which admits a left adjoint. On objects, this left adjoint sends a span

$$T \xleftarrow{f} U \xrightarrow{g} S$$

to the map $g\colon U \to S$. The same construction restricts to the relative slices used below:

$$\mathrm{Fin}^{\leq 1}_{/S} \longrightarrow \mathrm{Span}(\mathrm{Fin}^{\leq 1})_{/S}.$$

Proof. There is a morphism in $\mathrm{Span}(\mathrm{Fin})_{/S}$ from the displayed span to the forward map $g\colon U \to S$, represented by the span $T \xleftarrow{f} U \xrightarrow{\mathrm{id}_U} U$. Composition with this morphism identifies maps from $g\colon U \to S$ to a forward map $V \to S$ with maps from the original span to $V \to S$ in $\mathrm{Span}(\mathrm{Fin})_{/S}$. This is the required adjunction. □

Having established the case of monoids and groups, we now turn to commutative monoids and commutative groups.

Proposition 5.3.17. *Let C be a semiadditive ∞-category. Then the forgetful functors*

$$\mathrm{CMon}(C) \to \mathrm{Mon}(C) \to C$$

are equivalences. If C is additive, then also the forgetful functors

$$\mathrm{CGrp}(C) \to \mathrm{Grp}(C) \to C$$

are equivalences.

Proof. The equivalences $\mathrm{Mon}(C) \xrightarrow{\sim} C$ and $\mathrm{Grp}(C) \xrightarrow{\sim} C$ are Proposition 5.3.14. By Exercise 5.3.12, we have $\mathrm{CGrp}(C) \simeq \mathrm{CMon}(C) \times_{\mathrm{Mon}(C)} \mathrm{Grp}(C)$. The case for commutative groups therefore follows from that of commutative monoids, so it remains to show that the forgetful functor $\mathrm{CMon}(C) \to C$ is an equivalence.

The proof strategy is similar to, but more involved than, that of Proposition 5.3.14. Consider the following subcategories of Span(Fin):

$$* \hookrightarrow \mathrm{Fin}^{\leq 1} \hookrightarrow \mathrm{Span}(\mathrm{Fin}^{\leq 1}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}).$$

Here $\mathrm{Fin}^{\leq 1} \subseteq \mathrm{Fin}$ is the full subcategory on sets of cardinality at most 1, i.e. it has objects $\emptyset$ and $*$ and a single non-identity morphism $\emptyset \to *$. We claim that restriction along each of these inclusions defines equivalences

$$\begin{aligned}\mathrm{CMon}(C) = \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), C) &\xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}), C) \\ &\xrightarrow{\sim} \mathrm{Fun}^{*}(\mathrm{Span}(\mathrm{Fin}^{\leq 1}), C) \xrightarrow{\sim} \mathrm{Fun}^{*}(\mathrm{Fin}^{\leq 1}, C) \xrightarrow{\sim} C,\end{aligned}$$

with inverses given by right, left, right and left Kan extension, respectively. Here $\mathrm{Fun}^{*}(-,-)$ denotes those functors that send the empty set to the terminal object of C. For the first equivalence, this was proved in Proposition 5.3.9. The proof strategy for the other three equivalences is similar. We will spell out the details for the second equivalence, and leave the easier third and fourth equivalences to the reader.

Denoting the relevant inclusion by $i\colon \mathrm{Span}(\mathrm{Fin}^{\leq 1}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$, we need to show that the restriction functor

$$i^{*}\colon \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}), C) \to \mathrm{Fun}^{*}(\mathrm{Span}(\mathrm{Fin}^{\leq 1}), C)$$

is an equivalence. Note that by semiadditivity of C, a functor $F\colon \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to C$ satisfies the Segal condition if and only if its restriction $F|_{\mathrm{Fin}}\colon \mathrm{Fin} \to C$ is left Kan extended from the point. Similarly, a functor $G\colon \mathrm{Span}(\mathrm{Fin}^{\leq 1}) \to C$ satisfies $G(\emptyset) = *$ if and only if its restriction $G|_{\mathrm{Fin}^{\leq 1}}\colon \mathrm{Fin}^{\leq 1} \to C$ is left Kan extended from the point. We now claim that left Kan extension along i restricts to a functor

$$i_{!}\colon \mathrm{Fun}^{*}(\mathrm{Span}(\mathrm{Fin}^{\leq 1}), C) \to \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}), C).$$

To see this, let $G\colon \mathrm{Span}(\mathrm{Fin}^{\leq 1}) \to C$ be a functor satisfying $G(\emptyset) = *$. It will suffice to show that the restriction $i_{!}(G)|_{\mathrm{Fin}}$ of the left Kan extension of G is itself the left Kan extension of $G|_{\mathrm{Fin}^{\leq 1}}$, since then it must be Kan extended from the point. If we let $i'\colon \mathrm{Fin}^{\leq 1} \to \mathrm{Fin}$ denote the inclusion, there is a canonical map

$$i'_{!}(G|_{\mathrm{Fin}^{\leq 1}}) \to i_{!}(G)|_{\mathrm{Fin}}.$$

Using the pointwise formula for Kan extensions, Theorem 22.4.3, we may compute both sides as colimits of G over the relative slices of i' and i, and the previous comparison map is induced by the inclusion map $\mathrm{Fin}^{\leq 1}_{/S} \to \mathrm{Span}(\mathrm{Fin}^{\leq 1})_{/S}$ of relative slices, for $S \in \mathrm{Fin}$. Since this inclusion admits a left adjoint by Lemma 5.3.16, this functor is final by Corollary 22.5.7, and hence restriction along it does not change the value of the colimit.

Finally we observe that the Kan extension $i_{!}G$ has the same value on the point as G itself. From this one concludes that the unit $G \to i^{*}i_{!}G$ and counit $i_{!}i^{*}F \to F$ of the adjunction are both natural isomorphisms, and hence these two functors are inverse equivalences. □

5.3.4 Semiadditivity

One of the archetypical examples of a semiadditive ∞-category is the 1-category of commutative monoids: the product $A \times B$ of two commutative monoids is also the *co*product. In this section,

we will generalize this observation: given an arbitrary ∞-category C with finite products, the ∞-category $\mathrm{CMon}(C)$ is semiadditive. Combining this fact with the results from the previous section, we may conclude that the forgetful functor $\mathrm{CMon}(C) \to C$ is in fact *universal* among semiadditive ∞-categories equipped with a finite-product-preserving functor to C. Similarly, the forgetful functor $\mathrm{CGrp}(C) \to C$ is universal among finite-product-preserving functors $D \to C$ from an *additive* ∞-category.

We start with two preliminary lemmas:

Lemma 5.3.18. *Let C be an ∞-category with finite products. Then the ∞-category* $\mathrm{CMon}(C)$ *has finite products and the evaluation map* $\mathrm{ev}_T \colon \mathrm{CMon}(C) \to C$ *preserves finite products for each* $T \in \mathrm{Fin}$.

Proof. The functor category $\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$ admits finite products (Lemma 22.2.8) which are computed pointwise, and the subcategory $\mathrm{CMon}(C)$ is closed under finite products. □

Lemma 5.3.19. *Let C be an ∞-category with finite products. Then there exists a* cotensoring functor

$$(-)^{(-)} \colon \mathrm{Span}(\mathrm{Fin}) \times \mathrm{CMon}(C) \to \mathrm{CMon}(C), \qquad (S, X) \mapsto X^S,$$

defined by $X^S(T) := X(S \times T)$, which preserves finite products in both variables and whose restriction to $\{*\} \times \mathrm{CMon}(C)$ is the identity functor.

Proof. For a finite set S and a commutative monoid X we define the cotensoring by $X^S(T) := X(S \times T)$ for $T \in \mathrm{Fin}$. The functoriality in S and T comes from the observation that the product functor $- \times - \colon \mathrm{Fin} \times \mathrm{Fin} \to \mathrm{Fin}$ induces a functor on span categories by taking products of spans:

$$- \times - \colon \mathrm{Span}(\mathrm{Fin}) \times \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Span}(\mathrm{Fin}), \qquad (S, T) \mapsto S \times T.$$

(This is no longer the categorical product in $\mathrm{Span}(\mathrm{Fin})$!)

- Since we have a natural bijection $* \times T \cong T$, it is clear that the cotensoring is the identity on $\{*\} \times \mathrm{CMon}(C)$.
- Fixing $S \in \mathrm{Fin}$, the functor $(-)^S \colon \mathrm{CMon}(C) \to \mathrm{CMon}(C)$ preserves finite products: we may check this pointwise for every $T \in \mathrm{Fin}$, where it is by definition given by the evaluation $X \mapsto X(S \times T)$, which preserves finite products.
- Fixing $X \in \mathrm{CMon}(C)$, the functor $X^{(-)} \colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{CMon}(C)$ preserves finite products: again we may check this pointwise for every $T \in \mathrm{Fin}$, where we may write it as the composite

  $$\mathrm{Span}(\mathrm{Fin}) \xrightarrow{-\times T} \mathrm{Span}(\mathrm{Fin}) \xrightarrow{X} C.$$

  But X preserves finite products by assumption and $- \times T$ preserves finite products since the functor $- \times T \colon \mathrm{Fin} \to \mathrm{Fin}$ preserves finite *co*products: $\bigsqcup_{i=1}^n (S_i \times T) \xrightarrow{\sim} (\bigsqcup_{i=1}^n S_i) \times T$.

In particular we see that X^S is the S-fold product of X in $\mathrm{CMon}(C)$, explaining the notation. □

Proposition 5.3.20. *Let C be an ∞-category with finite products. Then the ∞-category* $\mathrm{CMon}(C)$ *is semiadditive. The subcategory* $\mathrm{CGrp}(C)$ *is even additive.*

Proof. We first show that $\mathrm{CMon}(C)$ is semiadditive. This amounts to two claims: first, $\mathrm{CMon}(C)$ has a zero object, and second, finite products are also coproducts.

For the first claim, note that $\mathrm{Span}(\mathrm{Fin})$ is pointed, with zero object given by the empty set. By Corollary 4.1.12, the ∞-category $\mathrm{Fun}_*(\mathrm{Span}(\mathrm{Fin}), C)$ is pointed, with zero object given by the constant functor $\mathrm{const}_*\colon \mathrm{Span}(\mathrm{Fin}) \to C$. Since $\mathrm{CMon}(C) \subseteq \mathrm{Fun}_*(\mathrm{Span}(\mathrm{Fin}), C)$ is the full subcategory spanned by the finite-product-preserving functors and contains const_*, this same object is also a zero object of $\mathrm{CMon}(C)$.

For the second claim, we show that for commutative monoids X and Y, the product $X \times Y$ equipped with the maps $(\mathrm{id}_X, 0)\colon X \to X \times Y$ and $(0, \mathrm{id}_Y)\colon Y \to X \times Y$ is also a coproduct. In other words, we need to show that the corresponding natural transformation

$$((\mathrm{id}_X, 0), (0, \mathrm{id}_Y))\colon (X, Y) \to (X \times Y, X \times Y) = \Delta(X \times Y)$$

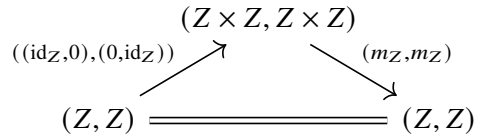
is the unit of an adjunction between the product functor and the diagonal functor $\Delta\colon \mathrm{CMon}(C) \to \mathrm{CMon}(C) \times \mathrm{CMon}(C)$ (with the product being the *left* adjoint!). By Proposition 22.1.2, it suffices to supplement this unit by a counit satisfying the triangle identities. We therefore have to produce for every third commutative monoid Z a compatible counit map

$$m_Z\colon Z \times Z \to Z$$

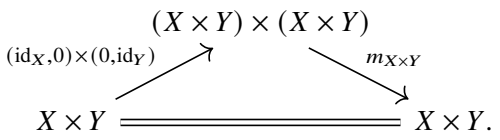
satisfying the triangle identities; this is the map that will a posteriori correspond to the fold map $Z \oplus Z \to Z$. Using the cotensoring from Lemma 5.3.19, we define m_Z as the composite

$$m_Z\colon Z \times Z \cong Z^{*\sqcup *} \xrightarrow{Z^{\nabla}} Z^* \cong Z,$$

where $\nabla\colon * \sqcup * \to *$ is the fold map in Fin, viewed as a span $(* \sqcup * \xleftarrow{\mathrm{id}} * \sqcup * \xrightarrow{\nabla} *)$. The triangle identities take the following form:



and



For the right-hand triangle, we use that the cotensoring preserves products, so that the map $m_{X\times Y}$ may be rewritten as the product map

$$m_X \times m_Y\colon (X \times X) \times (Y \times Y) \to X \times Y.$$

We then see that to produce both triangles, it suffices to produce for every $X \in \mathrm{CMon}(C)$ two natural identifications between the composites

$$X \xrightarrow{(\mathrm{id}_X, 0)} X \times X \xrightarrow{m_X} X, \qquad X \xrightarrow{(0, \mathrm{id}_X)} X \times X \xrightarrow{m_X} X$$

and the identity on X. For this, observe that we may write the zero map $0\colon \mathrm{const}_* \to X$ as the map $X^{\emptyset} \to X^*$ induced on cotensorings by the inclusion $\emptyset \hookrightarrow *$ in Fin. Consequently, under the isomorphism $X^* \times X^* \cong X^{*\sqcup *}$, the maps $(\mathrm{id}_X, 0)$ and $(0, \mathrm{id}_X)$ are induced by the two inclusions $\iota_1, \iota_2\colon * \hookrightarrow * \sqcup *$. The triangle identities then reduce to the observation that the two composites in Fin

$$* \xhookrightarrow{\iota_1} * \sqcup * \xrightarrow{\nabla} * \qquad \text{and} \qquad * \xhookrightarrow{\iota_2} * \sqcup * \xrightarrow{\nabla} *$$

are both the identity, which is immediate. This proves that $\mathrm{CMon}(C)$ is semiadditive.

Finally, $\mathrm{CGrp}(C) \subseteq \mathrm{CMon}(C)$ is closed under finite products and contains the zero object, so the biproducts constructed above restrict to $\mathrm{CGrp}(C)$. Their fold map is the algebraic addition m_M constructed above. Hence the additive shear map agrees with the algebraic shear map of Definition 5.3.6, which is invertible by definition. Thus $\mathrm{CGrp}(C)$ is additive. □

Remark 5.3.21. Observe that there are equivalences $\mathrm{CMon}(C_*) \simeq \mathrm{CMon}(C)_*$ and $\mathrm{CGrp}(C_*) \simeq \mathrm{CGrp}(C)_*$. Since $\mathrm{CMon}(C)$ and $\mathrm{CGrp}(C)$ are pointed, it follows from Lemma 4.1.9 that the forgetful functor $C_* \to C$ induces equivalences

$$\mathrm{CMon}(C_*) \xrightarrow{\sim} \mathrm{CMon}(C) \qquad \text{and} \qquad \mathrm{CGrp}(C_*) \xrightarrow{\sim} \mathrm{CGrp}(C).$$

Using a similar argument we find equivalences

$$\mathrm{Mon}(C_*) \xrightarrow{\sim} \mathrm{Mon}(C) \qquad \text{and} \qquad \mathrm{Grp}(C_*) \xrightarrow{\sim} \mathrm{Grp}(C).$$

With this at hand, we can now prove universal properties for the assignments $C \mapsto \mathrm{CMon}(C)$ and $C \mapsto \mathrm{CGrp}(C)$.

Notation 5.3.22. Let $\mathrm{Cat}_\infty^{\mathrm{prod}}$ be the subcategory of Cat_∞ spanned by those ∞-categories that have finite products and the finite-product-preserving functors. We further denote by

$$\mathrm{Cat}_\infty^{\mathrm{sadd}} \subseteq \mathrm{Cat}_\infty^{\mathrm{prod}} \qquad \text{and} \qquad \mathrm{Cat}_\infty^{\mathrm{add}} \subseteq \mathrm{Cat}_\infty^{\mathrm{prod}}$$

the full subcategories spanned by the semiadditive and additive ∞-categories, respectively.

Proposition 5.3.23. *The two functors*

$$\mathrm{CMon}(-)\colon \mathrm{Cat}_\infty^{\mathrm{prod}} \to \mathrm{Cat}_\infty^{\mathrm{sadd}} \qquad \textit{and} \qquad \mathrm{CGrp}(-)\colon \mathrm{Cat}_\infty^{\mathrm{prod}} \to \mathrm{Cat}_\infty^{\mathrm{add}}$$

are right adjoint to the respective inclusion functors.

Proof. We will check the (dual version of the) criterion from Proposition 22.7.9. We will treat the case for $\mathrm{CMon}(-)$; the case for $\mathrm{CGrp}(-)$ is identical. The forgetful functor defines a natural transformation $\varepsilon\colon \mathrm{CMon}(-) \to \mathrm{id}_{\mathrm{Cat}^{\mathrm{prod}}}$, and we must check that for every ∞-category C with finite products the two forgetful functors

$$\varepsilon_{\mathrm{CMon}(C)}\colon \mathrm{CMon}(\mathrm{CMon}(C)) \to \mathrm{CMon}(C) \qquad \text{and} \qquad \mathrm{CMon}(\varepsilon_C)\colon \mathrm{CMon}(\mathrm{CMon}(C)) \to \mathrm{CMon}(C)$$

are equivalences. Since $\mathrm{CMon}(C)$ is semiadditive by Proposition 5.3.20, the claim for $\varepsilon_{\mathrm{CMon}(C)}$ is an instance of Proposition 5.3.17. But now the case for $\mathrm{CMon}(\varepsilon_C)$ follows immediately: the swap map $\mathrm{Span}(\mathrm{Fin}) \times \mathrm{Span}(\mathrm{Fin}) \xrightarrow{\sim} \mathrm{Span}(\mathrm{Fin}) \times \mathrm{Span}(\mathrm{Fin})$ determines an equivalence

$$\mathrm{CMon}(\mathrm{CMon}(C)) \xrightarrow{\sim} \mathrm{CMon}(\mathrm{CMon}(C)), \qquad (M_n)_m \mapsto (M_m)_n,$$

and under this equivalence the forgetful functor $\varepsilon_{\mathrm{CMon}(C)}((M_n)_m) = (M_n)_1$ corresponds to the forgetful functor $\mathrm{CMon}(\varepsilon_C)((M_m)_n) = (M_1)_n$. □

Corollary 5.3.24. *Let A be a semiadditive ∞-category and let C be an ∞-category with finite products. The forgetful functor $\mathrm{CMon}(C) \to C$ induces an equivalence*

$$\mathrm{Fun}^\times(A, \mathrm{CMon}(C)) \xrightarrow{\sim} \mathrm{Fun}^\times(A, C).$$

Proof. By Yoneda, it suffices to show that this induces an equivalence on mapping animae out of every ∞-category E. This follows from the natural equivalences

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}_\infty}\big(E, \mathrm{Fun}^\times(A, \mathrm{CMon}(C))\big) &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{prod}}}\big(A, \mathrm{Fun}(E, \mathrm{CMon}(C))\big)\\ &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{prod}}}\big(A, \mathrm{CMon}(\mathrm{Fun}(E, C))\big)\\ &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{prod}}}\big(A, \mathrm{Fun}(E, C)\big)\\ &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}\big(E, \mathrm{Fun}^\times(A, C)\big),\end{aligned}$$

where the second equivalence uses pointwise finite products and the third is the adjunction from Proposition 5.3.23. □

As a useful consequence, we see that the hom animae of any semiadditive ∞-category admit canonical refinements to commutative monoids.

Proposition 5.3.25. *For a semiadditive ∞-category C, the hom functor* $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$ *lifts uniquely to a functor*

$$\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{CMon}(\mathrm{An})$$

which preserves direct sums in both variables. If C is additive, this lands in $\mathrm{CGrp}(\mathrm{An})$.

Proof. This follows directly from the universal property of $\mathrm{CMon}(\mathrm{An})$ established in Proposition 5.3.23; the proof is completely analogous to that of Proposition 4.4.4 and is omitted. □

5.4 The recognition principle for connective spectra

The recognition principle for infinite loop animae identifies connective spectra with commutative group objects in animae. In this section, we prove the equivalence

$$\boldsymbol{\Omega}^\infty(-)\colon \mathrm{Sp}_{\geq 0} \xrightarrow{\sim} \mathrm{CGrp}(\mathrm{An}).$$

It refines the recognition principle for loop spaces from Section 5.2, which identifies group objects in animae through the equivalence

$$\boldsymbol{\Omega}\colon \mathrm{An}_{*,\geq 1} \xrightarrow{\sim} \mathrm{Grp}(\mathrm{An})$$

with inverse given by the classifying anima construction $\mathbf{B}\colon \mathrm{Grp}(\mathrm{An}) \to \mathrm{An}_{*,\geq 1}$. We begin by recalling the definition of $\mathrm{Sp}_{\geq 0}$:

Definition 5.4.1. An anima X is called 0-*connected* if it is connected, i.e. if the set $\pi_0(X)$ has a single element. It is called n-*connected* for $n \geq 1$ if it is connected and the homotopy groups $\pi_k(X)$ are trivial whenever $k \leq n$. It is (-1)-*connected* whenever it is non-empty. We write

$$\mathrm{An}_{\geq n} \subseteq \mathrm{An}$$

for the full subcategory spanned by the $(n-1)$-connected animae. Thus $\mathrm{An}_{\geq 0}$ is the full subcategory of non-empty animae. Note that X is n-connected if and only if it is connected and ΩX is $(n-1)$-connected.

A spectrum X is called *connective* if, for every $n \geq 1$, its n-th space $X_n = \Omega^{\infty-n} X$ is $(n-1)$-connected. We write

$$\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$$

for the full subcategory spanned by the connective spectra. Connective spectra are also known as *infinite loop spaces*.

Remark 5.4.2. Since we have $\pi_i(X) \cong \pi_{n+i}(X_n)$ whenever $n + i \geq 0$, we see that a spectrum X is connective if and only if $\pi_i(X) = 0$ for $i < 0$.

The point of commutativity is that the delooping process can be repeated indefinitely: a commutative group G has a delooping $\mathbf{B}G$ which is again a commutative group, then a second delooping $\mathbf{B}^2 G$, and so on, and these deloopings assemble into a spectrum whose zeroth anima is G. Conversely, the underlying anima of every spectrum carries a canonical commutative group structure. We now make this precise, beginning with the delooping of commutative groups.

Lemma 5.4.3 (Group objects detected on path components)**.** *A monoid $M \in \mathrm{Mon}(\mathrm{An})$ is a group if and only if the ordinary monoid $\pi_0(M)$ is a group.*

Proof. If M is a group, then applying π_0 to its shear isomorphism shows that the shear map of $\pi_0(M)$ is a bijection, so $\pi_0(M)$ is a group. Conversely, suppose that $\pi_0(M)$ is a group. For every point $x \in M$, choose a point $y \in M$ whose component is inverse to that of x. The products xy and yx lie in the unit component, so multiplication by y is a homotopy inverse to multiplication by x. The shear map

$$(\mathrm{pr}_1, m) \colon M \times M \longrightarrow M \times M$$

is a map over the first factor whose fiber at x is multiplication by x. It is therefore an isomorphism, and hence M is a group. □

Proposition 5.4.4. *Applying* $\mathrm{CMon}(-)$ *to the adjunction* $\mathbf{B} \dashv \mathbf{\Omega}$ *produces an adjunction*

$$\mathbf{B} \colon \mathrm{CMon}(\mathrm{An}) \simeq \mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \rightleftarrows \mathrm{CMon}(\mathrm{An}_*) \overset{5.3.21}{\simeq} \mathrm{CMon}(\mathrm{An}) : \mathbf{\Omega}.$$

Moreover, both functors actually land in $\mathrm{CGrp}(\mathrm{An})$ *and induce for all $n \geq 0$ an equivalence*

$$\mathbf{B} \colon \mathrm{CGrp}(\mathrm{An}_{\geq n}) \underset{\sim}{\overset{\simeq}{\rightleftarrows}} \mathrm{CGrp}(\mathrm{An}_{\geq n+1}) : \mathbf{\Omega}.$$

Proof. Both $\mathbf{B}$ and $\mathbf{\Omega}$ preserve finite products: for $\mathbf{B}$ this is Lemma 5.1.16, while $\mathbf{\Omega}$ preserves them because it is a right adjoint. We may therefore apply $\mathrm{CMon}(-)$ to the adjunction.

For the first claim, it remains to argue that the forgetful functor $\mathrm{Mon}(\mathrm{An}) \to \mathrm{An}$ induces an equivalence

$$\mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \xrightarrow{\sim} \mathrm{CMon}(\mathrm{An}).$$

Since both $\mathrm{CMon}(-)$ and $\mathrm{Mon}(-)$ are defined as certain limit-preserving functors, there is an obvious equivalence $\mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \simeq \mathrm{Mon}(\mathrm{CMon}(\mathrm{An}))$, and under this equivalence the previous functor corresponds to the forgetful functor

$$\mathrm{Mon}(\mathrm{CMon}(\mathrm{An})) \to \mathrm{CMon}(\mathrm{An}).$$

But since CMon(An) is semiadditive by Proposition 5.3.20, this forgetful functor is an equivalence by Proposition 5.3.17.

Combining Lemma 5.4.3 with Remark 5.3.15, a commutative monoid $X \in \mathrm{CMon}(\mathrm{An})$ is a commutative group if and only if its monoid of path components $\pi_0(X)$ is a group. We check that both functors land in $\mathrm{CGrp}(\mathrm{An})$: the anima $\mathbf{B}M$ is connected, so $\pi_0(\mathbf{B}M) = *$ is trivially a group; and for $X \in \mathrm{CMon}(\mathrm{An}_*)$ we have $\pi_0(\mathbf{\Omega}X) = \pi_1(X)$, which is a group because loop spaces are grouplike. Hence the adjunction restricts to commutative groups, and by Theorem 5.2.1 we get that it restricts to an equivalence

$$\mathbf{B}\colon \mathrm{CGrp}(\mathrm{An}) \overset{5.3.15}{\simeq} \mathrm{CMon}(\mathrm{Grp}(\mathrm{An})) \underset{}{\overset{\simeq}{\rightleftarrows}} \mathrm{CMon}(\mathrm{An}_{\geq 1}) \simeq \mathrm{CGrp}(\mathrm{An}_{\geq 1}) :\mathbf{\Omega}.$$

Here we use Remark 5.3.21 to pass between commutative monoids in pointed animae and pointed commutative monoids in animae. The claim for arbitrary n follows because $\mathbf{B}G$ is n-connected if and only if $G \simeq \mathbf{\Omega}\mathbf{B}G$ is $(n-1)$-connected. □

Corollary 5.4.5 (Group completion)**.** *The inclusion* $\mathrm{CGrp}(\mathrm{An}) \hookrightarrow \mathrm{CMon}(\mathrm{An})$ *admits a left adjoint*

$$(-)^{\mathrm{grp}}\colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{An})$$

called group completion*. It is given on objects by* $M \mapsto \mathbf{\Omega}\mathbf{B}M$*.*

Proof. It follows immediately from the proposition that for every commutative group $G \in \mathrm{CGrp}(\mathrm{An})$, precomposition with the unit $M \to \mathbf{\Omega}\mathbf{B}M$ defines an equivalence

$$\mathrm{Hom}_{\mathrm{CGrp}(\mathrm{An})}(\mathbf{\Omega}\mathbf{B}M, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CMon}(\mathrm{An})}(M, G),$$

since both sides are equivalent to $\mathrm{Hom}_{\mathrm{CMon}(\mathrm{An}_*)}(\mathbf{B}M, \mathbf{B}G)$. □

After all this hard work, we are now able to prove the claim that connective spectra are commutative groups in An:

Theorem 5.4.6 (Recognition principle for connective spectra, cf. May [May72], Boardman and Vogt [BV73])**.** *The functor* $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$ *uniquely lifts to a finite-product-preserving functor*

$$\mathbf{\Omega}^\infty\colon \mathrm{Sp} \to \mathrm{CGrp}(\mathrm{An}).$$

This functor admits a fully faithful left adjoint

$$\mathbf{B}^\infty\colon \mathrm{CGrp}(\mathrm{An}) \hookrightarrow \mathrm{Sp}$$

whose image is precisely the subcategory $\mathrm{Sp}_{\geq 0}$ *of connective spectra.*

Proof. Since Ω^∞ preserves finite products and Sp is additive by Lemma 4.2.17, the fact that Ω^∞ uniquely lifts to a finite-product-preserving functor to $\mathrm{CGrp}(\mathrm{An})$ is an immediate consequence of the fact that $\mathrm{CGrp}(-)\colon \mathrm{Cat}_\infty^{\mathrm{prod}} \to \mathrm{Cat}_\infty^{\mathrm{add}}$ is right adjoint to the inclusion functor, see Proposition 5.3.23.

For the left adjoint, we use the equivalence $\mathbf{B}^n\colon \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\sim} \mathrm{CGrp}(\mathrm{An}_{\geq n})$ from Proposition 5.4.4, with inverse given by the iterated loop space functor $\mathbf{\Omega}^n$. For a commutative group G, we may now define the spectrum $\mathbf{B}^\infty G$ as follows:

$$\mathbf{B}^\infty G \quad := \quad (\mathbf{B}^n G, \sigma_n\colon \mathbf{B}^n G \xrightarrow{\sim} \mathbf{\Omega}\mathbf{B}^{n+1} G),$$

where the structure maps σ_n are the units of the adjunction $\mathbf{B} \dashv \mathbf{\Omega}$. To see that this is a left adjoint to $\mathbf{\Omega}^\infty \colon \mathrm{Sp} \to \mathrm{CGrp}(\mathrm{An})$, note that by additivity of Sp we get an equivalence $\mathrm{Sp} \simeq \mathrm{CGrp}(\mathrm{Sp}) \simeq \mathrm{Sp}(\mathrm{CGrp}(\mathrm{An}))$, so that we may write Sp as the following limit:

$$\mathrm{Sp} \quad \simeq \quad \lim(\cdots \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An})).$$

We may then compute

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sp}}(\mathbf{B}^\infty G, X) &\simeq \lim_n \mathrm{Hom}_{\mathrm{CGrp}(\mathrm{An})}(\mathbf{B}^n G, X_n) \\ &\simeq \lim_n \mathrm{Hom}_{\mathrm{CGrp}(\mathrm{An})}(G, \mathbf{\Omega}^n X_n) \\ &\simeq \lim_n \mathrm{Hom}_{\mathrm{CGrp}(\mathrm{An})}(G, X_0) \\ &\simeq \mathrm{Hom}_{\mathrm{CGrp}(\mathrm{An})}(G, X_0). \end{aligned}$$

Indeed, the structure isomorphisms $\sigma_n \colon X_n \xrightarrow{\sim} \Omega X_{n+1}$ induce compatible isomorphisms $\Omega^n X_n \xrightarrow{\sim} \Omega^{n+1} X_{n+1}$, identifying the resulting diagram constantly with X_0. Since all these equivalences are natural in G and X, this provides the adjunction. From this computation it also follows immediately that the unit $G \to \mathbf{\Omega}^\infty \mathbf{B}^\infty G$ of the adjunction is simply the identity map of G, hence is an isomorphism, showing that the functor $\mathbf{B}^\infty \colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{Sp}$ is fully faithful. The essential image is the subcategory of the above limit given by

$$\mathrm{CGrp}(\mathrm{An}) \quad \simeq \quad \lim(\cdots \xrightarrow[\simeq]{\Omega} \mathrm{CGrp}(\mathrm{An}_{\geq 2}) \xrightarrow[\simeq]{\Omega} \mathrm{CGrp}(\mathrm{An}_{\geq 1}) \xrightarrow[\simeq]{\Omega} \mathrm{CGrp}(\mathrm{An})),$$

which are precisely those spectra X such that X_n is $(n-1)$-connected for all $n \geq 1$, i.e. the connective spectra. □

Corollary 5.4.7. *The inclusion* $\mathrm{Sp}_{\geq 0} \hookrightarrow \mathrm{Sp}$ *admits a right adjoint* $\tau_{\geq 0} \colon \mathrm{Sp} \to \mathrm{Sp}_{\geq 0}$ *given by* $\tau_{\geq 0}(X) = \mathbf{B}^\infty \mathbf{\Omega}^\infty X$. □

Corollary 5.4.8. *There is a fully faithful functor*

$$\mathbf{B}^\infty \colon \mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set}) \hookrightarrow \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\mathbf{B}^\infty} \mathrm{Sp}$$

from the category of abelian groups into the ∞*-category of spectra. Here the first functor is induced by the fully faithful finite-product-preserving inclusion* $\mathrm{Set} \hookrightarrow \mathrm{An}$. □

Definition 5.4.9. The spectrum $\mathbf{B}^\infty A$ associated to an abelian group is called the *Eilenberg–MacLane spectrum* associated to A, and is classically denoted by HA. The resulting functor $H \colon \mathrm{Ab} \hookrightarrow \mathrm{Sp}$ is fully faithful. We will usually treat it as an inclusion and denote the Eilenberg–MacLane spectrum again by A, writing HA when we wish to emphasize the construction.

Remark 5.4.10. The agreement with the construction of Eilenberg–MacLane spectra from derived categories is explained in Remark 6.2.4.

5.5 Group completions as telescopes

By Corollary 5.4.5, every commutative monoid $M \in \mathrm{CMon}(\mathrm{An})$ admits a group completion

$$M \longrightarrow M^{\mathrm{grp}} = \mathbf{\Omega}\mathbf{B}M.$$

This is a universal construction, but in examples we often want a more explicit model for the underlying anima of M^{grp}.

For an ordinary commutative monoid A generated by an element x, the group A^{grp} admits a familiar description: it is obtained by formally inverting x. Equivalently, one may take the sequential colimit

$$A \xrightarrow{x+(-)} A \xrightarrow{x+(-)} A \xrightarrow{x+(-)} \cdots.$$

The result is $A[x^{-1}]$, which is already a group because x generates A.

In animae the same construction is subtler. The path components behave as expected, but the higher homotopy can remember the symmetry coherences of the commutative monoid. The telescope theorem below gives a useful condition under which the same formal-inversion telescope computes the group completion.

Definition 5.5.1 (Telescope)**.** Let $M \in \mathrm{CMon}(\mathrm{An})$ and let $x\colon * \to M$ be a point. We define the *telescope of M at x* as the sequential colimit

$$T(M,x) := \operatorname{colim}\big(M \xrightarrow{x+(-)} M \xrightarrow{x+(-)} M \xrightarrow{x+(-)} \cdots\big) \quad \in \quad \mathrm{An}.$$

Writing $\iota\colon M \to M^{\mathrm{grp}}$ for the group-completion map, the maps $m \mapsto \iota(m) - k\iota(x)$ from stage $k \geq 0$ are compatible with the transition maps and induce a comparison map

$$\gamma_{M,x}\colon T(M,x) \to M^{\mathrm{grp}}.$$

We use the following form of Nikolaus [Nik17, Proposition 6]. Nikolaus gives several equivalent hypotheses; the abelian fundamental-group condition used here is the version needed for complex K-theory.

Theorem 5.5.2 (Group completion theorem, telescope form)**.** *Let $M \in \mathrm{CMon}(\mathrm{An})$ and let $x\colon * \to M$ be a point. Assume that:*

(1) The class $[x]$ generates the commutative monoid $\pi_0(M)$;

(2) Each path component of $T(M,x)$ has abelian fundamental group.

Then the comparison map

$$\gamma_{M,x}\colon T(M,x) \to M^{\mathrm{grp}}$$

is an isomorphism of animae.

Proof. Under assumption (1), the singleton $\{[x]\}$ is a generating set for $\pi_0(M)$, and $T(M,x)$ is the object denoted M_∞ by Nikolaus. The claim is therefore the implication (4) $\Rightarrow$ (3) of Nikolaus [Nik17, Proposition 6]. □

Remark 5.5.3. The hypothesis on fundamental groups cannot be dropped. For the commutative monoid $\bigsqcup_{n\geq 0} \mathbf{B}\Sigma_n$, the telescope is $\mathbb{Z}\times\mathbf{B}\Sigma_\infty$, whereas the group completion is its plus construction $\mathbb{Z}\times\mathbf{B}\Sigma_\infty^+$; these differ already on π_1. See Nikolaus [Nik17, Proposition 6] for necessary and sufficient criteria.

Remark 5.5.4. Nikolaus also proves a homological group-completion theorem: after applying a multiplicative generalized homology theory, the map $M \to \mathbf{\Omega B} M$ becomes algebraic localization under suitable Ore-type hypotheses. Its proof uses localization of ring spectra and their module categories. We will not develop that result in this book; the telescope criterion above is the part needed for complex K-theory.

5.6 Finite deloopings (online only)

For completeness, we record a finite-stage version of the recognition principle. It is not needed for Theorem 5.4.6, but it explains how increasing coherence of the multiplication corresponds to the ability to deloop more often.

As an immediate corollary of the recognition principle for loop spaces we obtain a recognition principle for n-fold loop spaces.

Definition 5.6.1. Let C be an ∞-category with finite products. For $n \geq 0$ we iteratively define the ∞-category of *n-fold monoids in C*:

- For $n = 0$ we define $\mathrm{Mon}^{(0)}(C) := C_*$;
- For $n \geq 1$ we define $\mathrm{Mon}^{(n)}(C) := \mathrm{Mon}(\mathrm{Mon}^{(n-1)}(C))$.

We similarly define the ∞-category of *n-fold groups in C* by $\mathrm{Grp}^{(0)}(C) = C_*$ and $\mathrm{Grp}^{(n)}(C) := \mathrm{Grp}(\mathrm{Grp}^{(n-1)}(C))$.

Remark 5.6.2. For $n = 1$ note that we get $\mathrm{Mon}^{(1)}(C) \xrightarrow{\sim} \mathrm{Mon}(C)$ and $\mathrm{Grp}^{(1)}(C) \xrightarrow{\sim} \mathrm{Grp}(C)$ by Remark 5.3.21.

Remark 5.6.3. The n-fold monoids defined above are usually called *$\mathcal{E}_n$-monoids* in the literature. Indeed, if C is equipped with its cartesian symmetric monoidal structure, then $\mathrm{Mon}^{(n)}(C)$ can be described as the ∞-category of algebras over the little n-cubes ∞-operad $\mathcal{E}_n$; see [Lur17, Theorem 5.1.2.2 and Proposition 2.4.2.5]. We will not use this description here.

For general n, an n-fold monoid has n different multiplication operations that all commute with each other. In light of the Eckmann–Hilton argument these operations are all identified in the homotopy category $\mathrm{Ho}(C)$ and are homotopy commutative for $n \geq 2$. We should think of higher n as 'more commutativity'.

Proposition 5.6.4 (Recognition principle for iterated loop spaces)**.** *For every $n \geq 0$ there is an adjunction*

$$\mathbf{B}^n \colon \mathrm{Mon}^{(n)}(\mathrm{An}) \rightleftarrows \mathrm{An}_* : \mathbf{\Omega}^n$$

which restricts to an equivalence

$$\mathbf{B}^n \colon \mathrm{Grp}^{(n)}(\mathrm{An}) \xrightleftharpoons{\simeq} \mathrm{An}_{*,\geq n} : \mathbf{\Omega}^n.$$

Proof. Consider the adjunction $\mathbf{B} \colon \mathrm{Mon}(\mathrm{An}) \rightleftarrows \mathrm{An}_* : \mathbf{\Omega}$. Since both functors preserve finite products (see Lemma 5.1.16), we may apply the monoid construction repeatedly and paste the

resulting adjunctions. Using the equivalences $\mathrm{Mon}(C_*) \simeq \mathrm{Mon}(C)$ from Remark 5.3.21 at the intermediate stages gives the displayed adjunction $\mathbf{B}^n \dashv \mathbf{\Omega}^n$.

The recognition principle for loop spaces, Theorem 5.2.1, says that at each stage this adjunction restricts to an equivalence between group objects and connected pointed objects. Iterating raises connectivity by one at every stage, and therefore identifies $\mathrm{Grp}^{(n)}(\mathrm{An})$ with $\mathrm{An}_{*,\geq n}$. □

Corollary 5.6.5. *The ∞-category $\mathrm{Sp}_{\geq 0}$ is equivalent to the limit of the following diagram of ∞-categories:*

$$\cdots \to \mathrm{Grp}^{(n)}(\mathrm{An}) \to \cdots \to \mathrm{Grp}^{(2)}(\mathrm{An}) \to \mathrm{Grp}(\mathrm{An}) \to \mathrm{An}_*. \qquad \square$$

Proof. By definition, $\mathrm{Sp}_{\geq 0}$ is the full subcategory of Sp spanned by those spectra (X_n) such that the pointed anima X_n is $(n-1)$-connected for all $n \geq 1$, hence is given by the limit of the following diagram of ∞-categories:

$$\cdots \xrightarrow{\Omega} \mathrm{An}_{*,\geq n} \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathrm{An}_{*,\geq 2} \xrightarrow{\Omega} \mathrm{An}_{*,\geq 1} \xrightarrow{\Omega} \mathrm{An}_*.$$

The claim thus follows from the fact that these two diagrams are equivalent to each other, in light of the following commutative squares:

$$\begin{array}{ccc} \mathrm{An}_{*,\geq n} & \xrightarrow{\Omega} & \mathrm{An}_{*,\geq n-1} \\ {\scriptstyle \Omega^n}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \Omega^{n-1}} \\ \mathrm{Grp}^{(n)}(\mathrm{An}) & \xrightarrow{\mathrm{fgt}} & \mathrm{Grp}^{(n-1)}(\mathrm{An}). \end{array} \qquad \square$$

Exercises

Exercise 5.1. Let C be a classical 1-category. Show that the category $\mathrm{Mon}(C)$ is equivalent to the usual category of monoids in C:

(1) Given $X \in \mathrm{Mon}(C)$, show that the maps $e\colon * \to X_1$ and $m\colon X_1 \times X_1 \to X_1$ do indeed give X_1 the structure of a monoid;

(2) Given a monoid M in C, show that we obtain an object $X \in \mathrm{Mon}(C)$ by setting $X_n := M^n$, and by defining for $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ the map $\varphi^*\colon M^n \to M^m$ by

$$\varphi^*(x_1, \ldots, x_n) := (y_1, \ldots, y_m), \qquad y_i := \prod_{\varphi(i-1) < j \leq \varphi(i)} x_j.$$

(3) Show that these two constructions define the desired equivalence of categories.

Exercise 5.2. Show that the inclusion functor

$$\mathrm{Grp}(\mathrm{An}) \hookrightarrow \mathrm{Mon}(\mathrm{An})$$

admits a left adjoint

$$(-)^{\mathrm{grp}}\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{Grp}(\mathrm{An})$$

which on objects is given by $M \mapsto \mathbf{\Omega}\mathbf{B}M$. (This is called the *group completion functor*.)

Exercise 5.3. Show that the forgetful functor $\mathrm{Grp}(\mathrm{An}) \to \mathrm{An}_*$ admits a left adjoint

$$F^{\mathrm{Grp}}(-)\colon \mathrm{An}_* \to \mathrm{Grp}(\mathrm{An})$$

which is given on objects by $X \mapsto \mathbf{\Omega}\Sigma X$. (This is called the *free group functor*.)

Exercise 5.4 (Words and group completion)**.** Let X be an anima.

(1) Show that the forgetful functor $\mathrm{Mon}(\mathrm{An}) \to \mathrm{An}$ admits a left adjoint $F^{\mathrm{Mon}}\colon \mathrm{An} \to \mathrm{Mon}(\mathrm{An})$, called the *free monoid functor*, whose underlying anima is naturally

$$F^{\mathrm{Mon}}(X)_1 \cong \coprod_{n\geq 0} X^n.$$

Show that the monoid unit is the empty word, multiplication is concatenation, and the adjunction unit sends an element to the corresponding one-letter word.

(2) Show that $\pi_0(F^{\mathrm{Mon}}(X))$ is the ordinary free monoid on the set $\pi_0(X)$.

(3) Using only the relevant universal properties and the preceding exercises, construct canonical isomorphisms

$$F^{\mathrm{Mon}}(X)^{\mathrm{grp}} \cong F^{\mathrm{Grp}}(X_+) \cong \mathbf{\Omega}\Sigma(X_+),$$

where X_+ denotes X with a disjoint basepoint.

Hint for (1): Let $i\colon {\mathbf{\Delta}_{\mathrm{int}}}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ be the inclusion of the wide subcategory of inert maps, namely inclusions of intervals. Start with the diagram $R(X)_n = X^n$ on ${\mathbf{\Delta}_{\mathrm{int}}}^{\mathrm{op}}$, with restriction maps given by projections onto consecutive subtuples, and form $i_!R(X)$. Call a map in $\mathbf{\Delta}$ active if it preserves both endpoints. Use the unique active–inert factorization and Theorem 22.4.3 to show that

$$(i_!R(X))_m \cong \coprod_{\substack{n\geq 0 \\ \varphi\colon [m]\to[n] \text{ active}}} X^n.$$

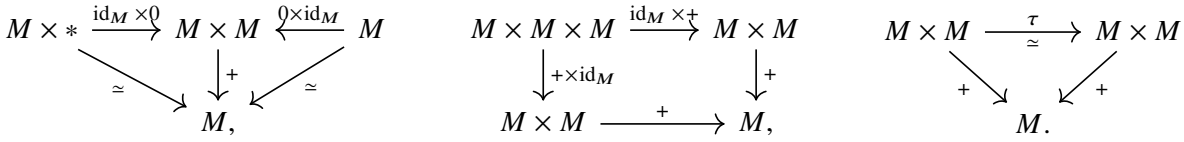
For the colimit calculation, show that the inclusion of active maps into the indexing category has a left adjoint and is therefore final. Identify the Segal maps using the integers $\varphi(r) - \varphi(r-1)$, and deduce the universal property from the Kan-extension adjunction and the fact that a Segal diagram on ${\mathbf{\Delta}_{\mathrm{int}}}^{\mathrm{op}}$ is determined by its value at $[1]$.

Exercise 5.5. Let x be an object of an ∞-category C. Denote by $\mathrm{Aut}_C(x)$ the subanima of $\mathrm{End}_C(x) := \mathrm{Hom}_C(x,x)$ spanned by the invertible endomorphisms $x \to x$. Show that composition of morphisms in C can be enhanced to a group structure on $\mathrm{Aut}_C(x)$.

Remark. One can also show that $\mathrm{End}_C(x)$ itself has the structure of a monoid via composition of morphisms, but this requires some theory we haven't introduced.

Exercise 5.6. Assume that C is a classical 1-category with finite products.

(1) Show that the category $\mathrm{CMon}(C)$ is equivalent to the category of commutative monoids in C defined in the usual way: an object M of C equipped with maps $+\colon M\times M \to M$ and $0\colon * \to M$ such that the following three diagrams commute:



(2) Combining part (1) with Exercise 5.1, deduce that in this case the forgetful functor $U\colon \mathrm{CMon}(C) \hookrightarrow \mathrm{Mon}(C)$ is fully faithful.

Exercise 5.7 (Group completion of the natural numbers)**.** Regard $\mathbb{N}$ as a discrete commutative monoid in animae. Show that its classifying anima $\mathbf{B}\mathbb{N}$ is equivalent to S^1, and deduce that the group-completion map identifies with the usual inclusion

$$\mathbb{N} \longrightarrow \mathbb{Z}\,.$$

More generally, compute the group completion of the free commutative monoid $\mathbb{N}^r$.

Exercise 5.8 (The connective-spectrum dictionary)**.** Let A be an abelian group, regarded as a discrete commutative group in animae. Show that the n-th anima of the associated Eilenberg–MacLane spectrum HA is a delooping $K(A, n)$. Deduce that $\Omega K(A, n+1) \cong K(A, n)$ and that HA has no nonzero homotopy groups outside degree 0.

6 Homological algebra

Homological algebra is the study of chain complexes and their derived categories. While its origins lie in classical algebra and algebraic topology, it is deeply intertwined with stable homotopy theory: the derived category of an abelian category is a stable ∞-category, and many of its key structures, like exact sequences and Ext-groups, are instances of general phenomena in stable ∞-categories.

The derived ∞-category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is obtained by inverting the quasi-isomorphisms of chain complexes. The general framework from Section 2.2 allows us to show that $\mathrm{D}(\mathcal{A})$ is stable. In the case of $\mathrm{D}(\mathbb{Z})$, Eilenberg–MacLane spectra provide the intrinsic construction of ordinary homology and cohomology promised in Chapter 3.

To recover the original abelian category from its derived category, we use a *t-structure*. A t-structure on a stable ∞-category comes with classes of *connective* and *coconnective* objects, which behave like connective and coconnective spectra. Their intersection turns out to be a 1-category, and in fact an abelian category, called the *heart* of the t-structure. The canonical t-structure on $\mathrm{D}(\mathcal{A})$ has heart $\mathcal{A}$. This gives a way to study classical algebra within a stable ∞-category.

From this perspective, Ext-groups are homotopy groups of mapping spectra, and Tor-groups arise from derived tensor products. Projective and flat resolutions make these constructions computable. Over a principal ideal domain, the Künneth formula and the universal coefficient theorems express the resulting groups in terms of homology, both for chain complexes and for the ordinary homology and cohomology of animae.

6.1 Derived ∞-categories

One of the foundational definitions of homological algebra is that of a *chain complex*: a sequence $\{C_n\}_{n\in\mathbb{Z}}$ of abelian groups, equipped with boundary maps $d_n\colon C_n \to C_{n-1}$ satisfying $d_{n-1}\circ d_n = 0$. For every chain complex $C_\bullet$, we may define its *n-th homology group* as the quotient of the kernel of $d_n\colon C_n \to C_{n-1}$ by the image of $d_{n+1}\colon C_{n+1} \to C_n$. One of the main examples of chain complexes in algebraic topology is the 'singular chain complex' $C_\bullet^{\mathrm{sing}}(X;\mathbb{Z})$ of a topological space X, whose homology groups recover the ordinary homology groups of X studied axiomatically in Section 3.3 (cf. Remark 3.3.2).

While chain complexes define a 1-category $\mathrm{Ch}(\mathbb{Z})$, this category is in some sense 'too rigid' from the perspective of homotopy theory. For example, while the homology groups $H_*(X;\mathbb{Z})$ of a topological space are homotopy invariants, the singular chain complex $C_\bullet^{\mathrm{sing}}(X;\mathbb{Z})$ itself is not homotopy invariant: the singular chain complexes of two homotopy equivalent (but non-homeomorphic) spaces are hardly ever isomorphic to each other. To resolve this issue, we may pass to the derived

category $\mathrm{D}(\mathbb{Z})$, which is obtained from $\mathrm{Ch}(\mathbb{Z})$ by inverting all *quasi-isomorphisms*: those morphisms of chain complexes that induce isomorphisms on homology groups.

In classical treatments, $\mathrm{D}(\mathbb{Z})$ is constructed as a triangulated category: a 1-category equipped with additional structure encoding exact sequences. However, this structure is somewhat unwieldy, and many natural constructions (such as limits and colimits, or functor categories) are difficult to perform in the triangulated setting. We will instead define $\mathrm{D}(\mathcal{A})$ as the ∞-categorical localization

$$\mathrm{D}(\mathcal{A}) \quad := \quad \mathrm{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}]$$

in the sense of Definition 1.5.19. The resulting ∞-category is stable, and the triangulated structure studied classically emerges as a shadow of the exact sequences coming from stability.

We now develop this theory systematically. After recalling the necessary background on abelian categories and their chain complexes in Section 6.1.1, we construct the derived ∞-category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ in Section 6.1.2 and establish its stability in Section 6.1.3.

6.1.1 Chain complexes in abelian categories

Notation 6.1.1. Let $\mathcal{A}$ be an additive 1-category. We refer to fibers and cofibers in $\mathcal{A}$ as *kernels* and *cokernels*, respectively:

$$\begin{array}{ccc} \ker(f) & \hookrightarrow & A \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ 0 & \hookrightarrow & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{coker}(f). \end{array}$$

If f is a monomorphism, we will use B/A as alternative notation for the cokernel.

Note that the map $\ker(f) \to A$ is automatically a monomorphism, while $B \twoheadrightarrow \mathrm{coker}(f)$ is automatically an epimorphism.

Definition 6.1.2. A 1-category $\mathcal{A}$ is called an *abelian category* if it is additive, has kernels and cokernels, and a nullsequence $A \overset{i}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C$ with i a monomorphism and p an epimorphism is a fiber sequence (i.e. $A \xrightarrow{\sim} \ker(p)$) if and only if it is a cofiber sequence (i.e. $\mathrm{coker}(i) \xrightarrow{\sim} C$). We will display such nullsequences as

$$0 \to A \overset{i}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C \to 0,$$

and refer to them as *short exact sequences*.

Example 6.1.3. The main example of an abelian category is the category Ab of abelian groups. More generally the category $\mathrm{LMod}_R(\mathrm{Ab})$ of left modules over an associative ring R is abelian.

Example 6.1.4. The opposite $\mathcal{A}^{\mathrm{op}}$ of an abelian category is again abelian.

The following are the basic properties of abelian categories we will need:

Proposition 6.1.5. *Let $\mathcal{A}$ be an abelian category.*

(1) *Every monomorphism $i\colon A \hookrightarrow B$ is the kernel of its cokernel, and every epimorphism $p\colon B \twoheadrightarrow C$ is the cokernel of its kernel.*

(2) *A morphism f is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

(3) *The category $\mathcal{A}$ admits all pullbacks and pushouts.*

(4) *Monomorphisms are closed under pushouts, and any pushout square along a monomorphism is also a pullback square.*

(5) *Epimorphisms are closed under pullbacks and any pullback square along an epimorphism is also a pushout square.*

(6) *Every morphism $f\colon A \to B$ factors uniquely*[13] *as an epimorphism followed by a monomorphism.*

The factorization from (6) is denoted by $A \twoheadrightarrow \mathrm{im}(f) \hookrightarrow B$, and we refer to the object $\mathrm{im}(f)$ as the *image* of f.

Proof. These are the standard exactness properties of an abelian category; see [Wei94, Section 1.2]. We only recall that the image factorization is obtained by setting

$$\mathrm{im}(f) := \ker(B \twoheadrightarrow \mathrm{coker}(f)),$$

and that a pushout of $f\colon A \to B$ along $g\colon A \to C$ is computed as the cokernel of $(f, -g)\colon A \to B \oplus C$. These descriptions give the remaining assertions and their duals. □

Definition 6.1.6. A nullsequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ is called *exact* if the map $\mathrm{im}(f) \to \ker(g)$ is an isomorphism.

Definition 6.1.7. A *chain complex in* $\mathcal{A}$ is a pair $(C_\bullet, d_\bullet)$ consisting of a $\mathbb{Z}$-graded object $C_\bullet = (C_n)_{n\in\mathbb{Z}}$, $C_n \in \mathcal{A}$, and a collection of 'boundary maps' $d_\bullet = (d_n\colon C_n \to C_{n-1})_{n\in\mathbb{Z}}$ satisfying the property that $d_{n-1} \circ d_n = 0$ for all n. We will often denote the chain complex simply by $C_\bullet$ or C.

A *chain map* (or *morphism of chain complexes*) $f_\bullet\colon C_\bullet \to D_\bullet$ consists of a collection of morphisms $f_n\colon C_n \to D_n$ that commute with the boundary maps, in the sense that $d^D_n \circ f_n = f_{n-1} \circ d^C_n$ for all n. There is a clear way to compose these, resulting in a 1-category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$.

Remark 6.1.8 (Cohomological grading of complexes)**.** We grade chain complexes *homologically*, so that the differential $d_n\colon C_n \to C_{n-1}$ *lowers* degree. Some references grade *cohomologically* instead, writing a complex as $(C^n)_{n\in\mathbb{Z}}$ with differential $d^n\colon C^n \to C^{n+1}$ *raising* degree. The two conventions record the same data, related by negating the index: a homologically graded complex $(C_\bullet, d_\bullet)$ becomes the cohomologically graded complex with $C^n := C_{-n}$ and $d^n := d_{-n}$, and conversely. We use homological grading throughout.

Example 6.1.9. Given an object $A \in \mathcal{A}$ and $k \in \mathbb{Z}$, we let $A[k]$ denote the chain complex consisting of A *concentrated in degree* k, meaning that $A[k]_k := A$ while $A[k]_n = 0$ for all $n \neq k$; all boundary maps of $A[k]$ are necessarily zero. This construction defines a fully faithful functor $(-)[k]\colon \mathcal{A} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ for all k.

[13] This is classically formulated as 'unique up to unique isomorphism'.

Example 6.1.10. Let $C_\bullet$ be a chain complex. Given $k \in \mathbb{Z}$, we define the *shift* $C[k]_\bullet$ of C by $C[k]_n := C_{n-k}$ and $d_n^{C[k]} := (-1)^k d_{n-k}^C$.

Remark 6.1.11. The sign $(-1)^k$ makes the shift compatible with our mapping-cone convention: the cone of $C \to 0$ is $C[1]$. Consequently, $C[1]$ represents the suspension of C in the derived ∞-category (see Theorem 6.1.30). Omitting the sign gives a naturally isomorphic chain complex, so it does not change the homology.

Definition 6.1.12. Given a chain complex $C_\bullet$ and an integer n, the *n-th homology* of $C_\bullet$ is the object

$$H_n(C) \quad := \quad \ker(d_n) \,/\, \mathrm{im}(d_{n+1}) \quad \in \quad \mathcal{A}.$$

Any chain map $f_\bullet \colon C_\bullet \to D_\bullet$ induces a map $H_n(f) \colon H_n(C) \to H_n(D)$ on n-th homology, resulting in a functor $H_n(-) \colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$.

Definition 6.1.13. A chain map $f_\bullet$ is called a *quasi-isomorphism* if the induced map $H_n(f) \colon H_n(C) \to H_n(D)$ is an isomorphism for all n.

Given two chain maps $f, g \colon C_\bullet \to D_\bullet$, a *chain homotopy* between f and g is a family of maps $(H_n \colon C_n \to D_{n+1})_{n\in\mathbb{Z}}$ satisfying $f_n - g_n = \partial^D \circ H_n + H_{n-1} \circ \partial^C$. We say f is a *chain homotopy equivalence* if there exists a chain map $g \colon D_\bullet \to C_\bullet$ and chain homotopies $g \circ f \sim \mathrm{id}_{C_\bullet}$ and $f \circ g \sim \mathrm{id}_{D_\bullet}$.

Definition 6.1.14 (Mapping cone)**.** The *mapping cone* of a chain map $f \colon C_\bullet \to D_\bullet$ is the chain complex $\mathrm{Cone}(f)$ with

$$\mathrm{Cone}(f)_n := D_n \oplus C_{n-1}, \qquad d^{\mathrm{Cone}(f)}(y, x) := (d^D(y) + f(x), -d^C(x)).$$

It fits into a natural short exact sequence $0 \to D_\bullet \to \mathrm{Cone}(f) \to C_\bullet[1] \to 0$.

Lemma 6.1.15 (Mapping-cone criterion)**.** *A chain map $f \colon C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if its mapping cone $\mathrm{Cone}(f)$ is acyclic.*

Proof. The long exact sequence in homology associated with the short exact sequence

$$0 \longrightarrow D_\bullet \longrightarrow \mathrm{Cone}(f) \longrightarrow C_\bullet[1] \longrightarrow 0$$

identifies the connecting maps with the maps on homology induced by f. The claim follows immediately from exactness. □

Exercise 6.1.16. Show that every chain homotopy equivalence is a quasi-isomorphism.

An important fact from homological algebra we will use is that short exact sequences of chain complexes result in long exact sequences of homology groups. We will state it as a black box; a proof may be found in [Wei94, Theorem 1.3.1].

Proposition 6.1.17. *Let $0 \to C_\bullet \overset{f}{\hookrightarrow} D_\bullet \overset{g}{\twoheadrightarrow} E_\bullet \to 0$ be a short exact sequence in $\mathrm{Ch}(\mathcal{A})$. Then there exist maps $\partial_n \colon H_n(E) \to H_{n-1}(C)$ for all n, and the sequence*

$$\cdots \xrightarrow{g} H_{n+1}(E) \xrightarrow{\partial_{n+1}} H_n(C) \xrightarrow{f} H_n(D) \xrightarrow{g} H_n(E) \xrightarrow{\partial_n} H_{n-1}(C) \xrightarrow{f} \ldots$$

in $\mathcal{A}$ is exact. □

When $\mathcal{A}$ is modules over a ring, the boundary map has the familiar elementwise description. Represent a class in $H_n(E)$ by a cycle $x \in E_n$, lift it to $y \in D_n$, and write $d(y) = f(z)$ for some $z \in C_{n-1}$. Then z is a cycle and $\partial_n([x]) = [z]$. The standard diagram chase shows that this is independent of all choices and proves exactness; see again [Wei94, Theorem 1.3.1].

Corollary 6.1.18. *Given a short exact sequence* $0 \to C_\bullet \overset{f}{\hookrightarrow} D_\bullet \overset{g}{\twoheadrightarrow} E_\bullet \to 0$*, the map* f *is a quasi-isomorphism if and only if* $E_\bullet$ *has trivial homology.* □

6.1.2 Derived ∞-categories

Fix an abelian category $\mathcal{A}$ for the remainder of this subsection.

Definition 6.1.19. We define the *derived* ∞*-category of* $\mathcal{A}$ as the (∞-categorical) localization of the 1-category $\mathrm{Ch}(\mathcal{A})$ of chain complexes at the class of quasi-isomorphisms:

$$\mathrm{D}(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

We will sometimes refer to objects of $\mathrm{D}(\mathcal{A})$ as *complexes*. This is slightly abusive terminology: objects of $\mathrm{D}(\mathcal{A})$ should best be thought of as 'chain complexes up to quasi-isomorphism' and in particular do not determine a well-defined chain complex in $\mathcal{A}$.

Example 6.1.20. Let R be an associative ring, and consider the abelian category $\mathcal{A} = \mathrm{RMod}_R(\mathrm{Ab})$ of right R-modules in the category of abelian groups. We denote the resulting derived ∞-category by $\mathrm{D}(R) := \mathrm{D}(\mathrm{RMod}_R(\mathrm{Ab}))$. In particular $\mathrm{D}(\mathbb{Z}) = \mathrm{D}(\mathrm{Ab})$.

Recall from Example 6.1.9 that $\mathrm{Ch}(\mathcal{A})$ contains $\mathcal{A}$ as a full subcategory, given by those chain complexes concentrated in degree 0. We will now show that the resulting functor $\mathcal{A} \to \mathrm{D}(\mathcal{A}), A \mapsto A[0]$ is still fully faithful. Note that this is not immediately clear, since the localization functor $\mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is far from fully faithful: for objects $A, B \in \mathcal{A}$ there are no nonzero chain maps $A[0] \to B[n]$ when $n > 0$ (the nonzero terms sit in different degrees), whereas the corresponding morphisms in $\mathrm{D}(\mathcal{A})$ compute the Ext-groups $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ (see Section 6.4), which are frequently nonzero.

We will proceed by showing that complexes can be 'truncated' both from below and from above: given a complex C, there is a universal approximation of C from the left by a complex concentrated in non-negative degrees, and from the right by a complex concentrated in non-positive degrees. By truncating from both directions, we can force the complex to be concentrated in degree 0. These truncation operations will descend to the level of derived ∞-categories, and will induce an equivalence between $\mathcal{A}$ and the full subcategory of $\mathrm{D}(\mathcal{A})$ on those complexes whose homology is concentrated in degree zero.

Notation 6.1.21. Given $k \in \mathbb{Z}$, we denote by $\mathrm{Ch}(\mathcal{A})_{\geq k} \subseteq \mathrm{Ch}(\mathcal{A})$ the full subcategory spanned by those chain complexes $C_\bullet$ satisfying $C_n = 0$ for $n < k$. Similarly, we write $\mathrm{Ch}(\mathcal{A})_{\leq k}$ for the full subcategory on those chain complexes satisfying $C_n = 0$ for $n > k$. We denote by

$$\mathrm{D}(\mathcal{A})_{\geq k} := \mathrm{Ch}(\mathcal{A})_{\geq k}[\{\text{quasi-isos}\}^{-1}], \qquad \mathrm{D}(\mathcal{A})_{\leq k} := \mathrm{Ch}(\mathcal{A})_{\leq k}[\{\text{quasi-isos}\}^{-1}]$$

the localizations of these subcategories at the quasi-isomorphisms.

Proposition 6.1.22. *For an integer k, the following hold:*

(1) *The inclusion* $\mathrm{Ch}(\mathcal{A})_{\geq k} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ *admits a right adjoint* $\tau_{\geq k} \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq k}$.

(2) *The induced functor* $\mathrm{D}(\mathcal{A})_{\geq k} \to \mathrm{D}(\mathcal{A})$ *is fully faithful and admits a right adjoint* $\tau_{\geq k} \colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})_{\geq k}$.

(3) *Dually, the inclusion* $\mathrm{Ch}(\mathcal{A})_{\leq k} \hookrightarrow \mathrm{Ch}(\mathcal{A})$ *admits a left adjoint* $\tau_{\leq k} \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\leq k}$.

(4) *The induced functor* $\mathrm{D}(\mathcal{A})_{\leq k} \to \mathrm{D}(\mathcal{A})$ *is fully faithful and admits a left adjoint* $\tau_{\leq k} \colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})_{\leq k}$.

Proof. (1) Given a chain complex C and an integer $k \in \mathbb{Z}$, we construct a new chain complex $\tau_{\geq k}(C)$ as follows:

$$\tau_{\geq k}(C)_n := \begin{cases} C_n & n > k \\ \ker(d_k \colon C_k \to C_{k-1}) & n = k \\ 0 & n < k. \end{cases}$$

The differentials $d_n \colon \tau_{\geq k}(C)_n \to \tau_{\geq k}(C)_{n-1}$ are the ones from $C_\bullet$ when $n > k+1$, are zero when $n \leq k$, and for $n = k+1$ are given by the canonical map $C_{k+1} \to \ker(d_k)$ induced by $d_{k+1} \colon C_{k+1} \to C_k$. This construction is functorial in C, defining a functor

$$\tau_{\geq k} \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq k}.$$

There is a canonical chain map $\varepsilon \colon \tau_{\geq k}(C) \to C$ given by the identity in degree $n > k$, by the zero map in degree $n < k$, and in degree $n = k$ by the inclusion $\ker(d_k) \hookrightarrow C_k$. It is clear that this chain map is a monomorphism. Furthermore, if D is a chain complex such that D_n vanishes for $n < k$, any chain map $f \colon D \to C$ uniquely factors through $\tau_{\geq k}(C)$: the commutative square

$$\begin{array}{ccc} D_k & \xrightarrow{f_k} & C_k \\ {\scriptstyle d_k}\downarrow & & \downarrow{\scriptstyle d_k} \\ 0 & \xrightarrow{f_{k-1}} & C_{k-1} \end{array}$$

guarantees that f_k factors through $\ker(d_k) \hookrightarrow C_k$. This shows that the transformation $\varepsilon \colon \tau_{\geq k} \to \mathrm{id}$ exhibits the functor $\tau_{\geq k} \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\geq k}$ as right adjoint to the inclusion.

(2) Observe that the counit $\varepsilon \colon \tau_{\geq k}(C) \to C$ of the adjunction from part (1) induces isomorphisms $H_n(\tau_{\geq k}(C)) \xrightarrow{\cong} H_n(C)$ for $n \geq k$. Since we have $H_n(\tau_{\geq k}(C)) = 0$ for $n < k$, it follows that the functor $\tau_{\geq k}$ sends quasi-isomorphisms to quasi-isomorphisms, hence induces a functor

$$\tau_{\geq k} \colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})_{\geq k}.$$

In a similar way, the unit and counit of the adjunction from part (1) induce natural transformations at the level of derived ∞-categories, exhibiting $\tau_{\geq k}$ as a right adjoint of the functor $\mathrm{D}(\mathcal{A})_{\geq k} \to \mathrm{D}(\mathcal{A})$ induced by the inclusion. Since the unit of the adjunction is a natural isomorphism, it follows that this inclusion is fully faithful, showing (2).

Parts (3) and (4) are similar, and can be seen as instances of (1) and (2) applied to $\mathcal{A}^{\mathrm{op}}$. The chain complex $\tau_{\leq k}(C)_\bullet$ may be explicitly given as follows:

$$\tau_{\leq k}(C)_n := \begin{cases} 0 & n > k \\ \operatorname{coker}(d_{k+1} \colon C_{k+1} \to C_k) & n = k \\ C_n & n < k. \end{cases}$$

□

Definition 6.1.23. Fix an integer $k \in \mathbb{Z}$. A complex A is called *k-connective* if $H_n(A) = 0$ for $n < k$. It is called *k-coconnective* if $H_n(A) = 0$ for $n > k$. When $k = 0$, we simply speak of *connective* and *coconnective* complexes.

Corollary 6.1.24. *The image of the inclusion $\mathrm{D}(\mathcal{A})_{\geq k} \hookrightarrow \mathrm{D}(\mathcal{A})$ from the previous proposition consists of the k-connective complexes, while the image of $\mathrm{D}(\mathcal{A})_{\leq k} \hookrightarrow \mathrm{D}(\mathcal{A})$ consists of the k-coconnective complexes.*

Proof. The essential image of the first inclusion consists of those complexes C for which the counit map $\tau_{\geq k}(C) \to C$ of the adjunction is an isomorphism in $\mathrm{D}(\mathcal{A})$, i.e. induces isomorphisms on homology. This happens precisely if the homology of C is trivial below degree k, i.e. if C is k-connective. The argument for the second inclusion is dual. □

Corollary 6.1.25. *Given a k-connective complex $A \in \mathrm{D}(\mathcal{A})_{\geq k}$ and a $(k-1)$-coconnective complex $B \in \mathrm{D}(\mathcal{A})_{\leq k-1}$, we have*

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B) = 0.$$

Proof. By adjunction we have $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, \tau_{\geq k}B) = 0$, where we use that $\tau_{\geq k}B = 0$ as its homology is trivial. □

We are now finally in a position to deduce the full faithfulness of $\mathcal{A} \hookrightarrow \mathrm{D}(\mathcal{A})$.

Proposition 6.1.26. *For every $k \in \mathbb{Z}$, the composite*

$$\mathcal{A} \xrightarrow{A \mapsto A[k]} \mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$$

is fully faithful, with essential image those complexes whose homology is concentrated in degree k.

Proof. Let $\mathrm{Ch}(\mathcal{A})_{=k} \subseteq \mathrm{Ch}(\mathcal{A})$ denote the subcategory of chain complexes $C_\bullet$ satisfying $C_n = 0$ for $n \neq k$. Then the assignment $A \mapsto A[k]$ induces an equivalence $\mathcal{A} \xrightarrow{\sim} \mathrm{Ch}(\mathcal{A})_{=k}$. The adjunction $\mathrm{Ch}(\mathcal{A})_{\geq k} \rightleftarrows \mathrm{Ch}(\mathcal{A})$ restricts to an adjunction $\mathcal{A} \simeq \mathrm{Ch}(\mathcal{A})_{=k} \rightleftarrows \mathrm{Ch}(\mathcal{A})_{\leq k}$, and since both functors preserve quasi-isomorphisms, we get an induced adjunction

$$\mathcal{A} \rightleftarrows \mathrm{D}(\mathcal{A})_{\leq k},$$

where the left adjoint is fully faithful as the unit is a natural isomorphism. The functor in question is now given by the composite

$$\mathcal{A} \hookrightarrow \mathrm{D}(\mathcal{A})_{\leq k} \hookrightarrow \mathrm{D}(\mathcal{A}),$$

hence is fully faithful as well. □

6.1.3 Stability of derived ∞-categories

We will now show that $\mathrm{D}(\mathcal{A})$ is a stable ∞-category, and that it has small (co)limits whenever $\mathcal{A}$ has small (co)products that interact nicely with the abelian structure. The main input will be to show that short exact sequences of chain complex in $\mathcal{A}$ result in exact sequences in $\mathrm{D}(\mathcal{A})$. To get a handle on fibers and cofibers in $\mathrm{D}(\mathcal{A})$, we will equip $\mathrm{Ch}(\mathcal{A})$ with the structure of an '∞-category with weak equivalences and (co)fibrations', as introduced in Definition 2.2.4, and then apply the results from Section 2.2.

Proposition 6.1.27. *Let $\mathcal{A}$ be an abelian category. Then the triple* $(\mathrm{Ch}(\mathcal{A}), \text{quasi-isos}, \text{monos})$ *is an ∞-category with weak equivalences and cofibrations. It admits functorial factorizations, and is homotopy cocomplete whenever $\mathcal{A}$ satisfies (AB3) and (AB4), i.e. $\mathcal{A}$ has small coproducts and monomorphisms are closed under small coproducts.*

Proof. We verify the (dualized) axioms from Definition 2.2.4:

(1) The 2-out-of-3 property for quasi-isomorphisms follows immediately from the 2-out-of-3 property for isomorphisms in $\mathcal{A}$.

(2) All isomorphisms are monomorphisms, and since $\mathrm{Ch}(\mathcal{A})$ is again an abelian category the monomorphisms are closed under compositions and pushouts; see Proposition 6.1.5. Note that every chain complex is cofibrant, since the map $0 \to C$ is a monomorphism for all $C \in \mathrm{Ch}(\mathcal{A})$.

(3) Consider a pushout square in $\mathrm{Ch}(\mathcal{A})$ of the form

$$\begin{array}{ccc} C_\bullet & \xrightarrow{f} & C'_\bullet \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle i'} \\ D_\bullet & \xrightarrow[f']{} & D'_\bullet, \end{array}$$

where i and i' are monomorphisms and i is a quasi-isomorphism. We need to show that also i' is a quasi-isomorphism. Let $E_\bullet$ be the cokernel of i and let $E'_\bullet$ be the cokernel of i'. Since i is a quasi-isomorphism, it follows from Corollary 6.1.18 that the homology of $E_\bullet$ is trivial. By the pasting law for pushout squares, we see that the induced map $E_\bullet \to E'_\bullet$ is an isomorphism of chain complexes, and so also the homology of $E'_\bullet$ is trivial. By applying Corollary 6.1.18 in the other direction, it follows that i' is a quasi-isomorphism.

(4) We now show that any chain map $f\colon C_\bullet \to D_\bullet$ factors functorially as a monomorphism followed by a quasi-isomorphism. Define the *mapping cylinder* $\mathrm{Cyl}(f)$ of f as the chain complex defined by $(\mathrm{Cyl}(f))_n = C_n \oplus C_{n-1} \oplus D_n$. The differential $d^{\mathrm{Cyl}(f)}\colon (\mathrm{Cyl}(f))_n \to (\mathrm{Cyl}(f))_{n-1}$ is given by

$$d^{\mathrm{Cyl}(f)} \quad := \quad \begin{pmatrix} d^C & \mathrm{id}_C & 0 \\ 0 & -d^C & 0 \\ 0 & -f & d^D \end{pmatrix},$$

i.e. in terms of elements we have $d^{\mathrm{Cyl}(f)}(c_n, c_{n-1}, d_n) := (d^C c_n + c_{n-1}, -d^C c_{n-1}, -f(c_{n-1}) + d^D d_n)$. There is a chain map $i_C\colon C_\bullet \to \mathrm{Cyl}(f)_\bullet$ given by $i_C(c_n) = (c_n, 0, 0)$, and a chain map $p\colon \mathrm{Cyl}(f)_\bullet \to D_\bullet$ given by $p(c_n, c_{n-1}, d_n) = f(c_n) + d_n$. Note that the constructions of $\mathrm{Cyl}(f)$, i_C and p are functorial in f and that $f = p \circ i_C$, so this provides a functorial factorization of f. Since i_C is clearly a monomorphism, it remains to show that p is a quasi-isomorphism. By Exercise 6.1.16,

it suffices to show that p is in fact a chain homotopy equivalence. We claim that the chain map $i_D \colon D_\bullet \to \mathrm{Cyl}(f)_\bullet$ given by $i_D(d_n) = (0, 0, d_n)$ is a chain homotopy inverse for p. It is clear that $p \circ i_D = \mathrm{id}_{D_\bullet}$, so it remains to construct a chain homotopy H between $i_D \circ p$ and $\mathrm{id}_{\mathrm{Cyl}(f)_\bullet}$. Such a chain homotopy is given by $H_n(c_n, c_{n-1}, d_n) := (0, c_n, 0)$.

We have thus shown that the triple $(\mathrm{Ch}(\mathcal{A}), \text{quasi-isos}, \text{monos})$ is an ∞-category with weak equivalences and cofibrations, and that it has functorial factorizations. Under the assumption that $\mathcal{A}$ satisfies (AB3) and (AB4), the category $\mathrm{Ch}(\mathcal{A})$ inherits arbitrary coproducts from $\mathcal{A}$, formed degreewise, and monomorphisms in $\mathrm{Ch}(\mathcal{A})$ are again closed under arbitrary coproducts. Moreover, since the coproduct functor $\bigoplus_I \colon \mathrm{Ch}(\mathcal{A})^I \to \mathrm{Ch}(\mathcal{A})$ is exact, it commutes with formation of homology, and it follows that an arbitrary coproduct of quasi-isomorphisms is again a quasi-isomorphism. This verifies the (dualized) conditions from Definition 2.2.10, showing that $\mathrm{Ch}(\mathcal{A})$ is homotopy cocomplete. □

Corollary 6.1.28. *Let $\mathcal{A}$ be an abelian category. The triple* $(\mathrm{Ch}(\mathcal{A}), \text{quasi-isos}, \text{epis})$ *is an ∞-category with weak equivalences and fibrations. It admits functorial factorizations, and is homotopy complete whenever $\mathcal{A}$ satisfies (AB3*) and (AB4*), i.e. $\mathcal{A}$ has small products and epimorphisms are closed under small products.*

Proof. This is a special case of Proposition 6.1.27 applied to the abelian category $\mathcal{A}^{\mathrm{op}}$, using that $\mathrm{Ch}(\mathcal{A}^{\mathrm{op}}) \simeq \mathrm{Ch}(\mathcal{A})^{\mathrm{op}}$. □

Corollary 6.1.29. *Let $\mathcal{A}$ be an abelian category.*

(1) The derived ∞-category $\mathrm{D}(\mathcal{A})$ *admits finite limits and finite colimits.*

(2) If $\mathcal{A}$ satisfies (AB3) and (AB4), then $\mathrm{D}(\mathcal{A})$ *admits small colimits. If it satisfies (AB3*) and (AB4*) then* $\mathrm{D}(\mathcal{A})$ *admits small limits.*

(3) The localization functor $\gamma \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ *preserves pushouts along monomorphisms and preserves pullbacks along epimorphisms.*

(4) For a short exact sequence $0 \to C_\bullet \hookrightarrow D_\bullet \twoheadrightarrow E_\bullet \to 0$ *in* $\mathrm{Ch}(\mathcal{A})$*, the induced sequence*

$$\gamma(C_\bullet) \to \gamma(D_\bullet) \to \gamma(E_\bullet)$$

in $\mathrm{D}(\mathcal{A})$ *is both a fiber sequence and a cofiber sequence.*

Proof. Parts (1)-(3) follow directly from applying Theorem 2.2.8, Theorem 2.2.11, and Corollary 2.2.9 to the structures established in Proposition 6.1.27 and Proposition 6.1.28. Part (4) is a specific case of (3). □

We are now ready to prove the main result of this section:

Theorem 6.1.30. *For any abelian category $\mathcal{A}$, the derived ∞-category* $\mathrm{D}(\mathcal{A})$ *is stable. Moreover, the localization functor* $\gamma \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ *sends short exact sequences to exact sequences.*

Proof. Assuming stability, the last claim is immediate from part (4) of the previous corollary. For stability, we saw in Corollary 6.1.29 that $\mathrm{D}(\mathcal{A})$ admits finite limits and finite colimits. Moreover, the localization functor $\gamma \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ preserves the zero object, so $\mathrm{D}(\mathcal{A})$ is pointed. By

Theorem 4.2.2, it remains to show that the unit and counit of the adjunction $\Sigma \dashv \Omega$ is an equivalence. By the universal property of $\gamma\colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$, this amounts to showing that for a chain complex $C_\bullet$, the unit $\gamma(C_\bullet) \to \Omega\Sigma\gamma(C_\bullet)$ and the counit $\Sigma\Omega\gamma(C_\bullet) \to \gamma(C_\bullet)$ are isomorphisms in $\mathrm{D}(\mathcal{A})$.

To this end, consider the chain complex $C'_\bullet$ defined by $C'_n := C_n \oplus C_{n-1}$, with differential $d^{C'}_n(c_n, c_{n-1}) = (d^C c_n + c_{n-1}, -d^C c_{n-1})$; note that this is the mapping cylinder of the map $C_\bullet \to 0$. The maps $f(c_n) := (c_n, 0)$ and $g(c_n, c_{n-1}) := c_{n-1}$ define a short exact sequence

$$0 \to C_\bullet \overset{f}{\hookrightarrow} C'_\bullet \overset{g}{\twoheadrightarrow} C[1]_\bullet \to 0,$$

and by part (4) of Corollary 6.1.29 the resulting commutative square

$$\begin{array}{ccc} \gamma(C_\bullet) & \xrightarrow{\gamma(f)} & \gamma(C'_\bullet) \\ {\scriptstyle 0}\downarrow & \ulcorner & \downarrow{\scriptstyle \gamma(g)} \\ 0 & \longrightarrow & \gamma(C[1]_\bullet) \end{array}$$

in $\mathrm{D}(\mathcal{A})$ is both a pullback square and a pushout square. Since the map $C'_\bullet \to 0$ is a quasi-isomorphism, we get $\gamma(C'_\bullet) \simeq 0$, so this square simultaneously exhibits $\gamma(C[1]_\bullet)$ as the suspension of $\gamma(C_\bullet)$ and $\gamma(C_\bullet)$ as the loops of $\gamma(C[1]_\bullet)$. We conclude that the maps $\gamma(C_\bullet) \to \Omega\Sigma\gamma(C_\bullet)$ and $\Sigma\Omega\gamma(C_\bullet) \to \gamma(C_\bullet)$ are isomorphisms in $\mathrm{D}(\mathcal{A})$. □

Remark 6.1.31. The proof shows that the localization functor $\mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ turns the explicit shift functor $(-)[n]$ from Example 6.1.10 into the abstract shift functor $[n]$ in the stable ∞-category $\mathrm{D}(\mathcal{A})$.

Corollary 6.1.32. *For a complex $A \in \mathrm{D}(\mathcal{A})$ and $n \in \mathbb{Z}$, there exists a functorial exact sequence in $\mathrm{D}(\mathcal{A})$ of the form*

$$\tau_{\geq n}A \to A \to \tau_{\leq n-1}A.$$

Proof. For a chain complex $C_\bullet \in \mathrm{Ch}(\mathcal{A})$, put $Q_\bullet := C_\bullet/\tau_{\geq n}C_\bullet$. There is a functorial short exact sequence

$$0 \to \tau_{\geq n}C_\bullet \hookrightarrow C_\bullet \twoheadrightarrow Q_\bullet \to 0.$$

The natural map $Q_\bullet \to \tau_{\leq n-1}C_\bullet$, given in degree $n-1$ by the quotient $C_{n-1} \to \mathrm{coker}(d_n)$ and by the identity below that degree, is a quasi-isomorphism. The theorem therefore turns the displayed short exact sequence into an exact sequence $\gamma\tau_{\geq n} \to \gamma \to \gamma\tau_{\leq n-1}$ in $\mathrm{Fun}(\mathrm{Ch}(\mathcal{A}), \mathrm{D}(\mathcal{A}))$. By the universal property of $\gamma\colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$, this descends to the desired exact sequence $\tau_{\geq n} \to \mathrm{id} \to \tau_{\leq n-1}$. □

6.2 Ordinary homology and cohomology via $\mathrm{D}(\mathbb{Z})$

We now construct the ordinary homology and cohomology theories that were used axiomatically in Chapter 3. The construction is intrinsic to animae: instead of choosing a topological model and taking singular chains, we use the universal property of An to produce chain and cochain complexes directly in the derived category $\mathrm{D}(\mathbb{Z})$. The bridge from chain complexes to spectra is provided by Eilenberg–MacLane spectra.

6.2.1 Eilenberg-MacLane spaces and spectra

Throughout this section we fix the abelian category $\mathcal{A} = \mathrm{Ab}$ of abelian groups and work in its derived ∞-category $\mathrm{D}(\mathbb{Z})$. By forming mapping spectra out of the complex $\mathbb{Z}[0]$, we may associate a spectrum to every complex:

Definition 6.2.1 (Eilenberg-MacLane spectrum)**.** Given a complex $A \in \mathrm{D}(\mathbb{Z})$, we define its *Eilenberg-MacLane spectrum* as the mapping spectrum

$$HA \quad := \quad \hom_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}[0], A) \quad \in \quad \mathrm{Sp}$$

Similarly, for $n \in \mathbb{Z}$ we define its *n-th Eilenberg MacLane anima* as

$$K(A,n) \quad := \quad \mathrm{Hom}_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}[-n], A) \quad \in \quad \mathrm{An}.$$

These define functors $H\colon \mathrm{D}(\mathbb{Z}) \to \mathrm{Sp}$ and $K(-,n)\colon \mathrm{D}(\mathbb{Z}) \to \mathrm{An}$.

In particular, every abelian group $A \in \mathrm{Ab}$ defines an Eilenberg-MacLane spectrum HA and an n-th Eilenberg MacLane anima $K(A,n)$ by regarding it as a chain complex concentrated in degree 0.

Eilenberg-MacLane spectra play an important role in stable homotopy theory. In the next subsection, we will use them to classify ordinary (co)homology theories with coefficients in an abelian group.

Exercise 6.2.2. Show that for an associative ring R the forgetful functor $\mathrm{RMod}_R(\mathrm{Ab}) \to \mathrm{Ab}$ induces a functor $\mathrm{D}(R) \to \mathrm{D}(\mathbb{Z})$. Show that the composite $\mathrm{D}(R) \to \mathrm{D}(\mathbb{Z}) \xrightarrow{H} \mathrm{Sp}$ may be identified with $\hom_{\mathrm{D}(R)}(R[0], -)$.

The main property of the Eilenberg-MacLane animae and spectra is that their homotopy groups recover the homology groups of A:

Proposition 6.2.3. *For a complex $A \in \mathrm{D}(\mathbb{Z})$ and integers $n \in \mathbb{Z}$, $k \geq 0$, there are natural isomorphisms of abelian groups*

$$\pi_k(K(A,n)) \quad \cong \quad H_{k-n}(A) \qquad \textit{and} \qquad \pi_n(HA) \quad \cong \quad H_n(A).$$

Remark 6.2.4. If A is an abelian group regarded as a complex concentrated in degree 0, then Proposition 6.2.3 shows that HA is connective and that $\Omega^\infty HA$ is the discrete commutative group A. Under the recognition equivalence of Theorem 5.4.6, it therefore corresponds to $A \in \mathrm{CGrp}(\mathrm{Set})$, giving a natural isomorphism

$$HA \xrightarrow{\sim} \mathbf{B}^\infty A.$$

Thus this construction agrees with the Eilenberg–MacLane spectrum of Definition 5.4.9. As there, we usually write A also for its Eilenberg–MacLane spectrum when A is an abelian group.

This has the following useful consequence:

Lemma 6.2.5. *The functor $H\colon \mathrm{D}(\mathbb{Z}) \to \mathrm{Sp}$ preserves colimits.*

Proof. Since it is exact, it remains to show it preserves arbitrary coproducts: the map $\bigoplus_{i\in I} H(A_i) \to H(\bigoplus_{i\in I} A_i)$ is an isomorphism of spectra. This may be tested at the level of homotopy groups. Combining with Proposition 6.2.3, this then follows from the fact that both $\pi_n(-)\colon \mathrm{Sp} \to \mathrm{Ab}$ and $H_n(-)\colon \mathrm{D}(\mathbb{Z}) \to \mathrm{Ab}$ preserve coproducts. □

To prove the proposition, we introduce an auxiliary ∞-category in which the complexes $\mathbb{Z}[n]$ corepresent homology directly. We will then transport this corepresentability across the further localization to $\mathrm{D}(\mathbb{Z})$.

Definition 6.2.6. Let $\mathcal{A}$ be an abelian category. We define the ∞-category $\mathcal{K}(\mathcal{A})$ as the localization

$$\mathcal{K}(\mathcal{A}) \quad := \quad \mathrm{Ch}(\mathcal{A})[\{\text{chain homotopy equivalences}^{-1}\}].$$

There are localization functors $\mathrm{Ch}(\mathcal{A}) \to \mathcal{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$.

A chain map $i\colon C_\bullet \to D_\bullet$ is called a *degreewise split monomorphism* if each component $i_n\colon C_n \to D_n$ is a split monomorphism in $\mathcal{A}$. The chosen retractions are not required to assemble into a chain map. Degreewise split epimorphisms are defined dually.

Proposition 6.2.7. *For every abelian category $\mathcal{A}$, the ∞-category $\mathcal{K}(\mathcal{A})$ is stable. Under the localization functor $\mathrm{Ch}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$, the suspension of a chain complex $C_\bullet$ is naturally isomorphic to the shifted complex $C[1]_\bullet$.*

Proof. The proof is the split analogue of the proofs of Proposition 6.1.27 and Theorem 6.1.30. Equip $\mathrm{Ch}(\mathcal{A})$ with chain homotopy equivalences as weak equivalences and degreewise split monomorphisms as cofibrations. The mapping-cylinder factorization shows that every map factors as a cofibration followed by a weak equivalence. The pushout axiom follows from the criterion that a degreewise split monomorphism is a chain homotopy equivalence precisely when its cokernel is contractible. The dual argument uses degreewise split epimorphisms, so the localization $\mathcal{K}(\mathcal{A})$ admits finite limits and colimits.

For a chain complex $C_\bullet$, the degreewise split short exact sequence

$$0 \to C_\bullet \to \mathrm{Cyl}(C_\bullet \to 0) \to C[1]_\bullet \to 0$$

is sent to a cofiber sequence in $\mathcal{K}(\mathcal{A})$. Its middle term is contractible, so it identifies suspension with the shift $[1]$. Thus suspension is an equivalence, with inverse induced by $[-1]$, and Theorem 4.2.2 implies that $\mathcal{K}(\mathcal{A})$ is stable. □

Lemma 6.2.8. *The homotopy category of $\mathcal{K}(\mathcal{A})$ is the category whose objects are chain complexes and whose morphisms are chain maps modulo chain homotopy.*

Proof. As in the proof of Corollary 2.3.7, this follows from the observation that a functor $\mathrm{Ch}(\mathcal{A}) \to C$ into a 1-category C inverts chain homotopy equivalences if and only if it identifies chain homotopic chain maps. □

Corollary 6.2.9. *For a chain complex $C_\bullet \in \mathrm{Ch}(\mathrm{Ab})$ and $n \in \mathbb{Z}$ there is a natural isomorphism*

$$\pi_0 \operatorname{Hom}_{\mathcal{K}(\mathrm{Ab})}(\mathbb{Z}[n], C_\bullet) \cong H_n(C).$$

In particular, the functor $\operatorname{Hom}_{\mathcal{K}(\mathrm{Ab})}(\mathbb{Z}[n], -)\colon \mathcal{K}(\mathrm{Ab}) \to \mathrm{An}$ corepresented by $\mathbb{Z}[n]$ inverts quasi-isomorphisms.

Proof. By Lemma 6.2.8, the left-hand side is the set of chain maps $\mathbb{Z}[n] \to C_\bullet$ up to chain homotopy. The first claim follows, since a chain map $\mathbb{Z}[n] \to C_\bullet$ is nothing but an element of $\ker(d_n \colon C_n \to C_{n-1})$ and two chain maps are chain homotopic if and only if their difference lies in the image of $d_{n+1} \colon C_{n+1} \to C_n$.

The second claim now follows from the stability of $\mathcal{K}(\mathrm{Ab})$ established in Proposition 6.2.7: for $k \geq 0$ we have

$$\pi_k \operatorname{Hom}_{\mathcal{K}(\mathrm{Ab})}(\mathbb{Z}[n], A) \cong \pi_0 \operatorname{Hom}_{\mathcal{K}(\mathrm{Ab})}(\mathbb{Z}[n+k], A) \cong H_{n+k}(A). \qquad \square$$

Lemma 6.2.10. *Let $\gamma \colon C \to C[W^{-1}]$ be the localization functor inverting a collection of morphisms W in some ∞-category C. Let $X \in C$ be an object with the property that $\operatorname{Hom}_C(X, -) \colon C \to \mathrm{An}$ inverts all morphisms in W. Then for every $Y \in C$ the map*

$$\operatorname{Hom}_C(X, Y) \to \operatorname{Hom}_{C[W^{-1}]}(\gamma X, \gamma Y)$$

is an isomorphism of animae.

Proof. The assumption on X guarantees that $\operatorname{Hom}_C(X, -)$ descends to a functor $C[W^{-1}] \to \mathrm{An}$. We need to show that this functor is corepresented by $\gamma X \in C[W^{-1}]$. But this follows immediately from the full faithfulness of $\gamma^* \colon \operatorname{Fun}(C[W^{-1}], \mathrm{An}) \hookrightarrow \operatorname{Fun}(C, \mathrm{An})$: for any other functor $F \colon C[W^{-1}] \to \mathrm{An}$ we have

$$\operatorname{Nat}_{C[W^{-1}] \to \mathrm{An}}(\operatorname{Hom}_C(X, -), F) \xrightarrow{\sim} \operatorname{Nat}_{C \to \mathrm{An}}(\operatorname{Hom}_C(X, -), F \circ \gamma) \xrightarrow{\sim} F(\gamma(X)). \qquad \square$$

We are now ready for the proof of Proposition 6.2.3:

Proof of Proposition 6.2.3. Combining Corollary 6.2.9 and Lemma 6.2.10, we see that for every complex $A \in \mathcal{K}(\mathrm{Ab})$ and $n \in \mathbb{Z}$, the canonical map

$$\operatorname{Hom}_{\mathcal{K}(\mathrm{Ab})}(\mathbb{Z}[-n], A) \to \operatorname{Hom}_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}[-n], A) = K(A, n)$$

is an isomorphism of animae. In particular, we have

$$\pi_k(K(A, n)) \cong \pi_k \operatorname{Hom}_{\mathcal{K}(\mathrm{Ab})}(\mathbb{Z}[-n], A) \cong \pi_0 \operatorname{Hom}_{\mathcal{K}(\mathrm{Ab})}(\mathbb{Z}[k-n], A) \cong H_{k-n}(A)$$

and thus

$$\pi_n(HA) \cong \pi_0 H(A[-n]) \cong \pi_0 K(A[-n], 0) \cong H_0(A[-n]) \cong H_n(A). \qquad \square$$

6.2.2 Ordinary homology and cohomology

Recall from Section 3.3 the axiomatization of ordinary (co)homology with coefficients in an abelian group A. We can now describe these as the (co)homology theory associated to the Eilenberg-MacLane spectrum HA of A. More generally, we obtain versions of ordinary (co)homology with coefficients in an arbitrary complex of abelian groups.

Definition 6.2.11 (Reduced ordinary (co)homology)**.** Let $A \in \mathrm{D}(\mathbb{Z})$ be a complex of abelian groups and let $X \in \mathrm{An}_*$ be a pointed anima. We define the *reduced ordinary homology of X with coefficients*

in A and the *reduced ordinary cohomology of X with coefficients in A* as the (co)homology theories represented by the Eilenberg-MacLane spectrum HA:

$$\widetilde{H}_n(X;A) \;:=\; HA_n(X) \;=\; \pi_n(\Sigma^\infty X \otimes HA)$$

and

$$\widetilde{H}^n(X;A) \;:=\; HA^n(X) \;=\; \pi_{-n}\hom(\Sigma^\infty X, HA).$$

We will mostly be interested in the case where $A \in \mathrm{Ab}$ is an abelian group, regarded as a complex concentrated in degree 0.

The groups $\widetilde{H}_*(X;A)$ and $\widetilde{H}^*(X;A)$ may alternatively be computed as the homology groups of certain chain complexes associated to X and A.

Construction 6.2.12 (Chain and cochain complexes)**.** Let $A \in \mathrm{D}(\mathbb{Z})$ be a complex. Since $\mathrm{D}(\mathbb{Z})$ admits all small limits and colimits by Corollary 6.1.29, the universal property of An from Corollary 1.8.11 implies that evaluation at the point induces equivalences

$$\mathrm{ev}_{\mathrm{pt}}\colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{An}, \mathrm{D}(\mathbb{Z})) \xrightarrow{\sim} \mathrm{D}(\mathbb{Z}) \qquad \text{and} \qquad \mathrm{ev}_{\mathrm{pt}}\colon \mathrm{Fun}^{\lim}(\mathrm{An}^{\mathrm{op}}, \mathrm{D}(\mathbb{Z})) \xrightarrow{\sim} \mathrm{D}(\mathbb{Z}).$$

In particular, the object $A \in \mathrm{D}(\mathbb{Z})$ determines two functors

$$C_*(-;A)\colon \mathrm{An} \to \mathrm{D}(\mathbb{Z}) \qquad \text{and} \qquad C^*(-;A)\colon \mathrm{An}^{\mathrm{op}} \to \mathrm{D}(\mathbb{Z}),$$

characterized uniquely by the properties that $C_*(-;A)$ preserves colimits, $C^*(-;A)$ preserves limits, and $C_*(\mathrm{pt};A) \simeq A \simeq C^*(\mathrm{pt};A)$.

For a commutative ring R, the same construction in $\mathrm{D}(R)$ defines chains and cochains with coefficients in any $A \in \mathrm{D}(R)$. Restriction of scalars along $\mathbb{Z} \to R$ identifies their underlying complexes in $\mathrm{D}(\mathbb{Z})$ with the constructions above.

For a pointed anima $(X,x) \in \mathrm{An}_*$, we define the *reduced chain complex* and *reduced cochain complex* as

$$\widetilde{C}_*(X;A) := \mathrm{cofib}(A \simeq C_*(\mathrm{pt};A) \xrightarrow{x_*} C_*(X;A))$$

and

$$\widetilde{C}^*(X;A) := \mathrm{fib}(C^*(X;A) \xrightarrow{x^*} C^*(\mathrm{pt};A) \simeq A).$$

Definition 6.2.13 (Ordinary (co)homology)**.** Given an anima $X \in \mathrm{An}$, a chain complex $A \in \mathrm{D}(\mathbb{Z})$, and an integer $n \in \mathbb{Z}$, we define the *ordinary homology* and *ordinary cohomology of X with coefficients in A* as

$$H_n(X;A) := H_n(C_*(X;A)) \qquad \text{and} \qquad H^n(X;A) := H_{-n}(C^*(X;A)).$$

When $A = \mathbb{Z}$ (regarded as a complex concentrated in degree 0), we simply write $H_n(X) := H_n(X;\mathbb{Z})$ and $H^n(X) := H^n(X;\mathbb{Z})$.

The next result shows that the spectrum-theoretic and chain complex definitions of ordinary (co)homology agree.

Proposition 6.2.14. *Let $A \in \mathrm{D}(\mathbb{Z})$ be a chain complex. There are natural equivalences of functors*

$$H(C_*(-;A)) \simeq \Sigma^\infty_+(-) \otimes HA \colon \mathrm{An} \to \mathrm{Sp}$$

and

$$H(C^*(-;A)) \simeq \hom(\Sigma^\infty_+(-), HA) \colon \mathrm{An}^{\mathrm{op}} \to \mathrm{Sp}.$$

In particular, for any anima X there are natural isomorphisms

$$H_n(X;A) \cong \pi_n(\Sigma^\infty_+ X \otimes HA) \qquad \text{and} \qquad H^n(X;A) \cong \pi_{-n} \hom(\Sigma^\infty_+ X, HA).$$

Proof. Since the Eilenberg-MacLane functor $H \colon \mathrm{D}(\mathbb{Z}) \to \mathrm{Sp}$ preserves colimits by Lemma 6.2.5, it follows that $H \circ C_*(-;A) \colon \mathrm{An} \to \mathrm{Sp}$ preserves colimits. On the other hand, the functor $\Sigma^\infty_+(-) \otimes HA$ also preserves colimits, since both Σ^∞_+ and $- \otimes HA$ do. Since both evaluate to HA on the point, it follows they are naturally equivalent. The equivalence $H(C^*(-;A)) \simeq \hom(\Sigma^\infty_+(-), HA)$ follows similarly, as both sides preserve limits.

The final statement follows by taking homotopy groups and using Proposition 6.2.3. □

Corollary 6.2.15. *For a pointed anima $X \in \mathrm{An}_*$ and a chain complex $A \in \mathrm{D}(\mathbb{Z})$, there are natural isomorphisms*

$$\widetilde{H}_n(X;A) \cong H_n(\widetilde{C}_*(X;A)) \qquad \text{and} \qquad \widetilde{H}^n(X;A) \cong H_{-n}(\widetilde{C}^*(X;A)).$$

Proof. We have

$$\widetilde{H}_n(X;A) = \pi_n(\Sigma^\infty X \otimes HA) \cong \pi_n(\mathrm{cofib}(\mathbb{S} \otimes HA \to \Sigma^\infty_+ X \otimes HA))$$

since $\Sigma^\infty X = \mathrm{cofib}(\mathbb{S} \to \Sigma^\infty_+ X)$ and $- \otimes HA$ is exact. By Proposition 6.2.14, this is isomorphic to

$$\pi_n(H(\mathrm{cofib}(A \to C_*(X;A)))) = \pi_n(H(\widetilde{C}_*(X;A))) = H_n(\widetilde{C}_*(X;A)).$$

The cohomological case is similar. □

Lemma 6.2.16. *For any anima X there is an isomorphism $H_0(X) \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$.*

Proof. The complexes $C_*(X;\mathbb{Z})$ are connective, since they are colimits of copies of $\mathbb{Z}$ and connective complexes are closed under colimits. The truncation adjunction shows that $H_0 \colon \mathrm{D}(\mathbb{Z})_{\geq 0} \to \mathrm{Ab}$ is left adjoint to the inclusion of Ab, so the composite $H_0(C_*(-;\mathbb{Z}))$ preserves colimits. The functor $X \mapsto \bigoplus_{\pi_0(X)} \mathbb{Z}$ also preserves colimits, being the composite of $\pi_0 \colon \mathrm{An} \to \mathrm{Set}$ and the free abelian group functor. Both functors take the point to $\mathbb{Z}$, so the universal property of An identifies them. □

For an abelian group A, singular chains and cochains from Remark 3.3.2 send weak homotopy equivalences to quasi-isomorphisms by [Hat02, Proposition 4.21], and hence define functors

$$C^{\mathrm{sing}}_*(-;A) \colon \mathrm{An} \to \mathrm{D}(\mathbb{Z}) \qquad \text{and} \qquad C^*_{\mathrm{sing}}(-;A) \colon \mathrm{An}^{\mathrm{op}} \to \mathrm{D}(\mathbb{Z}).$$

The chain-level Mayer–Vietoris argument of [Hat02, Proposition 2.21 and pp. 149–150, 203–204], applied to double mapping cylinders, shows that these functors carry pushouts of animae to pushouts and pullbacks, respectively. They also carry coproducts to coproducts and products, respectively, and therefore carry all colimits to colimits and limits. Since $C^{\mathrm{sing}}_*(\mathrm{pt};A) \simeq A[0] \simeq C^*_{\mathrm{sing}}(\mathrm{pt};A)$,

the universal property of An implies that these functors are naturally equivalent to $C_*(-;A)$ and $C^*(-;A)$ from Construction 6.2.12. Thus the intrinsic ordinary (co)homology constructed here agrees with singular (co)homology for animae coming from topological spaces.

Finally, for an abelian group A and the pointed anima S^0, we have

$$\widetilde{H}_n(S^0;A) \cong \pi_n(HA) \cong H_n(A) \qquad \text{and} \qquad \widetilde{H}^n(S^0;A) \cong \pi_{-n}(HA) \cong H_{-n}(A).$$

Both groups are therefore A when $n = 0$ and vanish otherwise. This verifies the dimension axiom and completes the construction of the ordinary theories postulated in Theorem 3.3.1.

6.2.3 The stable Hurewicz theorem

For a spectrum X and an abelian group A, we write

$$H_k(X;A) := \pi_k(X \otimes A) \qquad \text{and} \qquad H_k(X) := H_k(X;\mathbb{Z}).$$

For $X = \Sigma^\infty_+ Z$ with Z an anima, this recovers the ordinary homology of Z by Proposition 6.2.14. The unit map $\mathbb{S} \to \mathbb{Z}$ induces a natural map $X \simeq X \otimes \mathbb{S} \to X \otimes \mathbb{Z}$, and hence the *Hurewicz map*

$$h\colon \pi_k(X) \longrightarrow H_k(X).$$

Theorem 6.2.17 (Stable Hurewicz theorem)**.** *Let X be a spectrum and let $n \in \mathbb{Z}$ be such that $\pi_k(X) = 0$ for $k < n$. Then $H_k(X) = 0$ for $k < n$, the Hurewicz map $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism, and $h\colon \pi_{n+1}(X) \to H_{n+1}(X)$ is surjective.*

Proof. Let F be the fiber of the unit map $u\colon \mathbb{S} \to \mathbb{Z}$. Both $\mathbb{S}$ and $\mathbb{Z}$ are connective, $\pi_k(\mathbb{Z}) = 0$ for $k \neq 0$, and u induces an isomorphism $\pi_0(\mathbb{S}) \cong \mathbb{Z} \cong \pi_0(\mathbb{Z})$ by Example 4.4.25. The long exact sequence of homotopy groups therefore shows that $\pi_k(F) = 0$ for $k \leq 0$, so $F[-1]$ is connective. Since $X \otimes -$ is exact, we obtain a fiber sequence

$$X \otimes F \longrightarrow X \longrightarrow X \otimes \mathbb{Z}\,.$$

As $X[-n]$ and $F[-1]$ are connective, so is $(X \otimes F)[-n-1] \simeq X[-n] \otimes F[-1]$ by Lemma 8.2.1; that is, $\pi_k(X \otimes F) = 0$ for $k \leq n$. The claims now follow from the long exact sequence

$$\pi_k(X \otimes F) \longrightarrow \pi_k(X) \xrightarrow{h} H_k(X) \longrightarrow \pi_{k-1}(X \otimes F). \qquad \square$$

We call a spectrum X *bounded below* if there is an $n \in \mathbb{Z}$ such that $\pi_k(X) = 0$ for all $k < n$.

Corollary 6.2.18 (Homology Whitehead theorem)**.** *A morphism $f\colon X \to Y$ of bounded below spectra is an isomorphism if and only if it induces isomorphisms $H_k(X) \to H_k(Y)$ for all $k \in \mathbb{Z}$.*

Proof. One direction is clear. Conversely, let $C := \operatorname{cofib}(f)$. The long exact sequence of homotopy groups shows that C is bounded below. Since $- \otimes \mathbb{Z}$ is exact, $C \otimes \mathbb{Z} \simeq \operatorname{cofib}(f \otimes \mathbb{Z})$, and $f \otimes \mathbb{Z}$ is an isomorphism by assumption and Corollary 4.4.27. Hence $H_k(C) = 0$ for all k. If C were nonzero, then by Corollary 4.4.27 it would have a nonzero homotopy group, and there would be a smallest m with $\pi_m(C) \neq 0$. But Theorem 6.2.17 gives $\pi_m(C) \cong H_m(C) = 0$, a contradiction. Thus $C = 0$, so f is an isomorphism. $\square$

6.3 Connective objects and t-structures

In Section 6.1, we saw that the derived ∞-category $\mathrm{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ admits a natural decomposition into 'connective' and 'coconnective' parts. Specifically, a complex $A \in \mathrm{D}(\mathcal{A})$ is called connective if its homology vanishes in negative degrees, and coconnective if its homology vanishes in positive degrees. These two classes of objects interact in a structured way:

(1) (Closure under shifts) Connective complexes are closed under positive shifts, while coconnective complexes are closed under negative shifts.

(2) (Orthogonality) There are no nonzero maps from a connective complex to a (-1)-coconnective complex.

(3) (Decomposition) Every complex A sits in an exact sequence $\tau_{\geq 0}A \to A \to \tau_{\leq -1}A$ decomposing it into a connective and a (-1)-coconnective part.

The ∞-category Sp of spectra exhibits the same pattern: a spectrum X is called connective if $\pi_k(X) = 0$ for $k < 0$, and coconnective if $\pi_k(X) = 0$ for $k > 0$. These subcategories again satisfy properties (1)–(3).

Structures like these are omnipresent in stable homotopy theory and homological algebra. The axiomatic framework capturing this phenomenon is that of a *t-structure* on a stable ∞-category, introduced by Beĭlinson, Bernstein, and Deligne [BBD82]. One of the key features of a t-structure is its *heart*: the full subcategory of objects that are both connective and coconnective. The heart turns out to be an abelian 1-category, providing a bridge between the stable and abelian worlds. This allows many arguments from classical homological algebra to be carried out in any stable ∞-category equipped with a t-structure, even when that ∞-category does not arise as the derived category of an abelian category.

In Section 6.3.1 we give the definition of a t-structure and establish its basic properties, including the existence of truncation functors. In Section 6.3.2 we prove that the heart of a t-structure is an abelian category, and relate exact sequences in C to short exact sequences in the heart. In Section 6.3.3 we introduce homotopy group objects and establish the long exact sequence of homotopy groups. Finally, in Section 6.3.4 we discuss completeness and boundedness conditions that control how well objects are determined by their truncations.

6.3.1 Definition and basic properties

Definition 6.3.1 ([BBD82])**.** A *t-structure* on a stable ∞-category C consists of a pair $(C_{\geq 0}, C_{\leq 0})$ of full subcategories of C satisfying the following conditions:

(1) (Closure under shifts) We have $C_{\geq 0}[1] \subseteq C_{\geq 0}$ and $C_{\leq 0}[-1] \subseteq C_{\leq 0}$.

(2) (Orthogonality) If $X \in C_{\geq 0}$ and $Y \in C_{\leq 0}$, then $\mathrm{Hom}_C(X, Y[-1]) = 0$.

(3) (Decomposition) Every $X \in C$ sits in an exact sequence of the form

$$\tau_{\geq 0}X \to X \to \tau_{\leq -1}X,$$

with $\tau_{\geq 0}X \in C_{\geq 0}$ and $\tau_{\leq -1}X \in C_{\leq 0}[-1]$.

Objects in $C_{\geq 0}$ are called *connective* (with respect to the t-structure), while objects in $C_{\leq 0}$ are called *coconnective*. For $n \in \mathbb{Z}$ we set $C_{\geq n} := C_{\geq 0}[n]$ and $C_{\leq n} := C_{\leq 0}[n]$.

If D is another stable ∞-category equipped with a t-structure, then an exact functor $F\colon C \to D$ is called *left t-exact* if it sends $C_{\leq 0}$ to $D_{\leq 0}$, and hence by exactness sends $C_{\leq n}$ to $D_{\leq n}$ for all $n \in \mathbb{Z}$. Similarly, F is called *right t-exact* if it sends $C_{\geq 0}$ to $D_{\geq 0}$, or equivalently sends $C_{\geq n}$ to $D_{\geq n}$ for all n. We say F is *t-exact* if it is both left and right t-exact.

Remark 6.3.2 (Cohomological grading)**.** We index t-structures *homologically*, matching our convention for chain complexes (Remark 6.1.8). A reader preferring *cohomological* grading should negate the index throughout, writing $C^{\geq n} := C_{\leq -n}$ and $C^{\leq n} := C_{\geq -n}$, and likewise $\tau^{\geq n} := \tau_{\leq -n}$ and $\tau^{\leq n} := \tau_{\geq -n}$ for the truncation functors introduced below.

The two main examples of t-structures are the standard t-structure on derived categories and the Postnikov t-structure on spectra.

Proposition 6.3.3 (Standard t-structure on derived categories)**.** *Let $\mathcal{A}$ be an abelian category. The pair $(\mathrm{D}(\mathcal{A})_{\geq 0}, \mathrm{D}(\mathcal{A})_{\leq 0})$ defines a t-structure on $\mathrm{D}(\mathcal{A})$, called the* standard t-structure.

Proof. Property (1) is immediate from the definition of $\mathrm{D}(\mathcal{A})_{\geq n}$ and $\mathrm{D}(\mathcal{A})_{\leq n}$: a complex with homology concentrated in degrees ≥ 0 has, after a positive shift, homology concentrated in degrees ≥ 1, so in particular ≥ 0.

Property (2) was established in Corollary 6.1.25: if A is connective and B is (-1)-coconnective, then $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B) = 0$.

Property (3) is Corollary 6.1.32: for every complex A, there is an exact sequence $\tau_{\geq 0}A \to A \to \tau_{\leq -1}A$ in $\mathrm{D}(\mathcal{A})$. □

Proposition 6.3.4 (Postnikov t-structure on spectra)**.** *Define $\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ to be the full subcategory of* connective spectra*, i.e. those X with $\pi_k(X) = 0$ for $k < 0$, and $\mathrm{Sp}_{\leq 0} \subseteq \mathrm{Sp}$ the full subcategory of* coconnective spectra*, i.e. those with $\pi_k(X) = 0$ for $k > 0$. Then the pair $(\mathrm{Sp}_{\geq 0}, \mathrm{Sp}_{\leq 0})$ defines a t-structure on* Sp*, called the* Postnikov t-structure.

Proof. Property (1) is immediate from the behavior of homotopy groups under shifts: $\pi_k(X[1]) \cong \pi_{k-1}(X)$.

For property (2), let $X \in \mathrm{Sp}_{\geq 0}$ and $Y \in \mathrm{Sp}_{\leq 0}$. By the recognition principle for connective spectra (Theorem 5.4.6), we may write $X \simeq \mathbf{B}^\infty(A)$ for some $A \in \mathrm{CGrp}(\mathrm{An})$. By adjunction,

$$\mathrm{Hom}_{\mathrm{Sp}}(X, Y[-1]) \quad \cong \quad \mathrm{Hom}_{\mathrm{CGrp}(\mathrm{An})}(A, \Omega^\infty(Y[-1])).$$

But $\Omega^\infty(Y[-1])$ is contractible: its homotopy groups $\pi_k(\Omega^\infty(Y[-1])) \cong \pi_k(Y[-1]) \cong \pi_{k+1}(Y)$ vanish for all $k \geq 0$ since $Y \in \mathrm{Sp}_{\leq 0}$.

For property (3), given a spectrum X, we define $\tau_{\geq 0}X := \mathbf{B}^\infty\Omega^\infty X$. The counit of the adjunction $\mathbf{B}^\infty \dashv \Omega^\infty$ provides a map $\tau_{\geq 0}X \to X$. Define $\tau_{\leq -1}X$ as its cofiber. From the long exact sequence on homotopy groups (Proposition 4.4.30), one verifies that $\tau_{\geq 0}X$ is connective and $\tau_{\leq -1}X$ has homotopy groups concentrated in degrees ≤ -1. □

Example 6.3.5. If C admits a t-structure $(C_{\geq 0}, C_{\leq 0})$, then C^{op} can be given the *opposite t-structure*, in which $(C^{\mathrm{op}})_{\geq 0} := (C_{\leq 0})^{\mathrm{op}}$ and $(C^{\mathrm{op}})_{\leq 0} := (C_{\geq 0})^{\mathrm{op}}$.

Example 6.3.6. If C and D both admit t-structures, then so does their product $C \times D$ by setting $(C \times D)_{\geq 0} := C_{\geq 0} \times D_{\geq 0}$ and $(C \times D)_{\leq 0} := C_{\leq 0} \times D_{\leq 0}$.

We now show that the objects $\tau_{\geq 0}X$ and $\tau_{\leq -1}X$ appearing in axiom (3) are in fact functorial in X, arising from adjunctions.

Lemma 6.3.7. *Let $(C_{\geq 0}, C_{\leq 0})$ be a t-structure on C, and let $n \in \mathbb{Z}$.*

(1) *The inclusion $C_{\geq n} \hookrightarrow C$ admits a right adjoint $\tau_{\geq n} \colon C \to C_{\geq n}$.*

(2) *The inclusion $C_{\leq n} \hookrightarrow C$ admits a left adjoint $\tau_{\leq n} \colon C \to C_{\leq n}$.*

Proof. We prove (1); the proof of (2) is dual. By shifting, we may assume that $n = 0$; in general we may then set $\tau_{\geq n}(X) := \tau_{\geq 0}(X[-n])[n]$. Using the pointwise formula for adjunctions, it suffices to show that for every object $X \in C$ there exists some $\tau_{\geq 0}X$ equipped with a morphism $\tau_{\geq 0}X \to X$ inducing equivalences $\mathrm{Hom}_C(Y, \tau_{\geq 0}X) \xrightarrow{\sim} \mathrm{Hom}_C(Y, X)$ for every $Y \in C_{\geq 0}$. For this, we consider the exact sequence $\tau_{\geq 0}X \to X \to \tau_{\leq -1}X$ provided by axiom (3). This gives rise to a long exact sequence

$$\cdots \to \pi_{n+1} \mathrm{Hom}_C(Y, \tau_{\leq -1}X) \to \pi_n \mathrm{Hom}_C(Y, \tau_{\geq 0}X) \to \pi_n \mathrm{Hom}_C(Y, X) \to \pi_n \mathrm{Hom}_C(Y, \tau_{\leq -1}X) \to \dots$$

Since $\mathrm{Hom}_C(Y, \tau_{\leq -1}X) = 0$ by axiom (2), this implies that the map $\mathrm{Hom}_C(Y, \tau_{\geq 0}X) \to \mathrm{Hom}_C(Y, X)$ induces isomorphisms on all homotopy groups, hence is an equivalence. □

Corollary 6.3.8. *The subcategory $C_{\leq n}$ of C is closed under all limits that exist in C. The subcategory $C_{\geq n}$ of C is closed under all colimits that exist in C.* □

Corollary 6.3.9. *Assume C admits I-indexed colimits. Then so does the subcategory $C_{\leq n}$: they are computed as $\tau_{\leq n}(\mathrm{colim}_{i \in I} X_i)$. Dually, if C admits I-indexed limits, then so does $C_{\geq n}$ via the formula $\tau_{\geq n}(\lim_{i \in I} X_i)$.* □

Lemma 6.3.10. *Let C be a stable ∞-category equipped with a t-structure, and let $n, m \in \mathbb{Z}$.*

(1) *If $X \in C_{\leq m}$, then $\tau_{\geq n}X \in C_{\leq m}$.*

(2) *If $X \in C_{\geq n}$, then $\tau_{\leq m}X \in C_{\geq n}$.*

Proof. We prove (1) first. Let $X \in C_{\leq m}$. If $n > m$, then $\mathrm{Hom}_C(Z, X) = 0$ for every $Z \in C_{\geq n}$ by orthogonality. By adjunction, this gives $\mathrm{Hom}_C(Z, \tau_{\geq n}X) = 0$ for every $Z \in C_{\geq n}$. Taking $Z = \tau_{\geq n}X$, we see that the identity of $\tau_{\geq n}X$ is null, so $\tau_{\geq n}X$ is zero.

Assume instead that $n \leq m$. The truncation sequence

$$\tau_{\geq n}X \to X \to \tau_{\leq n-1}X$$

exhibits $\tau_{\geq n}X$ as the fiber of a morphism between objects of $C_{\leq m}$, since $C_{\leq n-1} \subseteq C_{\leq m}$. By Corollary 6.3.8, it follows that $\tau_{\geq n}X \in C_{\leq m}$.

The proof of (2) is dual. □

Warning 6.3.11. The notation $\tau_{\leq n}$ is sometimes also used for the notion of n-truncation in an ∞-category. This notation is *not* compatible with the left adjoint $\tau_{\leq n}$ from the lemma. In fact, the only n-truncated object in a stable ∞-category is the zero object. (Compare Lemma 6.3.18.)

Exercise 6.3.12. Let C and D have t-structures and let $F\colon C \rightleftarrows D :G$ be an adjunction. Show that F is right t-exact if and only if G is left t-exact.

Lemma 6.3.13. *Let $F\colon C \to D$ be a t-exact functor between two stable ∞-categories equipped with t-structures. Then for all $X \in C$ and $n \in \mathbb{Z}$, there are natural equivalences*

$$\tau_{\leq n}F(X) \xrightarrow{\sim} F(\tau_{\leq n}X) \qquad \text{and} \qquad F(\tau_{\geq n}X) \xrightarrow{\sim} \tau_{\geq n}F(X).$$

Proof. We prove the first equivalence; the second is dual. Since F is t-exact, applying F to the exact sequence $\tau_{\geq n+1}X \to X \to \tau_{\leq n}X$ gives another exact sequence

$$F(\tau_{\geq n+1}X) \to F(X) \to F(\tau_{\leq n}X)$$

where the first object lies in $D_{\geq n+1}$ and the last one in $D_{\leq n}$. It follows that the second map induces an isomorphism $\tau_{\leq n}F(X) \xrightarrow{\sim} F(\tau_{\leq n}X)$. □

6.3.2 The heart of a t-structure

Given a t-structure on C, a special role is played by the objects that are both connective and coconnective.

Definition 6.3.14. Let $(C_{\geq 0}, C_{\leq 0})$ be a t-structure on C. We define the *heart* of the t-structure to be the intersection

$$C^{\heartsuit} := C_{\geq 0} \cap C_{\leq 0}.$$

The key feature of t-structures is that the heart is always an abelian 1-category, and that there is a close relation between exact sequences in $C^{\heartsuit}$ (in the sense of homological algebra) and exact sequences in C (in the sense of stable homotopy theory).

Proposition 6.3.15. *Let C be a stable ∞-category equipped with a t-structure.*

(1) *The heart $C^{\heartsuit}$ is an abelian 1-category.*

(2) *A morphism $g\colon Y \to Z$ in $C^{\heartsuit}$ is an epimorphism in $C^{\heartsuit}$ if and only if its fiber in C lies in $C^{\heartsuit}$.*

(3) *A morphism $f\colon X \to Y$ in $C^{\heartsuit}$ is a monomorphism in $C^{\heartsuit}$ if and only if its cofiber in C lies in $C^{\heartsuit}$.*

(4) *Consider a commutative square in $C^{\heartsuit}$ of the form*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{k} & Z. \end{array}$$

(a) If g and h are epimorphisms in $C^{\heartsuit}$, then the square is a pullback square in $C^{\heartsuit}$ if and only if it is a pullback square in C.

(b) If f and k are monomorphisms in $C^\heartsuit$, then the square is a pushout square in $C^\heartsuit$ if and only if it is a pushout square in C.

Proof. For (1), first observe that $C^\heartsuit$ is a 1-category, in the sense that the hom animae $\mathrm{Hom}_{C^\heartsuit}(X,Y)$ are sets. Indeed, for $k \geq 1$, we have

$$\pi_k \mathrm{Hom}_{C^\heartsuit}(X,Y) \cong \pi_0 \mathrm{Hom}_C(X, Y[-k]) = 0$$

since $X \in C_{\geq 0}$ and $Y[-k] \in C_{\leq -k} \subseteq C_{\leq -1}$. Further, since both $C_{\leq 0}$ and $C_{\geq 0}$ are closed under direct sums in C, it is clear $C^\heartsuit$ is additive. We also observe that $C^\heartsuit$ admits kernels and cokernels, which are computed as

$$\ker(Y \to Z) = \tau_{\geq 0}(\mathrm{fib}(Y \to Z)) \qquad \text{and} \qquad \mathrm{coker}(X \to Y) = \tau_{\leq 0}(\mathrm{cofib}(X \to Y)).$$

Before finishing the proof of (1), let us first address the 'only if'-directions in (2) and (3). Since replacing C by C^{op} translates (2) into (3) and vice versa, it suffices to prove the 'only if'-direction in (2). To this end, let $g\colon Y \twoheadrightarrow Z$ be an epimorphism in $C^\heartsuit$. We need to show that $\mathrm{fib}(g) \in C_{\geq 0}$. (Since $C_{\leq 0}$ is closed under limits, the condition $\mathrm{fib}(g) \in C_{\leq 0}$ is automatic.) The assumption on g implies that the commutative square

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle =} \\ Z & \xrightarrow{=} & Z \end{array}$$

is a pushout square in $C^\heartsuit$. Since pushouts in $C^\heartsuit$ are computed by first forming the pushout in $C_{\geq 0}$ and then applying $\tau_{\leq 0}\colon C_{\geq 0} \to C^\heartsuit$, it follows that the map $Z \to Z \sqcup_Y Z$ obtained by forming the pushout in C induces an isomorphism $Z = \tau_{\leq 0} Z \xrightarrow{\sim} \tau_{\leq 0}(Z \sqcup_Y Z)$. Passing to cofibers then gives $\tau_{\leq 0}(\mathrm{cofib}(g)) = 0$, or equivalently $\mathrm{cofib}(g) \in C_{\geq 1}$. But then the relation $\mathrm{fib}(g) \simeq \mathrm{cofib}(g)[-1]$ implies $\mathrm{fib}(g) \in C_{\geq 0}$, which is what we needed to show.

We now prove (4). Again, (a) translates to (b) under replacing C by C^{op}, so it suffices to prove (a). By full faithfulness of the inclusion $C^\heartsuit \hookrightarrow C$, it is clear that if the square is a pullback in C then it is also a pullback in $C^\heartsuit$. Conversely, if the square is a pullback in $C^\heartsuit$, then it in particular induces an isomorphism $\ker(h) \xrightarrow{\sim} \ker(g)$ on kernels in $C^\heartsuit$. But by the direction of (2) that we just proved, these kernels agree with the fibers in C. We deduce that the induced map $\mathrm{fib}(h) \to \mathrm{fib}(g)$ is an isomorphism in C, which implies that the square is also a pullback square in C. This finishes the proof of (4).

We now return to (1), finishing the proof that $C^\heartsuit$ is abelian. Consider a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

in $C^\heartsuit$, and assume that f is a monomorphism in $C^\heartsuit$ and that g is an epimorphism in $C^\heartsuit$. We need to show that this square is a pullback square in $C^\heartsuit$ if and only if it is a pushout square in $C^\heartsuit$. But by (4) this is immediate from stability of C.

Finally, we prove the 'if'-directions in (2) and (3). As before, it suffices to do this for (2). Given a morphism $g\colon Y \to Z$ in $C^\heartsuit$ whose fiber lies in $C^\heartsuit$, we must show g is an epimorphism. Since

$C^\heartsuit$ is abelian, it suffices to show that its cokernel $\mathrm{coker}(g) = \tau_{\leq 0}(\mathrm{cofib}(g))$ is zero, i.e., that $\mathrm{cofib}(g) \in C_{\geq 1}$. But this follows from the relation $\mathrm{cofib}(g) \simeq \mathrm{fib}(g)[1]$ and the assumption $\mathrm{fib}(g) \in C_{\geq 0}$. □

Corollary 6.3.16. *Let C be a stable ∞-category equipped with a t-structure. For every short exact sequence*

$$0 \to X \hookrightarrow Y \twoheadrightarrow Z \to 0$$

in $C^\heartsuit$, the sequence $X \to Y \to Z$ is an exact sequence in C.

Proof. This is an immediate consequence of part (4) of Proposition 6.3.15. □

Corollary 6.3.17. *Let $F : C \to D$ be an exact functor between stable ∞-categories with t-structures. Assume F restricts to a functor $F : C^\heartsuit \to D^\heartsuit$. Then this restriction is an exact functor of abelian categories.*

Proof. Let $0 \to X \hookrightarrow Y \twoheadrightarrow Z \to 0$ be a short exact sequence in $C^\heartsuit$. By the previous corollary, the sequence $X \to Y \to Z$ is exact in C, hence induces an exact sequence $F(X) \to F(Y) \to F(Z)$ in D. Using the various parts of Proposition 6.3.15, we see:

(i) By (2), the morphism $F(Y) \to F(Z)$ is an epimorphism in $D^\heartsuit$, since its fiber $F(X)$ lies in $D^\heartsuit$;

(ii) By (3), the morphism $F(X) \to F(Y)$ is a monomorphism in $D^\heartsuit$, since its cofiber $F(Z)$ lies in $D^\heartsuit$;

(iii) By (4), the exact sequence $F(X) \to F(Y) \to F(Z)$ in D is also exact in $D^\heartsuit$.

We conclude that the sequence $0 \to F(X) \hookrightarrow F(Y) \twoheadrightarrow F(Z) \to 0$ is exact in $D^\heartsuit$, as desired. □

To identify the heart of a t-structure, the following observation is sometimes useful:

Lemma 6.3.18. *Let C be a stable ∞-category equipped with a t-structure, and let $X \in C_{\geq 0}$ be a connective object. Then X lies in the heart of C if and only if it is a 0-truncated object of $C_{\geq 0}$, in the sense that $\pi_k \mathrm{Hom}_{C_{\geq 0}}(Y, X) = 0$ for all $Y \in C_{\geq 0}$ and $k > 0$.*

Proof. First assume that $X \in C^\heartsuit$. Then for $Y \in C_{\geq 0}$ and $k > 0$ we have $\pi_k \mathrm{Hom}_C(Y, X) \simeq \pi_0 \mathrm{Hom}_C(Y, X[-k]) = 0$ as $X[-k] \in C_{\leq -k} \subseteq C_{\leq -1}$. Conversely, if X is 0-truncated in $C_{\geq 0}$, then we have $\pi_k \mathrm{Hom}_C(Y, X[-1]) \cong \pi_{k+1} \mathrm{Hom}_C(Y, X) \cong 0$ for all $Y \in C_{\geq 0}$ and $k \geq 0$. It follows that $X[-1] \in C_{\leq -1}$, and thus $X \in C^\heartsuit$. □

Corollary 6.3.19. *For a stable ∞-category equipped with a t-structure, the heart $C^\heartsuit$ is the full subcategory of 0-truncated objects in $C_{\geq 0}$.* □

We now identify the hearts of our two main examples of t-structures.

Example 6.3.20 (Heart of the standard t-structure)**.** Let $\mathcal{A}$ be an abelian category. The heart of the standard t-structure on $\mathrm{D}(\mathcal{A})$ is equivalent to $\mathcal{A}$. Indeed, the heart consists of those complexes whose homology is concentrated in degree zero, which by Proposition 6.1.26 is precisely the essential image of the fully faithful functor $\mathcal{A} \hookrightarrow \mathrm{D}(\mathcal{A})$.

Proposition 6.3.21 (Heart of the Postnikov t-structure)**.** *The functor* $\mathbf{B}^\infty\colon \mathrm{Ab} \hookrightarrow \mathrm{Sp}$ *induces an equivalence*

$$\mathbf{B}^\infty\colon \mathrm{Ab} \xrightarrow{\sim} \mathrm{Sp}^\heartsuit.$$

Proof. By the recognition principle for connective spectra (Theorem 5.4.6), the functor $\mathbf{B}^\infty$ induces an equivalence $\mathbf{B}^\infty\colon \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\sim} \mathrm{Sp}_{\geq 0}$. By Corollary 6.3.19, we may identify $\mathrm{Sp}^\heartsuit$ with the subcategory of $\mathrm{CGrp}(\mathrm{An})$ spanned by the 0-truncated objects. Note that an object in $\mathrm{CGrp}(\mathrm{An})$ is 0-truncated if and only if its underlying anima is 0-truncated, which in turn happens if and only if it is contained in the full subcategory $\mathrm{Set} \subseteq \mathrm{An}$. All in all, we get $\mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set}) \xrightarrow{\sim} \mathrm{CGrp}(\mathrm{An})_{\leq 0} \xrightarrow{\sim} \mathrm{Sp}^\heartsuit$. □

Remark 6.3.22. Combining Example 6.3.20 and Proposition 6.3.21, we see that the Eilenberg–MacLane functor $H\colon \mathrm{D}(\mathbb{Z}) \to \mathrm{Sp}$ from Definition 6.2.1 restricts to an equivalence $\mathrm{Ab} \xrightarrow{\sim} \mathrm{Sp}^\heartsuit$ on hearts. The formula $\pi_n(HA) \cong H_n(A)$ from Proposition 6.2.3 shows that H is t-exact.

More generally, for a connective ring spectrum R, the heart of the module category LMod_R can be identified with the category of discrete modules over $\pi_0(R)$. We will establish this in Section 8.3, after developing the necessary theory of ring spectra and their modules.

6.3.3 Homotopy group objects

We will now define the *homotopy group objects* of an object $X \in C$: the objects of the heart obtained by truncating X both from above and below. First observe that this operation is independent of the order of truncation:

Lemma 6.3.23. *Let C be a stable ∞-category equipped with a t-structure. For $n, m \in \mathbb{Z}$, the canonical natural transformation*

$$\tau_{\leq m}\tau_{\geq n} \to \tau_{\geq n}\tau_{\leq m}$$

is a natural isomorphism of functors $C \to C_{\leq m} \cap C_{\geq n}$.

Proof. By Lemma 6.3.10, both composites take values in $C_{\leq m} \cap C_{\geq n}$. If $n > m$, this intersection is zero by orthogonality, so both composites are zero. We may therefore assume that $n \leq m$.

Fixing $Y \in C$, the map $\tau_{\geq n}Y \to Y$ induces $\tau_{\leq m}\tau_{\geq n}Y \to \tau_{\leq m}Y$. Since the source is n-connective, this factors uniquely through a map

$$\tau_{\leq m}\tau_{\geq n}Y \to \tau_{\geq n}\tau_{\leq m}Y,$$

which is the 'canonical map' from the statement. To show it is an isomorphism, we may equivalently show that the cofiber of $\tau_{\leq m}\tau_{\geq n}Y \to \tau_{\leq m}Y$ is $(n-1)$-coconnective. To this end, we may compare the truncation sequences for $\tau_{\geq n}Y$ and Y to obtain a commutative diagram

$$\begin{array}{ccccc} \tau_{\geq m+1}\tau_{\geq n}Y & \longrightarrow & \tau_{\geq n}Y & \longrightarrow & \tau_{\leq m}\tau_{\geq n}Y \\ \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow \\ \tau_{\geq m+1}Y & \longrightarrow & Y & \longrightarrow & \tau_{\leq m}Y. \end{array}$$

Since $m \geq n$, the left vertical map is an isomorphism. By the pasting law for exact squares, it follows that the cofiber of the right vertical map is the cofiber of the middle vertical map. This cofiber is $\tau_{\leq n-1}Y$, which is $(n-1)$-coconnective, finishing the proof. □

Definition 6.3.24. Let C be a stable ∞-category equipped with a t-structure and let X be an object of C. For every $n \in \mathbb{Z}$, we define the *n-th homotopy group object* $\pi_n(X) \in C^\heartsuit$ by

$$\pi_n(X) := \tau_{\geq 0}\tau_{\leq 0}(X[-n]) \quad \in \quad C^\heartsuit.$$

Remark 6.3.25. In the context of homological algebra, for example when C is the derived ∞-category of an abelian category, the objects $\pi_n(X)$ are often denoted by $H_n(X)$ instead, emphasizing the analogy with the homology groups of a chain complex rather than the analogy with homotopy groups of a spectrum. Under the equivalence $\mathrm{D}(\mathcal{A})^\heartsuit \simeq \mathcal{A}$ from Example 6.3.20, the object $\pi_n(A) \in \mathcal{A}$ agrees with the n-th homology $H_n(A)$ of A viewed as a chain complex.

Lemma 6.3.26. *Let $F\colon C \to D$ be a t-exact functor between two stable ∞-categories equipped with t-structures. Then for all $X \in C$ and $n \in \mathbb{Z}$, there is a natural isomorphism $F(\pi_n X) \xrightarrow{\sim} \pi_n F(X)$.*

Proof. By Lemma 6.3.13 we have natural isomorphisms

$$F(\pi_n X) = F(\tau_{\geq 0}\tau_{\leq 0}(X[-n])) \xrightarrow{\sim} \tau_{\geq 0}\tau_{\leq 0}(F(X)[-n]) = \pi_n(F(X)). \qquad \square$$

Just like for spectra, exact sequences in C induce long exact sequences on homotopy groups:

Proposition 6.3.27 (Long exact sequence of homotopy groups)**.** *Let C be a stable ∞-category equipped with a t-structure. For every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in C, we obtain a long exact sequence of homotopy groups in $C^\heartsuit$:*

$$\cdots \to \pi_{n+1}(Z) \xrightarrow{\partial} \pi_n(X) \xrightarrow{\pi_n(f)} \pi_n(Y) \xrightarrow{\pi_n(g)} \pi_n(Z) \xrightarrow{\partial} \pi_{n-1}(X) \to \dots$$

Proof. Since every such exact sequence in C induces exact sequences $Z[-1] \to X \to Y$ and $Y \to Z \to X[1]$ and we have $\pi_n(Z[-1]) = \pi_{n+1}(Z)$ and $\pi_n(X[1]) = \pi_{n-1}(X)$, it suffices by induction to show that the sequence

$$\pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y) \xrightarrow{\pi_0(g)} \pi_0(Z)$$

is exact in $C^\heartsuit$. We will prove this in progressively more general cases.

Case 1: Assume that $X, Y, Z \in C_{\geq 0}$. We claim that in this case, the sequence

$$\pi_0(X) \to \pi_0(Y) \to \pi_0(Z) \to 0$$

is exact. To see this, it suffices to show that for every $W \in C^\heartsuit$, the induced sequence of abelian groups

$$0 \to \mathrm{Hom}_{C^\heartsuit}(\pi_0(Z), W) \to \mathrm{Hom}_{C^\heartsuit}(\pi_0(Y), W) \to \mathrm{Hom}_{C^\heartsuit}(\pi_0(X), W)$$

is exact. Since $Z \in C_{\geq 0}$, we have $\pi_0(Z) = \tau_{\leq 0}Z$, so by adjunction we obtain isomorphisms

$$\mathrm{Hom}_{C^\heartsuit}(\pi_0(Z), W) \cong \pi_0 \mathrm{Hom}_C(Z, W) \cong \pi_0 \hom_C(Z, W),$$

and similarly for X and Y. Furthermore, we have $\pi_1 \hom_C(X, W) \cong \pi_0 \hom_C(X[1], W) \cong 0$ as $\tau_{\leq 0}(X[1]) \cong 0$. The claim now follows from the long exact sequence on homotopy groups associated to the exact sequence of mapping spectra

$$\hom_C(Z, W) \to \hom_C(Y, W) \to \hom_C(X, W).$$

Case 2: Assume that $X \in C_{\geq 0}$, but Y and Z are arbitrary. For every $W \in C_{\leq -1}$, orthogonality and the exact sequence $X \to Y \to Z$ identify $\mathrm{Hom}_C(Z, W)$ with $\mathrm{Hom}_C(Y, W)$. By adjunction and the Yoneda lemma in $C_{\leq -1}$, the induced map $\tau_{\leq -1}Y \to \tau_{\leq -1}Z$ is therefore an isomorphism, so we get a cofiber sequence

$$0 \cong \tau_{\leq -1}X \to \tau_{\leq -1}Y \to \tau_{\leq -1}Z,$$

showing that $\tau_{\leq -1}Y \cong \tau_{\leq -1}Z$. Consider now the following diagram:

$$\begin{array}{ccccc}
\tau_{\geq 0}X & \longrightarrow & \tau_{\geq 0}Y & \longrightarrow & \tau_{\geq 0}Z \\
\downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tau_{\leq -1}Y & \xrightarrow{\cong} & \tau_{\leq -1}Z.
\end{array}$$

Since all three columns are exact and the bottom two rows are exact, it follows that the top row is also exact. Since $\pi_0(X) \cong \pi_0(\tau_{\geq 0}X)$, and similarly for Y and Z, the exactness of $\pi_0(X) \to \pi_0(Y) \to \pi_0(Z) \to 0$ has thus been reduced to Case 1.

Case 3: Assume that $Z \in C_{\leq 0}$, but X and Y are arbitrary. Using an argument dual to that of Cases 1 and 2, we obtain an exact sequence

$$0 \to \pi_0(X) \to \pi_0(Y) \to \pi_0(Z).$$

Case 4: Finally, we consider the case for general X, Y, Z. Let Q denote the cofiber of the composite $\tau_{\geq 0}X \to X \to Y$. We then have the following commutative diagram:

$$\begin{array}{ccccc}
\tau_{\geq 0}X & \longrightarrow & Y & \longrightarrow & Q \\
\downarrow & & \| & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\leq -1}X & \longrightarrow & 0 & \longrightarrow & (\tau_{\leq -1}X)[1].
\end{array}$$

The three rows are exact, as are the left two columns, and hence so is the third column. Applying Cases 2 and 3, we get exact sequences

$$\pi_0(X) \to \pi_0(Y) \twoheadrightarrow \pi_0(Q) \to 0 \qquad \text{and} \qquad 0 \to \pi_0(Q) \hookrightarrow \pi_0(Z) \to \pi_{-1}(X).$$

Gluing them together then gives the desired exact sequence $\pi_0(X) \to \pi_0(Y) \to \pi_0(Z)$. □

6.3.4 Complete and bounded t-structures

To study an object through its truncations, we need to know when these recover the original object. This leads to the following completeness and boundedness conditions.

Definition 6.3.28. A t-structure on C is *left complete* if the canonical functor

$$C \longrightarrow \lim_{n \geq 0} C_{\leq n}, \qquad X \longmapsto (\tau_{\leq n}X)_{n \geq 0},$$

where the transition functors are the truncations $\tau_{\leq n} \colon C_{\leq n+1} \to C_{\leq n}$, is an equivalence. It is *right complete* if the induced t-structure on C^{op} is left complete.

Thus left completeness requires both that every object is the limit of its Postnikov tower and that every compatible Postnikov tower arises from an object of C. Here a compatible Postnikov tower is a diagram $\cdots \to X_2 \to X_1 \to X_0$ with $X_n \in C_{\leq n}$ whose transition maps induce isomorphisms $\tau_{\leq n} X_{n+1} \xrightarrow{\sim} X_n$. In particular, left and right completeness imply, respectively, that the canonical maps

$$X \longrightarrow \lim_n \tau_{\leq n} X \qquad \text{and} \qquad \operatorname{colim}_n \tau_{\geq -n} X \longrightarrow X$$

are isomorphisms for every $X \in C$.

We say that the t-structure is *left separated* if every infinitely connective object $X \in C_{\geq \infty} := \bigcap_{n \in \mathbb{Z}} C_{\geq n}$ is zero. Dually, it is *right separated* if every infinitely coconnective object $X \in C_{\leq -\infty} := \bigcap_{n \in \mathbb{Z}} C_{\leq n}$ is zero.

Left separatedness says that no nonzero object is invisible to all upper truncations, while left completeness additionally ensures that every compatible Postnikov tower comes from a unique object. The corresponding statements for right separatedness and right completeness are dual. Under the hypotheses of the criterion below, separatedness and completeness agree.

Example 6.3.29. Let $\mathcal{A}$ be an abelian category. The standard t-structure on $\mathrm{D}(\mathcal{A})$ is left and right separated: an infinitely connective or infinitely coconnective complex has trivial homology and is therefore zero. If $\mathcal{A}$ has exact countable products, then the t-structure is left complete; if it has exact countable coproducts, then it is right complete. Indeed, the countable version of the construction underlying Corollary 6.1.29 supplies the required (co)products in $\mathrm{D}(\mathcal{A})$, and the criterion below applies.

Example 6.3.30. The Postnikov t-structure on Sp is left and right separated, since a spectrum with vanishing homotopy groups is zero. Homotopy groups commute with products and coproducts, so the criterion below shows that this t-structure is also left and right complete.

Lemma 6.3.31. *Let C be a stable ∞-category with a t-structure. Assume that C admits countable products and that $C_{\geq 0}$ is closed under countable products. Then the t-structure is left complete if and only if it is left separated. Dually, if C admits countable coproducts and $C_{\leq 0}$ is closed under countable coproducts, then the t-structure is right complete if and only if it is right separated.*

Proof. This is [Lur17, Proposition 1.2.1.19]. □

Definition 6.3.32. A t-structure on C is *right bounded* if every object $X \in C$ is contained in $C_{\geq n}$ for some $n \in \mathbb{Z}$. It is *left bounded* if every $X \in C$ is contained in $C_{\leq n}$ for some $n \in \mathbb{Z}$. It is *bounded* if it is both left and right bounded.

Definition 6.3.33. Let C be a stable ∞-category with a t-structure. We define the subcategory of *bounded objects* by

$$C^{\mathrm{b}} := \bigcup_{n \in \mathbb{N}} (C_{\geq -n} \cap C_{\leq n}) \subseteq C.$$

Observe that C^{b} inherits a t-structure from C which by construction is bounded. Moreover, C is bounded if and only if $C = C^{\mathrm{b}}$. One may similarly define subcategories

$$C^{-} := \bigcup_{n \in \mathbb{N}} C_{\geq -n} \subseteq C \qquad \text{and} \qquad C^{+} := \bigcup_{n \in \mathbb{N}} C_{\leq n} \subseteq C$$

of *bounded below* and *bounded above* objects, respectively.

Example 6.3.34. For the derived category $\mathrm{D}(\mathcal{A})$ of an abelian category, the subcategory $\mathrm{D}(\mathcal{A})^b$ is often denoted $\mathrm{D}^b(\mathcal{A})$ and called the *bounded derived category*. Similarly, $\mathrm{D}(\mathcal{A})^-$ is denoted $\mathrm{D}^-(\mathcal{A})$ and $\mathrm{D}(\mathcal{A})^+$ is denoted $\mathrm{D}^+(\mathcal{A})$.

Exercise 6.3.35. Let C and D be stable ∞-categories with t-structures, and assume that C is right bounded. Show that the restriction functor

$$\mathrm{Fun}^{t-\mathrm{rex}}(C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{rex}}(C_{\geq 0}, D_{\geq 0})$$

provides an equivalence between right t-exact functors $F\colon C \to D$ and right exact functors $C_{\geq 0} \to D_{\geq 0}$.

6.4 Projective objects and Ext-groups

In classical algebra, projective modules are the 'well-behaved' building blocks essential for homological algebra. Their definition makes sense in an arbitrary abelian category $\mathcal{A}$: an object P is called *projective* if the Hom functor $\mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathrm{Ab}$ is an exact functor of abelian categories. This definition has an immediate analogue in an arbitrary t-structure:

Definition 6.4.1. Let C be a stable ∞-category equipped with a t-structure. An object $P \in C$ is called *t-projective* if the mapping spectrum functor $\hom_C(P, -)\colon C \to \mathrm{Sp}$ is t-exact.

Observation 6.4.2. Every t-projective object P is connective. To see this, consider the truncation $\tau_{\leq -1}P$. By t-exactness, the mapping spectrum $\hom_C(P, \tau_{\leq -1}P)$ lies in $\mathrm{Sp}_{\leq -1}$, and it follows that

$$0 = \pi_0 \hom_C(P, \tau_{\leq -1}P) \cong \pi_0 \mathrm{Hom}_C(\tau_{\leq -1}P, \tau_{\leq -1}P),$$

where the last identification holds by adjunction. In particular, the identity on $\tau_{\leq -1}P$ can be identified with the zero map, forcing $\tau_{\leq -1}P = 0$, hence $P \in C_{\geq 0}$.

Observation 6.4.3. Conversely, if $P \in C_{\geq 0}$ is a connective object, then $\hom_C(P, -)\colon C \to \mathrm{Sp}$ is always left t-exact: if $X \in C_{\leq 0}$, then for $n > 0$ we have

$$\pi_n(\hom_C(P, X)) \cong \pi_0 \mathrm{Hom}_C(P, X[-n]) = 0,$$

since $P \in C_{\geq 0}$ while $X[-n] \in C_{\leq -1}$. So P is t-projective if and only if $\hom_C(P, -)$ is also *right* t-exact.

In the case $C = \mathrm{D}(\mathcal{A})$, this definition does indeed recover the usual notion of projectivity in $\mathcal{A}$:

Lemma 6.4.4. *Let $\mathcal{A}$ be an abelian category and let $M \in \mathcal{A}$. Then M is projective in $\mathcal{A}$ if and only if the complex $M[0] \in \mathrm{D}(\mathcal{A})$ is t-projective.*

Proof. Suppose first that $M[0]$ is t-projective, so that $\hom_{\mathrm{D}(\mathcal{A})}(M[0], -)\colon \mathrm{D}(\mathcal{A}) \to \mathrm{Sp}$ is t-exact. By Corollary 6.3.17, this functor restricts to an exact functor on hearts $\mathrm{D}(\mathcal{A})^\heartsuit \to \mathrm{Sp}^\heartsuit$, i.e., $\mathcal{A} \to \mathrm{Ab}$. This restricted functor is $\mathrm{Hom}_{\mathcal{A}}(M, -)$, showing that M is projective in $\mathcal{A}$.

Conversely, assume that $\mathrm{Hom}_{\mathcal{A}}(M,-)\colon \mathcal{A} \to \mathrm{Ab}$ is exact. Just as in Corollary 6.2.9, this implies that for any chain complex $C_\bullet$ in $\mathcal{A}$ we have an isomorphism

$$\pi_n \hom_{\mathcal{K}(\mathcal{A})}(M[0], C_\bullet) \cong \mathrm{Hom}_{\mathcal{A}}(M, H_n(C)).$$

In particular, the functor $\hom_{\mathcal{K}(\mathcal{A})}(M[0],-)\colon \mathcal{K}(\mathcal{A}) \to \mathrm{Sp}$ inverts quasi-isomorphisms. The same argument as in Lemma 6.2.10 shows that it is corepresented by $M[0] \in \mathrm{D}(\mathcal{A})$, giving a natural isomorphism $\hom_{\mathrm{D}(\mathcal{A})}(M[0],-) \cong \hom_{\mathcal{K}(\mathcal{A})}(M[0],-)$. It follows that $\pi_n \hom_{\mathrm{D}(\mathcal{A})}(M[0],X) \cong \mathrm{Hom}_{\mathcal{A}}(M, H_n(X))$ for all complexes $X \in \mathrm{D}(\mathcal{A})$, showing that $\hom_{\mathrm{D}(\mathcal{A})}(M[0],X)$ is connective whenever X is. □

Corollary 6.4.5. *Let $\mathcal{A}$ be an abelian category and let $M \in \mathcal{A}$ be projective. Then for every $X \in \mathrm{D}(\mathcal{A})$ and every $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\pi_n \hom_{\mathrm{D}(\mathcal{A})}(M[0], X) \cong \mathrm{Hom}_{\mathcal{A}}(M, H_n(X)).$$

Proof. This is the computation in the second half of the proof of Lemma 6.4.4. □

6.4.1 Ext-groups

We will next give a formulation of t-projectivity in terms of the Ext-groups in C.

Definition 6.4.6 (Ext-groups)**.** Let C be a stable ∞-category. For objects $X, Y \in C$ and $n \in \mathbb{Z}$, we define the *n-th Ext-group* to be

$$\mathrm{Ext}^n_C(X,Y) := \pi_{-n} \hom_C(X,Y) \simeq \pi_0 \mathrm{Hom}_C(X, Y[n]).$$

For objects $M, N \in \mathcal{A}$, regarded as complexes concentrated in degree zero, we abbreviate $\mathrm{Ext}^n_{\mathrm{D}(\mathcal{A})}(M,N)$ to $\mathrm{Ext}^n_{\mathcal{A}}(M,N)$.

Assume now that C comes equipped with a t-structure. For objects $M, N \in C^\heartsuit$, the Ext-groups $\mathrm{Ext}^n_C(M,N)$ admit a direct interpretation in small degrees:

Observation 6.4.7. For $n < 0$, we have $\mathrm{Ext}^n_C(M,N) = 0$, since $M \in C_{\geq 0}$ while $N[n] \in C_{\leq n} \subseteq C_{\leq -1}$.

Observation 6.4.8. For $n = 0$, we have $\mathrm{Ext}^0_C(M,N) = \pi_0 \mathrm{Hom}_C(M,N) = \mathrm{Hom}_{C^\heartsuit}(M,N)$, the set of morphisms in the heart.

The interpretation of $\mathrm{Ext}^1_C(M,N)$ in terms of extensions is developed in Exercise 6.5.

Observation 6.4.9 (Two-stage Postnikov extensions)**.** For $X \in C_{\geq 0} \cap C_{\leq 1}$, the truncation sequence takes the form

$$\pi_1(X)[1] \longrightarrow X \longrightarrow \pi_0(X) \xrightarrow{\kappa_X} \pi_1(X)[2].$$

The class $\kappa_X \in \mathrm{Ext}^2_C(\pi_0(X), \pi_1(X))$ is called the *first Postnikov k-invariant* of X. Conversely, for $M, N \in C^\heartsuit$ and a morphism $\kappa\colon M \to N[2]$, the object $X := \mathrm{fib}(\kappa)$ lies in $C_{\geq 0} \cap C_{\leq 1}$, with $\pi_0(X) \cong M$ and $\pi_1(X) \cong N$. Its truncation sequence is $N[1] \to X \to M \xrightarrow{\kappa} N[2]$, so it recovers κ. Thus taking fibers gives a bijection between $\mathrm{Ext}^2_C(M,N)$ and isomorphism classes of objects $X \in C_{\geq 0} \cap C_{\leq 1}$ equipped with identifications $\pi_0(X) \cong M$ and $\pi_1(X) \cong N$, where isomorphisms must respect these identifications. The zero class corresponds to $M \oplus N[1]$. Thus the k-invariant records the information needed to reconstruct X from its two homotopy group objects.

Exercise 6.4.10. Let $\mathcal{A}$ be an abelian category with enough projectives, meaning that every object admits an epimorphism from a projective object. For $M, N \in \mathcal{A}$ and $n \geq 2$, show that $\mathrm{Ext}^n_{\mathcal{A}}(M,N) = \mathrm{Ext}^n_{\mathrm{D}(\mathcal{A})}(M,N)$ can be described in terms of *degree n Yoneda extensions*: exact sequences in $\mathcal{A}$ of the form

$$0 \to N \to E_n \to E_{n-1} \to \cdots \to E_2 \to E_1 \to M \to 0.$$

Here extensions are identified under the equivalence relation generated by morphisms of exact sequences that are the identity on M and N. Hint: use Proposition 6.4.16 below.

We now record the fundamental long exact sequences for Ext-groups.

Observation 6.4.11 (Ext long exact sequence)**.** Every exact sequence $Y' \to Y \to Y''$ in C induces an exact sequence of mapping spectra $\hom_C(X,Y') \to \hom_C(X,Y) \to \hom_C(X,Y'')$, thus giving rise to a long exact sequence of homotopy groups of the form

$$\begin{array}{ccccc} & & \cdots & \longrightarrow & \mathrm{Ext}^{n-2}_C(X,Y'') \\ \longrightarrow & \mathrm{Ext}^{n-1}_C(X,Y') & \longrightarrow & \mathrm{Ext}^{n-1}_C(X,Y) & \longrightarrow \mathrm{Ext}^{n-1}_C(X,Y'') \\ \longrightarrow & \mathrm{Ext}^{n}_C(X,Y') & \longrightarrow & \mathrm{Ext}^{n}_C(X,Y) & \longrightarrow \mathrm{Ext}^{n}_C(X,Y'') \\ \longrightarrow & \mathrm{Ext}^{n+1}_C(X,Y') & \longrightarrow & \cdots & \end{array}$$

Similarly, every exact sequence $X' \to X \to X''$ in C induces a long exact sequence of the form

$$\cdots \to \mathrm{Ext}^{n-1}_C(X',Y) \to \mathrm{Ext}^n_C(X'',Y) \to \mathrm{Ext}^n_C(X,Y) \to \mathrm{Ext}^n_C(X',Y) \to \mathrm{Ext}^{n+1}_C(X'',Y) \to \ldots.$$

As a special case, given an object $N \in C^\heartsuit$ and a short exact sequence $M' \hookrightarrow P \twoheadrightarrow M$ in $C^\heartsuit$, the negative Ext-groups vanish, giving a one-sided long exact sequence of the form

$$\begin{aligned} 0 \to \mathrm{Hom}_{C^\heartsuit}(M,N) &\to \mathrm{Hom}_{C^\heartsuit}(P,N) \to \mathrm{Hom}_{C^\heartsuit}(M',N) \\ &\to \mathrm{Ext}^1_C(M,N) \to \mathrm{Ext}^1_C(P,N) \to \mathrm{Ext}^1_C(M',N) \\ &\to \mathrm{Ext}^2_C(M,N) \to \ldots. \end{aligned}$$

Using Ext-groups, we can give several equivalent characterizations of t-projectivity.

Proposition 6.4.12 ([Lur17, Proposition 7.2.2.6])**.** *Let C be a stable ∞-category equipped with a t-structure, and let $P \in C_{\geq 0}$ be a connective object. Then the following conditions are equivalent:*

(1) *The object P is t-projective;*

(2) *For every $Q \in C_{\geq 0}$ and every $i > 0$, the abelian group $\mathrm{Ext}^i_C(P,Q)$ is zero;*

(3) *For every $Q \in C_{\geq 0}$, the abelian group $\mathrm{Ext}^1_C(P,Q)$ is zero;*

(4) *Given an exact sequence*

$$N' \to N \to N''$$

in C with $N', N, N'' \in C_{\geq 0}$, the map $\mathrm{Ext}^0_C(P,N) \to \mathrm{Ext}^0_C(P,N'')$ is surjective.

If C is left complete, this is further equivalent to:

(5) *For every $Q \in C^{\heartsuit}$ and every $i > 0$, the abelian group $\mathrm{Ext}^i_C(P, Q)$ is zero.*

Proof. Recall from Observation 6.4.3 that P is t-projective if and only if the mapping spectrum functor $\hom_C(P, -) \colon C \to \mathrm{Sp}$ is right t-exact. By definition, this means that for any object $Q \in C_{\geq 0}$, the mapping spectrum $\hom_C(P, Q)$ has vanishing negative homotopy groups $\pi_{-i} \hom_C(P, Q) = \mathrm{Ext}^i_C(P, Q)$ for $i > 0$. This shows the equivalence (1) $\Leftrightarrow$ (2).

It is clear that (2) implies (3), and the implication (3) $\Rightarrow$ (2) follows by replacing Q by $Q[i-1]$ for all $i \geq 1$.

We next show that (3) $\Leftrightarrow$ (4). It is clear that (3) implies (4): the exact sequence $N' \to N \to N''$ induces a long exact sequence of the form

$$\cdots \to \mathrm{Ext}^0_C(P, N) \to \mathrm{Ext}^0_C(P, N'') \to \mathrm{Ext}^1_C(P, N') \to \dots,$$

so the vanishing of $\mathrm{Ext}^1_C(P, N')$ implies that the first map must be surjective. Assume now that (4) holds. Given a class $\eta \in \mathrm{Ext}^1(P, Q) = \pi_0 \mathrm{Hom}_C(P, Q[1])$, we obtain an exact sequence

$$P' \to P \xrightarrow{\eta} Q[1]$$

and thus a long exact sequence

$$\cdots \to \mathrm{Ext}^0_C(P, P') \to \mathrm{Ext}^0_C(P, P) \xrightarrow{\eta \circ -} \mathrm{Ext}^1_C(P, Q) \to \dots$$

As $P, P', Q \in C_{\geq 0}$, the first map is surjective, hence the second map is the zero map. Since it sends id_P to η, we conclude that $\eta = 0$. This shows that (3) $\Leftrightarrow$ (4).

Finally, suppose that C is left complete. It is clear that (2) implies (5). Conversely, assume (5). The exact sequences

$$(\pi_{n+1} Q)[n+1] \to \tau_{\leq n+1} Q \to \tau_{\leq n} Q$$

show that the towers of Ext-groups in each positive degree are eventually constant. Since left completeness gives $Q \simeq \lim_n \tau_{\leq n} Q$, the corresponding tower of mapping spectra computes $\hom_C(P, Q)$. Eventual constancy in adjacent degrees eliminates the derived-limit term and gives

$$\mathrm{Ext}^i_C(P, Q) \cong \mathrm{Ext}^i_C(P, \tau_{\leq 0} Q) = 0$$

for $i > 0$. This is the Postnikov-tower argument of [Lur17, Proposition 7.2.2.6]. □

Remark 6.4.13. More generally, an object P of an ∞-category D with geometric realizations is called *projective* if $\mathrm{Hom}_D(P, -)$ preserves geometric realizations [Lur09b, Definition 5.5.8.18]. If C is a stable ∞-category with a left-complete t-structure, this notion of projectivity in $C_{\geq 0}$ agrees with t-projectivity by [Lur17, Proposition 7.2.2.6] and Proposition 6.4.12.

6.4.2 Ext-groups via projective resolutions

As a consequence of Proposition 6.4.12, we may compute Ext-groups in terms of projective resolutions.

Definition 6.4.14. Let C be a stable ∞-category equipped with a t-structure, and let $M \in C^\heartsuit$ be an object in the heart of C. A *projective resolution* of M is a long exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \twoheadrightarrow M \to 0$$

in $C^\heartsuit$ such that each object P_n is t-projective.

Remark 6.4.15. When $C = \mathrm{D}(\mathcal{A})$ is the derived ∞-category of some abelian category $\mathcal{A}$, this recovers the usual notion of projective resolutions in $\mathcal{A}$ from homological algebra by Lemma 6.4.4.

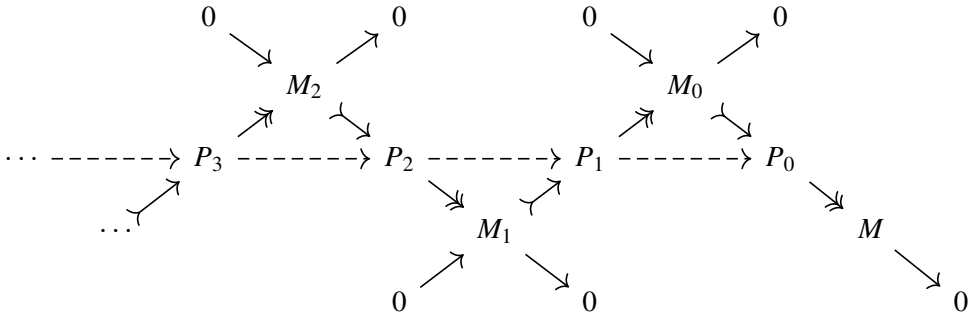
Proposition 6.4.16 (Ext-groups via projective resolutions)**.** *Let C be a stable ∞-category equipped with a t-structure, and let $P_\bullet$ be a projective resolution of $M \in C^\heartsuit$. Then for every other object $N \in C^\heartsuit$ and every $n \in \mathbb{Z}$ there is an isomorphism of abelian groups*

$$\mathrm{Ext}^n_C(M, N) \quad \cong \quad H^n(\mathrm{Hom}_{C^\heartsuit}(P_\bullet, N))$$

between the n-th Ext-group and the n-th cohomology group of the cochain complex $\mathrm{Hom}_{C^\heartsuit}(P_\bullet, N)$.

Proof. The claim is clear for $n < 0$ as both sides are zero: $\mathrm{Ext}^n_C(M, N) = \pi_0 \mathrm{Hom}_C(M, N[n]) = 0$ since $M \in C_{\geq 0}$ and $N[n] \in C_{\leq n} \subseteq C_{\leq -1}$.

For the remaining cases, we break the resolution into short exact sequences and use the vanishing of the higher Ext-groups of the P_k to splice the resulting long exact sequences together. Define auxiliary objects $M_{-1} := M$ and $M_k := \ker(P_k \twoheadrightarrow M_{k-1})$ for $k \geq 0$. The projective resolution then admits the following factorization in $C^\heartsuit$, where each of the diagonal sequences are exact:



For each of these short exact sequences, we obtain a long exact sequence on Ext-groups from Observation 6.4.11. Since the higher Ext-groups of the t-projective objects P_k vanish by Proposition 6.4.12, its initial part is

$$\begin{aligned} 0 \to \mathrm{Hom}_{C^\heartsuit}(M_{k-1}, N) \to \mathrm{Hom}_{C^\heartsuit}(P_k, N) \to \mathrm{Hom}_{C^\heartsuit}(M_k, N) \\ \to \mathrm{Ext}^1_C(M_{k-1}, N) \to 0, \end{aligned}$$

and its remaining terms give isomorphisms

$$\mathrm{Ext}^{i+1}_C(M_{k-1}, N) \cong \mathrm{Ext}^i_C(M_k, N) \qquad (i \geq 1).$$

Since the map $P_{k+1} \twoheadrightarrow M_k$ is an epimorphism, the induced map $\mathrm{Hom}_{C^\heartsuit}(M_k, N) \hookrightarrow \mathrm{Hom}_{C^\heartsuit}(P_{k+1}, N)$ is injective, and so the initial part of this sequence provides for each $k \geq 0$ an isomorphism

$$\mathrm{Hom}_{C^\heartsuit}(M_{k-1}, N) \quad \cong \quad \ker(\mathrm{Hom}_{C^\heartsuit}(P_k, N) \to \mathrm{Hom}_{C^\heartsuit}(P_{k+1}, N)).$$

The next part of the sequence then shows that $\mathrm{Ext}^1_C(M_{k-1}, N)$ is the cokernel of the map $\mathrm{Hom}_{C^\heartsuit}(P_k, N) \to \mathrm{Hom}_{C^\heartsuit}(M_k, N)$. Combining this with the previous identification of $\mathrm{Hom}_{C^\heartsuit}(M_k, N)$ as a kernel, this shows that

$$\mathrm{Ext}^1_C(M_{k-1}, N) \quad \cong \quad \frac{\ker(\mathrm{Hom}_{C^\heartsuit}(P_{k+1}, N) \to \mathrm{Hom}_{C^\heartsuit}(P_{k+2}, N))}{\mathrm{im}(\mathrm{Hom}_{C^\heartsuit}(P_k, N) \to \mathrm{Hom}_{C^\heartsuit}(P_{k+1}, N))} \quad = \quad H^{k+1}(\mathrm{Hom}_{C^\heartsuit}(P_\bullet, N)).$$

Observe that setting $k = 0$ in these two isomorphisms proves the statement of the proposition for $n = 0$ and $n = 1$. Iterating the displayed dimension-shifting isomorphisms gives $\mathrm{Ext}^n_C(M, N) \cong \mathrm{Ext}^1_C(M_{n-2}, N)$ for all $n \geq 1$. Setting $k = n - 1$ in the previous isomorphism thus gives

$$\mathrm{Ext}^n_C(M, N) \quad \cong \quad H^n(\mathrm{Hom}_{C^\heartsuit}(P_\bullet, N))$$

for all $n \geq 1$, finishing the proof. □

Example 6.4.17 (A nontrivial Postnikov extension)**.** Homology objects need not determine an object of a derived category. To see this, set $R := \mathbb{Z}/4$ and $K := R/(2) \cong \mathbb{Z}/2$, and consider the complex

$$A_\bullet := \big(0 \longrightarrow R \xrightarrow{2} R \longrightarrow 0\big)$$

in $\mathrm{D}(R)$, with the two copies of R in degrees 1 and 0. Its only nonzero homology groups are

$$H_1(A_\bullet) \cong K \qquad \text{and} \qquad H_0(A_\bullet) \cong K.$$

The periodic projective resolution $P_\bullet \to K$

$$\cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R \twoheadrightarrow K \longrightarrow 0$$

and Proposition 6.4.16 show that $\mathrm{Ext}^n_R(K, K) \cong K$ for every $n \geq 0$: after applying $\mathrm{Hom}_R(-, K)$, all differentials vanish. The truncation sequence of $A_\bullet$ identifies its k-invariant with the Yoneda 2-extension

$$0 \longrightarrow K \xrightarrow{j} R \xrightarrow{2} R \xrightarrow{q} K \longrightarrow 0,$$

where $q(r) = \bar{r}$ and $j(\bar{r}) = 2r$; see Exercise 6.4.10. This extension is obtained by splicing the short exact sequences

$$0 \longrightarrow K \xrightarrow{j} R \xrightarrow{2} 2R \longrightarrow 0, \qquad 0 \longrightarrow 2R \longrightarrow R \xrightarrow{q} K \longrightarrow 0,$$

coming from the kernel and cokernel of the differential of $A_\bullet$. Their connecting morphisms compose to the truncation map $K \to K[2]$. To compute its class, lift the resolution to this extension by the identity maps in degrees 0 and 1. Since $jq = 2$, the resulting degree-two cocycle is $q\colon P_2 = R \to K$. It is nonzero and cannot be a coboundary, so $\kappa_{A_\bullet}$ is the nonzero element of $\mathrm{Ext}^2_R(K, K)$.

By Observation 6.4.9, there are therefore exactly two isomorphism classes of objects in $\mathrm{D}(R)$ with homology K in degrees 0 and 1 and zero elsewhere: $K[0] \oplus K[1]$ and $A_\bullet$. Indeed, the two classes in $\mathrm{Ext}^2_R(K, K)$ give these objects, and K has no nontrivial R-module automorphisms, so forgetting the chosen identifications of their homology does not identify them.

6.5 Flat objects and Tor-groups

In classical algebra, flat modules form another important class of 'well-behaved' modules. An R-module N is called *flat* if the tensor product functor $- \otimes_R N$ is exact: short exact sequences of R-modules remain exact after tensoring with N. While projective modules are always flat, the converse does not hold in general. For example, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module but not projective.

Just as the notion of projective objects generalizes to t-projective objects in a stable ∞-category with t-structure, the notion of flat objects generalizes to the following setting.

Definition 6.5.1. Let C, D, and E be stable ∞-categories equipped with t-structures. Let $- \otimes - \colon C \times D \to E$ be a functor that is exact in each variable separately, and assume it restricts to $C_{\geq 0} \times D_{\geq 0} \to E_{\geq 0}$, i.e. the tensor product of connective objects is connective.

A connective object $N \in D_{\geq 0}$ is called *t-flat* (with respect to $\otimes$) if the functor $(-) \otimes N \colon C \to E$ is t-exact. Similarly, $M \in C_{\geq 0}$ is t-flat if the functor $M \otimes (-) \colon D \to E$ is t-exact.

Remark 6.5.2. By assumption, $- \otimes N$ is right t-exact, so N is t-flat if and only if it is additionally left t-exact.

Remark 6.5.3 (Motivation)**.** The three main examples we have in mind for this abstract setup are:

(1) The tensor product $\mathrm{Mod}_R \times \mathrm{Mod}_R \xrightarrow{\otimes_R} \mathrm{Mod}_R$ for a connective commutative ring spectrum R.

(2) More generally the relative tensor product $\mathrm{RMod}_R \times \mathrm{LMod}_R \xrightarrow{\otimes_R} \mathrm{Sp}$ for a connective associative ring spectrum R.

(3) The derived tensor product $\mathrm{D}^-(\mathcal{A}) \times \mathrm{D}^-(\mathcal{A}) \xrightarrow{\otimes^{\mathrm{L}}} \mathrm{D}^-(\mathcal{A})$ for a closed symmetric monoidal abelian category $\mathcal{A}$ with enough projectives and countable coproducts (constructed in Section 6.6).

We work in this generality for two reasons. First, it makes clear that the notion of flatness does not depend on the intricate coherences involved in the definition of Mod_R and its tensor product, discussed in Part II of this book. Second, it provides a single formulation that covers all relevant examples.

Exercise 6.5.4. Show that the t-flat objects in C and D are closed under finite direct sums and retracts.

The following exercise highlights the structural analogy between t-projectivity and t-flatness.

Exercise 6.5.5. Let C be a stable ∞-category with a t-structure. Setting $D := C^{\mathrm{op}}$ and $E := \mathrm{Sp}^{\mathrm{op}}$, the hom spectrum functor in C takes the form $\hom_C \colon C \times C^{\mathrm{op}} \to \mathrm{Sp}^{\mathrm{op}}$.

(1) Show that this functor satisfies the conditions of Definition 6.5.1: For $X \in C_{\geq 0}$ and $Y \in C_{\leq 0}$ we have $\hom_C(X, Y) \in \mathrm{Sp}_{\leq 0}$.

(2) Show that an object $P \in C$ is t-flat with respect to $\hom_C$ if and only if it is t-projective.

Just as the homotopy groups of mapping spectra give rise to Ext-groups, the homotopy groups of tensor products give rise to Tor-groups.

Definition 6.5.6 (Tor-groups)**.** In the situation of Definition 6.5.1, for objects $M \in C$ and $N \in D$, we define the *n-th Tor-group* to be

$$\mathrm{Tor}_n^{\otimes}(M,N) \quad := \quad \pi_n(M \otimes N) \quad \in \quad E^{\heartsuit}.$$

When $M \in C^{\heartsuit}$ and $N \in D^{\heartsuit}$ both lie in the heart, we also define $M \otimes^{\heartsuit} N := \mathrm{Tor}_0^{\otimes}(M,N) = \pi_0(M \otimes N)$, resulting in a functor

$$- \otimes^{\heartsuit} - \colon C^{\heartsuit} \times D^{\heartsuit} \to E^{\heartsuit}.$$

Since $M \otimes N \in E_{\geq 0}$, this agrees with $\tau_{\leq 0}(M \otimes N)$ and it follows that this functor is right exact in both variables.

Observation 6.5.7 (Tor long exact sequence)**.** Every exact sequence $N' \to N \to N''$ in D induces an exact sequence $M \otimes N' \to M \otimes N \to M \otimes N''$ in E (since $M \otimes (-)$ is exact), and thus a long exact sequence of homotopy groups of the form

$$\begin{aligned} \cdots \to \mathrm{Tor}_{n+1}^{\otimes}(M,N'') \to \mathrm{Tor}_n^{\otimes}(M,N') \to \mathrm{Tor}_n^{\otimes}(M,N) \\ \to \mathrm{Tor}_n^{\otimes}(M,N'') \to \mathrm{Tor}_{n-1}^{\otimes}(M,N') \to \dots. \end{aligned}$$

As a special case, given an object $N \in D^{\heartsuit}$ and a short exact sequence $M' \hookrightarrow M \twoheadrightarrow M''$ in $C^{\heartsuit}$, the negative Tor-groups vanish, giving a one-sided long exact sequence of the form

$$\begin{aligned} \cdots \to \mathrm{Tor}_1^{\otimes}(M,N) \to \mathrm{Tor}_1^{\otimes}(M'',N) \to \mathrm{Tor}_0^{\otimes}(M',N) \\ \to \mathrm{Tor}_0^{\otimes}(M,N) \to \mathrm{Tor}_0^{\otimes}(M'',N) \to 0. \end{aligned}$$

The same remarks apply to the other variable.

Exercise 6.5.8. The derived ∞-category $\mathrm{D}(\mathbb{Z})$ carries a symmetric monoidal structure $\otimes^{\mathbf{L}}$ with unit $\mathbb{Z}$; see Section 6.6. For an abelian group M and an integer $n \geq 1$, use the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

to show that $\mathrm{Tor}_1^{\otimes^{\mathbf{L}}}(M, \mathbb{Z}/n\mathbb{Z})$ is the subgroup of M consisting of the n-torsion elements, i.e., those $x \in M$ satisfying $nx = 0$.

We now give several equivalent characterizations of t-flatness, analogous to Proposition 6.4.12 for t-projective objects.

Proposition 6.5.9. *Assume that the t-structure on E is left separated. For a connective object $N \in D_{\geq 0}$, the following conditions are equivalent:*

(1) *The object N is t-flat;*

(2) *For every $M \in C_{\leq 0}$ and every $i > 0$, the object $\mathrm{Tor}_i^{\otimes}(M,N)$ is zero;*

(3) *For every $M \in C_{\leq 0}$, the object $\mathrm{Tor}_1^{\otimes}(M,N)$ is zero;*

Assume moreover that C is right complete, that the functor $(-) \otimes N$ preserves sequential colimits, and that $E_{\leq 0}$ is closed under sequential colimits. Then these conditions are further equivalent to:

(4) *For every $M \in C^\heartsuit$ and every $i > 0$, the object $\mathrm{Tor}_i^\otimes(M, N)$ is zero.*

Proof. The functor $(-) \otimes N$ is right t-exact by assumption, so N is t-flat if and only if it sends $C_{\leq 0}$ into $E_{\leq 0}$. Since E is left separated, an object $Y \in E$ is coconnective if and only if $\pi_i(Y) = 0$ for every $i > 0$. Indeed, under this vanishing assumption, the object $U := \tau_{\geq 1} Y$ satisfies $\tau_{\leq n} U = 0$ for every n: this is immediate for $n \leq 0$, and follows by induction from the truncation sequences for $n > 0$. Thus $U \in E_{\geq \infty}$, so $U = 0$. Consequently, preservation of coconnective objects is equivalent to condition (2). Conditions (2) and (3) are equivalent because, for $i > 0$,

$$\mathrm{Tor}_i^\otimes(M, N) \cong \mathrm{Tor}_1^\otimes(M[1-i], N),$$

and $M[1-i]$ is still coconnective.

It remains to compare with condition (4). For $M \in C_{\leq 0}$, each truncation $\tau_{\geq -n} M$ is obtained by finitely many extensions from shifts $(\pi_{-k} M)[-k]$ of objects of the heart. Condition (4) therefore implies that $(\tau_{\geq -n} M) \otimes N$ is coconnective. Right completeness and preservation of sequential colimits identify $M \otimes N$ with the colimit of these objects, which is coconnective because $E_{\leq 0}$ is closed under sequential colimits. Thus (4) implies (2); the converse is immediate. □

In practice, we may often reduce to the case $i = 1$ in condition (4). This is an instance of the following lemma:

Lemma 6.5.10. *Assume that C is right complete, that the t-structure on E is left separated, that $E_{\leq 0}$ is closed under sequential colimits, that $(-) \otimes N$ preserves sequential colimits, and that every object $M \in C^\heartsuit$ admits an epimorphism $P \twoheadrightarrow M$ from a t-flat object $P \in C^\heartsuit$. Then an object $N \in D^\heartsuit$ is t-flat if and only if $\mathrm{Tor}_1^\otimes(M, N) = 0$ for every $M \in C^\heartsuit$.*

Proof. The 'only if' is clear from Proposition 6.5.9. Conversely, we show by induction on i that $\mathrm{Tor}_i^\otimes(M, N) = 0$ for all $M \in C^\heartsuit$ and all $i > 0$. The case $i = 1$ holds by assumption. Assuming the claim in degree $i - 1$, consider an epimorphism $P \twoheadrightarrow M$ with P t-flat, and let M' be its kernel. Then the long exact sequence of Tor-groups contains

$$\cdots \to \mathrm{Tor}_i^\otimes(P, N) \to \mathrm{Tor}_i^\otimes(M, N) \to \mathrm{Tor}_{i-1}^\otimes(M', N) \to \mathrm{Tor}_{i-1}^\otimes(P, N) \to \ldots.$$

The first term vanishes because P is t-flat, and the third vanishes by the induction hypothesis applied to M'. Hence $\mathrm{Tor}_i^\otimes(M, N) = 0$, and Proposition 6.5.9 shows that N is t-flat. □

As a consequence of Proposition 6.5.9, we may compute Tor-groups in terms of flat resolutions.

Definition 6.5.11. Let $N \in D^\heartsuit$ be an object in the heart of D. A *flat resolution* of N is a long exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \twoheadrightarrow N \to 0$$

in $D^\heartsuit$ such that each object F_n is t-flat.

Proposition 6.5.12 (Tor-groups via flat resolutions)**.** *Let $F_\bullet$ be a flat resolution of $N \in D^\heartsuit$. Then for every object $M \in C^\heartsuit$ and every $n \in \mathbb{Z}$ there is an isomorphism*

$$\mathrm{Tor}_n^\otimes(M, N) \quad \cong \quad H_n(M \otimes^\heartsuit F_\bullet)$$

between the n-th Tor-group and the n-th homology group of the chain complex $M \otimes^\heartsuit F_\bullet$ in $E^\heartsuit$.

Proof. The proof is entirely dual to Proposition 6.4.16; we will therefore be rather brief. Set $N_{-1} := N$. The flat resolution splits up into short exact sequences of the form

$$0 \to N_n \rightarrowtail F_n \twoheadrightarrow N_{n-1} \to 0.$$

This provides a long exact sequence on Tor-groups by Observation 6.5.7, in which the higher Tor-groups of F_n vanish by Proposition 6.5.9. Using that $M \otimes^\heartsuit -\colon D^\heartsuit \to E^\heartsuit$ is right exact, one obtains isomorphisms

$$M \otimes^\heartsuit N_{k-1} \quad \cong \quad \operatorname{coker}(M \otimes^\heartsuit F_{k+1} \to M \otimes^\heartsuit F_k)$$

for all $k \geq 0$, which in turn give isomorphisms

$$\operatorname{Tor}_1^\otimes(M, N_{k-1}) \quad \cong \quad \frac{\ker(M \otimes^\heartsuit F_{k+1} \to M \otimes^\heartsuit F_k)}{\operatorname{im}(M \otimes^\heartsuit F_{k+2} \to M \otimes^\heartsuit F_{k+1})} \quad = \quad H_{k+1}(M \otimes^\heartsuit F_\bullet)$$

for all $k \geq 0$. The preceding cokernel formula for $k = 0$ gives the claim in degree zero. Finally, one can inductively show that $\operatorname{Tor}_n^\otimes(M, N) \cong \operatorname{Tor}_1^\otimes(M, N_{n-2})$ for all $n \geq 1$. Setting $k = n - 1$ in the previous isomorphism thus gives

$$\operatorname{Tor}_n^\otimes(M, N) \quad \cong \quad H_n(M \otimes^\heartsuit F_\bullet)$$

for all $n \geq 1$, finishing the proof. □

6.6 Derived functors

The derived ∞-category $\mathrm{D}(\mathcal{A})$ relates the stable and the abelian setting: exactness is better behaved in a stable ∞-category, while explicit computations are usually easier in an abelian category, and $\mathrm{D}(\mathcal{A})$ is stable and contains $\mathcal{A}$ as its heart. This relation may also be used to study functors on $\mathcal{A}$ itself. Consider a right t-exact functor $G\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ between derived categories. Given a short exact sequence

$$0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$$

in $\mathcal{A}$, we showed in Corollary 6.3.16 that this induces an exact sequence $A \to B \to C$ in $\mathrm{D}(\mathcal{A})$. Applying G, we obtain an exact sequence $G(A) \to G(B) \to G(C)$ in $\mathrm{D}(\mathcal{B})$, which in turn gives rise to a long exact sequence of homology groups:

$$\cdots \to G_1(B) \to G_1(C) \to G_0(A) \to G_0(B) \to G_0(C) \to 0,$$

where $G_n := H_n \circ G\colon \mathcal{A} \to \mathcal{B}$. In particular, the functor $G_0\colon \mathcal{A} \to \mathcal{B}$ is right exact, and the long exact sequence measures its failure to be left exact.

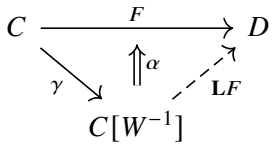
The goal of *derived functors* is to reverse this process: given a right exact functor $F\colon \mathcal{A} \to \mathcal{B}$, can we obtain it as the zeroth homology of some right t-exact functor $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$? One might try to obtain such an extension by simply applying F degreewise to chain complexes. However, this naive approach fails: the degreewise extension $\mathrm{Ch}(F)\colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ does not preserve quasi-isomorphisms unless F is exact. Instead, we could ask: does there exist a universal approximation of $\mathrm{Ch}(F)$ from the left by a functor that descends to $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$?

If $\mathcal{A}$ has enough projectives, we will see that such an approximation exists, at least at the level of bounded below complexes $\mathbf{L}F\colon \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$. It may be computed on an arbitrary bounded below complex C by first replacing C with a quasi-isomorphic projective complex P and then applying F: we have $\mathbf{L}F(C) \simeq F(P)$. Dually, a left exact functor gives rise to right derived functors.

6.6.1 Abstract derived functors

We begin by developing the abstract theory of derived functors in terms of universal properties. The key observation is that a derived functor should be a 'best approximation' to a given functor by one that factors through a localization. Our discussion here is inspired by [DHKS04, Sections 39-41], [Rie14, Chapter 2] and [Cis19, Section 7.5].

Definition 6.6.1. Let C be an ∞-category with a collection of morphisms W, and let $\gamma\colon C \to C[W^{-1}]$ be the localization functor. Let $F\colon C \to D$ be a functor. A functor $\mathbf{L}F\colon C[W^{-1}] \to D$ is called a *total left derived functor of* F if it comes equipped with a natural transformation $\alpha\colon \mathbf{L}F \circ \gamma \Rightarrow F$ of functors $C \to D$ that exhibits $\mathbf{L}F$ as a right Kan extension of F along γ:



Note that the functor $\mathbf{L}F$ is unique whenever it exists.

Dually, a *total right derived functor of* F is a total left derived functor of $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$, i.e. a functor $\mathbf{R}F\colon C[W^{-1}] \to D$ equipped with $\beta\colon F \Rightarrow \mathbf{R}F \circ \gamma$ which exhibits $\mathbf{R}F$ as a left Kan extension of F along γ.

Observation 6.6.2. A priori, the condition on $\mathbf{L}F$ to be a right Kan extension demands that for every functor $H\colon C[W^{-1}] \to D$, the composite

$$\mathrm{Nat}(H, \mathbf{L}F) \xrightarrow{\gamma^*} \mathrm{Nat}(H \circ \gamma, \mathbf{L}F \circ \gamma) \xrightarrow{\alpha \circ -} \mathrm{Nat}(H \circ \gamma, F)$$

is an equivalence. But since γ^* is always an equivalence by definition of localizations, this is equivalent to asking the second map to be an equivalence. We conclude that a transformation $\alpha\colon \mathbf{L}F \circ \gamma \Rightarrow F$ is a total left derived functor of F if and only if for every functor $G\colon C \to D$ that inverts all maps in W, the map

$$\alpha \circ -\colon \mathrm{Nat}(G, \mathbf{L}F \circ \gamma) \to \mathrm{Nat}(G, F)$$

is an equivalence.

In practice, total left derived functors can often be computed using *cofibrant replacements*. The idea is that while F may not preserve weak equivalences in general, it often does so when restricted to some full subcategory of cofibrant objects. The key ingredient then is the fact that cofibrant objects compute the localization: see Corollary 2.2.9. This turns out to be enough to produce the derived functor, which is the content of the following result of Cisinski [Cis19]. We will use it as a black box.

Proposition 6.6.3 (Cisinski [Cis19, p. 7.5.25])**.** *Let (C, W, I) be an ∞-category with weak equivalences and cofibrations, let $i\colon C_c \hookrightarrow C$ denote the full subcategory of cofibrant objects, and let $F\colon C \to D$ be a functor such that the restriction $F|_{C_c}\colon C_c \to D$ inverts the weak equivalences between cofibrant objects. Then F admits a total left derived functor*

$$\mathbf{L}F\colon C[W^{-1}] \to D.$$

It is uniquely characterized by the existence of an isomorphism $\mathbf{L}F \circ \gamma \circ i \simeq F \circ i$*. In particular, if* $q\colon Q \to X$ *is any weak equivalence with* Q *cofibrant, then*

$$\mathbf{L}F(\gamma X) \simeq F(Q).$$

Moreover, the structure transformation $\mathbf{L}F \circ \gamma \Rightarrow F$ *exhibits* $\mathbf{L}F$ *as an* absolute *right Kan extension: for every functor* $H\colon D \to E$*, the composite* $H \circ \mathbf{L}F$ *is a total left derived functor of* $H \circ F$*.*

Proof. We give the construction of $\mathbf{L}F$, and refer to [Cis19, p. 7.5.25] for the verification that it has the required universal property. (The reference treats the dual situation of an ∞-category with weak equivalences and fibrations, and constructs the total *right* derived functor of a functor which inverts the weak equivalences between *fibrant* objects.)

Write W_c for the weak equivalences between cofibrant objects. The restriction $F|_{C_c}$ descends to a functor $F_c\colon C_c[W_c^{-1}] \to D$. By part (1) of Corollary 2.2.9, the inclusion induces an equivalence $\bar{i}\colon C_c[W_c^{-1}] \xrightarrow{\sim} C[W^{-1}]$, and we may thus define

$$\mathbf{L}F \quad := \quad F_c \circ \bar{i}^{\,-1}\colon\ C[W^{-1}] \to D$$

by choosing an inverse of $\bar{i}$. By construction, the restriction of $\mathbf{L}F$ to the cofibrant objects is isomorphic to $F|_{C_c}$, that is, $\mathbf{L}F \circ \gamma \circ i \simeq F \circ i$. Since $\bar{i}$ is an equivalence, a functor out of $C[W^{-1}]$ is determined by its restriction along $\gamma \circ i$, which gives the asserted uniqueness. Finally, if $q\colon Q \to X$ is a weak equivalence with Q cofibrant, then $\gamma(q)$ is an isomorphism, and hence

$$\mathbf{L}F(\gamma X) \simeq \mathbf{L}F(\gamma Q) \simeq F(Q).$$

The construction of $\mathbf{L}F$ is also compatible with postcomposition: for every $H\colon D \to E$, the composite $H \circ \mathbf{L}F$ is obtained by applying the same construction to $H \circ F$. Its structure transformation is therefore again a right Kan extension by [Cis19, (7.5.25.3)], proving absoluteness. □

Remark 6.6.4. The substantive input is that the functor constructed on cofibrant objects is a right Kan extension of F along γ. This is [Cis19, Corollary 7.5.17], whose proof rests on the theory of finite direct diagrams and their Reedy fibrant replacements developed in [Cis19, Section 7.4].

Remark 6.6.5. The situation simplifies considerably when C admits a *functorial cofibrant replacement*: a functor $Q\colon C \to C_c$ together with a natural weak equivalence $q\colon i \circ Q \Rightarrow \mathrm{id}_C$. Such a replacement exists as soon as (C, W, I) admits functorial factorizations, by factoring the map $\emptyset \to X$ functorially in X. One may then simply take $\mathbf{L}F \circ \gamma \simeq F \circ Q$, with structure map $F(q)\colon F \circ Q \Rightarrow F$, and the universal property can be deduced directly from the naturality of q, without any of the machinery above. This is the route taken in [DHKS04, Sections 39-41] and [Rie14, Chapter 2], where such data is called a *left deformation*.

We will not assume functorial cofibrant replacements. For our application to bounded below complexes, it suffices that the abelian category has enough projectives.

Exercise 6.6.6. Let $F\colon C \to D$ be a functor which inverts all morphisms in W. By the universal property of $C[W^{-1}]$, F factors through a functor $\mathbf{L}F\colon C[W^{-1}] \to D$. Show that $\mathbf{L}F$ is a total left derived functor of F.

Lemma 6.6.7 (Products of cofibration structures)**.** *Let $(C_i, W_i, I_i)_{i\in K}$ be a finite collection of ∞-categories with weak equivalences and cofibrations. Then $\prod_{i\in K} C_i$, equipped with the componentwise weak equivalences and cofibrations, is again an ∞-category with weak equivalences and cofibrations. Its cofibrant objects are precisely the tuples of cofibrant objects. Consequently, a functor $F\colon \prod_{i\in K} C_i \to D$ which inverts weak equivalences between cofibrant objects admits a total left derived functor*

$$\mathbf{L}F\colon \prod_{i\in K} C_i[W_i^{-1}] \longrightarrow D.$$

Proof. The axioms for weak equivalences and cofibrations, including the factorization and pushout axioms, hold componentwise in the product, and the description of the cofibrant objects follows from the componentwise initial object. By currying and applying the universal property of localization successively in each variable, the canonical functor identifies the localization of the product with $\prod_{i\in K} C_i[W_i^{-1}]$. The final assertion now follows from Proposition 6.6.3. □

This is how derived functors of several variables, such as the derived tensor product of Definition 6.6.25, are obtained.

6.6.2 Derived functors between derived categories

We now apply the abstract framework to construct derived functors between derived ∞-categories. The key observation is that bounded below projective chain complexes play the role of cofibrant objects: quasi-isomorphisms between them are automatically chain homotopy equivalences, so any additive functor preserves such quasi-isomorphisms.

Definition 6.6.8 (Projective chain complex)**.** Let $\mathcal{A}$ be an abelian category. We write $\mathrm{Ch}^-(\mathcal{A}) \subseteq \mathrm{Ch}(\mathcal{A})$ for the full subcategory of bounded below chain complexes, those $C_\bullet$ for which $C_n = 0$ for all sufficiently small n. A chain complex $P_\bullet \in \mathrm{Ch}(\mathcal{A})$ is called *projective* if each object P_n is projective in $\mathcal{A}$, in the sense of Section 6.4.[14] We write $\mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A}) \subseteq \mathrm{Ch}^-(\mathcal{A})$ for the full subcategory of bounded below projective chain complexes.

Remark 6.6.9. The localization functor identifies

$$\mathrm{Ch}^-(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}] \xrightarrow{\sim} \mathrm{D}^-(\mathcal{A}),$$

where $\mathrm{D}^-(\mathcal{A})$ is the full subcategory of bounded below objects from Definition 6.3.33. Indeed, Proposition 6.1.22 identifies the localization of $\mathrm{Ch}(\mathcal{A})_{\geq m}$ with the full subcategory $\mathrm{D}(\mathcal{A})_{\geq m}$ of $\mathrm{D}(\mathcal{A})$ for every m. Taking the union over all m gives the displayed identification. We use it without further comment.

Lemma 6.6.10. *Let $P_\bullet$ be a bounded below projective chain complex with trivial homology, i.e., $H_n(P) = 0$ for all n. Then $P_\bullet$ is* contractible*: the identity $\mathrm{id}_{P_\bullet}$ is chain homotopic to zero.*

Proof. We construct a contracting homotopy $s_n\colon P_n \to P_{n+1}$ satisfying $d_{n+1}s_n + s_{n-1}d_n = \mathrm{id}_{P_n}$ by induction. Say $P_\bullet$ is concentrated in degrees $\geq m$; we set $s_n = 0$ for $n < m$.

[14] This is weaker than $P_\bullet$ being a projective object of the abelian category $\mathrm{Ch}(\mathcal{A})$: the latter are exactly the contractible complexes that are degreewise projective. Throughout, 'projective chain complex' always refers to the degreewise condition; some authors instead say *degreewise projective* or *complex of projectives*.

For the base case $n = m$: Since $P_{m-1} = 0$, the differential $d_m \colon P_m \to P_{m-1}$ is zero. The condition $H_m(P) = 0$ then says that $\ker(d_m)/\mathrm{im}(d_{m+1}) = P_m/\mathrm{im}(d_{m+1}) = 0$, meaning $d_{m+1} \colon P_{m+1} \to P_m$ is an epimorphism. Since P_m is projective, the identity id_{P_m} lifts along this epimorphism to give $s_m \colon P_m \to P_{m+1}$ with $d_{m+1}s_m = \mathrm{id}_{P_m}$. This satisfies our requirement since $s_{m-1} = 0$.

For the inductive step, suppose we have constructed s_k for all $k < n$ satisfying the homotopy relation. Define

$$\alpha_n \quad := \quad \mathrm{id}_{P_n} - s_{n-1}d_n \colon P_n \to P_n.$$

We claim that $\mathrm{im}(\alpha_n) \subseteq \mathrm{im}(d_{n+1})$. Indeed, using the induction hypothesis $d_n s_{n-1} + s_{n-2}d_{n-1} = \mathrm{id}_{P_{n-1}}$, we compute

$$d_n \circ \alpha_n \;=\; d_n - d_n s_{n-1} d_n \;=\; d_n - (\mathrm{id}_{P_{n-1}} - s_{n-2}d_{n-1})d_n \;=\; s_{n-2}d_{n-1}d_n \;=\; 0.$$

Thus $\mathrm{im}(\alpha_n) \subseteq \ker(d_n) = \mathrm{im}(d_{n+1})$, where the last equality uses $H_n(P) = 0$. Since P_n is projective and $d_{n+1} \colon P_{n+1} \to \mathrm{im}(d_{n+1})$ is an epimorphism, there exists $s_n \colon P_n \to P_{n+1}$ with $d_{n+1}s_n = \alpha_n = \mathrm{id}_{P_n} - s_{n-1}d_n$, as required. □

Proposition 6.6.11. *Let $P_\bullet$ and $Q_\bullet$ be bounded below projective chain complexes. Then any quasi-isomorphism $f \colon P_\bullet \to Q_\bullet$ is a chain homotopy equivalence.*

Proof. The mapping cone $\mathrm{Cone}(f)$ fits into a short exact sequence

$$0 \to Q_\bullet \to \mathrm{Cone}(f) \to P_\bullet[1] \to 0.$$

By Lemma 6.1.15, the complex $\mathrm{Cone}(f)$ is acyclic. It is again bounded below and projective, hence contractible by Lemma 6.6.10. Write the bottom-left component of a contracting homotopy

$$\mathrm{Cone}(f)_n = Q_n \oplus P_{n-1} \longrightarrow Q_{n+1} \oplus P_n = \mathrm{Cone}(f)_{n+1}$$

as $g_n \colon Q_n \to P_n$. The contracting-homotopy equation says that g is a chain map and that fg and gf are chain homotopic to the respective identity maps. Thus f is a chain homotopy equivalence. □

Corollary 6.6.12. *Let $F \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$ be a functor that preserves chain homotopies (for example, a functor induced by an additive functor $\mathcal{A} \to \mathcal{B}$). Then F preserves quasi-isomorphisms between bounded below projective chain complexes.* □

We now assume that $\mathcal{A}$ has enough projectives, meaning that every object of $\mathcal{A}$ admits an epimorphism from a projective object. This is enough to construct projective resolutions one complex at a time.

Proposition 6.6.13 (Existence of projective resolutions)**.** *Let $\mathcal{A}$ be an abelian category with enough projectives. Every bounded below chain complex $C_\bullet$ admits a degreewise epimorphic quasi-isomorphism*

$$q \colon P_\bullet \xrightarrow{\sim} C_\bullet$$

from a bounded below projective chain complex $P_\bullet$. Moreover, if $C_n = 0$ for all $n < m$, then $P_\bullet$ may be chosen with $P_n = 0$ for all $n < m$.

Proof. This is the standard bounded-below projective-replacement theorem. One construction takes the total complex of a Cartan–Eilenberg projective resolution; it produces a degreewise epimorphic quasi-isomorphism with the stated lower bound. See [Wei94, Lemma 5.7.2 and Theorem 10.4.8]. □

Lemma 6.6.14 (Lifting along a projective resolution)**.** *Let $q\colon Q_\bullet \to C_\bullet$ be a degreewise epimorphic quasi-isomorphism between bounded below chain complexes in an abelian category, and let $P_\bullet$ be a bounded below projective chain complex. Every chain map $f\colon P_\bullet \to C_\bullet$ admits a lift $\widehat{f}\colon P_\bullet \to Q_\bullet$ satisfying $q\widehat{f} = f$. Any two such lifts are chain homotopic through a chain homotopy h satisfying $qh = 0$.*

Proof sketch. Set $K_\bullet := \ker(q)$. The long exact sequence in homology shows that $K_\bullet$ is acyclic. Comparing boundaries, cycles, and homology shows that $q\colon Z_n(Q) \to Z_n(C)$ is an epimorphism. More generally, the map

$$Q_n \longrightarrow C_n \times_{Z_{n-1}(C)} Z_{n-1}(Q), \qquad x \longmapsto (qx, dx),$$

is an epimorphism. Starting below the common lower bound, projectivity of each P_n now constructs $\widehat{f_n}$ inductively. The difference of two lifts is a chain map $P_\bullet \to K_\bullet$, and the same induction, using the acyclicity of $K_\bullet$, constructs the required nullhomotopy. □

The projective complexes may now be characterized intrinsically as the cofibrant objects in a cofibration structure on bounded below chain complexes.

Proposition 6.6.15 (The projective cofibration structure)**.** *Let $\mathcal{A}$ be an abelian category with enough projectives. Equip $\mathrm{Ch}^-(\mathcal{A})$ with the quasi-isomorphisms as weak equivalences and the degreewise split monomorphisms with bounded below projective cokernel as cofibrations. This makes $\mathrm{Ch}^-(\mathcal{A})$ into an ∞-category with weak equivalences and cofibrations. Its cofibrant objects are precisely the bounded below projective chain complexes.*

Proof. The 2-out-of-3 property is clear. The stated cofibrations contain the isomorphisms and are closed under composition: the cokernel of a composite is a degreewise split extension of the two projective cokernels. They are closed under pushout, since a pushout is again degreewise split with the same cokernel. This also proves that pushouts of trivial cofibrations are weak equivalences, using Corollary 6.1.18.

It remains to verify the factorization axiom. Let $f\colon P_\bullet \to C_\bullet$ be a chain map with $P_\bullet$ cofibrant, i.e. bounded below projective. Choose a degreewise epimorphic projective resolution $q\colon Q_\bullet \xrightarrow{\sim} C_\bullet$ using Proposition 6.6.13. Since $P_\bullet$ is bounded below projective and q is a degreewise epimorphism with acyclic kernel, Lemma 6.6.14 provides a chain map $\widehat{f}\colon P_\bullet \to Q_\bullet$ with $q\widehat{f} = f$. Now consider the mapping-cylinder factorization of $\widehat{f}$ from the proof of Proposition 6.1.27:

$$P_\bullet \hookrightarrow \mathrm{Cyl}(\widehat{f})_\bullet \xrightarrow{\sim} Q_\bullet.$$

The first map is degreewise the inclusion $P_n \hookrightarrow P_n \oplus P_{n-1} \oplus Q_n$, hence a degreewise split monomorphism whose cokernel is degreewise $P_{n-1} \oplus Q_n$ and thus bounded below projective; so it is a cofibration. The second map is a chain homotopy equivalence, and composing it with q therefore exhibits f as a cofibration followed by a quasi-isomorphism.

Finally, $0 \to P_\bullet$ is a cofibration precisely when $P_\bullet$ is bounded below projective. □

Proposition 6.6.16. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then the inclusion $\mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A}) \hookrightarrow \mathrm{Ch}^-(\mathcal{A})$ induces an equivalence on localizations:*

$$\mathcal{K}^-_{\mathrm{proj}}(\mathcal{A}) := \mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A})[\text{chain homotopy equivalences}^{-1}] \quad \xrightarrow{\sim} \quad \mathrm{D}^-(\mathcal{A}).$$

Proof. By Proposition 6.6.15 and part (1) of Corollary 2.2.9, the inclusion of the cofibrant objects induces an equivalence after localizing at the quasi-isomorphisms. By Proposition 6.6.11, the quasi-isomorphisms between bounded below projective complexes are precisely the chain homotopy equivalences, which gives the displayed equivalence. □

We can now construct left derived functors between derived categories.

Construction 6.6.17 (Derived functors between derived categories)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, assume that $\mathcal{A}$ has enough projectives, and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Extending F degreewise gives a functor $\mathrm{Ch}^-(F)\colon \mathrm{Ch}^-(\mathcal{A}) \to \mathrm{Ch}^-(\mathcal{B})$ that preserves chain homotopies. By Corollary 6.6.12, the restriction

$$\mathrm{Ch}^-(F)|_{\mathrm{proj}}\colon \mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A}) \to \mathrm{Ch}^-(\mathcal{B})$$

preserves quasi-isomorphisms. Composing with the localization $\gamma_{\mathcal{B}}\colon \mathrm{Ch}^-(\mathcal{B}) \to \mathrm{D}^-(\mathcal{B})$, we obtain a functor that inverts quasi-isomorphisms between the cofibrant objects of the projective cofibration structure on $\mathrm{Ch}^-(\mathcal{A})$. By Proposition 6.6.3, it therefore admits a total left derived functor

$$\mathbf{L}F\colon \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$$

If $q\colon P_\bullet \xrightarrow{\sim} C_\bullet$ is any projective resolution, then

$$\mathbf{L}F(C_\bullet) \simeq F(P_\bullet).$$

Exercise 6.6.18. Let $P_\bullet \to C_\bullet$ be a quasi-isomorphism in $\mathrm{Ch}(\mathcal{A})_{\geq 0}$, with $P_\bullet$ projective. Show that $\mathbf{L}F(C_\bullet) \simeq F(P_\bullet)$ in $\mathrm{D}^-(\mathcal{B})$. If F is right exact, conclude that $\pi_0\mathbf{L}F(A) \cong F(A)$ for all $A \in \mathcal{A}$.

Lemma 6.6.19. *In the situation of Construction 6.6.17, the functor* $\mathbf{L}F\colon \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$ *is exact and right t-exact.*

Proof. For right t-exactness, let $C_\bullet$ be concentrated in degrees ≥ 0. The construction in Proposition 6.6.13 gives a projective resolution $P_\bullet \to C_\bullet$ which is also concentrated in degrees ≥ 0. Hence $\mathbf{L}F(C_\bullet) \simeq F(P_\bullet)$ is connective.

For exactness, it suffices to show that $\mathbf{L}F$ preserves cofiber sequences. Every morphism in $\mathrm{D}^-(\mathcal{A})$ can be represented by a chain map $f\colon P_\bullet \to Q_\bullet$ between bounded below projective complexes (by first replacing both source and target with projective resolutions). The cofiber in $\mathrm{D}^-(\mathcal{A})$ is represented by the mapping cone $\mathrm{Cone}(f)$, which is again a bounded below projective complex. Since F is additive, it preserves mapping cones: $F(\mathrm{Cone}(f)) \cong \mathrm{Cone}(F(f))$. We then compute

$$\mathbf{L}F(\mathrm{cofib}(f)) \simeq F(\mathrm{Cone}(f)) \cong \mathrm{Cone}(F(f)) \simeq \mathrm{cofib}(\mathbf{L}F(f)),$$

showing that $\mathbf{L}F$ preserves cofiber sequences. □

Remark 6.6.20. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. For an object $A \in \mathcal{A}$, viewed as a complex concentrated in degree zero, we may define functors $\mathbf{L}^nF\colon \mathcal{A} \to \mathcal{B}$ by

$$\mathbf{L}^nF(A) \quad := \quad H_n(\mathbf{L}F(A))$$

for $n \geq 0$. They may be computed as the homology groups of $F(P_\bullet)$, where $P_\bullet \to A$ is any choice of projective resolution of A. Moreover, applying $\mathbf{L}F$ to a short exact sequence $0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$

in $\mathcal{A}$ yields a cofiber sequence $\mathbf{L}F(A) \to \mathbf{L}F(B) \to \mathbf{L}F(C)$ in $\mathrm{D}^-(\mathcal{B})$ by Lemma 6.6.19; its long exact sequence on homology is the classical long exact sequence relating the derived functors $\mathbf{L}^n F$, and in particular the connecting homomorphisms of the latter come from the exactness of $\mathbf{L}F$. In classical treatments of homological algebra, one often merely considers the functors $\mathbf{L}^n F$ rather than the entire derived functor $\mathbf{L}F$.

Lemma 6.6.21. *Assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy $(AB3)$ and $(AB4)$, and that $F\colon \mathcal{A} \to \mathcal{B}$ preserves arbitrary coproducts. Then $\mathbf{L}F\colon \mathrm{D}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{B})$ preserves coproducts of uniformly bounded below families.*

Proof. A uniformly bounded below family of complexes has a coproduct in $\mathrm{Ch}^-(\mathcal{A})$, formed degreewise. The assumptions $(AB3)$ and $(AB4)$ imply that this coproduct preserves quasi-isomorphisms, and F commutes with it degreewise. The claim therefore follows by localization. □

The construction of derived functors can be generalized by replacing the target $\mathrm{D}^-(\mathcal{B})$ with a stable ∞-category C equipped with a left complete t-structure: any right exact functor $F\colon \mathcal{A} \to C^\heartsuit$ is recovered as the zeroth homology on $\mathcal{A}$ of a unique exact, right t-exact functor $\mathrm{D}^-(\mathcal{A}) \to C$ which sends the projective objects of $\mathcal{A}$ into the heart of C. In fact, this property completely characterizes the bounded below derived category:

Theorem 6.6.22 (Universal property of the bounded below derived category)**.** *Let $\mathcal{A}$ be an abelian category with enough projectives and let C be a stable ∞-category with a left complete t-structure. Then restriction along the inclusion $\mathcal{A} \hookrightarrow \mathrm{D}^-(\mathcal{A})$, followed by π_0, induces an equivalence of ∞-categories*

$$\begin{aligned} \mathrm{Fun}'(\mathrm{D}^-(\mathcal{A}), C) &\xrightarrow{\sim} \mathrm{Fun}^{\mathrm{rex}}(\mathcal{A}, C^\heartsuit), \\ G &\longmapsto \pi_0 \circ G|_{\mathcal{A}}, \end{aligned}$$

where the left-hand side denotes the full subcategory of exact functors $\mathrm{D}^-(\mathcal{A}) \to C$ that are right t-exact and send projective objects of $\mathcal{A}$ into the heart $C^\heartsuit$.

Proof. We refer to Lurie [Lur17, Theorem 1.3.3.2]. □

Given a right exact functor $F\colon \mathcal{A} \to C^\heartsuit$, the corresponding functor is denoted $\mathbf{L}F\colon \mathrm{D}^-(\mathcal{A}) \to C$, and referred to as the *left derived functor* of F. It comes with a natural isomorphism

$$\pi_0 \circ (\mathbf{L}F)|_{\mathcal{A}} \cong F.$$

For a projective object $P \in \mathcal{A}$, we have $\mathbf{L}F(P) \cong F(P)$, since $\mathbf{L}F(P)$ lies in the heart.

When $C = \mathrm{D}^-(\mathcal{B})$ for an abelian category $\mathcal{B}$ with enough projectives, this agrees with the functor constructed in Construction 6.6.17. Indeed, that functor is exact and right t-exact by Lemma 6.6.19, and sends each projective object $P \in \mathcal{A}$ to $F(P) \in \mathcal{B}$. Moreover, for $A \in \mathcal{A}$, a projective resolution beginning with $P_1 \to P_0 \twoheadrightarrow A$ gives natural isomorphisms

$$\pi_0 \mathbf{L}F(A) \cong \mathrm{coker}\big(F(P_1) \to F(P_0)\big) \cong F(A),$$

where the last isomorphism follows from right exactness of F. Thus the projective-resolution construction corresponds to F under the displayed equivalence, proving the agreement by uniqueness.

6.6.3 Derived tensor product and derived hom

We now apply the derived functor formalism to construct the derived tensor product and derived hom. Throughout this subsection, let $\mathcal{A}$ be a closed symmetric monoidal abelian category with enough projectives and countable coproducts, with tensor product $\otimes\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and internal hom $[-,-]\colon \mathcal{A}^{\mathrm{op}}\times\mathcal{A}\to\mathcal{A}$. These conditions are satisfied, for example, when $\mathcal{A}=\mathrm{RMod}_R$ for a commutative ring R.

Definition 6.6.23 (Tensor product of chain complexes)**.** The *tensor product* of chain complexes $C_\bullet, C'_\bullet\in \mathrm{Ch}(\mathcal{A})$ is the chain complex $(C\otimes C')_\bullet$ defined by

$$(C\otimes C')_n \;:=\; \bigoplus_{p+q=n} C_p\otimes C'_q$$

whose differential acts on the summand $C_p\otimes C'_q$ as the sum $d^C\otimes \mathrm{id}_{C'}+(-1)^p\,\mathrm{id}_C\otimes d^{C'}$ of the two canonical composite maps. Countable coproducts ensure that the displayed expression exists for arbitrary complexes. If both complexes are bounded below, each degree involves only finitely many nonzero summands, so finite direct sums suffice in that case. This makes $\mathrm{Ch}(\mathcal{A})$ into a symmetric monoidal category. Moreover, tensor products preserve chain homotopies: if $f\simeq g\colon C_\bullet\to D_\bullet$ via a homotopy h, then $f\otimes\mathrm{id}_{C'}\simeq g\otimes\mathrm{id}_{C'}$ via $h\otimes\mathrm{id}_{C'}$.

The tensor product of chain complexes does not preserve quasi-isomorphisms in general:

Example 6.6.24. Let $\mathcal{A}=\mathrm{Ab}$ and consider the quasi-isomorphism $f\colon C_\bullet\to D_\bullet$ where $C_\bullet$ is the complex $\mathbb{Z}\xrightarrow{\cdot 2}\mathbb{Z}$ concentrated in degrees 1 and 0, and $D_\bullet=\mathbb{Z}/2$ concentrated in degree 0. Then $\mathbb{Z}/2\otimes f$ is the map

$$(\mathbb{Z}/2\xrightarrow{0}\mathbb{Z}/2)\;\longrightarrow\;(0\to\mathbb{Z}/2),$$

which is not a quasi-isomorphism: the left-hand side has $H_1=\mathbb{Z}/2$, while the right-hand side has $H_1=0$.

Definition 6.6.25 (Derived tensor product)**.** We define the *derived tensor product*

$$-\otimes^{\mathbf{L}}-\colon\ \mathrm{D}^-(\mathcal{A})\times\mathrm{D}^-(\mathcal{A})\to\mathrm{D}^-(\mathcal{A})$$

as the total left derived functor of the composite

$$\mathrm{Ch}^-(\mathcal{A})\times\mathrm{Ch}^-(\mathcal{A})\xrightarrow{-\otimes-}\mathrm{Ch}^-(\mathcal{A})\longrightarrow\mathrm{D}^-(\mathcal{A}),$$

where the source is equipped with the product of the projective cofibration structures of Proposition 6.6.15 and Lemma 6.6.7. Indeed, on pairs of bounded below projective complexes, tensor product preserves quasi-isomorphisms, since these are chain homotopy equivalences and tensor product preserves chain homotopies in each variable. Consequently, if $P\to C$ and $Q\to D$ are any projective resolutions, then

$$C\otimes^{\mathbf{L}} D\simeq P\otimes Q.$$

As in Lemma 6.6.19, this functor is exact in both variables, and restricts to $\mathrm{D}(\mathcal{A})_{\geq 0}\times\mathrm{D}(\mathcal{A})_{\geq 0}\to\mathrm{D}(\mathcal{A})_{\geq 0}$. If $\mathcal{A}$ satisfies $(AB3)$ and $(AB4)$, then it also preserves coproducts of uniformly bounded below families in both variables.

For objects $M, N \in \mathcal{A}$, regarded as complexes concentrated in degree zero, we abbreviate

$$\mathrm{Tor}_n^{\mathcal{A}}(M,N) := H_n(M \otimes^{\mathbf{L}} N) = \mathrm{Tor}_n^{\otimes^{\mathbf{L}}}(M,N).$$

Remark 6.6.26. Suppose in addition that the monoidal unit of $\mathcal{A}$ is projective and that the tensor product of two projective objects is again projective. Then the derived tensor product equips $\mathrm{D}^-(\mathcal{A})$ with a symmetric monoidal structure. The construction, including the coherent symmetric monoidal structure and the identification of its tensor product with $- \otimes^{\mathbf{L}} -$, is given in Example 21.1.7.

In practice, it usually suffices to resolve only one of the two variables by a projective complex. This is a consequence of the following lemma.

An object $F \in \mathcal{A}$ is called *flat* if the functor $F \otimes (-) \colon \mathcal{A} \to \mathcal{A}$ is exact.

Lemma 6.6.27. *Let $F_\bullet$ be a bounded below chain complex such that each object F_i is flat. Then the functor*

$$F_\bullet \otimes - \colon \mathrm{Ch}^-(\mathcal{A}) \to \mathrm{Ch}^-(\mathcal{A})$$

preserves quasi-isomorphisms.

Proof. Given a quasi-isomorphism $f \colon C \to D$, we need to show that the map $F_\bullet \otimes f$ is again a quasi-isomorphism. To this end, filter $F_\bullet$ by the subcomplexes $F^{\leq n}$, where $(F^{\leq n})_k = F_k$ for $k \leq n$ and 0 otherwise. We have degreewise split short exact sequences $0 \to F^{\leq n-1} \to F^{\leq n} \to F_n[n] \to 0$. The map $F_n[n] \otimes f$ is a quasi-isomorphism since F_n is flat. By induction on n and the five lemma, each $F^{\leq n} \otimes f$ is a quasi-isomorphism. Since $F_\bullet, C_\bullet$ and $D_\bullet$ are all bounded below, the sequence $(F^{\leq n} \otimes C)_k$ is eventually constant with value $(F \otimes C)_k$ in every degree k, and similarly for D. The map $F_\bullet \otimes f$ is therefore a degreewise colimit of the maps $F^{\leq n} \otimes f$ along which homology is eventually constant, and so it is a quasi-isomorphism. □

Corollary 6.6.28. *Assume that all projective objects in $\mathcal{A}$ are flat.*

(1) *The derived tensor product may be computed by resolving only one variable: for any projective resolutions $P \to C$ and $Q \to D$, there are isomorphisms*

$$C \otimes^{\mathbf{L}} D \simeq P \otimes D \simeq C \otimes Q$$

in $\mathrm{D}^-(\mathcal{A})$.

(2) *Tor groups in $\mathcal{A}$ may be computed by resolving one variable: given objects $M, N \in \mathcal{A}$, we have*

$$\mathrm{Tor}_n^{\mathcal{A}}(M,N) \;\cong\; H_n(P_\bullet \otimes N)$$

for any projective resolution $P_\bullet \to M$.

(3) *For a flat object $F \in \mathcal{A}$, the complex $F[0]$ is a t-flat object of* $\mathrm{D}^-(\mathcal{A})$.

Proof. Part (1) is immediate from Lemma 6.6.27, as P and Q are bounded below flat chain complexes. Part (2) then follows, as $\mathrm{Tor}_n^{\mathcal{A}}(M,N)$ is defined as the n-th homology group of $M \otimes^{\mathbf{L}} N$. Part (3) also follows, as the derived tensor $F \otimes^{\mathbf{L}} -$ is quasi-isomorphic to the underived tensor $F \otimes -$, which preserves both connective and coconnective complexes. □

We now turn to the derived internal hom.

For the explicit construction below, we assume in addition that $\mathcal{A}$ admits countable products and that these are exact, that tensor products of projective objects are projective, and that projective objects are flat. These hypotheses ensure that bounded below projective complexes are K-projective and that tensoring with such a complex preserves quasi-isomorphisms. They hold, in particular, for modules over a commutative ring.

Definition 6.6.29 (Internal hom of chain complexes)**.** The *internal hom* of chain complexes $C_\bullet, D_\bullet \in \mathrm{Ch}(\mathcal{A})$ is the chain complex $\underline{\mathrm{Hom}}(C, D)_\bullet$ defined by

$$\underline{\mathrm{Hom}}(C, D)_n := \prod_{p \in \mathbb{Z}} [C_p, D_{p+n}]$$

with differential $(d\varphi)_p = d \circ \varphi_p - (-1)^n \varphi_{p-1} \circ d$ for $\varphi = (\varphi_p)_p \in \underline{\mathrm{Hom}}(C, D)_n$.

Lemma 6.6.30 (Tensor–hom adjunction for chain complexes)**.** *For chain complexes $A, B, C \in \mathrm{Ch}(\mathcal{A})$, there is a natural isomorphism of chain complexes*

$$\underline{\mathrm{Hom}}(A \otimes B, C) \cong \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(B, C)),$$

which induces the tensor-hom adjunction

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A \otimes B, C) \cong \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, \underline{\mathrm{Hom}}(B, C)).$$

Proof. Degreewise currying gives the first isomorphism, and direct substitution into the definitions shows that it commutes with the differentials. Applying $\mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, Z_0(-))$, where $\mathbb{1}$ is the monoidal unit, gives the adjunction on chain maps. □

Lemma 6.6.31. *If $P_\bullet$ is a bounded below projective chain complex, then the functor*

$$\underline{\mathrm{Hom}}(P, -) \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$$

preserves quasi-isomorphisms.

Proof. First observe that $[P_n, -] \colon \mathcal{A} \to \mathcal{A}$ is exact for every n. Indeed, exactness may be tested by mapping out of projective objects, and for projective objects Q we have

$$\mathrm{Hom}_{\mathcal{A}}(Q, [P_n, -]) \cong \mathrm{Hom}_{\mathcal{A}}(Q \otimes P_n, -),$$

which is exact because $Q \otimes P_n$ is projective. It follows that $\underline{\mathrm{Hom}}(P, A)$ is acyclic whenever A is acyclic: filter P by its bounded truncations and apply the preceding exactness degreewise. The transition maps between the resulting internal hom complexes are degreewise split epimorphisms, so exactness of countable products allows us to pass to their inverse limit. Applying this to the cone of a quasi-isomorphism proves the claim. □

Lemma 6.6.32. *Let $P_\bullet$ be a bounded below projective chain complex and let $X_\bullet$ be any chain complex. Then the localization functor induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P_\bullet, X_\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(P_\bullet, X_\bullet).$$

Proof. Let A be an acyclic complex. Every chain map $f\colon P \to A$ is nullhomotopic. Indeed, starting below the lower bound of P, we construct maps $h_n\colon P_n \to A_{n+1}$ inductively such that

$$f_n = d_A h_n + h_{n-1} d_P.$$

Once the preceding maps have been constructed, the map $f_n - h_{n-1}d_P$ lands in $Z_n(A)$: its composite with d_A is zero by the chain-map identity and the induction hypothesis. Since A is acyclic, the differential $A_{n+1} \twoheadrightarrow Z_n(A)$ is an epimorphism, and projectivity of P_n provides the required lift h_n.

The same argument applies to every shift $P[k]$. By stability of $\mathcal{K}(\mathcal{A})$ and Lemma 6.2.8, all homotopy groups of $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P, A)$ vanish, and this mapping anima is therefore contractible. The cone of a quasi-isomorphism is acyclic, so exactness of mapping spectra shows that $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P, -)$ inverts quasi-isomorphisms. Viewing $\mathrm{D}(\mathcal{A})$ as the localization of $\mathcal{K}(\mathcal{A})$ at the quasi-isomorphisms, the claim now follows from Lemma 6.2.10. □

Definition 6.6.33 (Derived hom)**.** Let $\mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A})$ denote the full subcategory of bounded below projective chain complexes. The internal hom restricts to a functor

$$\underline{\mathrm{Hom}}(-,-)\colon\ \mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A}).$$

It preserves quasi-isomorphisms in both variables: in the first variable this follows from Proposition 6.6.11, while in the second it follows from Lemma 6.6.31. Consequently, localization and Proposition 6.6.16 produce a functor

$$\mathbf{R}\,\underline{\mathrm{Hom}}(-,-)\colon\ \mathrm{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A}).$$

For any projective resolution $P \to C$, it is computed by

$$\mathbf{R}\,\underline{\mathrm{Hom}}(C, D) \simeq \underline{\mathrm{Hom}}(P, D).$$

Just as the derived tensor product computes Tor-groups, the derived hom computes Ext-groups, which ties the present discussion back to Section 6.4:

Observation 6.6.34. Let R be a commutative ring and let $M, N \in \mathrm{RMod}_R$, regarded as complexes concentrated in degree 0. Choosing a projective resolution $P_\bullet \to M$, the complex $\underline{\mathrm{Hom}}(P, N)$ has $\underline{\mathrm{Hom}}(P, N)_{-n} = [P_n, N]$ with the differential induced by that of $P_\bullet$; it is thus the cochain complex $\mathrm{Hom}_R(P_\bullet, N)$, placed in non-positive degrees. Taking homology, we obtain

$$H_{-n}\big(\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, N)\big) \ \cong\ H^n\big(\mathrm{Hom}_R(P_\bullet, N)\big) \ \cong\ \mathrm{Ext}^n_R(M, N),$$

where the second isomorphism is Proposition 6.4.16. In particular $\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, N)$ is coconnective, with $H_0 = \mathrm{Hom}_R(M, N)$.

The key relationship between the derived tensor product and derived hom is that they remain adjoint:

Proposition 6.6.35 (Derived tensor-hom adjunction)**.** *For $C, D \in \mathrm{D}^-(\mathcal{A})$ and $E \in \mathrm{D}(\mathcal{A})$, there is a natural equivalence*

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(C \otimes^{\mathbf{L}} D, E) \ \simeq\ \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(C, \mathbf{R}\,\underline{\mathrm{Hom}}(D, E)).$$

Proof. Choose a chain complex representing E. By Proposition 6.6.16, the ∞-category $\mathrm{D}^-(\mathcal{A})$ is equivalent to the localization of $\mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A})$ at chain homotopy equivalences. It therefore suffices to establish the adjunction for bounded below projective complexes $P_\bullet$ and $Q_\bullet$. In this case, we have

$$P \otimes^{\mathbf{L}} Q \simeq P \otimes Q \qquad \text{and} \qquad \mathbf{R}\,\underline{\mathrm{Hom}}(Q, E) \simeq \underline{\mathrm{Hom}}(Q, E).$$

The complex $P \otimes Q$ is again bounded below projective. Using Lemma 6.6.32 and the internal tensor-hom adjunction of Lemma 6.6.30, we obtain

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(P \otimes Q, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P \otimes Q, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P, \underline{\mathrm{Hom}}(Q, E)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(P, \underline{\mathrm{Hom}}(Q, E)),$$

as required. To justify the middle isomorphism of mapping animae, the chain-level adjunction $(-) \otimes Q \dashv \underline{\mathrm{Hom}}(Q, -)$ from Lemma 6.6.30 descends to $\mathcal{K}(\mathcal{A})$: both functors preserve chain homotopy equivalences, and their unit and counit descend along localization together with the triangle identities. The resulting adjunction gives the asserted isomorphism, naturally in P, Q, and E. □

Corollary 6.6.36. *Assume in addition that $\mathcal{A}$ satisfies $(AB3)$, $(AB4)$, $(AB3^*)$, and $(AB4^*)$. For every $C \in \mathrm{D}^-(\mathcal{A})$, the functor $\mathbf{R}\,\underline{\mathrm{Hom}}(C, -)\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ preserves all small limits. For every $E \in \mathrm{D}(\mathcal{A})$, the functor $\mathbf{R}\,\underline{\mathrm{Hom}}(-, E)$ sends colimits of uniformly bounded below diagrams in $\mathrm{D}^-(\mathcal{A})$ to limits.*

Proof. The functor $\mathbf{R}\,\underline{\mathrm{Hom}}(C, -) \simeq \underline{\mathrm{Hom}}(P, -)$, for P a projective resolution of C, is exact and satisfies $\underline{\mathrm{Hom}}(P, \prod_i E_i) \cong \prod_i \underline{\mathrm{Hom}}(P, E_i)$ degreewise. It therefore preserves all limits.

In the first variable, the functor is exact and sends coproducts of uniformly bounded below families to products: if P^i is a projective resolution of C^i, chosen with a common lower bound, then $\bigoplus_i P^i$ is a projective resolution of $\bigoplus_i C^i$, and

$$\underline{\mathrm{Hom}}\Big(\bigoplus_i P^i, E\Big) \cong \prod_i \underline{\mathrm{Hom}}(P^i, E)$$

degreewise. Together with exactness, this gives the assertion for uniformly bounded below colimits. □

6.7 Künneth and universal coefficient theorems

The derived tensor product and derived internal hom now give concrete computational consequences for ordinary homology and cohomology. We first record the algebraic input over principal ideal domains, then apply it to the chain complexes of animae constructed in Section 6.2.

6.7.1 Tor and Ext over principal ideal domains

For complexes of modules over a principal ideal domain, the homology groups of the derived tensor product and derived hom admit particularly explicit descriptions in terms of Tor and Ext groups of the homology of the individual complexes. The key simplification is that projective resolutions over a PID have length at most one, which allows for elementary proofs of the Künneth formula and universal coefficient theorem.

Proposition 6.7.1. *Let R be a principal ideal domain. Then every R-module M admits a projective resolution of length at most one:*

$$0 \to P_1 \to P_0 \to M \to 0.$$

In particular, $\operatorname{Tor}_n^R(M,N) = 0$ *and* $\operatorname{Ext}^n_R(M,N) = 0$ *for all R-modules M, N and all $n \geq 2$.*

Proof. Every R-module M admits a surjection from a free module: take $P_0 := R^{(M)}$, the free module on the underlying set of M, with the canonical surjection $\varepsilon\colon P_0 \twoheadrightarrow M$. The kernel $P_1 := \ker(\varepsilon)$ is a submodule of the free module P_0. Over a PID, every submodule of a free module is itself free, hence projective. This gives the desired resolution.

The vanishing of higher Tor and Ext groups follows from the formulas $\operatorname{Tor}_n^R(M,N) \cong H_n(P_\bullet \otimes_R N)$ and $\operatorname{Ext}^n_R(M,N) \cong H^n(\operatorname{Hom}_R(P_\bullet, N))$ of Corollary 6.6.28 and Proposition 6.4.16, which are both zero for $n \geq 2$ since $P_k = 0$ for $k \geq 2$. □

Theorem 6.7.2 (Algebraic Künneth formula)**.** *Let R be a principal ideal domain and let $A, B \in \mathrm{D}^-(R)$ be bounded below complexes of R-modules. Then for each $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(A) \otimes_R H_j(B) \to H_n(A \otimes^{\mathbf{L}}_R B) \to \bigoplus_{i+j=n-1} \operatorname{Tor}_1^R(H_i(A), H_j(B)) \to 0,$$

and this sequence splits (non-naturally).

Proof. We may represent the complexes A and B up to quasi-isomorphism by bounded below chain complexes C and C', respectively, such that the R-modules C_i and C'_i are free for all $i \in \mathbb{Z}$. Then the derived tensor product $A \otimes^{\mathbf{L}}_R B$ is represented by the algebraic tensor product $C \otimes_R C'$.

We first treat the special case where all the differentials in C are zero, so that $H_i(C) = C_i$ for all i. In this case, the differential on $C \otimes_R C'$ is given by $d(c \otimes c') = (-1)^i c \otimes dc'$ for $c \in C_i$, and the chain complex $C \otimes_R C'$ decomposes as the direct sum of the complexes $C_i[i] \otimes_R C'$. Since each C_i is free, the complex $C_i \otimes_R C'$ is a direct sum of copies of C', and hence

$$H_n(C_i[i] \otimes_R C') \;\cong\; C_i \otimes_R H_{n-i}(C') \;=\; H_i(C) \otimes_R H_{n-i}(C').$$

Summing over i yields an isomorphism $H_n(C \otimes_R C') \cong \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(C')$, which is the statement of the theorem since the Tor terms vanish ($H_i(C) = C_i$ being free, hence flat).

For the general case, let $Z_i \subseteq C_i$ and $B_i \subseteq C_i$ denote the cycles and boundaries in degree i, respectively. These give subchain complexes Z and B of C with trivial differentials. We have a short exact sequence of chain complexes

$$0 \to Z \to C \to B[1] \to 0$$

arising from the short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ in each degree. Each of these splits since B_{i-1} is free, being a submodule of the free module C_{i-1}. Because of this splitting, tensoring with C' preserves exactness, and we obtain a long exact sequence in homology:

$$\cdots \to H_n(Z \otimes_R C') \to H_n(C \otimes_R C') \to H_{n-1}(B \otimes_R C') \xrightarrow{i_{n-1}} H_{n-1}(Z \otimes_R C') \to \cdots$$

where $i_k\colon H_k(B \otimes_R C') \to H_k(Z \otimes_R C')$ is induced by the inclusion $B \hookrightarrow Z$.

Since Z and B are chain complexes with trivial differentials (and have free terms, being submodules of the free modules C_i), the special case treated before converts this to:

$$\cdots \xrightarrow{i_n} \bigoplus_{i+j=n} Z_i \otimes_R H_j(C') \to H_n(C \otimes_R C') \to \bigoplus_{i+j=n-1} B_i \otimes_R H_j(C') \xrightarrow{i_{n-1}} \bigoplus_{i+j=n-1} Z_i \otimes_R H_j(C') \to \cdots$$

The long exact sequence thus provides short exact sequences

$$0 \to \operatorname{coker}(i_n) \to H_n(C \otimes_R C') \to \ker(i_{n-1}) \to 0.$$

To identify $\operatorname{coker}(i_n)$ and $\ker(i_{n-1})$, observe that the short exact sequence $0 \to B_i \to Z_i \to H_i(C) \to 0$ functions as a free resolution of $H_i(C)$. The long exact sequence on Tor-groups between $H_i(C)$ and $H_j(C')$ takes the form

$$0 \to \operatorname{Tor}_1^R(H_i(C), H_j(C')) \to B_i \otimes_R H_j(C') \to Z_i \otimes_R H_j(C') \to H_i(C) \otimes_R H_j(C') \to 0.$$

By taking the direct sum over all j, this results in isomorphisms

$$\operatorname{coker}(i_n) \cong \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(C') \qquad \text{and} \qquad \ker(i_{n-1}) \cong \bigoplus_{i+j=n-1} \operatorname{Tor}_1^R(H_i(C), H_j(C')).$$

All maps used to construct this short exact sequence are functorial in chain maps. Through the projective-complex model of $\mathrm{D}^-(R)$ from Proposition 6.6.16, the sequence is therefore natural in A and B.

Finally, we show that the short exact sequence splits. Since each C_i is free, the short exact sequence $0 \to Z_i \to C_i \to B_{i-1} \to 0$ splits, and the quotient maps $Z_i \to H_i(C)$ extend to homomorphisms $C_i \to H_i(C)$. We similarly obtain homomorphisms $C'_j \to H_j(C')$. Viewing the sequences of homology groups $H_i(C)$ and $H_j(C')$ as chain complexes with trivial differentials, we obtain chain maps $C \to H(C)$ and $C' \to H(C')$. Their tensor product is a chain map $C \otimes_R C' \to H(C) \otimes_R H(C')$. Since the differentials on the target are trivial, the induced map on homology

$$H_n(C \otimes_R C') \to H_n(H(C) \otimes_R H(C')) = \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(C')$$

provides the desired splitting. □

An analogous argument yields the universal coefficient theorem for the derived hom:

Theorem 6.7.3 (Algebraic universal coefficient theorem)**.** *Let R be a principal ideal domain, let $A \in \mathrm{D}^-(R)$, and let M be an R-module, regarded as a complex concentrated in degree zero. For each $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \to \operatorname{Ext}_R^1(H_{n-1}(A), M) \to H_{-n}(\mathbf{R}\,\underline{\operatorname{Hom}}_R(A, M[0])) \to \operatorname{Hom}_R(H_n(A), M) \to 0,$$

and this sequence splits non-naturally.

Proof. Choose a bounded below complex C of free R-modules representing A. Its boundary modules are free because R is a PID. Thus [Wei94, Theorem 3.6.5] applies to C and M, and $\underline{\operatorname{Hom}}_R(C, M)$ represents $\mathbf{R}\,\underline{\operatorname{Hom}}_R(A, M[0])$. □

Example 6.7.4. For $R = \mathbb{Z}$, the Künneth formula specializes to the classical result for abelian groups. Given bounded below chain complexes A and B of abelian groups, the homology of the derived tensor product fits into the short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(A) \otimes_{\mathbb{Z}} H_j(B) \to H_n(A \otimes^{\mathbf{L}}_{\mathbb{Z}} B) \to \bigoplus_{i+j=n-1} \operatorname{Tor}^{\mathbb{Z}}_1(H_i(A), H_j(B)) \to 0.$$

The cyclic calculation in Exercise 6.5.8 illustrates how Tor detects torsion interactions. The formula shows that the homology of the derived tensor product is determined by the homology groups of A and B, up to an extension involving their torsion.

Example 6.7.5. Let R be a PID and let $A, B \in \mathrm{D}^-(R)$. If all homology groups $H_i(A)$ are flat (equivalently, torsion-free), then

$$H_n(A \otimes^{\mathbf{L}}_R B) \;\cong\; \bigoplus_{i+j=n} H_i(A) \otimes_R H_j(B).$$

Example 6.7.6. Let R be a PID, let $A \in \mathrm{D}^-(R)$, and let M be an R-module. If all homology groups $H_i(A)$ are projective (equivalently, free), then

$$H_{-n}(\mathbf{R}\,\underline{\mathrm{Hom}}_R(A, M[0])) \;\cong\; \mathrm{Hom}_R(H_n(A), M).$$

6.7.2 Künneth theorem

The Künneth theorem describes the homology of a product of spaces in terms of the homology of the factors. The key observation is that the chain complex of a product is the derived tensor product of the chain complexes of the factors.

Proposition 6.7.7. *Let R be a commutative ring. For animae $X, Y \in \mathrm{An}$, there is a natural equivalence*

$$C_*(X \times Y; R) \;\simeq\; C_*(X; R) \otimes^{\mathbf{L}}_R C_*(Y; R)$$

in $\mathrm{D}(R)$.

Proof. Chain complexes of animae are connective, so both sides are functors $\mathrm{An} \times \mathrm{An} \to \mathrm{D}(R)_{\geq 0}$, and the derived tensor product of Definition 6.6.25 is defined on them. It preserves colimits in each variable: it is exact and preserves coproducts of uniformly bounded below families, and every colimit of connective objects is of this form. Similarly, the functor $- \times - \colon \mathrm{An} \times \mathrm{An} \to \mathrm{An}$ preserves colimits in both variables, and $C_*(-; R)$ preserves colimits. It follows that both sides define functors $\mathrm{An} \times \mathrm{An} \to \mathrm{D}(R)_{\geq 0}$ preserving colimits in each variable. Both send $(\mathrm{pt}, \mathrm{pt})$ to $R[0]$, hence they are naturally equivalent by the universal property of An. □

Theorem 6.7.8 (Künneth theorem)**.** *Let R be a principal ideal domain and let $X, Y \in \mathrm{An}$ be animae. For each $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(X; R) \otimes_R H_j(Y; R) \to H_n(X \times Y; R) \to \bigoplus_{i+j=n-1} \operatorname{Tor}^R_1(H_i(X; R), H_j(Y; R)) \to 0,$$

and this sequence splits, though not naturally.

Proof. By Proposition 6.7.7, we have $H_n(X \times Y; R) \cong H_n(C_*(X; R) \otimes^{\mathbf{L}}_R C_*(Y; R))$. The result now follows immediately from the algebraic Künneth formula (Theorem 6.7.2). □

Corollary 6.7.9. *If R is a principal ideal domain and $H_i(X; R)$ is a free R-module for all i, then*

$$H_n(X \times Y; R) \;\cong\; \bigoplus_{i+j=n} H_i(X; R) \otimes_R H_j(Y; R)$$

for all $n \in \mathbb{Z}$ and all animae Y. □

6.7.3 Universal coefficient theorems

The universal coefficient theorems express the (co)homology with coefficients in a module in terms of the integral (co)homology. We begin with the homological version.

Proposition 6.7.10. *For an anima $X \in \mathrm{An}$ and a bounded below chain complex $A \in \mathrm{D}^-(\mathbb{Z})$, there is a natural equivalence*

$$C_*(X; A) \;\simeq\; C_*(X; \mathbb{Z}) \otimes^{\mathbf{L}}_{\mathbb{Z}} A$$

in $\mathrm{D}^-(\mathbb{Z})$.

Proof. Say A is concentrated in degrees $\geq m$. Both $C_*(-; A)$ and $C_*(-; \mathbb{Z}) \otimes^{\mathbf{L}}_{\mathbb{Z}} A$ then take values in $\mathrm{D}(\mathbb{Z})_{\geq m}$, which is closed under colimits in $\mathrm{D}(\mathbb{Z})$. Both preserve colimits: for the first this is the defining property from Construction 6.2.12, and for the second it follows as in Proposition 6.7.7, since $C_*(-; \mathbb{Z})$ is colimit-preserving with connective values and $- \otimes^{\mathbf{L}}_{\mathbb{Z}} A$ is exact and preserves coproducts of uniformly bounded below families. Finally, both send pt to A. By the universal property of An, they are naturally equivalent. □

Theorem 6.7.11 (Universal coefficient theorem for homology)**.** *Let $X \in \mathrm{An}$ be an anima and let A be an abelian group. For each $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \to H_n(X) \otimes_{\mathbb{Z}} A \to H_n(X; A) \to \mathrm{Tor}^{\mathbb{Z}}_1(H_{n-1}(X), A) \to 0,$$

and this sequence splits, though not naturally.

Proof. By Proposition 6.7.10, we have $H_n(X; A) \cong H_n(C_*(X; \mathbb{Z}) \otimes^{\mathbf{L}} A)$. Viewing A as a chain complex concentrated in degree 0, the statement is thus an instance of the algebraic Künneth formula (Theorem 6.7.2), using that $H_j(A) = 0$ for $j \neq 0$ and $H_0(A) = A$. □

For cohomology, we use the internal hom in $\mathrm{D}(\mathbb{Z})$ instead of the derived tensor product.

Proposition 6.7.12. *For an anima $X \in \mathrm{An}$ and a chain complex $A \in \mathrm{D}(\mathbb{Z})$, there is a natural equivalence*

$$C^*(X; A) \;\simeq\; \mathbf{R}\,\underline{\mathrm{Hom}}_{\mathbb{Z}}(C_*(X; \mathbb{Z}), A)$$

in $\mathrm{D}(\mathbb{Z})$.

Proof. The chain complexes $C_*(X; \mathbb{Z})$ are connective. Thus every diagram in the image of the colimit-preserving functor $C_*(-; \mathbb{Z})\colon \mathrm{An} \to \mathrm{D}(\mathbb{Z})_{\geq 0}$ is uniformly bounded below, and Corollary 6.6.36 shows that $\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathbb{Z}}(C_*(-; \mathbb{Z}), A)$ preserves limits as a functor on $\mathrm{An}^{\mathrm{op}}$. It sends pt to $\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}, A) \simeq A$, so the universal property of An identifies it with $C^*(-; A)$. □

Theorem 6.7.13 (Universal coefficient theorem for cohomology)**.** *Let $X \in \mathrm{An}$ be an anima and let A be an abelian group. For each $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), A) \to H^n(X; A) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X), A) \to 0,$$

and this sequence splits, though not naturally.

Proof. By Proposition 6.7.12, we have $H^n(X; A) \cong H_{-n}(\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathbb{Z}}(C_*(X; \mathbb{Z}), A))$. The result is therefore the case $R = \mathbb{Z}$, $M = A$ of Theorem 6.7.3. □

Corollary 6.7.14. *If $H_{n-1}(X)$ is a free abelian group, then*

$$H^n(X; A) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(X), A)$$

for all abelian groups A. □

Exercises

Exercise 6.1 (Two-term resolutions, Tor, and Ext)**.** For an integer $n \geq 1$, let $C(n)_\bullet$ be the chain complex of abelian groups with $\mathbb{Z}$ in degrees 1 and 0, differential

$$d_1 \colon C(n)_1 = \mathbb{Z} \xrightarrow{\cdot n} C(n)_0 = \mathbb{Z},$$

and zero in all other degrees.

(1) Compute the homology groups $H_k(C(n))$.

(2) Show that the evident map $C(n)_\bullet \to \mathbb{Z}/n\,\mathbb{Z}[0]$ is a quasi-isomorphism.

(3) Regard $C(n)_\bullet$ as a projective resolution of $\mathbb{Z}/n\,\mathbb{Z}$ and compute $\mathrm{Tor}^{\mathbb{Z}}_i(\mathbb{Z}/n\,\mathbb{Z}, A)$ and $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\,\mathbb{Z}, A)$ for every abelian group A and every $i \geq 0$.

(4) Specialize to $A = \mathbb{Z}/m\,\mathbb{Z}$ and express the answer in terms of $\gcd(m, n)$.

Exercise 6.2. Let C and D be stable ∞-categories with t-structures, let $F \colon C \to D$ be an exact functor, and let

$$0 \to A \hookrightarrow B \twoheadrightarrow C' \to 0$$

be a short exact sequence in the heart $C^\heartsuit$. Construct a long exact sequence in $D^\heartsuit$ of the form

$$\cdots \to \pi_{n+1}F(C') \to \pi_n F(A) \to \pi_n F(B) \to \pi_n F(C') \to \pi_{n-1}F(A) \to \ldots.$$

Show that if F is t-exact, this long exact sequence collapses to $0 \to F(A) \to F(B) \to F(C') \to 0$ in $D^\heartsuit$.

Exercise 6.3 (Truncating a two-term complex)**.** Let $\mathcal{A}$ be an abelian category and let $C_\bullet$ be the complex

$$\cdots \longrightarrow 0 \longrightarrow A_1 \xrightarrow{d} A_0 \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees 1 and 0. Describe $\tau_{\geq 1}C_\bullet$ and $\tau_{\leq 0}C_\bullet$ explicitly, and identify the canonical exact sequence

$$\tau_{\geq 1}C_\bullet \longrightarrow C_\bullet \longrightarrow \tau_{\leq 0}C_\bullet$$

in terms of $\ker(d)$ and $\mathrm{coker}(d)$.

Exercise 6.4 (Chains on spheres)**.** Use the construction of ordinary chains from Construction 6.2.12.

(1) Show that the reduced chain functor $\widetilde{C}_*(-;\mathbb{Z})\colon \mathrm{An}_* \to \mathrm{D}(\mathbb{Z})$ sends cofiber sequences of pointed animae to exact sequences in $\mathrm{D}(\mathbb{Z})$.

(2) Deduce that $\widetilde{C}_*(\Sigma X;\mathbb{Z}) \cong \widetilde{C}_*(X;\mathbb{Z})[1]$.

(3) Compute $\widetilde{C}_*(S^n;\mathbb{Z})$ and $C_*(S^n;\mathbb{Z})$ in $\mathrm{D}(\mathbb{Z})$.

Exercise 6.5 (Extensions and addition in Ext^1)**.** Let C be a stable ∞-category with a t-structure, and let $M, N \in C^\heartsuit$.

(1) Starting from a morphism $\varphi\colon M \to N[1]$, form its fiber and show that this gives an extension

$$0 \to N \to E_\varphi \to M \to 0$$

in $C^\heartsuit$.

(2) Starting from an extension of M by N, construct the corresponding morphism $M \to N[1]$.

(3) Check that these two constructions are inverse to each other on isomorphism classes.

(4) Describe the addition law on extensions which corresponds to the abelian group structure on $\mathrm{Ext}^1_C(M,N)$.

Exercise 6.6 (A derived functor from a homotopy invariant functor)**.** Let $F\colon \mathrm{Top} \to D$ be a functor to an ∞-category D which sends homotopy equivalences of topological spaces to isomorphisms.

(1) Using CW-complexes as cofibrant objects, explain how Proposition 6.6.3 produces a total left derived functor

$$\mathbf{L}F\colon \mathrm{An} \to D.$$

(2) Show that if X is a CW-complex, then $\mathbf{L}F(\Pi_\infty X) \cong F(X)$.

(3) Explain why, if F already sends weak homotopy equivalences to isomorphisms, then $\mathbf{L}F$ is just the functor induced by the universal property of the localization $\mathrm{Top} \to \mathrm{An}$.

For the next two exercises, you may use that $H_k(\mathbb{C}\mathrm{P}^m) \cong \mathbb{Z}$ for $k = 0, 2, \ldots, 2m$ and vanishes otherwise, while $H_0(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}$, $H_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/2\,\mathbb{Z}$, and all other homology groups of $\mathbb{R}\mathrm{P}^2$ vanish.

Exercise 6.7 (Künneth computations)**.** Use Theorem 6.7.8 to compute the integral homology groups of the following products:

(1) $S^m \times S^n$.

(2) $\mathbb{C}\mathrm{P}^m \times S^n$.

(3) $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$.

In each case, indicate whether the $\mathrm{Tor}_1^{\mathbb{Z}}$ term vanishes and why.

Exercise 6.8 (Universal coefficient computations)**.** Use Theorem 6.7.11 and Theorem 6.7.13.

(1) Compute $H_*(\mathbb{R}\mathrm{P}^2;\mathbb{Z}/2\,\mathbb{Z})$ from the integral homology of $\mathbb{R}\mathrm{P}^2$.

(2) Compute $H^*(\mathbb{R}\mathrm{P}^2;\mathbb{Z})$.

(3) Compute $H^*(\mathbb{R}\mathrm{P}^2;\mathbb{Z}/2\,\mathbb{Z})$ and identify the contribution of the $\mathrm{Ext}^1_{\mathbb{Z}}$ term.

7 Localization and completion

An abelian group can often be understood by separating its behavior at different primes. Its p-localization inverts all primes other than p, and its p-completion records compatible information modulo the powers p^n. For finitely generated groups, the arithmetic fracture square recovers the original group from its p-completion and its p-inversion. All three operations have analogues for spectra, and the aim of this chapter is to construct them and prove their basic properties.

The right level of generality is that of a *Bousfield localization*: given a spectrum E, we invert the maps that are isomorphisms on E-homology, so the resulting theory depends only on the features of a spectrum detected by E. Rationalization, p-localization and p-completion all arise this way, for suitable choices of E. Bousfield localizations are also the basic organizing device of chromatic homotopy theory, which we do not treat here.

Inverting a set of primes has a direct effect on homotopy groups: they are given by the algebraic localization. Rationally, Serre's theorem $\mathbb{S}_{\mathbb{Q}} \simeq \mathbb{Q}$ implies that every rational spectrum splits as a sum of shifted Eilenberg–MacLane spectra. Completion is more subtle: Theorem 7.3.10 expresses $\pi_*(X^\wedge_p)$ through the Ext- and Hom-groups of Chapter 6. The arithmetic fracture square then reassembles a spectrum from its p-completions and its rationalization.

7.1 Bousfield localizations

Bousfield localization isolates the information in a spectrum detected by a chosen homology theory.

Fix a spectrum E throughout this section. As discussed in Definition 4.4.21, it defines the *E-homology groups* of an anima:

$$X \mapsto E_*(X) := \pi_*(E \otimes \mathbb{S}[X]).$$

For example, taking $E = A$ for an abelian group A regarded as an Eilenberg–MacLane spectrum results in singular homology with coefficients in A, while taking $E = \mathbb{S}$ the sphere spectrum results in the stable homotopy groups.

The homology groups $E_*(X)$ will generally only see *some* aspects of the given anima X, and may completely ignore others. Consider for example the case of the projective space $X = \mathbb{R}\mathrm{P}^2$. If we considered the Eilenberg–MacLane spectrum $E = \mathbb{Q}$ of the rational numbers, its homology would give no information beyond that of a point: the reduced rational homology of $\mathbb{R}\mathrm{P}^2$ is trivial. But taking $E = \mathbb{Z}/2$ *does* capture interesting information about $\mathbb{R}\mathrm{P}^2$. The point of Bousfield localizations is to isolate those parts of an anima (or spectrum) which are 'seen by E'.

Definition 7.1.1. A morphism $f\colon X \to Y$ is called an *E-equivalence* if the induced map $E \otimes f\colon E \otimes X \to E \otimes Y$ is an isomorphism. We say that a spectrum X is *E-acyclic* if the map $X \to 0$ is an E-equivalence, i.e. if $E \otimes X$ is the zero spectrum.

Recall that a morphism of spectra is an isomorphism if and only if it induces isomorphisms on all homotopy groups. So if we write $E_*(X) := \pi_*(E \otimes X)$ for the E-homology of X, generalizing the definition for animae, we see that the E-equivalences are precisely those maps that induce isomorphisms on E-homology. Similarly, the E-acyclic spectra are those with trivial E-homology.

Example 7.1.2. For $E = \mathbb{Q}$, the E-equivalences are known as the *rational equivalences*: those that induce isomorphisms on rational homology.

We will now introduce a class of spectra that 'cannot distinguish between E-equivalent spectra':

Definition 7.1.3. A spectrum L is called *E-local* if for every E-equivalence $f\colon X \to Y$, the induced map $f^*\colon \hom(Y, L) \to \hom(X, L)$ on mapping spectra is an isomorphism. We denote by

$$\mathrm{Sp}_E \subseteq \mathrm{Sp}$$

the full subcategory spanned by the E-local spectra.

Equivalently, it suffices to require that f induce an isomorphism $\mathrm{Hom}_{\mathrm{Sp}}(Y, L) \to \mathrm{Hom}_{\mathrm{Sp}}(X, L)$ on hom animae for every E-equivalence $f\colon X \to Y$. Indeed, every shift of an E-equivalence is again an E-equivalence, and applying this condition to all shifts of f detects all homotopy groups of the map of mapping spectra.

Exercise 7.1.4. Show the following basic properties of E-local spectra:

(1) A spectrum L is E-local if and only if $\hom(X, L) = 0$ for every E-acyclic spectrum X;

(2) Given an exact sequence $X \to Y \to Z$, if two of these three spectra are E-local then so is the third;

(3) In particular, if L is E-local then so are all its shifts $L[n]$;

(4) A limit $\lim_i L_i$ of E-local spectra is E-local;

(5) A retract of an E-local spectrum is E-local.

Exercise 7.1.5. Show that for every spectrum Y, the mapping spectrum $\hom(E, Y)$ is E-local. Using this, show that for a morphism $f\colon X \to X'$ of spectra the following conditions are equivalent:

(1) The morphism $f\colon X \to X'$ is an E-equivalence;

(2) The induced map $f^*\colon \hom(X', L) \to \hom(X, L)$ is an isomorphism for all E-local spectra L;

(3) The induced map $f^*\colon \mathrm{Hom}_{\mathrm{Sp}}(X', L) \to \mathrm{Hom}_{\mathrm{Sp}}(X, L)$ is an isomorphism for all E-local spectra L.

Hint: use that we have $\pi_n \hom(X, Y) \cong \pi_0 \hom(X, Y[-n]) \cong \pi_0 \mathrm{Hom}_{\mathrm{Sp}}(X, Y[-n])$, and similarly for X'.

Lemma 7.1.6 ('E_*-Whitehead theorem')**.** *Let $f\colon L \to L'$ be a morphism between E-local spectra. Then f is an isomorphism if and only if it is an E-equivalence.*

Proof. By passing to fibers, we may equivalently show that an E-local spectrum L is zero if and only if it is E-acyclic. One direction is clear. For the other direction, assume L is E-acyclic. Then by assumption we have $\hom(L, L) = 0$, hence $\mathrm{Hom}_{\mathrm{Sp}}(L, L) = *$, hence the identity on L is the zero map. This implies that L itself is zero, as desired. □

We will now talk about the universal way of approximating an arbitrary spectrum by an E-local one:

Definition 7.1.7. A morphism $l\colon X \to X'$ of spectra is called an *E-localization* if X' is E-local and the morphism l is an E-equivalence.

Corollary 7.1.8. *Let X' be an E-local spectrum. A morphism $l\colon X \to X'$ is an E-localization if and only if for every E-local spectrum Y the induced map*

$$l^*\colon \mathrm{Hom}_{\mathrm{Sp}}(X', Y) \to \mathrm{Hom}_{\mathrm{Sp}}(X, Y)$$

is an isomorphism.

Proof. This is immediate from Exercise 7.1.5. □

Note that the conclusion of the corollary says that X' is a *left adjoint object* of X under the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$, in the sense of Definition 22.1.3. In particular, the E-localization of X is unique if it exists, and is then denoted by $L_E(X)$.

Theorem 7.1.9 (Bousfield [Bou79, Theorem 1.1])**.** *For every object X, there exists an E-localization $l\colon X \to L_E X$. In particular, the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$ admits a left adjoint*

$$L_E\colon \mathrm{Sp} \to \mathrm{Sp}_E$$

called the E-localization functor.

Proof. Let $\mathrm{Sp}^{E\text{-acyc}} \subseteq \mathrm{Sp}$ be the full subcategory of E-acyclic spectra. It is the kernel of the colimit-preserving functor

$$E \otimes -\colon \mathrm{Sp} \to \mathrm{Sp},$$

so it is presentable and its inclusion $i\colon \mathrm{Sp}^{E\text{-acyc}} \hookrightarrow \mathrm{Sp}$ preserves small colimits by Corollary 23.2.4. This kernel is stable, since $E \otimes -$ is exact. The adjoint functor theorem Theorem 23.2.5 therefore supplies a right adjoint Γ_E to i.

For a spectrum X, define $L_E X$ by the exact sequence

$$i\Gamma_E X \longrightarrow X \longrightarrow L_E X,$$

where the first map is the counit of the adjunction. Let A be an E-acyclic spectrum. Full faithfulness of i and the adjunction give isomorphisms

$$\mathrm{Hom}_{\mathrm{Sp}}(A, i\Gamma_E X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sp}^{E\text{-acyc}}}(A, \Gamma_E X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sp}}(A, X).$$

Their composite is the map induced by the counit. Applying the same argument to every shift of A, which is again E-acyclic, shows that $\hom(A, i\Gamma_E X) \to \hom(A, X)$ is an isomorphism of spectra. Exactness of $\hom(A, -)$ now gives $\hom(A, L_E X) = 0$, so $L_E X$ is E-local by Exercise 7.1.4. On the other hand, $E \otimes i\Gamma_E X = 0$, so applying $E \otimes -$ to the exact sequence shows that $X \to L_E X$ is an E-equivalence. Thus it is an E-localization. □

The proof packages the required size argument into the general theory of presentable ∞-categories. Bousfield's proof uses a transfinite size-reduction argument followed by a colocalization construction. The general notion of a Bousfield localization is discussed in Section 22.7.

7.2 Localization at a prime

In algebra, one commonly studies rings and modules through their *localizations*, in which one inverts certain non-zero elements in the ring. The terminology is inspired by the geometric perspective of this operation: it corresponds to studying the 'local' behavior of a variety/scheme around a point.

In the case of abelian groups, this operation corresponds to inverting collections of prime numbers. Given an abelian group A and a natural number n, there is a multiplication-by-n map $n\colon A \to A$, sending a to the n-fold sum na. Abelian groups for which this map is an isomorphism for certain n are usually easier to study than general abelian groups:

- If multiplication by n is an isomorphism for all $n > 0$, then A acquires a unique structure of a rational vector space, hence is simply a direct sum of copies of $\mathbb{Q}$.
- If for some prime p, multiplication by q is an isomorphism for all primes $q \neq p$, then A is called *p-local*. In particular, A does not admit q-torsion for any $q \neq p$.

More generally, for a set P of primes we write $A[P^{-1}]$ for the universal abelian group under A on which multiplication by every $p \in P$ is invertible. When P consists of all primes, this is the rationalization $A_{\mathbb{Q}}$ of A. When P consists of all primes except a single prime p, it is the p-localization $A_{(p)}$ of A. Some questions about A can be reduced to questions about $A_{\mathbb{Q}}$ or $A_{(p)}$.

The goal of this section is to introduce an analogous theory of localizations for spectra, and to exhibit it as a special case of the general formalism of Bousfield localizations introduced in the previous section.

Notation 7.2.1. Let C be a semiadditive ∞-category. We saw in Proposition 5.3.25 that the hom anima $\mathrm{Hom}_C(X, Y)$ admits a canonical structure of a commutative monoid for all $X, Y \in C$. In particular, there is for any $n \in \mathbb{N}$ a multiplication-by-n map

$$n \cdot -\colon \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_C(X, Y),$$

given explicitly by sending a map $f\colon X \to Y$ to the composite

$$X \xrightarrow{\Delta} \bigoplus_n X \xrightarrow{\oplus_n f} \bigoplus_n Y \xrightarrow{\nabla} Y.$$

For $f = \mathrm{id}_X\colon X \to X$, we refer to this map as *multiplication by n on X*, and denote it by

$$n\colon X \to X.$$

Note that when $C = \mathrm{Ab}$ is the category of abelian groups, this recovers the ordinary multiplication-by-n map.

We record the following compatibility because we will apply it both to spectra and to their homotopy groups.

Exercise 7.2.2. Show that any semiadditive functor $F\colon C \to D$ between semiadditive ∞-categories preserves the multiplication-by-n maps.

We now specialize to spectra. Let $P \subseteq \mathbb{N}$ be a set of primes. We say a spectrum X is *P-divisible* if multiplication by p on X is an isomorphism for every $p \in P$. We write

$$\mathrm{Sp}[P^{-1}] \subseteq \mathrm{Sp}$$

for the full subcategory of P-divisible spectra.

Definition 7.2.3. Consider a spectrum X.

- For $n > 0$, we define $X[\frac{1}{n}]$ as the colimit

$$X[\frac{1}{n}] \;:=\; \mathrm{colim}(X \xrightarrow{n} X \xrightarrow{n} X \xrightarrow{n} \dots) \quad \in \quad \mathrm{Sp}.$$

- If P consists of finitely many primes, we set $N := \prod_{p\in P} p$ and set

$$X[P^{-1}] \;:=\; X[\frac{1}{N}] \quad \in \quad \mathrm{Sp}.$$

- If P consists of infinitely many primes, say $p_1, p_2, p_3, \dots$, we define $X[P^{-1}]$ as the colimit

$$X[P^{-1}] \;:=\; \mathrm{colim}(X \xrightarrow{\cdot p_1} X \xrightarrow{\cdot p_1 p_2} X \xrightarrow{\cdot p_1 p_2 p_3} \dots) \quad \in \quad \mathrm{Sp}.$$

When P consists of all primes, we refer to $X[P^{-1}]$ as the *rationalization of X* and denote it by $X_{\mathbb{Q}}$.

When P consists of all primes except a single prime p, we refer to $X[P^{-1}]$ as the *p-localization of X*, and denote it by $X_{(p)}$.

Note that $X[P^{-1}]$ always comes with a canonical map $X \to X[P^{-1}]$, by including the first entry of the colimit diagram. Moreover, for every spectrum X, the functor $X \otimes -\colon \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits, and so there is an isomorphism

$$X[P^{-1}] \quad \cong \quad X \otimes \mathbb{S}[P^{-1}].$$

Lemma 7.2.4. *Let X be a spectrum. Then there is an isomorphism*

$$\pi_*(X[P^{-1}]) \cong \pi_*(X)[P^{-1}].$$

Proof. In each case of the definition, $X[P^{-1}]$ is a filtered colimit whose transition maps are multiplication by products of primes in P. By Exercise 7.2.2, the functors $\pi_n\colon \mathrm{Sp} \to \mathrm{Ab}$ carry these to the corresponding multiplication maps of abelian groups. Since they preserve filtered colimits by Lemma 4.4.28, and since the resulting colimit of abelian groups has the universal property defining localization, we obtain

$$\pi_*(X[P^{-1}]) \cong \pi_*(X)[P^{-1}]. \qquad \square$$

Corollary 7.2.5. *For every spectrum X, the spectrum $X[P^{-1}]$ is P-divisible.*

Proof. We need to show that for every prime $p \in P$, the map $\cdot p \colon X[P^{-1}] \to X[P^{-1}]$ is an isomorphism of spectra. We may check this on homotopy groups, where it becomes the statement that the map

$$p \colon \pi_*(X)[P^{-1}] \to \pi_*(X)[P^{-1}]$$

is an isomorphism. This holds by elementary properties of localization of abelian groups. □

Recall that an abelian group A is called *P-power torsion* if every element of A is annihilated by a positive integer all of whose prime factors lie in P. It is not difficult to see that A is P-power torsion if and only if $A[P^{-1}] = 0$. Together with Lemma 7.2.4, this implies:

Corollary 7.2.6. *A spectrum X is $\mathbb{S}[P^{-1}]$-acyclic if and only if the homotopy groups $\pi_*(X)$ are P-power torsion.* □

We next show that the assignment $X \mapsto X[P^{-1}]$ is a reflection onto the subcategory of P-divisible spectra.

Proposition 7.2.7. *Let X be a spectrum and let Y be a P-divisible spectrum. Then precomposition with the canonical map $X \to X[P^{-1}]$ induces an isomorphism of mapping spectra*

$$\hom(X[P^{-1}], Y) \xrightarrow{\sim} \hom(X, Y).$$

In particular, the inclusion $\mathrm{Sp}[P^{-1}] \hookrightarrow \mathrm{Sp}$ admits a left adjoint given by

$$(-)[P^{-1}] \colon \mathrm{Sp} \to \mathrm{Sp}[P^{-1}].$$

Proof. The mapping spectrum $\hom(X[P^{-1}], Y)$ is the limit obtained by applying $\hom(-, Y)$ to the defining diagram for $X[P^{-1}]$. The transition maps in this limit are given by precomposition with the corresponding multiplication maps on X. By bilinearity of composition in spectra, these maps agree with postcomposition by the corresponding multiplication maps on Y.

All transition maps are induced by multiplication by products of primes in P on Y, hence are isomorphisms since Y is P-divisible. The limit is therefore isomorphic to the first term $\hom(X, Y)$.

Passing to underlying mapping animae and using Corollary 7.2.5 gives the claim. □

The previous proposition establishes the functor $(-)[P^{-1}] \colon \mathrm{Sp} \to \mathrm{Sp}$ as the reflector onto the P-divisible spectra. We will now identify it with the E-localization functor for $E = \mathbb{S}[P^{-1}]$.

Lemma 7.2.8. *A spectrum X is P-divisible if and only if it is $\mathbb{S}[P^{-1}]$-local.*

Proof. First assume X is P-divisible. To see that X is $\mathbb{S}[P^{-1}]$-local, consider an $\mathbb{S}[P^{-1}]$-acyclic spectrum Y. We must show that $\hom(Y, X) = 0$. This follows from the following sequence of isomorphisms:

$$\hom(Y, X) \cong \hom(Y[P^{-1}], X) \cong \hom(0, X) \cong 0;$$

Here the first isomorphism follows from Proposition 7.2.7, and the second from the observation that $Y[P^{-1}] \cong Y \otimes \mathbb{S}[P^{-1}]$.

Conversely, assume that X is $\mathbb{S}[P^{-1}]$-local. To show that $p\colon X \to X$ is an isomorphism, it suffices by Lemma 7.1.6 to show that the map $p\colon X[P^{-1}] = X \otimes \mathbb{S}[P^{-1}] \to X \otimes \mathbb{S}[P^{-1}] = X[P^{-1}]$ is an isomorphism, which is the content of Corollary 7.2.5. □

Corollary 7.2.9. *For every spectrum X, the map $X \to X[P^{-1}]$ is an $\mathbb{S}[P^{-1}]$-localization of X.*

Proof. By Corollary 7.1.8, a map $X \to Y$ is an $\mathbb{S}[P^{-1}]$-localization if and only if it exhibits Y as a reflection into the subcategory of $\mathbb{S}[P^{-1}]$-local spectra. By Lemma 7.2.8, this subcategory agrees with the P-divisible spectra, and so the claim is an instance of Proposition 7.2.7. □

For $E = \mathbb{S}[P^{-1}]$, the previous corollary implies that the E-localization of X is given by the map $X \cong \mathbb{S} \otimes X \to L_E(\mathbb{S}) \otimes X$. Localizations of this form have a special name:

Definition 7.2.10. The localization functor $L_E\colon \mathrm{Sp} \to \mathrm{Sp}_E$ is called a *smashing localization* if the spectrum $L_E\,\mathbb{S} \otimes X$ is E-local for every spectrum X.

Exercise 7.2.11. Assume L_E is a smashing localization. Construct an isomorphism $L_E(X) \xrightarrow{\sim} L_E(\mathbb{S}) \otimes X$ for every spectrum X.

We next record the expected fact that reducing a spectrum modulo a power of p leaves only p-primary information, in the sense that all other primes act invertibly. The analogous claim for abelian groups is clear, since any prime different from p is invertible modulo p^n. The following proposition extends this claim to spectra by expressing each homotopy group of X/p^n as an extension of two groups killed by p^n, allowing us to reduce to the abelian setting.

Proposition 7.2.12. *Let X be a spectrum, let p be a prime number and let $n \geq 1$. Then the spectrum $X/p^n := \mathrm{cofib}(X \xrightarrow{\cdot p^n} X)$ is p-local.*

Proof. Let q be a prime number different from p. We must show that the map $q\colon X/p^n \to X/p^n$ is an isomorphism, which we may check on homotopy groups.

The long exact sequence on homotopy groups associated to the exact sequence $X \xrightarrow{\cdot p^n} X \to X/p^n$ contains, for every $m \in \mathbb{Z}$, a natural short exact sequence

$$0 \to \pi_m(X)/p^n \to \pi_m(X/p^n) \to \pi_{m-1}(X)[p^n] \to 0.$$

Here $\pi_m(X)/p^n$ denotes the cokernel of multiplication by p^n, and $\pi_{m-1}(X)[p^n]$ denotes the kernel of multiplication by p^n. By Exercise 7.2.2, homotopy groups carry multiplication by q on spectra to multiplication by q on abelian groups, so this natural short exact sequence is compatible with multiplication by q.

Now multiplication by q is an isomorphism on every abelian group killed by p^n: choose integers a, b with $aq + bp^n = 1$, and then multiplication by a is inverse to multiplication by q. It follows that multiplication by q is an isomorphism on the two outer terms of the short exact sequence above. By the short five lemma, it is therefore also an isomorphism on $\pi_m(X/p^n)$.

Since this holds for all m, multiplication by q is an isomorphism of spectra. As this is true for every prime $q \neq p$, the spectrum X/p^n is p-local. □

Remark 7.2.13. The previous argument shows more generally that X/N is P-divisible whenever $\gcd(q, N) = 1$ for every $q \in P$.

Theorem 7.2.14 (Serre)**.** *The rational sphere $\mathbb{S}_{\mathbb{Q}}$ is isomorphic to the Eilenberg–MacLane spectrum $\mathbb{Q}$ of the rational numbers.*

Proof. Consider the map $\mathbb{S} \to \mathbb{Q}$ corresponding to $1 \in \mathbb{Q} \cong \pi_0(\mathbb{Q})$. All primes act invertibly on the homotopy groups of $\mathbb{Q}$, so $\mathbb{Q}$ is $\mathbb{S}_{\mathbb{Q}}$-local by Lemma 7.2.8, and this map therefore uniquely refines to a map $\mathbb{S}_{\mathbb{Q}} \to \mathbb{Q}$. We have to show that this map induces isomorphisms on homotopy groups. By Lemma 7.2.4, the induced map on π_0 is the isomorphism $\mathbb{Q} \cong \pi_0(\mathbb{S}_{\mathbb{Q}}) \to \pi_0(\mathbb{Q}) \cong \mathbb{Q}$ which sends 1 to 1. It thus remains to show that $\pi_n(\mathbb{S}_{\mathbb{Q}}) = 0$ for all $n \neq 0$. For $n > 0$, the stable homotopy group $\pi_n(\mathbb{S}) = \pi_n^{\mathrm{st}}$ is finite by Serre's finiteness theorem [Ser51, Chapter V, Section 3, Proposition 3] and Freudenthal suspension, while for $n < 0$ it vanishes since $\mathbb{S}$ is connective. In either case it is killed by rationalization. □

Definition 7.2.15. We write $\mathrm{Sp}_{\mathbb{Q}} \subseteq \mathrm{Sp}$ for the full subcategory of *rational* spectra, i.e. the $\mathbb{S}_{\mathbb{Q}}$-local spectra, or equivalently the spectra on which all primes act invertibly. By Serre's theorem, these are also precisely the $\mathbb{Q}$-local spectra.

Proposition 7.2.16. *For every rational spectrum X, regarding its homotopy groups as Eilenberg–MacLane spectra, there is a noncanonical isomorphism*

$$\bigoplus_{n \in \mathbb{Z}} (\pi_n X)[n] \xrightarrow{\sim} X.$$

Proof. For every integer $n \in \mathbb{Z}$, we pick a basis of $\pi_n(X)$ as a rational vector space. An element of $\pi_n(X)$ corresponds to a map of spectra $\mathbb{S}[n] \to X$. Since X is rational, this uniquely extends to a map $\mathbb{Q}[n] = \mathbb{S}_{\mathbb{Q}}[n] \to X$. By taking a direct sum over all basis vectors, this induces a map $(\pi_n X)[n] \cong \bigoplus \mathbb{Q}[n] \to X$ which by construction induces the identity on $\pi_n(X)$. We now take the direct sum over all $n \in \mathbb{Z}$. Direct sums of spectra are filtered colimits of finite biproducts, so Lemma 4.4.28 shows that the resulting map $\bigoplus_{n \in \mathbb{Z}} (\pi_n X)[n] \to X$ induces an isomorphism on every homotopy group. It is therefore an isomorphism. □

7.3 Completion at a prime

Beyond localizations, there is another important operation in algebra, known as *p-completion* for a prime p. The process of p-completion is in some sense orthogonal to the process of inverting p. For example, if A is a finitely generated abelian group and $A^\wedge_p := \lim_n A/p^n A$ is its p-completion, then A can be recovered from its p-completion and its p-inversion via the *arithmetic fracture square*:[15] a pullback square in Ab of the form

$$\begin{array}{ccc} A & \longrightarrow & A^\wedge_p \\ \downarrow & \lrcorner & \downarrow \\ A[\tfrac{1}{p}] & \longrightarrow & A^\wedge_p[\tfrac{1}{p}]. \end{array}$$

The goal of this section is to construct an analogous p-completion operation on spectra, show that it is an instance of a Bousfield localization at some spectrum, compute the homotopy groups of a completion, and establish a spectral form of the arithmetic fracture square.

[15] The finite-generation hypothesis is essential here: for arbitrary abelian groups, the corresponding statement requires derived p-completion; see Section 7.3.2.

7.3.1 Completion as a Bousfield localization

Definition 7.3.1. Consider the *mod p Moore spectrum* $\mathbb{S}/p$, defined as the cofiber

$$\mathbb{S}/p \quad := \quad \mathrm{cofib}(p\colon \mathbb{S} \to \mathbb{S}).$$

We refer to an $\mathbb{S}/p$-local spectrum as *p-complete*. We write

$$\mathrm{Sp}_p^\wedge \quad \subseteq \quad \mathrm{Sp}$$

for the full subcategory of p-complete spectra, and denote the resulting localization functor by

$$(-)_p^\wedge := L_{\mathbb{S}/p}\colon \mathrm{Sp} \to \mathrm{Sp}_p^\wedge.$$

Observation 7.3.2. Note that $X \otimes \mathbb{S}/p \simeq X/p$ for every X. In particular, a spectrum X is $\mathbb{S}/p$-acyclic if and only if the map $p\colon X \to X$ is an isomorphism, i.e. X is $\mathbb{S}[\frac{1}{p}]$-local by Lemma 7.2.8.

Warning 7.3.3. An $\mathbb{S}[\frac{1}{p}]$-local spectrum should not be confused with a p-local spectrum: in the former, multiplication by p is an isomorphism; in the latter, this holds for every prime other than p. Moreover, unlike the prime-inversion localizations of the previous section, p-completion is not smashing in general.

We would like to get a more explicit description of p-completeness that makes its connection with p-completion in classical algebra clear.

Definition 7.3.4. We define the spectrum $\mathbb{S}/p^\infty$ as the cofiber of the map $\mathbb{S} \to \mathbb{S}[\frac{1}{p}]$:

$$\mathbb{S}/p^\infty \quad := \quad \mathrm{cofib}(\mathbb{S} \to \mathbb{S}[\frac{1}{p}] = \mathrm{colim}(\mathbb{S} \xrightarrow{\cdot p} \mathbb{S} \xrightarrow{\cdot p} \mathbb{S} \xrightarrow{\cdot p} \dots)).$$

By writing $\mathbb{S}$ as the colimit over the identity maps $\mathbb{S} \xrightarrow{\mathrm{id}} \mathbb{S}$, we may alternatively write $\mathbb{S}/p^\infty$ as the colimit over the bottom row of the following commutative diagram, where all vertical sequences are cofiber sequences:

$$\begin{array}{ccccccc}
\mathbb{S} & = & \mathbb{S} & = & \mathbb{S} & = & \dots \\
{\scriptstyle \cdot p}\downarrow & & \downarrow{\scriptstyle \cdot p^2} & & \downarrow{\scriptstyle \cdot p^3} & & \\
\mathbb{S} & \xrightarrow{\cdot p} & \mathbb{S} & \xrightarrow{\cdot p} & \mathbb{S} & \xrightarrow{\cdot p} & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{S}/p & \xrightarrow{\cdot p} & \mathbb{S}/p^2 & \xrightarrow{\cdot p} & \mathbb{S}/p^3 & \xrightarrow{\cdot p} & \dots.
\end{array}$$

(This also justifies the notation $\mathbb{S}/p^\infty$.)

Theorem 7.3.5. *For a spectrum X, the map*

$$X = \hom(\mathbb{S}, X) \to \hom(\mathbb{S}/p^\infty[-1], X)$$

induced by the map $\mathbb{S}/p^\infty[-1] \to \mathbb{S}$ exhibits its target as a p-completion of X, i.e.

$$X_p^\wedge \xrightarrow{\sim} \hom(\mathbb{S}/p^\infty[-1], X) \xrightarrow{\sim} \hom(\mathbb{S}/p^\infty, X)[1].$$

Proof. We have to show that $\hom(\mathbb{S}/p^\infty[-1], X)$ is p-complete and that the map $X \to \hom(\mathbb{S}/p^\infty[-1], X)$ is an $\mathbb{S}/p$-equivalence. For the latter, we may equivalently show that its fiber is $\mathbb{S}/p$-acyclic, or equivalently that its fiber is $\mathbb{S}[\frac{1}{p}]$-local by Observation 7.3.2. Because of the exact sequence $\mathbb{S}/p^\infty[-1] \to \mathbb{S} \to \mathbb{S}[\frac{1}{p}]$, this fiber is isomorphic to $\hom(\mathbb{S}[\frac{1}{p}], X)$. But this is clearly $\mathbb{S}[\frac{1}{p}]$-local: for any $\mathbb{S}[\frac{1}{p}]$-acyclic spectrum Y we have $\hom(Y, \hom(\mathbb{S}[\frac{1}{p}], X)) \simeq \hom(Y \otimes \mathbb{S}[\frac{1}{p}], X) \simeq \hom(0, X) = 0$.

We will now show that $\hom(\mathbb{S}/p^\infty[-1], X)$ is p-complete, i.e. $\mathbb{S}/p$-local. To this end, let A be an $\mathbb{S}/p$-acyclic spectrum, i.e. the map $p\colon A \to A$ is an isomorphism. We need to show that $0 = \hom(A, \hom(\mathbb{S}/p^\infty[-1], X)) \simeq \hom(A \otimes \mathbb{S}/p^\infty[-1], X)$. For this, it will suffice to show that $A \otimes \mathbb{S}/p^\infty = 0$. Since we may write $\mathbb{S}/p^\infty$ as a colimit of $\mathbb{S}/p^n$, we may similarly write

$$A \otimes \mathbb{S}/p^\infty \simeq \operatorname{colim}_n(A \otimes \mathbb{S}/p^n) \simeq \operatorname{colim}_n(A/p^n).$$

But as A is $\mathbb{S}/p$-acyclic, the map $p\colon A \to A$ and hence also $p^n\colon A \to A$ is an isomorphism, and so $A/p^n = 0$. This shows that $A \otimes \mathbb{S}/p^\infty \simeq 0$, as desired. □

As a corollary of the theorem, we may now describe the p-completion of X as a familiar limit:

Corollary 7.3.6. *For a spectrum X, its p-completion is given by the following limit:*

$$X^\wedge_p \quad \simeq \quad \lim(\cdots \to X/p^3 \to X/p^2 \to X/p).$$

Proof. The exact sequence $\mathbb{S}/p^n[-1] \to \mathbb{S} \xrightarrow{\cdot p^n} \mathbb{S}$ induces an exact sequence

$$\hom(\mathbb{S}, X) \xrightarrow{\cdot p^n} \hom(\mathbb{S}, X) \to \hom(\mathbb{S}/p^n[-1], X).$$

Since $\hom(\mathbb{S}, X) \simeq X$, it follows that $\hom(\mathbb{S}/p^n[-1], X) \simeq X/p^n$ for all n. Furthermore, under these isomorphisms the maps $\mathbb{S}/p^n \xrightarrow{\cdot p} \mathbb{S}/p^{n+1}$ induce the canonical maps $X/p^{n+1} \to X/p^n$, as these are the ones induced on vertical cofibers in the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\cdot p} & X \\ {\scriptstyle \cdot p^{n+1}}\downarrow & & \downarrow{\scriptstyle \cdot p^n} \\ X & = & X. \end{array}$$

Using Theorem 7.3.5, we may then compute that

$$\begin{aligned} X^\wedge_p &\simeq \hom(\mathbb{S}/p^\infty[-1], X) \simeq \hom(\operatorname{colim}_n \mathbb{S}/p^n[-1], X) \\ &\simeq \lim_n \hom(\mathbb{S}/p^n[-1], X) \simeq \lim_n X/p^n, \end{aligned}$$

as desired. □

Corollary 7.3.7. *Let p be a prime number. Then every p-complete spectrum is p-local.*

Proof. Since X is p-complete, the localization map $X \to X^\wedge_p$ is an $\mathbb{S}/p$-equivalence between $\mathbb{S}/p$-local spectra, hence is an isomorphism by Lemma 7.1.6. By Corollary 7.3.6, we may therefore write $X \cong \lim_n(X/p^n)$. Each X/p^n is p-local by Proposition 7.2.12. By Lemma 7.2.8, p-local spectra are $\mathbb{S}[P^{-1}]$-local for P the set of all primes different from p, so they are closed under limits by Exercise 7.1.4. □

Corollary 7.3.8. *Let p and q be two distinct primes. If X is both p-complete and q-complete, then X is the zero spectrum.*

Proof. The q-completeness of X means that X is $\mathbb{S}/q$-local. The p-completeness of X implies, via Corollary 7.3.7, that the map $q\colon X \to X$ is an isomorphism, hence that X is $\mathbb{S}/q$-acyclic. It then follows from Lemma 7.1.6 that $X = 0$. □

7.3.2 Homotopy groups of a p-completion

The inverse limit formula $X^\wedge_p \simeq \lim_k X/p^k$ does not immediately compute the homotopy groups of the p-completion, since homotopy groups need not preserve inverse limits. Instead, we use the mapping spectrum formula of Theorem 7.3.5. The telescope construction will give a resolution of $\mathbb{S}/p^\infty$ by direct sums of spheres which realizes a free resolution of the Prüfer group $\mathbb{Z}/p^\infty := \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. After applying $\hom(-, X)$, the resulting long exact sequence computes $\pi_*(X^\wedge_p)$ in terms of Hom and Ext.

Accordingly, for an abelian group A we write

$$L^p_0 A := \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A) \qquad \text{and} \qquad L^p_1 A := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$$

for its two *derived p-completion groups*. Throughout this subsection we write $\mathbb{Z}_p := \lim_k \mathbb{Z}/p^k$, with respect to the quotient maps, for the ring of p-adic integers. We first record a useful calculation.

Lemma 7.3.9. *Let A be an abelian group, and write $A[p^k] \subseteq A$ for the subgroup of elements annihilated by p^k. Then*

$$L^p_1 A \cong \lim_k A[p^k],$$
$$0 \longrightarrow \lim{}^1_k A[p^k] \longrightarrow L^p_0 A \longrightarrow \lim_k A/p^k A \longrightarrow 0,$$

where the transition maps on $A[p^k]$ are multiplication by p. If A is finitely generated, then $L^p_1 A = 0$ and $L^p_0 A \cong A \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Proof. Write $\mathbb{Z}/p^\infty \cong \mathrm{colim}_k \mathbb{Z}/p^k$ for the filtered colimit along the multiplication-by-p injections $\mathbb{Z}/p^k \xrightarrow{\cdot p} \mathbb{Z}/p^{k+1}$. Since filtered colimits are exact in Ab, the colimit in $\mathrm{D}(\mathbb{Z})$ of a filtered diagram of abelian groups concentrated in degree zero is again concentrated in degree zero, with degree-zero homology given by the colimit in Ab. Thus the displayed colimit also computes the colimit in $\mathrm{D}(\mathbb{Z})$. Mapping out of it gives

$$\hom_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}/p^\infty, A) \simeq \lim_k \hom_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}/p^k, A).$$

For every $k \geq 1$, the standard resolution of $\mathbb{Z}/p^k$ identifies

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^k, A) \cong A[p^k] \qquad \text{and} \qquad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^k, A) \cong A/p^k A.$$

Under these identifications, the transition maps are multiplication by p on the first tower and the quotient maps on the second. The two claims now follow by applying the Milnor sequence of Corollary 4.4.37 in degrees 0 and -1.

If A is finitely generated, its p-primary torsion is bounded, so the transition maps into every fixed stage of the first tower are eventually zero. It follows directly from the description in Definition 4.4.36 that both its limit and its first derived limit vanish. Finally, $\lim_k A/p^k A \cong A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for every finitely generated abelian group A. □

Theorem 7.3.10 (Homotopy groups of a p-completion)**.** *Let p be a prime and let X be a spectrum. Then for every $n \in \mathbb{Z}$ there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_n X) \to \pi_n(X^\wedge_p) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \to 0.$$

Proof. Let $F := \bigoplus_{k\geq 0} \mathbb{S}$, with summand inclusions $\iota_k \colon \mathbb{S} \to F$. Splicing the telescope sequence for $\mathbb{S}[\frac{1}{p}]$ from Lemma 4.4.34 with the defining exact sequence $\mathbb{S} \to \mathbb{S}[\frac{1}{p}] \to \mathbb{S}/p^\infty$, and then reindexing the source, gives an exact sequence

$$F \xrightarrow{d} F \longrightarrow \mathbb{S}/p^\infty,$$

where $d \circ \iota_0 = \iota_0$ and $d \circ \iota_{k+1} = \iota_k - p\iota_{k+1}$ for $k \geq 0$. On π_0 this realizes the free resolution

$$0 \longrightarrow \bigoplus_{k\geq 0} \mathbb{Z} \xrightarrow{d_*} \bigoplus_{k\geq 0} \mathbb{Z} \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0,$$

in which the k-th basis vector of the middle group maps to $p^{-k} + \mathbb{Z}$. Thus the exact sequence of spectra is obtained by realizing this free resolution by direct sums of spheres.

Apply $\hom(-, X)$ to the exact sequence above. For every $m \in \mathbb{Z}$, the universal property of the coproduct identifies

$$\pi_m \hom(F, X) \cong \mathrm{Hom}_{\mathbb{Z}}\Big(\bigoplus_{k\geq 0} \mathbb{Z}, \pi_m X\Big),$$

and the map induced by d agrees with precomposition by d_*. By Theorem 7.3.5, we have $X^\wedge_p \simeq \hom(\mathbb{S}/p^\infty, X)[1]$, so that $\pi_n(X^\wedge_p) \cong \pi_{n-1} \hom(\mathbb{S}/p^\infty, X)$. Write d^*_m for the endomorphism induced by d^* on $\pi_m \hom(F, X)$. The long exact sequence associated to the exact sequence $\hom(\mathbb{S}/p^\infty, X) \to \hom(F, X) \xrightarrow{d^*} \hom(F, X)$ then gives a short exact sequence

$$0 \to \mathrm{coker}(d^*_n) \to \pi_n(X^\wedge_p) \to \ker(d^*_{n-1}) \to 0.$$

By Proposition 6.4.16, its outer terms are

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_n X) = L^p_0(\pi_n X) \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{n-1} X) = L^p_1(\pi_{n-1} X),$$

respectively. This gives the asserted short exact sequence. □

Corollary 7.3.11. *Let X be a spectrum and let $n \in \mathbb{Z}$ be such that $\pi_n(X)$ and $\pi_{n-1}(X)$ are finitely generated. Then there is an isomorphism*

$$\pi_n(X^\wedge_p) \;\cong\; \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p\,.$$

Proof. By Lemma 7.3.9, the right-hand term of the short exact sequence of Theorem 7.3.10 vanishes, while its left-hand term is $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. □

Example 7.3.12. All homotopy groups of the sphere spectrum are finitely generated: this is $\pi_0(\mathbb{S}) = \mathbb{Z}$ together with the finiteness theorem of Serre [Ser51, Chapter V, Section 3, Proposition 3] used in the proof of Theorem 7.2.14. Applying Corollary 7.3.11, we obtain $\pi_0(\mathbb{S}^\wedge_p) \cong \mathbb{Z}_p$, while for $n \neq 0$ the group $\pi_n(\mathbb{S}^\wedge_p) \cong \pi_n(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is the p-primary part of the stable stem $\pi_n(\mathbb{S})$. Completing at p thus discards precisely the prime-to-p information in the stable stems.

Warning 7.3.13. The group $L_1^p A$ detects compatible systems of p-power torsion, and it need not vanish. For example, let $X := (\mathbb{Z}/p^\infty)$. Since the Prüfer group is injective, $L_0^p(\mathbb{Z}/p^\infty) = 0$, whereas Lemma 7.3.9 gives $L_1^p(\mathbb{Z}/p^\infty) \cong \mathbb{Z}_p$. Hence

$$X_p^\wedge \simeq \mathbb{Z}_p[1].$$

This example in particular shows that completion is not computed degreewise on homotopy groups: the p-completion of a nonzero spectrum concentrated in degree 0 may be concentrated in degree 1.

7.3.3 The arithmetic fracture square

We now return to the interaction between p-completion and inversion of p, and prove the spectral form of the arithmetic fracture square announced at the start of this section.

Lemma 7.3.14. *Let P be a set of primes and let X be a spectrum satisfying $X[P^{-1}] = 0$ and $X/p = 0$ for every $p \in P$. Then $X = 0$.*

Proof. For every $p \in P$, the condition $X/p = 0$ means that multiplication by p on X is an isomorphism. Thus X is P-divisible, hence $\mathbb{S}[P^{-1}]$-local by Lemma 7.2.8. On the other hand, the isomorphism $X[P^{-1}] \cong X \otimes \mathbb{S}[P^{-1}]$ shows that X is $\mathbb{S}[P^{-1}]$-acyclic. It follows from Lemma 7.1.6 that $X = 0$. □

Theorem 7.3.15 (The p-adic arithmetic fracture square)**.** *Let X be a spectrum. Then the following commutative square is a pullback square:*

$$\begin{array}{ccc} X & \longrightarrow & X_p^\wedge \\ \downarrow & \lrcorner & \downarrow \\ X[\tfrac{1}{p}] & \longrightarrow & X_p^\wedge[\tfrac{1}{p}]. \end{array}$$

Proof. We have to show that the map $X \to X[\frac{1}{p}] \times_{X_p^\wedge[\frac{1}{p}]} X_p^\wedge$ is an isomorphism, or equivalently that its fiber F is the zero spectrum. Note that $F[\frac{1}{p}] = 0$, since the square

$$\begin{array}{ccc} X[\tfrac{1}{p}] & \longrightarrow & X_p^\wedge[\tfrac{1}{p}] \\ \downarrow & & \downarrow \\ X[\tfrac{1}{p}][\tfrac{1}{p}] & \longrightarrow & X_p^\wedge[\tfrac{1}{p}][\tfrac{1}{p}] \end{array}$$

is a pullback square (as the two vertical maps are isomorphisms). We claim that also $F/p = 0$, so that $F = 0$ by Lemma 7.3.14, applied with $P = \{p\}$. To see this, we need to show that the commutative square

$$\begin{array}{ccc} X/p & \longrightarrow & X_p^\wedge/p \\ \downarrow & & \downarrow \\ X[\tfrac{1}{p}]/p & \longrightarrow & X_p^\wedge[\tfrac{1}{p}]/p \end{array}$$

is a pullback square. The two objects at the bottom are zero, since the maps $p\colon X[\frac{1}{p}] \to X[\frac{1}{p}]$ and $p\colon X_p^\wedge[\frac{1}{p}] \to X_p^\wedge[\frac{1}{p}]$ are isomorphisms by Corollary 7.2.5. In particular the bottom map is an isomorphism. But the top map is also an isomorphism, since the map $X \to X_p^\wedge$ is by definition an $\mathbb{S}/p$-equivalence. This finishes the proof. □

Exercise 7.3.16 (Arithmetic fracture square)**.** Show that for every spectrum X, the commutative squares

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^\wedge \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & \prod_p X_{(p)} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_{(p)}\right)_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} X_{(p)} & \longrightarrow & X_p^\wedge \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(X_p^\wedge\right)_{\mathbb{Q}} \end{array}$$

are pullback squares. In the first two squares, the products are taken over all prime numbers p; the third square is asserted for every prime p.

Hint: in each case, apply Lemma 7.3.14 to the fiber of the comparison map, taking P to be the set of all primes.

Exercises

Exercise 7.1. Let E and F be spectra. Show that the following two conditions are equivalent:

(1) Every E-acyclic spectrum X is also F-acyclic;

(2) Every F-local spectrum Y is also E-local.

Remark. Two spectra E and F are said to *have the same Bousfield class* if a spectrum X is E-acyclic if and only if it is F-acyclic. By the previous exercise, E and F have the same Bousfield class if and only if the subcategories $\mathrm{Sp}_E \subseteq \mathrm{Sp}$ and $\mathrm{Sp}_F \subseteq \mathrm{Sp}$ agree.

Exercise 7.2. Let $X \in \mathrm{Sp}_{\mathbb{Q}}$ be a rational spectrum. Assume that X is p-complete for some prime p. Show that X is the zero spectrum.

Exercise 7.3. Let E, F and X be spectra. Is it always true that there is an isomorphism $L_E L_F X \cong L_F L_E X$? If yes, provide a proof. If no, provide a counterexample.

Exercise 7.4 (Arithmetic localization of Eilenberg–MacLane spectra)**.** Let A be an abelian group and let P be a set of primes. Regard A as a spectrum. Prove that its localization satisfies

$$A[P^{-1}] \cong \mathbb{Z}[P^{-1}] \otimes_{\mathbb{Z}}^{\heartsuit} A.$$

Here the right-hand side is the ordinary tensor product of abelian groups, regarded as a spectrum.

Exercise 7.5 (Completion of basic Eilenberg–MacLane spectra)**.** Fix a prime p and an integer $r \geq 1$. Compute the homotopy groups of $\mathbb{Z}_p^\wedge$ and $(\mathbb{Z}/p^r)_p^\wedge$. Deduce isomorphisms

$$\mathbb{Z}_p^\wedge \cong \mathbb{Z}_p \qquad \text{and} \qquad (\mathbb{Z}/p^r)_p^\wedge \cong \mathbb{Z}/p^r.$$

Exercise 7.6 (Reconstructing the integers)**.** Write $\mathbb{Q}_p := \mathbb{Z}_p[1/p]$. Determine the p-adic arithmetic fracture square of $\mathbb{Z}$ and show that it is the Eilenberg–MacLane image of the pullback square of abelian groups

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}_p \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Z}[1/p] & \longrightarrow & \mathbb{Q}_p\,. \end{array}$$

Verify the pullback assertion directly inside $\mathbb{Q}_p$.

8 Ring spectra and modules

Ordinary cohomology carries more structure than its groups alone: the *cup product* makes $H^*(X;\mathbb{Z})$ into a graded ring. Complex K-theory, introduced in Chapter 9, has a product induced by tensor product of vector bundles. These products can be constructed from multiplications on the representing spectra. Given a multiplication $R \otimes R \to R$, the diagonal of X induces products

$$R^p(X) \otimes R^q(X) \longrightarrow R^{p+q}(X).$$

For the Eilenberg–MacLane spectrum $\mathbb{Z}$, this recovers the ordinary cup product.

To develop algebra over R, we need more than a multiplication which is unital and associative up to homotopy: the identifications expressing these properties must be accompanied by higher coherences. A spectrum with such a coherently unital and associative multiplication is called an *associative ring spectrum*; requiring coherent commutativity gives a *commutative ring spectrum*. We can then form ∞-categories of modules, take relative tensor products, and extend scalars along morphisms of ring spectra.

Since the sphere spectrum $\mathbb{S}$ is the unit for the tensor product of spectra, it inherits a unique commutative ring structure, and its modules are spectra themselves. We will also regard ordinary rings as ring spectra concentrated in degree 0, using the same letters for both. This extends our convention for abelian groups; we justify it in Corollary 8.2.3. For a commutative ring A, we will prove a symmetric monoidal equivalence

$$\mathrm{D}(A) \simeq \mathrm{Mod}_A$$

in Corollary 8.2.4, identifying the derived ∞-category from Chapter 6 with the ∞-category of A-module spectra. Under this equivalence, relative tensor products and mapping spectra recover classical Tor and Ext (Corollary 8.2.5). Thus module theory over ring spectra extends the homological algebra developed earlier in the book.

We begin in Section 8.1 with algebras, modules, and perfect modules. In Section 8.2, we construct Eilenberg–MacLane ring spectra and identify their modules with derived categories. For a connective ring spectrum R, we then construct the Postnikov t-structure on its module category and identify its heart with the ordinary modules over $\pi_0(R)$ (Section 8.3). Finally, Section 8.4 treats localization of commutative ring spectra. In particular, we construct the localization obtained by inverting a homogeneous element of $\pi_*(R)$; in Chapter 9, we use this to produce periodic complex K-theory by inverting the Bott class.

The coherent algebra used here is developed in Part II using ∞-operads. We collect the constructions and properties needed for this chapter in Proposition 8.1.5, which we take as given, so that no prior familiarity with ∞-operads is required. The online version also includes treatments of Ore localization, projective and flat modules, and Tor-amplitude.

8.1 Algebras and modules

The notions of algebras and modules make sense in every symmetric monoidal ∞-category. We first recall some examples of symmetric monoidal structures, and then state the results from Part II that we will use throughout the chapter.

Definition 8.1.1. A *symmetric monoidal ∞-category* is a commutative monoid in Cat_∞. In particular, it is an ∞-category C which comes equipped with a *tensor product functor*

$$- \otimes - \colon C \times C \to C$$

and a *monoidal unit* $\mathbb{1} \in C$, and the tensor product is coherently unital, associative and commutative. We denote this by $(C, \otimes, \mathbb{1})$, or simply C. Given symmetric monoidal ∞-categories C and D, a *symmetric monoidal functor* $F \colon C \to D$ is a morphism in $\mathrm{CMon}(\mathrm{Cat}_\infty)$. We write $\mathrm{Map}^{\otimes}(C, D)$ for the hom anima in $\mathrm{CMon}(\mathrm{Cat}_\infty)$.

Every classical symmetric monoidal category gives an example: a commutative monoid in the full subcategory $\mathrm{Cat}_1 \subseteq \mathrm{Cat}_\infty$ of 1-categories is precisely a symmetric monoidal category, and the inclusion preserves products.

We write $\mathrm{Fun}^{\otimes}(C, D)$ for the ∞-category of symmetric monoidal functors and monoidal natural transformations; its core is $\mathrm{Map}^{\otimes}(C, D)$. We will also use the more flexible notion of a *lax symmetric monoidal functor* $F \colon C \to D$. Concretely, such a functor is equipped with coherent structure maps

$$\mathbb{1}_D \longrightarrow F(\mathbb{1}_C), \qquad F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$$

compatible with the unit, associativity and symmetry. A lax symmetric monoidal functor is symmetric monoidal precisely when these structure maps are isomorphisms. We denote the ∞-category of lax symmetric monoidal functors by $\mathrm{Fun}^{\otimes\text{-lax}}(C, D)$. The formal constructions of these functor categories are given in Definitions 14.5.1 and 14.4.3.

Example 8.1.2. The following are examples of symmetric monoidal ∞-categories:

(1) The category Ab of abelian groups is symmetric monoidal under the ordinary tensor product, with unit $\mathbb{Z}$.

(2) The ∞-category Sp of spectra admits a symmetric monoidal structure $\otimes$ satisfying the property that for every spectrum X, the functor $X \otimes - \colon \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits. Furthermore, Sp is *universal* with this structure: if D is another symmetric monoidal stable ∞-category which admits colimits and in which the tensor product preserves colimits in both variables, then the ∞-category of colimit-preserving symmetric monoidal functors $\mathrm{Sp} \to D$ is contractible. We construct the symmetric monoidal structure in Chapter 16, and discuss its universal property in Section 18.2.

(3) Let R be a commutative ring. The derived ∞-category $\mathrm{D}(R)$ from Chapter 6 admits a symmetric monoidal structure with unit $R[0]$ and tensor product given by the derived tensor product of chain complexes. We construct it in Section 21.2.

(4) An ∞-category C with finite products admits the *cartesian monoidal structure* $(C, \times, *)$. Dually, finite coproducts give the *cocartesian monoidal structure* $(C, \amalg, \emptyset)$. These structures are constructed in Sections 15.2 and 15.3.

(5) If C is an ∞-category and D is symmetric monoidal, then $\mathrm{Fun}(C, D)$ admits the *pointwise symmetric monoidal structure*, with tensor product and unit computed pointwise.

(6) Let $(C, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category and let $D \subseteq C$ be a full subcategory which contains the monoidal unit and is closed under tensor products. Then D inherits a symmetric monoidal structure for which the inclusion $D \hookrightarrow C$ is symmetric monoidal. See Lemma 14.7.1 for the construction.

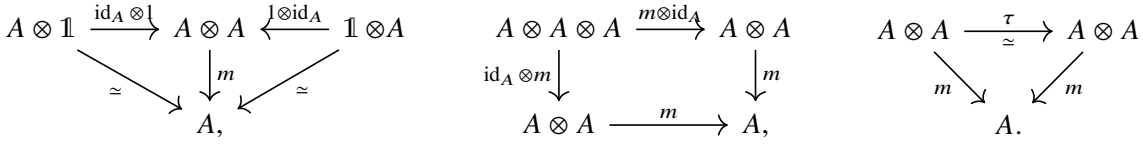
(7) Similar to (2), the ∞-categories An_*, $\mathrm{CMon}(\mathrm{An})$ and $\mathrm{CGrp}(\mathrm{An})$ admit universal symmetric monoidal structures for pointed, semiadditive, and additive ∞-categories. In particular, the functors

$$\mathrm{An} \xrightarrow{(-)_+} \mathrm{An}_* \to \mathrm{CMon}(\mathrm{An}) \xrightarrow{(-)^{\mathrm{grp}}} \mathrm{CGrp}(\mathrm{An}) \xhookrightarrow{\mathbf{B}^\infty} \mathrm{Sp}$$

are colimit-preserving and symmetric monoidal; see Proposition 16.6.4.

(8) For small symmetric monoidal C and cocomplete symmetric monoidal D whose tensor product preserves colimits separately in both variables, $\mathrm{Fun}(C, D)$ also admits the *Day convolution* structure; see Chapter 16.

Recall from classical algebra that a *commutative algebra* in a symmetric monoidal category C is an object A of C that comes equipped with a unit $1\colon \mathbb{1} \to A$ and a multiplication map $m\colon A \otimes A \to A$ such that the following diagrams commute:



In a symmetric monoidal ∞-category, these relations become identifications that must be accompanied by higher coherence data. The following definition encodes all of this data at once:

Definition 8.1.3 (Commutative algebra)**.** Let $(C, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category. We define a *commutative algebra in C* to be a symmetric monoidal functor

$$A\colon \mathrm{Fin} \to C,$$

where we equip the category Fin of finite sets with the cocartesian monoidal structure. We refer to the object $A(*)$ in C as the *underlying object of A*. We will frequently abuse notation and simply write A for $A(*)$.

We define the ∞-category of commutative algebras in C as

$$\mathrm{CAlg}(C) \quad := \quad \mathrm{Fun}^{\otimes}(\mathrm{Fin}, C).$$

Remark 8.1.4. For the cartesian monoidal structure on an ∞-category C with finite products, commutative algebras are precisely commutative monoids: $\mathrm{CAlg}(C, \times) \simeq \mathrm{CMon}(C)$ by Corollary 15.3.14.

For a finite set I, the unique map $I \to *$ induces a multiplication map

$$A(*)^{\otimes I} \simeq A(I) \longrightarrow A(*).$$

The empty set gives the unit, bijections give symmetry, and composition of maps of finite sets expresses compatibility of iterated multiplication. Functoriality includes the higher coherences between these identifications.

There are analogous coherent notions of associative algebras and of modules. We collect the part of their theory needed in this chapter.

Proposition 8.1.5 (Algebras and modules)**.** *Let C be a cocomplete symmetric monoidal ∞-category whose tensor product preserves colimits separately in both variables.*

(1) *There are ∞-categories $\mathrm{Alg}(C)$ and $\mathrm{CAlg}(C)$ of associative and commutative algebras in C. Their objects have underlying objects of C equipped with coherently unital and associative multiplication maps; for commutative algebras the multiplication is coherently commutative as well.*

(2) *For every $A \in \mathrm{Alg}(C)$ there are ∞-categories $\mathrm{LMod}_A(C)$ and $\mathrm{RMod}_A(C)$ of left and right A-modules. The forgetful functors to C detect isomorphisms, create every limit admitted by C and all colimits, and the free left A-module on $X \in C$ has underlying object $A \otimes X$. The forgetful functor $\mathrm{LMod}_{\mathbb{1}}(C) \to C$ is an equivalence. If C is stable, then $\mathrm{LMod}_A(C)$ and $\mathrm{RMod}_A(C)$ are stable.*

(3) *There is a relative tensor product*

$$- \otimes_A - \colon \mathrm{RMod}_A(C) \times \mathrm{LMod}_A(C) \longrightarrow C,$$

which preserves colimits separately in both variables. It is unital: for every left A-module M and right A-module N there are natural isomorphisms

$$A \otimes_A M \xrightarrow{\sim} M, \qquad N \otimes_A A \xrightarrow{\sim} N.$$

Commuting actions on the factors induce actions on the relative tensor product. In particular, if N also has a left B-action commuting with its right A-action, then $N \otimes_A M$ is a left B-module. Relative tensor products are coherently associative: for a right B-module L there is a natural isomorphism $(L \otimes_B N) \otimes_A M \simeq L \otimes_B (N \otimes_A M)$.

(4) *Every morphism $f \colon A \to B$ in $\mathrm{Alg}(C)$ induces a restriction functor*

$$f_* \colon \mathrm{LMod}_B(C) \longrightarrow \mathrm{LMod}_A(C),$$

which admits the extension-of-scalars functor

$$f^* := B \otimes_A - \colon \mathrm{LMod}_A(C) \longrightarrow \mathrm{LMod}_B(C)$$

as a left adjoint. There is an analogous adjunction for right modules, whose left adjoint is $- \otimes_A B$.

(5) *If $R \in \mathrm{CAlg}(C)$, then left and right R-modules agree, and the resulting ∞-category $\mathrm{Mod}_R(C)$ is symmetric monoidal under $- \otimes_R -$. For a morphism $f \colon R \to S$ in $\mathrm{CAlg}(C)$, the extension-of-scalars functor $f^* \colon \mathrm{Mod}_R(C) \to \mathrm{Mod}_S(C)$ is symmetric monoidal; for $R = \mathbb{1}$, this is the free module functor $- \otimes S \colon C \to \mathrm{Mod}_S(C)$. There are natural equivalences*

$$\mathrm{CAlg}(\mathrm{Mod}_R(C)) \simeq \mathrm{CAlg}(C)_{R/}.$$

(6) *Lax symmetric monoidal functors preserve algebra objects and their modules. In particular, a symmetric monoidal adjunction* $F\colon C \rightleftarrows D : G$ *induces adjunctions on algebras and on their module categories. If* $A \in \mathrm{Alg}(C)$ *and the unit* $A \to GF(A)$ *is an isomorphism, this gives an adjunction*

$$\mathrm{LMod}_A(C) \rightleftarrows \mathrm{LMod}_{F(A)}(D)$$

whose underlying functors are induced by F *and* G*. Right adjoints of symmetric monoidal functors are canonically lax symmetric monoidal.*

(7) *If* C *is presentable, then* $\mathrm{LMod}_A(C)$ *and* $\mathrm{RMod}_A(C)$ *are presentable for every* $A \in \mathrm{Alg}(C)$.

Proof. Algebras and modules are constructed in Section 17.1, and their limits and colimits are treated in Proposition 17.1.16. For relative tensor products and change of scalars, see Section 17.2 and Proposition 17.2.14. The symmetric monoidal structure for a commutative base and its properties are given in Theorem 17.2.16, Corollary 17.2.21, and Proposition 17.2.17. Finally, the assertions about adjunctions and presentability are Propositions 14.5.6 and 17.1.17. □

Definition 8.1.6 (Ring spectra and modules)**.** An *associative ring spectrum* is an associative algebra in $(\mathrm{Sp}, \otimes, \mathbb{S})$, and a *commutative ring spectrum* is a commutative algebra in Sp. Every algebra A is canonically a left and right module over itself via its multiplication $A \otimes A \to A$; this is called the *regular A-module*. We write

$$\mathrm{Alg} := \mathrm{Alg}(\mathrm{Sp}), \qquad \mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$$

for their ∞-categories. For an associative ring spectrum R, we abbreviate

$$\mathrm{LMod}_R := \mathrm{LMod}_R(\mathrm{Sp}), \qquad \mathrm{RMod}_R := \mathrm{RMod}_R(\mathrm{Sp}).$$

If R is commutative, we write $\mathrm{Mod}_R := \mathrm{Mod}_R(\mathrm{Sp})$ for its symmetric monoidal ∞-category of modules.

Unless commutativity is specified, a ring spectrum will mean an associative ring spectrum.

Example 8.1.7. The sphere spectrum $\mathbb{S}$ is a commutative ring spectrum, and $\mathrm{Mod}_{\mathbb{S}} \simeq \mathrm{Sp}$. Every ordinary ring A gives an associative ring spectrum, again denoted by A, which is commutative if A is commutative. We construct these Eilenberg–MacLane ring spectra in Section 8.2.

For every associative ring spectrum R, the ∞-category LMod_R is stable. When R is commutative, Mod_R also has a symmetric monoidal structure given by the relative tensor product over R.

Observation 8.1.8 (Graded homotopy rings)**.** The homotopy groups of an associative ring spectrum R form a graded ring. For $x \in \pi_p(R)$ and $y \in \pi_q(R)$, the product $xy \in \pi_{p+q}(R)$ is represented by

$$\mathbb{S}[p+q] \simeq \mathbb{S}[p] \otimes \mathbb{S}[q] \xrightarrow{x \otimes y} R \otimes R \xrightarrow{m} R.$$

The same construction using the action map $R \otimes M \to M$ makes $\pi_*(M)$ a graded left $\pi_*(R)$-module for every left R-module M, and similarly for right modules.

If R is commutative, then $\pi_*(R)$ is graded-commutative:

$$xy = (-1)^{pq} yx.$$

Indeed, interchanging $\mathbb{S}[p]$ and $\mathbb{S}[q]$ acts by $(-1)^{pq}$ on $\mathbb{S}[p+q]$. For nonnegative p, q, this is the degree of the permutation interchanging the two blocks of suspension coordinates; the general case follows by invertibility of suspension. See also Warning 4.2.11 and Observation 16.6.7.

Construction 8.1.9 (Cup products)**.** Let R be an associative ring spectrum and X an anima. The diagonal of X induces a map

$$\Delta\colon\ \mathbb{S}[X] \longrightarrow \mathbb{S}[X \times X] \simeq \mathbb{S}[X] \otimes \mathbb{S}[X].$$

For cohomology classes $x \in R^p(X)$ and $y \in R^q(X)$ represented by maps $x\colon\ \mathbb{S}[X] \to R[p]$ and $y\colon\ \mathbb{S}[X] \to R[q]$, the *cup product* $x \smile y \in R^{p+q}(X)$ is represented by

$$\mathbb{S}[X] \xrightarrow{\Delta} \mathbb{S}[X] \otimes \mathbb{S}[X] \xrightarrow{x \otimes y} R[p] \otimes R[q] \simeq (R \otimes R)[p+q] \xrightarrow{m[p+q]} R[p+q].$$

This gives $R^*(X)$ the structure of a graded ring, natural in X, with unit represented by $\mathbb{S}[X] \to \mathbb{S} \to R$. Associativity follows from associativity of m and coassociativity of the diagonal. If R is commutative, the same symmetry calculation as above gives $x \smile y = (-1)^{pq} y \smile x$. For a topological space, we apply the construction to its underlying anima.

The regular module generates an important finiteness class.

Definition 8.1.10 (Perfect module)**.** Let R be an associative ring spectrum. We denote by

$$\mathrm{Perf}(R) \quad \subseteq \quad \mathrm{LMod}_R$$

the smallest thick subcategory containing the regular left R-module R, where a full subcategory is called *thick* if it is stable and closed under retracts. A left R-module is called *perfect* if it belongs to $\mathrm{Perf}(R)$.

Thus a perfect module is obtained from finitely many shifts of R by iterating finite cofiber sequences and retracts. The compact-generation results from Chapter 23 identify this intrinsic finite construction with compactness.

Lemma 8.1.11 (Perfect modules and compact objects, [Lur17, Proposition 7.2.4.2])**.** *Let R be an associative ring spectrum.*

(1) *The regular left R-module R is compact.*

(2) *The ∞-category LMod_R is compactly generated by R.*

(3) *A left R-module is perfect if and only if it is compact.*

Proof. The functor $\mathrm{Hom}_{\mathrm{LMod}_R}(R, -)$ is naturally isomorphic to the composite

$$\mathrm{LMod}_R \xrightarrow{\mathrm{fgt}} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathrm{An},$$

which preserves filtered colimits. Thus R is compact. Its shifts jointly detect isomorphisms, since

$$\pi_n(M) \cong \pi_0 \mathrm{Hom}_{\mathrm{LMod}_R}(R[n], M).$$

Hence R is a compact generator of LMod_R. The characterization of compact objects in a compactly generated stable ∞-category from Proposition 23.3.4 now identifies the compact objects with the thick subcategory generated by R, which is $\mathrm{Perf}(R)$. □

8.2 Eilenberg–MacLane ring spectra and DGAs

This section explains how ordinary rings and differential graded algebras give rise to ring spectra. We begin with the Eilenberg–MacLane functor, using the symmetric monoidality of π_0 to obtain the required multiplicative structure.

Lemma 8.2.1. *The sphere spectrum is connective, and the tensor product of two connective spectra is connective. Thus* $\mathrm{Sp}_{\geq 0}$ *inherits a symmetric monoidal structure for which the inclusion into* Sp *is symmetric monoidal.*

Proof. By Example 8.1.2(7), infinite delooping $\mathbf{B}^{\infty}\colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{Sp}$ is symmetric monoidal. Its essential image is $\mathrm{Sp}_{\geq 0}$ by Theorem 5.4.6, so this subcategory contains the monoidal unit and is closed under tensor products. □

Proposition 8.2.2. *The functor* $\pi_0\colon \mathrm{Sp}_{\geq 0} \to \mathrm{Ab}$ *is symmetric monoidal, where* Ab *is equipped with the usual tensor product* $- \otimes^{\heartsuit} -$.

Proof. Under the symmetric monoidal equivalence $\mathrm{Sp}_{\geq 0} \simeq \mathrm{CGrp}(\mathrm{An})$ established in Part II (Corollary 16.6.5), this functor corresponds to the symmetric monoidal functor $\pi_0\colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{Set}) = \mathrm{Ab}$ of Proposition 16.4.5. □

Corollary 8.2.3. *The Eilenberg–MacLane functor* $H\colon \mathrm{Ab} \longrightarrow \mathrm{Sp}_{\geq 0}$ *is canonically lax symmetric monoidal, hence induces functors*

$$H\colon \mathrm{Alg}(\mathrm{Ab}) \longrightarrow \mathrm{Alg}(\mathrm{Sp}) \qquad \textit{and} \qquad H\colon \mathrm{CAlg}(\mathrm{Ab}) \longrightarrow \mathrm{CAlg}(\mathrm{Sp}).$$

Both functors are fully faithful, with essential image the ring spectra concentrated in degree 0*.*

Proof. Since H is right adjoint to the symmetric monoidal functor $\pi_0\colon \mathrm{Sp}_{\geq 0} \to \mathrm{Ab}$, it follows from Proposition 14.5.6 that H is canonically lax symmetric monoidal. Every lax symmetric monoidal functor preserves associative and commutative algebra objects, as does the symmetric monoidal inclusion $\mathrm{Sp}_{\geq 0} \hookrightarrow \mathrm{Sp}$. The induced adjunctions on algebras have counit an isomorphism, since $\pi_0 H \simeq \mathrm{id}$. Their units are isomorphisms precisely on algebras whose underlying spectra are concentrated in degree 0. This proves the assertions about full faithfulness and essential image. □

We will henceforth regard ordinary associative and commutative rings as ring spectra concentrated in degree 0, usually omitting H from the notation. Likewise, the lax symmetric monoidal functor H allows us to regard ordinary modules as module spectra concentrated in degree 0, denoted by the same letters. Examples include the commutative ring spectra $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{F}_2$. Ordinary tensor products will be denoted by $\otimes^{\heartsuit}$ to distinguish them from tensor products of spectra.

We next identify modules over Eilenberg–MacLane ring spectra with objects of the derived ∞-category. Let R be a commutative ring. The derived tensor product equips $\mathrm{D}(R)$ with a symmetric monoidal structure

$$- \otimes^{\mathbf{L}}_{R} -\colon\ \mathrm{D}(R) \times \mathrm{D}(R) \longrightarrow \mathrm{D}(R)$$

whose monoidal unit is $R[0]$. The ∞-category $\mathrm{D}(R)$ is stable and cocomplete, and the derived tensor product preserves colimits separately in both variables. This structure is constructed in Proposition 21.2.2, together with a lax symmetric monoidal refinement of the localization functor

$\mathrm{Ch}(R) \to \mathrm{D}(R)$. Moreover, $\mathrm{D}(R)$ is presentable and its unit $R[0]$ is a compact generator by Corollary 21.2.6 and Proposition 23.3.6.

The universal property of the symmetric monoidal structure on spectra recalled in Example 8.1.2 now produces a unique colimit-preserving symmetric monoidal functor

$$L_R \colon \mathrm{Sp} \longrightarrow \mathrm{D}(R).$$

It sends the sphere spectrum to the monoidal unit $R[0]$. Since $\mathrm{D}(R)$ and Sp are presentable, Theorem 23.2.5 gives a right adjoint to L_R. The universal enrichment of stable ∞-categories in spectra identifies this right adjoint as

$$H_R \colon \mathrm{D}(R) \longrightarrow \mathrm{Sp}, \qquad H_R(A) = \hom_{\mathrm{D}(R)}(R[0], A).$$

By Proposition 14.5.6, the right adjoint H_R has a canonical lax symmetric monoidal structure. It consequently induces functors

$$\mathrm{Alg}(\mathrm{D}(R)) \longrightarrow \mathrm{Alg}(\mathrm{Sp}) \qquad \text{and} \qquad \mathrm{CAlg}(\mathrm{D}(R)) \longrightarrow \mathrm{CAlg}(\mathrm{Sp}).$$

For $R = \mathbb{Z}$, the underlying functor of H_R is the Eilenberg–MacLane functor from Definition 6.2.1, since both are defined by the formula $A \mapsto \hom_{\mathrm{D}(\mathbb{Z})}(\mathbb{Z}[0], A)$.

The following identification is an application of symmetric monoidal Morita theory from Part II (Corollary 18.3.5). It identifies a presentably symmetric monoidal stable ∞-category whose unit is a compact generator with modules over the endomorphism ring spectrum of that unit.

Corollary 8.2.4 (Eilenberg–MacLane modules)**.** *For every commutative ring R, there is a symmetric monoidal equivalence*

$$\mathrm{D}(R) \xrightarrow{\simeq} \mathrm{Mod}_R(\mathrm{Sp}).$$

Under this equivalence, the functor $H_R = \hom_{\mathrm{D}(R)}(R[0], -)$ agrees with the forgetful functor $\mathrm{Mod}_R \to \mathrm{Sp}$, and the lax symmetric monoidal structure on H_R sends the unit $R[0]$ to the Eilenberg–MacLane ring spectrum R from Corollary 8.2.3.

Proof. Equip $\mathrm{D}(R)$ with the symmetric monoidal structure of Proposition 21.2.2. Its unit $R[0]$ is a compact generator by Corollary 21.2.6. Moreover,

$$\pi_n \hom_{\mathrm{D}(R)}(R[0], R[0]) \cong H_n(R[0])$$

by Corollary 6.4.5. This is R for $n = 0$ and zero otherwise. Thus the resulting commutative endomorphism ring spectrum is concentrated in degree 0. By Corollary 8.2.3, it is therefore determined by its ordinary commutative ring π_0. The multiplication on π_0 of the endomorphism ring spectrum is composition of endomorphisms of $R[0]$, hence the ordinary multiplication of R. It is therefore the Eilenberg–MacLane ring spectrum R. The result now follows from Corollary 18.3.5. The underlying spectrum of the module assigned to A by this equivalence is $\hom_{\mathrm{D}(R)}(R[0], A) = H_R(A)$. □

Corollary 8.2.5. *Let R be a commutative ring and let M and N be R-modules, regarded as complexes concentrated in degree 0. Write $\mathrm{Tor}_n^R(M, N)$ and $\mathrm{Ext}_R^n(M, N)$ for their classical Tor- and Ext-groups. Regard M and N also as discrete R-module spectra. For $n \geq 0$, the equivalence of Corollary 8.2.4 gives natural isomorphisms*

$$\mathrm{Tor}_n^R(M, N) \cong \pi_n(M \otimes_R N) \qquad \textit{and} \qquad \mathrm{Ext}_R^n(M, N) \cong \pi_{-n} \hom_R(M, N).$$

Proof. The equivalence $\mathrm{D}(R) \simeq \mathrm{Mod}_R$ preserves tensor products and mapping spectra. Thus $M \otimes_R N$ corresponds to $M[0] \otimes^{\mathbf{L}}_R N[0]$, and $\hom_R(M,N) \simeq \hom_{\mathrm{D}(R)}(M[0], N[0])$. The resolution formulas of Corollary 6.6.28 and Proposition 6.4.16 identify the indicated homotopy groups with classical Tor and Ext. □

We now turn to differential graded algebras. These give concrete examples of associative and commutative algebra objects in $\mathrm{D}(R)$, and hence, by applying H_R, of ring spectra.

Definition 8.2.6 (Differential graded algebra)**.** Let R be a commutative ring. An *associative differential graded R-algebra*, or *associative DGA over R*, is an associative algebra object in the symmetric monoidal category $\mathrm{Ch}(R)$ of chain complexes of R-modules. Concretely, it consists of a chain complex $A_\bullet$, a unit $R[0] \to A_\bullet$, and an associative multiplication

$$A_p \otimes_R A_q \longrightarrow A_{p+q}, \qquad x \otimes y \longmapsto xy,$$

which satisfies the Leibniz rule

$$d(xy) = d(x)y + (-1)^p x d(y)$$

for homogeneous $x \in A_p$ and $y \in A_q$.

A *commutative DGA over R* is a commutative algebra object in $\mathrm{Ch}(R)$. Equivalently, it is an associative DGA whose multiplication is graded commutative:

$$xy = (-1)^{pq} yx$$

for homogeneous $x \in A_p$ and $y \in A_q$.

Example 8.2.7. Every associative R-algebra A defines an associative DGA $A[0]$ concentrated in degree 0. If A is commutative, then $A[0]$ is a commutative DGA. As an example with non-zero differential, let $a \in R$ and consider the Koszul DGA

$$K_R(a) := (\Lambda_R(e), d), \qquad |e| = 1, \quad d(e) = a.$$

Here $\Lambda_R(e)$ denotes the exterior algebra on one generator, so this is a commutative DGA in homological degrees 1 and 0. Its homology is $H_0(K_R(a)) = R/(a)$ and $H_1(K_R(a)) = \mathrm{ann}_R(a)$. If a is a non-zero-divisor, the augmentation $K_R(a) \to R/(a)$ is therefore a quasi-isomorphism.

In Part II we show that the localization functor $\gamma\colon \mathrm{Ch}(R) \to \mathrm{D}(R)$ admits a lax symmetric monoidal refinement (Corollary 21.2.3). Since lax symmetric monoidal functors preserve algebra objects, it induces functors

$$\mathrm{Alg}(\mathrm{Ch}(R)) \longrightarrow \mathrm{Alg}(\mathrm{D}(R)) \qquad \text{and} \qquad \mathrm{CAlg}(\mathrm{Ch}(R)) \longrightarrow \mathrm{CAlg}(\mathrm{D}(R)).$$

Thus every associative or commutative DGA determines a corresponding derived algebra. The first functor is essentially surjective [Lur17, Proposition 7.1.4.6], and the second is essentially surjective when R is a $\mathbb{Q}$-algebra [Lur17, Proposition 7.1.4.11]. The commutative assertion does not hold over arbitrary base rings: strict commutative DGAs do not in general represent all commutative algebra objects of $\mathrm{D}(R)$.

Definition 8.2.8 (Eilenberg–MacLane ring spectrum)**.** Let A be an associative DGA over a commutative ring R. Its *Eilenberg–MacLane ring spectrum* is the associative ring spectrum

$$HA := H_R(\gamma A) \in \mathrm{Alg}(\mathrm{Sp}).$$

If A is a commutative DGA, the lax symmetric monoidal structure on $H_R \circ \gamma$ refines HA to a commutative ring spectrum. By Corollary 6.4.5, there are natural isomorphisms

$$\pi_n(HA) \cong H_n(A).$$

Under these identifications, the product of classes represented by cycles x and y is represented by the cycle xy. Thus $\pi_*(HA) \cong H_*(A)$ as graded rings.

Remark 8.2.9 (Agreement of the Eilenberg–MacLane constructions)**.** For an ordinary associative R-algebra A, the ring spectrum $H_R(A[0])$ is canonically isomorphic to the direct Eilenberg–MacLane ring spectrum HA from Corollary 8.2.3. Indeed, both are concentrated in degree 0, and their multiplications induce the ordinary multiplication of A on π_0. The classification of associative ring spectra concentrated in degree 0 from Corollary 8.2.3 therefore identifies them. If A is commutative, the same argument gives an isomorphism of commutative ring spectra.

8.3 Connective modules

For a connective ring spectrum R, the Postnikov truncations of an R-module inherit R-module structures. We will show that these define a t-structure on LMod_R, whose heart is the ordinary category of $\pi_0(R)$-modules. We first explain how such t-structures arise in a general symmetric monoidal setting.

Definition 8.3.1 (Connectivity-preserving symmetric monoidal structure)**.** Let C be a stably symmetric monoidal ∞-category, meaning that C is stable and its tensor product is exact separately in both variables, and equip C with a t-structure. We say that the symmetric monoidal structure on C is *connectivity-preserving* if the following two conditions are satisfied:

(1) The monoidal unit $\mathbb{1}$ of C is connective;

(2) For two connective objects $X, Y \in C_{\geq 0}$, the tensor product $X \otimes Y$ is again connective.

This condition is often expressed by saying that the t-structure is *compatible with the symmetric monoidal structure*. If the symmetric monoidal structure is connectivity-preserving, then $C_{\geq 0}$ inherits a symmetric monoidal structure making the inclusion $C_{\geq 0} \hookrightarrow C$ into a symmetric monoidal functor; this is proved in Part II in Lemma 14.7.1. As a consequence, the truncation functor $\tau_{\geq 0} \colon C \to C_{\geq 0}$ is canonically lax symmetric monoidal (see Proposition 14.5.6).

Definition 8.3.2. Let C be a stably symmetric monoidal ∞-category equipped with a t-structure for which the symmetric monoidal structure is connectivity-preserving. Let $R \in \mathrm{Alg}(C_{\geq 0})$ be a connective associative algebra in C. We say that a left R-module M is *connective* if its underlying object in C is connective, and similarly for coconnective modules. This defines full subcategories

$$\mathrm{LMod}_R(C)_{\geq 0}, \quad \mathrm{LMod}_R(C)_{\leq 0} \quad \subseteq \quad \mathrm{LMod}_R(C).$$

Lemma 8.3.3. *In the situation of the preceding definition, the pair*

$$\big(\mathrm{LMod}_R(C)_{\geq 0}, \mathrm{LMod}_R(C)_{\leq 0}\big)$$

defines a t-structure on $\mathrm{LMod}_R(C)$*. Its truncation functors are computed on underlying objects.*

Proof. The adjunction between the symmetric monoidal inclusion $C_{\geq 0} \hookrightarrow C$ and its lax symmetric monoidal right adjoint $\tau_{\geq 0}$ induces, after passing to algebras and modules and taking the fiber over R, an adjunction

$$\mathrm{LMod}_R(C_{\geq 0}) \rightleftarrows \mathrm{LMod}_R(C).$$

Its right adjoint is computed on underlying objects by $\tau_{\geq 0}$. Finite limits and colimits of modules are computed on underlying objects by Proposition 17.1.16. Since the forgetful functor detects isomorphisms, $\mathrm{LMod}_R(C)$ is stable. Closure under the required shifts follows from the corresponding property in C, so it remains to check orthogonality and decomposition. If X is connective and Y is coconnective, adjunction gives

$$\mathrm{Hom}_{\mathrm{LMod}_R(C)}(X, Y[-1]) \simeq \mathrm{Hom}_{\mathrm{LMod}_R(C_{\geq 0})}(X, \tau_{\geq 0}(Y[-1])) \simeq *.$$

Finally, for an arbitrary R-module X, the counit $\tau_{\geq 0}X \to X$ in modules has connective source, and its cofiber belongs to $\mathrm{LMod}_R(C)_{\leq -1}$ because this is true after forgetting to C. This gives the required decomposition. □

By Lemma 8.2.1, the symmetric monoidal structure on Sp is connectivity-preserving. We can therefore apply the preceding lemma to its Postnikov t-structure.

Corollary 8.3.4. *Let R be a connective associative ring spectrum. Then* LMod_R *admits a t-structure given by the connective and coconnective R-modules. This t-structure is both left and right complete, and every left R-module M satisfies*

$$M \simeq \mathrm{colim}(\tau_{\geq 0}M \to \tau_{\geq -1}M \to \dots) \qquad \text{and} \qquad M \simeq \lim(\dots \to \tau_{\leq 1}M \to \tau_{\leq 0}M).$$

The same statements hold for RMod_R*.*

Proof. The t-structure exists by Lemmas 8.2.1 and 8.3.3. Truncations are computed on underlying spectra, and the forgetful functor preserves limits and colimits and detects isomorphisms. Thus the two displayed formulas follow from the corresponding Postnikov formulas for spectra (Proposition 6.3.4 and Lemma 6.3.31), proving left and right completeness. □

Observation 8.3.5. Let R be a connective associative ring spectrum. Taking π_0 induces a colimit-preserving functor

$$\pi_0 \colon \mathrm{LMod}_{R,\geq 0} \longrightarrow \mathrm{LMod}_{\pi_0(R)}(\mathrm{Ab}).$$

Moreover, if M is a connective right R-module and N a connective left R-module, then $M \otimes_R N$ is connective, and the canonical map

$$\pi_0(M) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \longrightarrow \pi_0(M \otimes_R N), \qquad [x] \otimes [y] \longmapsto [x \otimes y],$$

is an isomorphism.

Indeed, the symmetric monoidal functors

$$\mathrm{Sp} \longleftarrow \mathrm{Sp}_{\geq 0} \xrightarrow{\pi_0} \mathrm{Ab}$$

preserve colimits. They therefore induce functors on algebras and modules and preserve relative tensor products by Proposition 17.2.12. The fully faithful inclusion on the left identifies modules in $\mathrm{Sp}_{\geq 0}$ with connective module spectra. Finally, colimits of modules are computed on underlying objects, so the induced π_0-functor preserves colimits as well. The analogous statement holds for right modules.

The following cellular description of connective modules will also identify the heart of the module t-structure.

Proposition 8.3.6 ([Lur17, Proposition 7.1.1.13(1)])**.** *Let R be a connective associative ring spectrum. Then the ∞-category $\mathrm{LMod}_{R,\geq 0}$ is the smallest subcategory of LMod_R which contains R and which is closed under colimits. The analogous statement holds for connective right R-modules.*

Proof. The connective modules contain R and are closed under colimits. Conversely, for a connective module M, we construct a sequence $M(0) \to M(1) \to \cdots$ over M by successively killing the homotopy groups of the fibers; we enforce this by requiring the fiber $K(i) := \mathrm{fib}(M(i) \to M)$ to be i-connective for each $i \geq 0$.

Choose $M(0) \to M$ from a coproduct of copies of R so that it is surjective on π_0; then $K(0)$ is connective. Given $M(i) \to M$, choose a map $g\colon F[i] \to K(i)$ which is surjective on π_i, where F is a coproduct of copies of R. Write $\widetilde{g}$ for its composite with $K(i) \to M(i)$, and form the pushout

$$\begin{array}{ccc} F[i] & \xrightarrow{\widetilde{g}} & M(i) \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & M(i+1). \end{array}$$

The nullhomotopy from the fiber sequence gives a map $M(i+1) \to M$. The resulting fiber sequence

$$F[i] \longrightarrow K(i) \longrightarrow K(i+1)$$

and surjectivity on π_i imply that $K(i+1)$ is $(i+1)$-connective.

Every $M(i)$ belongs to the subcategory generated by R under colimits, since nonnegative shifts and cofibers are colimits. The fiber of $\mathrm{colim}_i M(i) \to M$ is $\mathrm{colim}_i K(i)$, since filtered colimits are exact. For each j, the groups $\pi_j K(i)$ vanish for $i > j$, so Lemma 4.4.28 implies that this fiber is zero. Thus $M \simeq \mathrm{colim}_i M(i)$, as required. The same construction works for right modules. □

Lemma 8.3.7. *Let R be a connective associative ring spectrum. Then the functor π_0 from Observation 8.3.5 restricts to equivalences*

$$\mathrm{LMod}_R^{\heartsuit} \xrightarrow{\sim} \mathrm{LMod}_{\pi_0(R)}(\mathrm{Ab}), \qquad \mathrm{RMod}_R^{\heartsuit} \xrightarrow{\sim} \mathrm{RMod}_{\pi_0(R)}(\mathrm{Ab})$$

between the hearts of the module categories and the abelian categories of discrete modules over $\pi_0(R)$.

Proof. We prove the statement for left modules; the proof for right modules is identical. For essential surjectivity, write a discrete $\pi_0(R)$-module M as a coequalizer of free modules. The two maps between the free modules lift to maps between coproducts of copies of R, since homotopy classes of maps out of R are classified by elements of π_0. Form their coequalizer $\widetilde{M}$ in $\mathrm{LMod}_{R,\geq 0}$ and then apply $\tau_{\leq 0}$. Since π_0 preserves colimits of connective modules, we have $\pi_0(\widetilde{M}) \cong M$, and hence $\tau_{\leq 0}\widetilde{M}$ is a discrete R-module whose image under π_0 is M.

We will now show that the functor is fully faithful. Let $N \in \mathrm{LMod}_R^{\heartsuit}$, and consider the subcategory $C \subseteq \mathrm{LMod}_{R,\geq 0}$ spanned by those left R-modules M such that the map

$$\pi_0 \operatorname{Hom}_{\mathrm{LMod}_R}(M, N) \to \operatorname{Hom}_{\mathrm{LMod}_{\pi_0(R)}(\mathrm{Ab})}(\pi_0(M), \pi_0(N))$$

is an isomorphism. The subcategory C contains R. For connective M and discrete N, the hom anima $\operatorname{Hom}_{\mathrm{LMod}_R}(M, N)$ is discrete by t-structure orthogonality. Hence its π_0 carries colimits in M to limits of sets. The right-hand side does so as well, since π_0 preserves colimits of connective modules. Thus C is closed under colimits. It follows from Proposition 8.3.6 that $C = \mathrm{LMod}_{R,\geq 0}$, as desired. □

Notation 8.3.8. Let R be a discrete associative ring. We will from now on denote the categories $\mathrm{LMod}_R(\mathrm{Ab})$ and $\mathrm{RMod}_R(\mathrm{Ab})$ of discrete left and right R-modules by $\mathrm{LMod}_R^{\heartsuit}$ and $\mathrm{RMod}_R^{\heartsuit}$, respectively. This notation is justified by Lemma 8.3.7: these categories agree with the hearts of LMod_R and RMod_R, regarding R as an Eilenberg–MacLane ring spectrum.

Tensor products in Ab will from now on always be denoted by $\otimes^{\heartsuit}$; we reserve $\otimes$ for the tensor product in Sp.

Remark 8.3.9. For an arbitrary associative ring spectrum R, [Lur17, Proposition 1.4.4.11] gives a t-structure on LMod_R whose connective part is the smallest full subcategory containing R and closed under colimits and extensions. Its coconnective objects are precisely the modules whose underlying spectra are coconnective: orthogonality to $R[n]$ for $n > 0$ is equivalent to vanishing of the positive homotopy groups. If R is not connective, the connective objects of this t-structure need not have connective underlying spectra; R itself is an example.

8.4 Localizations of ring spectra

In classical algebra, given a commutative ring R and a multiplicative subset $S \subseteq R$, one can form the localization $R[S^{-1}]$, a new ring in which every element of S becomes invertible. We now develop the analogous construction for commutative ring spectra. This is the form needed in Chapter 9, where periodic complex K-theory is obtained by inverting the Bott element in connective complex K-theory.

We have already met a special case in Section 7.2: inverting a set of primes P on the sphere spectrum gives a mapping telescope $\mathbb{S}[P^{-1}]$, and tensoring with this telescope is the corresponding reflection. The same two features persist here. Localization is smashing, and localization at a single homogeneous element is computed by a telescope.

The online version also treats localization of associative ring spectra at multiplicative subsets satisfying an Ore condition.

8.4.1 The commutative case

Let R be a commutative ring spectrum. Its homotopy groups form a graded-commutative ring, and the homotopy groups of an R-module form a graded module over it.

Definition 8.4.1. Let $S \subseteq \pi_*(R)$ be a subset of homogeneous elements. We say that S is *multiplicative* if it contains the unit $1 \in \pi_0(R)$ and is closed under multiplication.

For a homogeneous element $s \in \pi_d(R)$, multiplication by s is already defined at the level of R-modules:

Observation 8.4.2. Choose a representative $s\colon R[d] \to R$ of an element of $\pi_d(R)$. Multiplication by s is the natural transformation of functors $\mathrm{Mod}_R \to \mathrm{Mod}_R$ given by

$$s\colon M[d] \simeq R[d] \otimes_R M \xrightarrow{s\otimes \mathrm{id}} R \otimes_R M \simeq M.$$

On homotopy groups this is the map $x \mapsto s \cdot x$.

Definition 8.4.3. Let $S \subseteq \pi_*(R)$ be a multiplicative subset of homogeneous elements. An R-module N is *S-local* if multiplication by every $s \in S$ is an isomorphism

$$s\colon N[|s|] \xrightarrow{\sim} N.$$

This condition is independent of the chosen representatives of the elements of S. Equivalently, multiplication by every $s \in S$ is an isomorphism on the graded group $\pi_*(N)$. We write $\mathrm{Mod}_R^{\mathrm{Loc}(S)} \subseteq \mathrm{Mod}_R$ for the full subcategory of S-local modules.

Proposition 8.4.4 (Commutative localization is smashing)**.** *Let R be a commutative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset of homogeneous elements.*

(1) *The inclusion $\mathrm{Mod}_R^{\mathrm{Loc}(S)} \hookrightarrow \mathrm{Mod}_R$ admits a left adjoint*

$$S^{-1}(-)\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R^{\mathrm{Loc}(S)}.$$

The subcategory $\mathrm{Mod}_R^{\mathrm{Loc}(S)}$ admits a symmetric monoidal structure with tensor product $\otimes_R$ and unit $S^{-1}R$. The localization functor is symmetric monoidal and the inclusion is lax symmetric monoidal.

(2) *The localization is* smashing*: for every $M \in \mathrm{Mod}_R$, the map*

$$M \simeq M \otimes_R R \xrightarrow{\mathrm{id}_M \otimes \eta_R} M \otimes_R S^{-1}R$$

exhibits $M \otimes_R S^{-1}R$ as the S-localization of M. In particular,

$$S^{-1}M \simeq M \otimes_R S^{-1}R$$

naturally in M.

Proof. For every $s \in S$ of degree d and every $n \in \mathbb{Z}$, consider the shifted multiplication map

$$s[n] \colon R[d+n] \longrightarrow R[n].$$

An R-module is local with respect to this small collection of morphisms precisely when it is S-local: allowing all shifts means that locality tests multiplication by s on every homotopy group. Since Mod_R is presentable by Proposition 17.1.17, Theorem 23.2.2(4) provides the asserted reflection $S^{-1}(-)$.

The symmetric monoidal ∞-category Mod_R is closed, as recalled after Corollary 17.2.19. If N is S-local and P is any R-module, then $\underline{\mathrm{Hom}}_R(P, N)$ is again S-local. Indeed, under the natural isomorphism

$$\underline{\mathrm{Hom}}_R(P, N)[d] \xrightarrow{\sim} \underline{\mathrm{Hom}}_R(P, N[d]),$$

multiplication by s is obtained by applying $\underline{\mathrm{Hom}}_R(P, -)$ to the isomorphism $s \colon N[d] \xrightarrow{\sim} N$. The internal-hom criterion Lemma 14.7.3 therefore gives the symmetric monoidal structure and the symmetric monoidality of the localization.

It remains to prove the smashing formula. The unit $\eta_R \colon R \to S^{-1}R$ is an S-local equivalence, and the proof of the internal-hom criterion shows that S-local equivalences remain so after tensoring with any R-module. Consequently,

$$\mathrm{id}_M \otimes \eta_R \colon M \longrightarrow M \otimes_R S^{-1}R$$

is an S-local equivalence. Its target is S-local: multiplication by s on $M \otimes_R S^{-1}R$ is obtained by tensoring id_M with the isomorphism $s \colon (S^{-1}R)[|s|] \xrightarrow{\sim} S^{-1}R$. A local equivalence from M to a local object exhibits that object as the reflection of M, which proves the claim. □

We can now localize in commutative ring spectra.

Corollary 8.4.5 (Localization of a commutative ring spectrum)**.** *In the situation of Proposition 8.4.4, the R-module $S^{-1}R$ carries a canonical structure of a commutative ring spectrum $R[S^{-1}]$, and the localization map refines to a morphism*

$$\eta \colon R \longrightarrow R[S^{-1}]$$

in $\mathrm{CAlg}(\mathrm{Sp})$. *It has the following universal property: for every commutative ring spectrum A, precomposition with η induces a monomorphism of animae*

$$- \circ \eta \colon \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Sp})}(R[S^{-1}], A) \hookrightarrow \mathrm{Hom}_{\mathrm{CAlg}(\mathrm{Sp})}(R, A)$$

whose image consists of those maps $f \colon R \to A$ for which $\pi_*(f)$ carries every element of S to a unit of $\pi_*(A)$.

Proof. By Proposition 8.4.4, the reflection

$$S^{-1}(-) \colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R^{\mathrm{Loc}(S)}$$

is a symmetric monoidal Bousfield localization. The Part II result Corollary 17.2.19 therefore equips $S^{-1}R$ with a commutative R-algebra structure and characterizes it as initial among commutative R-algebras whose underlying modules are S-local. Under the equivalence

$$\mathrm{CAlg}(\mathrm{Mod}_R) \simeq \mathrm{CAlg}(\mathrm{Sp})_{R/},$$

this gives the asserted morphism $\eta\colon R \to R[S^{-1}]$ of commutative ring spectra.

Let $f\colon R \to A$ be a morphism of commutative ring spectra. Its underlying R-module is S-local if and only if $\pi_*(f)$ sends every $s \in S$ to a unit. One direction is immediate. Conversely, suppose that A is S-local and let $s' = \pi_*(f)(s)$ have degree d. Since multiplication by s' is bijective, there is an element $u \in \pi_{-d}(A)$ such that $s'u = 1$. Since $s'(us' - 1) = 0$, injectivity of multiplication by s' gives $us' = 1$. Thus s' is a unit.

The initial property of $S^{-1}R$ among commutative R-algebras with local underlying module now gives the claimed factorization property. Conversely, every morphism that factors through $R[S^{-1}]$ inverts S, since the image of S in $\pi_*(R[S^{-1}])$ consists of units and ring morphisms preserve units. □

Finally, we record the computation used for Bott inversion.

Lemma 8.4.6 (Inverting a single element)**.** *Let R be a commutative ring spectrum, let $s \in \pi_d(R)$ be a homogeneous element, and let $S := \{1, s, s^2, \dots\}$ be the multiplicative subset it generates. Then for every $M \in \mathrm{Mod}_R$ the canonical map exhibits the telescope*

$$M[s^{-1}] := \operatorname{colim}\big(M \xrightarrow{s} M[-d] \xrightarrow{s} M[-2d] \xrightarrow{s} \cdots\big)$$

as the S-localization of M. On homotopy groups it is given by graded localization:

$$\pi_*(M[s^{-1}]) \cong \pi_*(M)[s^{-1}].$$

Proof. The transition maps are shifts of the module map of Observation 8.4.2, so the telescope is a colimit in Mod_R. Since homotopy groups preserve filtered colimits by Lemma 4.4.28, we have

$$\pi_k(M[s^{-1}]) \cong \operatorname{colim}\big(\pi_k(M) \xrightarrow{s\cdot} \pi_{k+d}(M) \xrightarrow{s\cdot} \pi_{k+2d}(M) \longrightarrow \cdots\big),$$

which is the degree-k part of $\pi_*(M)[s^{-1}]$. In particular, multiplication by s on $M[s^{-1}]$ is an isomorphism, so the telescope is S-local.

Let N be any S-local R-module. Mapping out of the telescope gives

$$\mathrm{Hom}_{\mathrm{Mod}_R}(M[s^{-1}], N) \simeq \lim_j \mathrm{Hom}_{\mathrm{Mod}_R}(M[-jd], N).$$

Every transition map in this limit is an equivalence, because under the shift adjunction it is induced by the isomorphism $s\colon N[d] \xrightarrow{\sim} N$. Evaluation at the zeroth term therefore gives an equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_R}(M[s^{-1}], N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_R}(M, N),$$

induced by precomposition with the canonical map $M \to M[s^{-1}]$. This is the universal property of the S-localization. □

Corollary 8.4.7 (Homotopy groups of a localization)**.** *Let R be a commutative ring spectrum, let $S \subseteq \pi_*(R)$ be a multiplicative subset of homogeneous elements, and let M be an R-module. There is a natural isomorphism of graded modules*

$$\pi_*(S^{-1}M) \cong S^{-1}\pi_*(M).$$

For $M = R$, this identifies $\pi_*(R[S^{-1}])$ with the graded ring $S^{-1}\pi_*(R)$.

Proof. For a finite subset $F \subseteq S$, successively applying Lemma 8.4.6 gives an F-localization $M[F^{-1}]$ with homotopy groups $\pi_*(M)[F^{-1}]$. Each step preserves locality for the elements already inverted, since their actions commute with the localization maps and filtered colimits preserve isomorphisms. The universal property makes these localizations functorial in F.

The filtered colimit of $M[F^{-1}]$ over the finite subsets of S is S-local: for each $s \in S$, the subsets containing s are cofinal. Mapping into an S-local module identifies the hom anima out of each term with that out of M, so this colimit is $S^{-1}M$. Taking homotopy groups gives the asserted formula. For $M = R$, the identification respects multiplication and hence is an isomorphism of graded rings. □

Remark 8.4.8. Combining Corollary 8.4.5 with Lemma 8.4.6 for $M = R$, the telescope

$$R[s^{-1}] = \operatorname{colim}\big(R \xrightarrow{s} R[-d] \xrightarrow{s} R[-2d] \longrightarrow \cdots\big)$$

is formed in Mod_R and acquires its commutative ring structure from its identification with the localization. The map $R \to R[s^{-1}]$ is a morphism of commutative ring spectra, and

$$\pi_*(R[s^{-1}]) \cong \pi_*(R)[s^{-1}].$$

8.4.2 Ore localization (online only)

We now extend localization to associative ring spectra. Module localization exists for every multiplicative subset $S \subseteq \pi_*(R)$. Under the left Ore condition introduced below, its kernel consists precisely of the S-nilpotent modules, and for ordinary rings it recovers the classical construction of fractions.

Definition 8.4.9 (Localization of a ring spectrum)**.** Let R be a ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset of homogeneous elements. The *localization of R at S*, denoted $R[S^{-1}]$, is a ring spectrum equipped with a map of ring spectra $\eta\colon R \to R[S^{-1}]$ that satisfies the following property: for every ring spectrum A, precomposition with η induces a monomorphism of animae

$$-\circ\eta\colon \operatorname{Hom}_{\mathrm{Alg}}(R[S^{-1}], A) \hookrightarrow \operatorname{Hom}_{\mathrm{Alg}}(R, A)$$

whose image consists of those algebra maps $f\colon R \to A$ for which $\pi_*(f)$ carries every element of S to a unit of the graded ring $\pi_*(A)$.

In other words: an algebra map $f\colon R \to A$ factors through η if and only if it inverts S, and in that case the factorization is unique. Note that $R[S^{-1}]$ is unique, if it exists. In an analogous way, we may define localization at the level of modules. Let us first introduce some terminology:

Definition 8.4.10. Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset of homogeneous elements.

(1) A left R-module M is called *S-nilpotent* if for every element $x \in \pi_n(M)$, there exists an element $s \in S$ such that $s \cdot x = 0$ in $\pi_*(M)$.

(2) A left R-module N is called *S-local* if, for every element $s \in S$ of degree d and every $k \in \mathbb{Z}$, left multiplication by s induces an isomorphism

$$s \cdot -\colon \pi_k(N) \xrightarrow{\sim} \pi_{k+d}(N).$$

We denote by $\mathrm{LMod}_R^{S\text{-nil}}$ and $\mathrm{LMod}_R^{\mathrm{Loc}(S)}$ the full subcategories of LMod_R spanned by the S-nilpotent and S-local modules, respectively.

When R is commutative, the second condition agrees with the definition given in the previous subsection. For a general associative ring spectrum, however, left multiplication by s on N need not be a morphism of left R-modules, which explains the formulation in terms of homotopy groups. Equivalently, the underlying map of spectra $s\colon N[|s|] \to N$ is an isomorphism for every $s \in S$. There is also an intrinsic reformulation in LMod_R: right multiplication by $s \in \pi_d(R)$ defines a morphism $\cdot s\colon R[d] \to R$ of left R-modules, and N is S-local precisely when it is local with respect to all shifts of these morphisms.

Both conditions are formulated purely in terms of the graded $\pi_*(R)$-module $\pi_*(M)$, and this makes them stable under all the operations we will need:

Observation 8.4.11. The subcategories $\mathrm{LMod}_R^{S\text{-nil}}$ and $\mathrm{LMod}_R^{\mathrm{Loc}(S)}$ are stable subcategories of LMod_R, closed under all small colimits.

Proof. Both subcategories contain zero and are closed under shifts. Consider an exact sequence $M' \to M \to M''$. If M' and M are S-local, the five lemma applied to multiplication by each $s \in S$ on the long exact sequence shows that M'' is S-local.

If M' and M are S-nilpotent, take $x \in \pi_n(M'')$ and choose $s \in S$ annihilating its boundary in $\pi_{n-1}(M')$. Then sx lifts to an element $y \in \pi_{n+|s|}(M)$. Choosing $t \in S$ with $ty = 0$ gives $(ts)x = 0$. Thus M'' is S-nilpotent. These arguments use only vanishing, so the suspension signs in the boundary maps do not affect them.

Both subcategories are therefore stable and closed under finite colimits. They are closed under filtered colimits because homotopy groups preserve these colimits: invertibility of each action passes to the colimit, and every element in a filtered colimit comes from a stage where it is annihilated by an element in S. Arbitrary coproducts are filtered colimits of finite coproducts. Closure under coproducts and finite colimits now implies closure under all small colimits. □

Definition 8.4.12. Let R be a ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset of homogeneous elements. Given a left R-module M, its *S-localization* is an S-local left R-module $S^{-1}M$ equipped with a morphism of left R-modules $\eta\colon M \to S^{-1}M$ satisfying the property that for every other S-local left R-module N, precomposition with η induces an equivalence of animae

$$-\circ\eta\colon \mathrm{Hom}_{\mathrm{LMod}_R}(S^{-1}M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LMod}_R}(M, N).$$

The localization exists for every multiplicative subset S: the local objects are those local with respect to the small collection of all shifts of $\cdot s\colon R[|s|] \to R$, so presentability gives a reflection by Theorem 23.2.2(4). The left Ore condition below will identify the kernel of this reflection with the S-nilpotent modules.

Definition 8.4.13 ([Lur17, Definition 7.2.3.1, Remark 7.2.3.7])**.** Let A_* be a graded associative ring and let $S \subseteq A_*$ be a set of homogeneous elements which contains the unit and is closed under multiplication. We say that S satisfies the *left Ore condition* if the following hold:

(a) For every pair of elements $x \in A_*$ and $s \in S$, there exist elements $y \in A_*$ and $t \in S$ such that $tx = ys$.

(b) If $xs = 0$ for some $x \in A_*$ and $s \in S$, then there exists some $t \in S$ such that $tx = 0$.

It suffices to check these conditions for homogeneous x, in which case y in part (a) may also be chosen homogeneous. An ordinary ring is regarded as concentrated in degree zero.

This seemingly purely algebraic condition can be conveniently rephrased in terms of S-nilpotent modules. For an element $s \in S$ of degree d, we denote by R/Rs the cofiber of the map of left R-modules $\cdot s\colon R[d] \to R$ given by right multiplication by s, i.e., the cofiber of the following composite

$$\cdot s\colon R[d] \simeq R \otimes \mathbb{S}^d \xrightarrow{\mathrm{id}\otimes s} R \otimes R \xrightarrow{m} R.$$

Lemma 8.4.14 ([Lur17, Lemma 7.2.3.11])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset of homogeneous elements. The following conditions are equivalent:*

(1) *The set S satisfies the left Ore condition.*

(2) *For every element $s \in S$, the left R-module R/Rs is S-nilpotent.*

Proof. For each homogeneous $s \in S$, the long exact sequence of $R[d] \xrightarrow{\cdot s} R \to R/Rs$ expresses $\pi_*(R/Rs)$ as an extension of the cokernel of right multiplication by s by a shift of its kernel. Condition (a) says precisely that every element of this cokernel is annihilated on the left by an element of S: the class of x is killed by t exactly when $tx = ys$ for some y. Condition (b) says the same for the kernel.

Elementwise S-nilpotence is preserved by submodules, quotients, and extensions. For the last assertion, first annihilate the image of an element in the quotient, then annihilate its lift in the submodule; the product of the two annihilators lies in S. Consequently, $\pi_*(R/Rs)$ is S-nilpotent if and only if both the kernel and cokernel are. This proves the equivalence. □

The previous lemma says that under the Ore condition the modules R/Rs are S-nilpotent. In fact they generate all S-nilpotent modules, in the explicit sense of the following lemma. This lemma is the main technical input of this section, and we use it as a black box.

Lemma 8.4.15 ([Lur17, Lemma 7.2.3.13])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset satisfying the left Ore condition. Then every S-nilpotent left R-module M can be written as the colimit of a sequence of left R-modules*

$$0 = M_0 \to M_1 \to M_2 \to \cdots$$

in which the fiber of each map $M_i \to M_{i+1}$ is a coproduct of modules of the form $(R/Rs_\alpha)[n_\alpha]$ with $s_\alpha \in S$ and $n_\alpha \in \mathbb{Z}$. In particular, $\mathrm{LMod}_R^{S\text{-nil}}$ is generated under colimits by the set of objects

$$\{\, (R/Rs)[n] \mid s \in S,\, n \in \mathbb{Z} \,\}.$$

Proposition 8.4.16 ([Lur17, Proposition 7.2.3.14])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset satisfying the left Ore condition. Let N be a left R-module. Then the following conditions are equivalent:*

(1) *The module N is S-local.*

(2) *For every element $s \in S$ and every integer n, the hom anima* $\mathrm{Hom}_{\mathrm{LMod}_R}((R/Rs)[n], N)$ *is contractible.*

(3) *For every S-nilpotent module $M \in \mathrm{LMod}_R$, the hom anima* $\mathrm{Hom}_{\mathrm{LMod}_R}(M, N)$ *is contractible.*

Proof. Let $s \in S$ be an element of degree d. Applying the mapping spectrum functor $\hom_{\mathrm{LMod}_R}(-, N)$ to the exact sequence

$$R[d] \xrightarrow{\cdot s} R \to R/Rs$$

and using the identifications $\hom_{\mathrm{LMod}_R}(R, N) \simeq N$ and $\hom_{\mathrm{LMod}_R}(R[d], N) \simeq N[-d]$ gives an exact sequence of spectra

$$\hom_{\mathrm{LMod}_R}(R/Rs, N) \longrightarrow N \xrightarrow{s\cdot -} N[-d].$$

The second map is left multiplication by s: precomposition with right multiplication $\cdot s \colon R[d] \to R$ sends a map $f \colon R \to N$ to the map determined by $f(s) = s \cdot f(1)$.

Condition (1) says precisely that the second map is an isomorphism for every $s \in S$, which by exactness is equivalent to the vanishing of $\hom_{\mathrm{LMod}_R}(R/Rs, N)$. Since a spectrum vanishes if and only if all the animae $\Omega^\infty(-[-n])$ do, this is in turn equivalent to $\mathrm{Hom}_{\mathrm{LMod}_R}((R/Rs)[n], N)$ being contractible for all n. This proves (1) $\Leftrightarrow$ (2).

To prove the implication (3) $\Rightarrow$ (2), we observe that since S satisfies the left Ore condition, the module R/Rs is S-nilpotent for every $s \in S$, as established in Lemma 8.4.14. As $\mathrm{LMod}_R^{S\text{-nil}}$ is closed under shifts (Observation 8.4.11), condition (3) applied to $M = (R/Rs)[n]$ immediately yields condition (2).

For (2) $\Rightarrow$ (3), the modules M for which $\mathrm{Hom}_{\mathrm{LMod}_R}(M, N)$ is contractible are closed under colimits. By Lemma 8.4.15, they therefore include every S-nilpotent module. □

Under the left Ore condition, every module decomposes into an S-nilpotent part and an S-local part:

Proposition 8.4.17 ([Lur17, Proposition 7.2.3.17, Remark 7.2.3.18])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset satisfying the left Ore condition. Then for every left R-module M, there exists an exact sequence in* LMod_R

$$M' \to M \to M'',$$

where M' is S-nilpotent and M'' is S-local.

Proof. By Lemma 8.4.15, the ∞-category $\mathrm{LMod}_R^{S\text{-nil}}$ is the localizing subcategory generated by the set of objects $\{(R/Rs)[n] \mid s \in S, n \in \mathbb{Z}\}$. It is therefore presentable by Proposition 23.2.7(2). Its inclusion into LMod_R preserves colimits by Observation 8.4.11, so Theorem 23.2.5 gives a right adjoint G. For any module M, the counit of this adjunction gives a map $G(M) \to M$. We define $M' := G(M)$ and we let M'' be the cofiber of this map. It remains to show that M'' is S-local, for which we use criterion (3) from Proposition 8.4.16. Let N be an S-nilpotent left R-module. Applying $\hom_{\mathrm{LMod}_R}(N, -)$ provides an exact sequence of spectra

$$\hom_{\mathrm{LMod}_R}(N, M') \to \hom_{\mathrm{LMod}_R}(N, M) \to \hom_{\mathrm{LMod}_R}(N, M'').$$

Both functors in the adjunction are exact, so the adjunction is enriched in spectra. The first map is therefore an isomorphism, and it follows that $\hom_{\mathrm{LMod}_R}(N, M'') = 0$, as desired. □

Remark 8.4.18. The pair $(\mathrm{LMod}_R^{S\text{-nil}}, \mathrm{LMod}_R^{\mathrm{Loc}(S)})$ is a *semiorthogonal decomposition*: every module admits the functorial decomposition above, and the hom anima from an S-nilpotent module to an S-local module is contractible. Both subcategories are stable. Thus this pair also defines a t-structure in the sense of Definition 6.3.1, with zero heart.

Corollary 8.4.19. *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ satisfy the left Ore condition. In the decomposition $M' \to M \to M''$ of Proposition 8.4.17, the map $M \to M''$ exhibits M'' as $S^{-1}M$. In particular, the fiber of the localization map is S-nilpotent.*

Proof. By Proposition 8.4.17 there is an exact sequence $M' \to M \to M''$ of left R-modules such that M' is S-nilpotent and M'' is S-local. We claim that M'' is an S-localization. Indeed, for any other S-local left R-module N, mapping into N provides a fiber sequence of animae

$$\mathrm{Hom}_{\mathrm{LMod}_R}(M'', N) \to \mathrm{Hom}_{\mathrm{LMod}_R}(M, N) \to \mathrm{Hom}_{\mathrm{LMod}_R}(M', N).$$

But since M' is S-nilpotent, the anima $\mathrm{Hom}_{\mathrm{LMod}_R}(M', N)$ is contractible by part (3) of Proposition 8.4.16. It follows that the first map is an equivalence, as desired. □

The proof exhibits the fiber of the localization map $\eta_M \colon M \to S^{-1}M$ as the S-nilpotent part M' of M. This gives the following recognition criterion:

Observation 8.4.20. Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ satisfy the left Ore condition. Then:

(1) For $M \in \mathrm{LMod}_R$ we have $S^{-1}M = 0$ if and only if M is S-nilpotent.

(2) A map $f \colon M \to N$ of left R-modules becomes an isomorphism after S-localization if and only if $\mathrm{cofib}(f)$ is S-nilpotent.

(3) If T is an S-local module and $f \colon M \to T$ is a map with S-nilpotent cofiber, then f exhibits T as the S-localization of M.

Proof. (1) Write $M' \to M \to S^{-1}M$ for the decomposition of Proposition 8.4.17. If M is S-nilpotent, then the counit $M' = G(M) \to M$ of the coreflection is an isomorphism, so its cofiber $S^{-1}M$ vanishes. Conversely, if $S^{-1}M = 0$ then $M' \to M$ is an isomorphism, so M is S-nilpotent.

(2) The functor $S^{-1}(-)$ is a left adjoint between stable ∞-categories, hence preserves cofibers. Thus $S^{-1}(f)$ is an isomorphism if and only if $S^{-1}\,\mathrm{cofib}(f) \simeq \mathrm{cofib}(S^{-1}f)$ vanishes, which by (1) happens if and only if $\mathrm{cofib}(f)$ is S-nilpotent.

(3) By (2) the map $S^{-1}(f) \colon S^{-1}M \to S^{-1}T$ is an isomorphism, and the unit $\eta_T \colon T \to S^{-1}T$ is an isomorphism because T is already S-local. Naturality of the unit gives $\eta_T \circ f = S^{-1}(f) \circ \eta_M$. It follows that the isomorphism $\eta_T^{-1} \circ S^{-1}(f) \colon S^{-1}M \to T$ identifies f with the localization map $\eta_M \colon M \to S^{-1}M$. □

The existence of S-localizations of R-modules can be used to deduce the existence of localizations of ring spectra. Regarding R as a module over itself, applying the S-localization functor gives an S-local R-module $S^{-1}R \in \mathrm{LMod}_R$. This object can be endowed with the structure of an associative ring spectrum, which we denote by $R[S^{-1}]$.

Proposition 8.4.21 ([Lur17, Remark 7.2.3.26, Proposition 7.2.3.27])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset satisfying the left Ore condition. The S-localization $S^{-1}R$ admits the structure of an associative ring spectrum, which we denote by $R[S^{-1}]$. Furthermore, the canonical map $R \to S^{-1}R$ can be enhanced to a morphism of associative ring spectra*

$$\varphi\colon R \to R[S^{-1}].$$

This morphism exhibits $R[S^{-1}]$ as an S-localization of R, in the sense of Definition 8.4.9.

We use this result as a black box, taking it directly from the cited reference. Constructing the coherently associative multiplication on $S^{-1}R$ is the one step of the theory for which the module-theoretic arguments above do not suffice.

Finally, this abstract homotopical construction recovers the classical construction of fractions when applied to an ordinary associative ring and module.

Proposition 8.4.22 ([Lur17, Proposition 7.2.3.20 and the proof of Proposition 7.2.3.5])**.** *Let R be an ordinary associative ring and let $S \subseteq R$ be a multiplicative subset satisfying the left Ore condition. For every ordinary left R-module M, its localization $S^{-1}M$ in module spectra is discrete and identifies naturally with the classical module of left fractions.*

8.5 Projective and flat modules (online only)

Over a connective ring spectrum, projective modules are retracts of free modules, and flat modules are filtered colimits of finitely generated free modules. We deduce these descriptions from the theory of t-projective and t-flat objects in Sections 6.4 and 6.5. We then study Tor-amplitude, which bounds the degrees that can occur after tensoring with a discrete module, and show that perfect modules have finite Tor-amplitude.

8.5.1 Projective modules

Definition 8.5.1 (Ext-groups for modules)**.** Let R be an associative ring spectrum and let $M, N \in \mathrm{LMod}_R$ be left R-modules. We define the *n-th Ext-group* of M and N over R to be

$$\mathrm{Ext}^n_R(M,N) \quad := \quad \mathrm{Ext}^n_{\mathrm{LMod}_R}(M,N) \quad = \quad \pi_{-n}\,\mathrm{hom}_R(M,N) \quad \simeq \quad \pi_0\,\mathrm{Hom}_{\mathrm{LMod}_R}(M,N[n]).$$

These are the Ext-groups from Definition 6.4.6 specialized to the stable ∞-category LMod_R.

The long exact sequence from Observation 6.4.11 applies to these groups. When R is connective and M, N are discrete, Observations 6.4.7 and 6.4.8 show that negative Ext-groups vanish and that $\mathrm{Ext}^0_R(M,N)$ is the ordinary Hom group over $\pi_0(R)$.

Definition 8.5.2 (Projective module)**.** Let R be a connective associative ring spectrum. A left R-module P is *projective* if it is a t-projective object of LMod_R in the sense of Definition 6.4.1; that is, if the mapping spectrum functor $\mathrm{hom}_R(P,-)\colon \mathrm{LMod}_R \to \mathrm{Sp}$ is t-exact.

Observation 8.5.3. Projective modules are connective by Observation 6.4.2. The characterizations of t-projective objects from Proposition 6.4.12 specialize to the following equivalent conditions for a connective left R-module P:

(1) P is projective;

(2) For every connective left R-module Q and every $i > 0$, we have $\mathrm{Ext}^i_R(P, Q) = 0$;

(3) For every connective left R-module Q, we have $\mathrm{Ext}^1_R(P, Q) = 0$;

(4) Given an exact sequence $N' \to N \to N''$ of connective left R-modules, the induced map $\mathrm{Ext}^0_R(P, N) \to \mathrm{Ext}^0_R(P, N'')$ is surjective.

Since the t-structure on LMod_R is left complete (Corollary 8.3.4), these are further equivalent to:

(5) For every $Q \in \mathrm{LMod}^{\heartsuit}_R$ and every $i > 0$, we have $\mathrm{Ext}^i_R(P, Q) = 0$.

The projective R-modules admit a simple description as retracts of free R-modules.

Definition 8.5.4 (Free module)**.** Let R be an associative ring spectrum. A left R-module M is *free* if M is a (possibly infinite) coproduct of (unshifted) copies of R, viewed as a left module over itself. A free left R-module is *finitely generated* if it is isomorphic to a finite coproduct of copies of R.

Remark 8.5.5. This is a special case of the free-module construction from Section 17.1.2. There one allows modules of the form $R \otimes X$ for an arbitrary spectrum X; here $X = \bigoplus_{i \in I} \mathbb{S}$ is required to be a coproduct of unshifted copies of the sphere spectrum.

Proposition 8.5.6 ([Lur17, Proposition 7.2.2.7])**.** *Let R be a connective associative ring spectrum, and let P be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module P is projective.*

(2) *There exists a free R-module M such that P is a retract of M.*

Proof. Suppose first that P is projective. We can choose a map of left R-modules $p\colon N \to P$, where N is free, such that the induced map $\pi_0(p)\colon \pi_0(N) \to \pi_0(P)$ is surjective. (For instance, we can take N to be the free module generated by the set $\pi_0(P)$.) Letting N' be the fiber of p, it follows from the long exact sequence on homotopy groups that N' is connective as well. The exact sequence $N' \to N \to P$ of connective R-modules induces by Observation 6.4.11 a sequence

$$\mathrm{Ext}^0_R(P, N') \to \mathrm{Ext}^0_R(P, N) \to \mathrm{Ext}^0_R(P, P) \to \mathrm{Ext}^1_R(P, N').$$

Since P is projective, the last term vanishes by criterion (3) of Observation 8.5.3. In particular, there exists a map $s\colon P \to N$ such that $p \circ s = \mathrm{id}_P$, exhibiting P as a retract of the free R-module N.

For the converse, we observe that the collection of projective left R-modules is stable under retracts. It will therefore suffice to show that every free left R-module is projective. We check criterion (2) from Observation 8.5.3. Since $\mathrm{Ext}^i_R(\bigoplus_I R, Q) \cong \prod_I \mathrm{Ext}^i_R(R, Q)$, we may assume $P = R$ is free on a single generator. But then

$$\mathrm{Ext}^i_R(R, Q) = \pi_0 \mathrm{Hom}_{\mathrm{LMod}_R}(R, Q[i]) \simeq \pi_0 \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, Q[i]) = \pi_{-i}(Q).$$

Since Q is connective, this is zero for $i > 0$, as desired. □

Definition 8.5.7. Let R be a connective associative ring spectrum. A left R-module is *finitely generated projective* if it is a retract of a finite coproduct of copies of R.

For a projective module P, this is equivalent to finite generation of $\pi_0(P)$ over $\pi_0(R)$. Indeed, finitely many generators give a map from a finite free module that is surjective on π_0, and the preceding proof shows that this map splits. The converse follows by applying π_0 to a retraction.

Remark 8.5.8. If R is an ordinary associative ring, it follows from the proposition that the projective R-module spectra are precisely the ordinary projective R-modules, regarded as discrete module spectra.

8.5.2 Flat modules

We now turn to flat modules. Recall from Definition 6.5.1 the general notion of t-flatness with respect to a tensor product $C \times D \to E$ of stable ∞-categories with t-structures. We will specialize this to the relative tensor product of modules over a ring spectrum.

Definition 8.5.9 (Tor-groups for modules)**.** Let R be an associative ring spectrum. For a right R-module M and a left R-module N, we define the *n-th Tor-group* of M and N over R to be

$$\mathrm{Tor}_n^R(M, N) \quad := \quad \pi_n(M \otimes_R N) \quad \in \quad \mathrm{Ab}.$$

These are the Tor-groups from Definition 6.5.6 specialized to the relative tensor product $\mathrm{RMod}_R \times \mathrm{LMod}_R \xrightarrow{\otimes_R} \mathrm{Sp}$.

Observation 8.5.10. Let R be a connective associative ring spectrum. By Observation 8.3.5, the relative tensor product of a connective right R-module M and a connective left R-module N is connective, and

$$\mathrm{Tor}_0^R(M, N) = \pi_0(M \otimes_R N) \cong \pi_0(M) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N).$$

In particular, $\mathrm{Tor}_n^R(M, N) = 0$ for $n < 0$ whenever M and N are connective.

Observation 8.5.11 (Tor long exact sequence)**.** Specializing Observation 6.5.7, every exact sequence of right R-modules induces a long exact sequence of Tor-groups after tensoring with a fixed left R-module. The same holds for exact sequences in the left-module variable.

Definition 8.5.12 (Flat module)**.** Let R be a connective associative ring spectrum. A connective left R-module N is said to be *flat* if the functor $(-) \otimes_R N \colon \mathrm{RMod}_R \to \mathrm{Sp}$ is t-exact.

Remark 8.5.13. By Observation 8.5.10, the functor $(-) \otimes_R N$ is automatically right t-exact when N is connective. Thus a connective left R-module N is flat if and only if this functor is left t-exact: for every coconnective right R-module M, the spectrum $M \otimes_R N$ is coconnective.

We now give several complementary tests for flatness. Conditions (2) and (3) describe the homotopy groups of a flat module and of its tensor products, while conditions (4) and (5) reduce flatness to discrete modules. The final two conditions recover the classical criterion for an ordinary ring.

Theorem 8.5.14 (Characterization of flat modules, [Lur17, Proposition 7.2.2.13, Theorem 7.2.2.15])**.** *Let R be a connective associative ring spectrum and let N be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module N is flat.*

(2) *For every right R-module M and $n \in \mathbb{Z}$, the following map is an isomorphism:*

$$\pi_n(M) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \xrightarrow{\sim} \pi_n(M \otimes_R N), \qquad [x] \otimes [y] \mapsto [x \otimes y].$$

(3) *The discrete $\pi_0(R)$-module $\pi_0(N)$ is flat, and for each integer n the map*

$$\pi_n(R) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \to \pi_n(N), \qquad [r] \otimes [x] \mapsto [rx]$$

is an isomorphism of abelian groups.

(4) *For every discrete right R-module M, the spectrum $M \otimes_R N$ is discrete. Equivalently, $\mathrm{Tor}^R_i(M, N) = 0$ for all discrete M and all $i > 0$.*

(5) *The left $\pi_0(R)$-module $\pi_0(R) \otimes_R N$ is discrete and flat.*

If R and N are both discrete, these conditions are further equivalent to:

(6) *For every injection $M' \hookrightarrow M$ of discrete right R-modules, the induced map $M' \otimes^{\heartsuit}_R N \to M \otimes^{\heartsuit}_R N$ is injective.*

(7) *For every discrete right R-module M, we have $\mathrm{Tor}^R_1(M, N) = 0$.*

Proof. We first show that the t-structural conditions are equivalent. Apply Proposition 6.5.9 to the relative tensor product $\mathrm{RMod}_R \times \mathrm{LMod}_R \to \mathrm{Sp}$. The t-structure on RMod_R is right complete by Corollary 8.3.4, relative tensor products preserve sequential colimits, and $\mathrm{Sp}_{\leq 0}$ is closed under sequential colimits. Moreover, $M \otimes_R N$ is connective when M and N are connective by Observation 8.5.10. Thus the heart criterion of Proposition 6.5.9 gives (1) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (2): A t-exact functor commutes with homotopy group objects by Lemma 6.3.26. On hearts, $(-) \otimes_R N$ is identified with $(-) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N)$ by Observation 8.5.10, giving the required formula.

(2) $\Rightarrow$ (3): Taking $M = R$ gives the displayed isomorphisms in (3). To prove that $\pi_0(N)$ is flat, consider a short exact sequence $0 \to M' \to M \to M'' \to 0$ of ordinary right $\pi_0(R)$-modules, regarded as discrete right R-modules. Tensoring the associated exact sequence with N and applying (2) gives a short exact sequence

$$0 \to M' \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \to M \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \to M'' \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \to 0.$$

Hence $\pi_0(N)$ is flat over $\pi_0(R)$.

(3) $\Rightarrow$ (2): The right R-modules for which the formula holds in every degree form a localizing subcategory. Indeed, they are closed under shifts and coproducts, and closure under cofibers follows from the five lemma applied to the long exact sequences, using flatness of $\pi_0(N)$. By assumption this subcategory contains R, so it is all of RMod_R by the right-module version of Lemma 8.1.11.

(2) $\Rightarrow$ (1): The formula shows that $(-) \otimes_R N$ preserves both connective and coconnective modules.

(4) $\Longleftrightarrow$ (5): Let $A := \pi_0(R)$ and let $L := A \otimes_R N$. Every discrete right R-module is equivalently a discrete right A-module, and for such an M, associativity of relative tensor products gives a natural isomorphism

$$M \otimes_A L \xrightarrow{\sim} M \otimes_R N.$$

If (4) holds, then taking $M = A$ shows that L is discrete. Moreover, the displayed isomorphism shows that $M \otimes_A L$ is discrete for every discrete right A-module M. Thus L satisfies (4) as an A-module, and hence is flat by the equivalence between (1) and (4) already proved above.

Conversely, if L is discrete and flat as an A-module, then for every discrete right R-module M, regarded as a discrete right A-module, the spectrum $M \otimes_A L$ is discrete. By the displayed isomorphism, so is $M \otimes_R N$, which is (4).

For discrete R and N, Lemma 6.5.10 gives (1) $\Leftrightarrow$ (7), since every discrete right module is a quotient of a free one. Vanishing of $\mathrm{Tor}_1^R(-, N)$ implies (6) by the Tor long exact sequence. Conversely, given a free presentation $0 \to K \to F \to M \to 0$, that sequence identifies $\mathrm{Tor}_1^R(M, N)$ with the kernel of $K \otimes_R^{\heartsuit} N \to F \otimes_R^{\heartsuit} N$, which vanishes under (6). □

Observation 8.5.15. Let R be a connective associative ring spectrum. Condition (3) of Theorem 8.5.14 immediately gives the following facts:

(a) The subcategory of LMod_R spanned by the flat R-modules is closed under coproducts, retracts, and filtered colimits.

(b) Every free module is flat. Combining this with (a) and Proposition 8.5.6, it follows that every projective module is flat.

(c) If A is an ordinary associative ring, then a connective left A-module N is flat if and only if N is discrete and $\pi_0 N$ is a flat A-module in the classical sense, i.e., satisfies condition (6) of Theorem 8.5.14.

Lemma 8.5.16. *Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$ be a connective commutative ring spectrum, and let M and N be flat R-modules. Then $M \otimes_R N$ is again flat.*

Proof. Connectivity and coconnectivity are detected on underlying spectra by Corollary 8.3.4. Thus $(-) \otimes_R M$ and $(-) \otimes_R N$ are t-exact endofunctors of Mod_R. Their composite is $(-) \otimes_R (M \otimes_R N)$ by associativity, so $M \otimes_R N$ is flat. □

Observation 8.5.17. Every free R-module is flat (Observation 8.5.15), so in particular the monoidal unit R is flat. Combined with Lemma 8.5.16, this shows that the full subcategory $\mathrm{Mod}_R^{\flat} \subseteq \mathrm{Mod}_{R,\geq 0}$ spanned by the flat R-modules contains the unit and is closed under the tensor product. It therefore inherits a symmetric monoidal structure by the Part II result Lemma 14.7.1.

For connective ring spectra, flat modules admit a characterization analogous to Lazard's theorem in ordinary algebra. It expresses them as filtered colimits of the simplest flat modules, the finitely generated free ones.

Theorem 8.5.18 (Lazard's theorem for spectra, [Lur17, Theorem 7.2.2.15])**.** *Let R be a connective associative ring spectrum and let N be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module N can be obtained as a filtered colimit of finitely generated free left R-modules.*

(2) *The left R-module N can be obtained as a filtered colimit of finitely generated projective left R-modules.*

(3) *The left R-module N is flat.*

Proof sketch. The implication (1) $\Rightarrow$ (2) is immediate because every free module is projective. The implication (2) $\Rightarrow$ (3) follows from Observation 8.5.15: projective modules are flat, and flat modules are closed under filtered colimits. The reverse implication is the substantive content of [Lur17, Theorem 7.2.2.15], which we use as a black box. Its proof shows that the ∞-category of maps $F \to N$, with F finitely generated and free, is filtered and that the canonical map from the colimit of this diagram to N is an isomorphism. The filteredness is ultimately reduced to the classical Lazard theorem over $\pi_0(R)$. □

8.5.3 Tor-amplitude

We conclude with two boundedness properties of perfect modules. The first is immediate from their construction out of the unit.

Lemma 8.5.19. *Let R be a connective associative ring spectrum and let M be a perfect R-module in the sense of Definition 8.1.10. Then M is bounded below: there exists an integer n such that $\pi_i(M) = 0$ for all $i < n$.*

Proof. The ring spectrum R is bounded below, and the bounded below R-modules form a thick subcategory of LMod_R: they are closed under shifts, cofibers, and retracts. The claim follows from the definition of a perfect module. □

Tor-amplitude gives a uniform bound after tensoring with discrete modules. By Theorem 8.5.14, flat modules are precisely the connective modules of Tor-amplitude contained in $[0, 0]$.

Definition 8.5.20 (Tor-amplitude)**.** Let R be a connective associative ring spectrum and let M be a left R-module. We say that M has *Tor-amplitude contained in* $[a, b]$ for integers $a \leq b$ if for every discrete right R-module N, the homotopy groups of $N \otimes_R M$ are concentrated in degrees $[a, b]$, that is, $\mathrm{Tor}_i^R(N, M) = 0$ whenever $i < a$ or $i > b$. If such a and b exist, M is said to have *finite Tor-amplitude*.

The last three statements below make this bound effective for perfect modules. A perfect module concentrated in a single Tor degree is a shifted finitely generated projective module; in general, one can successively remove the bottom Tor degree.

For commutative R, these properties appear in Antieau and Gepner [AG14, Proposition 2.13]. The following proofs also apply to associative R.

Proposition 8.5.21. *Let R be a connective associative ring spectrum and let M and N be left R-module spectra.*

(1) *If M has Tor-amplitude contained in $[a, b]$, then the shift $M[k]$ has Tor-amplitude contained in $[a + k, b + k]$ for all $k \in \mathbb{Z}$.*

(2) *If M is perfect, then M has finite Tor-amplitude.*

(3) *If $R \to S$ is a morphism of connective associative ring spectra and M has Tor-amplitude contained in $[a, b]$, then the left S-module $S \otimes_R M$ has Tor-amplitude contained in $[a, b]$.*

(4) *If M and N have Tor-amplitude contained in $[a, b]$, then the fiber and cofiber of a map $M \to N$ have Tor-amplitude contained in $[a - 1, b]$ and $[a, b + 1]$.*

(5) *If M is perfect with Tor-amplitude contained in $[0, b]$, then M is connective and $\pi_0 M \cong \pi_0(\pi_0(R) \otimes_R M)$.*

(6) *If M is perfect with Tor-amplitude contained in $[a, a]$, then M is isomorphic to $P[a]$ for a finitely generated projective left R-module P.*

(7) *If M is perfect with Tor-amplitude contained in $[a, b]$ and $a < b$, then there exists an exact sequence $P[a] \to M \to Q$ with P finitely generated projective and Q perfect with Tor-amplitude contained in $[a + 1, b]$. When $a = b$, part (6) gives such a sequence with $Q = 0$.*

Proof. Part (1) follows directly from the definition. For (2), observe that R has Tor-amplitude contained in $[0, 0]$ and that the left R-modules of finite Tor-amplitude form a thick subcategory of LMod_R: they are closed under shifts and retracts, while closure under cofibers follows from the long exact sequence of Tor-groups (Observation 8.5.11). For (3), if T is a discrete right S-module, then

$$T \otimes_S (S \otimes_R M) \simeq T \otimes_R M.$$

Part (4) follows from the same long exact sequence of Tor-groups.

For (5), suppose $M \neq 0$ and let k be its lowest nonzero homotopy degree, which exists by Lemma 8.5.19. Applying Observation 8.5.10 to the connective module $M[-k]$ gives

$$\pi_k(\pi_0(R) \otimes_R M) \cong \pi_k(M) \neq 0.$$

The amplitude bound forces $k \geq 0$, so M is connective. The assertion about π_0 follows from the same tensor-product formula. For $M = 0$, both claims are immediate.

For (6), shift to $a = 0$. By (5), M is connective, and the discrete-module test in Theorem 8.5.14 makes it flat. By Theorem 8.5.18, M is a filtered colimit of finitely generated free modules. Since M is compact by Lemma 8.1.11, its identity factors through one of these modules, exhibiting M as a retract of it.

For (7), shift to $a = 0 < b$. Then M is connective by (5). Write $\pi_0(M)$ as the filtered union of its finitely generated submodules A_i. Since M is compact and the inclusion of discrete modules preserves filtered colimits, the truncation map $M \to \pi_0(M)$ factors through some A_i. On π_0, this forces $A_i = \pi_0(M)$. Choose a finite free module P and a map $P \to M$ surjective on π_0, and let Q be its cofiber. For any discrete right R-module T, the map

$$\pi_0(T \otimes_R P) \to \pi_0(T \otimes_R M)$$

is surjective by Observation 8.5.10 and right exactness of ordinary tensor products. Moreover, $T \otimes_R P$ is discrete. The long exact sequence therefore puts $T \otimes_R Q$ in degrees $[1, b]$. Finally, Q is perfect as the cofiber of a map between perfect modules. □

Example 8.5.22. Perfectness, projectivity, and flatness impose different conditions. Over $\mathbb{Z}$, the module $(\mathbb{Z}/p)$ for a prime p is perfect, being the cofiber of multiplication by p on $\mathbb{Z}$. It has Tor-amplitude contained in $[0, 1]$, but is not flat: tensoring this cofiber sequence with $(\mathbb{Z}/p)$ gives a nonzero homotopy group in degree 1.

In the other direction, $\bigoplus_{n\geq 0} \mathbb{Z}$ is projective and flat, but not perfect. Indeed, it is the filtered colimit of its finite partial sums, and its identity cannot factor through any of them, so it is not compact.

Exercises

Exercise 8.1 (Perfect modules and base change)**.** Let $f\colon R \to S$ be a morphism of associative ring spectra.

(1) Show that extension of scalars $S \otimes_R -\colon \mathrm{LMod}_R \to \mathrm{LMod}_S$ sends perfect R-modules to perfect S-modules.

(2) Assume that R and S are commutative and let $x \in \pi_0(R)$. Write R/x for the cofiber of multiplication by x on R. Show that R/x is perfect and that

$$S \otimes_R (R/x) \cong S/f(x).$$

Exercise 8.2 (Relative tensor products and mapping spectra)**.** Let $m, n \geq 1$ and put $d := \gcd(m, n)$.

(1) Compute $\pi_* \hom_{\mathbb{Z}}((\mathbb{Z}/m), (\mathbb{Z}/n))$. Show that these groups vanish outside degrees 0 and -1, and are isomorphic to $\mathbb{Z}/d$ in each of those degrees.

(2) Compute $\pi_*((\mathbb{Z}/m) \otimes_{\mathbb{Z}} (\mathbb{Z}/n))$. Show that these groups vanish outside degrees 0 and 1, and are isomorphic to $\mathbb{Z}/d$ in each of those degrees.

Hint: Apply the two functors to the cofiber sequence defining $(\mathbb{Z}/m)$.

Exercise 8.3 (Graded localization)**.** Let R be a commutative ring spectrum with $\pi_*(R) \cong A[u]$, where A is a nonzero ordinary commutative ring concentrated in degree 0 and u has degree 2.

(1) Compute $\pi_*(R[u^{-1}])$. Show that multiplication by u gives an isomorphism $R[u^{-1}][2] \xrightarrow{\sim} R[u^{-1}]$, and that $R[u^{-1}]$ is not connective.

(2) Let R/u be the cofiber of multiplication by u:

$$R[2] \xrightarrow{u} R \to R/u.$$

Show that R/u is a nonzero perfect R-module, compute its homotopy groups, and show that $R[u^{-1}] \otimes_R (R/u) = 0$.

Exercise 8.4 (Projectivity and flatness of the discrete truncation)**.** Let R be a connective associative ring spectrum, and regard $\pi_0(R)$ as a left R-module via the canonical morphism $R \to \pi_0(R)$. Show that the following conditions are equivalent:

(1) The ring spectrum R is discrete.

(2) The left R-module $\pi_0(R)$ is projective.

(3) The left R-module $\pi_0(R)$ is flat.

Deduce that $\mathbb{Z}$ is neither projective nor flat as a module over the sphere spectrum, even though $\pi_0(\mathbb{Z})$ is free of rank one over $\pi_0(\mathbb{S})$.

Exercise 8.5 (Finite projective filtrations)**.** Let R be a connective associative ring spectrum and let M be a perfect left R-module of Tor-amplitude contained in $[a, b]$. Show that there is a filtration

$$0 = M_{a-1} \to M_a \to \cdots \to M_b = M$$

with $\mathrm{cofib}(M_{i-1} \to M_i) \cong P_i[i]$ for finitely generated projective left R-modules P_i. *Hint:* Induct on $b - a$ using Proposition 8.5.21, and pull back a filtration of the cofiber along $M \to Q$.

9 Complex K-theory

Formal differences of complex vector bundles over X, with addition induced by direct sum, form the group $K^0(X)$. By Bott periodicity, these groups form a 2-periodic cohomology theory, called *complex K-theory*. Our aim in this chapter is to construct that theory and its representing spectrum from the geometry of vector bundles.

The origins of K-theory lie in the Riemann–Roch theorem. Grothendieck introduced groups of virtual algebraic vector bundles to express the additivity underlying that theorem. Combining this idea with Bott's periodicity theorem for the stable unitary group [Bot59], Atiyah and Hirzebruch [AH59] developed topological K-theory in the late 1950s.

We begin with complex vector bundles and their classification by maps into Grassmannians in Sections 9.1 and 9.2. For a compact Hausdorff space X, the group $K^0(X)$ consists of formal differences of vector bundles, with addition induced by direct sum. Stabilizing the Grassmannians gives a space BU and a natural identification

$$K^0(X) \cong [X, \mathbb{Z}\times BU].$$

We then introduce the Bott class and state classical Bott periodicity in Section 9.3, which we use to construct the spectrum.

We therefore consider the topological groupoid of finite-dimensional complex vector spaces and their isomorphisms. Its underlying anima is

$$\mathrm{Vect}(\mathrm{pt})^{\simeq} \simeq \coprod_{n\geq 0} \mathbf{B}U(n),$$

with a coherently commutative addition given by direct sum. Applying homotopical group completion and the recognition principle of Chapter 5 gives the connective complex K-theory spectrum ku, whose underlying infinite loop anima is $\mathbb{Z}\times\mathbf{B}U$. The Bott class determines an element $\beta \in \pi_2\,\mathrm{ku}$, and inverting it produces the periodic spectrum

$$\mathrm{KU} = \mathrm{ku}[\beta^{-1}].$$

We carry out these constructions in Section 9.4.

The commutative ring structures on these spectra come from the tensor product of vector bundles; we use the commutative semiring structure of Proposition 9.4.2. Finally, we construct the Chern character, which identifies complex K-theory with periodic ordinary cohomology after rationalization on finite cell complexes; see Corollary 9.4.22.

9.1 Vector bundles

We begin with complex vector bundles, their behavior under pullback, and their classification over spheres by clutching functions.

Definition 9.1.1. Let X be a topological space. A *finite-rank complex vector bundle* over X is a continuous map $p\colon E \to X$, together with a complex vector space structure on every fiber $E_x := p^{-1}(x)$, satisfying the following local triviality condition: every point $x \in X$ admits an open neighborhood U, an integer $k \geq 0$, and a homeomorphism

$$\varphi\colon U \times \mathbb{C}^k \xrightarrow{\cong} p^{-1}(U)$$

over U whose restriction $\{y\} \times \mathbb{C}^k \to E_y$ is complex-linear for every $y \in U$. We will usually simply call this a *complex vector bundle*.

The rank function

$$\mathrm{rk}_E\colon X \longrightarrow \mathbb{N}, \qquad x \longmapsto \dim_{\mathbb{C}}(E_x),$$

is locally constant. We say that E has *rank* k if this function is constant with value k; in general, E has rank k on the clopen subspace $\mathrm{rk}_E^{-1}(k)$.

Given another complex vector bundle $q\colon E' \to X$, a *morphism of vector bundles* $E \to E'$ is a continuous map $f\colon E \to E'$ over X whose restriction $E_x \to E'_x$ is complex-linear for every $x \in X$. We write $\pi_0\mathrm{Vect}(X)$ for the set of isomorphism classes of finite-rank complex vector bundles over X, and $\pi_0\mathrm{Vect}_k(X)$ for the subset represented by bundles of rank k.

The notation $\pi_0\mathrm{Vect}(X)$ anticipates the anima of vector bundles introduced in Section 9.4. We write $\underline{\mathbb{C}}^n := X \times \mathbb{C}^n$ for the trivial rank n bundle over X. Given a continuous map $f\colon Y \to X$ and a vector bundle $E \to X$, its *pullback* is the bundle $f^*E := Y \times_X E \to Y$: pulling back local trivializations of E gives local trivializations of f^*E.

Let X be a topological space, and let $p_1\colon E_1 \to X$ and $p_2\colon E_2 \to X$ be two complex vector bundles over X. Their *direct sum* is defined as the fiber product

$$E_1 \oplus E_2 \quad := \quad E_1 \times_X E_2,$$

equipped on each fiber with the usual direct-sum vector space structure. We leave it to the reader to check that this is again a vector bundle; see Exercise 9.1. The zero bundle $\underline{0} := X \times \mathbb{C}^0$ is the unit for direct sum. Together with the usual associativity and symmetry isomorphisms, this gives an abelian monoid structure on $\pi_0\mathrm{Vect}(X)$.

We will also use the *tensor product* of two vector bundles E_1 and E_2. This is the vector bundle $E_1 \otimes E_2 \to X$ whose fiber over $x \in X$ is $E_{1,x} \otimes_{\mathbb{C}} E_{2,x}$; local trivializations of E_1 and E_2 induce local trivializations of $E_1 \otimes E_2$. Pullback along maps of spaces is compatible with direct sums and tensor products, and tensor product distributes over direct sums up to isomorphism. Thus tensor product induces a multiplication on $\pi_0\mathrm{Vect}(X)$, compatible with its direct-sum monoid structure.

9.1.1 Homotopy invariance of vector bundles

We will use the following classical homotopy invariance theorem for vector bundles over paracompact spaces.

Definition 9.1.2. A topological space X is called *paracompact* if for every open cover $\{U_i\}_{i\in I}$ of X there exists a *partition of unity subordinate to this cover*, i.e. a collection of continuous maps $\rho_\alpha \colon X \to [0,1]$ satisfying the following three conditions:

(1) The family $(\rho_\alpha)_\alpha$ is locally finite: every point of X has a neighborhood on which all but finitely many ρ_α vanish.

(2) The resulting sum satisfies $\sum_\alpha \rho_\alpha(x) = 1$ for all x.

(3) For each α, the support $\operatorname{supp}(\rho_\alpha) := \overline{\rho_\alpha^{-1}((0,1])}$ is contained in one of the open sets U_i.

For Hausdorff spaces, this is equivalent to the usual definition in terms of locally finite refinements of open covers.

Example 9.1.3. The following classes of topological spaces are paracompact:

- Every compact Hausdorff space is paracompact [Hat17, Proposition 1.18].
- Every direct limit, with the colimit topology, of an increasing sequence $X_1 \subseteq X_2 \subseteq \dots$ of compact Hausdorff spaces is paracompact [Hat17, Proposition 1.19].
- Every CW-complex is paracompact [Hat17, Proposition 1.20].

Theorem 9.1.4 (Homotopy invariance of vector bundles, Hatcher [Hat17, Theorem 1.6 and Corollary 1.8])**.** *Let $p \colon E \to X$ be a complex vector bundle and let $f_0, f_1 \colon Y \to X$ be homotopic continuous maps. If Y is paracompact Hausdorff, then the pullback bundles f_0^*E and f_1^*E are isomorphic.*

In particular, every homotopy equivalence $f \colon X \to Y$ between paracompact Hausdorff spaces induces a bijection

$$f^* \colon \pi_0\mathrm{Vect}(Y) \xrightarrow{\cong} \pi_0\mathrm{Vect}(X).$$

Remark 9.1.5. In particular, every complex vector bundle over a contractible paracompact Hausdorff space is trivial, since pullback along the map to a point induces a bijection on isomorphism classes.

9.1.2 Clutching functions

If a rank n vector bundle is trivial over two open subsets covering its base, the two trivializations differ on their intersection by a continuous map to $\mathrm{GL}_n(\mathbb{C})$. Such transition maps describe how to glue the trivial bundles together.

For spheres, we can instead glue along the boundaries of the closed hemispheres. Write $S^m = D^m_+ \cup_{S^{m-1}} D^m_-$, where $m \geq 1$. Given a continuous map $g \colon S^{m-1} \to \mathrm{GL}_n(\mathbb{C})$, called a *clutching function*, form

$$E_g := \big((D^m_+ \times \mathbb{C}^n) \sqcup (D^m_- \times \mathbb{C}^n)\big) \big/ \big((x,v)_+ \sim (x, g(x)v)_-\big), \qquad x \in S^{m-1}.$$

The projection $E_g \to S^m$ is a rank n vector bundle. Conversely, every rank n vector bundle over S^m is obtained in this way, since its restrictions to the two hemispheres are trivial by Theorem 9.1.4. The construction induces a bijection

$$[S^{m-1}, \mathrm{GL}_n(\mathbb{C})] \xrightarrow{\cong} \pi_0\mathrm{Vect}_n(S^m)$$

from the set of free homotopy classes; see Hatcher [Hat17, Proposition 1.11].

The group $\mathrm{GL}_n(\mathbb{C})$ is path-connected, as we verify in Exercise 9.1.6. For $m \geq 2$, free homotopy classes therefore agree with based homotopy classes: replacing $g(x)$ by $g(x_0)^{-1}g(x)$ gives a based map in the same free homotopy class, and applying this normalization to a homotopy makes it based. Thus the set above is $\pi_{m-1}(\mathrm{GL}_n(\mathbb{C}))$. For $m = 1$, it is a singleton.

Exercise 9.1.6 (Vector bundles over the circle)**.** Show that the topological group $\mathrm{GL}_n(\mathbb{C})$ is path-connected. Deduce that every rank n complex vector bundle over S^1 is trivial, so that

$$\pi_0 \mathrm{Vect}(S^1) \cong \mathbb{N}\,.$$

Exercise 9.1.7 (Vector bundles over the two-sphere)**.** Compute that $\pi_1(\mathrm{GL}_1(\mathbb{C})) \cong \mathbb{Z}$, with $m \in \mathbb{Z}$ represented by the clutching function $g_m \colon S^1 \to \mathbb{C}^\times$, $z \mapsto z^m$, and construct an explicit representative L_m for the corresponding line bundle over S^2. Next, let $U(n)$ be the group of unitary $(n\times n)$-matrices, and show that the inclusion $U(n) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is a homotopy equivalence. Using the fiber sequence

$$U(n-1) \hookrightarrow U(n) \twoheadrightarrow S^{2n-1},$$

where the projection sends a matrix to its last column, show that the block inclusion $U(1) \hookrightarrow U(n)$ induces an isomorphism on π_1 for all $n \geq 1$. Deduce that $\pi_1(\mathrm{GL}_n(\mathbb{C})) \cong \mathbb{Z}$. Conclude that $\pi_0 \mathrm{Vect}_k(S^2) \cong \mathbb{Z}$ for all $k \geq 1$, with every rank k complex vector bundle over S^2 isomorphic to

$$L_m \oplus \underline{\mathbb{C}}^{k-1}$$

for a unique $m \in \mathbb{Z}$.

9.2 Classifying vector bundles

We now construct the infinite Grassmannian and its universal vector bundle, and explain how maps into it classify complex vector bundles. We then identify its underlying anima with $\mathbf{B}U(k)$ and pass to the stable Grassmannian BU.

9.2.1 The infinite Grassmannian

Definition 9.2.1 (Infinite Grassmannian)**.** Let $\mathbb{C}^\infty$ denote the colimit in Top of the complex vector spaces $\mathbb{C}^n$ along the standard inclusions $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$. For $n \geq k$, let $\mathrm{Gr}_k(\mathbb{C}^n)$ be the Grassmannian of k-dimensional linear subspaces of $\mathbb{C}^n$, with its usual topology. We define the *infinite Grassmannian of k-planes* as

$$\mathrm{Gr}_k(\mathbb{C}^\infty) \quad := \quad \bigcup_{n \geq k} \mathrm{Gr}_k(\mathbb{C}^n),$$

equipped with the *colimit topology*: a subset $U \subseteq \mathrm{Gr}_k(\mathbb{C}^\infty)$ is open if and only if $U \cap \mathrm{Gr}_k(\mathbb{C}^n)$ is open for every n.

Similarly, let $V_k(\mathbb{C}^n)$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{C}^n$. The *infinite Stiefel manifold*

$$V_k(\mathbb{C}^\infty) \quad := \quad \bigcup_{n \geq k} V_k(\mathbb{C}^n)$$

is likewise given the colimit topology. Its points may equivalently be regarded as linear isometric embeddings $\mathbb{C}^k \hookrightarrow \mathbb{C}^\infty$. Sending a frame to the subspace it spans defines a surjective map

$$\pi\colon V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty), \qquad (\varphi\colon \mathbb{C}^k \hookrightarrow \mathbb{C}^\infty) \mapsto \varphi(\mathbb{C}^k),$$

whose restriction at every finite stage is the usual quotient map $V_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$.

The standard Schubert cell structures on the finite Grassmannians are compatible with the inclusions above. Consequently $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is a cell complex whose topology is the colimit topology just defined; see Hatcher [Hat17, Proposition 1.17 and p. 34].

Definition 9.2.2. For $n \geq k$, let

$$E_{k,n} \quad := \quad \{(V, x) \in \mathrm{Gr}_k(\mathbb{C}^n) \times \mathbb{C}^n \mid x \in V\}$$

be the *canonical rank k bundle* over $\mathrm{Gr}_k(\mathbb{C}^n)$. We define

$$E_{\mathrm{univ}} \quad := \quad \bigcup_{n \geq k} E_{k,n}$$

with the colimit topology, and let $p_{\mathrm{univ}}\colon E_{\mathrm{univ}} \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ be the projection $(V, x) \mapsto V$.

In the continuity arguments below, we use that product with a locally compact Hausdorff space preserves colimits in Top. In particular, maps from products of our colimit spaces with $\mathbb{C}^k$, $U(k)$, or $[0, 1]$ can be checked for continuity at each finite stage.

Lemma 9.2.3. *The map p_{univ} is a complex vector bundle of rank k.*

Proof. Fix a k-plane $V \subseteq \mathbb{C}^\infty$ and let U_V be the set of k-planes W for which the orthogonal projection $W \to V$ is an isomorphism. Its intersection with every finite Grassmannian is open, so U_V is open in the colimit topology. Every $W \in U_V$ is the graph of a unique linear map $a_W\colon V \to V^\perp$, depending continuously on W. The maps

$$U_V \times V \longrightarrow p_{\mathrm{univ}}^{-1}(U_V), \qquad (W, v) \longmapsto (W, v + a_W(v))$$

are local trivializations of p_{univ}: they and their inverses are continuous at finite stages, and the observation above gives continuity on the colimit. □

Let X be a paracompact Hausdorff space. By Theorem 9.1.4, pulling back E_{univ} along continuous maps defines a map

$$[X, \mathrm{Gr}_k(\mathbb{C}^\infty)] \to \pi_0 \mathrm{Vect}_k(X), \qquad [f] \mapsto f^*(E_{\mathrm{univ}}). \tag{9.1}$$

The following classical theorem expresses that p_{univ} is the *universal* rank k complex vector bundle:

Theorem 9.2.4 (Hatcher [Hat17, Theorem 1.16 and p. 31])**.** *For a paracompact Hausdorff space X, the map* (9.1) *is a bijection.*

Because of this theorem, $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is known as the *classifying space for rank k complex vector bundles*.

Proof sketch of Theorem 9.2.4. We sketch Hatcher's argument. For a rank k vector bundle $p\colon E \to X$, the following are in natural bijection:

(1) A pair (f, φ) consisting of a continuous map $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ and an isomorphism $\varphi\colon E \cong f^*(E_{\mathrm{univ}})$;

(2) A continuous map $g\colon E \to \mathbb{C}^\infty$ whose restriction to each fiber is a $\mathbb{C}$-linear injection.

Indeed, an isomorphism $E \cong f^*(E_{\mathrm{univ}})$ gives such a map by projecting $f^*(E_{\mathrm{univ}}) \subseteq X \times \mathbb{C}^\infty$ to $\mathbb{C}^\infty$. Conversely, a fiberwise linear injection g determines $f(x) := g(E_x) \subseteq \mathbb{C}^\infty$, and the evident map $E \to E_{\mathrm{univ}}$ identifies E with $f^*(E_{\mathrm{univ}})$.

Surjectivity is therefore reduced to constructing a map $g\colon E \to \mathbb{C}^\infty$ which is linear and injective on each fiber. Starting from a trivializing cover, one uses the countable-cover lemma for paracompact spaces to obtain a countable trivializing cover $(U_i)_{i\geq 1}$ together with a locally finite subordinate partition of unity $(\rho_i)_{i\geq 1}$. Each U_i is a disjoint union of open subsets of the original cover, so the bundle is trivial over U_i. Let $g_i\colon E|_{U_i} \to \mathbb{C}^k$ be the coordinate map of a trivialization. The maps

$$v \longmapsto \rho_i(p(v))g_i(v),$$

extended by zero outside U_i, assemble into a map $E \to (\mathbb{C}^k)^\infty \cong \mathbb{C}^\infty$. Local finiteness ensures that this map is continuous and takes values in the algebraic direct sum, while the partition-of-unity condition makes it injective on every fiber.

For injectivity, suppose two classifying maps $f_0, f_1\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ pull back E_{univ} to isomorphic bundles. Let $g_0, g_1\colon E \to \mathbb{C}^\infty$ be the corresponding fiberwise injections, and let $j_0, j_1\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ send e_i to e_{2i-1} and e_{2i}, respectively. The straight-line interpolations from the identity to j_0 and j_1 remain injective, as one sees from the first nonzero coordinate of a vector. They give homotopies from g_i to $j_i g_i$ through fiberwise injections. Since j_0 and j_1 have complementary images, $(1-t)j_0 g_0 + t j_1 g_1$ is also fiberwise injective. These homotopies induce a homotopy from f_0 to f_1.

For the countable-cover lemma, see Hatcher [Hat17, Lemma 1.21]; the remaining continuity details are part of the proof of Hatcher [Hat17, Theorem 1.16]. □

9.2.2 Classifying animae revisited

Recall that in Construction 5.1.14 we defined the *classifying anima* $\mathbf{B}G$ of a group object $G \in \mathrm{Grp}(\mathrm{An})$. To identify the underlying anima of the infinite Grassmannian, we develop a general recognition criterion for classifying animae. The same criterion will be reused for real vector bundles in Chapter 10.

Recall from Corollary 2.3.20 that the functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ preserves finite products, hence induces a functor

$$\Pi_\infty\colon \mathrm{Grp}(\mathrm{Top}) \to \mathrm{Grp}(\mathrm{An})$$

from the category of topological groups to the ∞-category of group objects in animae.

Definition 9.2.5. For a topological group G, we define its *classifying anima* $\mathbf{B}G$ as

$$\mathbf{B}G \quad := \quad \mathbf{B}\Pi_\infty(G) \quad \in \quad \mathrm{An}.$$

We will now discuss a concrete way of recognizing topological spaces whose underlying anima is isomorphic to $\mathbf{B}G$.

Definition 9.2.6 (Principal bundle)**.** Let G be a topological group and let B be a topological space. The *trivial G-principal bundle* is the projection map $B \times G \to B$, where we equip $B \times G$ with the canonical right G-action given by $(b, g)g' := (b, gg')$.

A *G-principal bundle* is a continuous map $\pi\colon P \to B$ such that P is equipped with a continuous right G-action such that:

(1) The right G-action on P preserves the fibers of π: we have $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$;

(2) The map π is a *locally trivial bundle*: around every point $b \in B$ there is an open neighborhood $b \in U \subseteq B$ such that the restriction $\pi^{-1}(U) \to U$ of π is a trivial G-principal bundle, i.e. there exists a G-equivariant homeomorphism $\varphi\colon U \times G \xrightarrow{\cong} \pi^{-1}(U)$ over U:

$$\begin{array}{ccccc} U \times G & \xrightarrow[\cong]{\varphi} & \pi^{-1}(U) & \hookrightarrow & P \\ & \searrow^{\mathrm{pr}_1} & \downarrow{\scriptstyle \pi|_U} & & \downarrow{\scriptstyle \pi} \\ & & U & \hookrightarrow & B. \end{array}$$

We refer to P as the *total space*, to G as the *structure group*, and to B as the *base space*.

Proposition 9.2.7 (Principal bundles and vector bundles)**.** *Every principal $\mathrm{GL}_n(\mathbb{C})$-bundle $P \to B$ defines a complex vector bundle of rank n over B by taking*

$$E := P \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n = (P \times \mathbb{C}^n)_{/(p,Av)\sim(pA,v)},$$

where the equivalence relation identifies the pair (p, Av) with (pA, v) for all $p \in P$, $v \in \mathbb{C}^n$ and $A \in \mathrm{GL}_n(\mathbb{C})$. This construction defines a bijection between the set of isomorphism classes of principal $\mathrm{GL}_n(\mathbb{C})$-bundles and the set of isomorphism classes of rank n vector bundles.

Proof. Local trivializations of P induce local trivializations of the associated bundle. Conversely, the frame bundle $\mathrm{Fr}(E) \to B$ of a rank n vector bundle is a principal $\mathrm{GL}_n(\mathbb{C})$-bundle. The natural map $\mathrm{Fr}(E) \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n \to E$ is an isomorphism, and the map

$$P \longrightarrow \mathrm{Fr}(P \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n), \qquad p \longmapsto \big(v \mapsto [p, v]\big)$$

is an isomorphism of principal bundles. Thus the two constructions are inverse on isomorphism classes. □

Remark 9.2.8. Since $\pi\colon P \to B$ is locally trivial, hence locally a Serre fibration, it follows from Theorem 2.3.13 that every principal bundle is a Serre fibration. In particular, the functor Π_∞ preserves pullbacks along principal bundles by Proposition 2.3.19.

Lemma 9.2.9. *Let $\pi\colon P \to B$ be a principal G-bundle. The maps*

$$\Phi_n\colon P \times G^n \longrightarrow \underbrace{P \times_B \cdots \times_B P}_{n+1 \text{ factors}}, \qquad (p, g_1, \ldots, g_n) \longmapsto (p, pg_1, \ldots, pg_1 \cdots g_n)$$

define an isomorphism from the action bar construction to the Čech nerve $\check{C}(\pi)$.

Proof. For $n = 1$, local trivializations identify Φ_1 with the homeomorphism $(b, g, h) \mapsto (b, g, gh)$, whose inverse is $(b, g, h') \mapsto (b, g, g^{-1}h')$. Iterating this gives a homeomorphism in every degree.

In the action bar construction, the first face acts on p by g_1, the interior faces multiply adjacent g_i, and the last face drops g_n. Under Φ_n, these operations delete the corresponding partial product. Similarly, inserting the identity as a degeneracy repeats a partial product. Thus the maps Φ_n commute with the simplicial operators. □

We obtain the following criterion.

Proposition 9.2.10. *Let $\pi\colon P \to B$ be a principal G-bundle such that P is contractible.*

(1) The Čech nerve of $\Pi_\infty(\pi)\colon \Pi_\infty(P) \to \Pi_\infty(B)$ is isomorphic to $\Pi_\infty(G) \in \mathrm{Grp}(\mathrm{An})$;

(2) In particular, passing to colimits provides an isomorphism of animae

$$\Pi_\infty(B) \xrightarrow{\sim} \mathbf{B}G.$$

Proof. By Remark 9.2.8, the functor Π_∞ preserves the pullbacks defining the Čech nerve of π. Applying Lemma 9.2.9 therefore identifies $\check{C}(\Pi_\infty\pi)$ with the simplicial anima $[n] \mapsto \Pi_\infty(P) \times \Pi_\infty(G)^n$. Projection onto $\Pi_\infty(G)^n$ is compatible with the simplicial operators and is an isomorphism in every degree, since P is contractible. This proves (1).

Since π is surjective, $\Pi_\infty(\pi)$ is surjective on path components and hence an effective epimorphism. Thus $\Pi_\infty(B)$ is the colimit of its Čech nerve. By (1), this colimit is $\mathbf{B}G$, proving (2). □

9.2.3 The infinite Grassmannian as a classifying anima

We now return to the infinite Grassmannian: we wish to apply Proposition 9.2.10 to the map $\pi\colon V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ to conclude that $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is a geometric model for the classifying anima of the unitary group $U(k)$, the group of complex isometries $\mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^k$.

Lemma 9.2.11. *The projection map $\pi\colon V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is a principal $U(k)$-bundle. Here $U(k)$ acts on $V_k(\mathbb{C}^\infty)$ by precomposition: given $A \in U(k)$ and $\varphi\colon \mathbb{C}^k \hookrightarrow \mathbb{C}^\infty$, we define φA as the composite $\mathbb{C}^k \xrightarrow{A} \mathbb{C}^k \xrightarrow{\varphi} \mathbb{C}^\infty$.*

Proof. Let $W \in \mathrm{Gr}_k(\mathbb{C}^\infty)$, and define $U \subseteq \mathrm{Gr}_k(\mathbb{C}^\infty)$ to be the subspace of those k-planes V for which the orthogonal projection $W \to V$ is an isomorphism. This is open: its intersection with every finite-dimensional stage $\mathrm{Gr}_k(\mathbb{C}^n)$ is open, and $\mathrm{Gr}_k(\mathbb{C}^\infty)$ has the colimit topology. We claim that there exists a $U(k)$-equivariant homeomorphism

$$\varphi\colon U \times U(k) \xrightarrow{\cong} \pi^{-1}(U)$$

over U, which will suffice to finish the proof. To construct φ, fix a k-frame $(w_1, \ldots, w_k)$ for W. Given any k-plane $V \in U$, the projection $W \to V$ is an isomorphism, hence sends the k-frame $(w_1, \ldots, w_k)$ to some basis for V. Using Gram-Schmidt orthonormalization (which is a continuous operation), we may turn this into a k-frame $(v_1, \ldots, v_k)$ for V. Define

$$\varphi(V, A) := (v_1, \ldots, v_k) \circ A.$$

Two orthonormal frames of a k-plane differ by a unique unitary matrix, whose entries are their pairwise inner products. This gives the inverse of φ. Both maps are continuous at finite stages, and hence on the colimit by the observation above. □

Remark 9.2.12. The principal $U(k)$-bundle $V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is the orthonormal frame bundle of E_{univ}, and its associated vector bundle $V_k(\mathbb{C}^\infty) \times_{U(k)} \mathbb{C}^k$ is E_{univ}. Extending its structure group along $U(k) \hookrightarrow \mathrm{GL}_k(\mathbb{C})$ gives the full frame bundle of Proposition 9.2.7.

Lemma 9.2.13. *The topological space $V_k(\mathbb{C}^\infty)$ is contractible.*

Proof. Consider the space $\widetilde{V}_k(\mathbb{C}^\infty) := \mathrm{colim}_n \widetilde{V}_k(\mathbb{C}^n)$ of all linear injections $\mathbb{C}^k \hookrightarrow \mathbb{C}^\infty$, with the colimit topology. Equivalently, $\widetilde{V}_k(\mathbb{C}^\infty)$ is the topological space of k-tuples $(v_1, \ldots, v_k)$ of linearly independent vectors in $\mathbb{C}^\infty$ which are not necessarily orthogonal to each other. Observe that the inclusion $V_k(\mathbb{C}^\infty) \hookrightarrow \widetilde{V}_k(\mathbb{C}^\infty)$ is a homotopy equivalence: the Gram-Schmidt orthogonalization procedure provides a deformation retract $\widetilde{V}_k(\mathbb{C}^\infty) \to V_k(\mathbb{C}^\infty)$.

It will thus suffice to show that $\widetilde{V}_k(\mathbb{C}^\infty)$ is contractible. To this end, consider the 'shift map' $T^k \colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ given on basis vectors by $T^k(e_n) := e_{n+k}$. This induces a map $T^k \colon \widetilde{V}_k(\mathbb{C}^\infty) \to \widetilde{V}_k(\mathbb{C}^\infty)$. Note that this map is homotopic to the constant map on $(e_1, \ldots, e_k)$ via the straight-line homotopy:

$$H(v, t) := (1 - t)(e_1, \ldots, e_k) + tT^k(v).$$

This is well-defined since the first k components of the vectors $T^k(v_i)$ are all zero. Similarly, the map T^k is homotopic to the identity on $\widetilde{V}_k(\mathbb{C}^\infty)$, again via the straight-line homotopy:

$$H(v, t) := (1 - t)v + tT^k(v).$$

For $t < 1$, the map $(1-t)\,\mathrm{id} + tT^k$ is injective, as one sees from the first nonzero coordinate of a vector; for $t = 1$, this follows from injectivity of T^k. Thus the homotopy preserves linear independence. Both homotopies are continuous because their restrictions to each finite stage land in a finite stage and product with $[0, 1]$ preserves colimits.

It follows that the identity on $\widetilde{V}_k(\mathbb{C}^\infty)$ is homotopic to a constant map, and hence this space is contractible. □

Writing $U(k)$ also for the underlying anima of the topological group $U(k)$, we get:

Corollary 9.2.14. *The underlying anima of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is isomorphic to the classifying anima $\mathbf{B}U(k)$.*

Proof. In light of Lemma 9.2.11 and Lemma 9.2.13, this is an instance of Proposition 9.2.10. □

Corollary 9.2.15. *Let X be a paracompact Hausdorff space. There is a natural bijection*

$$\left[X, \coprod_{k=0}^{\infty} \mathrm{Gr}_k(\mathbb{C}^\infty)\right] \xrightarrow{\cong} \pi_0 \mathrm{Vect}(X).$$

If X is a cell complex, this bijection can equivalently be written as

$$\pi_0 \,\mathrm{Hom}_{\mathrm{An}}\left(\Pi_\infty X, \coprod_{k=0}^{\infty} \mathbf{B}U(k)\right) \xrightarrow{\cong} \pi_0 \mathrm{Vect}(X).$$

Proof. Both sides of the first bijection decompose according to locally constant functions $r\colon X \to \mathbb{N}$. For such a function, let $X_k := r^{-1}(k)$. These subspaces are clopen, and hence paracompact Hausdorff, and they form a topological disjoint-union decomposition

$$X = \bigsqcup_{k\geq 0} X_k.$$

A continuous map $X \to \bigsqcup_{k\geq 0} \mathrm{Gr}_k(\mathbb{C}^\infty)$ with component function r amounts to a family of continuous maps $X_k \to \mathrm{Gr}_k(\mathbb{C}^\infty)$. Similarly, a vector bundle with rank function r amounts to a family of rank k vector bundles over the X_k. The first bijection therefore follows on each clopen stratum from Theorem 9.2.4.

The Schubert cell structures above also make $\bigsqcup_{k\geq 0} \mathrm{Gr}_k(\mathbb{C}^\infty)$ a cell complex. If X is a cell complex as well, then Corollaries 2.3.7 and 9.2.14 identifies the source of the first bijection with the source of the second. □

Remark 9.2.16. More generally, for every topological group G there exists a topological space BG together with a principal G-bundle $EG \to BG$ whose total space EG is contractible. In [Hus93, Section 4.12], it is shown that Milnor's construction of BG classifies principal G-bundles on paracompact Hausdorff spaces. Since EG is contractible, Proposition 9.2.10 implies that the underlying anima of BG is the classifying anima $\mathbf{B}G$. Thus the terminology 'classifying anima' is compatible with the classical terminology 'classifying space'.

9.2.4 The stable Grassmannian

To pass from vector bundles to K-theory, we use the stable Grassmannian BU, the colimit of the Grassmannians $\mathrm{Gr}_k(\mathbb{C}^\infty)$ along the maps $V \mapsto V \oplus \mathbb{C}$. We first record two point-set facts about sequential colimits of closed embeddings, which allow us to compute maps from compact spaces into BU and to identify its underlying anima.

Lemma 9.2.17. *Let*

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \dots$$

be a sequence of closed embeddings of T_1 topological spaces, and let $Y := \mathrm{colim}_n Y_n$ be the resulting colimit in Top. *Then every compact subset $K \subseteq Y$ is contained in some Y_n. In particular, every continuous map $f\colon X \to Y$ from a compact space factors through some Y_n.*

Proof. Assume for contradiction that K is not contained in any Y_n, and choose $x_n \in K \setminus Y_n$. Set $A := \{x_n \mid n \geq 0\}$. This set is infinite: if it were finite, all its points would lie in some common stage Y_N, contradicting $x_n \notin Y_n$ for $n \geq N$. For every m, the intersection $A \cap Y_m$ is contained in the finite set $\{x_0, \dots, x_{m-1}\}$, hence is closed in the T_1-space Y_m. The same holds for every subset of A. By the definition of the colimit topology, every subset of A is therefore closed in Y. Thus A is an infinite closed discrete subspace of K, contradicting compactness of K.

Each inclusion $Y_n \to Y$ is a closed embedding: a closed subset of Y_n is closed in every later stage and hence in Y. Thus, if $f(X)$ lies in Y_n, the factorization through Y_n is continuous. The last claim now follows by applying the first to the compact subset $f(X) \subseteq Y$. □

Proposition 9.2.18. *In the situation of Lemma 9.2.17, the canonical morphism*

$$\mathrm{colim}_n\, \Pi_\infty(Y_n) \longrightarrow \Pi_\infty(Y)$$

is an isomorphism of animae.

Proof. We use the Whitehead theorem for animae. The compactness lemma, applied to the point and to the interval, shows that the canonical morphism induces a bijection on path components. Fix a basepoint represented by some $y \in Y_N$. For every $q \geq 1$, compactness of S^q and $S^q \times [0,1]$ shows that maps $S^q \to Y$ and homotopies between them factor through finite stages. Consequently,

$$\operatorname{colim}_{n \geq N} \pi_q(Y_n, y) \xrightarrow{\cong} \pi_q(Y, y).$$

By Remark 2.4.19, these are the homotopy groups of the corresponding underlying animae. Homotopy groups of pointed animae commute with filtered colimits, so the left-hand side is also the q-th homotopy group of $\operatorname{colim}_n \Pi_\infty(Y_n)$ at y. The claim now follows from Proposition 2.4.22. □

Definition 9.2.19. We define the *stable Grassmannian*

$$BU := \operatorname{colim}\left(\operatorname{Gr}_0(\mathbb{C}^\infty) \xrightarrow{V \mapsto V \oplus \mathbb{C}} \operatorname{Gr}_1(\mathbb{C}^\infty) \xrightarrow{V \mapsto V \oplus \mathbb{C}} \operatorname{Gr}_2(\mathbb{C}^\infty) \to \cdots\right),$$

where the transition maps are induced by the standard isometric identification $\mathbb{C}^\infty \oplus \mathbb{C} \cong \mathbb{C}^\infty$. The Schubert cell structures make these transition maps inclusions of subcomplexes, so their colimit BU is again a cell complex; see Hatcher [Hat17, p. 34]. In particular, the transition maps are closed embeddings, and we equip BU with the colimit topology.

Proposition 9.2.20. *Let $U := \operatorname{colim}_n U(n)$ be the stable unitary group, where the transition maps are block-sum with the identity on $\mathbb{C}$. Then there is an isomorphism of animae*

$$\Pi_\infty(BU) \simeq \mathbf{B}U.$$

Proof. The principal bundles $V_n(\mathbb{C}^\infty) \to \operatorname{Gr}_n(\mathbb{C}^\infty)$ identify $\Pi_\infty \operatorname{Gr}_n(\mathbb{C}^\infty)$ with $\mathbf{B}U(n)$ by Corollary 9.2.14. Adjoining the new coordinate vector defines a map $V_n(\mathbb{C}^\infty) \to V_{n+1}(\mathbb{C}^\infty)$ over Grassmannian stabilization, equivariant for the block-sum inclusion $U(n) \to U(n+1)$. Thus these identifications are compatible with stabilization. Since the Grassmannian stabilization maps are closed embeddings between Hausdorff spaces, Proposition 9.2.18 gives

$$\Pi_\infty(BU) \simeq \operatorname{colim}_n \mathbf{B}U(n).$$

The classifying-anima equivalence $\mathbf{B}\colon \operatorname{Grp}(\mathrm{An}) \xrightarrow{\sim} \mathrm{An}_{*,\geq 1}$ preserves colimits. Since connected pointed animae are closed under filtered colimits, the filtered colimit of the $\mathbf{B}U(n)$ agrees whether computed in An, An_*, or $\mathrm{An}_{*,\geq 1}$. Moreover, filtered colimits in $\operatorname{Grp}(\mathrm{An})$ are computed on underlying animae, since finite limits commute with filtered colimits in An. Applying Proposition 9.2.18 also to the closed embeddings $U(n) \hookrightarrow U(n+1)$ therefore gives

$$\operatorname{colim}_n \mathbf{B}U(n) \simeq \mathbf{B}(\operatorname{colim}_n \Pi_\infty U(n)) \simeq \mathbf{B}\Pi_\infty(U),$$

which is the asserted classifying anima $\mathbf{B}U$. □

Remark 9.2.21. The Grassmannian and Stiefel-manifold results above have real analogues with $\mathbb{C}$ replaced by $\mathbb{R}$ and $U(k)$ by $O(k)$, with the same proofs. In particular, $\operatorname{Gr}_k(\mathbb{R}^\infty)$ classifies rank k real vector bundles over paracompact Hausdorff spaces, $V_k(\mathbb{R}^\infty)$ is contractible, and $\Pi_\infty \operatorname{Gr}_k(\mathbb{R}^\infty) \simeq \mathbf{B}O(k)$. We will use these facts in Chapter 10.

9.3 Complex K-theory groups

Direct sum makes the isomorphism classes of complex vector bundles into an abelian monoid. By adjoining additive inverses, we obtain the groups $K^0(X)$. For compact Hausdorff X, stable complements give an identification $K^0(X) \cong [X, \mathbb{Z}\times BU]$. Bott periodicity identifies $\widetilde{K}^0(X)$ with $\widetilde{K}^0(S^2 \wedge X)$ for pointed compact Hausdorff X.

Definition 9.3.1 (Complex K-theory)**.** Let X be a compact Hausdorff space. We define the *complex K-theory group* $K^0(X)$ as the Grothendieck group of the abelian monoid $\pi_0\mathrm{Vect}(X)$:

$$K^0(X) := (\pi_0\mathrm{Vect}(X))^{\mathrm{grp}}.$$

In other words, $K^0(X)$ consists of formal differences $[E] - [F]$ of complex vector bundles, subject to the relation $[E_1] - [F_1] = [E_2] - [F_2]$ if $E_1 \oplus F_2 \oplus G \cong F_1 \oplus E_2 \oplus G$ for some vector bundle G over X.

Example 9.3.2. For $X = \mathrm{pt}$, a vector bundle is simply a complex vector space, which up to isomorphism is determined by its dimension. We thus see that the abelian monoid $\pi_0\mathrm{Vect}(\mathrm{pt})$ is isomorphic to $\mathbb{N}$, and hence $K^0(\mathrm{pt}) \cong \mathbb{Z}$.

The abelian group $K^0(X)$ is contravariantly functorial in X: given a continuous map $f\colon X \to Y$, pullback along f determines a map

$$f^*\colon \pi_0\mathrm{Vect}(Y) \to \pi_0\mathrm{Vect}(X),$$

which induces a map $f^*\colon K^0(Y) \to K^0(X)$ on Grothendieck groups.

Tensor product distributes over direct sum, so it makes $\pi_0\mathrm{Vect}(X)$ into a commutative semiring. Its multiplication extends uniquely to a commutative ring structure on $K^0(X)$, with unit the class of the trivial line bundle. Pullback maps are morphisms of rings. The details are left to Exercise 9.1.

Corollary 9.3.3. *Homotopic maps between compact Hausdorff spaces induce the same pullback on K-theory. In particular, every homotopy equivalence $f\colon X \to Y$ between compact Hausdorff spaces induces an isomorphism*

$$f^*\colon K^0(Y) \xrightarrow{\cong} K^0(X)$$

of commutative rings.

Proof. This follows from Theorem 9.1.4 by applying group completion. □

Definition 9.3.4 (Reduced K-theory)**.** Let X be a compact Hausdorff space. We define the abelian group $\widetilde{K}^0(X)$ as the cokernel of the map $\mathbb{Z} \cong K^0(\mathrm{pt}) \to K^0(X)$. This is called the *reduced K-theory group* of X.

If X has a basepoint x, restriction to x splits the map from $\mathbb{Z}$, giving an isomorphism $\widetilde{K}^0(X) \cong \ker(K^0(X) \to K^0(\{x\}))$. It sends the class of α modulo constant trivial classes to $\alpha - \mathrm{rk}_x(\alpha)[\underline{\mathbb{C}}]$, where $\mathrm{rk}_x(\alpha)$ is its virtual rank at x.

Example 9.3.5 (Complex K-theory of the circle and the two-sphere)**.** There are isomorphisms

$$K^0(S^1) \cong \mathbb{Z}, \qquad K^0(S^2) \cong \mathbb{Z}\times\mathbb{Z},$$

and hence $\widetilde{K}^0(S^1) = 0$ and $\widetilde{K}^0(S^2) \cong \mathbb{Z}$. Identifying S^2 with $\mathbb{C}\mathrm{P}^1$, the class $[H] - [\underline{\mathbb{C}}]$ of the tautological line bundle generates $\widetilde{K}^0(S^2)$.

Proof. The calculation for S^1 follows from Exercise 9.1.6. For S^2, the clutching classification of Exercise 9.1.7 identifies a positive-rank vector bundle with its rank and the winding number of the determinant of its clutching function. These invariants are additive under direct sum, so group completion gives $K^0(S^2) \cong \mathbb{Z}\times\mathbb{Z}$. The second factor is the reduced group, and the clutching function of H has winding number 1 up to sign. □

We next show that every class in $K^0(X)$ can be written as $[E] - [\underline{\mathbb{C}}^N]$.

Theorem 9.3.6 (Stable complements, Hatcher [Hat17, Proposition 1.4 and proof of Proposition 2.1])**.** *Let X be a compact Hausdorff space. For every complex vector bundle $E \to X$ there exists another vector bundle $E' \to X$ such that the direct sum $E \oplus E'$ is isomorphic to a trivial vector bundle $\underline{\mathbb{C}}^N := X \times \mathbb{C}^N$ for some $N \in \mathbb{N}$.*

Proof sketch. Choose finitely many local trivializations and a subordinate partition of unity. Multiplying the local coordinate functions by the partition-of-unity functions embeds E into a trivial bundle $X \times \mathbb{C}^N$. After choosing a Hermitian inner product on this trivial bundle, the orthogonal complements of the fibers of E assemble to a vector bundle E' with $E \oplus E' \cong \underline{\mathbb{C}}^N$. □

Corollary 9.3.7. *For every compact Hausdorff space X, there is a surjective morphism of monoids $\pi_0\mathrm{Vect}(X) \to \widetilde{K}^0(X)$.*

Proof. Represent a class in $\widetilde{K}^0(X)$ by a difference $[E] - [F]$. Choose a vector bundle F' such that $F \oplus F' \cong \underline{\mathbb{C}}^N$ by Theorem 9.3.6. Modulo classes of trivial bundles, the given class is then represented by the honest vector bundle $E \oplus F'$. □

9.3.1 Representability of K^0

We now show that $K^0(X)$ is represented by $\mathbb{Z}\times BU$. First, recall how direct sum of vector bundles defines addition on $[X, BU]$. For compact Hausdorff X, any two maps $X \to BU$ factor through Grassmannians by Lemma 9.2.17 and hence classify vector bundles E and F by Theorem 9.2.4. Their sum is the classifying map of $E \oplus F$, viewed as a map to BU. This is well-defined: homotopic maps classify isomorphic bundles after stabilization, and adding a trivial summand to either bundle only stabilizes their direct sum. Together with pointwise addition on $[X, \mathbb{Z}]$, this defines addition on $[X, \mathbb{Z}\times BU]$.

Proposition 9.3.8. *Let X be a compact Hausdorff space. There is a natural isomorphism of abelian groups*

$$[X, \mathbb{Z}\times BU] \xrightarrow{\cong} K^0(X).$$

Proof. Both sides take finite disjoint unions to products. Since a continuous integer-valued function on X has finite image, we may work on clopen pieces where the integer coordinate or virtual rank is constant. Let $f\colon X \to \{m\} \times BU$ be continuous. By Lemma 9.2.17, its BU-coordinate factors through some $\mathrm{Gr}_n(\mathbb{C}^\infty)$. If E is the classified bundle, we assign to f the class

$$[E] - n[\underline{\mathbb{C}}] + m[\underline{\mathbb{C}}] \in K^0(X).$$

Passing from $\mathrm{Gr}_n(\mathbb{C}^\infty)$ to $\mathrm{Gr}_{n+1}(\mathbb{C}^\infty)$ replaces E by $E \oplus \underline{\mathbb{C}}$, and hence leaves $[E] - n[\underline{\mathbb{C}}] + m[\underline{\mathbb{C}}]$ unchanged. If two maps are homotopic, the homotopy has constant $\mathbb{Z}$-coordinate on $X \times [0, 1]$ and,

again by compactness, factors through a common Grassmannian stage after stabilization. Pulling back the universal bundle along the two ends gives isomorphic bundles by Theorem 9.1.4. Thus we get a well-defined natural map

$$\Phi_X \colon [X, \mathbb{Z}\times BU] \to K^0(X).$$

For surjectivity, let $\alpha = [E] - [F] \in K^0(X)$ have constant virtual rank m. By Theorem 9.3.6, choose F' and N such that $F \oplus F' \cong \underline{\mathbb{C}}^N$. Then

$$\alpha = [E \oplus F'] - N[\underline{\mathbb{C}}].$$

The bundle $E \oplus F'$ has rank $N + m$, so Theorem 9.2.4 classifies it by a map $g \colon X \to \mathrm{Gr}_{N+m}(\mathbb{C}^\infty) \subseteq BU$. Viewed as a map $X \to \{m\} \times BU$, this maps under Φ_X to

$$[E \oplus F'] - (N + m)[\underline{\mathbb{C}}] + m[\underline{\mathbb{C}}] = [E] - [F] = \alpha.$$

This proves surjectivity.

For injectivity, suppose $f_0, f_1 \colon X \to \mathbb{Z}\times BU$ have the same image under Φ_X. Their images in $K^0(X)$ have the same rank function, so the $\mathbb{Z}$-coordinates of f_0 and f_1 agree. On a piece where their common integer coordinate is m, choose factorizations $f_i \colon X \to \mathrm{Gr}_{n_i}(\mathbb{C}^\infty)$ classifying bundles E_i. The equality $\Phi_X(f_0) = \Phi_X(f_1)$ gives

$$[E_0] - n_0[\underline{\mathbb{C}}] = [E_1] - n_1[\underline{\mathbb{C}}]$$

in $K^0(X)$. By the definition of the Grothendieck group, there exists a vector bundle F and an isomorphism

$$E_0 \oplus \underline{\mathbb{C}}^{n_1} \oplus F \cong E_1 \oplus \underline{\mathbb{C}}^{n_0} \oplus F.$$

Choose F' with $F \oplus F' \cong \underline{\mathbb{C}}^q$ by Theorem 9.3.6. After stabilizing f_0 by $n_1 + q$ steps and f_1 by $n_0 + q$ steps, the two maps land in the same Grassmannian $\mathrm{Gr}_{n_0+n_1+q}(\mathbb{C}^\infty)$ and classify isomorphic vector bundles. By Theorem 9.2.4, they are homotopic. This proves injectivity.

The construction is compatible with addition. If maps through $\{m\}\times\mathrm{Gr}_n(\mathbb{C}^\infty)$ and $\{m'\}\times\mathrm{Gr}_{n'}(\mathbb{C}^\infty)$ classify bundles E and E', then their sum classifies $E \oplus E'$ at stage $n + n'$ and has $\mathbb{Z}$-coordinate $m + m'$. Therefore

$$[E \oplus E'] - (n + n')[\underline{\mathbb{C}}] + (m + m')[\underline{\mathbb{C}}]$$

equals the sum of the two associated K^0-classes. Thus Φ_X is an isomorphism of abelian groups. □

9.3.2 The Bott class and Bott periodicity

Definition 9.3.9 (Bott class)**.** Let $H \to \mathbb{C}\mathrm{P}^1$ be the tautological line bundle. The *Bott class* is

$$\beta := [H] - [\underline{\mathbb{C}}] \in \widetilde{K}^0(\mathbb{C}\mathrm{P}^1).$$

By Example 9.3.5, this class generates $\widetilde{K}^0(\mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}$.

For compact Hausdorff spaces X and Y, the *external product* is defined by $\alpha \boxtimes \gamma := \mathrm{pr}_X^*\,\alpha \cdot \mathrm{pr}_Y^*\,\gamma \in K^0(X \times Y)$. For vector bundles, this is induced by $\mathrm{pr}_X^*\,E \otimes \mathrm{pr}_Y^*\,F$.

If X and Y are pointed, regard their reduced groups as kernels of restriction to the basepoints. For reduced classes α and γ, the external product restricts to zero on $X \vee Y \subseteq X \times Y$. The classical exactness result of Hatcher [Hat17, pp. 53–54] identifies the kernel of this restriction with $\widetilde{K}^0(X \wedge Y)$: the quotient map $q\colon X \times Y \to X \wedge Y$ induces an injection

$$q^*\colon \widetilde{K}^0(X \wedge Y) \hookrightarrow K^0(X \times Y)$$

whose image is precisely that kernel. Thus $\mathrm{pr}_X^*\,\alpha \cdot \mathrm{pr}_Y^*\,\gamma$ lifts uniquely to a class in $\widetilde{K}^0(X \wedge Y)$. This defines the *reduced external product*

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \longrightarrow \widetilde{K}^0(X \wedge Y),$$

also denoted $\alpha \boxtimes \gamma$. The periodicity of complex K-theory rests on the following classical theorem.

Theorem 9.3.10 (Bott periodicity for $\widetilde{K}^0$, Hatcher [Hat17, Theorem 2.11])**.** *For every pointed compact Hausdorff space X, external multiplication by the Bott class induces an isomorphism*

$$\widetilde{K}^0(X) \xrightarrow[\cong]{\alpha \mapsto \alpha \boxtimes \beta} \widetilde{K}^0(X \wedge \mathbb{C}\mathrm{P}^1) \cong \widetilde{K}^0(\Sigma^2 X).$$

9.4 Complex K-theory spectra

The classical construction above uses only isomorphism classes of vector bundles. To construct a representing spectrum, we now incorporate the automorphisms of vector bundles and all their homotopies, and then apply the homotopical group completion theorem. This produces connective complex K-theory. Inverting the Bott class then produces its periodic counterpart.

9.4.1 The anima of vector bundles

Let X be a topological space, and write $\mathrm{Vect}^{\mathrm{disc}}(X)$ for the 1-category of finite-rank complex vector bundles over X and bundle maps, with discrete hom sets.

Definition 9.4.1. The *moduli anima of complex vector bundles over X* is

$$\mathrm{Vect}(X)^{\simeq} := \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Vect}^{\mathrm{disc}}(X \times |\Delta^n|)^{\simeq},$$

where the colimit is formed in An and the simplicial maps are given by pullback.

Bundles over $X \times |\Delta^n|$ describe families parametrized by simplices. For compact Hausdorff X, this construction models the homotopy coherent nerve of the topologically enriched groupoid of vector bundles and bundle isomorphisms. Direct sum and tensor product supply addition and multiplication. The following proposition includes their distributivity and higher coherences; here $\mathrm{CRig}(\mathrm{An}) = \mathrm{CAlg}(\mathrm{CMon}(\mathrm{An}))$ is the ∞-category of commutative semiring animae.

Proposition 9.4.2. *Direct sum and tensor product refine Definition 9.4.1 to a functor*

$$\mathrm{Vect}(-)^{\simeq}\colon \mathrm{Top}^{\mathrm{op}} \longrightarrow \mathrm{CRig}(\mathrm{An}).$$

The defining colimit can also be formed in $\mathrm{CRig}(\mathrm{An})$*. If X is paracompact Hausdorff, the inclusion of the zeroth simplicial degree induces a natural bijection*

$$\pi_0 \mathrm{Vect}^{\mathrm{disc}}(X)^{\simeq} \xrightarrow{\cong} \pi_0 \mathrm{Vect}(X)^{\simeq}.$$

Corollary 9.4.3. *The construction of Definition 9.4.1, applied to finite-rank real vector bundles, defines a functor*

$$\mathrm{Vect}_{\mathbb{R}}(-)^{\simeq}\colon \mathrm{Top}^{\mathrm{op}} \longrightarrow \mathrm{CRig}(\mathrm{An})$$

whose underlying anima is computed by the corresponding colimit in An.

The coherent constructions and proofs are given in Section 9.5, using the algebraic results of Part II. For a point, the components and the semiring operations admit the following explicit description.

Proposition 9.4.4. *The underlying anima of* $\mathrm{Vect}(\mathrm{pt})^{\simeq}$ *is the disjoint union* $\bigsqcup_{n\geq 0}\mathbf{B}U(n)$. *Under this isomorphism, the addition map*

$$\left(\bigsqcup_{k=0}^{\infty}\mathbf{B}U(k)\right)\times\left(\bigsqcup_{l=0}^{\infty}\mathbf{B}U(l)\right)\to\left(\bigsqcup_{m=0}^{\infty}\mathbf{B}U(m)\right)$$

is given on the (k,l)*-component by the map* $\mathbf{B}U(k)\times\mathbf{B}U(l)\to\mathbf{B}U(k+l)$ *induced by the block-sum inclusion*

$$U(k)\times U(l)\hookrightarrow U(k+l),\qquad (A,B)\mapsto\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}.$$

The multiplication map is given on the (k,l)*-component by the map* $\mathbf{B}U(k)\times\mathbf{B}U(l)\to\mathbf{B}U(kl)$ *induced by the tensor-product representation*

$$U(k)\times U(l)\longrightarrow U(kl),\qquad (A,B)\longmapsto A\otimes B.$$

In particular, the induced semiring structure on $\pi_0\mathrm{Vect}(\mathrm{pt})\cong\mathbb{N}$ *is the usual one.*

Proof. By Definition 9.4.1 and Proposition 9.4.2, the rank n component of $\mathrm{Vect}(\mathrm{pt})^{\simeq}$ is the geometric realization of the simplicial groupoid whose value at $[q]$ is the full subgroupoid

$$\left(\mathrm{Vect}^{\mathrm{disc}}(|\Delta^q|)^{\simeq}\right)_{\mathrm{rk}=n}$$

spanned by the rank n bundles. Every rank n vector bundle over $|\Delta^q|$ is trivial. Consequently, the full subgroupoid on the standard trivial bundle $|\Delta^q|\times\mathbb{C}^n$ is an equivalent one-object groupoid. Its automorphism group is

$$\mathrm{Hom}_{\mathrm{Top}}(|\Delta^q|,\mathrm{GL}_n(\mathbb{C})),$$

and these groups assemble into the singular simplicial group of $\mathrm{GL}_n(\mathbb{C})$. Geometric realization preserves finite products by Lemma 22.6.12, so it commutes with the formation of classifying animae of simplicial groups. Since the realization of the singular simplicial group is the underlying group anima of $\mathrm{GL}_n(\mathbb{C})$, the realization of these one-object groupoids is $\mathbf{B}\,\mathrm{GL}_n(\mathbb{C})$. Taking the disjoint union over all ranks therefore gives

$$\mathrm{Vect}(\mathrm{pt})^{\simeq}\simeq\bigsqcup_{n\geq 0}\mathbf{B}\,\mathrm{GL}_n(\mathbb{C}).$$

The inclusions $U(n)\hookrightarrow\mathrm{GL}_n(\mathbb{C})$ are homotopy equivalences by polar decomposition and commute with block sums and tensor products. They therefore identify $\mathbf{B}\,\mathrm{GL}_n(\mathbb{C})$ with $\mathbf{B}U(n)$ compatibly with both operations.

Indeed, direct sum sends $(\mathbb{C}^k,\mathbb{C}^l)$ to $\mathbb{C}^{k+l}$ and induces the block-sum inclusion on automorphism groups. Similarly, tensor product sends $(\mathbb{C}^k,\mathbb{C}^l)$ to $\mathbb{C}^k\otimes\mathbb{C}^l\cong\mathbb{C}^{kl}$ and induces the tensor-product representation on automorphism groups. Passing to classifying animae gives the stated formulas. □

9.4.2 Connective complex K-theory

The definition of the complex K-theory groups via group completion admits the following homotopical refinement.

Definition 9.4.5. We denote by

$$\mathrm{BUP} := (\mathrm{Vect}(\mathrm{pt})^{\simeq})^{\mathrm{grp}} \quad \in \quad \mathrm{CGrp}(\mathrm{An})$$

the group completion of $\mathrm{Vect}(\mathrm{pt})^{\simeq}$ with respect to direct sum, using the left adjoint $(-)^{\mathrm{grp}}$ from Corollary 5.4.5. We define the *connective complex K-theory spectrum* $\mathrm{ku} \in \mathrm{Sp}_{\geq 0}$ to be the connective spectrum corresponding to BUP under the equivalence $\mathrm{Sp}_{\geq 0} \simeq \mathrm{CGrp}(\mathrm{An})$ of Theorem 5.4.6.

While group completion is abstract in general, the telescope form of Theorem 5.5.2 gives an explicit description here.

Corollary 9.4.6. *The underlying anima of* BUP *is* $\mathbb{Z} \times \mathbf{B}U$*:*

$$\mathrm{BUP} \simeq (\mathrm{Vect}(\mathrm{pt})^{\simeq})^{\mathrm{grp}} \simeq \mathbb{Z} \times \mathbf{B}U.$$

Proof. By Proposition 9.4.4, the commutative monoid $M = \mathrm{Vect}(\mathrm{pt})^{\simeq}$ has underlying anima $\bigsqcup_{k \geq 0} \mathbf{B}U(k)$, with addition induced by block sum. Let $x \in M$ be the point corresponding to the one-dimensional vector space $\mathbb{C}$. The telescope $T(M, x)$ is obtained by repeatedly adding this line. Its component of virtual rank $m \in \mathbb{Z}$ is the colimit

$$\mathrm{colim}_{n \geq \max(0, -m)} \mathbf{B}U(m + n),$$

which is canonically $\mathbf{B}U$. Thus $T(M, x) \simeq \mathbb{Z} \times \mathbf{B}U$.

The two hypotheses of Theorem 5.5.2 are immediate. The class of x is the generator $1 \in \pi_0(M) \cong \mathbb{N}$. The fundamental group of $\mathbf{B}U$ is $\pi_0(U)$, which is trivial because the stable unitary group is connected. Hence $T(M, x) \to M^{\mathrm{grp}}$ is an isomorphism. □

Proposition 9.4.7. *The connective complex K-theory spectrum* ku *admits a canonical structure of commutative ring spectrum.*

Symmetric monoidal group completion sends the commutative semiring anima $\mathrm{Vect}(\mathrm{pt})^{\simeq}$ to the commutative ring spectrum ku. The coherent construction is given in Section 9.5; the symmetric monoidal group-completion functor is constructed in Proposition 16.6.4.

Corollary 9.4.8. *Let X be a compact Hausdorff space which has the homotopy type of a CW-complex. There is a natural isomorphism of commutative rings*

$$K^0(X) \cong \mathrm{ku}^0(\Pi_\infty(X)).$$

Proof. Since X has CW homotopy type and $\mathbb{Z} \times BU$ is a cell complex, Corollary 2.3.7 identifies continuous homotopy classes with morphisms of their underlying animae. Combining Propositions 9.3.8 and 9.2.20 and Corollary 9.4.6 with the adjunction $\mathbb{S}[-] \dashv \Omega^\infty$ gives

$$K^0(X) \cong [X, \mathbb{Z} \times BU] \cong \pi_0 \mathrm{Hom}_{\mathrm{An}}(\Pi_\infty X, \mathrm{BUP}) \cong \mathrm{ku}^0(\Pi_\infty X).$$

It remains to check multiplication. Set $M := \mathrm{Vect}(\mathrm{pt})^{\simeq}$. The unit map $M \to M^{\mathrm{grp}} = \mathrm{BUP}$ is a morphism of commutative semiring objects, since group completion is symmetric monoidal by Proposition 16.6.4. For a vector bundle E over X, its classifying map to M therefore determines a class in $\mathrm{ku}^0(\Pi_\infty X)$, and Proposition 9.4.4 shows that direct sum and tensor product of bundles map to the sum and product of the corresponding classes. This semiring map

$$\pi_0 \mathrm{Vect}(X) \longrightarrow \mathrm{ku}^0(\Pi_\infty X)$$

extends uniquely to a ring map from $K^0(X)$. Under the description $M^{\mathrm{grp}} \simeq \mathbb{Z} \times \mathbf{B}U$ of Corollary 9.4.6, this extension is the isomorphism displayed above: an honest vector bundle is sent to its stable classifying map. Hence that isomorphism is multiplicative. □

For a pointed compact Hausdorff space X of the homotopy type of a CW-complex, naturality with respect to the basepoint shows that the preceding isomorphism identifies the kernels of the restriction maps to the point. It therefore restricts to a natural isomorphism

$$\widetilde{K}^0(X) \cong \widetilde{\mathrm{ku}}^0(\Pi_\infty X).$$

In particular, for $n \geq 0$ we obtain $\widetilde{K}^0(S^n) \cong \pi_n \mathrm{ku}$.

For finite cell complexes X and Y with basepoints that are vertices, these reduced isomorphisms are compatible with external products. Indeed, if $E \to X$ and $F \to Y$ are vector bundles, then multiplication of their classifying maps classifies

$$E \boxtimes F := \mathrm{pr}_X^* E \otimes \mathrm{pr}_Y^* F \longrightarrow X \times Y.$$

Thus the unreduced isomorphism preserves external products. Under the stated hypotheses, $X \wedge Y$ is again a finite cell complex and $\Pi_\infty(X \wedge Y) \simeq \Pi_\infty X \wedge \Pi_\infty Y$. The reduced product in ku therefore pulls back along $X \times Y \to X \wedge Y$ to the ordinary external product. The injectivity of this pullback in classical K-theory, established in Section 9.3, shows that the reduced isomorphism preserves products as well.

Choose a pointed homeomorphism $\mathbb{C}\mathrm{P}^1 \cong S^2$ under which $c_1(H)$ corresponds to the generator of $\widetilde{H}^2(S^2;\mathbb{Z})$ determined by suspension. Here $c_1(H)$ is the first Chern class of the tautological line bundle; see Milnor and Stasheff [MS74]. Under this identification and the preceding reduced isomorphism, the Bott class of Definition 9.3.9 gives a class

$$\beta \in \pi_2 \mathrm{ku}\,.$$

Construction 9.4.9 (The Bott self-map)**.** Choose a morphism $\beta\colon \mathbb{S}[2] \to \mathrm{ku}$ representing the Bott class, or equivalently a morphism $\beta\colon \mathbb{S} \to \mathrm{ku}[-2]$. Multiplication by this class defines a morphism of spectra

$$b\colon \mathrm{ku} \simeq \mathbb{S} \otimes \mathrm{ku} \xrightarrow{\beta \otimes \mathrm{id}} \mathrm{ku}[-2] \otimes \mathrm{ku} \xrightarrow{\mu[-2]} \mathrm{ku}[-2],$$

where μ denotes the multiplication on ku. We call b the *Bott map*.

By the compatibility of the preceding isomorphism with reduced external products, the map

$$b_*\colon \pi_n \mathrm{ku} \longrightarrow \pi_{n+2} \mathrm{ku}$$

for $n \geq 0$ is reduced external multiplication with the classical Bott class.

Corollary 9.4.10 (Bott periodicity for ku)**.** *The Bott map induces an isomorphism*

$$b_*\colon \pi_n \mathrm{ku} \xrightarrow{\cong} \pi_{n+2} \mathrm{ku}$$

for every $n \geq 0$*.*

Proof. Apply Theorem 9.3.10 to $X = S^n$. Under the isomorphism above,

$$\pi_n \mathrm{ku} \cong \widetilde{\mathrm{ku}}^0(S^n) \cong \widetilde{K}^0(S^n),$$

the map b_* is external multiplication by the Bott class. The claim therefore follows from classical Bott periodicity. □

9.4.3 Periodic complex K-theory

The connective spectrum ku admits a 2-periodic variant KU, obtained by inverting the Bott class.

Definition 9.4.11. The *periodic complex K-theory spectrum* is the colimit

$$\mathrm{KU} := \mathrm{ku}[\beta^{-1}] := \mathrm{colim}\left(\mathrm{ku} \xrightarrow{b} \mathrm{ku}[-2] \xrightarrow{b[-2]} \mathrm{ku}[-4] \to \cdots\right) \quad \in \quad \mathrm{Sp}.$$

We write $\iota\colon \mathrm{ku} \to \mathrm{KU}$ for the canonical map into the colimit.

Corollary 9.4.12. *The periodic complex K-theory spectrum* KU *admits a canonical structure of commutative ring spectrum, and the map* $\mathrm{ku} \to \mathrm{KU}$ *is a morphism of commutative ring spectra.*

Proof. By Corollary 8.4.5, we may invert β in the commutative ring spectrum ku. By Lemma 8.4.6, the underlying spectrum and localization map are KU and ι from Definition 9.4.11. □

Lemma 9.4.13. *For every* $k \in \mathbb{Z}$ *there is a natural isomorphism*

$$\pi_k \mathrm{KU} \cong \mathrm{colim}\left(\pi_k \mathrm{ku} \xrightarrow{b_*} \pi_{k+2} \mathrm{ku} \xrightarrow{b_*} \pi_{k+4} \mathrm{ku} \to \cdots\right).$$

Proof. By Lemma 4.4.28, the functor π_k preserves the filtered colimit defining KU. Moreover $\pi_k(\mathrm{ku}[-2j]) = \pi_{k+2j}\,\mathrm{ku}$, and the transition maps are induced by the Bott map. □

Theorem 9.4.14. *The map* $\iota\colon \mathrm{ku} \to \mathrm{KU}$ *exhibits* ku *as the connective cover of* KU*: it induces an isomorphism on* π_k *for* $k \geq 0$*, and hence an isomorphism*

$$\mathrm{ku} \xrightarrow{\sim} \tau_{\geq 0}\,\mathrm{KU}\,.$$

Proof. For $k \geq 0$, Lemma 9.4.13 writes $\pi_k \mathrm{KU}$ as a filtered colimit whose transition maps

$$b_*\colon \pi_{k+2j}\,\mathrm{ku} \to \pi_{k+2j+2}\,\mathrm{ku}$$

are all isomorphisms by Corollary 9.4.10. Hence $\pi_k(\iota)$ is an isomorphism for $k \geq 0$. Since ku is connective, the map ι factors through the connective cover as

$$\mathrm{ku} \xrightarrow{\bar{\iota}} \tau_{\geq 0}\,\mathrm{KU} \to \mathrm{KU}$$

by Corollary 5.4.7. The map $\bar{\iota}$ is an isomorphism on all homotopy groups, because both source and target are connective and we have already checked the non-negative homotopy groups. It is therefore an isomorphism by Corollary 4.4.27. □

Corollary 9.4.15. *For every anima X and every integer $k \leq 0$, the map* $\mathrm{ku} \to \mathrm{KU}$ *induces an isomorphism*

$$\mathrm{ku}^k(X) \xrightarrow{\cong} \mathrm{KU}^k(X).$$

Proof. For every connective spectrum Y, the isomorphism $\mathrm{ku} \simeq \tau_{\geq 0}\,\mathrm{KU}$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{Sp}}(Y, \mathrm{ku}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sp}}(Y, \mathrm{KU}).$$

Apply this to $Y = \Sigma^{-k}\,\mathbb{S}[X]$, which is connective for $k \leq 0$. □

Proposition 9.4.16 (Periodicity)**.** *There is an isomorphism* $\Sigma^2\,\mathrm{KU} \simeq \mathrm{KU}$.

Proof. The spectrum $\mathrm{KU}[-2]$ is the colimit of the diagram defining KU with its first term omitted:

$$\mathrm{ku}[-2] \xrightarrow{b[-2]} \mathrm{ku}[-4] \to \cdots.$$

This subdiagram is cofinal in the original sequential diagram, so $\mathrm{KU}[-2] \simeq \mathrm{KU}$. Applying Σ^2 gives the desired isomorphism. □

Remark 9.4.17. The periodicity isomorphism of Proposition 9.4.16 is multiplication by the image of β in $\pi_2\,\mathrm{KU}$. Thus that image is invertible in the graded ring $\pi_*(\mathrm{KU})$.

Corollary 9.4.18. *There are isomorphisms of graded rings*

$$\pi_*\,\mathrm{ku} \cong \mathbb{Z}[\beta] \qquad \textit{and} \qquad \pi_*\,\mathrm{KU} \cong \mathbb{Z}[\beta^{\pm 1}],$$

with $|\beta| = 2$*. Equivalently,*

$$\pi_k(\mathrm{KU}) \cong \begin{cases} \mathbb{Z} & k \textit{ even}, \\ 0 & k \textit{ odd}. \end{cases}$$

Proof. The description $\mathrm{BUP} \simeq \mathbb{Z} \times \mathbf{B}U$ of Corollary 9.4.6 gives $\pi_0\,\mathrm{ku} \cong \mathbb{Z}$ and $\pi_1\,\mathrm{ku} = 0$. Multiplication by β induces isomorphisms $\pi_n\,\mathrm{ku} \to \pi_{n+2}\,\mathrm{ku}$ for $n \geq 0$ by Corollary 9.4.10. Since ku is connective, this proves $\pi_*\,\mathrm{ku} \cong \mathbb{Z}[\beta]$. The calculation for $\mathrm{KU} = \mathrm{ku}[\beta^{-1}]$ now follows from Lemma 7.2.4. □

9.4.4 Rational K-theory and the Chern character

The homotopy-group calculation and Proposition 7.2.16 give an abstract noncanonical isomorphism of spectra

$$\mathrm{KU}_{\mathbb{Q}} \simeq \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}[2n].$$

The chosen Bott class allows us to refine this to an isomorphism of commutative ring spectra. Let $\mathbb{Q}[\beta^{\pm 1}]$ denote the commutative DGA with zero differential on an invertible generator β in degree 2, and let $H\,\mathbb{Q}[\beta^{\pm 1}]$ be its Eilenberg–MacLane ring spectrum in the sense of Section 8.2. Its underlying spectrum is $\bigoplus_{n \in \mathbb{Z}} \mathbb{Q}[2n]$. The canonical map

$$\bigoplus_{n \in \mathbb{Z}} \mathbb{Q}[2n] \longrightarrow \prod_{n \in \mathbb{Z}} \mathbb{Q}[2n]$$

is an isomorphism, since it induces an isomorphism on every homotopy group.

Proposition 9.4.19 (Rational periodic complex K-theory)**.** *A choice of representative of the rationalized Bott class induces an isomorphism of commutative $\mathbb{Q}$-algebras*

$$H\,\mathbb{Q}[\beta^{\pm 1}] \xrightarrow{\sim} \mathrm{KU}_{\mathbb{Q}}$$

which sends the Laurent polynomial generator β to the rationalized Bott class. Its homotopy class is independent of the chosen representative.

Proof. By the smashing description of rationalization and Theorem 7.2.14, we have $\mathrm{KU}_{\mathbb{Q}} \simeq \mathrm{KU} \otimes \mathbb{Q}$, so $\mathrm{KU}_{\mathbb{Q}}$ is canonically a commutative $\mathbb{Q}$-algebra. Under the symmetric monoidal equivalence $\mathrm{D}(\mathbb{Q}) \simeq \mathrm{Mod}_{\mathbb{Q}}$ of Corollary 8.2.4, the polynomial algebra $\mathbb{Q}[\beta]$ is the free commutative $\mathbb{Q}$-algebra on a generator in degree 2. Indeed, the r-th derived symmetric power of $\mathbb{Q}[2]$ is $\mathbb{Q}[2r]$: the permutation action is trivial, and higher homology of finite groups over $\mathbb{Q}$ vanishes. A chosen representative $\mathbb{Q}[2] \to \mathrm{KU}_{\mathbb{Q}}$ of the rationalized Bott class therefore induces a morphism of commutative $\mathbb{Q}$-algebras

$$H\,\mathbb{Q}[\beta] \longrightarrow \mathrm{KU}_{\mathbb{Q}}\,.$$

The canonical map $H\,\mathbb{Q}[\beta] \to H\,\mathbb{Q}[\beta^{\pm 1}]$ exhibits its target as $(H\,\mathbb{Q}[\beta])[\beta^{-1}]$: indeed, the induced map from this localization is an isomorphism on homotopy groups by Lemma 8.4.6. Since the Bott class is invertible in $\pi_*(\mathrm{KU}_{\mathbb{Q}})$, the preceding morphism therefore extends uniquely to $H\,\mathbb{Q}[\beta^{\pm 1}]$. It induces the isomorphism

$$\mathbb{Q}[\beta^{\pm 1}] \xrightarrow{\cong} \pi_*(\mathrm{KU}_{\mathbb{Q}})$$

from Corollary 9.4.18 and Lemma 7.2.4, and hence is an isomorphism of commutative ring spectra. Any two representatives of the Bott class lie in the same component of the mapping anima. The free-algebra and localization adjunctions preserve this identification, so the induced algebra maps have the same homotopy class. □

Definition 9.4.20 (Chern character)**.** The *Chern character* is the stable multiplicative cohomology operation induced by the composite

$$\mathrm{KU} \longrightarrow \mathrm{KU}_{\mathbb{Q}} \xrightarrow{\simeq} H\,\mathbb{Q}[\beta^{\pm 1}],$$

where the second map is the inverse of an isomorphism constructed in Proposition 9.4.19. The induced operation on cohomology groups is independent of the chosen representative of the Bott class. Thus, for every anima X and every $k \in \mathbb{Z}$, it gives a natural morphism

$$\mathrm{ch}\colon\ \mathrm{KU}^k(X) \longrightarrow \prod_{n \in \mathbb{Z}} H^{k+2n}(X; \mathbb{Q}).$$

Remark 9.4.21. Let $x = c_1(L)$ for the universal line bundle L over $\mathbb{C}\mathrm{P}^\infty$. Its classifying map defines a class in $\mathrm{KU}^0(\Pi_\infty \mathbb{C}\mathrm{P}^\infty)$, whose Chern character is a power series $F(x) \in \mathbb{Q}[[x]]$. Multiplicativity under tensor product of line bundles gives $F(x + y) = F(x)F(y)$. The unit gives $F(0) = 1$, and our normalization on $\mathbb{C}\mathrm{P}^1$ gives $F'(0) = 1$. Differentiating with respect to y and setting $y = 0$ gives $F' = F$, so $F(x) = e^x$. By naturality,

$$\mathrm{ch}(L) = e^{c_1(L)}$$

for every line bundle on a finite cell complex. The splitting principle then identifies our operation with the classical Chern character. We do not develop Chern classes or the splitting principle here; see Milnor and Stasheff [MS74] and Hatcher [Hat17, pp. 109–110 and Proposition 4.2].

Corollary 9.4.22. *Let X be a finite anima and let $k \in \mathbb{Z}$. Then the Chern character induces an isomorphism*

$$\mathrm{KU}^k(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{Z}} H^{k+2n}(X;\mathbb{Q}).$$

Proof. Since X is finite, the functor $\hom(\mathbb{S}[X], -)\colon \mathrm{Sp} \to \mathrm{Sp}$ commutes with rationalization and with direct sums. Applying it to Proposition 9.4.19 gives

$$\hom(\mathbb{S}[X], \mathrm{KU})_{\mathbb{Q}} \simeq \hom(\mathbb{S}[X], \mathrm{KU}_{\mathbb{Q}}) \simeq \bigoplus_{n\in\mathbb{Z}} \hom(\mathbb{S}[X], \mathbb{Q}[2n]).$$

The resulting map is induced by the Chern character of Definition 9.4.20. Since X is finite, only finitely many of the groups $H^{k+2n}(X;\mathbb{Q})$ are nonzero, so the product in its definition agrees with the direct sum above. Passing to π_{-k} gives the claim. □

9.5 Coherence for vector bundles (online only)

Let X be a topological space. We write $\mathrm{Vect}^{\mathrm{disc}}(X)$ for the 1-category whose objects are the finite-rank complex vector bundles over X and whose morphisms are the morphisms of vector bundles over X. The superscript disc indicates that we are not yet recording the topology on spaces of bundle maps. Since vector bundles are stable under base change along continuous maps, this gives a contravariant functor

$$\mathrm{Vect}^{\mathrm{disc}}\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Cat}.$$

We show that $\mathrm{Vect}^{\mathrm{disc}}(X)$ can be equipped with a symmetric monoidal structure via tensor product of complex vector bundles. We construct this structure explicitly using Lurie's equivalent model of symmetric monoidal ∞-categories over Fin_*.

Construction 9.5.1. For a finite pointed set S_+, we write S for the complement of the basepoint. Define a 1-category

$$\mathrm{Vect}^{\otimes,\mathrm{disc}}(X)$$

equipped with a functor $p_X\colon \mathrm{Vect}^{\otimes,\mathrm{disc}}(X) \to \mathrm{Fin}_*$ as follows. An object over S_+ is an S-indexed family $(V_s)_{s\in S}$ of vector bundles over X. A morphism over a pointed map $\alpha\colon S_+ \to T_+$ from $(V_s)_{s\in S}$ to $(W_t)_{t\in T}$ consists of, for every $t \in T$, a map over X

$$\varphi_t\colon \prod_{s\in\alpha^{-1}(t)} V_s \longrightarrow W_t$$

which is continuous and complex multilinear on each fiber. Here the product is the fiber product over X, and if $\alpha^{-1}(t) = \emptyset$ we interpret the source as X, so that φ_t is the same as a section of W_t.

Composition is substitution of multilinear maps. Thus, if $\beta\colon T_+ \to U_+$ is another pointed map and $\psi_u\colon \prod_{t\in\beta^{-1}(u)} W_t \to Z_u$ is the corresponding family of multilinear maps, the composite over $\beta\alpha$ is given by

$$(v_s)_{s\in(\beta\alpha)^{-1}(u)} \longmapsto \psi_u\left(\left(\varphi_t((v_s)_{s\in\alpha^{-1}(t)})\right)_{t\in\beta^{-1}(u)}\right).$$

This is again multilinear in all variables. The identity morphism over id_{S_+} is given by the identity maps $V_s \to V_s$.

Proposition 9.5.2. *The functor*

$$p_X \colon \mathrm{Vect}^{\otimes,\mathrm{disc}}(X) \to \mathrm{Fin}_*$$

is a cocartesian fibration of 1-categories. It therefore defines a symmetric monoidal 1-category whose underlying category is $\mathrm{Vect}^{\mathrm{disc}}(X)$ *and whose tensor product is the tensor product of vector bundles.*

Proof. Let $\alpha \colon S_+ \to T_+$ be a pointed map and let $(V_s)_{s\in S}$ be an object over S_+. For each $t \in T$, choose the tensor product

$$W_t := \bigotimes_{s\in\alpha^{-1}(t)} V_s,$$

with the convention that the tensor product over the empty set is the trivial line bundle $\underline{\mathbb{C}}$. Let

$$\mu_t \colon \prod_{s\in\alpha^{-1}(t)} V_s \to W_t$$

be the universal multilinear map. Local trivializations show that W_t is again a vector bundle and that μ_t is continuous. These maps define a morphism

$$(V_s)_{s\in S} \longrightarrow (W_t)_{t\in T}$$

over α.

We claim that this morphism is p_X-cocartesian. Let $\beta \colon T_+ \to U_+$ be another pointed map, and let $(Z_u)_{u\in U}$ be an object over U_+. By definition, a morphism from $(W_t)_{t\in T}$ to $(Z_u)_{u\in U}$ over β is a family of multilinear maps

$$\prod_{t\in\beta^{-1}(u)} W_t \to Z_u.$$

Precomposition with the maps μ_t identifies this set with the set of multilinear maps

$$\prod_{s\in(\beta\alpha)^{-1}(u)} V_s \to Z_u,$$

for all $u \in U$, by the universal property of the tensor products W_t. This is the required universal property of a cocartesian morphism.

The fiber over S_+ is visibly $\mathrm{Vect}^{\mathrm{disc}}(X)^S$, so the Segal maps are isomorphisms. Thus p_X is a symmetric monoidal category in Lurie's model. Taking nerves gives the corresponding cocartesian fibration of ∞-categories, and the equivalence of ∞-operad models from Proposition 19.4.8 carries it to the desired symmetric monoidal ∞-category over Span(Fin). Its tensor product is obtained by pushing forward along the active map $\{1,2\}_+ \to \{1\}_+$, hence is the usual tensor product of vector bundles. □

Tensor product of vector bundles preserves finite direct sums in each variable, so the preceding semiring construction applies.

Corollary 9.5.3. *For every topological space X, the groupoid*

$$\mathrm{Vect}^{\mathrm{disc}}(X)^{\simeq}$$

of finite-rank complex vector bundles and bundle isomorphisms is naturally an object of $\mathrm{CRig}(\mathrm{An})$. *This structure is contravariantly functorial in X by pullback.*

Proof. Direct sum gives finite coproducts in the small 1-category $\mathrm{Vect}^{\mathrm{disc}}(X)$, with initial object the rank-zero bundle $\underline{0}$. Tensor product distributes over direct sums in each variable, so Corollary 20.3.4 applies to the symmetric monoidal structure of Proposition 9.5.2. This construction is natural in symmetric monoidal functors which preserve finite coproducts. Since pullback preserves direct sums, tensor products, and trivial bundles, it gives the asserted functoriality in X. □

The 1-category $\mathrm{Vect}^{\mathrm{disc}}(X)$ discards the natural compact-open topologies on its hom sets. Let us first explain how the familiar topological enrichment motivates the construction below. Suppose that X is compact Hausdorff. The spaces $\mathrm{Iso}_X(E, E')$ of bundle isomorphisms, with their compact-open topologies, form the hom spaces of a topologically enriched groupoid. Its homotopy coherent nerve, in the sense of Construction 25.3.2, remembers the underlying anima of each of these spaces of isomorphisms by Proposition 25.3.4.

The simplicial resolution used to construct this nerve has the same vector bundles as objects in every degree, while its morphisms in degree n are the continuous maps

$$\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, \mathrm{Iso}_X(E, E')).$$

The total spaces of vector bundles over a compact Hausdorff space are locally compact Hausdorff, so the compact-open exponential law identifies this set with

$$\mathrm{Iso}_{X\times|\Delta^n|}\left(\mathrm{pr}_X^* E, \mathrm{pr}_X^* E'\right).$$

Thus the groupoid in simplicial degree n is equivalent to the full subgroupoid of $\mathrm{Vect}^{\mathrm{disc}}(X \times |\Delta^n|)^{\simeq}$ spanned by the bundles pulled back from X. Every vector bundle over $X \times |\Delta^n|$ is isomorphic to one of this form by Theorem 9.1.4. Consequently the inclusion of this full subgroupoid is an equivalence. For compact Hausdorff X, the colimit below is therefore the standard simplicial model for the homotopy coherent nerve of the topologically enriched groupoid of vector bundles and bundle isomorphisms. We use the same formula as our definition for an arbitrary topological space.

The coherent direct sums and tensor products therefore refine the simplicial diagram in Definition 9.4.1 to a diagram in $\mathrm{CRig}(\mathrm{An})$.

In this realization, a bundle over $X \times [0, 1]$ gives an identification between its endpoint restrictions; bundles over $X \times |\Delta^n|$ supply the higher identifications. Direct sums and tensor products are defined degreewise.

Proof. Functoriality follows from pullback. The ∞-category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted by Proposition 22.6.9, and the forgetful functors

$$\mathrm{CRig}(\mathrm{An}) \longrightarrow \mathrm{CMon}(\mathrm{An}) \longrightarrow \mathrm{An}$$

create sifted colimits. For the second functor, this follows by regarding $\mathrm{CMon}(\mathrm{An})$ as the full subcategory of $\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), \mathrm{An})$ spanned by the finite-product-preserving functors: pointwise sifted colimits preserve this condition by Lemma 22.6.11. For the first functor, use $\mathrm{CRig}(\mathrm{An}) = \mathrm{CAlg}(\mathrm{CMon}(\mathrm{An}))$, the fact that $\mathrm{CMon}(\mathrm{An})$ is presentably symmetric monoidal by Theorem 18.2.5, and [Lur17, Corollary 3.2.3.2]. This proves the assertion about the underlying anima.

Applying π_0 to the geometric realization gives the coequalizer of the two endpoint maps from simplicial degree 1 to degree 0. If X is paracompact Hausdorff, the endpoint restrictions of every vector bundle over $X \times [0, 1]$ are isomorphic by Theorem 9.1.4. Thus the two maps agree, and the coequalizer is $\pi_0\mathrm{Vect}^{\mathrm{disc}}(X)^{\simeq}$. □

Proof. The constructions and proofs of Propositions 9.5.2 and 9.4.2 and Corollary 9.5.3 apply verbatim with $\mathbb{C}$ replaced by $\mathbb{R}$. □

We now specialize to $X = \mathrm{pt}$. Recall from Section 9.4.2 that the group completion of $\mathrm{Vect}(\mathrm{pt})^{\simeq}$ is the infinite loop anima underlying connective complex K-theory. The semiring structure above supplies its multiplication and all its coherences.

Proof. By Proposition 9.4.2, we have

$$\mathrm{Vect}(\mathrm{pt})^{\simeq} \in \mathrm{CAlg}(\mathrm{CMon}(\mathrm{An})).$$

By Proposition 16.6.4, the composite

$$\mathrm{CMon}(\mathrm{An}) \xrightarrow{(-)^{\mathrm{grp}}} \mathrm{CGrp}(\mathrm{An}) \xhookrightarrow{\mathbf{B}^{\infty}} \mathrm{Sp}$$

is symmetric monoidal. Applying it to the commutative algebra object $\mathrm{Vect}(\mathrm{pt})^{\simeq}$ gives a commutative algebra object of Sp, i.e. a commutative ring spectrum. Under Theorem 5.4.6, the functor $\mathbf{B}^{\infty}\colon \mathrm{CGrp}(\mathrm{An}) \to \mathrm{Sp}_{\geq 0}$ is the inverse of the recognition equivalence, and $(-)^{\mathrm{grp}}$ is the same group-completion left adjoint used in Section 9.4.2. Its underlying spectrum is therefore ku. □

Exercises

Exercise 9.1 (Direct sums, tensor products, and K-theory)**.** Let E_1 and E_2 be complex vector bundles over a topological space X.

(1) Show that the fiber product $E_1 \times_X E_2$, with its fiberwise direct-sum structure, is again a vector bundle. Verify that direct sum makes $\pi_0\mathrm{Vect}(X)$ into an abelian monoid.

(2) Construct the tensor-product bundle $E_1 \otimes E_2$, whose fiber over $x \in X$ is $(E_1)_x \otimes_{\mathbb{C}} (E_2)_x$. Show that tensor product is unital, commutative, associative, and distributive over direct sum on isomorphism classes, making $\pi_0\mathrm{Vect}(X)$ into a commutative semiring. Verify that direct sum and tensor product commute with pullback.

(3) Assume now that X is compact Hausdorff. Deduce that multiplication extends uniquely across group completion to a commutative ring structure on $K^0(X)$, and that pullback makes K^0 a contravariant functor from compact Hausdorff spaces to commutative rings.

Exercise 9.2 (Stable equivalence and reduced K-theory)**.** Let X be a compact Hausdorff space and consider the morphism of abelian monoids

$$\pi_0\mathrm{Vect}(X) \to K^0(X) \to \widetilde{K}^0(X).$$

(1) Show that two vector bundles E and E' have the same image in $\widetilde{K}^0(X)$ if and only if there are $n, m \geq 0$ and an isomorphism

$$E \oplus \underline{\mathbb{C}}^n \cong E' \oplus \underline{\mathbb{C}}^m.$$

(2) For the discrete space $X = \{1, \dots, r\}$, where $r \geq 1$, compute $K^0(X)$ and $\widetilde{K}^0(X)$. Identify the map $\mathbb{Z} \to K^0(X)$ and the isomorphism from $\widetilde{K}^0(X)$ to the kernel of restriction to the basepoint 1. Show that if X is disconnected, $\widetilde{K}^0(X)$ can contain classes whose virtual rank is nonzero on some component.

Exercise 9.3 (Pointwise operations on classifying maps)**.** Let X be a cell complex. Using the commutative semiring structure on

$$\mathrm{Vect}(\mathrm{pt})^{\simeq} \simeq \coprod_{k \geq 0} \mathbf{B}U(k),$$

show that $\mathrm{Hom}_{\mathrm{An}}(\Pi_\infty X, \mathrm{Vect}(\mathrm{pt})^{\simeq})$ admits a canonical commutative semiring structure. Under the classification of Corollary 9.2.15, identify addition and multiplication on its set of components with direct sum and tensor product of vector bundles.

Exercise 9.4 (Connective and periodic K-theory of spheres)**.** For every $k \in \mathbb{Z}$ and $n \geq 0$, compute the groups and the natural map

$$\widetilde{\mathrm{ku}}^k(S^n) \longrightarrow \widetilde{\mathrm{KU}}^k(S^n).$$

When these groups are nonzero, identify generators as powers of the Bott class under the suspension isomorphisms. Explain which negative powers occur in periodic K-theory and why they have no connective counterpart.

Exercise 9.5 (The rational Chern character of projective space)**.** Let $m \geq 0$, let $H \to \mathbb{C}\mathrm{P}^m$ be the tautological line bundle, and put $x = c_1(H)$ and $u = [H] - [\underline{\mathbb{C}}] \in \mathrm{KU}^0(\mathbb{C}\mathrm{P}^m)$. Show that

$$\mathrm{ch}(u) = e^x - 1 \in H^*(\mathbb{C}\mathrm{P}^m; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^{m+1}).$$

Deduce that the map

$$\mathbb{Q}[t]/(t^{m+1}) \longrightarrow \mathrm{KU}^0(\mathbb{C}\mathrm{P}^m) \otimes \mathbb{Q}, \qquad t \longmapsto u \otimes 1,$$

is a well-defined isomorphism of commutative $\mathbb{Q}$-algebras. Compute $\mathrm{KU}^k(\mathbb{C}\mathrm{P}^m) \otimes \mathbb{Q}$ for every $k \in \mathbb{Z}$.

10 Thom spectra and bordism theory

When is a closed smooth manifold the boundary of a compact smooth manifold? More generally, we may ask whether two closed smooth n-manifolds M and N are *bordant*, meaning that their disjoint union is the boundary of a compact smooth $(n+1)$-manifold. In this chapter, we relate these geometric questions to stable homotopy theory through the construction of *Thom spectra*.

The foundations of bordism theory were laid by Thom [Tho54]. Thom determined the unoriented bordism ring and characterized which closed smooth manifolds are boundaries in terms of their Stiefel–Whitney numbers. His work also addressed Steenrod's problem of representing homology classes by maps from closed manifolds: every mod 2 homology class of a finite polyhedron can be represented in this way, whereas integral homology classes need not be representable by oriented manifolds.

The connection with homotopy theory builds on Pontryagin's description of maps between spheres in terms of framed manifolds; see [Pon59] for an account. After stabilization, this identifies framed bordism groups with the stable stems $\pi_*(\mathbb{S})$. Thom's construction allows one to dispense with the framings by replacing the spheres with the *Thom spaces* of universal vector bundles. As the rank varies, these Thom spaces assemble into the unoriented bordism spectrum MO.

Given a space X, we may also equip our manifolds with maps to X, producing the unoriented bordism groups $\Omega_n^O(X)$. For a cell complex X, the Pontryagin–Thom theorem (Theorem 10.3.6) gives a natural isomorphism

$$\Omega_n^O(X) \cong \pi_n\big(\mathrm{MO} \otimes \mathbb{S}[X]\big).$$

This identifies the bordism groups with the unreduced homology groups $\mathrm{MO}_n(X)$ of the homology theory represented by MO. The systematic study of bordism and its associated cohomology theory was developed by Atiyah [Ati61a], working with oriented manifolds. Other choices of structure on the stable normal bundle similarly lead to bordism spectra, including the oriented bordism spectrum MSO and the complex bordism spectrum MU.

To construct these spectra, we regard Thom spectra as *twisted suspension spectra*. For a rank n vector bundle $E \to X$, fiberwise one-point compactification gives a family of pointed n-spheres indexed by $\Pi_\infty(X)$, whose colimit describes the underlying anima of the Thom space under suitable hypotheses on X; see Proposition 10.2.6. For a trivialized bundle, this is the constant family with value S^n, and its colimit is $\Sigma^n(X_+)$. Passing to spectra leads to the definition of a Thom spectrum as the colimit of a family of sphere spectra, following Ando et al. [And+14].

Stabilization also allows us to extend the construction to virtual vector bundles. Fiberwise one-point compactification carries direct sums to smash products, and after passing to spectra these become tensor products of invertible objects. Group completion therefore gives the stable J-homomorphism,

which associates a family of sphere spectra to each virtual vector bundle. In particular, this construction gives the Thom spectrum of the negative tangent bundle, which we use in Atiyah duality in Section 11.3.

Finally, this point of view leads to a notion of orientation with respect to an arbitrary commutative ring spectrum R. For a rank n bundle, an R-orientation identifies the associated family of sphere spectra, after tensoring with R, with the constant family $R[n]$. Taking colimits gives the Thom isomorphism; we recover its classical homological form after passing to homotopy groups. We develop this directly for virtual bundles in Section 10.4.

The compatibility of one-point compactification with direct sums also gives rise to multiplicative structures on bordism spectra. The technical proof that one-point compactification is coherently symmetric monoidal is postponed to Section 10.5, which may be skipped on a first reading.

10.1 Thom spaces and the collapse construction

Let us make Pontryagin's construction from the introduction precise, so as to see what has to be modified in the absence of a framing. Suppose a closed n-manifold M is embedded in $\mathbb{R}^{n+k}$ and that its normal bundle has been trivialized, so that M admits a tubular neighborhood identified with $M \times \mathbb{R}^k$. Collapsing everything outside this tubular neighborhood gives a pointed map

$$S^{n+k} \longrightarrow (M \times \mathbb{R}^k)^+ \cong M_+ \wedge S^k \longrightarrow S^k,$$

where the last arrow is induced by the map $M \to \mathrm{pt}$. After stabilization, this gives an element of the stable stem $\pi_n(\mathbb{S}) = \pi_0 \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}[n], \mathbb{S})$. Without a framing, the collapse map still exists, but its target must account for the varying normal directions. Thom's remedy is to replace the target S^k by a pointed space built out of the normal bundle itself, obtained by compactifying each fiber separately and then gluing all of the resulting points at infinity into a single basepoint.

Construction 10.1.1 (Fiberwise one-point compactification)**.** Let $p\colon E \to X$ be a real vector bundle of rank n. The linear action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$ extends to a continuous action on its one-point compactification $S^n = \mathbb{R}^n \cup \{\infty\}$ fixing ∞, since $\|gv\| \geq \|v\|/\|g^{-1}\|$. We define the *fiberwise one-point compactification* of E as the associated bundle

$$S^E := \mathrm{Fr}(E) \times_{\mathrm{GL}_n(\mathbb{R})} S^n,$$

where $\mathrm{Fr}(E)$ is the frame bundle of E; compare Proposition 9.2.7 and its real analogue. Concretely, S^E is obtained from E by one-point compactifying each fiber, and it restricts to $U \times S^n$ over any open subset $U \subseteq X$ over which E is trivial.

The space S^E comes equipped with a map $S^E \to X$. Since the action fixes ∞, the 'points at infinity' assemble into a continuous section $s_\infty\colon X \to S^E$.

Definition 10.1.2 (Thom space)**.** Let $p\colon E \to X$ be a real vector bundle. We define the *Thom space* $\mathrm{Th}(E)$ as the quotient of S^E by the points at infinity:

$$\mathrm{Th}(E) := S^E / s_\infty(X).$$

We regard this quotient as a pointed topological space; when X is empty, this is pt by convention.

Remark 10.1.3. Given a metric on E, one obtains a homeomorphic description of $\mathrm{Th}(E)$ as the quotient $D(E)/S(E)$, where $D(E)$ and $S(E)$ denote the unit disk and sphere bundles with respect to the metric. The homeomorphism is given by sending a vector v to $v/(1+\|v\|)$ in each fiber. This is often taken as the definition, but has the disadvantage of requiring an auxiliary choice of metric.

Remark 10.1.4. If X is compact Hausdorff, the inclusion of E induces a homeomorphism $E^+ \cong \mathrm{Th}(E)$. Indeed, a neighborhood of the collapsed section in $\mathrm{Th}(E)$ has compact complement because S^E is compact Hausdorff, so the quotient topology at the basepoint is the one-point compactification topology.

The Thom space of a trivial bundle is an n-fold suspension:

Example 10.1.5 (Suspension as Thom space)**.** Let $E = X \times \mathbb{R}^n$ be the trivial rank n vector bundle over X. Then $\mathrm{Th}(E) \cong X_+ \wedge S^n \cong \Sigma^n(X_+)$.

For a general rank n bundle E, the Thom space $\mathrm{Th}(E)$ differs from $\Sigma^n(X_+)$ only in the way the fiber spheres are glued to one another. We will therefore think of $\mathrm{Th}(E)$ as a *twisted suspension* of X_+, twisted by E.

For example, the Thom space of the Möbius line bundle over S^1 is $\mathbb{R}\mathrm{P}^2$, whereas the trivial line bundle has Thom space $S^1_+ \wedge S^1$. We leave the identification with $\mathbb{R}\mathrm{P}^2$ to Exercise 10.3.

Example 10.1.6. Let $E \to X$ and $E' \to X'$ be vector bundles, and suppose that at least one of X and X' is compact Hausdorff. Their product $E \times E' \to X \times X'$ is a vector bundle, called their *external direct sum*. There is a natural isomorphism of pointed spaces

$$\mathrm{Th}(E \times E') \cong \mathrm{Th}(E) \wedge \mathrm{Th}(E').$$

By symmetry, assume that X' is compact Hausdorff; the case of an empty base is immediate. Both Thom-space constructions collapse precisely the pairs in $S^E \times S^{E'}$ with an infinite coordinate. It therefore suffices to check that the natural surjections

$$S^E \times S^{E'} \longrightarrow \mathrm{Th}(E \times E'), \qquad S^E \times S^{E'} \longrightarrow \mathrm{Th}(E) \wedge \mathrm{Th}(E')$$

are quotient maps.

For the first, the map to $S^{E \times E'}$ is locally the product of an identity with $S^n \times S^m \to S^{n+m}$. This is a closed surjection with compact fibers, so its products are quotient maps. Composing with the quotient map to $\mathrm{Th}(E \times E')$ proves the claim.

For the second, write q_E and $q_{E'}$ for the quotient maps to $\mathrm{Th}(E)$ and $\mathrm{Th}(E')$. The map $q_E \times \mathrm{id}_{S^{E'}}$ is a quotient map because $S^{E'}$ is compact Hausdorff, while $\mathrm{id}_{\mathrm{Th}(E)} \times q_{E'}$ is a quotient map because $q_{E'}$ is a closed surjection with compact fibers. Composing these with the quotient map $\mathrm{Th}(E) \times \mathrm{Th}(E') \to \mathrm{Th}(E) \wedge \mathrm{Th}(E')$ proves the claim.

Construction 10.1.7 (Functoriality of Thom spaces)**.** Let $p\colon E \to X$ be a vector bundle and let $f\colon X' \to X$ be a continuous map. The pullback bundle comes with a commutative diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\widetilde{f}} & E \\ {\scriptstyle f^*(p)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X. \end{array}$$

Since $\widetilde{f}$ is a linear isomorphism on each fiber, it extends to a continuous map $S^{f^*(E)} \to S^E$ preserving the sections at infinity, and so induces a pointed continuous map

$$\mathrm{Th}(\widetilde{f})\colon\ \mathrm{Th}(f^*(E)) \longrightarrow \mathrm{Th}(E).$$

These maps are compatible with identities and composition in f. An isomorphism $F \cong f^*(E)$ therefore also gives a map $\mathrm{Th}(F) \to \mathrm{Th}(E)$.

We record two facts about the topology of Thom spaces.

Lemma 10.1.8. *Let $E \to X$ be a real vector bundle.*

(1) If X is a cell complex, then $(S^E, s_\infty(X))$ is a relative cell complex, with one $(m+n)$-cell for each m-cell of X over which E has rank n.

(2) If $f\colon X' \to X$ is a homotopy equivalence between paracompact Hausdorff spaces, then the induced map $\mathrm{Th}(f^*E) \to \mathrm{Th}(E)$ is a pointed homotopy equivalence.

Proof. For part (1), choose a cellular filtration (X_α) of X. We construct a relative cellular filtration on S^E with stages $s_\infty(X) \cup S^E|_{X_\alpha}$, starting with the whole section at infinity. Consider a cell attached to X_α with characteristic map $\chi\colon D^m \to X_{\alpha+1}$. Pulling E back along χ gives a trivial bundle by the real version of Theorem 9.1.4, so pulling back S^E gives $D^m \times S^n$. The corresponding $(m+n)$-cell in S^E has characteristic map

$$D^{m+n} \cong D^m \times D^n \longrightarrow D^m \times S^n \longrightarrow S^E.$$

Its boundary lands in the preceding stage: $\partial D^m \times D^n$ maps to $S^E|_{X_\alpha}$, while $D^m \times \partial D^n$ maps to $s_\infty(X)$. The two descriptions agree on their intersection. The interior maps homeomorphically onto the vectors in E over the new open cell, so each new m-cell of the base contributes one relative $(m+n)$-cell. Moreover, at each attachment and each limit stage, the colimit topology agrees with the bundle topology; this can be checked locally in charts, using that product with the compact Hausdorff sphere S^n preserves quotient maps. This proves part (1).

Part (2) is left as Exercise 10.1. □

We now carry out the collapse construction in the generality we need; it is used again in the proof of Atiyah duality in Section 11.3.

Construction 10.1.9 (Pontryagin–Thom collapse map)**.** Let $i\colon N \hookrightarrow N'$ be a smooth embedding of smooth manifolds without boundary, with N compact, and let $\nu(i)$ denote its normal bundle. Choose a fiberwise metric and a tubular embedding $j\colon D(\nu(i)) \hookrightarrow N'$ of the closed unit disk bundle, whose differential along the zero section induces the identity on $\nu(i)$; such an embedding is obtained by shrinking a tubular neighborhood and rescaling the metric. Identifying $D(\nu(i))$ with its image in N', we obtain a pointed map

$$\mathrm{PT}(i)\colon N'_+ \to D(\nu(i))/S(\nu(i)) \cong \mathrm{Th}(\nu(i))$$

which restricts to the quotient map on $D(\nu(i))$ and sends the complement of its interior to the basepoint; the identification of the target uses Remark 10.1.3. The map $\mathrm{PT}(i)$ is called the *Pontryagin–Thom collapse map*. Its homotopy class is independent of the auxiliary choices: metrics can be

joined by interpolation, and normalized tubular embeddings are isotopic after shrinking, by the proof of Hirsch [Hir76, Chapter 4, Theorem 5.3]. These isotopies induce pointed homotopies of the collapse maps. If N' is equipped with a basepoint away from N, we may choose the tubular neighborhood of N to avoid this basepoint, giving a pointed map

$$\mathrm{PT}(i)\colon N' \longrightarrow \mathrm{Th}(\nu(i)).$$

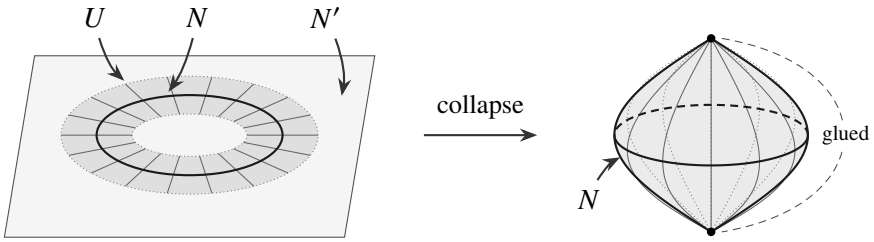


Figure 10.1: The Pontryagin–Thom collapse map of an embedding $N \hookrightarrow N'$, which crushes the complement of a tubular neighborhood of N to the basepoint. In the case drawn, N is a circle in a surface with trivial normal bundle. Cutting the band U along N exhibits $\mathrm{Th}(\nu(i))$ as two cones on N glued along N, that is, as a sphere with equator N whose two cone points have both become the basepoint.

For a framed embedding, the normal bundle is trivialized and $\mathrm{Th}(\nu(i)) \cong \Sigma^k(N_+)$, so that we recover Pontryagin's construction. In general the target depends on the normal bundle $\nu(i)$. The next step is to make it independent of the manifold by passing to the universal bundle.

Example 10.1.10. Let us consider the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^\infty)$ of k-dimensional subspaces of $\mathbb{R}^\infty$. By the real analogue of the Grassmannian classification theorem recorded in Remark 9.2.21, this space is a model for the classifying space $BO(k)$ and classifies rank k real vector bundles over paracompact Hausdorff spaces. It is a cell complex by its Schubert cell decomposition; see Hatcher [Hat17, Proposition 1.17 and p. 34]. The universal bundle γ_k over $\mathrm{Gr}_k(\mathbb{R}^\infty)$ has as total space

$$E_k = \{(V, v) \in \mathrm{Gr}_k(\mathbb{R}^\infty) \times \mathbb{R}^\infty \mid v \in V\}$$

with the projection map sending (V, v) to V. The fiber over a point $V \in \mathrm{Gr}_k(\mathbb{R}^\infty)$ is the k-dimensional vector space V itself. By Remark 10.1.3, the Thom space $\mathrm{Th}(\gamma_k)$ is the space of pairs (V, v) where $v \in V$ has length ≤ 1, modulo those pairs where v has length exactly 1.

Every rank k vector bundle over a paracompact Hausdorff space is pulled back from γ_k by Remark 9.2.21, so its Thom space maps to $\mathrm{Th}(\gamma_k)$ by Construction 10.1.7. Composing with the collapse map, an embedding $N \hookrightarrow N'$ as in Construction 10.1.9, with normal bundle of rank k, therefore gives a map $N'_+ \to \mathrm{Th}(\gamma_k)$. As k varies, the Thom spaces $\mathrm{Th}(\gamma_k)$ form a prespectrum:

Construction 10.1.11 (The prespectrum $\mathrm{MO}^{\mathrm{pre}}$)**.** Consider the canonical inclusion

$$\mathrm{Gr}_k(\mathbb{R}^\infty) \hookrightarrow \mathrm{Gr}_{k+1}(\mathbb{R} \times \mathbb{R}^\infty) \cong \mathrm{Gr}_{k+1}(\mathbb{R}^\infty), \quad V \mapsto \mathbb{R} \oplus V.$$

The pullback of γ_{k+1} along this inclusion is $\underline{\mathbb{R}} \oplus \gamma_k$, and in particular we obtain a pointed continuous map

$$\Sigma(\mathrm{Th}(\gamma_k)) \cong \mathrm{Th}(\underline{\mathbb{R}} \oplus \gamma_k) \to \mathrm{Th}(\gamma_{k+1}).$$

Taking the topological adjoint and then passing to underlying pointed animae gives the structure map

$$\sigma_k \colon \Pi_\infty \operatorname{Th}(\gamma_k) \longrightarrow \Pi_\infty \Omega \operatorname{Th}(\gamma_{k+1}) \cong \Omega \Pi_\infty \operatorname{Th}(\gamma_{k+1}).$$

Here we use the identification of topological loop spaces with loops of animae from Section 2.4. These maps turn the sequence $(\Pi_\infty \operatorname{Th}(\gamma_k))$ into a prespectrum, which we denote by $\mathrm{MO}^{\mathrm{pre}}$.

Applying spectrification (Proposition 4.3.29) to $\mathrm{MO}^{\mathrm{pre}}$ produces a spectrum, and this is the classical construction of MO going back to Thom [Tho54]. In the next section we define Thom spectra by colimits of families of sphere spectra. This construction also applies to virtual bundles, and Theorem 10.2.24 identifies the resulting MO with the spectrum constructed here.

10.2 Thom spectra

The construction of the previous section assigns Thom spaces to vector bundles. For applications such as Atiyah duality, we also need a Thom construction for virtual vector bundles. The normal bundle ν of an embedding $M \hookrightarrow \mathbb{R}^{n+k}$ of a closed n-manifold is not an invariant of M: enlarging the ambient space changes it by trivial summands. What is intrinsic is the *stable normal bundle*, the virtual vector bundle

$$\nu - \underline{\mathbb{R}^k} \quad = \quad \underline{\mathbb{R}^n} - T_M$$

of virtual rank 0, which is classified by a map $M \to \mathbf{B}O$. Closely related is the virtual bundle $-T_M = \nu - \underline{\mathbb{R}^{n+k}}$ of virtual rank $-n$, which occurs in Atiyah duality in Section 11.3. These virtual bundles need not be represented by vector bundles, so Definition 10.1.2 does not apply to them in general.

To extend the construction, we describe the underlying anima of a Thom space as the colimit of the family of pointed spheres associated to a vector bundle. After stabilization, these spheres become invertible spectra, allowing us to extend the construction to virtual bundles.

10.2.1 Thom animae

We start by making precise the claim that Thom spaces are colimits of parametrized families of spheres.

Definition 10.2.1 (Spherical fibration)**.** Let B be an anima and let $n \geq 0$. A *rank n spherical fibration over B* is a functor $\xi \colon B \to \mathrm{An}_*$ which lands in the full subcategory spanned by the n-sphere S^n. A *spherical fibration* is a rank n spherical fibration for some n.

Definition 10.2.2 (Thom anima)**.** We define the *Thom anima* of a spherical fibration $\xi \colon B \to \mathrm{An}_*$ as the pointed anima given by its colimit:

$$\operatorname{Th}(\xi) \quad := \quad \operatorname{colim}(\xi \colon B \to \mathrm{An}_*).$$

Recall from Corollary 24.2.8 the straightening-unstraightening equivalence $\mathrm{An}_{/B} \simeq \operatorname{Fun}(B, \mathrm{An})$. Passing to pointed objects, this provides an equivalence

$$\operatorname{Fun}(B, \mathrm{An}_*) \;\simeq\; \operatorname{Fun}(B, \mathrm{An})_* \;\simeq\; (\mathrm{An}_{/B})_*,$$

so that a rank n spherical fibration over B is the same as a pair (p, s) of a morphism $p\colon A \to B$ of animae and a section $s\colon B \to A$ of p such that each fiber A_b of p is isomorphic to the n-sphere S^n.

Remark 10.2.3 (Thom anima as a pushout)**.** Suppose that a functor $\xi\colon B \to \mathrm{An}_*$ corresponds to a pointed object $B \xrightarrow{s} A \to B$ of $\mathrm{An}_{/B}$. Then

$$\mathrm{colim}_B \xi \;\simeq\; A \amalg_B *,$$

where the pushout is formed in An and pointed by the map from $*$. In particular, if ξ is a spherical fibration, this pushout is its Thom anima.

Indeed, the colimit of the underlying functor $B \to \mathrm{An}$ is A by Proposition 24.5.2: its unstraightening is the anima A, so the localization in that result does not change it. The colimit of the constant basepoint diagram is B. The formula now follows by computing colimits in pointed objects as pushouts.

Example 10.2.4 (Sphere bundle)**.** Let X be a topological space and consider an n-dimensional sphere bundle $Y \to X$ over X, i.e. a fiber bundle whose typical fiber is S^n and whose structure group is the homeomorphism group $\mathrm{Homeo}_*(S^n)$. The fixed basepoint of S^n defines a section of $Y \to X$. After applying Π_∞, this bundle with section corresponds to a rank n spherical fibration over $\Pi_\infty(X)$: by Proposition 2.3.19 and Corollary 2.4.15, the fibers of $\Pi_\infty(Y) \to \Pi_\infty(X)$ are the spheres S^n.

Example 10.2.5 (Spherical fibration of vector bundle)**.** Given a rank n vector bundle $p\colon E \to X$, we may apply the previous example to the fiberwise one-point compactification $S^E \to X$ from Construction 10.1.1. We denote the resulting spherical fibration by

$$\xi_E\colon \Pi_\infty(X) \to \mathrm{An}_*.$$

Informally, ξ_E is the functor which sends a point $x \in X$ to $\Pi_\infty(S^{E_x})$, where S^{E_x} is the one-point compactification of the fiber E_x of p over x.

We can now identify the underlying pointed anima of a Thom space.

Proposition 10.2.6. *Let $p\colon E \to X$ be a vector bundle over a paracompact Hausdorff space of the homotopy type of a cell complex. Then the underlying pointed anima of its Thom space is isomorphic to the Thom anima of ξ_E:*

$$\Pi_\infty(\mathrm{Th}(E)) \quad\cong\quad \mathrm{Th}(\xi_E).$$

Proof. Choose a homotopy equivalence $f\colon X' \to X$ with X' a cell complex. By Lemma 10.1.8, the induced map $\mathrm{Th}(f^*E) \to \mathrm{Th}(E)$ is a pointed homotopy equivalence. On the other hand, ξ_{f^*E} is the pullback of ξ_E along the isomorphism $\Pi_\infty(f)$, so their colimits are isomorphic. We may therefore assume that X is a cell complex.

Under pointed straightening, ξ_E corresponds to the infinity section $X \to S^E$, so we obtain

$$\mathrm{Th}(\xi_E) \simeq \Pi_\infty(S^E) \amalg_{\Pi_\infty(X)} *$$

by Remark 10.2.3. Since the infinity section is a relative cell complex by Lemma 10.1.8, we may apply Proposition 2.3.9 to its pushout along $X \to *$ to compute

$$\Pi_\infty(S^E) \amalg_{\Pi_\infty(X)} * \simeq \Pi_\infty\big(S^E / s_\infty(X)\big) = \Pi_\infty(\mathrm{Th}(E)).$$ □

10.2.2 Thom spectra

We now replace pointed spheres by shifts of the sphere spectrum and take colimits in spectra.

Definition 10.2.7 (Stable spherical fibration)**.** Let $\mathrm{Pic}(\mathrm{Sp}) \subseteq \mathrm{Sp}^{\simeq}$ denote the full subanima spanned by shifts $\mathbb{S}[n]$ of the sphere spectrum. For an anima A, we define a *stable spherical fibration on A* to be a functor $\xi\colon A \to \mathrm{Sp}$ which lands in $\mathrm{Pic}(\mathrm{Sp})$. For an integer $n \in \mathbb{Z}$ we say that ξ has *rank n* if it lands in the full subanima of $\mathrm{Pic}(\mathrm{Sp})$ spanned by $\mathbb{S}[n]$. In general, the rank may vary between connected components of A.

The shifts of the sphere spectrum are precisely the *invertible spectra*: those spectra X for which there is a spectrum X^{-1} and an isomorphism $X \otimes X^{-1} \cong \mathbb{S}$; see Rognes [Rog08, Example 3.3.2] and also Exercise 11.7. Thus this definition of $\mathrm{Pic}(\mathrm{Sp})$ agrees with the general definition of the Picard anima in Definition 11.1.6 below. Tensor product restricts to a symmetric monoidal structure on $\mathrm{Pic}(\mathrm{Sp})$, with unit $\mathbb{S}$ and inverse to $\mathbb{S}[n]$ given by $\mathbb{S}[-n]$. In particular, $\mathrm{Pic}(\mathrm{Sp})$ is a commutative group in animae.

If $\xi\colon A \to \mathrm{An}_*$ is a rank n spherical fibration, then $\Sigma^\infty\xi$ is a rank n stable spherical fibration. In particular, a rank n vector bundle $E \to X$ determines the stable spherical fibration $\Sigma^\infty\xi_E$ on $\Pi_\infty(X)$.

Definition 10.2.8 (Thom spectrum)**.** Given a stable spherical fibration $\xi\colon A \to \mathrm{Sp}$, we define its *Thom spectrum* as its colimit:

$$\mathrm{th}(\xi) \quad := \quad \mathrm{colim}(\xi\colon A \to \mathrm{Sp}).$$

For a vector bundle $E \to X$, we write $\mathrm{th}(E) := \mathrm{th}(\Sigma^\infty\xi_E)$ for its Thom spectrum. If X is paracompact Hausdorff and has the homotopy type of a cell complex, we obtain

$$\mathrm{th}(E) \quad \cong \quad \Sigma^\infty \mathrm{Th}(\xi_E) \quad \cong \quad \Sigma^\infty\Pi_\infty \mathrm{Th}(E),$$

where the first isomorphism follows since Σ^∞ preserves colimits, and the second is Proposition 10.2.6. Throughout, Th denotes a Thom space or Thom anima and th a Thom spectrum.

Example 10.2.9 (Constant spherical fibrations)**.** Let A be an anima and let $n \in \mathbb{Z}$. The Thom spectrum of the constant functor $\xi\colon A \to \mathrm{Sp}$ with value $\mathbb{S}[n]$ is

$$\mathrm{th}(\xi) \simeq \mathbb{S}[A][n].$$

Indeed, shifts commute with colimits, and the colimit of the constant functor with value $\mathbb{S}$ is $\mathbb{S}[A]$. For $n = 0$, this is the suspension spectrum of A with a disjoint basepoint. Allowing negative n gives its desuspensions.

Construction 10.2.10 (Multiplicative structures on Thom spectra)**.** Let A be a small commutative monoid in animae, regarded as a symmetric monoidal ∞-category, and let $\xi\colon A \to \mathrm{Pic}(\mathrm{Sp}) \subseteq \mathrm{Sp}$ be a symmetric monoidal functor. We equip the Thom spectrum $\mathrm{th}(\xi)$ with a canonical commutative ring spectrum structure, following Antolín-Camarena and Barthel [AB19]. The multiplication is induced by addition in A:

$$\begin{aligned}\mathrm{th}(\xi) \otimes \mathrm{th}(\xi) &\simeq \mathrm{colim}_{(a,b)\in A\times A}\big(\xi(a) \otimes \xi(b)\big)\\ &\simeq \mathrm{colim}_{(a,b)\in A\times A}\,\xi(a+b) \longrightarrow \mathrm{colim}_{a\in A}\,\xi(a) = \mathrm{th}(\xi).\end{aligned}$$

The unit is $\mathbb{S} \simeq \xi(0) \to \mathrm{th}(\xi)$.

To obtain the coherent commutative algebra structure, we use the Day convolution monoidal structure on $\mathrm{Fun}(A, \mathrm{Sp})$ from Corollary 16.2.2. The universal property of Day convolution from Lemma 16.1.4 identifies the underlying lax symmetric monoidal functor of ξ with a commutative algebra in $\mathrm{Fun}(A, \mathrm{Sp})$. The colimit functor $\mathrm{colim}_A \colon \mathrm{Fun}(A, \mathrm{Sp}) \to \mathrm{Sp}$ is pointwise left Kan extension along the unique symmetric monoidal functor $A \to *$, so it is symmetric monoidal by Lemma 16.2.5; here $\mathrm{Fun}(*, \mathrm{Sp})$ is identified with Sp equipped with its usual tensor product. It therefore carries ξ to a commutative algebra in spectra whose underlying spectrum is $\mathrm{colim}_A \xi = \mathrm{th}(\xi)$.

Remark 10.2.11. More generally, an $\mathcal{E}_n$-map $A \to \mathrm{Pic}(\mathrm{Sp})$ gives its Thom spectrum an $\mathcal{E}_n$-algebra structure; see Antolín-Camarena and Barthel [AB19, Corollary 3.2]. The operadic Day convolution used for such extensions is discussed in Remark 16.1.9.

The symmetric monoidal stable spherical fibrations used below arise from the stable J-homomorphism, which we discuss next. We apply the construction to bordism spectra in Construction 10.2.26.

10.2.3 The J-homomorphism

One-point compactification sends direct sums of vector spaces to smash products of pointed spheres. We will use this compatibility to construct the J-homomorphism: after passing to suspension spectra, it gives a symmetric monoidal functor from vector spaces to invertible spectra, which extends to virtual vector spaces by group completion.

For a topological space X, let $\mathrm{Vect}^{\mathrm{disc}}_{\mathbb{R}}(X)$ denote the 1-category of finite-rank real vector bundles over X and morphisms of vector bundles. As in Chapter 9, we consider the *moduli anima of real vector bundles over X*,

$$\mathrm{Vect}_{\mathbb{R}}(X)^{\simeq} \quad := \quad \mathrm{colim}_{[q]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Vect}^{\mathrm{disc}}_{\mathbb{R}}(X \times |\Delta^q|)^{\simeq}.$$

By Corollary 9.4.3, this defines a functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{CRig}(\mathrm{An})$ whose underlying anima is computed by the same colimit; here we only use the underlying commutative monoids. When $X = \mathrm{pt}$, we refer to $\mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq}$ as the *moduli anima of finite-dimensional real vector spaces*.

The following proposition expresses one-point compactification as a symmetric monoidal functor. Its proof occupies Section 10.5.

Proposition 10.2.12 (One-point compactification)**.** *For every topological space X, fiberwise one-point compactification defines a symmetric monoidal functor*

$$S^{(-)}_X \colon \mathrm{Vect}_{\mathbb{R}}(X)^{\simeq} \longrightarrow (\mathrm{An}_{/\Pi_\infty(X)})_* \simeq \mathrm{Fun}(\Pi_\infty(X), \mathrm{An}_*).$$

Here the source has direct sum and the target has pointwise smash product under the displayed equivalence. These functors are natural in X as symmetric monoidal functors. For $X = \mathrm{pt}$, the source identifies with $\bigsqcup_{n\geq 0} \mathbf{B}O(n)$, with addition induced by block sum. On the rank n component, $S^{(-)}_{\mathrm{pt}}$ is classified by the standard pointed action of $O(n)$ on $S^{\mathbb{R}^n}$.

We also write $S^{(-)}$ for the component $S^{(-)}_{\mathrm{pt}}$ when the base is a point.

Definition 10.2.13. Write $O := \operatorname{colim}_n O(n)$ for the stable orthogonal group and $\mathbf{B}O$ for its classifying anima. We define

$$\mathrm{BOP} \;:=\; \big(\mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq}\big)^{\mathrm{grp}} \quad \in \quad \mathrm{CGrp}(\mathrm{An})$$

to be the group completion of the commutative monoid of finite-dimensional real vector spaces. The telescope calculation of Corollary 9.4.6, with real vector spaces in place of complex ones, identifies its underlying anima as

$$\mathrm{BOP} \;\simeq\; \mathbb{Z}\times\mathbf{B}O.$$

The only additional point in applying Theorem 5.5.2 is that $\pi_1(\mathbf{B}O) \cong \pi_0(O) \cong C_2$ is abelian. The virtual-rank-zero component has the presentation $\mathbf{B}O \cong \operatorname{colim}_n \mathbf{B}O(n)$, where the transition maps add a trivial line. Under this presentation, the map from the n-th stage to the virtual-rank-zero component sends V to $V - \mathbb{R}^n$.

Construction 10.2.14 (Stable J-homomorphism)**.** Applying reduced suspension spectra to Proposition 10.2.12 gives a symmetric monoidal functor

$$\mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq} \xrightarrow{S^{(-)}} \mathrm{An}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}.$$

Its image lies in $\mathrm{Pic}(\mathrm{Sp})$, since $\Sigma^\infty S^V \cong \mathbb{S}[\dim(V)]$ is invertible. Since tensor product of spectra turns $\mathrm{Pic}(\mathrm{Sp})$ into a commutative group in animae, the resulting functor to $\mathrm{Pic}(\mathrm{Sp})$ extends uniquely over group completion, giving a symmetric monoidal functor

$$J\colon\ \mathrm{BOP} \longrightarrow \mathrm{Pic}(\mathrm{Sp}).$$

We call J the *stable J-homomorphism*. We use the same notation for its restriction to any component of BOP, in particular for

$$J\colon \mathbf{B}O \simeq \{0\}\times\mathbf{B}O \longrightarrow \mathrm{Pic}(\mathrm{Sp}).$$

Forgetting the complex structure defines a symmetric monoidal functor from complex to real vector bundles for direct sum, compatible with pullback. Applying the simplicial resolutions above therefore gives a map of commutative monoids $\mathrm{Vect}(\mathrm{pt})^{\simeq} \to \mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq}$. Group completion gives a map of commutative groups $\mathrm{BUP} \to \mathrm{BOP}$, which doubles virtual rank. The complex analogue of J is the composite $\mathrm{BUP} \to \mathrm{BOP} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp})$.

Remark 10.2.15 (Relation to the classical J-homomorphism)**.** The connected component of $\mathbb{S}$ in $\mathrm{Pic}(\mathrm{Sp})$ is isomorphic to $\mathbf{B}\,\mathrm{GL}_1(\mathbb{S})$, where $\mathrm{GL}_1(\mathbb{S}) := \mathrm{Aut}_{\mathrm{Sp}}(\mathbb{S})$ is the automorphism group of the sphere spectrum. The restriction of J to the virtual-rank-zero component lands in this connected component. Passing to loops gives a group homomorphism $\Pi_\infty(O) \to \mathrm{GL}_1(\mathbb{S})$ in An, obtained by one-point compactifying orthogonal transformations and stabilizing. Its effect on positive homotopy groups is the classical stable J-homomorphism $\pi_n(O) \to \pi_n(\mathbb{S})$; see Ravenel [Rav86, Definition 1.1.12].

10.2.4 Virtual vector bundles

Since J is defined on all of BOP, it assigns stable spherical fibrations, and hence Thom spectra, to virtual vector bundles of arbitrary rank.

Definition 10.2.16 (Virtual vector bundle)**.** Let A be an anima. A *virtual vector bundle* on A is a map of animae

$$V\colon A \longrightarrow \mathrm{BOP}\,.$$

Its *rank* is the composite $\mathrm{rk}(V)\colon A \to \mathbb{Z}$ with the projection to $\mathbb{Z}$, and we say that V has rank $d \in \mathbb{Z}$ if this composite is constant with value d. A virtual vector bundle on a topological space X means one on $\Pi_\infty(X)$.

Since BOP is a commutative group in animae, so is the anima $\mathrm{Hom}_{\mathrm{An}}(A, \mathrm{BOP})$ of virtual vector bundles on A. We write $V + W$ for the sum and $-V$ for the inverse; ranks add, and $\mathrm{rk}(-V) = -\mathrm{rk}(V)$.

Example 10.2.17 (The class of a vector bundle)**.** Let X be a paracompact Hausdorff space and let $E \to X$ be a rank n vector bundle, classified by a map $f\colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$. Combining the identification $\Pi_\infty(\mathrm{Gr}_n(\mathbb{R}^\infty)) \xrightarrow{\sim} \mathbf{B}O(n)$ of Proposition 9.2.10 with the group-completion map $\mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq} \to \mathrm{BOP}$, we obtain a virtual vector bundle

$$[E]\colon \Pi_\infty(X) \xrightarrow{\Pi_\infty(f)} \mathbf{B}O(n) \subseteq \mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq} \longrightarrow \mathrm{BOP}$$

of rank n. For the trivial bundle $\underline{\mathbb{R}^N}$ this is the constant map with value $\mathbb{R}^N$.

This construction is compatible with direct sums. Indeed, under the identifications $\Pi_\infty(\mathrm{Gr}_n(\mathbb{R}^\infty)) \cong \mathbf{B}O(n)$, the Whitney-sum map corresponds to the map induced by the block-sum homomorphism $O(n) \times O(m) \to O(n+m)$. To see this, observe that the orthonormal frame bundle of $E \oplus E'$ is obtained from the product over X of the orthonormal frame bundles of E and E' by extension of structure group along this homomorphism. Since the addition on $\mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq}$ is induced by block sum, it follows that

$$[E \oplus E'] \cong [E] + [E'].$$

Construction 10.2.18 (Thom spectrum of a virtual vector bundle)**.** Let V be a virtual vector bundle on an anima A. Composing with the J-homomorphism gives a stable spherical fibration

$$\xi_V \quad := \quad J \circ V\colon A \longrightarrow \mathrm{Pic}(\mathrm{Sp}).$$

If V has rank d, then so does ξ_V, since J carries the class of $\mathbb{R}$ to $\mathbb{S}[1]$. We define the *Thom spectrum* of V as

$$\mathrm{th}(V) \quad := \quad \mathrm{th}(\xi_V) \quad = \quad \mathrm{colim}\big(J \circ V\colon A \to \mathrm{Sp}\big).$$

Remark 10.2.19. As J is a homomorphism of commutative groups in animae, the assignment $V \mapsto \xi_V$ carries sums to pointwise tensor products and inverses to pointwise inverses:

$$\xi_{V+W} \cong \xi_V \otimes \xi_W \qquad \text{and} \qquad \xi_{-V} \cong \xi_V^{-1}.$$

For bundles over the same base, this does not in general give an isomorphism $\mathrm{th}(V + W) \cong \mathrm{th}(V) \otimes \mathrm{th}(W)$: the tensor product on the right is computed by a colimit over $A \times A$. External sums do satisfy this formula; see Exercise 10.2.

We next show that this construction extends our previous definition of Thom spectra for vector bundles. We first identify the associated families of pointed spheres. As in Construction 10.1.1, a pointed action of a topological group G on a topological space Y, together with a principal G-bundle $P \to X$, gives an associated pointed bundle $P \times_G Y \to X$. Passing to underlying animae and applying pointed straightening produces a functor $\Pi_\infty(X) \to \mathrm{An}_*$. For a universal principal bundle, we refer to the resulting functor $\mathbf{B}G \to \mathrm{An}_*$ as the functor classified by the pointed G-action on Y.

Construction 10.2.20 (One-point compactification of vector spaces)**.** By Proposition 10.2.12, the rank n component of $\mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq}$ is isomorphic to $\mathbf{B}O(n)$. We denote the restriction of one-point compactification to this component by

$$J_n^{\mathrm{un}} \colon \mathbf{B}O(n) \longrightarrow \mathrm{An}_*.$$

It is the functor classified by the standard pointed action of $O(n)$ on $S^{\mathbb{R}^n}$. Consequently, the restriction of J along the rank n component $\mathbf{B}O(n) \to \mathrm{BOP}$ is $\Sigma^\infty J_n^{\mathrm{un}}$. Its restriction along the stable map

$$\mathbf{B}O(n) \longrightarrow \{0\} \times \mathbf{B}O \subseteq \mathrm{BOP}, \qquad V \longmapsto V - \mathbb{R}^n,$$

is $\Sigma^\infty J_n^{\mathrm{un}}[-n]$.

Lemma 10.2.21. *Let X be a paracompact Hausdorff space and let $f \colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ be a continuous map. Let $p \colon E \to X$ be the rank n vector bundle classified by f. Then the map $\xi_E \colon \Pi_\infty(X) \to \mathrm{An}_*$ is given by the composite*

$$\Pi_\infty(X) \xrightarrow{\Pi_\infty(f)} \Pi_\infty(\mathrm{Gr}_n(\mathbb{R}^\infty)) \xrightarrow{\sim} \mathbf{B}O(n) \xrightarrow{J_n^{\mathrm{un}}} \mathrm{An}_*,$$

where the middle isomorphism is the one from Proposition 9.2.10.

Proof. Since $E \cong f^*\gamma_n$, naturality in Proposition 10.2.12 gives $\xi_E \cong \xi_{\gamma_n} \circ \Pi_\infty(f)$. The universal bundle γ_n is associated to the universal principal $O(n)$-bundle $V_n(\mathbb{R}^\infty) \to \mathrm{Gr}_n(\mathbb{R}^\infty)$, so its fiberwise one-point compactification is the associated pointed sphere bundle. Under $\Pi_\infty(\mathrm{Gr}_n(\mathbb{R}^\infty)) \cong \mathbf{B}O(n)$, its straightening is therefore J_n^{un}, by the final assertion of that proposition. □

Applying suspension spectra and taking colimits now identifies the two Thom spectra. Subtracting a trivial bundle gives a desuspension, a fact we will use in the proof of Atiyah duality in Section 11.3.

Lemma 10.2.22. *Let X be a paracompact Hausdorff space and let $E \to X$ be a rank n vector bundle. Then there is an isomorphism $\xi_{[E]} \cong \Sigma^\infty(\xi_E)$ of stable spherical fibrations, and hence an isomorphism $\mathrm{th}([E]) \cong \mathrm{th}(E)$. More generally, for every $N \geq 0$ we have $\xi_{[E]-[\underline{\mathbb{R}^N}]} \cong \Sigma^\infty(\xi_E)[-N]$ and therefore*

$$\mathrm{th}\left([E] - [\underline{\mathbb{R}^N}]\right) \quad \cong \quad \mathrm{th}(E)[-N].$$

If X also has the homotopy type of a cell complex, this is isomorphic to $\Sigma^{\infty-N}\Pi_\infty \mathrm{Th}(E)$.

Proof. By Construction 10.2.20, the restriction of J to the rank n component is $\Sigma^\infty J_n^{\mathrm{un}}$. The first isomorphism is therefore given by Lemma 10.2.21, and passing to colimits gives $\mathrm{th}([E]) \cong \Sigma^\infty \mathrm{Th}(\xi_E) \cong \mathrm{th}(E)$, since Σ^∞ preserves colimits.

The virtual vector bundle $[\underline{\mathbb{R}^N}]$ is constant with value $\mathbb{R}^N$, so $\xi_{[\underline{\mathbb{R}^N}]}$ is constant with value $\Sigma^\infty S^{\mathbb{R}^N} \cong \mathbb{S}[N]$. By Remark 10.2.19 we thus obtain

$$\xi_{[E]-[\underline{\mathbb{R}^N}]} \cong \Sigma^\infty(\xi_E) \otimes \mathbb{S}[-N] \cong \Sigma^\infty(\xi_E)[-N].$$

Passing to colimits gives the asserted shift of Thom spectra. Under the additional cell-complex hypothesis, Proposition 10.2.6 identifies $\mathrm{th}(E)$ with $\Sigma^\infty\Pi_\infty \mathrm{Th}(E)$, giving the final assertion. □

10.2.5 Bordism spectra

We now apply the Thom-spectrum construction to bordism spectra. We start with MO, which comes from the virtual-rank-zero component of BOP, and relate it to the prespectrum $\mathrm{MO}^{\mathrm{pre}}$ constructed in Section 10.1.

Definition 10.2.23 (Unoriented bordism spectrum)**.** We define the *unoriented bordism spectrum* MO as the Thom spectrum of the restriction of J to the virtual-rank-zero component:

$$\mathrm{MO} \quad := \quad \mathrm{th}(J|_{\mathbf{B}O}) \quad = \quad \mathrm{colim}\,\big(J\colon \mathbf{B}O \to \mathrm{Sp}\big).$$

Theorem 10.2.24. *The spectrum associated to the prespectrum* $\mathrm{MO}^{\mathrm{pre}}$ *of Construction 10.1.11 via Proposition 4.3.29 is* MO*:*

$$(\mathrm{MO}^{\mathrm{pre}})^{\mathrm{sp}} \cong \mathrm{MO}\,.$$

Proof. We first identify the levels and structure maps of the classical prespectrum with the unstable Thom construction. We then pass to the colimit presentation of the virtual-rank-zero component of BOP.

Step 1: Consider the functor $\xi_{\gamma_k}\colon \Pi_\infty(\mathrm{Gr}_k(\mathbb{R}^\infty)) \to \mathrm{An}_*$ associated to γ_k. Applying Lemma 10.2.21 to the identity of $\mathrm{Gr}_k(\mathbb{R}^\infty)$ identifies ξ_{γ_k} with J^{un}_k. In particular, Proposition 10.2.6 provides an isomorphism

$$\mathrm{MO}^{\mathrm{pre}}_k = \Pi_\infty(\mathrm{Th}(\gamma_k)) \cong \mathrm{Th}(J^{\mathrm{un}}_k)$$

of pointed animae. Write i_k for the inclusion of Grassmannians used in Construction 10.1.11. The identification $i_k^*\gamma_{k+1} \cong \underline{\mathbb{R}} \oplus \gamma_k$ gives, by the compatibility of one-point compactification with direct sums, an isomorphism

$$J^{\mathrm{un}}_{k+1}|_{\mathbf{B}O(k)} \cong \Sigma J^{\mathrm{un}}_k.$$

The isomorphisms of Proposition 10.2.6 are natural for pullback maps of bundles. Under fiberwise one-point compactification, adjoining a trivial line gives fiberwise suspension; taking colimits gives suspension of the Thom anima. Hence, under the direct-sum identifications of Construction 10.1.11, the adjoints $\Sigma\,\mathrm{MO}^{\mathrm{pre}}_k \to \mathrm{MO}^{\mathrm{pre}}_{k+1}$ of the structure maps correspond to

$$\Sigma\,\mathrm{Th}(J^{\mathrm{un}}_k) \cong \mathrm{Th}\left(J^{\mathrm{un}}_{k+1}|_{\mathbf{B}O(k)}\right) \longrightarrow \mathrm{Th}(J^{\mathrm{un}}_{k+1}).$$

Step 2: In the presentation $\mathbf{B}O \simeq \mathrm{colim}_{k\in\mathbb{N}}\,\mathbf{B}O(k)$ of Definition 10.2.13, the map from the k-th stage sends V to $V - \mathbb{R}^k$. The restriction of J to this stage is therefore $\Sigma^\infty J^{\mathrm{un}}_k[-k]$ by Construction 10.2.20. Applying Theorem 22.2.11 gives

$$\mathrm{MO} \cong \mathrm{colim}_k\,\Sigma^\infty\,\mathrm{Th}(J^{\mathrm{un}}_k)[-k].$$

By Step 1, the transition maps $\Sigma^\infty\,\mathrm{Th}(J^{\mathrm{un}}_k)[-k] \to \Sigma^\infty\,\mathrm{Th}(J^{\mathrm{un}}_{k+1})[-(k+1)]$ are induced by the structure maps of $\mathrm{MO}^{\mathrm{pre}}$ after applying suspension spectra and the indicated shifts. Thus Proposition 4.3.29 identifies this colimit with $(\mathrm{MO}^{\mathrm{pre}})^{\mathrm{sp}}$. □

More generally, a map $B \to \mathrm{BOP}$ defines a Thom spectrum.

Definition 10.2.25 (The Thom spectrum MB)**.** Let $\varphi\colon B \to \mathrm{BOP}$ be a map of animae. We define its associated Thom spectrum MB by

$$MB := \operatorname{colim}\left(B \xrightarrow{\varphi} \mathrm{BOP} \xrightarrow{J} \mathrm{Sp}\right).$$

Equivalently, φ is a virtual vector bundle on B, and $MB = \mathrm{th}(\varphi)$ in the notation of Construction 10.2.18. The notation suppresses φ, which will always be clear from the context. We allow B to be an arbitrary anima. For the Pontryagin–Thom theorem of Section 10.3, we will specialize to maps which land in the virtual-rank-zero component $\mathbf{B}O \subseteq \mathrm{BOP}$.

Construction 10.2.26 (Multiplicative structures on bordism spectra)**.** Let B be a commutative monoid in animae and let $\varphi\colon B \longrightarrow \mathrm{BOP}$ be a morphism of commutative monoids. Since the J-homomorphism is symmetric monoidal, so is the composite

$$B \xrightarrow{\varphi} \mathrm{BOP} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp}) \subseteq \mathrm{Sp}.$$

Consequently, Construction 10.2.10 equips MB with a canonical commutative ring spectrum structure.

Example 10.2.27 (Classical bordism spectra)**.** A continuous group homomorphism $G \to O$ induces a map $\mathbf{B}G \to \mathbf{B}O$, and we write MG for the resulting Thom spectrum. Taking $B = \mathbf{B}O$ with φ the inclusion $\mathbf{B}O \hookrightarrow \mathrm{BOP}$ recovers MO. The composite

$$\mathbf{B}U \longrightarrow \mathbf{B}O \hookrightarrow \mathrm{BOP},$$

where the first map is induced by realification, defines the *complex bordism spectrum* MU. Further classical examples are

$$\mathrm{MSO}, \quad \mathrm{MSU}, \quad \mathrm{MSp}, \quad \text{and} \quad \mathrm{MSpin}.$$

For MO and MU, the defining maps to BOP are restrictions of morphisms of commutative groups constructed above, so Construction 10.2.26 equips these Thom spectra with commutative ring structures. The other classical Thom spectra in the display also admit commutative ring structures, but their construction requires compatible infinite-loop refinements of the maps $\mathbf{B}G \to \mathbf{B}O$, which we do not develop here.

Example 10.2.28 (Periodic bordism spectra)**.** The full group completions give the *periodic unoriented and complex bordism spectra*

$$\mathrm{MOP} := \mathrm{th}\left(J\colon\ \mathrm{BOP} \to \mathrm{Sp}\right) \qquad \text{and} \qquad \mathrm{MUP} := \mathrm{th}\left(\mathrm{BUP} \longrightarrow \mathrm{BOP} \xrightarrow{J} \mathrm{Sp}\right),$$

respectively. Both are commutative ring spectra by Construction 10.2.26. Restricting these defining diagrams to their virtual-rank-zero components recovers MO and MU. Translation by $r[\mathbb{R}]$ and $r[\mathbb{C}]$ identifies the rank-zero components with the rank r components, while shifting the associated stable spherical fibrations by r and $2r$, respectively. Taking colimits componentwise then gives isomorphisms of underlying spectra

$$\mathrm{MOP} \simeq \bigoplus_{r\in\mathbb{Z}} \mathrm{MO}[r] \qquad \text{and} \qquad \mathrm{MUP} \simeq \bigoplus_{r\in\mathbb{Z}} \mathrm{MU}[2r].$$

Reindexing these sums gives

$$\mathrm{MOP} \cong \mathrm{MOP}[1] \qquad \text{and} \qquad \mathrm{MUP} \cong \mathrm{MUP}[2],$$

which explains the terminology.

10.3 Bordism and the Pontryagin–Thom theorem

We now return to the classification of manifolds up to bordism. Given a map $B \to \mathbf{B}O$ and a topological space X, we may consider bordism classes of manifolds equipped with a normal B-structure and a map to X. For a cell complex X, the Pontryagin–Thom theorem identifies these groups with the homology groups represented by MB. We state this theorem as a black box and describe the collapse map which induces this identification.

We fix a map $\varphi\colon B \to \mathbf{B}O$ throughout.

Definition 10.3.1 (Normal B-structure)**.** Let M be a compact smooth n-manifold. Its *stable normal bundle* $\underline{\mathbb{R}^n} - T_M$ is a virtual vector bundle of rank 0, with classifying map $\nu_M\colon \Pi_\infty(M) \to \mathbf{B}O$. A *normal B-structure* on M is a lift of this map along φ:

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow{\scriptstyle \varphi} \\ \Pi_\infty(M) & \xrightarrow[\nu_M]{} & \mathbf{B}O. \end{array}$$

Concretely, this consists of a map $\sigma\colon \Pi_\infty(M) \to B$ together with an identification $\varphi \circ \sigma \simeq \nu_M$. If M has boundary, choose a splitting $T_M|_{\partial M} \cong \underline{\mathbb{R}}_{\text{out}} \oplus T_{\partial M}$, where the first summand is trivialized by an outward-pointing vector field. This gives an identification

$$(\underline{\mathbb{R}^n} - T_M)|_{\partial M} \cong \underline{\mathbb{R}^{n-1}} - T_{\partial M}.$$

The choices form a contractible space, so a normal B-structure on M restricts to one on ∂M. A diffeomorphism of manifolds with normal B-structures is understood to be equipped with an identification in the anima of such lifts.

Example 10.3.2. For $B = \mathbf{B}O$ with φ the identity, a normal B-structure is no data at all. For $B = \mathbf{B}SO$ it is an orientation of the stable normal bundle, and for $B = \mathrm{pt}$ it is a trivialization of it, that is, a stable framing of M.

Definition 10.3.3 (B-bordism)**.** Let X be a topological space and let $n \geq 0$. A *B-bordism cycle of degree n over X* is a triple (M, f, σ) consisting of a closed smooth n-manifold M, a continuous map $f\colon M \to X$, and a normal B-structure σ on M. Two cycles (M_0, f_0, σ_0) and (M_1, f_1, σ_1) are *bordant* if there exist compact smooth $(n+1)$-manifolds W_0 and W_1 with normal B-structures, together with maps $F_i\colon W_i \to X$, and a diffeomorphism

$$M_0 \amalg \partial W_0 \;\cong\; M_1 \amalg \partial W_1$$

of manifolds with normal B-structures over X.

We write $\Omega_n^B(X)$ for the set of bordism classes. Disjoint union defines an addition, with the empty manifold as zero. For $n < 0$, we set $\Omega_n^B(X) = 0$.

Remark 10.3.4. This is the formulation of Switzer [Swi75, Definition 12.21]. Bordism is an equivalence relation: reflexivity takes $W_0 = W_1 = \varnothing$, symmetry is built into the statement, and transitivity follows by taking disjoint unions. It is compatible with disjoint union, making $\Omega_n^B(X)$ a commutative monoid. The cylinder $M \times [0,1]$, equipped with a normal B-structure restricting to σ at one end, gives an additive inverse to $[(M, f, \sigma)]$ at the other end, with its induced boundary structure and the map f. Thus $\Omega_n^B(X)$ is an abelian group; see Switzer [Swi75, Lemma 12.23].

A map $g\colon X \to Y$ induces a homomorphism $g_*\colon \Omega^B_n(X) \to \Omega^B_n(Y)$ by postcomposition. The map to the homology theory represented by MB is given by the collapse construction.

Construction 10.3.5 (Pontryagin–Thom transformation)**.** Let (M, f, σ) be a B-bordism cycle of degree n over X. The closed smooth manifold M has the homotopy type of a finite cell complex by [Mil63, Theorem 3.5 and Corollary 6.7], so the Thom-spectrum constructions above apply to its vector bundles. Choose a smooth embedding $i\colon M \hookrightarrow \mathbb{R}^{n+k}$ for some k, compactify the ambient space to S^{n+k} with basepoint at infinity, and let ν be the rank k normal bundle. Applying Σ^∞ to the pointed collapse map of Construction 10.1.9 and shifting gives a map of spectra

$$\mathbb{S}[n] \longrightarrow \mathrm{th}(\nu)[-k].$$

The identification $T_M \oplus \nu \cong \underline{\mathbb{R}^{n+k}}$ gives an isomorphism $[\nu] - [\underline{\mathbb{R}^k}] \cong [\underline{\mathbb{R}^n}] - [T_M]$, so by Lemma 10.2.22 the target is the Thom spectrum of the stable normal bundle. The normal B-structure identifies the stable spherical fibration of $\nu - \underline{\mathbb{R}^k}$ with $J \circ \varphi \circ \sigma$. The map $(\sigma, \Pi_\infty(f))\colon \Pi_\infty(M) \to B \times \Pi_\infty(X)$ therefore induces

$$\mathrm{th}(\nu)[-k] \cong \mathrm{colim}_{\Pi_\infty(M)}(J \circ \varphi \circ \sigma) \longrightarrow \mathrm{colim}_{B\times\Pi_\infty(X)}(J \circ \varphi \circ \mathrm{pr}_B) \cong MB \otimes \mathbb{S}[X].$$

The arrow is induced by restricting the colimit cocone along $(\sigma, \Pi_\infty(f))$; the final isomorphism follows by first taking the colimit over B. The composite determines an element of $\pi_n(MB \otimes \mathbb{S}[X])$. The resulting class is independent of the choices, depends only on the bordism class of (M, f, σ), and is additive under disjoint union; these standard facts are part of the Pontryagin–Thom theorem as treated in Switzer [Swi75, Theorem 12.30 and Proposition 12.35]. We thus obtain a natural homomorphism

$$\mathrm{PT}\colon \Omega^B_n(X) \longrightarrow \pi_n(MB \otimes \mathbb{S}[X]).$$

Theorem 10.3.6 (Pontryagin–Thom theorem)**.** *For every cell complex X and every $n \in \mathbb{Z}$, the Pontryagin–Thom transformation is a natural isomorphism*

$$\Omega^B_n(X) \xrightarrow{\cong} \pi_n(MB \otimes \mathbb{S}[X]).$$

Thus $\Omega^B_n(X)$ is naturally isomorphic to the unreduced homology group $MB_n(X)$.

This is the generalized Pontryagin–Thom theorem for normal B-structures. We use it as a black box; see Stong [Sto68, Chapter II] and Switzer [Swi75, Theorem 12.30, p. 230, and Proposition 12.35, p. 232]. The theorem in Switzer treats the coefficient groups, and the proposition allows maps to a space X. For $n < 0$, both sides vanish: the left-hand side does so by definition, while $MB \otimes \mathbb{S}[X]$ is connective because $J \circ \varphi$ takes values in connective spectra and connective spectra are closed under colimits and tensor products.

Remark 10.3.7 (Relation to the classical formulation)**.** The classical formulation uses a sequence of spaces over $BO(k)$, which we may obtain from $\varphi\colon B \to \mathbf{B}O$ by forming the pullbacks $B_k := B \times_{\mathbf{B}O} \mathbf{B}O(k)$, where the map $\mathbf{B}O(k) \to \mathbf{B}O$ classifies $\gamma_k - \underline{\mathbb{R}^k}$. Universality of colimits in An gives $B \cong \mathrm{colim}_k B_k$. Choose a topological fibration $T \to BO$ representing φ and form the strict pullbacks $T_k := T \times_{BO} BO(k)$. These give compatible fibration models for the maps $B_k \to \mathbf{B}O(k)$. Letting E_k be the pullback of the universal rank k bundle (and using CW replacements to compute Thom spectra), the argument of Theorem 10.2.24 gives

$$MB \cong \mathrm{colim}_k \mathrm{th}(E_k)[-k] \cong \mathrm{colim}_k \Sigma^\infty \Pi_\infty \mathrm{Th}(E_k)[-k].$$

Thus MB is the spectrum associated to the classical sequence of Thom spaces.

Our definition of normal B-structures also gives the classical bordism groups. Compact smooth manifolds and their bordisms have finite cell-complex homotopy type, so their stable lifting data, including the identifications on boundaries, can be represented at finite stages B_k. After stabilization, the corresponding bundles can be identified with normal bundles of embeddings. The fibration models $T_k \to BO(k)$ represent these lifting data, including relative homotopies on boundaries, by strict lifts and homotopies through lifts. Choices of stabilized embeddings do not change the resulting bordism classes. Replacing B by $B \times \Pi_\infty(X)$ incorporates maps to X.

Remark 10.3.8. The spectrum MB also represents a cohomology theory $MB^*(X)$, traditionally called *cobordism*. We do not develop a geometric cocycle model for it; see Atiyah [Ati61a] for the relation with bordism.

Remark 10.3.9 (Coefficient rings)**.** The Pontryagin–Thom theorem turns computations of bordism groups into computations of homotopy groups of spectra. For MO and MU, Construction 10.2.26 constructs the multiplication on the Thom spectrum from Whitney sum. Under the Pontryagin–Thom isomorphism, this multiplication corresponds to cartesian product of bordism classes. We will not prove this geometric compatibility here.

Thom computed the unoriented coefficient ring as

$$\pi_*(\mathrm{MO}) \cong \mathbb{F}_2[a_i \mid i \geq 1,\ i \notin \{2^j - 1 \mid j \geq 1\}], \qquad |a_i| = i;$$

see Thom [Tho54]. The complex bordism spectrum has coefficient ring

$$\pi_*(\mathrm{MU}) \cong \mathbb{Z}[x_1, x_2, \ldots], \qquad |x_i| = 2i,$$

a theorem of Milnor and Novikov; see Ravenel [Rav86]. By Theorem 10.3.6, the underlying graded group is the complex bordism group, whose cycles are manifolds equipped with stable complex normal structures. Granting the multiplicative compatibility above, multiplication corresponds to cartesian product.

Example 10.3.10 (Framed bordism)**.** For $B = \mathrm{pt}$ with the constant map to $\mathbf{B}O$, we have $MB = \mathbb{S}$. By Example 10.3.2 the cycles are then stably framed manifolds, and the Pontryagin–Thom theorem becomes

$$\Omega_n^{\mathrm{fr}} := \Omega_n^{\mathrm{pt}}(\mathrm{pt}) \cong \pi_n(\mathbb{S}),$$

recovering Pontryagin's construction from the beginning of Section 10.1. In degree one, the nonzero class is represented by a stably framed circle that does not bound a framed surface. Under Pontryagin–Thom, it corresponds to the stable Hopf map η of Example 4.4.26.

10.4 Orientations and the Thom isomorphism

Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ be a commutative ring spectrum. A virtual vector bundle V of rank d on an anima A determines a family of R-modules $\xi_V[-d] \otimes R$, each isomorphic to R. An R-orientation is a trivialization of this family. Taking colimits will give the Thom isomorphism. We follow Ando et al. [And+14, Section 2.6]; for the classical account, see May [May99a, Chapter 23, Section 5].

Definition 10.4.1 (R-lines)**.** We denote by

$$\mathrm{Line}_R \subseteq \mathrm{Mod}_R^{\simeq}$$

the full subanima spanned by those R-modules that are isomorphic to R. Its objects are called *R-lines*.

Remark 10.4.2. The group $\mathrm{Aut}_R(R)$ of R-linear automorphisms of R is also denoted by $\mathrm{GL}_1(R)$. The pointed connected anima Line_R is its classifying anima: $\mathrm{Line}_R \simeq \mathbf{B}\,\mathrm{Aut}_R(R)$.

Example 10.4.3. For an ordinary commutative ring k, the group $\mathrm{Aut}_{Hk}(Hk)$ is the discrete group $k^\times$. In particular,

$$\mathrm{Line}_{\mathbb{F}_2} \simeq * \qquad \text{and} \qquad \mathrm{Line}_{\mathbb{Z}} \simeq \mathbf{B}C_2.$$

Definition 10.4.4 (Local systems and trivializations)**.** A *local system of R-lines* on an anima A is a functor $\xi\colon A \to \mathrm{Line}_R$. A *trivialization* of ξ is an isomorphism $\xi \cong R_A$, where R_A is the constant functor with value R.

Definition 10.4.5 (Orientation)**.** Let V be a virtual vector bundle of rank d on an anima A. Its associated local system of R-lines is

$$\xi_{V,R}\colon A \xrightarrow{\xi_V[-d]} \mathrm{Line}_{\mathbb{S}} \xrightarrow{-\otimes R} \mathrm{Line}_R\,.$$

Here $\xi_V[-d]$ lands in the virtual-rank-zero component $\mathrm{Line}_{\mathbb{S}} \subseteq \mathrm{Pic}(\mathrm{Sp})$. An *$R$-orientation* of V is a trivialization $\xi_{V,R} \cong R_A$.

Theorem 10.4.6 (Thom isomorphism)**.** *Let V be an R-oriented virtual vector bundle of rank d on an anima A. Then there is an isomorphism of R-modules*

$$\mathrm{th}(V) \otimes R \cong (\mathbb{S}[A] \otimes R)[d],$$

and an isomorphism of spectra

$$\hom(\mathrm{th}(V), R) \cong \hom(\mathbb{S}[A][d], R).$$

Proof. Write $R[A] := \mathbb{S}[A] \otimes R$. Passing to colimits in the chosen trivialization $\xi_{V,R} \cong R_A$ gives

$$\mathrm{th}(V)[-d] \otimes R \cong \mathrm{colim}_A\, \xi_{V,R} \cong \mathrm{colim}_A\, R_A \cong R[A],$$

which proves the first isomorphism. The second follows by applying $\hom_R(-, R)$ and using the free-forgetful adjunction between spectra and R-modules. □

An R-orientation of V induces a morphism of R-modules

$$\mathrm{th}(V)[-d] \otimes R \cong \mathrm{colim}_A\, \xi_{V,R} \cong R[A] \longrightarrow R,$$

where the last map is the augmentation. By adjunction, this corresponds to a map

$$u_V\colon\ \mathrm{th}(V) \longrightarrow R[d],$$

whose homotopy class is called the *Thom class* of the oriented virtual bundle V.

For vector bundles, we now relate R-orientations to the classical notion of a Thom class. Let $E \to X$ be a rank n vector bundle over a paracompact Hausdorff space of the homotopy type of a cell complex. By Lemma 10.2.22, we have

$$\mathrm{th}([E]) \cong \Sigma^\infty \Pi_\infty \mathrm{Th}(E).$$

Thus a map $u\colon \mathrm{th}([E]) \to R[n]$ determines a class in $\widetilde{R}^n(\mathrm{Th}(E))$, whose restriction to each fiber lies in the free rank-one $\pi_0(R)$-module $\widetilde{R}^n(S^{E_x})$.

Proposition 10.4.7. *Sending an R-orientation to its associated map $u_{[E]}$ identifies the anima of R-orientations of $[E]$ with the full subanima of*

$$\mathrm{Hom}_{\mathrm{Sp}}(\mathrm{th}([E]), R[n])$$

spanned by maps whose restrictions to the fibers represent generators. In particular, homotopy classes of R-orientations correspond bijectively to classes $\theta \in \widetilde{R}^n(\mathrm{Th}(E))$ whose restrictions to the fibers are generators.

Proof. The free-forgetful adjunction and the universal property of the colimit give isomorphisms of animae

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}}(\mathrm{th}([E]), R[n]) &\cong \mathrm{Hom}_{\mathrm{Mod}_R}(\mathrm{th}([E])[-n] \otimes R, R)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(\Pi_\infty(X), \mathrm{Mod}_R)}(\xi_{[E],R}, R_{\Pi_\infty(X)}).\end{aligned}$$

A natural transformation on the right is an isomorphism precisely when each of its components is an isomorphism. After choosing coordinates on E_x, the homotopy class of the component at x is multiplication by an element of $\pi_0(R)$; it is invertible precisely when that element is a unit, or equivalently when the restriction generates $\widetilde{R}^n(S^{E_x})$. Restricting to these components therefore gives the anima of trivializations of $\xi_{[E],R}$. The inverse sends a trivialization to its colimit followed by the augmentation, which is the construction of the Thom class above. □

Corollary 10.4.8 (Classical Thom isomorphism)**.** *Let $E \to X$ be a rank n vector bundle over a paracompact Hausdorff space of the homotopy type of a cell complex, equipped with an R-Thom class θ. Then*

$$\widetilde{R}_m(\mathrm{Th}(E)) \cong R_{m-n}(X) \qquad \textit{and} \qquad \widetilde{R}^m(\mathrm{Th}(E)) \cong R^{m-n}(X).$$

The inverse of the cohomological isomorphism is the Thom homomorphism $x \mapsto x \smile \theta$.

Proof. To get the two isomorphisms, apply Theorem 10.4.6 to $V = [E]$ and use Proposition 10.4.7 and Lemma 10.2.22. To identify the cohomological map, consider the Thom diagonal

$$\Delta_E\colon \mathrm{Th}(E) \longrightarrow X_+ \wedge \mathrm{Th}(E), \qquad [v] \longmapsto \big(p(v), [v]\big).$$

It sends the basepoint to the basepoint. We also write $\Delta_E\colon \mathrm{th}([E]) \to \mathbb{S}[X] \otimes \mathrm{th}([E])$ for the induced map of spectra. If $u\colon \mathrm{th}([E]) \to R[n]$ represents θ, the composite

$$\mathrm{th}([E]) \xrightarrow{\Delta_E} \mathbb{S}[X] \otimes \mathrm{th}([E]) \xrightarrow{\mathrm{id}\otimes u} \mathbb{S}[X] \otimes R[n]$$

extends R-linearly to the Thom isomorphism $\mathrm{th}([E]) \otimes R \cong R[X][n]$. Indeed, on the sphere over $a \in X$, the diagonal records a, while u gives the fiberwise orientation. The R-linear extension

is therefore the chosen trivialization followed by the inclusion of $R[n]$ at a into $R[X][n]$. These maps form the cocone obtained from the orientation, so the assertion follows by taking colimits. Consequently, for $x \in R^{m-n}(X)$ the inverse cohomological isomorphism is represented by

$$\mathrm{th}([E]) \xrightarrow{\Delta_E} \mathbb{S}[X] \otimes \mathrm{th}([E]) \xrightarrow{x \otimes u} R[m-n] \otimes R[n] \xrightarrow{\mu} R[m].$$

This is, by definition, the cup product $x \smile \theta$ formed with the Thom diagonal. □

Lemma 10.4.9 (Orientations of inverse virtual bundles)**.** *Let V be a virtual vector bundle of fixed rank on A. An R-orientation of V canonically induces an R-orientation of $-V$.*

Proof. Since $\xi_{-V} \cong \xi_V^{-1}$ by Remark 10.2.19 and $\mathrm{rk}(-V) = -\mathrm{rk}(V)$, there is an isomorphism $\xi_{-V,R} \cong \xi_{V,R}^{-1}$. Inverting a trivialization of $\xi_{V,R}$ gives the required trivialization. □

Example 10.4.10. Since $\mathrm{Line}_{\mathbb{F}_2}$ is contractible, every virtual vector bundle of fixed rank has a unique $\mathbb{F}_2$-orientation. For $R = \mathbb{Z}$ and a rank n real vector bundle $E \to X$ over a paracompact Hausdorff space of cell-complex homotopy type, the local system $\xi_{[E],\mathbb{Z}}$ is classified by

$$\Pi_\infty(X) \longrightarrow \mathbf{B}O(n) \xrightarrow{\mathbf{B}\det} \mathbf{B}C_2.$$

Indeed, an orthogonal transformation acts on the top integral cohomology of its one-point compactification by its determinant. Thus $\xi_{[E],\mathbb{Z}}$ is the classical orientation local system, and an $\mathbb{Z}$-orientation is a classical orientation. It exists precisely when E is orientable, that is, when the displayed composite $\Pi_\infty(X) \to \mathbf{B}C_2$ is nullhomotopic. Classically, this is the vanishing of the first Stiefel–Whitney class $w_1(E)$.

Example 10.4.11 (The universal MO-orientation)**.** The universal rank-zero virtual bundle on $\mathbf{B}O$ is canonically MO-oriented. For a point $W \in \mathbf{B}O$, i.e. a virtual vector space of rank 0, let $j_W \colon J(W) \to \mathrm{MO}$ be the colimit map. Multiplication on MO gives an MO-linear map

$$J(W) \otimes \mathrm{MO} \xrightarrow{j_W \otimes \mathrm{id}} \mathrm{MO} \otimes \mathrm{MO} \xrightarrow{\mu} \mathrm{MO}\,.$$

By Construction 10.2.10, its underlying map of spectra is

$$J(W) \otimes \mathrm{MO} \cong \mathrm{colim}_{V \in \mathbf{B}O}\, J(W+V) \cong \mathrm{MO}\,.$$

The second map is an isomorphism because translation $W + (-) \colon \mathbf{B}O \to \mathbf{B}O$ is an isomorphism of animae. The colimit cocone makes these MO-linear isomorphisms natural in W, so they trivialize the associated local system of MO-lines. Pulling back along the classifying map of $[E] - [\underline{\mathbb{R}^n}]$ gives a canonical MO-orientation of a rank n real vector bundle E.

Remark 10.4.12 (The mod 2 homology of MO)**.** The universal rank-zero virtual bundle on $\mathbf{B}O$ has a unique $\mathbb{F}_2$-orientation. The Thom isomorphism gives

$$\mathbb{F}_2 \otimes \mathrm{MO} \;\cong\; \mathbb{F}_2[\mathbf{B}O]$$

of $\mathbb{F}_2$-modules, and hence

$$H_*(\mathrm{MO}; \mathbb{F}_2) \cong H_*(\mathbf{B}O; \mathbb{F}_2).$$

The mod 2 cohomology of $\mathbf{B}O$ is the polynomial algebra $\mathbb{F}_2[w_1, w_2, \ldots]$ on the Stiefel–Whitney classes. The Thom isomorphism is a first step towards the calculation of $\pi_*(\mathrm{MO})$ in Remark 10.3.9; the splitting of MO into shifts of $\mathbb{F}_2$ and the coefficient-ring multiplication require further input.

10.5 Coherence for one-point compactification (online only)

We now prove Proposition 10.2.12. We first construct fiberwise one-point compactification on the ordinary groupoids of vector bundles, compatibly with direct sums and pullback. We then pass to underlying animae, apply the construction over $X \times |\Delta^q|$, and take the colimit defining the moduli anima. The passage to animae uses the formalism of pointed ∞-operads from Part II. This section may be skipped on a first reading.

10.5.1 Fiberwise compactification

Fix a topological space X. Let $\mathrm{Bun}(X)$ be the 1-category of locally trivial fiber bundles over X and continuous maps over X. Its terminal object is $X \to X$, and its finite products are fiber products over X. Let $\mathrm{Bun}_{*,\mathrm{c}}(X)$ be the 1-category of locally trivial pointed bundles over X with compact Hausdorff fibers and section-preserving maps. That is, the local trivializations carry the distinguished section to the basepoint of the fiber.

For $K, L \in \mathrm{Bun}_{*,\mathrm{c}}(X)$ with sections s_K and s_L, define the fiberwise wedge

$$K \vee_X L := \big(K \times_X s_L(X)\big) \cup \big(s_K(X) \times_X L\big) \subseteq K \times_X L.$$

The fiberwise smash product is the pushout

$$K \wedge_X L := (K \times_X L) \amalg_{K \vee_X L} X$$

in topological spaces over X, pointed by the last term. In each fiber this collapses the wedge to a point. Over a common pointed trivializing open subset $U \subseteq X$, the quotient map is

$$\mathrm{id}_U \times q \colon U \times (K_0 \times L_0) \longrightarrow U \times (K_0 \wedge L_0).$$

Here $q \colon K_0 \times L_0 \to K_0 \wedge L_0$ is the quotient map. Since it is a closed surjection with compact fibers, its product with U is a quotient map. It follows that $K \wedge_X L$ is locally trivial with compact Hausdorff pointed fiber $K_0 \wedge L_0$. Iterated fiberwise smash products are obtained by collapsing tuples with a basepoint coordinate. The associativity, unit and symmetry maps therefore satisfy the coherence identities, giving a symmetric monoidal structure on $\mathrm{Bun}_{*,\mathrm{c}}(X)$ with unit $X \times S^0$. These constructions commute with pullback, as can be checked in local trivializations.

Lemma 10.5.1 (Fiberwise one-point compactification)**.** *Fiberwise one-point compactification defines a symmetric monoidal functor*

$$\mathrm{Vect}_{\mathbb{R}}^{\mathrm{disc}}(X)^{\simeq} \longrightarrow \mathrm{Bun}_{*,\mathrm{c}}(X)^{\simeq}, \qquad E \longmapsto S^E,$$

natural in X under pullback.

Proof. A linear isomorphism extends uniquely to a pointed homeomorphism of one-point compactifications. The local description of Construction 10.1.1 shows that this extension is continuous in families. It therefore gives a functor on groupoids of vector bundles, commuting with pullback.

The natural homeomorphisms $S^{V \oplus W} \cong S^V \wedge S^W$ give, in bundle charts, isomorphisms

$$S^{E \oplus F} \cong S^E \wedge_X S^F, \qquad S^{0_X} \cong X \times S^0,$$

where 0_X denotes the zero vector bundle on X. The associativity, unit and symmetry diagrams reduce away from infinity to those for direct sums. Since all maps preserve the infinity sections, these diagrams commute everywhere. □

10.5.2 Passage to animae

We next pass from pointed bundles over X to families of pointed animae over $\Pi_\infty(X)$. Taking underlying animae preserves fiber products of bundles. The construction of pointed ∞-operads turns this compatibility into a lax symmetric monoidal functor for smash products; we will then show that its structure maps are isomorphisms on sphere bundles.

We use the following consequence of the construction of pointed ∞-operads. A finite-product-preserving functor $F\colon C \to D$ induces a morphism

$$(\mathcal{M}_{(C,\times)})_* \longrightarrow (\mathcal{M}_{(D,\times)})_*$$

by Proposition 15.3.6 and Theorem 20.2.10. On pointed objects, this sends $(* \to K)$ to $(* \to F(K))$. If D admits finite colimits and its cartesian product preserves them separately in each variable, the target is the multimorphism operad of $(D_*, \wedge)$ by Lemma 20.4.2. This construction is functorial in F: the cartesian symmetric monoidal refinement is functorial by the equivalence of Proposition 15.3.6, and $(-)_*$ is a functor. It gives a lax symmetric monoidal functor whenever the source pointed ∞-operad is represented by a symmetric monoidal ∞-category.

For $C = \mathrm{Bun}(X)$, we only need its full pointed suboperad on $\mathrm{Bun}_{*,\mathrm{c}}(X)$. An operation from $K_1, \dots, K_r$ to L is a bundle map

$$K_1 \times_X \cdots \times_X K_r \longrightarrow L$$

which sends a tuple with any coordinate in the distinguished section to the distinguished section of L. Such maps factor uniquely through the fiberwise smash product. Nullary operations are sections of L, equivalently pointed maps $X \times S^0 \to L$. Thus this full suboperad is represented by $(\mathrm{Bun}_{*,\mathrm{c}}(X), \wedge_X)$.

Lemma 10.5.2 (Passage to underlying animae)**.** *Taking underlying animae of fiberwise one-point compactifications defines symmetric monoidal functors*

$$\xi_X\colon \mathrm{Vect}_{\mathbb{R}}^{\mathrm{disc}}(X)^{\simeq} \longrightarrow \mathrm{Fun}(\Pi_\infty(X), \mathrm{An}_*), \qquad E \longmapsto \xi_E,$$

natural in X under pullback.

Proof. Taking underlying animae of total spaces and projections gives

$$\Pi_{\infty,X}\colon \mathrm{Bun}(X) \longrightarrow \mathrm{An}_{/\Pi_\infty(X)}.$$

This functor preserves finite products: the terminal bundle maps to the terminal object, and for bundles K, L the canonical map

$$\Pi_\infty(K \times_X L) \longrightarrow \Pi_\infty(K) \times_{\Pi_\infty(X)} \Pi_\infty(L)$$

is an isomorphism. Indeed, bundle projections are Serre fibrations by Theorem 2.3.13, so Proposition 2.3.19 applies.

These functors are natural under pullback. More precisely, the categories $\mathrm{Bun}(X)$ form a cartesian fibration over Top. Taking underlying animae gives a functor from this fibration to the pullback along Π_∞ of the codomain fibration $\mathrm{Ar}(\mathrm{An}) \to \mathrm{An}$. It preserves cartesian morphisms, since a pullback square along a bundle projection becomes a pullback square of animae. Cartesian straightening therefore gives a natural transformation of finite-product-preserving functors.

Pointing now gives lax symmetric monoidal functors on $\mathrm{Bun}_{*,c}(X)$, naturally in X. Straightening preserves finite products and colimits, so its pointed refinement identifies the target with $\mathrm{Fun}(\Pi_\infty(X), \mathrm{An}_*)$, equipped with pointwise smash product, by Lemma 20.4.2. Composing with Lemma 10.5.1 gives natural lax symmetric monoidal functors ξ_X.

It remains to check that their structure maps are isomorphisms. This can be checked at each point $x \in X$. Pullback to x identifies the fibers of the realized bundles with the underlying animae of their topological fibers, again because bundle projections are Serre fibrations. The binary structure map at x is consequently

$$\Pi_\infty(S^{E_x}) \wedge \Pi_\infty(S^{F_x}) \longrightarrow \Pi_\infty(S^{E_x} \wedge S^{F_x}) \cong \Pi_\infty(S^{E_x \oplus F_x}).$$

Choose coordinates identifying the two compactifications with S^m and S^n. The inclusion

$$(S^m \times \{\infty\}) \cup (\{\infty\} \times S^n) \hookrightarrow S^m \times S^n$$

is a relative cell complex, as are the inclusions used to form the wedge. Thus Corollary 2.3.20 and Proposition 2.3.9 identify the displayed structure map with an isomorphism. The unit structure map is $S^0 \cong \Pi_\infty(S^0)$. Hence the functors ξ_X are symmetric monoidal, and their naturality gives the asserted transformation. □

10.5.3 Passage to the moduli anima

We now apply the preceding lemma over $X \times |\Delta^q|$. Pullback along the projections identifies the resulting targets with the constant simplicial diagram at $\mathrm{Fun}(\Pi_\infty(X), \mathrm{An}_*)^{\simeq}$, allowing us to take the colimit defining the moduli anima.

Proof of Proposition 10.2.12. Put

$$D(X) := \mathrm{Fun}(\Pi_\infty(X), \mathrm{An}_*)^{\simeq},$$

regarded as a commutative monoid in animae under pointwise smash product. Since the source of ξ_X is a groupoid, Lemma 10.5.2 gives a natural map

$$\mathrm{Vect}_{\mathbb{R}}^{\mathrm{disc}}(X)^{\simeq} \longrightarrow D(X) \qquad \text{in } \mathrm{CMon}(\mathrm{An}).$$

Applying it over $X \times |\Delta^q|$ gives a map of simplicial diagrams

$$\mathrm{Vect}_{\mathbb{R}}^{\mathrm{disc}}(X \times |\Delta^\bullet|)^{\simeq} \longrightarrow D(X \times |\Delta^\bullet|).$$

On the other hand, pullback along the projections gives

$$\mathrm{const}_{D(X)} \longrightarrow D(X \times |\Delta^\bullet|).$$

Every component of this map is an isomorphism, since $X \times |\Delta^q| \to X$ is a homotopy equivalence. Inverting it and composing with the preceding map gives a cocone in $\mathrm{CMon}(\mathrm{An})$. By Corollary 9.4.3, the universal property of the colimit in $\mathrm{CMon}(\mathrm{An})$ therefore gives

$$S_X^{(-)} \colon \mathrm{Vect}_{\mathbb{R}}(X)^{\simeq} \longrightarrow D(X) \subseteq \mathrm{Fun}(\Pi_\infty(X), \mathrm{An}_*).$$

This functor sends E to ξ_E, as seen in degree zero, and is natural in X, since both maps of simplicial diagrams are natural in X.

It remains to identify the restriction of $S^{(-)}_{\mathrm{pt}}$ to $\mathbf{B}O(n)$ with the functor classified by the standard orthogonal action. The proof of Proposition 9.4.4, with $\mathbb{R}$ in place of $\mathbb{C}$, identifies the rank n component of $\mathrm{Vect}_{\mathbb{R}}(\mathrm{pt})^{\simeq}$ with $\mathbf{B}\,\mathrm{GL}_n(\mathbb{R}) \simeq \mathbf{B}O(n)$, compatibly with block sums. In degree q, an automorphism of the trivial bundle is a continuous map $g\colon |\Delta^q| \to \mathrm{GL}_n(\mathbb{R})$. Fiberwise compactification sends it to

$$(t, v) \longmapsto (t, g(t)v), \qquad (t, \infty) \longmapsto (t, \infty).$$

These formulas commute with the simplicial operators and composition. They identify the restriction to $\mathbf{B}\,\mathrm{GL}_n(\mathbb{R})$ with the functor obtained by realizing the singular simplicial action on $S^{\mathbb{R}^n}$. Restricting along $O(n) \to \mathrm{GL}_n(\mathbb{R})$ gives the corresponding functor for the orthogonal action.

To identify this with the associated-bundle description used in Section 10.2.4, take the universal principal $O(n)$-bundle $P = V_n(\mathbb{R}^\infty) \to B = \mathrm{Gr}_n(\mathbb{R}^\infty)$. Its total space is contractible, and

$$S^{\gamma_n} \cong P \times_{O(n)} S^{\mathbb{R}^n}.$$

Over P this pointed bundle is trivial, and under $P \times_B P \cong P \times O(n)$ its transition maps are the standard orthogonal action. By Lemma 9.2.9 and Proposition 9.2.10, taking underlying animae identifies the Čech nerve of $P \to B$ with the bar construction for $O(n)$. The transition maps therefore give the same action as the simplicial formulas above, including their compatibility with composition. Descent for animae (Theorem 5.2.5) identifies the resulting pointed object over $\mathbf{B}O(n)$ with the underlying pointed object of $S^{\gamma_n} \to B$. Its straightening is, by definition, the functor classified by the pointed orthogonal action. This proves the final assertion of the proposition. □

Exercises

Exercise 10.1 (Homotopy invariance of Thom spaces)**.** Let $f\colon X' \to X$ be a homotopy equivalence between paracompact Hausdorff spaces and let $E \to X$ be a real vector bundle. Prove part (2) of Lemma 10.1.8: the induced map $\mathrm{Th}(f^*E) \to \mathrm{Th}(E)$ is a pointed homotopy equivalence. *Hint:* Use homotopy invariance of vector bundles over cylinders, and observe that it suffices for both composites of the induced Thom maps to be pointed homotopic to homeomorphisms.

Exercise 10.2 (External sums and Thom spectra)**.** Let $V\colon A \to \mathrm{BOP}$ and $W\colon B \to \mathrm{BOP}$ be virtual vector bundles. Define their external sum as the composite

$$V \boxplus W\colon A \times B \xrightarrow{V \times W} \mathrm{BOP} \times \mathrm{BOP} \xrightarrow{+} \mathrm{BOP}\,.$$

Show that there is a natural isomorphism

$$\mathrm{th}(V \boxplus W) \cong \mathrm{th}(V) \otimes \mathrm{th}(W).$$

Exercise 10.3 (The Möbius bundle)**.** Let $L \to S^1$ be the Möbius real line bundle.

(1) Identify $\mathrm{Th}(L)$ with $\mathbb{R}\mathrm{P}^2$ as a pointed space.

(2) Show that L is not $\mathbb{Z}$-orientable.

(3) Construct its canonical $\mathbb{F}_2$-orientation. Writing $a \in H^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ for the standard generator, show that the Thom class is a and that the Thom isomorphism sends the generators of $H^0(S^1;\mathbb{F}_2)$ and $H^1(S^1;\mathbb{F}_2)$ to a and a^2, respectively.

Exercise 10.4 (Collapse of a framed point)**.** Let $n \geq 1$. Embed a point in $\mathbb{R}^n$ and equip its normal bundle with the standard framing. Write down the Pontryagin–Thom collapse map

$$S^n \longrightarrow \mathrm{Th}(\mathbb{R}^n) \cong S^n.$$

Show that, with the standard choices of orientation, this map has degree 1. Explain how reversing one vector in the framing changes the degree.

Exercise 10.5 (Low-dimensional unoriented bordism)**.** Compute $\Omega_0^O(*)$ and $\Omega_1^O(*)$. Compare your answer with $\pi_0(\mathrm{MO})$ and $\pi_1(\mathrm{MO})$.

Exercise 10.6 (Thom isomorphism for the tautological line)**.** Let $\gamma^1 \to \mathbb{R}\mathrm{P}^n$ be the tautological real line bundle. Compute the reduced mod-2 cohomology of $\mathrm{Th}(\gamma^1)$ as a graded $\mathbb{F}_2$-vector space. Identify the Thom class and its restrictions to the fibers.

11 Duality phenomena

Poincaré duality identifies the cohomology of a closed oriented n-manifold M with its homology in complementary degrees:

$$H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z}).$$

In this chapter, we explain Poincaré duality from the viewpoint of stable homotopy theory and extend it to generalized homology and cohomology theories. We use the Thom spectra and orientations of Chapter 10.

Spanier and Whitehead [SW55] developed a duality theory for finite polyhedra by studying their complements in spheres. Their construction lifts classical homological duality to a relation between stable homotopy classes of maps. In modern terms, every finite spectrum X has a *dual spectrum* DX, and for every spectrum E there are natural isomorphisms

$$E^k(X) \cong E_{-k}(DX).$$

For a closed smooth manifold M, we would like a geometric description of the dual of its suspension spectrum $\mathbb{S}[M] = \Sigma^\infty M_+$. Milnor and Spanier [MS60] obtained such a description using the Thom space of the normal bundle of an embedding into Euclidean space. Atiyah [Ati61b] developed this further into a duality theorem for Thom complexes. Using virtual bundles, the resulting *Atiyah duality* takes the form

$$D(\mathbb{S}[M]) \cong \mathrm{th}(-T_M).$$

Indeed, if $M \hookrightarrow \mathbb{R}^N$ has normal bundle ν, then $[\nu] - [\underline{\mathbb{R}^N}] = -T_M$, so the desuspended normal Thom spectrum can be expressed in terms of the tangent bundle alone.

Atiyah duality on its own requires no orientation. Given a commutative ring spectrum R and an R-orientation of the tangent bundle, we may apply the Thom isomorphism of Theorem 10.4.6 and obtain an isomorphism

$$R^k(M) \cong R_{n-k}(M).$$

For $R = \mathbb{Z}$, this recovers classical Poincaré duality. For other ring spectra, the required orientation depends on the cohomology theory, as discussed in Section 10.4.

We develop these results using duality in symmetric monoidal ∞-categories, following the categorical approach to topological duality systematically developed by Dold and Puppe [DP80]. We begin with dualizable objects in Section 11.1 and their relation to finiteness in Section 11.2: for modules over a commutative ring spectrum, dualizability is equivalent to both compactness and perfectness. We specialize to spectra in Section 11.2.1, and in Section 11.3 we prove Atiyah duality by constructing evaluation and coevaluation maps using Pontryagin–Thom collapse maps. Finally,

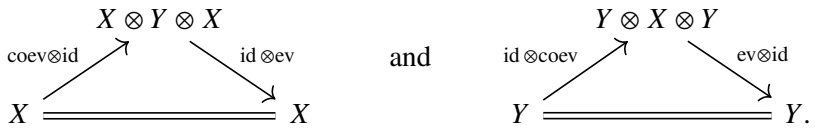
in Section 11.4, we deduce generalized Poincaré duality and identify its isomorphism with cap product by the fundamental class.

For a historical account of these developments, we refer to Becker and Gottlieb [BG99].

11.1 Symmetric monoidal duality

The evaluation pairing between a finite-dimensional vector space and its linear dual has an analogue in any symmetric monoidal ∞-category. Throughout this section, we fix a symmetric monoidal ∞-category $(C, \otimes, \mathbb{1})$.

Definition 11.1.1 (Dualizable object)**.** Consider objects X and Y of C. We say that a morphism $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ *exhibits Y as a dual to X* if there exists a morphism $\mathrm{coev}\colon \mathbb{1} \to X \otimes Y$ such that the following triangle diagrams commute:



In this case we refer to ev as the *evaluation map* and to coev as the *coevaluation map*, and call (Y, ev) a *duality datum* for X. We say that an object X is *dualizable* if it admits such a duality datum.

Example 11.1.2 (Linear duality)**.** Let V be a finite-dimensional vector space over a field k, and write $V^* = \mathrm{Hom}_k(V, k)$. In the symmetric monoidal 1-category $(\mathrm{Vect}_k, \otimes_k, k)$, the evaluation pairing $\mathrm{ev}\colon V^* \otimes_k V \to k$ exhibits V^* as a dual to V. For a basis $v_1, \dots, v_r$ with dual basis $v_1^*, \dots, v_r^*$, the coevaluation is given by

$$\mathrm{coev}\colon k \to V \otimes_k V^*, \qquad 1 \mapsto \sum_{i=1}^{r} v_i \otimes v_i^*.$$

This tensor corresponds to id_V under the natural isomorphism $V \otimes_k V^* \cong \mathrm{Hom}_k(V, V)$, so it is independent of the basis. The triangle identities follow from the dual-basis identities.

Example 11.1.3 (Finitely generated projective modules)**.** Let R be an ordinary commutative ring. In the category of ordinary R-modules with tensor product $\otimes_R$, the dualizable objects are precisely the finitely generated projective modules.

Indeed, such a module P is a retract of R^r, so there are elements $p_i \in P$ and linear maps $\lambda_i\colon P \to R$ with $p = \sum_{i=1}^{r} \lambda_i(p) p_i$ for every $p \in P$. The evaluation pairing with $P^* = \mathrm{Hom}_R(P, R)$ and the coevaluation $1 \mapsto \sum_i p_i \otimes \lambda_i$ satisfy the triangle identities.

Conversely, if Q is a dual of P, write $\mathrm{coev}(1) = \sum_{i=1}^{r} p_i \otimes q_i$ and set $\lambda_i(p) = \mathrm{ev}(q_i \otimes p)$. The triangle identity gives $p = \sum_i \lambda_i(p) p_i$. Thus the maps $P \to R^r$, $p \mapsto (\lambda_i(p))_i$, and $R^r \to P$, $(a_i)_i \mapsto \sum_i a_i p_i$, exhibit P as a retract of R^r.

The following proposition characterizes duality by an adjunction: the evaluation map exhibits $- \otimes Y$ as right adjoint to $- \otimes X$. In particular, it implies that the duality datum (Y, ev) is unique if it exists.

Proposition 11.1.4. *Consider objects X and Y of C and consider a morphism $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ in C. The following conditions are equivalent:*

(1) The morphism $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ *exhibits* Y *as a dual to* X.

(2) For all objects Z *and* W *of* C*, the composite*

$$\mathrm{Hom}_C(W, Z \otimes Y) \xrightarrow{-\otimes X} \mathrm{Hom}_C(W \otimes X, Z \otimes Y \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(W \otimes X, Z)$$

is an equivalence.

(3) The map in (2) is an equivalence for $(W, Z) = (\mathbb{1}, X)$ *and for* $(W, Z) = (Y, \mathbb{1})$.

Proof. We first show that (1) implies (2). Let $\mathrm{coev}\colon \mathbb{1} \to X \otimes Y$ be the corresponding coevaluation map. It then follows directly from the triangle identities that an inverse to the map in (2) is given by the composite

$$\mathrm{Hom}_C(W \otimes X, Z) \xrightarrow{-\otimes Y} \mathrm{Hom}_C(W \otimes X \otimes Y, Z \otimes Y) \xrightarrow{-\circ \mathrm{coev}} \mathrm{Hom}_C(W, Z \otimes Y).$$

The implication (2)⇒(3) is immediate. For (3)⇒(1), the first test produces the coevaluation and one triangle identity; the second detects the other triangle identity. Indeed, the test $(W, Z) = (\mathbb{1}, X)$ gives a map $\mathrm{coev}\colon \mathbb{1} \to X \otimes Y$ whose image is id_X, so the composite

$$X \xrightarrow{\mathrm{coev}\otimes \mathrm{id}} X \otimes Y \otimes X \xrightarrow{\mathrm{id}\otimes \mathrm{ev}} X$$

is identified with id_X. The test $(W, Z) = (Y, \mathbb{1})$ gives an equivalence

$$\mathrm{Hom}_C(Y, Y) \xrightarrow{-\otimes X} \mathrm{Hom}_C(Y \otimes X, Y \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(Y \otimes X, \mathbb{1}).$$

Both id_Y and $(\mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev})$ map to ev: for the latter, this follows from the first triangle identity and the commutative diagram

$$\begin{array}{ccccc} Y \otimes X & \xrightarrow{\mathrm{id}\otimes \mathrm{coev}\otimes \mathrm{id}} & Y \otimes X \otimes Y \otimes X & \xrightarrow{\mathrm{ev}\otimes \mathrm{id}\otimes \mathrm{id}} & Y \otimes X \\ & \searrow & \downarrow{\scriptstyle \mathrm{id}\otimes \mathrm{id}\otimes \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} \\ & & Y \otimes X & \xrightarrow{\mathrm{ev}} & \mathbb{1}\,. \end{array}$$

The two maps are therefore identified, proving the second triangle identity. □

Thus being a duality datum is a property of the evaluation map ev: once condition (2) holds, the required coevaluation and triangle identities follow. We will use this criterion below to establish duality by constructing and analyzing an evaluation map.

Convention 11.1.5. Let X be a dualizable object of C. By condition (2) of Proposition 11.1.4, its dual represents the functor $\mathrm{Hom}_C(- \otimes X, \mathbb{1})\colon C^{\mathrm{op}} \to \mathrm{An}$. In particular, the dual is unique, and we denote it by $X^\vee$. An object representing this functor need not be a dual when X is not known to be dualizable; see the discussion of weak duals in Section 11.1.1.

Definition 11.1.6 (Invertible object)**.** An object X of C is called *invertible* if there exists another object X^{-1} and an isomorphism $X \otimes X^{-1} \cong \mathbb{1}$. Equivalently, X is invertible if the functor $X \otimes -\colon C \to C$ is an equivalence.

The full subanima $\mathrm{Pic}(C) \subseteq C^{\simeq}$ spanned by the invertible objects is called the *Picard anima* of C.

Lemma 11.1.7. *Every invertible object is dualizable.*

Proof. Let X be an invertible object and let X^{-1} be an inverse with isomorphism $\mathrm{ev}\colon X^{-1} \otimes X \xrightarrow{\sim} \mathbb{1}$. We claim that ev exhibits X^{-1} as a dual to X. Using Proposition 11.1.4 we must show that for all objects Z and W of C the composite

$$\mathrm{Hom}_C(W, Z \otimes X^{-1}) \xrightarrow{-\otimes X} \mathrm{Hom}_C(W \otimes X, Z \otimes X^{-1} \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(W \otimes X, Z)$$

is an equivalence. But both maps are equivalences: the first because the functor $- \otimes X\colon C \to C$ is an equivalence, and the second because ev is an isomorphism. □

In particular, the unit is dualizable, with itself as dual.

Lemma 11.1.8. *Symmetric monoidal functors preserve dualizable objects and their duals. Moreover, if $X, Y \in C$ are dualizable, then $X \otimes Y$ is dualizable with dual $X^\vee \otimes Y^\vee$.*

Proof. A symmetric monoidal functor preserves evaluation and coevaluation maps and the triangle identities. For the second claim, apply this observation to the tensor-product functor $C \times C \to C$, which is symmetric monoidal for the componentwise tensor product on $C \times C$, and to the dualizable object (X, Y). □

11.1.1 Internal homs

When C admits *internal homs*, every object X has a candidate dual $D(X) = \underline{\mathrm{Hom}}(X, \mathbb{1})$. Dualizability requires that the natural maps $Z \otimes D(X) \to \underline{\mathrm{Hom}}(X, Z)$ be isomorphisms for all Z. We make this precise below and use it to prove that dualizable objects form a thick subcategory.

Definition 11.1.9. Given two objects $X, Z \in C$, an object $\underline{\mathrm{Hom}}(X, Z) \in C$ equipped with a map $\mathrm{ev}\colon \underline{\mathrm{Hom}}(X, Z) \otimes X \to Z$ is said to *exhibit* $\underline{\mathrm{Hom}}(X, Z)$ *as an internal hom from X to Z* if for every third object W the composite

$$\mathrm{Hom}_C(W, \underline{\mathrm{Hom}}(X, Z)) \xrightarrow{-\otimes X} \mathrm{Hom}_C(W \otimes X, \underline{\mathrm{Hom}}(X, Z) \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(W \otimes X, Z)$$

is an equivalence.

Remark 11.1.10. Internal homs are unique if they exist. When they exist for every Z, the assignment $Z \mapsto \underline{\mathrm{Hom}}(X, Z)$ defines a right adjoint to $- \otimes X\colon C \to C$. A symmetric monoidal ∞-category admitting all internal homs is also called *closed*.

We now provide a formulation of duality in terms of internal homs.

Lemma 11.1.11. *Let X and Y be objects and let* $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ *be a morphism in C. Then the following conditions are equivalent:*

(1) The morphism ev *exhibits Y as a dual to X.*

(2) For every object Z of C, the morphism $\mathrm{id}_Z \otimes \mathrm{ev}\colon Z \otimes Y \otimes X \to Z$ *exhibits $Z \otimes Y$ as an internal hom* $\underline{\mathrm{Hom}}(X, Z)$.

(3) The functor $X \otimes -\colon C \to C$ admits a right adjoint $\underline{\mathrm{Hom}}(X, -)\colon C \to C$, and for every object Z of C the map

$$Z \otimes Y \to \underline{\mathrm{Hom}}(X, Z)$$

induced by the morphism $\mathrm{id} \otimes \mathrm{ev}\colon Z \otimes Y \otimes X \to Z$ *is an isomorphism in C.*

Proof. The equivalence between (1) and (2) is a reformulation of Proposition 11.1.4. The equivalence between (2) and (3) is clear. □

There is often an a priori candidate for the dual of X, sometimes called the *weak dual*:

Definition 11.1.12. Assume that C admits internal homs. We define the *internal duality functor* $D\colon C^{\mathrm{op}} \to C$ as

$$D(-) := \underline{\mathrm{Hom}}(-, \mathbb{1})\colon C^{\mathrm{op}} \to C.$$

If X is dualizable, the evaluation map $\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1}$ induces an isomorphism $X^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, \mathbb{1}) = D(X)$.

Corollary 11.1.13. *Let X be an object of C for which the internal hom functor $\underline{\mathrm{Hom}}(X, -)\colon C \to C$ exists. Then X is dualizable if and only if for every $Z \in C$ the natural map $Z \otimes D(X) \to \underline{\mathrm{Hom}}(X, Z)$ adjoint to the composite $Z \otimes D(X) \otimes X \xrightarrow{Z \otimes \mathrm{ev}} Z \otimes \mathbb{1} \simeq Z$ is an isomorphism.*

Lemma 11.1.14. *Assume that C admits internal homs. For $X, Y, Z \in C$, there is a natural currying isomorphism*

$$\underline{\mathrm{Hom}}(X \otimes Y, Z) \simeq \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z)).$$

Proof. This follows from the Yoneda lemma, and the observation that for every $W \in C$ there are natural equivalences

$$\begin{aligned} \mathrm{Hom}_C(W, \underline{\mathrm{Hom}}(X \otimes Y, Z)) &\simeq \mathrm{Hom}_C(W \otimes X \otimes Y, Z) \\ &\simeq \mathrm{Hom}_C(W \otimes X, \underline{\mathrm{Hom}}(Y, Z)) \\ &\simeq \mathrm{Hom}_C(W, \underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y, Z))). \end{aligned}$$

□

Remark 11.1.15. The objects called dualizable here are also known as *strongly dualizable*. A useful criterion is given in Lewis, May, and Steinberger [LMS86, Chapter III, Definition 1.1]: if C is closed, then X is dualizable precisely when the morphism $\mathbb{1} \to \underline{\mathrm{Hom}}(X, X)$ adjoint to id_X factors through the natural map $X \otimes D(X) \to \underline{\mathrm{Hom}}(X, X)$.

Definition 11.1.16. A *stably symmetric monoidal ∞-category* is a symmetric monoidal ∞-category C which is stable and whose tensor product $- \otimes -\colon C \times C \to C$ is exact in both variables.

Lemma 11.1.17. *Let C be a stably symmetric monoidal ∞-category which admits internal homs. Then the collection of dualizable objects forms a thick subcategory of C, i.e. it is a stable subcategory closed under retracts.*

Proof. By Corollary 11.1.13, an object X is dualizable if and only if the natural map

$$Z \otimes D(X) \to \underline{\mathrm{Hom}}(X, Z)$$

is an isomorphism for every $Z \in C$. By the universal property of internal homs and the exactness of tensor products, $\underline{\mathrm{Hom}}(-, Z)$ carries finite colimits to finite limits. Thus both sides are exact functors $C^{\mathrm{op}} \to C$, and the objects on which this natural map is an isomorphism form a stable subcategory. This subcategory is also closed under retracts, since isomorphisms are closed under retracts. □

11.2 Dualizability and compactness

For vector spaces, compactness and dualizability both mean finite-dimensionality. In general, we show that dualizable objects are compact whenever the unit is compact, and give a criterion for the converse. For modules over a commutative ring spectrum, both conditions characterize the perfect modules.

Lemma 11.2.1. *Let C be a symmetric monoidal ∞-category admitting filtered colimits. If its unit $\mathbb{1}$ is compact, then every dualizable object is compact.*

Proof. Let X be a dualizable object in C, and let $X^\vee$ denote its dual. Then the functor $X \otimes - \colon C \to C$ admits a right adjoint of the form $X^\vee \otimes - \colon C \to C$. In particular, we have

$$\mathrm{Hom}_C(X, -) \simeq \mathrm{Hom}_C(X \otimes \mathbb{1}, -) \simeq \mathrm{Hom}_C(\mathbb{1}, X^\vee \otimes -).$$

Since X is also a dual of $X^\vee$, the functor $X^\vee \otimes -$ is itself a left adjoint and preserves filtered colimits. Compactness of $\mathbb{1}$ therefore implies that the displayed functor preserves filtered colimits. □

For the converse, we use that dualizable objects form a thick subcategory. It therefore suffices to generate the compact objects from dualizable ones.

Lemma 11.2.2 (cf. Hovey, Palmieri, and Strickland [HPS97])**.** *Let C be a presentably symmetric monoidal stable ∞-category in the sense of Definition 16.2.8, and let $\mathcal{G}$ be a small collection of compact generators of C. If every object of $\mathcal{G}$ is dualizable, then every compact object of C is dualizable.*

Proof. The tensor product admits internal homs by Theorem 23.2.5. By the compact-generation theorem Proposition 23.3.4, the compact objects form the thick subcategory generated by $\mathcal{G}$. The claim now follows from Lemma 11.1.17. □

Corollary 11.2.3. *Let C be a presentably symmetric monoidal stable ∞-category, and let $\mathcal{G}$ be a small collection of compact generators. Then the compact objects of C are precisely its dualizable objects if and only if the unit $\mathbb{1}$ is compact and every object of $\mathcal{G}$ is dualizable.*

Proof. If the unit is compact and every object of $\mathcal{G}$ is dualizable, the claim follows from Lemmas 11.2.1 and 11.2.2. Conversely, if the compact and dualizable objects coincide, then the unit is compact because it is always dualizable, while every object of $\mathcal{G}$ is dualizable because it is compact. □

We apply this criterion to modules over a commutative ring spectrum R. Recall from Definition 8.1.10 that the perfect modules form the thick subcategory generated by R.

Proposition 11.2.4. *Let R be a commutative ring spectrum. An R-module is perfect if and only if it is dualizable in the symmetric monoidal ∞-category* $(\mathrm{Mod}_R, \otimes_R)$.

Proof. By Proposition 8.1.5, the ∞-category Mod_R is presentably symmetric monoidal and stable. Its unit R is a compact generator by Lemma 8.1.11, and is dualizable. Thus Corollary 11.2.3 identifies the dualizable modules with the compact modules, which are precisely the perfect modules. □

11.2.1 Spanier–Whitehead duality

Taking $R = \mathbb{S}$ in Proposition 11.2.4, we see that a spectrum is dualizable if and only if it is compact. We relate this to the following notion of finiteness.

Definition 11.2.5 (Finite spectrum)**.** A spectrum is *finite* if it lies in the smallest full stable subcategory of Sp containing the sphere spectrum $\mathbb{S}$.

One can show that the finite spectra are precisely the spectra of the form $\Sigma^\infty Y[-n]$ for a finite pointed cell complex Y and some $n \geq 0$. By Proposition 23.3.4, the compact spectra are precisely the retracts of finite spectra. For spectra, taking retracts does not enlarge this class:

Lemma 11.2.6. *A retract of a finite spectrum is finite.*

Proof. A finite spectrum is bounded below and has finitely generated integral homology concentrated in finitely many degrees. These properties pass to retracts. We show that any spectrum X with these properties is finite.

Choose $a \leq b$ such that X is a-connective and $H_i(X) = 0$ for $i > b$. Set $X_a = X$. For $i = a, \dots, b$, choose generators of $H_i(X_i)$ and lift them, by Theorem 6.2.17, to a map $\mathbb{S}[i]^{\oplus r_i} \to X_i$. Its cofiber X_{i+1} is $(i+1)$-connective, still has finitely generated homology, and satisfies $H_j(X_{i+1}) \cong H_j(X_i)$ for $j > i + 1$.

Thus X_{b+1} has homology only in degree $b + 1$, where it is the kernel of $\mathbb{Z}^{r_b} \to H_b(X_b)$ and hence free of finite rank. A basis lifts to a homology isomorphism $\mathbb{S}[b+1]^{\oplus r} \to X_{b+1}$, which is an isomorphism by Corollary 6.2.18. Reversing the finitely many cofiber sequences shows that X is finite. □

Corollary 11.2.7. *A spectrum is finite if and only if it is compact, and if and only if it is dualizable.*

Proof. The compact spectra are the retracts of finite spectra, so the first equivalence follows from Lemma 11.2.6. The second is the case $R = \mathbb{S}$ of Proposition 11.2.4 and Lemma 8.1.11. □

Corollary 11.2.8. *Let X be a finite anima. Then the spectrum $\mathbb{S}[X]$ is dualizable.*

Proof. The collection of animae X for which $\mathbb{S}[X]$ is dualizable is closed under finite colimits and contains $* \in \mathrm{An}$; consequently it contains all finite animae. □

The dual of a finite spectrum is called its *Spanier–Whitehead dual*. Tensoring with this dual computes the internal hom out of the original spectrum, so duality interchanges homology and cohomology. For spectra X and E, we use the notation

$$E_k(X) := \pi_k(E \otimes X) \qquad \text{and} \qquad E^k(X) := \pi_{-k} \hom(X, E).$$

For $X = \mathbb{S}[A]$, this recovers the conventions for the E-(co)homology of an anima A.

Lemma 11.2.9. *Let X and E be spectra and assume that X is dualizable. Then there are natural isomorphisms*

$$E_*(X^\vee) \cong E^{-*}(X) \qquad \textit{and} \qquad E^*(X^\vee) \cong E_{-*}(X).$$

Proof. The first isomorphism follows from

$$E_*(X^\vee) \cong \pi_*(E \otimes X^\vee) \cong \pi_*(\hom(X, E)) \cong E^{-*}(X).$$

The second follows by applying the first to $X^\vee$, whose dual is X. □

11.3 Atiyah duality

Let M be a closed smooth n-manifold. By Morse theory, M has the homotopy type of a finite cell complex [Mil63, Theorem 3.5 and Corollary 6.7], so its suspension spectrum $\mathbb{S}[M]$ is dualizable by Corollary 11.2.8. Atiyah duality identifies its dual geometrically:

$$D(\mathbb{S}[M]) \cong \mathrm{th}(-T_M).$$

Here $-T_M$ is the inverse of the virtual tangent bundle $[T_M]$ and has virtual rank $-n$; its Thom spectrum is defined as in Construction 10.2.18. We will prove this identification by constructing evaluation and coevaluation maps using Pontryagin–Thom collapse maps [MS60; Ati61b].

To construct these maps, we choose an embedding $M \hookrightarrow \mathbb{R}^N$ with normal bundle ν. The identification $T_M \oplus \nu \cong \underline{\mathbb{R}^N}$ expresses $\mathrm{th}(-T_M)$ as the desuspension $\mathrm{th}(\nu)[-N]$. We first recall why such an embedding exists.

Lemma 11.3.1. *Let M be a compact smooth n-manifold. Then there exists a natural number N and a smooth embedding $\varphi\colon M \hookrightarrow \mathbb{R}^N$.*

Proof. By compactness, we may choose finitely many coordinate charts $\varphi_i\colon U_i \to \mathbb{R}^n$, smaller open subsets V_i which still cover M and satisfy $\overline{V_i} \subseteq U_i$, and smooth functions $\rho_i\colon M \to [0,1]$ supported in U_i and strictly positive on V_i. Extend $\rho_i\varphi_i$ by zero outside U_i and consider

$$\varphi := (\rho_i\varphi_i, \rho_i)_{i=1}^k\colon M \longrightarrow (\mathbb{R}^n \times \mathbb{R})^k.$$

If $x \neq y$, choose i with $x \in V_i$. Equality of the ith pair of coordinates would force $\rho_i(y) = \rho_i(x) > 0$, hence $y \in U_i$, and then $\varphi_i(y) = \varphi_i(x)$, a contradiction. Thus φ is injective. The same coordinate pair shows that $d\varphi$ is injective at x: if it killed a tangent vector, then both $d\rho_i$ and $d(\rho_i\varphi_i) = \rho_i d\varphi_i + \varphi_i d\rho_i$ would vanish, and hence so would $d\varphi_i$. Thus φ is an injective immersion. Since M is compact, it is a smooth embedding. □

Remark 11.3.2. If $n > 0$, the *Whitney embedding theorem* shows that N can be chosen to be $2n$ in the previous lemma.

For the remainder of this section, we will fix once and for all a smooth embedding $\varphi\colon M \hookrightarrow \mathbb{R}^N$. We will denote by $\nu := \nu_\varphi \to M$ the normal bundle of this embedding, i.e., the cokernel of the map $T_M \to \varphi^*(T_{\mathbb{R}^N}) \cong \underline{\mathbb{R}^N}$ of vector bundles over M:

$$T_M \to \underline{\mathbb{R}^N} \to \nu.$$

By choosing a metric on the vector bundle $\underline{\mathbb{R}^N}$ one obtains an orthogonal decomposition

$$\underline{\mathbb{R}^N} \cong T_M \oplus \nu.$$

Passing to classes of virtual vector bundles and using Example 10.2.17, this gives $[T_M]+[\nu] \cong [\underline{\mathbb{R}^N}]$, so that $-T_M$ is the virtual vector bundle $[\nu] - [\underline{\mathbb{R}^N}]$ of rank $-n$. Hence Lemma 10.2.22 identifies

$$\mathrm{th}(-T_M) \cong \mathrm{th}(\nu)[-N] \cong \Sigma^{\infty-N}(\mathrm{Th}(\nu)). \tag{11.1}$$

We will construct evaluation and coevaluation maps

$$\mathrm{ev}\colon \mathrm{th}(-T_M) \otimes \mathbb{S}[M] \to \mathbb{S} \qquad \text{and} \qquad \mathrm{coev}\colon \mathbb{S} \to \mathbb{S}[M] \otimes \mathrm{th}(-T_M).$$

We first construct maps of pointed spaces, then specify the symmetries used to obtain these spectrum-level maps. Write τ for the symmetry of a smash or tensor product.

Construction 11.3.3. For a continuous map $f\colon X \to Y$ and a vector bundle $E \to Y$, we abbreviate the map of Construction 10.1.7 as

$$f_* := \mathrm{Th}(\widetilde{f})\colon \mathrm{Th}(f^*E) \to \mathrm{Th}(E).$$

Example 11.3.4. For the terminal map $p\colon M \to \mathrm{pt}$ and the bundle $\mathbb{R}^N \to \mathrm{pt}$, this gives

$$p_*\colon \mathrm{Th}(\underline{\mathbb{R}^N}) \cong S^N \wedge M_+ \longrightarrow S^N.$$

It is induced by the projection $\mathbb{R}^N \times M \to \mathbb{R}^N$.

Example 11.3.5. The diagonal $\Delta\colon M \to M \times M$ satisfies $\Delta^*(\nu \times \underline{0}) \cong \nu$. It therefore induces the Thom diagonal

$$\Delta_*\colon \mathrm{Th}(\nu) \to \mathrm{Th}(\nu \times \underline{0}) \cong \mathrm{Th}(\nu) \wedge M_+.$$

This sends a vector $v \in \nu_m$ to $v \wedge m$ and the basepoint to the basepoint.

Recall the Pontryagin–Thom collapse map of Construction 10.1.9. We also need its twisted version.

Construction 11.3.6 (Twisted Pontryagin–Thom collapse map)**.** Let $i\colon N \hookrightarrow N'$ be a smooth embedding of compact smooth manifolds without boundary, and let $E \to N'$ be a smooth vector bundle. The twisted collapse map has the form

$$\mathrm{PT}(i, E)\colon \mathrm{Th}(E) \to \mathrm{Th}\big(\nu(i) \oplus i^*E\big).$$

The composite of i with the zero section $N' \hookrightarrow E$ has normal bundle $\nu(i) \oplus i^*E$, using the canonical splitting of TE along the zero section. Choose a normalized tubular embedding of a closed disk bundle as in Construction 10.1.9. Its image is compact, so the collapse map on E is constant outside a compact subset and extends continuously to E^+. The identification $E^+ \cong \mathrm{Th}(E)$ of Remark 10.1.4 gives the desired map. Normalized tubular isotopies give homotopies of these extensions, so its based homotopy class is independent of the choices.

Example 11.3.7. Our embedding $\varphi\colon M \hookrightarrow \mathbb{R}^N$ gives a smooth embedding into S^N whose image avoids infinity. Choosing the tubular neighborhood to avoid infinity gives the pointed collapse map

$$\mathrm{PT}(\varphi)\colon S^N \to \mathrm{Th}(\nu).$$

Example 11.3.8. We identify the normal bundle of $\Delta\colon M \hookrightarrow M \times M$ with T_M by

$$(T_M \oplus T_M)/T_M \xrightarrow{\cong} T_M, \qquad [(u, v)] \mapsto v - u,$$

where the denominator is the diagonal subbundle. Identify ν with the orthogonal complement of $d\varphi(T_M)$ in $\underline{\mathbb{R}^N}$. The twist $E = \underline{0} \times \nu$ then gives the specific trivialization

$$\nu(\Delta) \oplus \Delta^* E \longrightarrow \underline{\mathbb{R}^N}, \qquad ([(u, v)], w) \longmapsto d\varphi(v - u) + w.$$

With this trivialization, the twisted collapse map is

$$\mathrm{PT}(\Delta, \underline{0} \times \nu)\colon M_+ \wedge \mathrm{Th}(\nu) \cong \mathrm{Th}(\underline{0} \times \nu) \longrightarrow \mathrm{Th}(\underline{\mathbb{R}^N}) \cong S^N \wedge M_+.$$

The point-set evaluation collapses away from the diagonal, combines its normal directions with ν using this trivialization, and then forgets the point of M:

$$\mathrm{ev}_0\colon M_+ \wedge \mathrm{Th}(\nu) \xrightarrow{\mathrm{PT}(\Delta, \underline{0} \times \nu)} S^N \wedge M_+ \xrightarrow{p_*} S^N.$$

The point-set coevaluation collapses a tubular neighborhood of M and records the basepoint of the resulting normal vector:

$$\mathrm{coev}_0\colon S^N \xrightarrow{\mathrm{PT}(\varphi)} \mathrm{Th}(\nu) \xrightarrow{\Delta_*} \mathrm{Th}(\nu) \wedge M_+.$$

We use the convention $\Sigma^{-N}(-) = \mathbb{S}[-N] \otimes -$ when passing to spectra. Thus the evaluation and coevaluation are the composites

$$\begin{aligned} \mathrm{ev}\colon \mathrm{th}(-T_M) \otimes \mathbb{S}[M] &\cong \Sigma^{-N}\Sigma^\infty(\mathrm{Th}(\nu) \wedge M_+) \\ &\xrightarrow{\Sigma^{-N}\Sigma^\infty \tau_{\mathrm{Th}(\nu), M_+}} \Sigma^{-N}\Sigma^\infty(M_+ \wedge \mathrm{Th}(\nu)) \\ &\xrightarrow{\Sigma^{-N}\Sigma^\infty \mathrm{ev}_0} \mathbb{S}, \end{aligned}$$

$$\begin{aligned} \mathrm{coev}\colon \mathbb{S} \cong \Sigma^{-N}\Sigma^\infty S^N &\xrightarrow{\Sigma^{-N}\Sigma^\infty \mathrm{coev}_0} \mathrm{th}(-T_M) \otimes \mathbb{S}[M] \\ &\xrightarrow{\tau_{\mathrm{th}(-T_M), \mathbb{S}[M]}} \mathbb{S}[M] \otimes \mathrm{th}(-T_M). \end{aligned}$$

Note that the last symmetry in the coevaluation moves the desuspension factor as well as the Thom spectrum.

To finish the proof of Atiyah duality, it remains to show that these maps satisfy the space-level analogues of the triangle identities. We will use the following standard compatibilities of the Pontryagin–Thom construction. We formulate them only for compact manifolds without boundary, which is the case needed below.

Proposition 11.3.9 (Pontryagin–Thom calculus)**.** *Pontryagin–Thom collapse maps have the following properties, up to based homotopy.*

(1) Isotopy invariance. *An isotopy identifies the normal bundles of its endpoint embeddings through the normal bundle over the parameter interval. Under this identification, the collapse maps are homotopic. The same holds for twisted collapse maps when the twisting bundle is fixed.*

(2) Compatibility with composition. *Let $N \overset{i}{\hookrightarrow} P \overset{j}{\hookrightarrow} Q$ be smooth embeddings of compact manifolds, and let E be a smooth vector bundle over Q. After choosing a splitting of the normal-bundle sequence*

$$0 \to \nu(i) \to \nu(ji) \to i^*\nu(j) \to 0,$$

the composite

$$\mathrm{Th}(E) \xrightarrow{\mathrm{PT}(j,E)} \mathrm{Th}(\nu(j) \oplus j^*E) \xrightarrow{\mathrm{PT}(i,\nu(j)\oplus j^*E)} \mathrm{Th}(\nu(i) \oplus i^*\nu(j) \oplus (ji)^*E)$$

agrees up to homotopy with $\mathrm{PT}(ji, E)$ under the resulting identification of target Thom spaces.

(3) Transverse base change. *Let $f\colon M_1 \to M_2$ be a smooth map between compact smooth manifolds without boundary and let $i_2\colon N_2 \hookrightarrow M_2$ be a closed smooth submanifold. Assume that f is transverse to N_2, and set $N_1 := f^{-1}(N_2)$, with inclusion $i_1\colon N_1 \hookrightarrow M_1$ and induced map $\bar{f}\colon N_1 \to N_2$. Then N_1 is a smooth submanifold and there is a natural isomorphism $\nu(i_1) \cong \bar{f}^*\nu(i_2)$. Under this identification, the square*

$$\begin{array}{ccc} (M_1)_+ & \xrightarrow{f_+} & (M_2)_+ \\ {\scriptstyle \mathrm{PT}(i_1)}\downarrow & & \downarrow{\scriptstyle \mathrm{PT}(i_2)} \\ \mathrm{Th}(\nu(i_1)) & \xrightarrow{\bar{f}_*} & \mathrm{Th}(\nu(i_2)) \end{array}$$

commutes up to based homotopy. More generally, for a smooth vector bundle E over M_2, the square

$$\begin{array}{ccc} \mathrm{Th}(f^*E) & \xrightarrow{f_*} & \mathrm{Th}(E) \\ {\scriptstyle \mathrm{PT}(i_1, f^*E)}\downarrow & & \downarrow{\scriptstyle \mathrm{PT}(i_2, E)} \\ \mathrm{Th}(\nu(i_1) \oplus i_1^* f^* E) & \xrightarrow{\bar{f}_*} & \mathrm{Th}(\nu(i_2) \oplus i_2^* E) \end{array}$$

commutes up to based homotopy.

Proof sketch. Choose Riemannian metrics and tubular neighborhoods. An isotopy of embeddings has a tubular neighborhood over the parameter interval, which gives the homotopy in (1). For (2), choose nested tubular neighborhoods for $N \subseteq P \subseteq Q$; collapsing successively then agrees with collapsing the resulting tubular neighborhood of N in Q. For (3), transversality identifies the normal bundle of N_1 with $\bar{f}^*\nu(i_2)$. In tubular coordinates, rescaling the normal directions deforms f to its normal derivative. After shrinking the collapse radius, this gives a homotopy constant outside a fixed compact neighborhood of N_1. Applying the same argument to $f^*E \to E$ and the zero-section embeddings of N_1 and N_2 proves twisted base change; the isotopy and composition arguments likewise apply in the total space of E. □

The evaluation map ev has a homotopy class independent of the metrics and normalized tubular neighborhoods used in its construction.

Theorem 11.3.10 (Atiyah duality)**.** *The map* ev *exhibits* $\mathrm{th}(-T_M)$ *as a dual of* $\mathbb{S}[M]$*, and hence determines an isomorphism:*

$$D(\mathbb{S}[M]) \cong \mathrm{th}(-T_M).$$

Proof. Write $a_N \colon S^N \to S^N$ for the one-point compactification of $-\mathrm{id}_{\mathbb{R}^N}$, of degree $(-1)^N$. With the symmetries in the definitions of ev and coev, the triangle identities follow from the point-set homotopies

$$(\mathrm{id} \wedge \mathrm{ev}_0) \circ (\mathrm{coev}_0 \wedge \mathrm{id}) \simeq (\mathrm{id}_{\mathrm{Th}(\nu)} \wedge a_N) \circ \tau_{S^N, \mathrm{Th}(\nu)},$$

$$(\mathrm{ev}_0 \wedge \mathrm{id}) \circ (\mathrm{id} \wedge \mathrm{coev}_0) \simeq \tau_{M_+, S^N}.$$

The sources are $S^N \wedge \mathrm{Th}(\nu)$ and $M_+ \wedge S^N$, respectively. The symmetry interchanging $\mathbb{S}[-N]$ and $\mathbb{S}[N]$ acts by $(-1)^{-N^2} = (-1)^N$; compare Warning 4.2.11. In the first triangle this cancels the map induced by a_N; the second triangle has no remaining sphere interchange. We verify the second identity by a graph isotopy, and the first by the same method together with a rotation of the two normal-bundle summands.

We start with the first composite. Consider the following pullback square:

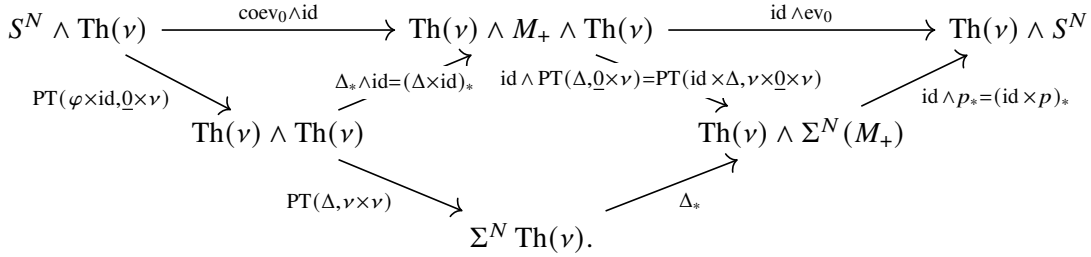
$$\begin{array}{ccc} M & \xhookrightarrow{\Delta} & M \times M \\ {\scriptstyle\Delta}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{id}\times\Delta} \\ M \times M & \xhookrightarrow{\Delta\times\mathrm{id}} & M \times M \times M. \end{array}$$

At (m, m), the map induced by $\Delta \times \mathrm{id}$ into the normal bundle of $\mathrm{id} \times \Delta$ is

$$T_m M \oplus T_m M \longrightarrow T_m M, \qquad (v, w) \longmapsto w - v.$$

It is surjective, proving transversality, and agrees with our normal-bundle convention for the diagonal. Apply transverse base change in Proposition 11.3.9 with twist $\nu \times \underline{0} \times \nu$.

For embeddings into $S^N \times M$ or $M \times S^N$, we choose tubular neighborhoods avoiding the fiber at infinity. Compactness permits this throughout the graph isotopies below, so the collapse maps descend to the reduced quotients. We use these reduced maps without further notation. Together with the definitions of ev_0 and coev_0, transverse base change gives the following homotopy-commutative diagram:



The top row is the first triangle composite. The upper two triangles commute by the definitions of coev_0 and ev_0, and the central diamond commutes by transverse base change. At the bottom vertex, the identification $\mathrm{Th}(T_M \oplus \nu \oplus \nu) \cong S^N \wedge \mathrm{Th}(\nu)$ uses $(t, v, w) \mapsto (d\varphi(t) + w, v)$; the outgoing pushforward places the ν-factor first and records the point of M over which it lies. Compatibility with composition identifies the successive collapses with the twisted collapse for the graph embedding

$$j \colon M \xhookrightarrow{\Delta} M \times M \xhookrightarrow{\varphi \times \mathrm{id}} S^N \times M,$$

with twist $\underline{0} \times \nu$. To identify the whole composite, we must also track the bundle isomorphism used at the bottom vertex. For the isotopy $j_s = (s\varphi, \mathrm{id})$, identify the normal bundle with $\underline{\mathbb{R}^N}$ by

$$[(z,u)] \longmapsto z - s\, d\varphi(u).$$

At $s = 1$, write the normal coordinate as $\eta = z - d\varphi(u) = d\varphi(t) + v$, with $v \in \nu_m$. The convention $[(u_1,u_2)] \mapsto u_2 - u_1$ for the diagonal sends the tangential component to $-t$. Including the right diagonal maps, the triangle composite is therefore the graph collapse followed by the Thom map of

$$F\colon \underline{\mathbb{R}^N} \oplus \nu \longrightarrow \nu \oplus \underline{\mathbb{R}^N}, \qquad (d\varphi(t) + v, w) \longmapsto (v, -d\varphi(t) + w).$$

Transport this identification along j_s. At $s = 0$, the collapse for the zero graph is homotopic to the identity on $S^N \wedge \mathrm{Th}(\nu)$ before applying the Thom map of F. The base map in the right diagonal is $(\mathrm{id} \times p) \circ \Delta = \mathrm{id}_M$, so no further base map remains.

The bundle isomorphisms

$$(d\varphi(t) + v, w) \longmapsto \big(\cos\theta\, v + \sin\theta\, w,\ -d\varphi(t) - \sin\theta\, v + \cos\theta\, w\big), \qquad 0 \le \theta \le \pi/2,$$

join F to $(\eta, w) \mapsto (w, -\eta)$. Each map is an isomorphism: it is a rotation on the two normal summands and negation on the tangent summand. Their Thom maps give the first required homotopy.

For the second composite, apply the transposed pullback square. Take $i_2 = \Delta \times \mathrm{id}$, $f = \mathrm{id} \times \Delta$ and use the twist $E = \underline{0} \times \nu \times \underline{0}$ over $M \times M \times M$. Transverse base change and compatibility with composition identify the composite $(\mathrm{ev}_0 \wedge \mathrm{id}) \circ (\mathrm{id} \wedge \mathrm{coev}_0)$ with the composite of the Pontryagin–Thom collapse map for the graph embedding

$$(\mathrm{id}, \varphi)\colon M \hookrightarrow M \times S^N$$

and the Thom-space pushforward induced by $(p \times \mathrm{id}) \circ \Delta = \mathrm{id}_M$. For the isotopy $(\mathrm{id}, s\varphi)$, the normal identification $[(u,z)] \mapsto z - s\, d\varphi(u)$ agrees at $s = 1$ with our convention for the diagonal and the splitting $T_M \oplus \nu \cong \underline{\mathbb{R}^N}$. At $s = 0$, the reduced collapse is homotopic to the identity on $M_+ \wedge S^N$. Placing the sphere factor first gives τ_{M_+, S^N}, as required. □

11.4 Poincaré duality

For a closed oriented smooth n-manifold M, classical Poincaré duality gives an isomorphism

$$- \cap [M]\colon H^{n-k}(M) \xrightarrow{\sim} H_k(M)$$

by cap product with the fundamental class. We now derive this from Atiyah duality and extend it to other cohomology theories. Atiyah duality identifies the dual of $\mathbb{S}[M]$ with $\mathrm{th}(-T_M)$. An E-orientation of T_M gives a Thom isomorphism for this spectrum; combined with Atiyah duality, this yields an isomorphism between E-homology and shifted E-cohomology.

Theorem 11.4.1 (Poincaré duality)**.** *Let E be a commutative ring spectrum, and let M be a closed smooth n-manifold equipped with an E-orientation of the virtual tangent bundle $[T_M]$, or equivalently of $-T_M$ by Lemma 10.4.9. Then there is an isomorphism of spectra*

$$\mathbb{S}[M] \otimes E \cong \hom(\mathbb{S}[M], E[n]).$$

In particular, for every $k \in \mathbb{Z}$ there is an isomorphism

$$E_k(M) \cong E^{n-k}(M).$$

Proof. By Theorem 11.3.10, the dual of the spectrum $\mathbb{S}[M]$ in Sp is $\mathrm{th}(-T_M)$, and in particular we obtain an isomorphism of spectra

$$\mathbb{S}[M] \otimes E \cong \hom(\mathrm{th}(-T_M), E).$$

By Lemma 10.4.9, the given orientation of $[T_M]$ induces an orientation of $-T_M$. Since this virtual bundle has rank $-n$, the cohomological Thom isomorphism of Theorem 10.4.6 gives

$$\hom(\mathrm{th}(-T_M), E) \cong \hom(\mathbb{S}[M][-n], E) \cong \hom(\mathbb{S}[M], E[n]).$$

This proves the first claim. Applying $\pi_k(-)$ gives the second. □

The preceding proof is the abstract form of Poincaré duality. We next relate it to the classical fundamental-class formulation. The orientation of $-T_M$ gives a Thom class $u_{-T_M}\colon \mathrm{th}(-T_M) \to E[-n]$. Define the *$E$-fundamental class*

$$[M]_E \in E_n(M) = \pi_n(E \otimes \mathbb{S}[M])$$

to be represented, after shifting, by the composite

$$\mathbb{S} \xrightarrow{\mathrm{coev}} \mathbb{S}[M] \otimes \mathrm{th}(-T_M) \xrightarrow{\mathrm{id}\otimes u_{-T_M}} \mathbb{S}[M] \otimes E[-n].$$

Using the embedding $M \hookrightarrow \mathbb{R}^N$ chosen in the proof of Theorem 11.3.10, this is concretely the class determined by the Pontryagin–Thom collapse $S^N \to \mathrm{Th}(\nu)$, followed by the diagonal $\Delta_*\colon \mathrm{Th}(\nu) \to \mathrm{Th}(\nu) \wedge M_+$ and then the Thom class of the normal bundle on the first factor. For $E = \mathbb{Z}$, restricting this construction to a coordinate neighborhood of $m \in M$ shows that its image in $H_n(M, M \setminus \{m\}; \mathbb{Z})$ is the local generator determined by the orientation. Thus it is the classical fundamental class.

Proposition 11.4.2. *The inverse of the isomorphism in Theorem 11.4.1 is cap product with $[M]_E$:*

$$- \cap [M]_E\colon E^{n-k}(M) \xrightarrow{\sim} E_k(M).$$

Proof. Represent a cohomology class by a map $a\colon \mathbb{S}[M] \to E[n-k]$. Cap product with $[M]_E$ is represented, after shifting, by the composite

$$\begin{aligned}\mathbb{S} &\xrightarrow{[M]_E} \mathbb{S}[M] \otimes E[-n] \xrightarrow{\mathbb{S}[\Delta]\otimes\mathrm{id}} \mathbb{S}[M] \otimes \mathbb{S}[M] \otimes E[-n] \\ &\xrightarrow{\mathrm{id}\otimes a\otimes\mathrm{id}} \mathbb{S}[M] \otimes E[n-k] \otimes E[-n] \xrightarrow{\mathrm{id}\otimes\mu} \mathbb{S}[M] \otimes E[-k].\end{aligned}$$

Let $\delta\colon \mathrm{th}(-T_M) \to \mathbb{S}[M] \otimes \mathrm{th}(-T_M)$ be the Thom diagonal. The construction of coev gives

$$(\mathbb{S}[\Delta] \otimes \mathrm{id}) \circ \mathrm{coev} \simeq (\mathrm{id}\otimes\delta) \circ \mathrm{coev}\colon \mathbb{S} \longrightarrow \mathbb{S}[M] \otimes \mathbb{S}[M] \otimes \mathrm{th}(-T_M).$$

Indeed, both sides apply the Pontryagin–Thom collapse and record twice the point of M over which the normal vector lies. Substituting the definition of $[M]_E$ therefore rewrites the cap product as

$$\mathbb{S} \xrightarrow{\mathrm{coev}} \mathbb{S}[M] \otimes \mathrm{th}(-T_M) \xrightarrow{\mathrm{id}\otimes b} \mathbb{S}[M] \otimes E[-k],$$

where

$$b\colon \mathrm{th}(-T_M) \xrightarrow{\delta} \mathbb{S}[M] \otimes \mathrm{th}(-T_M) \xrightarrow{a\otimes u_{-T_M}} E[n-k] \otimes E[-n] \xrightarrow{\mu} E[-k].$$

As in the proof of Corollary 10.4.8, the map $(\mathrm{id} \otimes u_{-T_M}) \circ \delta$ extends E-linearly to the Thom isomorphism: on each fiber it applies the chosen trivialization and records the point of M over which the fiber lies. Consequently, b is the image of a under the inverse cohomological Thom isomorphism. The triangle identities identify $b \mapsto (\mathrm{id} \otimes b) \circ \mathrm{coev}$ with the inverse of $\mathbb{S}[M] \otimes E \cong \hom(\mathrm{th}(-T_M), E)$, after shifting by $-k$. The displayed cap product is therefore the inverse of the composite used in Theorem 11.4.1. □

Example 11.4.3. For $E = \mathbb{F}_2$, every real vector bundle is E-orientable because $\mathrm{Line}_{\mathbb{F}_2} \simeq *$. Thus every closed smooth manifold satisfies mod-2 Poincaré duality. For $E = \mathbb{Z}$, Example 10.4.10 identifies an $\mathbb{Z}$-orientation with a classical orientation. This recovers integral Poincaré duality for oriented manifolds.

Example 11.4.4. The manifold $\mathbb{R}P^2$ is not integrally orientable, and integral Poincaré duality indeed fails: $H_0(\mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}$, whereas $H^2(\mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2$. With $\mathbb{F}_2$-coefficients, it is orientable and Poincaré duality holds.

Remark 11.4.5. A Spin^c-structure on M determines a KU-orientation of $[T_M]$ through the construction of Atiyah, Bott, and Shapiro [ABS64, Theorem 12.3(ii)], after stabilization. Moreover, Example 10.4.11 provides a canonical MO-orientation of every real vector bundle. Thus every closed smooth manifold satisfies MO-Poincaré duality.

Remark 11.4.6. For a compact smooth n-manifold M with boundary, equipped with an E-orientation of $[T_M]$, Poincaré–Lefschetz duality gives

$$E_k(M) \cong E^{n-k}(M, \partial M).$$

This uses the boundary version of Atiyah duality, which we have not proved here:

$$D\big(\Sigma^\infty \Pi_\infty (M/\partial M)\big) \cong \mathrm{th}(-T_M);$$

see Atiyah [Ati61b, Proposition 3.2]. Tensoring with E and applying the Thom isomorphism gives the displayed identification of the homology of M with its relative cohomology.

Exercises

Exercise 11.1 (Duality for the cartesian monoidal structure)**.** Let C be an ∞-category with finite products, equipped with its cartesian symmetric monoidal structure. Show that an object of C is dualizable if and only if it is terminal. Deduce the corresponding statement for the cocartesian monoidal structure on an ∞-category with finite coproducts.

Exercise 11.2 (Duality and cofiber sequences)**.** Let C be a stably symmetric monoidal ∞-category.

(1) Show that the zero object is dualizable.

(2) Let $f\colon X \to Y$ be a morphism between dualizable objects. Show that $\mathrm{cofib}(f)$ is dualizable with dual $\mathrm{fib}(f^\vee\colon Y^\vee \to X^\vee)$.

(3) Deduce that the full subcategory of dualizable objects is closed under finite colimits. Apply this to $C = \mathrm{Sp}$ to give a direct proof of Corollary 11.2.8.

Hint: Use the adjunction characterization in Proposition 11.1.4 and the exactness of tensor products.

Exercise 11.3 (The dual of a quotient module)**.** Let R be a commutative ring spectrum, let $x \in \pi_0(R)$, and let R/x be the cofiber of multiplication by x on R. The R-module R/x is perfect and hence dualizable. Compute its R-linear dual, showing that

$$\hom_R(R/x, R) \cong (R/x)[-1].$$

Exercise 11.4 (Duals of finite wedges of spheres)**.** Show that $D(\mathbb{S}[n]) \cong \mathbb{S}[-n]$ for every $n \in \mathbb{Z}$. Deduce that

$$D\left(\bigoplus_{i=1}^{r} \mathbb{S}[n_i]\right) \cong \bigoplus_{i=1}^{r} \mathbb{S}[-n_i].$$

Show that the analogous formula for an infinite wedge has a product on the right, and that an infinite wedge of shifted spheres is not dualizable.

Exercise 11.5 (Atiyah duality for a sphere)**.** Let $n \geq 0$. Use the standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$ and its trivial normal line bundle to identify $\mathrm{th}(-T_{S^n})$. Show that Atiyah duality gives

$$D(\mathbb{S}[S^n]) \cong \mathbb{S} \oplus \mathbb{S}[-n].$$

Compare this with the splitting $\mathbb{S}[S^n] \cong \mathbb{S} \oplus \mathbb{S}[n]$.

Exercise 11.6 (Consequences of Poincaré duality)**.** Let M be a closed oriented smooth n-manifold.

(1) Show that the pairing

$$H^k(M;\mathbb{Q}) \otimes_{\mathbb{Q}} H^{n-k}(M;\mathbb{Q}) \longrightarrow \mathbb{Q}, \qquad a \otimes b \longmapsto \langle a \smile b, [M] \rangle,$$

is nondegenerate.

(2) Deduce that the Betti numbers satisfy $b_k(M) = b_{n-k}(M)$, and that M has Euler characteristic zero if n is odd.

(3) Use mod-2 Poincaré duality to show that every closed smooth odd-dimensional manifold has Euler characteristic zero, without assuming orientability.

Exercise 11.7 (Invertible spectra)**.** Show that every invertible spectrum is isomorphic to $\mathbb{S}[n]$ for a unique $n \in \mathbb{Z}$. Conclude that the definitions of $\mathrm{Pic}(\mathrm{Sp})$ in Definitions 11.1.6 and 10.2.7 agree. Let X be an invertible spectrum with inverse Y, and proceed as follows:

(1) Show that X is compact. Deduce that X is bounded below and that $\bigoplus_k H_k(X)$ is a finitely generated abelian group.

(2) Let k be a field. Show that $H_*(X \otimes Y; k) \cong H_*(X; k) \otimes_k H_*(Y; k)$, and conclude that $H_*(X; k)$ is one-dimensional.

(3) Deduce that $H_*(X) \cong \mathbb{Z}$, concentrated in a single degree n.

(4) Show that $X \cong \mathbb{S}[n]$.

Part II

Operads and higher algebra

Introduction to Part II

In Definition 8.1.1, we defined symmetric monoidal ∞-categories as commutative monoids in Cat_∞. This definition packages the tensor product and all its coherences into a single object, but constructing that object requires more than specifying a tensor product: we must also supply its coherent associativity, symmetry, and unit data. Even for the cartesian monoidal structure on An, it is not immediately clear how to do this directly. In this part, we develop the theory of ∞-*operads*, which gives us a more explicit way to construct symmetric monoidal structures and to describe algebras and modules in them.

Our approach differs from the one in Lurie's foundational text [Lur17] in that we define ∞-operads as functors to the *span category* $\mathrm{Span}(\mathrm{Fin})$ of finite sets, rather than to the category Fin_* of finite pointed sets. While the two approaches are equivalent (see Section 19.4), the span-based perspective has the advantage that all the required conditions can be formulated in terms of finite products, which is conceptually cleaner.

The first chapters develop the language of ∞-operads and use it to construct cartesian and cocartesian monoidal structures, Day convolution, and the tensor product of spectra. With these constructions in place, we establish the theory of algebras and modules used in Part I (Chapter 17). Tensor products of presentable ∞-categories then allow us to characterize ∞-categories of modules by universal properties and to study when different algebras have equivalent module categories (Chapter 18). The remaining chapters develop general operadic constructions, including envelopes, stabilization, and localization. These also supply the multiplicative structures on derived categories needed in Chapter 8.

Prerequisites. This part makes heavy use of the theory of cocartesian fibrations and the straightening/unstraightening equivalence from Chapter 24. The construction of span categories in Chapter 13 also relies on the presentation of ∞-categories by complete Segal animae from Chapter 25. Readers unfamiliar with either topic are encouraged to consult the corresponding chapter in Part III as needed.

12 Operads: the heuristics

This chapter develops the intuition behind the definition of ∞-operads before we turn to the abstract theory. Roughly speaking, an ∞-operad encodes a type of 'algebraic structure'; for example, there will be an ∞-operad for commutative algebras, one for associative algebras, one for left modules over an associative algebra, and so on.

In classical algebra, a non-colored operad encodes collections of n-ary operations $A^{\otimes n} \to A$, together with their composition. Colored operads allow several types of inputs and outputs, with operations of the form $A_1 \otimes \cdots \otimes A_n \to B$. To pass to ∞-categories, we need a way to encode the coherences between these operations. We therefore reformulate a colored operad in terms of its symmetric monoidal envelope. This categorical description will guide the definition of an ∞-operad.

12.1 Non-colored operads

Let us start by thinking about how to describe some algebraic structure on an object A of a symmetric monoidal category $(C, \otimes, \mathbb{1})$. Often, such algebraic structures are given in terms of a collection of *n-ary operations* for natural numbers $n \geq 0$, i.e., maps of the form $A^{\otimes n} \to A$ from the n-fold tensor product of A to itself. For example, if A is an associative algebra object in C, we have the multiplication map $\cdot\colon A^{\otimes 2} \to A, (x, y) \mapsto xy$. Also the unit is of this form: when $n = 0$, an n-fold tensor product of A is by convention the monoidal unit $\mathbb{1} \in C$, and we often want to treat the unit of the algebra A as a 0-ary operation $1\colon \mathbb{1} \to A$.

Observe that operations may be combined to obtain more of them. For example, assume that for natural numbers $k, n_1, \ldots, n_k \geq 0$ we are given operations on A of the form

$$P\colon A^{\otimes k} \to A, \qquad Q_i\colon A^{\otimes n_i} \to A, \quad i = 1, \ldots, k.$$

Then we may combine them into an n-ary operation $P(Q_1, \ldots, Q_k)$ for $n := \sum_{i=1}^{k} n_i$ by forming the following composite:

$$A^{\otimes n} \cong \bigotimes_{i=1}^{k} A^{\otimes n_i} \xrightarrow{\bigotimes_{i=1}^{k} Q_i} \bigotimes_{i=1}^{k} A = A^{\otimes k} \xrightarrow{P} A.$$

We will refer to the operation $P(Q_1, \ldots, Q_k)$ as the *composition* of P with $(Q_1, \ldots, Q_k)$. In similar fashion we may also always consider the *identity operation* $\mathrm{id}\colon A \to A, a \mapsto a$. Finally, we may permute the inputs of an n-ary operation by any permutation $\sigma \in \Sigma_n$:

$$(P\sigma)(x_1, \ldots, x_n) := P(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

If we apply this process to our associative algebra A, we find that there are precisely $n!$ formal ways of constructing an n-ary operation: for every permutation $\sigma \in \Sigma_n := \mathrm{Aut}_{\mathrm{Set}}(\{1, \dots, n\})$, we obtain a multiplication map $m_\sigma \colon A^{\otimes n} \to A$ given by $m_\sigma(x_1, \dots, x_n) := x_{\sigma(1)} \cdots x_{\sigma(n)}$. For a particular algebra, some of these maps may of course agree.

The notion of a (non-colored, symmetric, non-enriched) *operad* is trying to axiomatize the collections of possible operations one may have, together with their various compositions and permutations:

Definition 12.1.1 ([May72; BV73])**.** A *non-colored operad* $\mathcal{O}$ consists of the following data:

- A set $\mathcal{O}(n)$ of n-ary operations, for every $n \geq 0$;
- A *composition map*

$$-\circ-\colon \mathcal{O}(k) \times \prod_{i=1}^{k} \mathcal{O}(n_i) \to \mathcal{O}(n_1 + \cdots + n_k), \quad (P, Q_1, \dots, Q_k) \mapsto P(Q_1, \dots, Q_k)$$

  for all natural numbers $k, n_1, \dots, n_k \geq 0$;
- An *identity operation* $\mathrm{id} \in \mathcal{O}(1)$;
- A right action of the symmetric group Σ_n on $\mathcal{O}(n)$ for every $n \geq 0$.

These data are subject to the following conditions:

- Composition is *unital* and *associative*:

$$P(\mathrm{id}, \dots, \mathrm{id}) = P = \mathrm{id}(P),$$
$$R(P_1(Q_1^1, \dots, Q_{k_1}^1), \dots, P_l(Q_1^l, \dots, Q_{k_l}^l)) = (R(P_1, \dots, P_l))(Q_1^1, \dots, Q_{k_1}^1, Q_1^2, \dots, Q_{k_l}^l).$$

- Composition is compatible with the permutation actions; see Exercise 12.1 for details.

The following are some examples of non-colored operads:

Example 12.1.2. The *associative operad* $\mathcal{A}ssoc$ has $\mathcal{A}ssoc(n) := \Sigma_n$, i.e., there is one n-ary operation for every permutation $\sigma \in \Sigma_n$.

Example 12.1.3. The *commutative operad* $\mathcal{C}omm$ has $\mathcal{C}omm(n) := *$, i.e., there is a *single* n-ary operation for every $n \geq 0$.

Example 12.1.4. There is the *trivial operad* $\mathcal{T}riv$, whose only operation is the identity operation:

$$\mathcal{T}riv(1) = \{\mathrm{id}\}, \qquad \mathcal{T}riv(n) = \emptyset \quad \text{for} \quad n \neq 1.$$

Example 12.1.5. The *pointed operad* $\mathcal{E}_0$ has only one non-identity operation:

$$\mathcal{E}_0(1) = \{\mathrm{id}\}, \qquad \mathcal{E}_0(0) = \{\eta\}, \qquad \mathcal{E}_0(n) = \emptyset \quad \text{for} \quad n \geq 2.$$

Example 12.1.6. For every field K there is an operad enriched in K-vector spaces called the *Lie operad*, denoted $\mathcal{L}ie$, which captures the operations present in a Lie algebra. The vector space $\mathcal{L}ie(n)$ is the subspace of the free Lie algebra $L(x_1, \dots, x_n)$ spanned by the 'multilinear' elements: formal sums of iterated Lie brackets that use each variable x_i exactly once. Lie algebras are algebras over $\mathcal{L}ie$ in the enriched sense; forgetting the enrichment gives an underlying set-valued operad.

Example 12.1.7. For every $k \geq 0$, there is a topological operad[16] called the *little k-cubes operad*, often denoted $\mathcal{E}_k$. Its n-ary operations are the so-called 'rectilinear' embeddings $\bigsqcup_{i=1}^{n}[0,1]^k \to [0,1]^k$ of a disjoint union of n copies of the k-dimensional cube into the k-dimensional cube. Here 'rectilinear' means that on each of the n components it is given by an affine function, i.e., maps of the form

$$[0,1]^k \to [0,1]^k, \quad (x_i)_{i=1}^k \mapsto (a_i x_i + b_i)_{i=1}^k, \qquad a_i > 0,$$

which allow us to scale and shift the cube but not rotate or distort it in any other way. The composition in $\mathcal{E}_k$ is given by composition of rectilinear embeddings. For $k = 0$, this recovers the pointed operad $\mathcal{E}_0$ described above.

Remark 12.1.8. For $k = 1$, ordering the little intervals from left to right defines an operad morphism $\mathcal{E}_1 \to \mathcal{A}ssoc$. For every $n \geq 0$, its fibers are the contractible components of $\mathcal{E}_1(n)$, so the map $\mathcal{E}_1(n) \to \mathcal{A}ssoc(n)$ is a homotopy equivalence. In particular, after passing from sets and topological spaces to animae (as we will do when working with ∞-operads) there is no longer a distinction between $\mathcal{E}_1$ and $\mathcal{A}ssoc$. In more classical setups, for example when working with model categories, the homotopically correct operad to work with is often $\mathcal{E}_1$. This is the reason why in older literature you will often find the phrase '$\mathcal{E}_1$-algebras' in places where we would say 'associative algebra'.

Example 12.1.9 (Endomorphism operad)**.** For an object A of a symmetric monoidal category C, we may form the *endomorphism operad* $\mathcal{E}nd_C(A)$, given by $\mathcal{E}nd_C(A)(n) := \mathrm{Hom}_C(A^{\otimes n}, A)$. Composition is induced by composition in C and the associativity isomorphisms for tensor products. The right action of Σ_n is given by precomposition with the symmetry isomorphisms of $A^{\otimes n}$, using the convention from the display above.

Given the structure an operad has, there is a clear notion of *morphisms* between them: if $\mathcal{O}$ and $\mathcal{P}$ are operads, then an operad morphism $f\colon \mathcal{O} \to \mathcal{P}$ is a collection of maps $f(n)\colon \mathcal{O}(n) \to \mathcal{P}(n)$ that preserves the identity operation, the composition of operations and the permutation of operations.

Definition 12.1.10. Let $\mathcal{O}$ be a non-colored operad and let C be a symmetric monoidal category. We define an *$\mathcal{O}$-algebra in C* as a pair (A, f_A) consisting of an object A of C together with an operad morphism $f_A\colon \mathcal{O} \to \mathcal{E}nd_C(A)$. In particular, f_A consists of an n-ary operation $P_A\colon A^{\otimes n} \to A$ for every $P \in \mathcal{O}(n)$ satisfying the condition that composition of operations in $\mathcal{O}$ corresponds to the composition of operations on A.

The basic examples of algebras over these operads are worked out in Exercise 12.3 at the end of the chapter.

Remark 12.1.11. Note that $\mathcal{C}omm$ is the *terminal* operad: there is precisely one operad map $\mathcal{O} \to \mathcal{C}omm$ given by the unique maps $\mathcal{O}(n) \to \mathcal{C}omm(n) = *$. Also note that $\mathcal{T}riv$ is the *initial* operad: there is precisely one operad map $\mathcal{T}riv \to \mathcal{O}$ sending the unique operation id of $\mathcal{T}riv$ to the identity operation id $\in \mathcal{O}(1)$.

12.2 Colored operads

While the notion of non-colored operads introduced previously is elegant, it is insufficient for our purposes. To illustrate this, let us discuss the algebraic structure of an associative algebra A acting

[16] This means that we remember the topology on each set of operations $\mathcal{O}(n)$.

on some module M. To encode such an action, we need a map in C of the form

$$\mathrm{act}\colon A \otimes M \to M, \qquad (a, m) \mapsto am,$$

and its source is not a tensor-power of a single object. More generally, given a collection of objects (A_x) in C we would like to have some notion of operad which can deal with operations of the form

$$P\colon A_{y_1} \otimes \cdots \otimes A_{y_n} \to A_x$$

for certain choices of indices $y_1, \ldots, y_n, x$. This leads to the following 'multicolored' version of Definition 12.1.1:

Definition 12.2.1. A *colored operad* (or *multicategory*) $\mathcal{O}$ consists of the following structure:

- There is a set of *colors* (sometimes also called the *objects* of $\mathcal{O}$);
- For colors $x, y_1, \ldots, y_n$ of $\mathcal{O}$, there is a set $\mathcal{O}((y_1, \ldots, y_n); x)$ of *multimorphisms* from $(y_1, \ldots, y_n)$ to x. We refer to x as the *output color* and to the y_i as the *input colors*;
- If we are additionally given colors z^i_j of $\mathcal{O}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$, there is a *composition map*

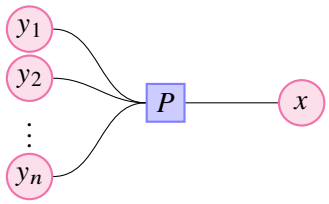
$$\circ\colon \mathcal{O}((y_1, \ldots, y_n); x) \times \prod_{i=1}^{n} \mathcal{O}((z^i_1, \ldots, z^i_{k_i}); y_i) \to \mathcal{O}((z^1_1, \ldots, z^1_{k_1}, \ldots, z^n_1, \ldots, z^n_{k_n}); x).$$

- For every color x there is an identity operation $\mathrm{id}_x \in \mathcal{O}((x); x)$;
- Composition should be unital and associative like before;
- For every permutation $\sigma \in \Sigma_n$ there is a 'permutation isomorphism'

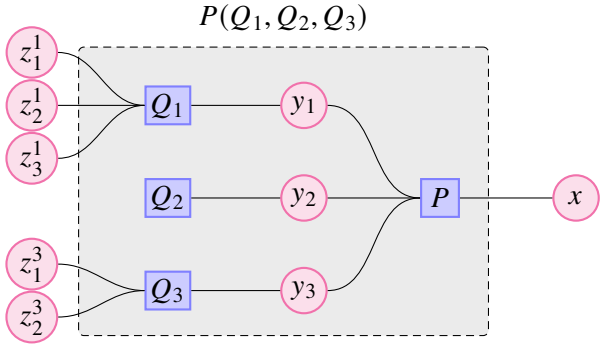
$$\mathcal{O}((y_1, \ldots, y_n); x) \xrightarrow{\sim} \mathcal{O}((y_{\sigma(1)}, \ldots, y_{\sigma(n)}); x), \qquad P \mapsto P\sigma,$$

  which is compatible with the multiplication in the group Σ_n and is furthermore compatible with composition in $\mathcal{O}$. We leave a precise formulation of this condition to the reader; see Exercise 12.1.

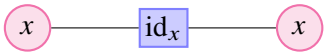
We may pictorially think of a multimorphism P from $(y_1, \ldots, y_n)$ to x as follows:



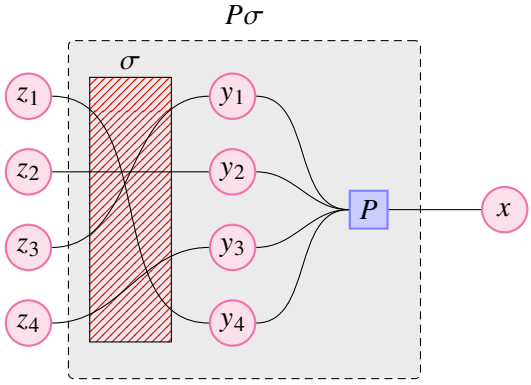
The composition of multimorphisms may then be illustrated as follows (with $n = k_1 = 3$, $k_2 = 0$ and $k_3 = 2$):



The identity multimorphism will be pictured as follows:



Finally, we have the permutation operation on multimorphisms, which we may picture as follows (with $z_i := y_{\sigma(i)}$):



Definition 12.2.2. If $\mathcal{P}$ is another colored operad, then a *morphism* $f : \mathcal{O} \to \mathcal{P}$ of colored operads consists of a function f assigning to each color of $\mathcal{O}$ a color of $\mathcal{P}$ and for all colors $x, y_1, \ldots, y_n$ of $\mathcal{O}$ a map

$$f : \mathcal{O}((y_1, \ldots, y_n); x) \to \mathcal{P}((fy_1, \ldots, fy_n); fx)$$

on multimorphisms, in such a way that f preserves identity operations, composition of operations and permutations.

Let us go through some examples of colored operads:

Example 12.2.3 (From non-colored to colored)**.** Every non-colored operad $\mathcal{O}$ may be regarded as a colored operad with a single color $*$ by setting

$$\mathcal{O}(\underbrace{(*, \ldots, *)}_{n \text{ times}}; *) := \mathcal{O}(n).$$

From now on, we will abuse notation and regard non-colored operads as colored operads in this way. In particular, we obtain colored operads $\mathcal{A}ssoc$, $\mathcal{C}omm$, $\mathcal{L}ie$ and $\mathcal{E}_k$.

Warning 12.2.4. While *Comm* is still terminal as a colored operad, $\mathcal{T}riv$ is no longer initial as a colored operad: the initial colored operad has an empty set of colors.

Example 12.2.5 (From colored to non-colored)**.** Conversely, for every colored operad O and every color x of O we obtain a non-colored operad O_x by setting

$$O_x(n) := O(\underbrace{(x, \dots, x)}_{n \text{ times}}; x).$$

The operad O_x comes with an obvious inclusion map $O_x \hookrightarrow O$ of colored operads.

Example 12.2.6 (Trivial operad on a 1-category)**.** Let C be a 1-category. We define a colored operad $\mathcal{T}riv_C$, called the *trivial operad generated by* C, as follows: its colors are the objects of C, and its multimorphisms are given by

$$\mathcal{T}riv_C((y_1, \dots, y_n); x) \quad := \quad \begin{cases} \mathrm{Hom}_C(y_1, x) & \text{if } n = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

The identity operations are the identity morphisms of C, the composition is the composition in C, and the permutation operations are trivial since Σ_1 is the trivial group. In other words, $\mathcal{T}riv_C$ has no operations besides the unary ones, and these are the morphisms of C.

The trivial operad $\mathcal{T}riv$ is the special case $C = *$, while $C = \emptyset$ gives the initial colored operad from the warning above.

Example 12.2.7 (Multimorphism operad)**.** Let C be a symmetric monoidal category. We define a colored operad $\mathcal{M}_C$, called the *multimorphism operad*[17] *of* C, as follows:

- The colors of $\mathcal{M}_C$ are the objects of C.
- For objects $x, y_1, \dots, y_n$ of C, the multimorphisms in $\mathcal{M}_C$ are given by maps from the tensor product:
$$\mathcal{M}_C((y_1, \dots, y_n); x) \quad := \quad \mathrm{Hom}_C(y_1 \otimes \cdots \otimes y_n; x);$$
- The identity operations are given by the identity maps $\mathrm{id}_x \in \mathrm{Hom}_C(x, x)$;
- The composition of operations is given by the composition in C;
- The permutation operations are induced by the isomorphisms $y_1 \otimes \cdots \otimes y_n \simeq y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}$ coming from the symmetry isomorphisms in C.

Note that for an object A of C, the non-colored operad $(\mathcal{M}_C)_A$ obtained by applying Example 12.2.5 is the endomorphism operad $\mathcal{E}nd_C(A)$ introduced in Example 12.1.9.

Example 12.2.8 (Module operad)**.** There is the *module operad* $\mathcal{M}od$ which has two colors a and m, so that $\mathcal{M}od^{\simeq} = \{a, m\}$. The operations are given as follows:

[17] This operad is usually not given a name in the literature. Some sources refer to it as the 'colored endomorphism operad of C', generalizing the terminology from Example 12.1.9. The terminology 'multimorphism operad' was suggested to the author by Jan Steinebrunner.

- If the output color is a, the operations should encode a commutative algebra structure, so we only want to allow operations all of whose input colors are a as well:

$$\mathcal{M}od((x_1,\dots,x_n);a) := \begin{cases} * & \text{if } x_j = a \text{ for all } j = 1,\dots,n, \\ \emptyset & \text{otherwise,} \end{cases}$$

- If the output color is m, the operations should encode the action of the commutative algebra on a module, so we only want operations that have precisely one input color that is m, while all others are a:

$$\mathcal{M}od((x_1,\dots,x_n);m) := \begin{cases} * & \text{if for some } i \text{ we have } x_i = m \text{ and } x_j = a \text{ for } j \neq i, \\ \emptyset & \text{otherwise.} \end{cases}$$

The composition is uniquely determined.

Note that restricting the operations to the color a gives the commutative operad, while restricting the operations to the color m gives the trivial operad:

$$\mathcal{M}od_a \cong \mathcal{C}omm \qquad \text{and} \qquad \mathcal{M}od_m \cong \mathcal{T}riv.$$

Example 12.2.9 (Left module operad)**.** There is also a non-commutative version of the module operad, capturing the structure of a left module over an associative algebra. We will denote it by $\mathcal{LM}od$ and call it the *left module operad*. It again has two colors, $\mathcal{LM}od^{\simeq} = \{a, m\}$. The operations are defined as follows:

- If the output color is a, we again only want operations all of whose input colors are a, but we should allow them to be ordered arbitrarily:

$$\mathcal{LM}od((x_1,\dots,x_n);a) := \begin{cases} \{\text{linear orders on } \{1,\dots,n\}\} & \text{if } x_j = a \text{ for all } j = 1,\dots,n, \\ \emptyset & \text{otherwise.} \end{cases}$$

- If the output color is m, we should again only allow operations such that precisely one of the inputs is m. To allow for permutations of the other inputs, let us write $I_m := \{i \in \{1,\dots,n\} \mid x_i = m\}$ for the set of indices corresponding to module inputs, and let $I_a := \{j \in \{1,\dots,n\} \mid x_j = a\}$ be the set of indices corresponding to algebra inputs. We then set

$$\mathcal{LM}od((x_1,\dots,x_n);m) := \begin{cases} \{\text{linear orders on } I_a\} & \text{if } |I_m| = 1 \text{ (which implies } |I_a| = n-1), \\ \emptyset & \text{if } |I_m| \neq 1. \end{cases}$$

  Note that this set has cardinality $(n-1)!$.

Composition substitutes an ordered list into each algebra input and concatenates the resulting lists in the order specified by the outer operation. For an operation with output color m, the ordered lists substituted into the algebra inputs are followed by the ordered list of algebra inputs belonging to the operation substituted into the unique module input; the unique module input itself is carried along. The symmetric group action transports these linear orders along the induced bijections between the sets of algebra input positions. These rules are unital, associative and compatible with the symmetric group actions.

As in the commutative case, restricting operations to the color a recovers the underlying algebra operad, while restricting to the color m yields the trivial operad:

$$\mathcal{LM}od_a \cong \mathcal{A}ssoc \qquad \text{and} \qquad \mathcal{LM}od_m \cong \mathcal{T}riv.$$

Just as in the non-colored case, we may define algebras over operads in terms of operad morphisms:

Definition 12.2.10. Let $\mathcal{O}$ be a colored operad and let C be a symmetric monoidal category. An *$\mathcal{O}$-algebra in C* is a morphism $\mathcal{O} \to \mathcal{M}_C$ of colored operads.

For a non-colored operad, this agrees with the preceding definition because $(\mathcal{M}_C)_A = \mathcal{E}nd_C(A)$ for every object $A \in C$.

Example 12.2.11. For specific instances of $\mathcal{O}$, the $\mathcal{O}$-algebras often have specialized names:

- For $\mathcal{O} = \mathcal{A}ssoc$ these are called *associative algebras*;
- For $\mathcal{O} = \mathcal{C}omm$ they are *commutative algebras*;
- For $\mathcal{O} = \mathcal{E}_k$ they are called *$\mathcal{E}_k$-algebras*;
- For $\mathcal{O} = \mathcal{M}od$, an $\mathcal{O}$-algebra is called a *module*. Restricting along the inclusion $\mathcal{C}omm \hookrightarrow \mathcal{M}od$ gives the *underlying commutative algebra* of the module, generically denoted A. Restricting along the inclusion $\mathcal{T}riv \hookrightarrow \mathcal{M}od$ gives the *underlying object*, generically denoted M.
- Similarly, $\mathcal{O} = \mathcal{LM}od$ encodes *left modules*: pairs (A, M) where A is an *associative* algebra rather than a commutative one.

12.3 Encoding the coherences: the envelope of an operad

The notion of ∞-operad we are after is a homotopical version of the notion of colored operad introduced in the previous section: the set of colors should be replaced by an anima, and similarly the sets $\mathcal{O}((y_1, \dots, y_n); x)$ of multimorphisms should be replaced by animae. This leads to some subtleties regarding higher coherences: the unitality and associativity conditions required in Definition 12.2.1 should now become additional *data*, and this data should again satisfy further higher compatibilities. Writing down all the required structure explicitly would be impractical.

Fortunately, there is a way to circumvent this difficulty: the unitality, associativity and permutation laws required in an operad $\mathcal{O}$ can be reformulated in terms of a certain symmetric monoidal category $\mathrm{Env}(\mathcal{O})$ called the *envelope* of the operad. This alternative description of operads can immediately be adapted to the ∞-categorical setting, thus providing a definition of ∞-operads. Let us therefore begin by describing this envelope construction in the classical setting.

We start with an observation regarding the formulation of colored operads: while we have formulated the notion of multimorphisms only for ordered tuples $(y_1, \dots, y_n)$ of colors, it is actually more natural to formulate them for *unordered* tuples:

Definition 12.3.1. Given a set S, an *unordered tuple* of S is a finite set I with a map of sets $x\colon I \to S$. We usually denote such unordered tuples by $\{x_i\}_{i\in I}$. Given two unordered tuples $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ of S, we can form their *unordered concatenation*

$$\{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J} \quad := \quad \{z_k\}_{k\in K}, \qquad\qquad K := I \sqcup J, \quad z_i := x_i, \quad z_j := y_j.$$

More generally, if we are given a finite family $\{x_i^j\}_{i\in I_j}$ of unordered tuples for $j \in J$, then their unordered concatenation is $\{z_k\}_{k\in K}$ with $K := \bigsqcup_{j\in J} I_j$ and $z_{(j,i)} := x_i^j$.

Convention 12.3.2. Given a color x of $\mathcal{O}$ and an unordered tuple of colors $\{y_i\}_{i\in I}$, we define the set

$$\mathcal{O}(\{y_i\}_{i\in I}; x)$$

of multimorphisms from $\{y_i\}_{i\in I}$ to x as follows. Take pairs consisting of a bijection $\{1,\dots,n\} \xrightarrow{\sim} I$ and an operation with inputs in that order, and identify pairs using the symmetric actions. For any specified ordering, the resulting set identifies canonically with the corresponding operation set. Under this convention, the composition map in the operad $\mathcal{O}$ has the form

$$\circ\colon \mathcal{O}(\{y_i\}_{i\in I}; x) \times \prod_{i\in I} \mathcal{O}(\{z_j^i\}_{j\in J_i}; y_i) \to \mathcal{O}(\{z_j^i\}_{(i,j)\in\bigsqcup_{i\in I} J_i}; x).$$

The permutation isomorphism then says that for every bijection $\sigma\colon J \xrightarrow{\cong} I$, there is an isomorphism

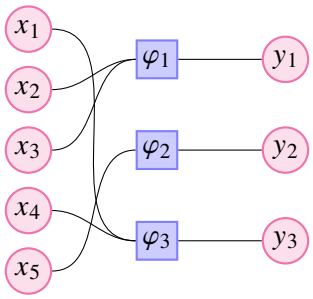
$$\mathcal{O}(\{y_i\}_{i\in I}; x) \xrightarrow{\cong} \mathcal{O}(\{y_{\sigma(j)}\}_{j\in J}; x).$$

This transport is strictly functorial in bijections, and it commutes with substitution by the equivariance axioms.

Henceforth, we shall use the unordered notation for colored operads.

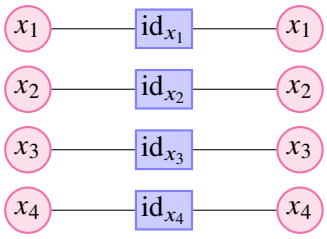
Definition 12.3.3. Let $\mathcal{O}$ be a colored operad. We define a symmetric monoidal category $\mathrm{Env}(\mathcal{O})$, called the *envelope* of $\mathcal{O}$.

- The objects of $\mathrm{Env}(\mathcal{O})$ are unordered tuples $\{x_i\}_{i\in I}$ of colors of $\mathcal{O}$;
- A morphism $\Phi\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ in $\mathrm{Env}(\mathcal{O})$ consists of a morphism of finite sets $f\colon I \to J$ together with a multimorphism $\varphi_j \in \mathcal{O}(\{x_i\}_{i\in I_j}; y_j)$ for every $j \in J$. Here we write $I_j := f^{-1}(j)$ for the preimage of j under f. We will visualize this data as follows:



  Here the function f is indicated by the lines from left to right, so that $f(1) = f(4) = 3$, $f(2) = f(3) = 1$ and $f(5) = 2$.

- The identity of $\{x_i\}_{i\in I}$ is given by taking $f = \mathrm{id}_I \colon I \to I$, and taking $\varphi_j = \mathrm{id}_{x_j} \in O(\{x_i\}_{i\in\{j\}}; x_j)$:



- The composition of $\Phi = (f, \varphi_j) \colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ and $\Psi = (g, \psi_k) \colon \{y_j\}_{j\in J} \to \{z_k\}_{k\in K}$ is given by $\Psi \circ \Phi = (h, \xi_k)$ where $h := g \circ f \colon I \to K$ is the composition of g and f, and for every $k \in K$ the multimorphism $\xi_k \in O(\{x_i\}_{i\in I_k}; z_k)$ is the image of $(\psi_k, (\varphi_j)_{j\in J_k})$ under the composition map

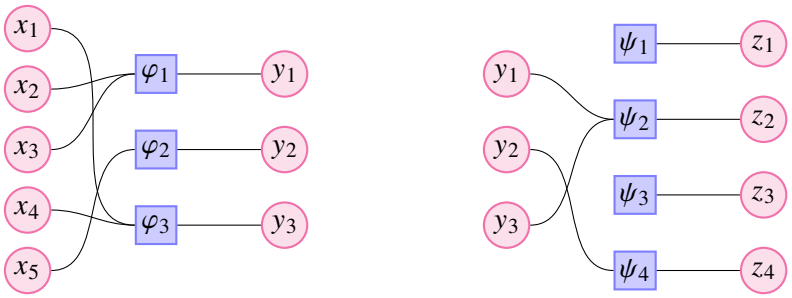
$$\circ \colon O(\{y_j\}_{j\in J_k}; z_k) \times \prod_{j\in J_k} O(\{x_i\}_{i\in I_j}; y_j) \to O(\{x_i\}_{i\in I_k}; z_k);$$

  here we are implicitly using that $I_k = (g \circ f)^{-1}(k) = f^{-1}(g^{-1}(k))$ may be written as the disjoint union of the sets $I_j = f^{-1}(j)$ for $j \in J_k = g^{-1}(k)$.

- Unitality and associativity of composition in $\mathrm{Env}(O)$ amount to the unitality and associativity conditions for colored operads.

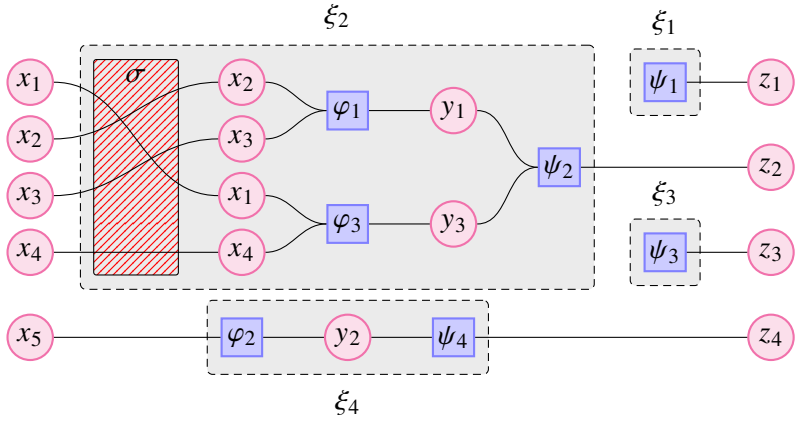
The category $\mathrm{Env}(O)$ admits a symmetric monoidal structure given by unordered concatenation of tuples.

The composition in $\mathrm{Env}(O)$ may look somewhat difficult when written in formulas, but it becomes clear what one needs to do if one visually represents the morphisms using diagrams. For illustration, let us examine the composition of the following two morphisms in $\mathrm{Env}(O)$:



First, we determine the composite map $h := g \circ f \colon \{1, \ldots, 5\} \to \{1, \ldots, 4\}$; in this case, this is given by $h(1) = h(2) = h(3) = h(4) = 2$ and $h(5) = 4$. Note that the preimages $h^{-1}(1)$ and $h^{-1}(3)$ are empty, so that ξ_1 and ξ_3 will be nullary operations. The multimorphism ξ_2 will have x_1, x_2, x_3 and x_4 as input colors, and is obtained by composing ψ_2 with (φ_1, φ_3), where we need to remember which of the four inputs should go into φ_1 and which ones go into φ_3. Finally, the multimorphism ξ_4 has only x_5 as input color, and is simply the composite $\psi_4 \circ \varphi_2$. The resulting composite morphism

can be illustrated as follows:



Note: we draw the permutation σ here to match the ordered tuple convention used in Section 12.2, since the nodes must be drawn in some fixed order on the page. When working with unordered tuples, the σ would not appear: the composition of multimorphisms simply remembers that its inputs are indexed by the disjoint union of $\{2,3\}$ and $\{1,4\}$ without fixing any order between them.

Example 12.3.4. The envelope $\mathrm{Env}(\mathit{Comm})$ of the commutative operad is the category Fin of finite sets. The symmetric monoidal structure is given by disjoint union.

The construction $\mathcal{O} \mapsto \mathrm{Env}(\mathcal{O})$ is functorial in $\mathcal{O}$ and is left adjoint to the multimorphism-operad construction $C \mapsto \mathcal{M}_C$: operad morphisms $\mathcal{O} \to \mathcal{M}_C$ correspond naturally to symmetric monoidal functors $\mathrm{Env}(\mathcal{O}) \to C$. Verification of these assertions is left to the reader. In particular, since Comm is terminal and $\mathrm{Env}(\mathit{Comm}) = \mathrm{Fin}$, every envelope comes with a symmetric monoidal functor

$$p\colon \mathrm{Env}(\mathcal{O}) \longrightarrow \mathrm{Env}(\mathit{Comm}) = \mathrm{Fin}.$$

The nice feature about the envelope construction is that it allows us to formulate the unitality and associativity conditions for composition of multimorphisms in $\mathcal{O}$ in terms of unitality and associativity of morphisms in an actual category. In particular, this could potentially provide a convenient way to generalize the definition of operads to the ∞-categorical setting. The main question we would then need to address is: what structure do we need on a symmetric monoidal category C to guarantee that it is of the form $\mathrm{Env}(\mathcal{O})$ for some operad $\mathcal{O}$? We identify the three crucial properties that $\mathrm{Env}(\mathcal{O})$ has:

(1) The colors of $\mathcal{O}$ are precisely the objects that are sent to a one-point set $\{i\}$ under the symmetric monoidal functor $p\colon \mathrm{Env}(\mathcal{O}) \to \mathrm{Fin}$.

(2) Given an object $X \in \mathrm{Env}(\mathcal{O})$ with $p(X) = I$, we may uniquely write X as an I-tuple $\{x_i\}_{i\in I}$ for some colors $x_i \in p^{-1}(\{i\})$. Similarly, the morphisms in the fiber $p^{-1}(I)$ are I-tuples of maps $\{f_i\colon x_i \to y_i\}_{i\in I}$. In other words, the fiber over I is the I-fold product of the fiber over the one-point set.

(3) The morphisms into an arbitrary tuple $\{y_j\}_{j\in J}$ are completely determined by morphisms into 1-tuples: If we consider two unordered tuples $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$, then the fiber of the induced map

$$\mathrm{Hom}_{\mathrm{Env}(\mathcal{O})}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) \xrightarrow{p} \mathrm{Hom}_{\mathrm{Fin}}(I, J)$$

over some map $f\colon I \to J$ is given by the product

$$\prod_{j \in J} \mathrm{Hom}_{\mathrm{Env}(\mathcal{O})}(\{x_i\}_{i \in I_j}, y_j).$$

The first property explains how to recover the colors from p, while the remaining two characterize the morphisms and tensor products in the envelope. One can verify that a symmetric monoidal category C equipped with a symmetric monoidal functor $p\colon C \to \mathrm{Fin}$ is of the form $\mathrm{Env}(\mathcal{O})$ for some $\mathcal{O}$ if and only if it satisfies the corresponding analogues of conditions (2) and (3):

(2') For every finite set I, the tensor product in C defines an equivalence

$$\prod_{i \in I} C_{\{i\}} \xrightarrow{\sim} C_I,$$

where C_I is the fiber of p over $I \in \mathrm{Fin}$;

(3') For every finite collection $(I_j)_{j \in J}$ of finite sets and every collection of objects $X_j \in p^{-1}(I_j)$ and $y_j \in p^{-1}(\{j\})$, the commutative square

$$\begin{array}{ccc} \prod_{j \in J} \mathrm{Hom}_C(X_j, y_j) & \xrightarrow{\bigotimes_{j \in J}} & \mathrm{Hom}_C(\bigotimes_{j \in J} X_j, \bigotimes_{j \in J} y_j) \\ \downarrow & & \downarrow p \\ * & \longrightarrow & \mathrm{Hom}_{\mathrm{Fin}}(\bigsqcup_{j \in J} I_j, J) \end{array}$$

is a pullback square, where the bottom map hits the map $\bigsqcup_{j \in J} I_j \to \bigsqcup_{j \in J} * = J$ induced by the maps $I_j \to *$.

Exercise 12.3.5. Given a symmetric monoidal category C with $p\colon C \to \mathrm{Fin}$ satisfying (2') and (3'), construct a colored operad $\mathcal{O}$ and a symmetric monoidal equivalence $C \xrightarrow{\sim} \mathrm{Env}(\mathcal{O})$ over Fin.

12.4 Writing concatenations as products

The characterization of envelopes of operads obtained in the previous section is neat, and immediately generalizes to the ∞-categorical setting. Nevertheless, this is not the definition of ∞-operads that usually appears in the literature. One inconvenience about this characterization is that it requires us to equip an ∞-category with a symmetric monoidal structure, which is a task substantially more involved than for classical categories since we can no longer 'just write it down by hand'. It would be convenient if we could avoid reliance on symmetric monoidal structures.

Fortunately, this is possible: there is a variation of the envelope construction in which the concatenation of tuples becomes the *categorical product*, and hence does not need to be recorded as additional data. We have already seen this idea in action in our definition of commutative monoids in Chapter 5. Recall from Definition 8.1.3 that a commutative algebra in a symmetric monoidal ∞-category C is encoded by a symmetric monoidal functor

$$\mathrm{Fin} \to C,$$

so its definition uses the given tensor product on C. By contrast, if C admits finite products, a commutative monoid in C is encoded by a product-preserving functor

$$\mathrm{Span}(\mathrm{Fin}) \to C.$$

The additional left-pointing morphisms ensure that disjoint union in Fin becomes the categorical product in Span(Fin), so no separately specified symmetric monoidal structure on C is needed.

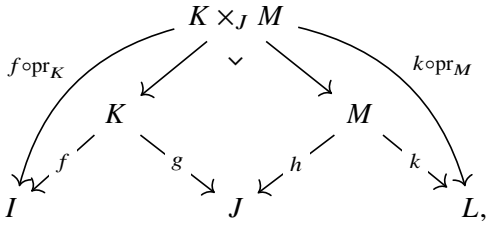
We apply the same idea to the envelope $\mathrm{Env}(\mathcal{O})$: by adjoining left-pointing morphisms, we obtain a variant of $\mathrm{Env}(\mathcal{O})$ in which concatenation of tuples is the categorical product, and therefore need not be recorded separately. Note that this only concerns how the operad structure is recorded; it does not require the ∞-categories in which we later consider $\mathcal{O}$-algebras to be cartesian monoidal. As for Span(Fin), the construction produces a $(2,1)$-category, since composition is defined using pullbacks and is therefore associative only up to canonical isomorphism.

Construction 12.4.1. Let $\mathcal{O}$ be a colored operad. We describe a (2,1)-category $\mathcal{O}^{\otimes}$ as follows:

- The objects of $\mathcal{O}^{\otimes}$ are unordered tuples $\{x_i\}_{i\in I}$;
- A morphism $\{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ in $\mathcal{O}^{\otimes}$ consists of a span

  $$I \xleftarrow{f} K \xrightarrow{g} J$$

  of finite sets together with a multimorphism $\varphi_j \in \mathcal{O}(\{x_{f(k)}\}_{k\in K_j}; y_j)$ for every $j \in J$; here we write $K_j := g^{-1}(j)$ for the preimage of j under g. Together with the canonical notion of isomorphisms between such labeled spans, these form a groupoid $\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J})$.
- The identity of $\{x_i\}_{i\in I}$ is given by taking $f = g = \mathrm{id}_I : I \to I$, and taking $\varphi_j = \mathrm{id}_{x_j} \in \mathcal{O}(\{x_i\}_{i\in\{j\}}; x_j)$.
- The composition of $(f, g, \varphi_j) : \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ and $(h, k, \psi_l) : \{y_j\}_{j\in J} \to \{z_l\}_{l\in L}$ is given by $(f \circ \mathrm{pr}_K, k \circ \mathrm{pr}_M, \xi_l)$ where pr_K and pr_M are the projection maps in the following pullback diagram

  and for every $l \in L$ the multimorphism $\xi_l \in \mathcal{O}(\{x_{f(u)}\}_{(u,m)\in(K\times_J M)_l}; z_l)$ is the image of $(\psi_l, (\varphi_{h(m)})_{m\in M_l})$ under the composition map

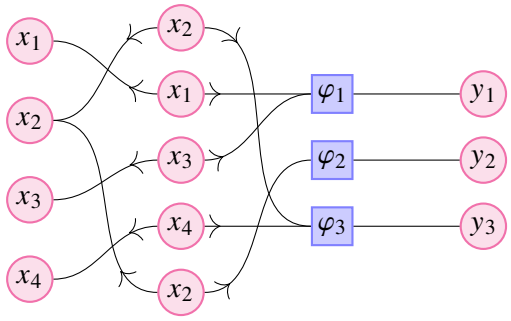
  $$\circ : \mathcal{O}(\{y_{h(m)}\}_{m\in M_l}; z_l) \times \prod_{m\in M_l} \mathcal{O}(\{x_{f(u)}\}_{u\in K_{h(m)}}; y_{h(m)}) \to \mathcal{O}(\{x_{f(u)}\}_{(u,m)\in(K\times_J M)_l}; z_l).$$

- The operad axioms make composition unital and associative up to the canonical isomorphisms induced by pullbacks of spans.

Lemma 12.4.2. *The construction $O \mapsto O^{\otimes}$ defines a functor from colored operads to $(2,1)$-categories over* Span(Fin). *The $(2,1)$-category $O^{\otimes}$ admits finite products, given by unordered concatenation of tuples, and the structure functor $O^{\otimes} \to$ Span(Fin) preserves them.*

Proof sketch. We describe the product universal property in the labeled-span picture; the coherent construction and verification are given in Proposition 14.2.4. The operad axioms and the canonical associativity isomorphisms for pullbacks make the composition law unital and associative. A morphism into an unordered concatenation of tuples is uniquely a collection of morphisms into its factors, since both the indexing span and its labeling operations decompose according to the corresponding partition of the target. Thus concatenation satisfies the product universal property. The same description for $Comm^{\otimes} =$ Span(Fin) shows that the structure functor preserves these products. □

This time, morphisms in $O^{\otimes}$ may be visualized as follows:



The first part of the data consists of two maps $f\colon \{1,\dots,5\} \to \{1,\dots,4\}$ and $g\colon \{1,\dots,5\} \to \{1,2,3\}$; in this case f is given by $f(1) = f(5) = 2$, $f(2) = 1$, $f(3) = 3$ and $f(4) = 4$, while g is given by $g(2) = g(3) = 1$, $g(5) = 2$ and $g(1) = g(4) = 3$. In particular, the preimages of 1, 2 and 3 under g are $\{2,3\}$, $\{5\}$ and $\{1,4\}$, respectively. The second part of the data consists of multimorphisms $\varphi_1\colon (x_{f(2)}, x_{f(3)}) \to y_1$, $\varphi_2\colon (x_{f(5)}) \to y_2$ and $\varphi_3\colon (x_{f(1)}, x_{f(4)}) \to y_3$.

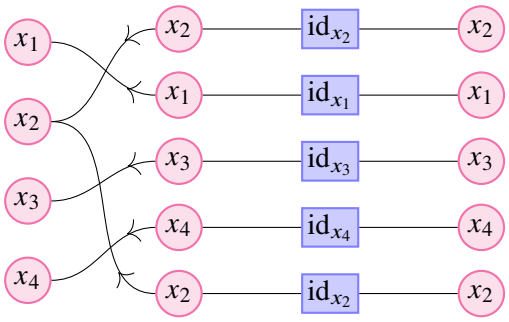
For $O = Comm$ we get $Comm^{\otimes} =$ Span(Fin). Since the construction of $O^{\otimes}$ is functorial in O, we obtain a functor

$$p\colon O^{\otimes} \to Comm^{\otimes} = \text{Span(Fin)}$$

which preserves finite products. As in the previous section, we may ask: can we characterize those $(2,1)$-categories over Span(Fin) that are of the form $O^{\otimes}$ for some operad? The fiber decomposition from (2') remains valid, now in terms of categorical products. The product universal property decomposes morphisms according to the factors of their target, but it does not by itself isolate the corresponding factors in the source. Moreover, the morphisms over right-pointing spans do not determine the morphisms over a general span $I \leftarrow K \to J$. To reduce the general case to the right-pointing case, we need distinguished lifts of left-pointing spans.

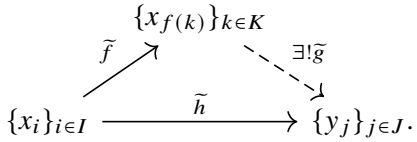
To reduce the general case to the special case, consider an object $\{x_i\}_{i\in I}$ of $O^{\otimes}$ and consider a morphism $f\colon K \to I$ of finite sets. We may then form the morphism $\widetilde{f}\colon \{x_i\}_{i\in I} \to \{x_{f(k)}\}_{k\in K}$ in $O^{\otimes}$ whose underlying span of finite sets is $I \xleftarrow{f} K \xrightarrow{=} K$ and whose multimorphisms $\varphi_k \in O(\{x_{f(k)}\}, x_{f(k)})$ are the identity morphisms $\mathrm{id}_{x_{f(k)}}$. This morphism $\widetilde{f}$ may be visualized as

follows:



This morphism has the following special feature, which is the shadow of the general notion of a p-cocartesian morphism that we will meet in Definition 24.1.1:

(3") Consider objects $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ in $O^{\otimes}$, and consider a span $I \xleftarrow{f} K \xrightarrow{g} J$. Given a morphism $\widetilde{h}\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ in $O^{\otimes}$ lifting[18] the given span, there exists a unique morphism $\widetilde{g}\colon \{x_{f(k)}\}_{k\in K} \to \{y_j\}_{j\in J}$ which lifts the span $K \xleftarrow{=} K \xrightarrow{g} J$ and satisfies $\widetilde{h} = \widetilde{g}\circ\widetilde{f}$:



Indeed, the lift $\widetilde{h}$ is encoded by certain multimorphisms $\varphi_j \in O(\{x_{f(k)}\}_{k\in K_j}; y_j)$, and this is exactly the data that determines the lift $\widetilde{g}$. The full cocartesian condition is the pullback-square condition appearing as condition (3) below.

In a sense, we see that these morphisms $\widetilde{f}$ in $O^{\otimes}$ do not really contribute very much apart from encoding the categorical product structure. These morphisms, and more generally the cocartesian morphisms lying over left-pointing spans, are called *inert morphisms*. In contrast, the morphisms in $O^{\otimes}$ living over *right*-pointing spans are the ones that encode the actual operad structure, and are called the *active morphisms*.

Under the product description of Lemma 12.4.2, the morphism $\widetilde{f}$ is the map whose k-th component is the projection onto $x_{f(k)}$.

We have now achieved what we were after: the entire data of the operad O is encoded by the $(2,1)$-category $O^{\otimes}$ together with its functor to Span(Fin). The structural properties we have identified still make sense for an ∞-category $O^{\otimes}$ equipped with a functor $p_O\colon O^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$. Let us summarize them:

(1) The ∞-category $O^{\otimes}$ admits finite products, and the functor p_O preserves finite products;

(2) For every finite set I, the product in $O^{\otimes}$ defines an equivalence

$$\prod_{i\in I} O^{\otimes}_{\{i\}} \xrightarrow{\sim} O^{\otimes}_I.$$

Here we write $O^{\otimes}_I$ for the fiber of p_O over the set I.

[18] This means that the image of $\widetilde{h}$ under p is the given span $I \xleftarrow{f} K \xrightarrow{g} J$.

(3) For a morphism $f\colon J \to I$ in Fin and objects $X_i \in \mathcal{O}^{\otimes}$, the map $\widetilde{f}\colon \prod_{i\in I} X_i \to \prod_{j\in J} X_{f(j)}$ whose j-th component is the projection to $X_{f(j)}$ is $p_{\mathcal{O}}$-*cocartesian*, in the sense that for every $Y \in \mathcal{O}^{\otimes}$ the following square is a pullback square:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\prod_{j\in J} X_{f(j)}, Y) & \xrightarrow{-\circ\widetilde{f}} & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\prod_{i\in I} X_i, Y) \\ \downarrow{\scriptstyle p_{\mathcal{O}}} & & \downarrow{\scriptstyle p_{\mathcal{O}}} \\ \mathrm{Hom}_{\mathrm{Span(Fin)}}(p_{\mathcal{O}}(\prod_{j\in J} X_{f(j)}), p_{\mathcal{O}}(Y)) & \xrightarrow{-\circ p_{\mathcal{O}}(\widetilde{f})} & \mathrm{Hom}_{\mathrm{Span(Fin)}}(p_{\mathcal{O}}(\prod_{i\in I} X_i), p_{\mathcal{O}}(Y)). \end{array}$$

These three conditions will define an ∞-*operad* in Definition 14.1.1, after we have constructed the span category rigorously.

Remark 12.4.3. Our definition of ∞-operad is a priori different from that of Lurie [Lur17], who instead works with functors $p_{\mathcal{O}}$ whose target is the category Fin_* of finite pointed sets. To relate the two approaches, recall from Lemma 5.3.8 that Fin_* is equivalent to the subcategory $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ of $\mathrm{Span}(\mathrm{Fin})$ in which the left-pointing morphisms are required to be injective. We will show in Section 19.4 below that pullback along the resulting inclusion $\mathrm{Fin}_* \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ induces an equivalence between our ∞-category of ∞-operads and Lurie's one.

One advantage of working over Fin_* is that it is a strict 1-category. In Lurie's model, the category $\mathcal{O}^{\otimes}$ from Construction 12.4.1 also becomes a strict 1-category, and can therefore be directly imported into the world of ∞-categories, showing that every colored operad gives rise to an ∞-operad. In Proposition 14.2.4, we instead construct the span-based ∞-operad directly by labeling the pullback diagrams used to construct span categories.

On the other hand, there are a variety of reasons to work over $\mathrm{Span}(\mathrm{Fin})$ rather than Fin_*. First, the span approach makes clear the orthogonal roles played by the backwards and forwards maps; this separation is less visible in Fin_*. Second, allowing only injective backwards maps breaks the property that concatenation of unordered tuples is a categorical product, so conditions on $\mathcal{O}^{\otimes}$ need to be formulated in a slightly more complicated way. And third, the span perspective generalizes more easily to other settings, making it easier to connect the theory developed here to concepts like Mackey functors, normed ∞-categories and six-functor formalisms. For example, it allows us to treat the theory of (co)cartesian monoidal structures as an instance of Barwick's more general unfurling construction in Chapter 15.

Remark 12.4.4 (Other models for ∞-operads)**.** The span-based model used in this book and Lurie's model over Fin_* from Section 19.4 are not the only available descriptions of ∞-operads. In the *dendroidal* approach, introduced by Moerdijk and Weiss [MW07], the simplex category is replaced by a category of trees. The presentation of ∞-operads by complete dendroidal Segal animae is due to Cisinski and Moerdijk [CM13]. A direct comparison with Lurie's model is given by Hinich and Moerdijk [HM24b]; see [HM22] for a comprehensive account of dendroidal homotopy theory.

The classical description of one-colored operads as associative algebras in symmetric sequences also admits a homotopy-coherent generalization due to Haugseng [Hau22]. More generally, this construction treats varying animae of colors by means of a double ∞-category of symmetric collections; see [Hau23a] for a general introduction to ∞-operads. A related viewpoint describes pinned ∞-operads by analytic monads on slices $\mathrm{An}_{/S}$, with the animae S of colors allowed to vary [GHK22; Hau23b]. This description is compared directly with both Lurie's model and the span-based model through

their Lawvere theories by Haugseng [Hau23b]. More general Segal-type formalisms, notably the theory of algebraic patterns developed by Chu and Haugseng [CH21], provide a common framework for these and many similar homotopy-coherent algebraic structures. The general theory of envelopes for algebraic patterns developed by Barkan, Haugseng, and Steinebrunner [BHS22] contains our construction of operadic envelopes in Section 19.3 as a special case. We will not otherwise use the alternative presentations discussed in this remark.

Before studying ∞-operads in more detail, we will give a proper introduction to span ∞-categories and their basic properties in the next chapter. We return to ∞-operads and their algebras in Chapter 14.

Exercises

Exercise 12.1 (Equivariance conditions)**.** In the definition of an operad, we have not specified how the permutation actions on $O(n)$ interact with the composition structure. In this exercise, you will formulate the missing conditions.

Let O be a non-colored operad, and consider operations $P \in O(k)$ and $Q_i \in O(n_i)$ for $i = 1, \dots, k$.

(1) Consider a permutation $\sigma \in \Sigma_k$. Using the motivating example of the endomorphism operad $O = \mathcal{E}nd_C(A)$, write down a rule for expressing the operation

$$(P\sigma)(Q_1, \dots, Q_k)$$

as an operation of the form $P(R_1, \dots, R_k)\tau$ for certain operations R_i and a certain permutation $\tau \in \Sigma_n$, where $n := \sum_{i=1}^k n_i$.

(2) Consider permutations $\sigma_i \in \Sigma_{n_i}$ for all $i = 1, \dots, k$. Again using $O = \mathcal{E}nd_C(A)$ as motivating example, write down a rule for expressing the operation

$$P(Q_1\sigma_1, \dots, Q_k\sigma_k)$$

as an operation of the form $P(Q_1, \dots, Q_k)\tau$ for a certain permutation $\tau \in \Sigma_n$, where $n := \sum_{i=1}^k n_i$.

(3) Formulate the corresponding equivariance conditions for a colored operad. In doing so, keep track of the reindexing of the input colors under each permutation.

Exercise 12.2 (The associative operad)**.** Write out the composition maps and the Σ_n-actions in the case of the associative operad $\mathcal{A}ssoc$.

Exercise 12.3. Let C be a symmetric monoidal category. Show that:

- $\mathcal{T}riv$-algebras are simply objects of C;
- $\mathcal{E}_0$-algebras are 'pointed objects' of C, i.e., objects A equipped with a 'unit map' $\eta\colon \mathbb{1} \to A$.
- $\mathcal{C}omm$-algebras are objects A equipped with maps $\eta\colon \mathbb{1} \to A$ and $m\colon A \otimes A \to A$ which are unital, associative and commutative.
- $\mathcal{A}ssoc$-algebras are similar but without commutativity.

Exercise 12.4 (Universal property of trivial operads)**.** Let $\mathcal{O}$ be a colored operad. Show that the colors of $\mathcal{O}$ together with the sets $\mathcal{O}((y);x)$ of unary operations form a 1-category $U(\mathcal{O})$, called the *underlying category* of $\mathcal{O}$. Show furthermore that for every 1-category C, restriction to the unary operations defines a bijection between the morphisms of colored operads $\mathcal{T}riv_C \to \mathcal{O}$ and the functors $C \to U(\mathcal{O})$, where $\mathcal{T}riv_C$ is the trivial operad of Example 12.2.6.

Exercise 12.5. Let $\mathcal{O}$ be a colored operad, and let C be a symmetric monoidal category. Recall that we defined an *$\mathcal{O}$-algebra* as a morphism of colored operads $A\colon \mathcal{O} \to \mathcal{M}_C$. Spell out in explicit terms all the data contained in such an $\mathcal{O}$-algebra. In the special case of $\mathcal{O} = \mathcal{LM}od$, verify that this structure amounts to a pair (A, M) of an associative algebra A in C together with a left module M over it.

Remark: Saying 'left' versus 'right' is just a convention; the same operad also encodes right actions.

Exercise 12.6. Compute the envelope $\mathrm{Env}(\mathcal{A}ssoc)$ of the associative operad, and show that it may be described as follows:

- The objects are finite sets I;
- The morphisms $I \to J$ are tuples $(f, \preceq_j)$ consisting of a morphism $f\colon I \to J$ of finite sets and a collection of linear orders $\preceq_j$ on each of the preimages $f^{-1}(j)$ of f;
- The symmetric monoidal structure is disjoint union.

Exercise 12.7. Show that the assignment $\mathcal{O} \mapsto \mathrm{Env}(\mathcal{O})$ is functorial: every morphism of operads $\mathcal{O} \to \mathcal{P}$ induces a symmetric monoidal functor $\mathrm{Env}(\mathcal{O}) \to \mathrm{Env}(\mathcal{P})$. Deduce from the terminal morphism $\mathcal{O} \to \mathcal{C}omm$ the symmetric monoidal functor $p\colon \mathrm{Env}(\mathcal{O}) \to \mathrm{Fin}$ used above.

Exercise 12.8. Show that, for an operad $\mathcal{O}$ and a symmetric monoidal category C, restriction to singleton tuples induces a natural equivalence between the category of symmetric monoidal functors $\mathrm{Env}(\mathcal{O}) \to C$, with monoidal natural transformations, and the category of $\mathcal{O}$-algebras in C, with maps compatible with all operations.

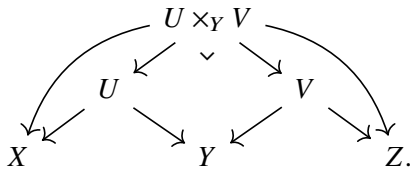
13 Span categories

This chapter introduces span categories, which provide the organizing language for the approach to higher algebra taken in this book. They offer a convenient way to encode algebraic structures in settings such as *spectral Mackey functors* [Bar17], *normed* ∞*-categories* [BH21], and *six-functor formalisms* [Sch25; HM24a]. Their use in higher algebra is not yet standard, in part because Lurie's foundational text [Lur17] takes a different approach, so we give a detailed development of their construction.

Let C be an ∞-category with pullbacks, and let C_L and C_R be collections of morphisms of C closed under composition and pullback. Then the *span category* $\mathrm{Span}(C, C_L, C_R)$ associated with this triple is an ∞-category whose objects are the objects of C, while the morphisms from X to Y are given by *spans*

$$\begin{array}{ccccc} & & U & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} & \\ X & & & & Y, \end{array}$$

where $f \in C_L$ and $g \in C_R$. The identity morphism is the *identity span* $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. The composition of a span $X \leftarrow U \rightarrow Y$ with a span $Y \leftarrow V \rightarrow Z$ is given by forming pullbacks:



To turn this description into an ∞-category, we must account for the higher coherences of composition by pullbacks. We follow the construction of Barwick [Bar17] and Haugseng, Hebestreit, Linskens, and Nuiten [HHLN23b].

13.1 Constructing span categories

So far, we could reduce all of the constructions of ∞-categories we encountered to a handful of elementary constructions, like pullbacks, subcategories and functor categories. The situation for span categories is different: while the heuristic description given before tells us what the objects and morphisms should be, it is not immediately clear how to assemble these into an ∞-category.

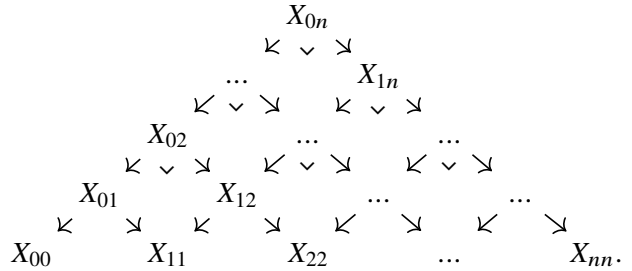
The technique we will use is that of *complete Segal animae*, discussed in detail in Chapter 25. Let us briefly recall the main idea. Given an ∞-category C, we may consider for every $n \geq 0$

the anima $\mathrm{Map}([n], C)$ of 'strings of n composable morphisms in C'. Using the functoriality of mapping animae from Remark 1.8.4, sending $[n]$ to $\mathrm{Map}([n], C)$ for all n provides a simplicial anima $N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$, called the *nerve of* C. A major insight due to Rezk is that we may use this assignment to identify ∞-categories with suitable simplicial animae. More precisely, the resulting nerve functor

$$N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$$

from ∞-categories to simplicial animae is fully faithful. The essential image of this inclusion is given by the *complete Segal animae*, a class of simplicial animae satisfying analogues of the Segal Axiom (Proposition 1.4.5) and the Rezk Axiom (Axiom G).

For the purpose of constructing span categories, we will use this process in reverse: if we manage to write down a simplicial anima $\mathrm{NSpan}(C, C_L, C_R)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ encoding strings of n composable morphisms in the span category for all n, and if we manage to verify that it is a complete Segal anima, then we have managed to construct our desired ∞-category $\mathrm{Span}(C, C_L, C_R)$. This leads to the next question: what data should be contained in a functor $[n] \to \mathrm{Span}(C, C_L, C_R)$? Such a functor should first of all encode n composable spans. Furthermore, it should provide a compatible family of choices for their compositions. Given that we want composition in the span category to be given by pullback, we conclude that $\mathrm{NSpan}(C, C_L, C_R)_n$ should be the anima of diagrams in C of the following form:



Here all the left-pointing morphisms should be in the given class C_L, all the right-pointing morphisms should be in C_R, and all squares should be pullback squares.

In the remainder of this section, we will make the construction of the simplicial anima $\mathrm{NSpan}(C, C_L, C_R)$ more precise, and verify that it is indeed a complete Segal anima. We will also prove various basic properties about the resulting ∞-categories of spans. Our exposition closely follows that of Haugseng, Hebestreit, Linskens, and Nuiten [HHLN23b].

Definition 13.1.1. A subcategory $C' \subseteq C$ is called *wide* if it contains all objects of C.

We will usually think of a wide subcategory C' as specifying a collection of morphisms in C closed under composition. Since a collection of morphisms is a union of components of $\mathrm{Map}([1], C)$, every wide subcategory contains all isomorphisms of C.

Definition 13.1.2 ([Bar17])**.** An *adequate triple* (C, C_L, C_R) consists of an ∞-category C equipped with two wide subcategories $C_L, C_R \subseteq C$ satisfying the condition that for every morphism $l\colon X \to Y$ in C_L and $r\colon Y' \to Y$ in C_R, there exists a pullback square of the form

$$\begin{array}{ccc} X' & \xrightarrow{r'} & X \\ {\scriptstyle l'}\downarrow & \lrcorner & \downarrow{\scriptstyle l} \\ Y' & \xrightarrow{r} & Y, \end{array} \tag{13.1}$$

and we have $l' \in C_L$ and $r' \in C_R$. We refer to C_L as the *class of backwards morphisms* and to C_R as the *class of forward morphisms*.

Given another adequate triple (D, D_L, D_R), a *morphism of adequate triples* is a functor $F\colon C \to D$ that sends C_L to D_L and C_R to D_R, and sends every pullback square of the form (13.1), with $l \in C_L$ and $r \in C_R$, to a pullback square in D. We denote by

$$\mathrm{AdTrip} \quad \subseteq \quad \mathrm{Fun}(\sqcup, \mathrm{Cat}_\infty)$$

the non-full subcategory spanned by those diagrams $C_L \hookrightarrow C \hookleftarrow C_R$ in Cat_∞ that correspond to adequate triples (C, C_L, C_R) and those morphisms that correspond to morphisms of adequate triples.

Example 13.1.3. For any wide subcategory C' of an ∞-category C, we have the adequate triples $(C, C', C^{\simeq})$ and $(C, C^{\simeq}, C')$. In particular, for any ∞-category C we always have the adequate triples $(C, C, C^{\simeq})$ and $(C, C^{\simeq}, C)$.

Example 13.1.4. If C admits all pullbacks, we have the adequate triple (C, C, C) where we take all maps to be both forward and backward maps. This gives rise to a fully faithful embedding

$$\mathrm{Cat}_\infty^{\mathrm{pb}} \hookrightarrow \mathrm{AdTrip},$$

where $\mathrm{Cat}_\infty^{\mathrm{pb}} \subseteq \mathrm{Cat}_\infty$ is the subcategory spanned by those ∞-categories which admit pullbacks, and those functors which preserve pullbacks.

Example 13.1.5. If (C, C_L, C_R) is an adequate triple, then so is (C, C_R, C_L). The resulting endomorphism of AdTrip will be denoted

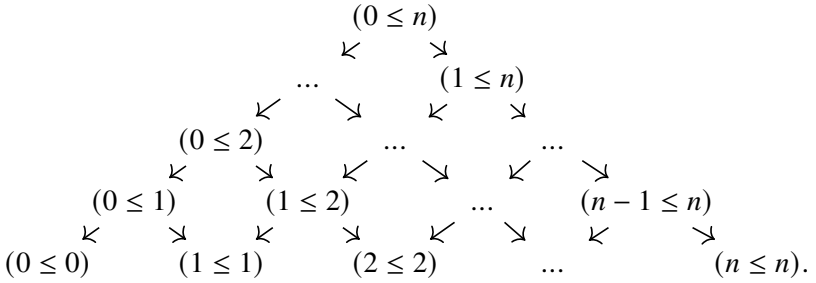
$$(-)_{\mathrm{rev}}\colon \mathrm{AdTrip} \to \mathrm{AdTrip}\,.$$

We will now introduce the indexing diagrams that classify the functors $[n] \to \mathrm{Span}(C, C_L, C_R)$.

Definition 13.1.6 (Twisted arrow category)**.** For $n \geq 0$, let $\mathrm{Tw}^r([n])$ be the partially ordered set of tuples (i, j) with $0 \leq i \leq j \leq n$, where the partial order is given by

$$(i, j) \leq (k, l) \qquad \text{if and only if} \qquad i \leq k \quad \text{and} \quad l \leq j.$$

We will generally denote the pair (i, j) by $(i \leq j)$. The poset looks as follows:



Given a morphism of posets $\varphi\colon [n] \to [m]$, we obtain a map $\mathrm{Tw}^r(\varphi)\colon \mathrm{Tw}^r([n]) \to \mathrm{Tw}^r([m])$ by sending $(i \leq j)$ to $(\varphi(i) \leq \varphi(j))$. This assignment is clearly functorial, resulting in a functor

$$\mathrm{Tw}^r\colon \mathbf{\Delta} \to \mathrm{Poset} \subseteq \mathrm{Cat} \subseteq \mathrm{Cat}_\infty.$$

Exercise 13.1.7. Show that the functors $s\colon \mathrm{Tw}^r([n]) \to [n], (i \leq j) \mapsto i$ and $t\colon \mathrm{Tw}^r([n]) \to [n]^{\mathrm{op}}, (i \leq j) \mapsto j$ are cartesian fibrations.

Lemma 13.1.8. *For integers $0 \leq i \leq k \leq l \leq j \leq n$, the commutative square*

$$\begin{array}{ccc} & (i \leq j) & \\ \swarrow & & \searrow \\ (i \leq l) & & (k \leq j) \\ \searrow & & \swarrow \\ & (k \leq l) & \end{array}$$

is a pullback square in $\mathrm{Tw}^r([n])$.

Proof. Given $0 \leq i' \leq j' \leq n$ we need to show that we have $(i', j') \leq (i, j)$ if and only if $(i', j') \leq (i, l)$ and $(i', j') \leq (k, j)$. This is clear: in both cases this amounts to asking that for both of the relations $i' \leq i$ and $j \leq j'$. □

Corollary 13.1.9. *The poset* $\mathrm{Tw}^r([n])$ *is an adequate triple, with* $\mathrm{Tw}^r([n])_L$ *consisting of morphisms of the form* $(i \leq j) \leq (i \leq l)$ *and* $\mathrm{Tw}^r([n])_R$ *consisting of morphisms of the form* $(i \leq j) \leq (k \leq j)$.

Proof. The two classes are closed under composition, and every cospan consisting of a morphism in $\mathrm{Tw}^r([n])_L$ and a morphism in $\mathrm{Tw}^r([n])_R$ has the form treated in Lemma 13.1.8. □

From now on, we will equip $\mathrm{Tw}^r([n])$ with the adequate triple structure from Corollary 13.1.9. Note that a morphism of posets $\varphi\colon [n] \to [m]$ induces a morphism of adequate triples $\mathrm{Tw}^r(\varphi)\colon \mathrm{Tw}^r([n]) \to \mathrm{Tw}^r([m])$, resulting in a functor

$$\mathrm{Tw}^r\colon \mathbf{\Delta} \to \mathrm{AdTrip}\,.$$

Observe that the diagrams in C we drew in our informal description of the functors $[n] \to \mathrm{Span}(C, C_L, C_R)$ given in the introduction to this chapter are precisely the morphisms of adequate triples $\mathrm{Tw}^r([n]) \to (C, C_L, C_R)$. This leads to the following definition:

Construction 13.1.10. Let (C, C_L, C_R) be an adequate triple. We define the simplicial anima $\mathrm{NSpan}(C, C_L, C_R) \in \mathrm{sAn}$ as the following composite:

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Tw}^r} \mathrm{AdTrip}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{AdTrip}}(-,(C,C_L,C_R))} \mathrm{An}.$$

In other words, for $n \geq 0$ we consider the subanima

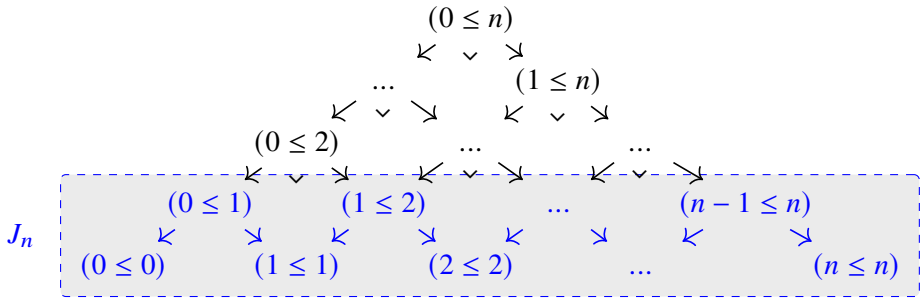
$$\mathrm{NSpan}(C, C_L, C_R)_n \subseteq \mathrm{Map}(\mathrm{Tw}^r([n]), C)$$

consisting of the morphisms of adequate triples. This construction is natural in the adequate triple, hence defines a functor

$$\mathrm{NSpan}\colon \mathrm{AdTrip} \to \mathrm{sAn}.$$

Proposition 13.1.11 (Barwick [Bar17, Proposition 5.6], Haugseng, Hebestreit, Linskens, and Nuiten [HHLN23b, Theorem 2.13])**.** *For every adequate triple (C, C_L, C_R) the simplicial anima* $\mathrm{NSpan}(C, C_L, C_R)$ *is a complete Segal anima.*

Proof. We start by proving the Segal condition. Let us write $X := \mathrm{NSpan}(C, C_L, C_R)$ for ease of notation. For every $n \geq 0$, consider the subposet $J_n \subseteq \mathrm{Tw}^r([n])$ spanned by the objects $(i \leq j)$ with $j \leq i + 1$:



Note that a functor $\mathrm{Tw}^r([n]) \to C$ preserves the pullback squares from Lemma 13.1.8 if and only if it is right Kan extended along the inclusion $J_n \hookrightarrow \mathrm{Tw}^r([n])$: if we compute this Kan extension row by row, this follows from the pointwise formula for Kan extensions (Theorem 22.4.3). Concretely, right Kan extension from J_n fills the diagram by iterated pullbacks, and hence encodes the successive composites of the original string of spans.

We now use the adequacy of (C, C_L, C_R). Starting with a functor $J_n \to C$ that sends the left-pointing morphisms to C_L and the right-pointing morphisms to C_R, construct its right Kan extension by induction on $j - i$. At each step, the value at $(i \leq j)$ is obtained by pulling back the diagram formed by the previously constructed values at $(i \leq j - 1)$, $(i + 1 \leq j)$ and $(i + 1 \leq j - 1)$. The adequate-triple axiom guarantees that this pullback exists and that its new left- and right-pointing legs again lie in C_L and C_R, respectively. Thus restriction to J_n induces an inclusion of animae

$$X_n = \mathrm{NSpan}(C, C_L, C_R)_n \hookrightarrow \mathrm{Map}(J_n, C)$$

whose image consists precisely of those functors $J_n \to C$ sending the left-pointing morphisms to C_L and the right-pointing morphisms to C_R. We now observe that the Segal maps $e_i \colon [1] \hookrightarrow [n]$ induce inclusions $\mathrm{Tw}^r([1]) \hookrightarrow J_n$ which assemble into an equivalence

$$\mathrm{Tw}^r([1]) \sqcup_{\mathrm{Tw}^r([0])} \mathrm{Tw}^r([1]) \sqcup_{\mathrm{Tw}^r([0])} \cdots \sqcup_{\mathrm{Tw}^r([0])} \mathrm{Tw}^r([1]) \xrightarrow{\sim} J_n,$$

as may be checked directly in posets: the objects $(i \leq i)$ along which we glue are maximal in the adjacent copies of $\mathrm{Tw}^r([1])$, so the pushout creates no additional relations. In particular, we get

$$\mathrm{Map}(J_n, C) \xrightarrow{\sim} \mathrm{Map}(\mathrm{Tw}^r([1]), C) \times_{C^{\simeq}} \mathrm{Map}(\mathrm{Tw}^r([1]), C) \times_{C^{\simeq}} \cdots \times_{C^{\simeq}} \mathrm{Map}(\mathrm{Tw}^r([1]), C).$$

Under this equivalence, the image of the restriction map $X_n \hookrightarrow \mathrm{Map}(J_n, C)$ is identified with the target of the Segal map

$$(e_i^*)_{i=1}^n \colon X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

showing that X satisfies the Segal condition.

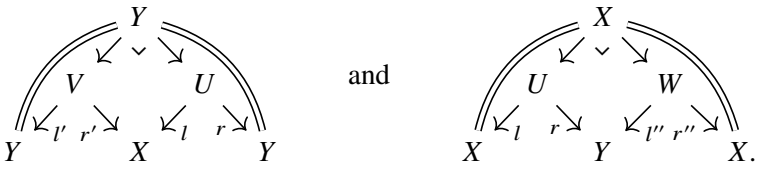
For completeness, first note that the functor

$$C^{\simeq} = X_0 \to X_1 \subseteq \mathrm{Map}(\mathrm{Tw}^r([1]), C)$$

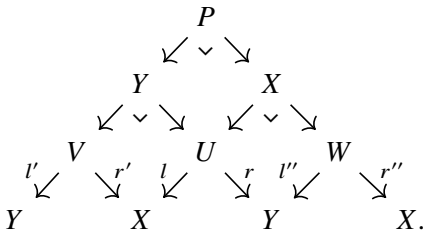
is an inclusion of animae. Indeed, since $\mathrm{Tw}^r([1])$ has an initial object we have $\| \mathrm{Tw}^r([1]) \| \simeq *$, and thus the constant diagram functor $C \to \mathrm{Fun}(\mathrm{Tw}^r([1]), C)$ may be identified with the fully

faithful inclusion $\mathrm{Fun}(\|\mathrm{Tw}^r([1])\|, C) \hookrightarrow \mathrm{Fun}(\mathrm{Tw}^r([1]), C)$. It thus remains to show that a span $X \xleftarrow{l} U \xrightarrow{r} Y$ is an isomorphism in the Segal anima $\mathrm{NSpan}(C, C_L, C_R)$ if and only if both l and r are isomorphisms in C.

We first show that an invertible span has invertible legs. Let $Y \xleftarrow{l'} V \xrightarrow{r'} X$ be a left inverse, and let $Y \xleftarrow{l''} W \xrightarrow{r''} X$ be a right inverse. The two diagrams below exhibit these inverses; pulling them back against each other will produce the isomorphisms needed for a 2-out-of-6 argument. The identities $(l, r) \circ (l', r') = \mathrm{id}$ and $(l'', r'') \circ (l, r) = \mathrm{id}$ are exhibited by commutative diagrams in C of the following form:

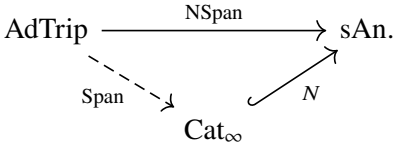


Putting them into one single diagram and forming another pullback, we then obtain:



Since isomorphisms in C are closed under pullbacks, the maps $P \to V$ and $P \to W$ are isomorphisms, so by the 2-out-of-6 property all morphisms along the outer two edges of the diagram are isomorphisms. By the 2-out-of-3 property, then so are the maps $r' \colon V \to X$ and $l'' \colon W \to Y$, hence by pullback also $Y \to U$ and $X \to U$. Finally, another instance of the 2-out-of-3 property shows that $l \colon U \to X$ and $r \colon U \to Y$ are isomorphisms. Conversely, if l and r are isomorphisms, then the span is isomorphic in X_1 to the identity span on U, and hence lies in the image of $X_0 \to X_1$. This proves completeness. □

Notation 13.1.12. By Proposition 13.1.11 and Theorem 25.1.10, the functor $\mathrm{NSpan} \colon \mathrm{AdTrip} \to \mathrm{sAn}$ factors uniquely as



We refer to the resulting ∞-category $\mathrm{Span}(C, C_L, C_R)$ as the *span category* of the adequate triple. When it is convenient to give the adequate triple a name, say $\mathcal{C} = (C, C_L, C_R)$, we also write $\mathrm{Span}(\mathcal{C})$ for its span category. Since $\mathrm{NSpan}(C, C_L, C_R)$ is complete and its degree-zero anima is $C^{\simeq}$, the groupoid core of $\mathrm{Span}(C, C_L, C_R)$ identifies naturally with $C^{\simeq}$.

If C admits pullbacks, we write

$$\mathrm{Span}(C) := \mathrm{Span}(C, C, C)$$

for the span category associated with the adequate triple (C, C, C). In particular, $\mathrm{Span}(\mathrm{Fin})$ is the rigorous construction of the span category of finite sets promised in Construction 5.3.1.

A useful special case was already identified in Lemma 5.3.8: a span of finite sets whose left leg is injective may be regarded as a partially defined map, and this gives an equivalence

$$\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \simeq \mathrm{Fin}_*.$$

Here, as in Lemma 5.3.8, we denote classes of maps of finite sets by labels rather than by wide subcategories, so that $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) = \mathrm{Span}(\mathrm{Fin}, \mathrm{Fin}_{\mathrm{inj}}, \mathrm{Fin})$ for the wide subcategory $\mathrm{Fin}_{\mathrm{inj}} \subseteq \mathrm{Fin}$ of injections. This description will be used in the comparison with Lurie's model.

Lemma 13.1.13. *Let (C, C_L, C_R) be an adequate triple and consider objects $X, Y \in C$. Then there is an equivalence*

$$\mathrm{Hom}_{\mathrm{Span}(C,C_L,C_R)}(X,Y) \quad \simeq \quad ((C_L)_{/X})^{\simeq} \times_{C^{\simeq}} ((C_R)_{/Y})^{\simeq}.$$

Proof. Observe that the twisted arrow category $\mathrm{Tw}^r([1])$ is the walking span $\ulcorner$ from Definition 1.4.1. We may then compute:

$$\begin{aligned}\mathrm{Map}([1], \mathrm{Span}(C, C_L, C_R)) &\simeq \mathrm{NSpan}(C, C_L, C_R)_1 \\ &\simeq \mathrm{Hom}_{\mathrm{AdTrip}}(\ulcorner, (C, C_L, C_R)) \\ &\simeq \mathrm{Map}([1], C_L) \times_{\mathrm{ev}_0, C^{\simeq}, \mathrm{ev}_0} \mathrm{Map}([1], C_R).\end{aligned}$$

The hom anima $\mathrm{Hom}_{\mathrm{Span}(C,C_L,C_R)}(X,Y)$ is then given as the fiber over (X,Y) of the map

$$\mathrm{Map}([1], C_L) \times_{\mathrm{ev}_0, C^{\simeq}, \mathrm{ev}_0} \mathrm{Map}([1], C_R) \xrightarrow{(\mathrm{ev}_1, \mathrm{ev}_1)} C^{\simeq} \times C^{\simeq},$$

which is indeed equivalent to $((C_L)_{/X})^{\simeq} \times_{C^{\simeq}} ((C_R)_{/Y})^{\simeq}$. □

Thus the case $n = 1$ of Construction 13.1.10 recovers the informal description from the beginning of the chapter: the objects are those of C, morphisms are spans with left leg in C_L and right leg in C_R, and identities are the identity spans. Under the Segal equivalence, the face map $d_1 \colon \mathrm{NSpan}(C, C_L, C_R)_2 \to \mathrm{NSpan}(C, C_L, C_R)_1$ sends a pullback grid to its outer span. Consequently, composition in $\mathrm{Span}(C, C_L, C_R)$ is given by pullback.

Corollary 13.1.14. *Suppose that $C'_L \subseteq C_L$ and $C'_R \subseteq C_R$ are wide subcategories such that both (C, C'_L, C'_R) and (C, C_L, C_R) are adequate triples. Then the induced functor*

$$\mathrm{Span}(C, C'_L, C'_R) \hookrightarrow \mathrm{Span}(C, C_L, C_R)$$

is a wide subcategory inclusion.

Proof. The induced map on groupoid cores identifies with the identity of $C^{\simeq}$. Moreover, Lemma 13.1.13 identifies the induced maps on hom animae with inclusions of components. Thus the functor is a wide subcategory inclusion. □

Let us now prove various basic properties of span categories.

Lemma 13.1.15. *There is a natural equivalence* $\mathrm{Span}(C, C_L, C_R)^{\mathrm{op}} \simeq \mathrm{Span}(C, C_R, C_L)$*, in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{AdTrip} & \xrightarrow{(-)_{\mathrm{rev}}} & \mathrm{AdTrip} \\ {\scriptstyle \mathrm{Span}}\downarrow & & \downarrow{\scriptstyle \mathrm{Span}} \\ \mathrm{Cat}_\infty & \xrightarrow{(-)^{\mathrm{op}}} & \mathrm{Cat}_\infty. \end{array}$$

Proof. By definition of $(-)^{\mathrm{op}} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$, there is a commutative square

$$\begin{array}{ccc} \mathrm{Cat}_\infty & \xrightarrow{(-)^{\mathrm{op}}} & \mathrm{Cat}_\infty \\ {\scriptstyle N}\uparrow & & \uparrow{\scriptstyle N} \\ \mathrm{sAn} & \xrightarrow{(-)^{\mathrm{op}}} & \mathrm{sAn}, \end{array}$$

where the bottom functor is given by precomposition with $(-)^{\mathrm{op}} \colon \mathbf{\Delta} \xrightarrow{\sim} \mathbf{\Delta}$. We thus need to produce a natural equivalence $\mathrm{NSpan}(C, C_L, C_R)^{\mathrm{op}} \simeq \mathrm{NSpan}(C, C_R, C_L)$ of simplicial animae. Unwinding definitions, this amounts to producing a natural equivalence

$$\mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]^{\mathrm{op}}), (C, C_L, C_R)) \simeq \mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]), (C, C_R, C_L)),$$

natural in $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ and (C, C_L, C_R). Equivalently, we must produce a natural equivalence $\mathrm{Tw}^r([n]^{\mathrm{op}}) \simeq \mathrm{Tw}^r([n])_{\mathrm{rev}}$. Such an equivalence is given by sending an object $(i \geq j)$ in $\mathrm{Tw}^r([n]^{\mathrm{op}})$ to the object $(j \leq i)$ in $\mathrm{Tw}^r([n])$. □

The span category $\mathrm{Span}(C, C_L, C_R)$ contains C_R as a wide subcategory on the right-pointing spans $X \xleftarrow{=} X \xrightarrow{r} Y$. Similarly it contains C_L^{op} as a wide subcategory on the left-pointing spans $X \xleftarrow{l} Y \xrightarrow{=} Y$. This is made precise by the following result:

Lemma 13.1.16. *Let* $C_L, C_R \subseteq C$ *be wide subcategories. Then the span categories of the adequate triples* $(C, C^{\simeq}, C_R)$ *and* $(C, C_L, C^{\simeq})$ *from Example 13.1.3 are given by:*

$$\mathrm{Span}(C, C^{\simeq}, C_R) \simeq C_R \qquad \textit{and} \qquad \mathrm{Span}(C, C_L, C^{\simeq}) \simeq C_L^{\mathrm{op}}.$$

In particular, there are canonical inclusions

$$C_R \simeq \mathrm{Span}(C, C^{\simeq}, C_R) \hookrightarrow \mathrm{Span}(C, C_L, C_R) \qquad \textit{and} \qquad C_L^{\mathrm{op}} \simeq \mathrm{Span}(C, C_L, C^{\simeq}) \hookrightarrow \mathrm{Span}(C, C_L, C_R).$$

Proof. It suffices to prove the first equivalence; the second one follows by combining the first one with Lemma 13.1.15. For $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, consider the source functor $s \colon \mathrm{Tw}^r([n]) \to [n]$, $(i \leq j) \mapsto i$. Note that this functor admits a fully faithful left adjoint $[n] \to \mathrm{Tw}^r([n])$ sending $i \in [n]$ to $(i \leq n) \in \mathrm{Tw}^r([n])$. Indeed, there is a morphism $(i \leq n) \to (k \leq l)$ in $\mathrm{Tw}^r([n])$ if and only if $i \leq k$. It thus follows from Lemma 22.7.8 that restriction along s induces an inclusion of animae

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C_R) \subseteq \mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C) \hookrightarrow \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Tw}^r([n]), C)$$

whose essential image consists of those functors $\mathrm{Tw}^r[n] \to C$ that send backwards morphisms to isomorphisms and forward morphisms into C_R. Since these are precisely the morphisms of adequate triples into $(C, C^{\simeq}, C_R)$, we obtain a natural equivalence

$$N(C_R)_n = \mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C_R) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]), (C, C^{\simeq}, C_R)) = \mathrm{NSpan}(C, C^{\simeq}, C_R)_n.$$

This shows that $N(C_R) \simeq \mathrm{NSpan}(C, C^{\simeq}, C_R)$, and hence that $C_R \simeq \mathrm{Span}(C, C^{\simeq}, C_R)$ by full faithfulness of the nerve functor. □

The construction of span categories is summarized conceptually by the following adjunction. We will not need the adjunction itself below, but it explains the role of the right twisted-arrow construction in the definition of Span.

Proposition 13.1.17 ([HHLN23b, Theorem 2.18])**.** *The right twisted-arrow construction* $\mathrm{Tw}^r : \mathbf{\Delta} \to \mathrm{AdTrip}$ *extends to a functor*

$$\mathrm{Tw}^r : \mathrm{Cat}_\infty \to \mathrm{AdTrip},$$

and this extension is left adjoint to the span construction:

$$\mathrm{Tw}^r : \mathrm{Cat}_\infty \rightleftarrows \mathrm{AdTrip} : \mathrm{Span}\,.$$

Lemma 13.1.18. *The* ∞*-category* AdTrip *admits limits and filtered colimits, and the inclusion* $\mathrm{AdTrip} \hookrightarrow \mathrm{Fun}(\sqcup, \mathrm{Cat}_\infty)$ *preserves limits and filtered colimits.*

Proof. We start with the case for limits. Recall that limits in $\mathrm{Fun}(\sqcup, \mathrm{Cat}_\infty)$ are computed pointwise. Given a functor $C\colon I \to \mathrm{AdTrip}$, we thus need to show that the triple $(C, C_L, C_R) := (\lim_i C(i), \lim_i C(i)_L, \lim_i C(i)_R)$ is again adequate. Note that subcategory inclusions are closed under limits in Cat_∞: a functor $D \hookrightarrow D'$ is a subcategory inclusion if and only if the commutative square

$$\begin{array}{ccc} D & = & D \\ \| & & \downarrow \\ D & \longrightarrow & D' \end{array}$$

is a pullback square, and limits commute with pullback squares. Furthermore, *wide* subcategory inclusions are closed under limits: an inclusion $D \hookrightarrow D'$ is a wide subcategory if and only if the induced map $D^{\simeq} \to D'^{\simeq}$ on groupoid cores is an equivalence of animae, and $(-)^{\simeq}$ preserves limits.

Consider a cospan $X \xrightarrow{l} X'' \xleftarrow{r} X'$ in $C = \lim_i C(i)$ with $l \in C_L$ and $r \in C_R$. Since the transition functors preserve the pullbacks specified by the adequate-triple structures, the pointwise pullbacks in the $C(i)$ assemble to an object of C. The identity

$$\mathrm{Hom}_C(W, -) \simeq \lim_i \mathrm{Hom}_{C(i)}(W_i, -)$$

shows that this object is the pullback in C, and the pointwise criterion for membership in C_L and C_R shows that its legs lie in the required classes. Thus (C, C_L, C_R) is an adequate triple. Furthermore, a functor into C is a morphism of adequate triples if and only if its composite with each projection $C \to C(i)$ is, so (C, C_L, C_R) is a limit in AdTrip.

For filtered colimits, note that subcategory inclusions are closed under filtered colimits in Cat_∞ since filtered colimits commute with pullbacks by Lemma 25.2.2. Since every object in a filtered colimit of ∞-categories $D(i)$ comes from one of the $D(i)$, it follows that a filtered colimit of wide subcategory inclusions is again wide. Any cospan in $C = \mathrm{colim}_i C(i)$ lifts to a cospan in some stage $C(j)$. Its pullback exists there by adequacy, and the transition functors carry this pullback to the corresponding pullbacks at every later stage. Mapping animae in a filtered colimit of ∞-categories are filtered colimits of the mapping animae, and finite limits of animae commute with

filtered colimits. It follows that the resulting square in C remains a pullback. Thus (C, C_L, C_R) is an adequate triple. A functor out of C is a morphism of adequate triples if and only if its composite with each $C(j) \to C$ is, so (C, C_L, C_R) is a colimit in AdTrip. □

Proposition 13.1.19. *The functor* $\mathrm{Span}\colon \mathrm{AdTrip} \to \mathrm{Cat}_\infty$ *preserves limits and filtered colimits.*

Proof. The inclusion $N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$ preserves limits and filtered colimits (see Lemma 25.2.2 for the latter claim). It will thus suffice to show that $\mathrm{NSpan}\colon \mathrm{AdTrip} \to \mathrm{sAn}$ preserves these (co)limits. Since these are computed pointwise, this boils down to showing that the functor

$$\mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]), -)\colon \mathrm{AdTrip} \to \mathrm{An}$$

preserves them. The case for limits is clear, since the functor is corepresented. For filtered colimits, we use again that filtered colimits commute with pullbacks, so that we may reduce to the cases $n = 0$ and $n = 1$ via the Segal condition. Note that the ∞-categories $\mathrm{Tw}^r([0]) = *$ and $\mathrm{Tw}^r([1]) = \ulcorner$ are finite, so $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Tw}^r([n]), -)\colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves filtered colimits by Corollary 25.2.3. This proves the claim for $n = 0$. For $n = 1$, one observes that the subanimae of morphisms of adequate triples are closed under filtered colimits. □

13.2 Adjunctions of span categories

Before studying products and coproducts in span categories, we develop a general criterion for an adjunction between adequate triples to induce an adjunction between their span categories.

Construction 13.2.1. Consider adequate triples (C, C_L, C_R) and (D, D_L, D_R) and let $F, G\colon C \to D$ be morphisms of adequate triples. Let $\alpha\colon F \Rightarrow G$ be a natural transformation, and assume that the following two conditions are satisfied:

(1) The morphism $\alpha(X)\colon F(X) \to G(X)$ lies in D_R for every object X of C;

(2) For every morphism $l\colon Z \to X$ in C_L the commutative square

$$\begin{array}{ccc} F(Z) & \xrightarrow{\alpha(Z)} & G(Z) \\ {\scriptstyle F(l)}\downarrow & & \downarrow{\scriptstyle G(l)} \\ F(X) & \xrightarrow{\alpha(X)} & G(X) \end{array}$$

is a pullback square in D.

We will construct a natural transformation

$$\mathrm{Span}^{\mathrm{R}}(\alpha)\colon \mathrm{Span}(F) \implies \mathrm{Span}(G)$$

of functors $\mathrm{Span}(C, C_L, C_R) \to \mathrm{Span}(D, D_L, D_R)$ which is objectwise given by the forward spans

$$F(X) \xleftarrow{=} F(X) \xrightarrow{\alpha(X)} G(X)$$

for $X \in C$. For this, write $\mathcal{C} := (C, C_L, C_R)$ and $\mathcal{D} := (D, D_L, D_R)$, and equip $[1]$ with the adequate triple structure $([1], [1]^{\simeq}, [1])$. Then the product $\mathcal{C} \times [1]$ is the adequate triple $(C \times$

$[1], C_L \times [1]^{\simeq}, C_R \times [1])$. Regarding α as a functor $C \times [1] \to D$, conditions (1) and (2) say that it is a morphism of adequate triples. Indeed, the only additional pullback squares to check are sent to pastings of images under F or G of adequate pullback squares with the naturality squares from (2). We therefore obtain a functor

$$\mathrm{Span}(C \times [1]) \to \mathrm{Span}(\mathcal{D}).$$

Using that $\mathrm{Span}(-)$ preserves products by Proposition 13.1.19 and that $\mathrm{Span}([1], [1]^{\simeq}, [1]) \simeq [1]$ by Lemma 13.1.16, this functor takes the form

$$\mathrm{Span}(C) \times [1] \to \mathrm{Span}(\mathcal{D}).$$

The restrictions to 0 and 1 are $\mathrm{Span}(F)$ and $\mathrm{Span}(G)$, so this produces the desired natural transformation $\mathrm{Span}^{\mathrm{R}}(\alpha)$.

Remark 13.2.2. The previous construction can be dualized, if we assume instead that the morphism $\alpha(X)\colon F(X) \to G(X)$ is in D_L for every $X \in C$ and that for every morphism $r\colon Z \to Y$ in C_R the commutative square

$$\begin{array}{ccc} F(Z) & \xrightarrow{\alpha(Z)} & G(Z) \\ {\scriptstyle F(r)}\downarrow & & \downarrow{\scriptstyle G(r)} \\ F(Y) & \xrightarrow{\alpha(Y)} & G(Y) \end{array}$$

is a pullback square in D. By applying Lemma 13.1.15 to the previous construction, we deduce that α induces a natural transformation

$$\mathrm{Span}^{\mathrm{L}}(\alpha)\colon \mathrm{Span}(G) \implies \mathrm{Span}(F)$$

of functors $\mathrm{Span}(C, C_L, C_R) \to \mathrm{Span}(D, D_L, D_R)$ given on an object $X \in C$ by the backwards span

$$G(X) \xleftarrow{\alpha(X)} F(X) \xrightarrow{=} F(X).$$

Lemma 13.2.3. *Given a third morphism of adequate triples $H\colon (C, C_L, C_R) \to (D, D_L, D_R)$, let $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ be natural transformations satisfying the conditions (1) and (2) from Construction 13.2.1. Then we have natural equivalences*

$$\mathrm{Span}^{\mathrm{R}}(\mathrm{id}_F) \simeq \mathrm{id}_{\mathrm{Span}(F)} \qquad \textit{and} \qquad \mathrm{Span}^{\mathrm{R}}(\beta \circ \alpha) \simeq \mathrm{Span}^{\mathrm{R}}(\beta) \circ \mathrm{Span}^{\mathrm{R}}(\alpha).$$

The construction is also compatible with whiskering: if $K\colon (B, B_L, B_R) \to (C, C_L, C_R)$ and $J\colon (D, D_L, D_R) \to (E, E_L, E_R)$ are morphisms of adequate triples, then

$$\mathrm{Span}^{\mathrm{R}}(\alpha K) \simeq \mathrm{Span}^{\mathrm{R}}(\alpha) \circ \mathrm{Span}(K) \qquad \textit{and} \qquad \mathrm{Span}^{\mathrm{R}}(J\alpha) \simeq \mathrm{Span}(J) \circ \mathrm{Span}^{\mathrm{R}}(\alpha).$$

The dual claims for $\mathrm{Span}^{\mathrm{L}}(-)$ also hold.

Proof. Composition and identities of natural transformations are formed using the posets $[n]$ for $n = 0, 2$. Whiskering is given by pre- and postcomposition with morphisms of adequate triples. The claims therefore follow from the naturality in both $[n] \in \mathbf{\Delta}$ and $C = (C, C_L, C_R) \in \mathrm{AdTrip}$ of the equivalence $\mathrm{Span}(C \times [n]) \simeq \mathrm{Span}(C) \times [n]$, where $[n]$ carries the adequate triple structure $([n], [n]^{\simeq}, [n])$. □

Corollary 13.2.4. *Let (C, C_L, C_R) and (D, D_L, D_R) be adequate triples. Let $F\colon C \rightleftarrows D : G$ be an adjunction of ∞-categories and assume that F and G are morphisms of adequate triples.*

(1) Assume that the unit map $\eta_X\colon X \to G(F(X))$ is in C_R for all $X \in C$ and the counit map $\varepsilon_Y\colon F(G(Y)) \to Y$ is in D_R for all $Y \in D$. Furthermore, assume that for every morphism $l\colon Z \to X$ in C_L and every morphism $l'\colon Z' \to X'$ in D_L, the commutative squares

$$\begin{array}{ccc} Z & \xrightarrow{\eta_Z} & G(F(Z)) \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle G(F(l))} \\ X & \xrightarrow{\eta_X} & G(F(X)) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F(G(Z')) & \xrightarrow{\varepsilon_{Z'}} & Z' \\ {\scriptstyle F(G(l'))}\downarrow & & \downarrow{\scriptstyle l'} \\ F(G(X')) & \xrightarrow{\varepsilon_{X'}} & X' \end{array}$$

are pullback squares. Then F and G induce an adjunction

$$\mathrm{Span}(F)\colon \mathrm{Span}(C, C_L, C_R) \rightleftarrows \mathrm{Span}(D, D_L, D_R) : \mathrm{Span}(G).$$

(2) Dually, assume that the unit map $\eta_X\colon X \to G(F(X))$ is in C_L for all $X \in C$ and the counit map $\varepsilon_Y\colon F(G(Y)) \to Y$ is in D_L for all $Y \in D$. Furthermore, assume that for every morphism $r\colon Z \to Y$ in C_R and every morphism $r'\colon Z' \to Y'$ in D_R, the commutative squares

$$\begin{array}{ccc} Z & \xrightarrow{\eta_Z} & G(F(Z)) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle G(F(r))} \\ Y & \xrightarrow{\eta_Y} & G(F(Y)) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F(G(Z')) & \xrightarrow{\varepsilon_{Z'}} & Z' \\ {\scriptstyle F(G(r'))}\downarrow & & \downarrow{\scriptstyle r'} \\ F(G(Y')) & \xrightarrow{\varepsilon_{Y'}} & Y' \end{array}$$

are pullback squares. Then F and G induce an adjunction

$$\mathrm{Span}(G)\colon \mathrm{Span}(D, D_L, D_R) \rightleftarrows \mathrm{Span}(C, C_L, C_R) : \mathrm{Span}(F).$$

Proof. We prove (1), as (2) is dual. By Construction 13.2.1 and Lemma 13.2.3, the unit $\eta\colon \mathrm{id} \to GF$ and counit $\varepsilon\colon FG \to \mathrm{id}$ induce transformations

$$\mathrm{Span}^{\mathrm{R}}(\eta)\colon \mathrm{Span}(\mathrm{id}) \to \mathrm{Span}(G) \circ \mathrm{Span}(F) \qquad \text{and} \qquad \mathrm{Span}^{\mathrm{R}}(\varepsilon)\colon \mathrm{Span}(F) \circ \mathrm{Span}(G) \to \mathrm{id}\,.$$

Compatibility with vertical composition and whiskering from Lemma 13.2.3 shows that these transformations satisfy the triangle identities. They therefore exhibit the asserted adjunction by Proposition 22.1.2. □

13.3 Products and coproducts in span categories

We now apply the previous results on adjunctions of span categories to study products and coproducts in span categories.

Definition 13.3.1. An ∞-category C is called *extensive* if it admits finite coproducts and, for every $n \geq 0$ and every collection of objects $X_1, \ldots, X_n$, the coproduct functor

$$\prod_{i=1}^{n} C_{/X_i} \longrightarrow C_{/\coprod_i X_i}$$

is an equivalence.

We record the two consequences of this definition that will be needed below.

Lemma 13.3.2. *Let C be an extensive ∞-category.*

(1) Finite coproducts of pullback squares in C are pullback squares;

(2) For every morphism $r\colon X \to Y$, the commutative squares

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{r\sqcup r} & Y \sqcup Y \\ {\scriptstyle\nabla}\downarrow & & \downarrow{\scriptstyle\nabla} \\ X & \xrightarrow{r} & Y \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \emptyset & = & \emptyset \\ \downarrow & & \downarrow \\ X & \xrightarrow{r} & Y \end{array} \tag{13.2}$$

are pullback squares.

Proof. A pullback square whose lower right corner is Y_i is equivalently a pullback square in the slice category $C_{/Y_i}$. Under the equivalence

$$\prod_i C_{/Y_i} \longrightarrow C_{/\coprod_i Y_i}$$

from Definition 13.3.1, a collection of such squares is sent to its coproduct. Since equivalences preserve pullbacks, this proves part (1).

For part (2), recall that a right adjoint to postcomposition between slice categories computes pullback along the relevant morphism. Under the equivalence

$$C_{/Y} \times C_{/Y} \longrightarrow C_{/Y\sqcup Y},$$

postcomposition with $\nabla\colon Y \sqcup Y \to Y$ identifies with the coproduct functor $C_{/Y} \times C_{/Y} \to C_{/Y}$. Its right adjoint is the diagonal functor, so pullback along ∇ sends r to $r \sqcup r$. The counit square is the first square in (13.2).

For the second square, the case $n = 0$ of Definition 13.3.1 says that $C_{/\emptyset}$ is terminal. Postcomposition along $\emptyset \to Y$ selects the initial object $\emptyset \to Y$ of $C_{/Y}$ and is therefore left adjoint to the unique functor $C_{/Y} \to C_{/\emptyset}$. Thus pullback along $\emptyset \to Y$ sends r to the unique object of $C_{/\emptyset}$, and its counit square is the second square in (13.2). □

For restricted span categories we need a relative version of extensivity that only involves the two classes of morphisms in an adequate triple.

Definition 13.3.3. An adequate triple (C, C_L, C_R) is called *weakly extensive* if the following conditions are satisfied:

(1) The ∞-category C admits finite coproducts;

(2) The coproduct functor $C \times C \to C$ is a morphism of adequate triples (i.e., morphisms in C_L and C_R are closed under coproducts, and coproducts of pullback squares of morphisms in C_L along morphisms in C_R are again pullback squares);

(3) For every object $X \in C$, the morphisms $\emptyset \to X$ and $\nabla\colon X \sqcup X \to X$ are in C_L;

(4) For every morphism $r\colon X \to Y$ in C_R, the two squares in (13.2) are pullback squares.

We say that the triple is *weakly coextensive* if (C, C_R, C_L) is weakly extensive, and we say it is *extensive* if it is both weakly extensive and weakly coextensive.

The relation between the intrinsic and relative notions is summarized by the following observation.

Lemma 13.3.4. *Let C be an extensive ∞-category and let (C, C_L, C_R) be an adequate triple such that C_L and C_R are closed under finite coproducts.*

(1) If the morphisms $\emptyset \to X$ and $\nabla\colon X \sqcup X \to X$ lie in C_L for every $X \in C$, then the triple is weakly extensive.

(2) If the morphisms $\emptyset \to X$ and $\nabla\colon X \sqcup X \to X$ lie in C_R for every $X \in C$, then the triple is weakly coextensive.

Consequently, if these morphisms lie in both classes, then the triple is extensive.

Proof. Part (1) of Lemma 13.3.2 shows that the coproduct functor preserves the pullback squares required by adequacy, while part (2) verifies condition (4) of Definition 13.3.3. Together with the closure hypotheses and the assumption on the fold and initial maps, this verifies conditions (1)–(4) of that definition and proves part (1). Interchanging C_L and C_R proves part (2), and the final assertion follows by combining the two parts. □

Example 13.3.5. The categories Fin and Fin_G, for a finite group G, are extensive. Since they admit pullbacks, Lemma 13.3.4 shows that the triples (Fin, Fin, Fin) and $(\mathrm{Fin}_G, \mathrm{Fin}_G, \mathrm{Fin}_G)$ are extensive.

Example 13.3.6. The ∞-category Top is extensive, and so is the triple (Top, Top, Top).

Example 13.3.7. If a full subcategory $C \subseteq \mathrm{An}$ is closed under finite coproducts and pullbacks, then C and the triple (C, C, C) are extensive.

Lemma 13.3.8. *Let (C, C_L, C_R) be an adequate triple.*

(1) If the triple is weakly extensive, then $\mathrm{Span}(C, C_L, C_R)$ admits finite products, and the inclusion $C_L^{\mathrm{op}} \hookrightarrow \mathrm{Span}(C, C_L, C_R)$ preserves finite products;

(2) If the triple is weakly coextensive, then $\mathrm{Span}(C, C_L, C_R)$ admits finite coproducts, and the inclusion $C_R \hookrightarrow \mathrm{Span}(C, C_L, C_R)$ preserves finite coproducts;

(3) If the triple is extensive, then $\mathrm{Span}(C, C_L, C_R)$ is semiadditive.

Proof. For part (1), note that C has binary coproducts if and only if the diagonal functor $\Delta\colon C \to C \times C$ admits a left adjoint $\sqcup\colon C \times C \to C$. The counit of the adjunction is the fold map $\nabla\colon X \sqcup X \to X$ and the unit is the pair of canonical inclusions $(X \hookrightarrow X \sqcup Y, Y \hookrightarrow X \sqcup Y)$. Conditions (1) to (4) supply the hypotheses of part (2) of Corollary 13.2.4, so this adjunction induces an adjunction

$$\mathrm{Span}(\mathcal{C}) \rightleftarrows \mathrm{Span}(\mathcal{C} \times \mathcal{C}) \simeq \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}),$$

where $\mathcal{C} := (C, C_L, C_R)$ and $\mathcal{C} \times \mathcal{C} = (C \times C, C_L \times C_L, C_R \times C_R)$. Let us spell out the hypotheses of Corollary 13.2.4. The coproduct functor is a morphism of adequate triples by condition (2), while

the diagonal functor preserves the relevant classes and pullbacks componentwise. The fold map lies in C_L by condition (3). Each coprojection, for example

$$X = X \sqcup \emptyset \xrightarrow{\mathrm{id}_X \sqcup (\emptyset \to Y)} X \sqcup Y,$$

lies in C_L by conditions (2) and (3). The naturality square for the fold map is the first pullback square in condition (4). The naturality squares for the coprojections decompose as coproducts of an identity square and the second pullback square in condition (4), so they are pullbacks by condition (2). This proves that the displayed adjunction exists, and hence that $\mathrm{Span}(C, C_L, C_R)$ admits binary products. For the terminal object, we similarly regard the initial object $\emptyset$ of C as a left adjoint to the functor $C \to *$. Conditions (3) and (4) allow us to apply Corollary 13.2.4 and obtain an adjunction $\mathrm{Span}(C, C_L, C_R) \rightleftarrows \mathrm{Span}(*) \simeq *$ exhibiting $\emptyset$ as a terminal object of $\mathrm{Span}(C, C_L, C_R)$.

It remains to prove the claim about the inclusion $C_L^{\mathrm{op}} \hookrightarrow \mathrm{Span}(C, C_L, C_R)$. The subcategory C_L admits finite coproducts computed as in C. Indeed, the coprojections lie in C_L by the preceding argument. Given morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ in C_L, their copairing factors as

$$X \sqcup Y \xrightarrow{f \sqcup g} Z \sqcup Z \xrightarrow{\nabla} Z$$

and therefore lies in C_L. Conversely, a morphism $X \sqcup Y \to Z$ in C_L restricts to morphisms in C_L because the coprojections lie in C_L. The initial object is initial in C_L by condition (3). Under the inclusion $C_L^{\mathrm{op}} \hookrightarrow \mathrm{Span}(C, C_L, C_R)$, the resulting product projections are the backwards spans associated with the coprojections, which are the product projections constructed above. Thus this inclusion preserves finite products.

Part (2) is an immediate consequence of (1) using $\mathrm{Span}(C, C_L, C_R)^{\mathrm{op}} \simeq \mathrm{Span}(C, C_R, C_L)$. For part (3), note first that the initial object $\emptyset$ in C becomes both initial and terminal in $\mathrm{Span}(C, C_L, C_R)$, and that the zero morphism between objects X and Y is given by the span

$$X \leftarrow \emptyset \to Y.$$

Also notice that the coproduct $X \sqcup Y$ in C becomes both the product and the coproduct in $\mathrm{Span}(C, C_L, C_R)$, via the spans

$$X \sqcup Y \leftarrow X \xrightarrow{=} X, \qquad X \sqcup Y \leftarrow Y \xrightarrow{=} Y, \qquad X \xleftarrow{=} X \hookrightarrow X \sqcup Y, \qquad Y \xleftarrow{=} Y \hookrightarrow X \sqcup Y.$$

It thus remains to compute the four composites and show that they are id_X, 0, 0 and id_Y. The required pullback squares follow from the extensivity assumptions. The first square below is the coproduct of the identity square on X and the empty pullback square from condition (4) for the morphism $\emptyset \to Y$, which lies in C_R by weak coextensivity; the second is obtained symmetrically. The third is the coproduct of the empty pullback square from condition (4) for id_X and the identity square on the morphism $\emptyset \to Y$. Condition (2) guarantees that these coproducts are again pullback squares. Thus we have:

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & X \sqcup Y, \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & X \sqcup Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & X \sqcup Y. \end{array}$$

□

Corollary 13.3.9. *The* ∞*-category* $\mathrm{Span}(\mathrm{Fin})$ *is semiadditive, with biproducts given by disjoint unions of finite sets.*

Proof. Apply part (3) of Lemma 13.3.8 to the extensive triple $(\mathrm{Fin}, \mathrm{Fin}, \mathrm{Fin})$. This recovers the direct argument of Lemma 5.3.3. □

Exercises

Exercise 13.1 (Examples of extensive categories)**.** Show that the following categories are extensive:

(1) The category Fin_G of finite G-sets, for any group G.

(2) The categories Top and Sch of topological spaces and schemes, respectively.

Exercise 13.2. Let C be an ∞-category with finite coproducts. Show that C is extensive if and only if it satisfies the following two conditions:

- *Coproducts in C are disjoint*: for objects X and Y, the commutative square

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \sqcup Y \end{array}$$

  is a pullback square.

- *Coproduct decompositions are stable under pullback*: given a morphism $f\colon Z \to X \sqcup Y$, the map $Z_X \sqcup Z_Y \to Z$ is an isomorphism, where Z_X and Z_Y are defined by the following two pullback squares:

$$\begin{array}{ccc} Z_X & \longrightarrow & Z \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \hookrightarrow & X \sqcup Y \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Z_Y & \longrightarrow & Z \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y & \hookrightarrow & X \sqcup Y. \end{array}$$

Exercise 13.3. Show that in Definition 13.3.1 it suffices to require the coproduct functor on slice categories to be an equivalence for $n = 0$ and $n = 2$.

Exercise 13.4 (The span nerve in low degrees)**.** Let (C, C_L, C_R) be an adequate triple.

(1) Describe explicitly the animae $\mathrm{NSpan}(C, C_L, C_R)_n$ for $0 \leq n \leq 3$.

(2) Show that the face map

$$d_1\colon \mathrm{NSpan}(C, C_L, C_R)_2 \to \mathrm{NSpan}(C, C_L, C_R)_1$$

sends a pair of composable spans to their composite formed by pullback.

(3) For three composable spans with apexes U_{01}, U_{12}, and U_{23}, explain how an element of $\mathrm{NSpan}(C, C_L, C_R)_3$ determines the canonical isomorphism

$$(U_{01} \times_{X_1} U_{12}) \times_{X_2} U_{23} \simeq U_{01} \times_{X_1} (U_{12} \times_{X_2} U_{23})$$

that witnesses the associativity of composition.

Exercise 13.5 (Spans of finite sets as matrices)**.** Let I and J be finite sets. Given a span $I \leftarrow K \rightarrow J$, let $K_{ji} \subseteq K$ be the fiber over $(i, j) \in I \times J$.

(1) Show that there is an equivalence

$$\mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J) \simeq \prod_{(i,j)\in I\times J} \mathrm{Fin}^{\simeq}.$$

(2) Deduce that the isomorphism classes of morphisms $I \to J$ in Span(Fin) are naturally identified with matrices $(a_{ji})_{(j,i)\in J\times I}$ over $\mathbb{N}$, where $a_{ji} := |K_{ji}|$.

(3) Show that composition of spans corresponds to matrix multiplication and that identity spans correspond to identity matrices. Explain what information in the hom animae is forgotten by passing to matrices over $\mathbb{N}$.

Exercise 13.6 (Adjunctions for the two trivial span structures)**.** Let $F\colon C \rightleftarrows D : G$ be an adjunction.

(1) Equip C and D with the adequate triples $(C, C^{\simeq}, C)$ and $(D, D^{\simeq}, D)$. Show that the induced adjunction of span categories identifies with $F \dashv G$.

(2) Repeat the argument for the adequate triples $(C, C, C^{\simeq})$ and $(D, D, D^{\simeq})$. Explain why the induced adjunction is the opposite adjunction $G^{\mathrm{op}} \dashv F^{\mathrm{op}}$.

14 Basics on ∞-operads

With the construction of span categories in place, we can now develop the definition of an ∞-operad proposed in Chapter 12. Our aim is to describe algebraic structures in symmetric monoidal ∞-categories, together with the morphisms that preserve them. For this, we must relate the operadic description of operations to our earlier definition of a symmetric monoidal ∞-category as a commutative monoid in Cat_∞.

The central construction is the *multimorphism operad* $\mathcal{M}_C$ of a symmetric monoidal ∞-category C, whose multimorphism animae are the $\mathrm{Hom}_C(x_1 \otimes \cdots \otimes x_n, y)$. It turns symmetric monoidal ∞-categories into ∞-operads, and thereby supplies both of the notions we will use constantly: an *$\mathcal{O}$-algebra* in C is an operad map $\mathcal{O} \to \mathcal{M}_C$, and a *lax symmetric monoidal functor* $C \to D$ is an operad map $\mathcal{M}_C \to \mathcal{M}_D$. From this chapter onwards we make constant use of cocartesian fibrations and the straightening/unstraightening equivalence of Chapter 24.

The operadic description also allows us to construct monoidal structures on suitable subcategories and Bousfield localizations, including those used in Part I.

14.1 Definition and basic properties

We begin by recalling the three conditions obtained in Chapter 12, now using the span category constructed in Chapter 13. We then define morphisms of ∞-operads and the ∞-category they form.

Definition 14.1.1. An *∞-operad* (or: Span(Fin)-*∞-operad*) is a pair $\mathcal{O} = (\mathcal{O}^\otimes, p_\mathcal{O})$ consisting of an ∞-category $\mathcal{O}^\otimes$ equipped with a functor $p_\mathcal{O} \colon \mathcal{O}^\otimes \to \mathrm{Span}(\mathrm{Fin})$ satisfying the following conditions:

(1) The ∞-category $\mathcal{O}^\otimes$ admits finite products, and the functor $p_\mathcal{O}$ preserves finite products;

(2) For every finite set I, the product in $\mathcal{O}^\otimes$ defines an equivalence

$$\prod_{i \in I} \mathcal{O}^\otimes_{\{i\}} \xrightarrow{\sim} \mathcal{O}^\otimes_I,$$

where $\mathcal{O}^\otimes_I$ denotes the fiber of $p_\mathcal{O}$ over the set I;

(3) For a morphism $f \colon J \to I$ in Fin and objects $X_i \in \mathcal{O}^\otimes$, the map $\widetilde{f} \colon \prod_{i \in I} X_i \to \prod_{j \in J} X_{f(j)}$ whose j-th component is the projection to $X_{f(j)}$ is $p_\mathcal{O}$-cocartesian.

Here cocartesian morphisms are understood in the sense of Definition 24.1.1. The objects X_i in condition (3) are not required to be colors; this more general form will be used in Proposition 14.1.12.

If $\mathcal{P} = (\mathcal{P}^{\otimes}, p_{\mathcal{P}})$ is another ∞-operad, then a *morphism of ∞-operads* $f\colon \mathcal{O} \to \mathcal{P}$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{\quad f^{\otimes} \quad} & \mathcal{P}^{\otimes} \\ & {\scriptstyle p_{\mathcal{O}}} \searrow \quad \swarrow {\scriptstyle p_{\mathcal{P}}} & \\ & \mathrm{Span}(\mathrm{Fin}) & \end{array}$$

such that the functor $f^{\otimes}$ preserves finite products.

If we write $\mathrm{Cat}_{\infty}^{\mathrm{prod}}$ for the (very large) ∞-category of ∞-categories that admit finite products and functors that preserve finite products, we obtain a full subcategory

$$\mathrm{Op}_{\infty} \quad \subseteq \quad (\mathrm{Cat}_{\infty}^{\mathrm{prod}})_{/\mathrm{Span}(\mathrm{Fin})}$$

spanned by the ∞-operads. Given two ∞-operads $\mathcal{O}$ and $\mathcal{P}$, we will denote by $\mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{O}, \mathcal{P})$ the ∞-category of morphisms of ∞-operads from $\mathcal{O}$ to $\mathcal{P}$, defined as the following fiber:

$$\begin{array}{ccc} \mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{O}, \mathcal{P}) & \longrightarrow & \mathrm{Fun}^{\times}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) \\ \downarrow & \lrcorner & \downarrow {\scriptstyle p_{\mathcal{P}} \circ -} \\ \{p_{\mathcal{O}}\} & \hookrightarrow & \mathrm{Fun}^{\times}(\mathcal{O}^{\otimes}, \mathrm{Span}(\mathrm{Fin})). \end{array}$$

In certain contexts, morphisms of ∞-operads $\mathcal{O} \to \mathcal{P}$ are also known as *$\mathcal{O}$-algebras in $\mathcal{P}$*, and a common alternative notation for the ∞-category $\mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{O}, \mathcal{P})$ is $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$.

Example 14.1.2. The *commutative ∞-operad Comm* is given by $Comm^{\otimes} = \mathrm{Span}(\mathrm{Fin})$ and $p_{Comm} = \mathrm{id}\colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Span}(\mathrm{Fin})$. This is the terminal ∞-operad.

Example 14.1.3. The *∞-operad $\mathcal{E}_0$ of pointed objects* is given by

$$\mathcal{E}_0^{\otimes} := \mathrm{Span}(\mathrm{Fin}, \mathrm{all}, \mathrm{inj}) \qquad \text{and} \qquad p_{\mathcal{E}_0} = \mathrm{incl}\colon \mathrm{Span}(\mathrm{Fin}, \mathrm{all}, \mathrm{inj}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}).$$

The finite-product conditions in the definition of an ∞-operad follow from part (1) of Lemma 13.3.8; the fibers of $p_{\mathcal{E}_0}$ are contractible, and the backwards spans remain cocartesian under the inclusion into Span(Fin).

Observation 14.1.4. The ∞-category Op_{∞} admits finite products. The terminal object is $Comm$, and the product of ∞-operads $\mathcal{O}$ and $\mathcal{P}$ has total category

$$(\mathcal{O} \times \mathcal{P})^{\otimes} = \mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathrm{Fin})} \mathcal{P}^{\otimes}.$$

Indeed, finite products, the decomposition of the fibers, and the required cocartesian lifts are all computed componentwise in this fiber product.

Warning 14.1.5. As mentioned in Remark 12.4.3, our definition of ∞-operads does not agree with the standard one in the literature, due to Lurie, which we will discuss in Section 19.4 below. However, the difference is mostly inconsequential for practical purposes, as most constructions with Lurie's definition go through with our definition. The main difference is that with our definition we can state various conditions on the ∞-category $\mathcal{O}^{\otimes}$ in terms of finite products rather than via a Segal condition.

Terminology 14.1.6. We write $U(\mathcal{O}) := \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ for the *underlying* ∞*-category* of $\mathcal{O}$, where $\langle 1 \rangle := \{1\}$. Its objects are the colors of $\mathcal{O}$, and its morphisms are the unary operations. Restriction to this fiber defines a functor $U\colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty$. We denote the groupoid core of $U(\mathcal{O})$ by

$$\mathcal{O}^{\simeq} \quad := \quad U(\mathcal{O})^{\simeq}$$

and refer to it as the *anima of colors of* $\mathcal{O}$. Unlike the classical set of colors, this anima also remembers the invertible unary morphisms between them. For a finite set I, condition (2) guarantees that every object in $\mathcal{O}^{\otimes}_I$ may be uniquely written as a product $\prod_{i\in I} x_i$ for some colors $x_i \in \mathcal{O}^{\simeq}$. In analogy with the classical case, we may also denote such a product as an unordered tuple $\{x_i\}_{i\in I}$. Given another color $y \in \mathcal{O}^{\simeq}$, we define the *anima of multimorphisms in* $\mathcal{O}$ from $\{x_i\}_{i\in I}$ to y as the anima of morphisms in $\mathcal{O}^{\otimes}$ that map to the span $I \overset{=}{\leftarrow} I \to \langle 1 \rangle$:

$$\begin{array}{ccc} \mathcal{O}(\{x_i\}_{i\in I}; y) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, y) \\ \downarrow & \lrcorner & \downarrow p_{\mathcal{O}} \\ * & \longrightarrow & \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, \langle 1 \rangle). \end{array}$$

A morphism $\varphi\colon X \to Y$ in $\mathcal{O}^{\otimes}$ whose image in Span(Fin) is a forward map $I \overset{=}{\leftarrow} I \overset{g}{\to} J$ is called an *active morphism*. We say φ is an *inert morphism* if it is $p_{\mathcal{O}}$-cocartesian and its image in Span(Fin) is a backwards map $I \overset{f}{\leftarrow} J \overset{=}{\to} J$. We want to think of the active morphisms as the ones that encode the operad structure, and of the inert morphisms as merely capturing the product structure in $\mathcal{O}^{\otimes}$.

14.1.1 Suboperads and fully faithful morphisms

Construction 14.1.7 (Suboperads)**.** Let $\mathcal{O}$ be an ∞-operad, and choose a subanima $\mathcal{P}^{\simeq} \subseteq \mathcal{O}^{\simeq}$ of colors. For each tuple of chosen colors $\{x_i\}_{i\in I}$ and chosen color y, choose a subanima

$$\mathcal{P}(\{x_i\}_{i\in I}; y) \subseteq \mathcal{O}(\{x_i\}_{i\in I}; y).$$

Assume that these choices are invariant under bijections of the input sets and isomorphisms of the chosen colors, contain the identity operations, and are closed under substitution of multimorphisms.

These data define a *suboperad* $\mathcal{P} \subseteq \mathcal{O}$. Its total category has the tuples of chosen colors as objects. A morphism $\{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ over a span $I \overset{u}{\leftarrow} K \overset{v}{\to} J$ belongs to $\mathcal{P}^{\otimes}$ if each of its component operations belongs to

$$\mathcal{P}(\{x_{u(k)}\}_{k\in v^{-1}(j)}; y_j), \qquad j \in J.$$

Here the component operation is obtained by projecting to y_j and factoring through the cocartesian reindexing map to $\{x_{u(k)}\}_{k\in v^{-1}(j)}$. The component operations of a composite are substitutions of those of its factors. Thus the specified morphisms are closed under identities and composition, resulting in a subcategory $\mathcal{P}^{\otimes} \subseteq \mathcal{O}^{\otimes}$.

We now check the three conditions of Definition 14.1.1. Products are given by concatenation of tuples: the product projections have identity component operations, and a morphism into a concatenation belongs to $\mathcal{P}^{\otimes}$ precisely when all its projections do. The projection to Span(Fin) therefore preserves finite products. Over an identity span, the component operations are unary, giving the required equivalence $\mathcal{P}^{\otimes}_I \simeq U(\mathcal{P})^I$. Finally, the reindexing maps belong to $\mathcal{P}^{\otimes}$, since their component operations are identities. The ambient cocartesian isomorphisms identify morphisms

with the same component operations, so they restrict to $\mathcal{P}^{\otimes}$. This proves condition (3) for colors; decomposing each X_i into colors gives the general case. Hence $\mathcal{P}$ is an ∞-operad, and its inclusion preserves finite products.

If we choose all multimorphisms between the chosen colors, then $\mathcal{P}^{\otimes}$ is the full subcategory of $\mathcal{O}^{\otimes}$ on their tuples. We call $\mathcal{P}$ the *full suboperad of $\mathcal{O}$ spanned by $\mathcal{P}^{\simeq}$*.

Definition 14.1.8. A morphism of ∞-operads $f\colon \mathcal{O} \to \mathcal{P}$ is called *fully faithful* if for every finite set I and every collection of colors $\{x_i\}_{i\in I}$ and y of $\mathcal{O}$, the induced map

$$\mathcal{O}(\{x_i\}_{i\in I}; y) \to \mathcal{P}(\{f(x_i)\}_{i\in I}; f(y))$$

is an equivalence of animae.

The inclusion of a full suboperad is fully faithful. To relate this condition to full faithfulness of the functor on total categories, we describe morphisms in $\mathcal{O}^{\otimes}$ in terms of multimorphisms.

Notation 14.1.9. Given a functor $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ and a morphism $\alpha\colon I \to J$ in $\mathrm{Span}(\mathrm{Fin})$, we write

$$\mathrm{Hom}^{\alpha}_{\mathcal{O}^{\otimes}}(X,Y) \qquad := \qquad \mathrm{fib}_{\alpha}\big(\, p_{\mathcal{O}}\colon \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X,Y) \to \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J)\,\big)$$

for all $X, Y \in \mathcal{O}^{\otimes}$ with $p_{\mathcal{O}}(X) \cong I$ and $p_{\mathcal{O}}(Y) \cong J$.

Lemma 14.1.10. *Let $\mathcal{O}$ be an ∞-operad and $\alpha = (I \xleftarrow{u} K \xrightarrow{v} J)$ a span of finite sets. Taking products of multimorphisms and precomposing with the inert map $\{x_i\}_{i\in I} \to \{x_{u(k)}\}_{k\in K}$ gives an isomorphism of animae*

$$\prod_{j\in J} \mathcal{O}(\{x_{u(k)}\}_{k\in v^{-1}(j)}; y_j) \xrightarrow{\sim} \mathrm{Hom}^{\alpha}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}),$$

natural in $\mathcal{O}$, for all families of colors $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$.

Proof. Factor α as the backwards span $I \xleftarrow{u} K \xrightarrow{=} K$ followed by the active span $\gamma = (K \xleftarrow{=} K \xrightarrow{v} J)$. By condition (3) of Definition 14.1.1 and preservation of products by $p_{\mathcal{O}}$, we obtain

$$\begin{aligned}
\mathrm{Hom}^{\alpha}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) &\simeq \mathrm{Hom}^{\gamma}_{\mathcal{O}^{\otimes}}(\{x_{u(k)}\}_{k\in K}, \{y_j\}_{j\in J}) \\
&\simeq \prod_{j\in J} \mathrm{Hom}^{\rho_j\circ\gamma}_{\mathcal{O}^{\otimes}}(\{x_{u(k)}\}_{k\in K}, y_j) \\
&\simeq \prod_{j\in J} \mathcal{O}(\{x_{u(k)}\}_{k\in v^{-1}(j)}; y_j),
\end{aligned}$$

where $\rho_j\colon J \to \{j\}$ is the product projection in $\mathrm{Span}(\mathrm{Fin})$. The last isomorphism uses condition (3) again, applied to $v^{-1}(j) \hookrightarrow K$. The inverse of this composite is the map in the statement. Naturality follows because morphisms of ∞-operads preserve products and their projections. □

It follows that a morphism of ∞-operads $f\colon \mathcal{O} \to \mathcal{P}$ is fully faithful if and only if $f^{\otimes}$ is fully faithful. Indeed, the maps on hom animae of the total categories lie over the same hom anima in $\mathrm{Span}(\mathrm{Fin})$, and Lemma 14.1.10 identifies their maps on fibers with products of the maps on multimorphism animae. Conversely, full faithfulness of $f^{\otimes}$ implies full faithfulness on multimorphisms by taking fibers over active spans with singleton target.

Lemma 14.1.11. *Let $f\colon \mathcal{O} \to \mathcal{P}$ be a morphism of ∞-operads. If f is fully faithful and the induced functor $U(\mathcal{O}) \to U(\mathcal{P})$ is essentially surjective, then f is an equivalence of ∞-operads.*

Proof. By the preceding observation, $f^{\otimes}$ is fully faithful. Every object of $\mathcal{P}^{\otimes}$ is a finite product of colors, each of which is in the image of f by assumption. Since $f^{\otimes}$ preserves finite products, it is essentially surjective as well. Thus $f^{\otimes}$ is an equivalence over Span(Fin), and hence f is an equivalence of ∞-operads. □

14.1.2 An alternative characterization of ∞-operads

To recognize an ∞-operad, it is often easier to construct cocartesian lifts than to establish the finite-product conditions directly. The following characterization allows us to do this.

Proposition 14.1.12. *Consider a functor $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$. Then $(\mathcal{O}^{\otimes}, p_{\mathcal{O}})$ is an ∞-operad if and only if the following conditions are satisfied:*

(i) $\mathcal{O}^{\otimes}$ admits all $p_{\mathcal{O}}$-cocartesian lifts of backwards morphisms $I \xleftarrow{f} J \xrightarrow{=} J$ in Span(Fin)*;*

(ii) For every finite set I, the functor

$$\mathcal{O}^{\otimes}_I \to \prod_{i \in I} \mathcal{O}^{\otimes}_{\{i\}}$$

given by cocartesian transport along the maps $\rho_i\colon I \hookleftarrow \{i\} \xrightarrow{=} \{i\}$ in Span(Fin) *is an equivalence, where the transport functors are supplied by Proposition 24.1.4;*

(iii) Consider an object $Y \in \mathcal{O}^{\otimes}$ with image $J := p_{\mathcal{O}}(Y) \in$ Span(Fin) and cocartesian lifts $Y \to Y_j$ over the maps ρ_j in Span(Fin). *Then the maps $Y \to Y_j$ exhibit Y as a product $\prod_{j \in J} Y_j$ in $\mathcal{O}^{\otimes}$.*

Proof. First suppose that $\mathcal{O}$ is an ∞-operad. Every object is a product of colors, and condition (3) of Definition 14.1.1 says that reindexing such a product gives a cocartesian lift of the corresponding backwards span. This proves (i). In particular, the product projections are cocartesian lifts of the ρ_i. Thus cocartesian transport along the ρ_i is inverse to the product functor, proving (ii), and uniqueness of cocartesian lifts gives (iii).

Conversely, assume (i)–(iii). By (ii), any family of colors $\{x_i\}_{i \in I}$ is obtained by cocartesian transport from an object over I. By (iii), the resulting maps exhibit this object as $\prod_{i \in I} x_i$. Thus the inverse of the equivalence in (ii) is the product functor, giving condition (2) of the definition.

For arbitrary objects $Y^1, \dots, Y^n$, write $T_i := p_{\mathcal{O}}(Y^i)$ and decompose each Y^i into its color factors Y^i_t. Assembling all these colors gives an object Y over $\coprod_i T_i$ which, by associativity of products, is a product of the Y^i. Cocartesian transport of Y along the backwards inclusion $\coprod_i T_i \hookleftarrow T_i$ has the same color factors as Y^i, since cocartesian morphisms compose. By (ii), its target identifies with Y^i, and by (iii) the resulting map $Y \to Y^i$ is the product projection. Its image under $p_{\mathcal{O}}$ is therefore the corresponding product projection in Span(Fin). This proves that finite products exist and are preserved by $p_{\mathcal{O}}$, giving condition (1).

For condition (3), first consider colors x_i and a map $f\colon J \to I$. Choose a cocartesian lift $\prod_{i \in I} x_i \to Y$ of the backwards span associated with f, followed by cocartesian lifts $Y \to Y_j$ of the projections ρ_j. Their composites lift the projections selecting $x_{f(j)}$, so uniqueness of cocartesian lifts identifies them with the product projections $\prod_{i \in I} x_i \to x_{f(j)}$. By (iii), the maps $Y \to Y_j$ exhibit Y as $\prod_{j \in J} x_{f(j)}$.

Hence the projection-induced reindexing map is cocartesian. For arbitrary X_i, write $X_i \simeq \prod_{t \in T_i} x_{i,t}$ with $T_i = p_{\mathcal{O}}(X_i)$ and apply the case of colors to

$$\coprod_{j \in J} T_{f(j)} \longrightarrow \coprod_{i \in I} T_i, \qquad (j,t) \longmapsto (f(j),t).$$

Under the product decompositions, this gives precisely the required map $\prod_{i \in I} X_i \to \prod_{j \in J} X_{f(j)}$. □

It is sometimes convenient to use the notation from Notation 14.1.9 to give the following alternative formulation of part (iii).

Lemma 14.1.13. *Given a functor $p_{\mathcal{O}} \colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ and cocartesian morphisms $Y \to Y_j$ over the maps ρ_j, these maps exhibit Y as a product of the Y_j in $\mathcal{O}^{\otimes}$ if and only if for every $X \in \mathcal{O}^{\otimes}_I$ and every $\alpha \in \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J)$ they induce an equivalence*

$$\mathrm{Hom}^{\alpha}_{\mathcal{O}^{\otimes}}(X,Y) \to \prod_{j \in J} \mathrm{Hom}^{\rho_j \circ \alpha}_{\mathcal{O}^{\otimes}}(X,Y_j).$$

Proof. The maps $Y \to Y_j$ exhibit Y as a product if and only if for every $X \in \mathcal{O}^{\otimes}_I$ the top map in the following square is an equivalence:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X,Y) & \longrightarrow & \prod_{j \in J} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X,Y_j) \\ {\scriptstyle p_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle p_{\mathcal{O}}} \\ \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J) & \longrightarrow & \prod_{j \in J} \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,\{j\}). \end{array}$$

Since the bottom map is an equivalence, this is equivalent to the square being a pullback square, which is in turn equivalent to the condition that the top map induces isomorphisms on all fibers over $\alpha \in \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J)$. □

Corollary 14.1.14. *If $\mathcal{O}$ is an ∞-operad, then the inert morphisms in $\mathcal{O}^{\otimes}$ are precisely the morphisms of the form $\widetilde{f} \colon \prod_{i \in I} x_i \to \prod_{j \in J} x_{f(j)}$ for maps $f \colon J \to I$ in* Fin *and colors $x_i \in \mathcal{O}^{\simeq}$.*

Similarly, if $\mathcal{P}$ is another ∞-operad, and $\varphi^{\otimes} \colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a functor over Span(Fin)*, then $\varphi^{\otimes}$ preserves finite products (i.e., corresponds to a morphism $\varphi \colon \mathcal{O} \to \mathcal{P}$ of ∞-operads) if and only if it preserves the inert morphisms.*

Proof. By assumption $\widetilde{f}$ is inert for every f. Conversely, we saw in the proof of Proposition 14.1.12 that for every backwards span $I \xleftarrow{f} J \xrightarrow{=} J$ and every $X \in \mathcal{O}^{\otimes}_I$ the map $\widetilde{f} \colon X \simeq \prod_{i \in I} X_i \to \prod_{j \in J} X_{f(j)}$ is a cocartesian lift of f starting in X, so by uniqueness it follows that every cocartesian lift is of this form.

If $\varphi^{\otimes}$ preserves finite products, it in particular preserves the maps $\widetilde{f} \colon \prod_{i \in I} X_i \to \prod_{j \in J} X_{f(j)}$, hence all inert morphisms. Conversely, if $\varphi^{\otimes}$ preserves inert morphisms, it sends the projections from a tuple of colors to cocartesian lifts of the same spans ρ_j. By condition (iii) of Proposition 14.1.12, these exhibit the image of the tuple as the product of the images of its colors. Since finite products are given by concatenation of tuples, $\varphi^{\otimes}$ preserves them. □

Remark 14.1.15. Let $O = (O^{\otimes}, p_O)$ be an ∞-operad. Then every morphism $\varphi\colon X \to Z$ in $O^{\otimes}$ factors uniquely as

$$X \xrightarrow{\widetilde{f}} Y \xrightarrow{\psi} Z,$$

where $\widetilde{f}$ is an inert morphism and ψ is an active morphism. To see this, let us denote by $I \xleftarrow{f} K \xrightarrow{g} J$ the image of φ under p. This span may be factored uniquely as a composite of a backwards span $I \xleftarrow{f} K \xrightarrow{=} K$ followed by a forward span $K \xleftarrow{=} K \xrightarrow{g} J$. We then let $\widetilde{f}\colon X \to Y$ be the unique cocartesian lift of f with domain X. It follows that there is a unique morphism $\psi\colon Y \to Z$ over g such that $\psi \circ \widetilde{f} = \varphi$.

The argument above produces a factorization and shows that any two of them are isomorphic. With more care, it in fact shows that the factorization is unique, in the sense that the *anima of factorizations* of φ is contractible. We will deduce this in Section 19.1 from the general theory of *factorization systems* on an ∞-category, where we show that the inert and active morphisms define such a system on $O^{\otimes}$.

14.2 Classical colored operads

We can now make the construction of Construction 12.4.1 precise. The idea is to repeat the construction of the span category, adding the operation labels to its pullback diagrams. The resulting simplicial anima is Segal, but need not be complete. The associated category construction from Construction 1.8.12 supplies the desired ∞-category without changing its hom animae.

Construction 14.2.1 (Labeled pullback diagrams)**.** Let O be a classical colored operad with sets of operations. We use the finite-set-indexed operations and their transport along bijections from Convention 12.3.2. We first construct a simplicial object in the strict category of groupoids and functors.

For $n \geq 0$, define a groupoid $N_{\mathrm{span}}(O)_n$ as follows. An object consists of:

(1) A diagram $S\colon \mathrm{Tw}^r([n]) \to \mathrm{Fin}$ belonging to $\mathrm{NSpan}(\mathrm{Fin})_n$, as in Construction 13.1.10. Write $S_{ij} = S(i \leq j)$. The induced maps $S_{ik} \to S_{ij} \times_{S_{jj}} S_{jk}$ are bijections for $i \leq j \leq k$. The structure maps of S give spans

$$S_{ii} \xleftarrow{f_{ij}} S_{ij} \xrightarrow{g_{ij}} S_{jj}.$$

(2) For each i, a family $\{x^i_s\}_{s \in S_{ii}}$ of colors of O.

(3) For every $i \leq j$ and $t \in S_{jj}$, an operation

$$\varphi_{ij,t} \in O\big(\{x^i_{f_{ij}(u)}\}_{u \in g_{ij}^{-1}(t)}; x^j_t\big).$$

For $i = j$ this is required to be $\mathrm{id}_{x^i_t}$. For $i \leq j \leq k$ and $t \in S_{kk}$, we require that $\varphi_{ik,t}$ agrees with the composite operation

$$\varphi_{jk,t} \circ \big(\varphi_{ij,f_{jk}(v)}\big)_{v \in g_{jk}^{-1}(t)}$$

under the bijection $\{(u,v) \in S_{ij} \times_{S_{jj}} S_{jk} \mid g_{jk}(v) = t\} \cong \{w \in S_{ik} \mid g_{ik}(w) = t\}$.

A morphism is a natural isomorphism of the finite-set diagrams that preserves the diagonal colors and carries each operation label to the corresponding label by transport of its input set. In particular, it does not include arbitrary invertible unary operations between the diagonal colors.

For a monotone map $\alpha\colon [m] \to [n]$, define $\alpha^*\colon N_{\mathrm{span}}(O)_n \to N_{\mathrm{span}}(O)_m$ by restriction along $\mathrm{Tw}^r(\alpha)$, with colors $x^{\alpha(i)}$ and labels $\varphi_{\alpha(i)\alpha(j),t}$. Pullback squares remain pullback squares under this restriction, and the two conditions in (3) remain valid. Note that we have strict equalities $(\alpha\beta)^* = \beta^*\alpha^*$ and $\mathrm{id}^* = \mathrm{id}$, so that we obtain a strict functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the strict category of groupoids. Composing with the functor that regards groupoids as animae gives a simplicial anima, which we also denote by $N_{\mathrm{span}}(O)$, and a forgetful map $N_{\mathrm{span}}(O) \to \mathrm{NSpan}(\mathrm{Fin})$. A morphism of classical operads acts on colors and operation labels, commuting with restriction and forgetting labels. Thus $N_{\mathrm{span}}(O)$ is functorial in O, and the forgetful map is natural.

Lemma 14.2.2. *The simplicial anima $N_{\mathrm{span}}(O)$ is Segal. Its hom anima from $X = \{x_i\}_{i\in I}$ to $Y = \{y_j\}_{j\in J}$ is the groupoid of spans $I \xleftarrow{f} K \xrightarrow{g} J$ labeled by operations*

$$\varphi_j \in O(\{x_{f(k)}\}_{k\in g^{-1}(j)}; y_j).$$

Morphisms in this groupoid are bijections of K over I and J that transport the labels to one another.

Proof. The forgetful map gives a commutative square of Segal maps, with n factors on the right:

$$\begin{array}{ccc} N_{\mathrm{span}}(O)_n & \longrightarrow & N_{\mathrm{span}}(O)_1 \times_{N_{\mathrm{span}}(O)_0} \cdots \times_{N_{\mathrm{span}}(O)_0} N_{\mathrm{span}}(O)_1 \\ \downarrow & & \downarrow \\ \mathrm{NSpan}(\mathrm{Fin})_n & \xrightarrow{\simeq} & \mathrm{NSpan}(\mathrm{Fin})_1 \times_{\mathrm{NSpan}(\mathrm{Fin})_0} \cdots \times_{\mathrm{NSpan}(\mathrm{Fin})_0} \mathrm{NSpan}(\mathrm{Fin})_1. \end{array}$$

The bottom map is an isomorphism by Proposition 13.1.11, so it suffices to check that the induced maps on fibers are isomorphisms. For fixed $S \in \mathrm{NSpan}(\mathrm{Fin})_n$, the left fiber consists of families of colors indexed by each finite set S_{ii} for $0 \le i \le n$, and compatible operation labels on all S_{ij}. Since fibers commute with fiber products, the right fiber consists of the same color assignments on each S_{ii}, and labels only on the consecutive spans $S_{i-1,i-1} \leftarrow S_{i-1,i} \to S_{ii}$. The induced map that forgets the remaining operations is a bijectio: the diagonal labels are identities, and under the induced bijections

$$S_{ij} \xrightarrow{\sim} S_{i,i+1} \times_{S_{i+1,i+1}} \cdots \times_{S_{j-1,j-1}} S_{j-1,j}$$

we recover the labels on S_{ij} by iterated substitution. This proves the Segal condition.

Finally, the hom anima from X to Y is the fiber of $N_{\mathrm{span}}(O)_1 \to N_{\mathrm{span}}(O)_0 \times N_{\mathrm{span}}(O)_0$ at (X, Y), which we may identify with the groupoid of labeled spans from X to Y, as stated. □

Construction 14.2.3 (The associated category)**.** Set $O^{\otimes} := \mathrm{ac}(N_{\mathrm{span}}(O))$. Applying ac to the forgetful map gives a functor

$$p_O\colon O^{\otimes} \longrightarrow \mathrm{ac}(\mathrm{NSpan}(\mathrm{Fin})) \simeq \mathrm{Span}(\mathrm{Fin}),$$

where the last equivalence is supplied by Proposition 13.1.11 and Theorem 25.1.10. By Proposition 25.1.9, every object of $O^{\otimes}$ is isomorphic to a tuple, and its hom animae between tuples are those of Lemma 14.2.2. Naturality of the completion map identifies p_O on these hom animae with forgetting the labels. Composition is given by pullback of spans followed by substitution on labels. All in all, $O^{\otimes}$ is the $(2,1)$-category described in Construction 12.4.1.

Proposition 14.2.4. *The construction $\mathcal{O} \mapsto (\mathcal{O}^{\otimes}, p_{\mathcal{O}})$ defines a functor from classical colored operads to ∞-operads. The underlying ∞-category of the associated ∞-operad is naturally equivalent to the classical category $U(\mathcal{O})$ of colors and unary operations. For a span $\alpha = (I \xleftarrow{f} K \xrightarrow{g} J)$ and families of colors $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$, there is an isomorphism*

$$\mathrm{Hom}^{\alpha}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) \quad \simeq \quad \prod_{j\in J} \mathcal{O}(\{x_{f(k)}\}_{k\in g^{-1}(j)}; y_j).$$

In particular, its multimorphism animae are the classical operation sets, regarded as discrete animae.

Proof. By Lemma 14.2.2, the map

$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) \to \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, J)$$

on hom animae forgets the operation labels on a span. Its fiber over α is the displayed product of operation sets, since there is a unique way to transport labels along a bijection of spans.

We show that concatenation gives binary products in $\mathcal{O}^{\otimes}$, with projections given by the backwards spans for the summand inclusions labeled by the identity. To see this, consider tuples $\{x_i\}_{i\in I}$, $\{y_j\}_{j\in J}$ and $\{z_k\}_{k\in K}$. The projections give a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}, \{y_j\} \sqcup \{z_k\}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}, \{y_j\}) \times \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}, \{z_k\}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, J \sqcup K) & \xrightarrow[\simeq]{} & \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, J) \times \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, K), \end{array}$$

in which the bottom map is an isomorphism because disjoint union is the product in Span(Fin). From the computation of the first paragraph, it is also clear that the induced map on fibers is a bijection showing that $\{y_j\} \sqcup \{z_k\}$ is a product in $\mathcal{O}^{\otimes}$ of $\{y_j\}$ and $\{z_k\}$. In analogous fashion one shows that the empty tuple is the terminal object, so that $\mathcal{O}^{\otimes}$ admits finite products. It is also clear from these computations that $p_{\mathcal{O}}$ preserves finite products.

Next we identify fiber $\mathcal{O}^{\otimes}_I$ over a finite set I with the product $U(\mathcal{O})^I$. There is a map of simplicial animae $N(U(\mathcal{O})^I) \to N_{\mathrm{span}}(\mathcal{O})$ given in degree n by sending a string of n composable morphisms in $U(\mathcal{O})^I$ to the constant pullback diagram $\mathrm{const}_I \colon \mathrm{Tw}^r([n]) \to \mathrm{Fin}$ labeled by the composites of the given unary operations. Since the underlying pullback diagram is constant on I, passing to associated ∞-categories results in a functor

$$U(\mathcal{O})^I \longrightarrow \mathcal{O}^{\otimes}_I.$$

This functor is clearly essentially surjective, and it is fully faithful by the earlier identification of the hom animae for $\alpha = \mathrm{id}_I$. We conclude it is an equivalence.

For $I = \langle 1 \rangle$ this identifies the underlying category as claimed. Under these equivalences, concatenation is the equivalence required in condition (2) of Definition 14.1.1.

Next we verify the full cocartesian universal property. Let $u \colon K \to I$ be a map in Fin, let $\{x_i\}_{i\in I}$ be a family of colors, and let $\widehat{b} \colon \{x_i\}_{i\in I} \to \{x_{u(k)}\}_{k\in K}$ be the lift of $b = (I \xleftarrow{u} K \xrightarrow{=} K)$ whose labels are identities. For any tuple $Y = \{y_j\}_{j\in J}$, consider the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_{u(k)}\}_{k\in K}, Y) & \xrightarrow{-\circ\widehat{b}} & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Span(Fin)}}(K, J) & \xrightarrow{-\circ b} & \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, J). \end{array}$$

For a span $\beta = (K \xleftarrow{a} L \xrightarrow{h} J)$, the induced map from the fiber over β to the fiber over $\beta \circ b$ is identified with the identity on

$$\prod_{j \in J} O(\{x_{u(a(l))}\}_{l \in h^{-1}(j)}; y_j),$$

since composition with $\widehat{b}$ leaves the operation labels unchanged by the unit axiom. Thus the square is a pullback, showing that $\widehat{b}$ is p_O-cocartesian.

Finally, let $f\colon B \to A$ and let X_a be arbitrary objects of $O^{\otimes}$, represented by tuples indexed by T_a. The map in condition (3) of Definition 14.1.1 is the identity-labeled backwards span associated with

$$\coprod_{b \in B} T_{f(b)} \longrightarrow \coprod_{a \in A} T_a, \qquad (b, t) \longmapsto (f(b), t).$$

The preceding argument proves it cocartesian.

The functor induced by a morphism of classical operads preserves concatenation, its projections, and the empty tuple, hence finite products. □

Remark 14.2.5. The simplicial anima $N_{\mathrm{span}}(O)$ need not be complete: non-identity invertible unary operations are not represented by identifications in $N_{\mathrm{span}}(O)_0$. The associated category construction producing $O^{\otimes}$ adds these identifications without changing the hom animae.

Further note that the hom animae of $O^{\otimes}$ need not be discrete. For example, $N_{\mathrm{span}}(\mathcal{C}omm) = \mathrm{NSpan}(\mathrm{Fin})$, so $\mathcal{C}omm^{\otimes} \simeq \mathrm{Span}(\mathrm{Fin})$, whose hom groupoids include the automorphisms of spans.

Corollary 14.2.6. *The classical colored operads $\mathcal{A}ssoc$, $\mathcal{LM}od$, and $\mathcal{M}od$ determine ∞-operads whose multimorphism animae are their classical operation sets. The same construction applies to the bimodule operad of Definition 17.2.1.*

Proof. Apply Proposition 14.2.4. □

Remark 14.2.7. By Corollary 14.2.6, the classical associative operad determines the associative ∞-operad $\mathcal{A}ssoc$. The topological little n-cubes operad of Example 12.1.7 similarly determines an ∞-operad $\mathcal{E}_n$; see [Lur17, Section 5.1.0] for the construction. For $n = 1$, the operad morphism $\mathcal{E}_1 \to \mathcal{A}ssoc$ described after Example 12.1.7 is an equivalence of ∞-operads by [Lur17, Example 5.1.0.7].

14.3 Finite families and trivial operads

The simplest ∞-operads are those without any operations beyond the unary ones, generalizing the trivial colored operads of Example 12.2.6. Given an ∞-category C, we will construct an ∞-operad $\mathcal{T}riv_C$ with underlying ∞-category C and multimorphism animae

$$\mathcal{T}riv_C(\{x_i\}_{i \in I}; y) \quad \simeq \quad \begin{cases} \mathrm{Hom}_C(x_{i_0}, y) & \text{if } I = \{i_0\} \text{ is a singleton;} \\ \emptyset & \text{otherwise,} \end{cases}$$

with composition given by composition in C. There is no operadic structure left to specify: operations of arity $\neq 1$ do not exist, and the unary operations are simply the morphisms of C. To construct $\mathcal{T}riv_C$, note that multimorphisms from $\{x_i\}_{i \in I}$ to y live over the active span $I \xleftarrow{=} I \to \langle 1 \rangle$, which

lies in the subcategory $\mathrm{Fin}^{\mathrm{op}} \subseteq \mathrm{Span}(\mathrm{Fin})$ precisely when I is a singleton. We should therefore look for an ∞-operad whose structure map factors through the inclusion $\mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$.

The construction is based on the following ∞-category of finite families, which will reappear in Section 15.2 as the basis for cocartesian monoidal structures.

Construction 14.3.1 (Finite families)**.** Let C be an ∞-category. There is a unique finite-product-preserving functor

$$C^{(-)} \colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

which sends the one-point set to C; explicitly, it sends a finite set I to the ∞-category C^I of I-indexed families and a map $f \colon I \to J$ to the restriction functor $f^* \colon C^J \to C^I$. Equivalently, this functor is the right Kan extension of C from the one-point set. We write

$$q \colon \mathrm{Fin}(C) \to \mathrm{Fin}$$

for its cartesian unstraightening. Thus objects of $\mathrm{Fin}(C)$ are finite unordered tuples $\{x_i\}_{i \in I}$, while morphisms $\{x_i\}_{i \in I} \to \{y_j\}_{j \in J}$ are pairs $(f, (\varphi_i)_{i \in I})$ consisting of a map $f \colon I \to J$ and morphisms $\varphi_i \colon x_i \to y_{f(i)}$ in C for all $i \in I$.

We record the following basic observations about the ∞-category of finite families:

- The fiber of q over $\langle 1 \rangle$ is equivalent to C, and the resulting inclusion $i \colon C \hookrightarrow \mathrm{Fin}(C)$ is fully faithful.
- A morphism $(f, (\varphi_i)_{i \in I}) \colon \{x_i\}_{i \in I} \to \{y_j\}_{j \in J}$ is q-cartesian if and only if each φ_i is an isomorphism in C. We denote by

  $$\mathrm{Fin}(C)_{\mathrm{ct}} \subseteq \mathrm{Fin}(C)$$

  the wide subcategory spanned by the q-cartesian morphisms.
- The ∞-category $\mathrm{Fin}(C)$ admits finite coproducts, given by concatenation of unordered tuples, and the functor $q \colon \mathrm{Fin}(C) \to \mathrm{Fin}$ preserves finite coproducts. More precisely, given a finite collection of objects $X_a \in \mathrm{Fin}(C)$ for $a \in A$, we may uniquely write each X_a as $\{x_j^a\}_{j \in J_a}$ for some finite set J_a, and then the coproduct is given by $X = \{x_j^a\}_{(a,j) \in \coprod_{a \in A} J_a}$, with the maps $X_a \to X$ given by the q-cartesian morphisms over the inclusions $J_a \hookrightarrow \coprod_{a \in A} J_a$. This follows from Proposition 24.2.10.

Together with the fully faithful inclusion i, these coproducts exhibit $\mathrm{Fin}(C)$ as the free finite-coproduct completion of C:

Lemma 14.3.2. *Let C be an ∞-category. For every ∞-category D with finite coproducts, restriction along the fully faithful inclusion $i \colon C \hookrightarrow \mathrm{Fin}(C)$ induces an equivalence*

$$i^* \colon \mathrm{Fun}^{\amalg}(\mathrm{Fin}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D),$$

where $\mathrm{Fun}^{\amalg}(-,-)$ denotes the full subcategory of functors preserving finite coproducts.

Proof. For a finite family $\{x_k\}_{k\in K}$, its relative slice under i is

$$C \times_{\mathrm{Fin}(C)} \mathrm{Fin}(C)_{/\{x_k\}_{k\in K}} \simeq \coprod_{k\in K} C_{/x_k}.$$

Each $C_{/x_k}$ has terminal object id_{x_k}. Thus every functor $F\colon C \to D$ admits a left Kan extension along i, given by

$$(i_!F)(\{x_k\}_{k\in K}) \simeq \coprod_{k\in K} F(x_k).$$

This formula shows that $i_!F$ preserves finite coproducts. The unit of $i_! \dashv i^*$ is an isomorphism because i is fully faithful. For a finite-coproduct-preserving functor G, the counit is the canonical isomorphism $\coprod_{k\in K} G(x_k) \xrightarrow{\sim} G(\{x_k\}_{k\in K})$. Hence $i_!$ and i^* are inverse equivalences. □

Construction 14.3.3 (Trivial operads)**.** Let C be an ∞-category. Applying Construction 14.3.1 to C^{op} gives a cartesian fibration $q\colon \mathrm{Fin}(C^{\mathrm{op}}) \to \mathrm{Fin}$; passing to opposite categories, we obtain a functor $q^{\mathrm{op}}\colon \mathrm{Fin}(C^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Fin}^{\mathrm{op}}$. We define

$$\mathcal{T}\mathit{riv}_C^{\otimes} := \mathrm{Fin}(C^{\mathrm{op}})^{\mathrm{op}}, \qquad p_{\mathcal{T}\mathit{riv}_C}\colon \mathrm{Fin}(C^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{q^{\mathrm{op}}} \mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin}),$$

and refer to $\mathcal{T}\mathit{riv}_C$ as the *trivial operad generated by* C. Unwinding Construction 14.3.1, the objects of $\mathcal{T}\mathit{riv}_C^{\otimes}$ are finite unordered tuples $\{x_i\}_{i\in I}$ of objects of C, while a morphism $\{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ lying over the backwards span $I \xleftarrow{g} J \xrightarrow{=} J$ is a collection of morphisms $x_{g(j)} \to y_j$ in C. The opposite in the definition ensures that the underlying ∞-category is C itself, and not C^{op}.

Remark 14.3.4. Equivalently, $q^{\mathrm{op}}\colon \mathcal{T}\mathit{riv}_C^{\otimes} \to \mathrm{Fin}^{\mathrm{op}}$ is the *cocartesian* unstraightening of the functor $C^{(-)}$ from Construction 14.3.1: it is a cocartesian fibration and its cocartesian straightening $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ preserves finite products and sends the point to C.

Lemma 14.3.5. *Let C be an ∞-category. Then $\mathcal{T}\mathit{riv}_C$ is an ∞-operad with underlying ∞-category $U(\mathcal{T}\mathit{riv}_C) \simeq C$, and its multimorphism animae are the ones described at the beginning of this section. Moreover, for every ∞-operad $\mathcal{O}$, restriction to the underlying ∞-categories induces an equivalence*

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}\mathit{riv}_C, \mathcal{O}) \xrightarrow{\sim} \mathrm{Fun}(C, U(\mathcal{O})).$$

Proof. Let us first check that $\mathcal{T}\mathit{riv}_C$ is an ∞-operad. By Construction 14.3.1, the ∞-category $\mathrm{Fin}(C^{\mathrm{op}})$ admits finite coproducts and q preserves them, so $\mathcal{T}\mathit{riv}_C^{\otimes} = \mathrm{Fin}(C^{\mathrm{op}})^{\mathrm{op}}$ admits finite products and q^{op} preserves them; as the inclusion $\mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ preserves finite products by part (1) of Lemma 13.3.8, the same is true for $p_{\mathcal{T}\mathit{riv}_C}$. Since the products of $\mathcal{T}\mathit{riv}_C^{\otimes}$ are the concatenations of tuples, the fiber over a finite set I is C^I, whose decomposition $C^I \simeq \prod_{i\in I} C$ is the one required in the definition of an ∞-operad.

For the final condition, consider objects $X_i = \{x_a^i\}_{a\in A_i}$ of $\mathcal{T}\mathit{riv}_C^{\otimes}$ for $i \in I$ and a map $f\colon J \to I$ of finite sets. Read in $\mathrm{Fin}(C^{\mathrm{op}})$, the morphism $\widetilde{f}\colon \prod_{i\in I} X_i \to \prod_{j\in J} X_{f(j)}$ is the map $\coprod_{j\in J} X_{f(j)} \to \coprod_{i\in I} X_i$ which restricts on the j-th summand to the inclusion of $X_{f(j)}$; it lies over the map $\bigsqcup_{j\in J} A_{f(j)} \to \bigsqcup_{i\in I} A_i$, $(j,a) \mapsto (f(j),a)$, and all of its components in C^{op} are identities. It is therefore q-cartesian, and hence $\widetilde{f}$ is q^{op}-cocartesian. It remains $p_{\mathcal{T}\mathit{riv}_C}$-cocartesian after composing with the inclusion $\mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$: every morphism of $\mathcal{T}\mathit{riv}_C^{\otimes}$ lies over a backwards span, and composing a non-backwards span with a backwards span is again non-backwards, so the mapping animae over the remaining spans are empty on both sides.

The description of the multimorphism animae follows: the active span $I \xleftarrow{=} I \to \langle 1 \rangle$ lies in $\mathrm{Fin}^{\mathrm{op}}$ only when I is a singleton, in which case it is the identity of $\langle 1 \rangle$. Thus $\mathcal{T}\mathit{riv}_C(\{x_i\}_{i \in I}; y)$ is empty unless $I = \{i_0\}$, in which case it is the anima of morphisms $x_{i_0} \to y$ in the fiber $(\mathcal{T}\mathit{riv}_C)^{\otimes}_{\langle 1 \rangle} \simeq C$.

We turn to the universal property, which will follow from the universal property of $\mathrm{Fin}(C^{\mathrm{op}})$. Indeed, applying Lemma 14.3.2 to C^{op} and passing to opposite categories, we see that restriction along the inclusion $j \colon C \hookrightarrow \mathcal{T}\mathit{riv}_C^{\otimes}$ of the fiber over $\langle 1 \rangle$ induces an equivalence

$$j^* \colon \mathrm{Fun}^{\times}(\mathcal{T}\mathit{riv}_C^{\otimes}, E) \xrightarrow{\sim} \mathrm{Fun}(C, E)$$

for every ∞-category E that admits finite products. This equivalence is natural in E, so applying it to $E = O^{\otimes}$ and to $E = \mathrm{Span}(\mathrm{Fin})$ yields a commutative square

$$\begin{array}{ccc} \mathrm{Fun}^{\times}(\mathcal{T}\mathit{riv}_C^{\otimes}, O^{\otimes}) & \xrightarrow[\simeq]{j^*} & \mathrm{Fun}(C, O^{\otimes}) \\ {\scriptstyle p_O \circ -}\downarrow & & \downarrow{\scriptstyle p_O \circ -} \\ \mathrm{Fun}^{\times}(\mathcal{T}\mathit{riv}_C^{\otimes}, \mathrm{Span}(\mathrm{Fin})) & \xrightarrow[\simeq]{j^*} & \mathrm{Fun}(C, \mathrm{Span}(\mathrm{Fin})). \end{array}$$

The composite $p_{\mathcal{T}\mathit{riv}_C} \circ j$ is the constant functor at $\langle 1 \rangle$, so passing to the fibers over $p_{\mathcal{T}\mathit{riv}_C}$ and $\mathrm{const}_{\langle 1 \rangle}$ gives equivalences

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}\mathit{riv}_C, O) \xrightarrow{\sim} \mathrm{Fun}(C, O^{\otimes}) \times_{\mathrm{Fun}(C, \mathrm{Span}(\mathrm{Fin}))} \{\mathrm{const}_{\langle 1 \rangle}\} \simeq \mathrm{Fun}(C, U(O)),$$

where the second equivalence holds because $\mathrm{Fun}(C, -)$ preserves pullbacks. By construction, this equivalence is given by restriction to the underlying ∞-categories. □

Note that the construction $C \mapsto \mathcal{T}\mathit{riv}_C$ is functorial in C and thus defines a functor $\mathcal{T}\mathit{riv}_{(-)} \colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty$. It may be characterized as follows:

Corollary 14.3.6. *The functor $\mathcal{T}\mathit{riv}_{(-)} \colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty$ is left adjoint to the functor*

$$U \colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty, \qquad O \mapsto U(O),$$

and it is fully faithful.

Proof. The equivalence of Lemma 14.3.5 is natural in both variables. Passing to groupoid cores gives the adjunction, whose unit is the isomorphism $C \xrightarrow{\sim} U(\mathcal{T}\mathit{riv}_C)$. Thus $\mathcal{T}\mathit{riv}_{(-)}$ is fully faithful. □

Example 14.3.7. Taking $C = *$ recovers the *trivial operad*

$$\mathcal{T}\mathit{riv} := \mathcal{T}\mathit{riv}_*, \qquad \mathcal{T}\mathit{riv}^{\otimes} = \mathrm{Fin}^{\mathrm{op}}, \qquad p_{\mathcal{T}\mathit{riv}} = \mathrm{incl} \colon \mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin}).$$

Taking $C = \emptyset$ gives the *empty ∞-operad* $\mathcal{T}\mathit{riv}_\emptyset$, with $\mathcal{T}\mathit{riv}_\emptyset^{\otimes} = *$ and $p_{\mathcal{T}\mathit{riv}_\emptyset} \colon * \xhookrightarrow{\langle 0 \rangle} \mathrm{Span}(\mathrm{Fin})$ the inclusion of the empty set $\langle 0 \rangle$. Since $\mathcal{T}\mathit{riv}_{(-)}$ is a left adjoint by Corollary 14.3.6, it preserves initial objects; as $\emptyset$ is initial in Cat_∞, we conclude that $\mathcal{T}\mathit{riv}_\emptyset$ is the initial ∞-operad.

Lemma 14.3.8. *For every ∞-operad O, there is a canonical equivalence*

$$\mathcal{T}\mathit{riv} \times O \simeq \mathcal{T}\mathit{riv}_{U(O)}$$

of ∞-operads.

Proof. Recall that $\mathcal{T}riv^{\otimes} = \mathrm{Fin}^{\mathrm{op}}$ and that $p_{\mathcal{T}riv}\colon \mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ is the inclusion. The product operad $\mathcal{T}riv \times \mathcal{O}$ is therefore represented by the pullback

$$\begin{array}{ccc} \mathrm{Fin}^{\mathrm{op}} \times_{\mathrm{Span}(\mathrm{Fin})} \mathcal{O}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_{\mathcal{O}}} \\ \mathrm{Fin}^{\mathrm{op}} & \hookrightarrow & \mathrm{Span}(\mathrm{Fin}). \end{array}$$

By the defining properties of an ∞-operad, the left vertical map is a cocartesian fibration whose cocartesian straightening preserves finite products and sends $\langle 1 \rangle$ to $U(\mathcal{O})$. It is therefore equivalent to

$$\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}, \qquad I \mapsto U(\mathcal{O})^{I}.$$

By Remark 14.3.4, its cocartesian unstraightening is

$$q^{\mathrm{op}}_{U(\mathcal{O})}\colon \mathrm{Fin}(U(\mathcal{O})^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Fin}^{\mathrm{op}},$$

which is the total category of $\mathcal{T}riv_{U(\mathcal{O})}$. □

Remark 14.3.9. More generally, restricting an arbitrary ∞-operad $\mathcal{O}$ to the subcategory $\mathrm{Fin}^{\mathrm{op}} \subseteq \mathrm{Span}(\mathrm{Fin})$ discards precisely the operations of arity $\neq 1$: by Lemma 14.3.8, the pullback $\mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Fin}^{\mathrm{op}}$ is the total category of $\mathcal{T}riv_{U(\mathcal{O})}$. The resulting projection $\mathcal{T}riv_{U(\mathcal{O})} \to \mathcal{O}$ induces the identity on underlying ∞-categories, and is thus the counit of the adjunction of Corollary 14.3.6.

14.4 Symmetric monoidal ∞-categories and algebras over ∞-operads

Let $\mathcal{O}$ be a colored operad and let C be a symmetric monoidal 1-category. Recall that an *$\mathcal{O}$-algebra in C* is defined as an operad map $\mathcal{O} \to \mathcal{M}_C$ into the *multimorphism operad* of C, whose multimorphisms are given by

$$\mathcal{M}_C(\{x_i\}_{i \in I}; y) \quad := \quad \mathrm{Hom}_C(\bigotimes_{i \in I} x_i, y);$$

see Definition 12.2.10. The goal of this section is to define the analogous notion of algebras over ∞-operads in the ∞-categorical setting.

14.4.1 Symmetric monoidal ∞-categories as ∞-operads

We start by constructing the multimorphism operad $\mathcal{M}_C$ of a symmetric monoidal ∞-category C. Recall from Definition 8.1.1 that a symmetric monoidal ∞-category is a commutative monoid in Cat_{∞}, i.e. a finite-product-preserving functor $C\colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_{\infty}$. These naturally form an ∞-category that we denote by

$$\mathrm{Cat}^{\otimes}_{\infty} \quad := \quad \mathrm{CMon}(\mathrm{Cat}_{\infty}).$$

Since the opposite-category involution $(-)^{\mathrm{op}}\colon \mathrm{Cat}_{\infty} \to \mathrm{Cat}_{\infty}$ preserves finite products, postcomposition with it defines an involution on $\mathrm{Cat}^{\otimes}_{\infty}$. For a symmetric monoidal ∞-category $(C, \otimes)$, its image under this involution is called the *opposite symmetric monoidal ∞-category* and is denoted $(C, \otimes)^{\mathrm{op}}$.

Given symmetric monoidal ∞-categories C and D, write $\mathrm{Map}^{\otimes}(C, D)$ for their hom anima in $\mathrm{Cat}^{\otimes}_{\infty}$.

Example 14.4.1 (Terminal ∞-category)**.** The terminal ∞-category $*$ admits a unique symmetric monoidal structure, given by the constant functor $\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty, S \mapsto *$.

Example 14.4.2 (Pointwise monoidal structure)**.** If C is a symmetric monoidal ∞-category and I is an ∞-category, then $\mathrm{Fun}(I, C)$ admits a symmetric monoidal structure, called the *pointwise monoidal structure*. It is given by the composite

$$\mathrm{Span}(\mathrm{Fin}) \xrightarrow{(C,\otimes)} \mathrm{Cat}_\infty \xrightarrow{\mathrm{Fun}(I,-)} \mathrm{Cat}_\infty.$$

This is functorial in both I and C, resulting in a functor

$$\mathrm{Fun}(-,-)\colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty^{\otimes} \to \mathrm{Cat}_\infty^{\otimes}.$$

The symmetric monoidal functors and their monoidal natural transformations themselves form an ∞-category:

Definition 14.4.3. Let C and D be symmetric monoidal ∞-categories. We define $\mathrm{Fun}^{\otimes}(C, D)$ to be the ∞-category whose associated complete Segal anima is

$$N(\mathrm{Fun}^{\otimes}(C, D))_n := \mathrm{Map}^{\otimes}(C, \mathrm{Fun}([n], D)),$$

where $\mathrm{Fun}([n], D)$ carries the pointwise symmetric monoidal structure. This is a complete Segal anima because $\mathrm{Map}^{\otimes}(C, -)$ preserves limits; see Proposition 1.8.7.

Given a symmetric monoidal ∞-category $C\colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$, we may unstraighten this to a cocartesian fibration $p_C\colon C^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$. The next result implies that this is always an ∞-operad.

Lemma 14.4.4. *Let $p_O\colon O^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ be a cocartesian fibration. Then $(O^{\otimes}, p_O)$ is an ∞-operad if and only if the cocartesian straightening of p_O*

$$\mathrm{Str}^{\mathrm{cc}}(p_O)\colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$$

preserves finite products.

Proof. We check conditions (i)–(iii) of Proposition 14.1.12. Condition (i) is part of the assumption on p_O. Since the spans $\rho_i\colon I \hookleftarrow \{i\} \xrightarrow{=} \{i\}$ exhibit I as the product of the one-element sets in $\mathrm{Span}(\mathrm{Fin})$, condition (ii) is equivalent to preservation of finite products by $\mathrm{Str}^{\mathrm{cc}}(p_O)$.

Assume (ii), and let $Y \to Y_j$ be cocartesian lifts of the projections ρ_j for $Y \in O_J^{\otimes}$. For $X \in O_I^{\otimes}$ and a span $\alpha\colon I \to J$, we have

$$\begin{aligned}
\mathrm{Hom}^{\alpha}_{O^{\otimes}}(X, Y) &\simeq \mathrm{Hom}_{O_J^{\otimes}}(\alpha_! X, Y) \\
&\simeq \prod_{j \in J} \mathrm{Hom}_{O^{\otimes}_{\{j\}}}((\rho_j)_! \alpha_! X, Y_j) \\
&\simeq \prod_{j \in J} \mathrm{Hom}^{\rho_j \circ \alpha}_{O^{\otimes}}(X, Y_j).
\end{aligned}$$

The first and last isomorphisms follow from the universal property of cocartesian transport, and the middle one from (ii). Compatibility of cocartesian transport with composition identifies this composite with the map induced by the $Y \to Y_j$. Thus Lemma 14.1.13 gives (iii). □

Definition 14.4.5. Given a symmetric monoidal ∞-category C, we denote its cocartesian unstraightening by

$$C^{\otimes} \quad := \quad \mathrm{Un}^{\mathrm{cc}}(C\colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty).$$

By the previous lemma, the resulting cocartesian fibration $p_C\colon C^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ defines an ∞-operad, which we will denote by $\mathcal{M}_C := (C^{\otimes}, p_C)$ and refer to as the *multimorphism operad* associated to C. By functoriality of unstraightening, this defines a functor

$$\mathcal{M}\colon \mathrm{Cat}_\infty^{\otimes} \to \mathrm{Op}_\infty\,.$$

Morphisms in its essential image are studied in the next section.

Lemma 14.4.6. *Given $C \in \mathrm{Cat}_\infty^{\otimes}$ and objects $x_1, \dots, x_n, y \in C^{\simeq}$, there is a natural equivalence*

$$\mathcal{M}_C((x_1, \dots, x_n); y) \simeq \mathrm{Hom}_C(x_1 \otimes \cdots \otimes x_n, y).$$

Proof. Let α denote the active span $\langle n\rangle \xleftarrow{=} \langle n\rangle \to \langle 1\rangle$. Cocartesian transport along α sends $(x_1, \dots, x_n)$ to $x_1 \otimes \cdots \otimes x_n$. The universal property of this cocartesian lift gives

$$\mathrm{Hom}^{\alpha}_{C^{\otimes}}((x_1, \dots, x_n), y) \simeq \mathrm{Hom}_C(x_1 \otimes \cdots \otimes x_n, y),$$

and the left-hand side is the indicated multimorphism anima. □

Observation 14.4.7. Cocartesian transport in $C^{\otimes}$ is compatible with finite products. More precisely, suppose that $\alpha_j\colon I_j \to J_j$ are morphisms in $\mathrm{Span}(\mathrm{Fin})$ and that $\varphi_j\colon X_j \to Y_j$ are p_C-cocartesian lifts of α_j. Since the straightening of p_C preserves finite products, the product

$$\prod_j \varphi_j\colon \prod_j X_j \longrightarrow \prod_j Y_j$$

is a p_C-cocartesian lift of the product $\prod_j \alpha_j$. In particular, cocartesian transport along a forward span is computed separately over the elements of its target.

We now give a criterion for an ∞-operad to arise from a symmetric monoidal ∞-category: its multimorphism functors must be corepresentable, and the universal multimorphisms must remain universal under substitution.

Definition 14.4.8. Let O be an ∞-operad. A multimorphism $\varphi\colon \{x_i\}_{i\in I} \to y$ *corepresents multimorphisms out of* $\{x_i\}_{i\in I}$ if composition with φ induces an isomorphism

$$\mathrm{Hom}_{U(O)}(y, z) \xrightarrow{\sim} O(\{x_i\}_{i\in I}; z)$$

for every color z. The pair (y, φ) is unique if it exists; we then denote its target by $\bigotimes_{i\in I} x_i$ and refer to φ as the *universal multimorphism* for the tuple.

Suppose that every finite tuple of colors admits a universal multimorphism. For a map $f\colon I \to J$ of finite sets, substituting the universal multimorphisms for the families $\{x_i\}_{i\in f^{-1}(j)}$ into the universal multimorphism for their targets gives a multimorphism

$$\{x_i\}_{i\in I} \longrightarrow \bigotimes_{j\in J} \bigotimes_{i\in f^{-1}(j)} x_i.$$

By the universal property of $\bigotimes_{i\in I} x_i$, this determines a canonical map

$$\bigotimes_{i\in I} x_i \longrightarrow \bigotimes_{j\in J} \bigotimes_{i\in f^{-1}(j)} x_i.$$

Lemma 14.4.9. *Let $\mathcal{O}$ be an ∞-operad. Suppose that every finite tuple of colors admits a universal multimorphism and that the canonical map*

$$\bigotimes_{i\in I} x_i \longrightarrow \bigotimes_{j\in J} \bigotimes_{i\in f^{-1}(j)} x_i$$

is an isomorphism for every map $f\colon I \to J$ of finite sets and every family $\{x_i\}_{i\in I}$. Then $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration, corresponding to a symmetric monoidal structure on $U(\mathcal{O})$.

Proof. For a forward span $\alpha = (I \xleftarrow{=} I \xrightarrow{f} J)$ and colors $\{x_i\}_{i\in I}$, put $y_j := \bigotimes_{i\in f^{-1}(j)} x_i$. The universal multimorphisms define a lift $\varphi\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ of α. We show that it is cocartesian by checking that, for every span $\beta = (J \xleftarrow{g} K \xrightarrow{h} L)$ and colors $\{z_l\}_{l\in L}$, composition with φ induces an isomorphism

$$\mathrm{Hom}^{\beta}_{\mathcal{O}^{\otimes}}(\{y_j\},\{z_l\}) \longrightarrow \mathrm{Hom}^{\beta\circ\alpha}_{\mathcal{O}^{\otimes}}(\{x_i\},\{z_l\}).$$

By Lemma 14.1.10, this is the product over $l \in L$ of the substitution maps

$$\mathcal{O}(\{y_{g(k)}\}_{k\in h^{-1}(l)}; z_l) \longrightarrow \mathcal{O}(\{x_i\}_{(i,k)\in I\times_J h^{-1}(l)}; z_l).$$

Under the corepresenting isomorphisms, each is precomposition with the canonical map

$$\bigotimes_{(i,k)\in I\times_J h^{-1}(l)} x_i \longrightarrow \bigotimes_{k\in h^{-1}(l)} y_{g(k)},$$

which is an isomorphism by the hypothesis applied to the projection $I \times_J h^{-1}(l) \to h^{-1}(l)$. Thus φ is cocartesian.

Backwards spans already admit cocartesian lifts. Since every span is a backwards span followed by a forward span and cocartesian morphisms compose, $p_{\mathcal{O}}$ is a cocartesian fibration. By Lemma 14.4.4, its straightening preserves finite products and hence gives the asserted symmetric monoidal structure. □

14.4.2 Algebras over ∞-operads

Definition 14.4.10. Let $\mathcal{O}$ be an ∞-operad and let C be a symmetric monoidal ∞-category. An *$\mathcal{O}$-algebra in C* is an operad morphism $\mathcal{O} \to \mathcal{M}_C$. We write

$$\mathrm{Alg}_{\mathcal{O}}(C) := \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{M}_C)$$

for the ∞-category of $\mathcal{O}$-algebras in C.

Example 14.4.11 (Commutative/associative algebras)**.** Algebras over the ∞-operads $\mathcal{C}omm$ and $\mathcal{A}ssoc$ are referred to as *commutative algebras* and *associative algebras*, respectively, and their ∞-categories are denoted by

$$\mathrm{CAlg}(C) := \mathrm{Alg}_{\mathcal{C}omm}(C) \qquad \text{and} \qquad \mathrm{Alg}(C) := \mathrm{Alg}_{\mathcal{A}ssoc}(C).$$

Example 14.4.12 (Trivial algebras)**.** Let D be an ∞-category. Applying Lemma 14.3.5 to the multimorphism operad $\mathcal{M}_C$, we see that algebras over the trivial operad $\mathcal{T}riv_D$ correspond to functors $D \to C$:

$$\mathrm{Alg}_{\mathcal{T}riv_D}(C) \xrightarrow{\sim} \mathrm{Fun}(D, C).$$

In particular, taking $D = *$ shows that a $\mathcal{T}riv$-algebra is simply an object of C:

$$\mathrm{Alg}_{\mathcal{T}riv}(C) \xrightarrow{\sim} C.$$

Taking $D = \emptyset$ shows that the ∞-category of $\mathcal{T}riv_{\emptyset}$-algebras is terminal.

Lemma 14.4.13 (Algebras in pointwise monoidal structure)**.** *Let C be a symmetric monoidal ∞-category, let I be an ∞-category, and let O be an ∞-operad. Then an O-algebra in* $\mathrm{Fun}(I, C)$ *with the pointwise monoidal structure is the same data as a functor $I \to \mathrm{Alg}_O(C)$: there is an equivalence of ∞-categories*

$$\mathrm{Alg}_O(\mathrm{Fun}(I, C)) \xrightarrow{\sim} \mathrm{Fun}(I, \mathrm{Alg}_O(C)).$$

Proof. The pointwise symmetric monoidal structure of Example 14.4.2 is classified by the composite

$$\mathrm{Span}(\mathrm{Fin}) \xrightarrow{\mathrm{Str}^{\mathrm{cc}}(p_C)} \mathrm{Cat}_\infty \xrightarrow{\mathrm{Fun}(I,-)} \mathrm{Cat}_\infty.$$

By Proposition 24.2.4, its total category is $\mathrm{Span}(\mathrm{Fin}) \times_{\mathrm{Fun}(I,\mathrm{Span}(\mathrm{Fin}))} \mathrm{Fun}(I, C^\otimes)$. Consequently, before imposing the finite-product condition, a functor over $\mathrm{Span}(\mathrm{Fin})$ from $O^\otimes$ to this total category amounts to a functor

$$O^\otimes \longrightarrow \mathrm{Fun}(I, C^\otimes)$$

whose composite with $\mathrm{Fun}(I, p_C)$ is the constant I-diagram on p_O. Currying identifies these with functors

$$I \longrightarrow \mathrm{Fun}_{/\mathrm{Span}(\mathrm{Fin})}(O^\otimes, C^\otimes).$$

Under this identification, preservation of finite products is pointwise in I: the functor into the pointwise multimorphism operad is an operad map precisely when the corresponding functor $O^\otimes \to C^\otimes$ is an operad map at every object of I. Thus the equivalence restricts to the asserted equivalence of algebra categories. □

14.5 Lax symmetric monoidal functors

The definition of symmetric monoidal ∞-categories as commutative monoids in Cat_∞ makes it easy to talk about symmetric monoidal functors. It is a priori less clear how to define *lax* symmetric monoidal functors: functors $F\colon C \to D$ that come equipped with *lax structure maps*

$$F(X_1) \otimes \cdots \otimes F(X_n) \to F(X_1 \otimes \cdots \otimes X_n),$$

for $X_1, \ldots, X_n \in C$, that are coherently compatible with the symmetric monoidal structures of C and D. The operadic perspective on symmetric monoidal ∞-categories provides a quick definition:

Definition 14.5.1. Let C and D be symmetric monoidal ∞-categories. A *lax symmetric monoidal functor* $C \to D$ is a morphism of ∞-operads $\mathcal{M}_C \to \mathcal{M}_D$. An *oplax symmetric monoidal functor* is a morphism of ∞-operads $\mathcal{M}_{C^{\mathrm{op}}} \to \mathcal{M}_{D^{\mathrm{op}}}$. We denote by

$$\mathrm{Cat}_\infty^{\otimes\text{-lax}} \quad \subseteq \quad \mathrm{Op}_\infty$$

the full subcategory spanned by the (multimorphism operads of) symmetric monoidal ∞-categories. We write

$$\mathrm{Fun}^{\otimes\text{-lax}}(C, D) \quad := \quad \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{M}_C, \mathcal{M}_D)$$

for the ∞-category of lax symmetric monoidal functors from C to D.

The ∞-category $\mathrm{Fun}^{\otimes}(C, D)$ identifies with the full subcategory of $\mathrm{Fun}^{\otimes\text{-lax}}(C, D)$ spanned by the lax symmetric monoidal functors whose structure maps are isomorphisms. Indeed, these are precisely the functors over Span(Fin) that preserve cocartesian morphisms, so the identification follows from straightening. Passing to groupoid cores shows that $\mathcal{M}\colon \mathrm{Cat}_\infty^{\otimes} \to \mathrm{Op}_\infty$ is a (non-full) subcategory inclusion.

Let us unwind how to recover the lax structure maps. An operad map $\mathcal{M}_C \to \mathcal{M}_D$ is encoded by a commutative triangle of the form

$$\begin{array}{ccc} C^{\otimes} & \xrightarrow{\quad F^{\otimes} \quad} & D^{\otimes} \\ & {\scriptstyle p_C} \searrow \quad \swarrow {\scriptstyle p_D} & \\ & \mathrm{Span(Fin)}. & \end{array}$$

Passing to fibers over $\langle 1\rangle$ gives an underlying functor $F\colon C \to D$. The fact that $F^{\otimes}$ preserves finite products means that on objects it is given as $F^{\otimes}(X_1, \dots, X_n) = (F(X_1), \dots, F(X_n))$, and this also fully specifies $F^{\otimes}$ on inert maps. To understand what $F^{\otimes}$ does on active maps, it suffices to consider the case $(X_1, \dots, X_n) \to X_1 \otimes \cdots \otimes X_n$, for which the image in $D^{\otimes}$ takes the form of a map

$$(F(X_1), \dots, F(X_n)) \to F(X_1 \otimes \cdots \otimes X_n)$$

lifting the span $\langle n\rangle \xleftarrow{=} \langle n\rangle \to \langle 1\rangle$. This map uniquely factors through the cocartesian morphism $(F(X_1), \dots, F(X_n)) \to F(X_1) \otimes \cdots \otimes F(X_n)$, resulting in the expected lax structure map.

For oplax symmetric monoidal functors the structure maps point the other way.

Construction 14.5.2. For an ∞-operad $\mathcal{O}$, the construction $C \mapsto \mathrm{Alg}_{\mathcal{O}}(C)$ defines a functor

$$\mathrm{Alg}_{\mathcal{O}}(-) := \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}, -)\colon \mathrm{Cat}_\infty^{\otimes\text{-lax}} \to \mathrm{Cat}_\infty.$$

Given a lax symmetric monoidal functor $F\colon C \to D$, the induced functor $\mathrm{Alg}_{\mathcal{O}}(C) \to \mathrm{Alg}_{\mathcal{O}}(D)$ is given by postcomposition with $\mathcal{M}_C \to \mathcal{M}_D$. Its functoriality follows directly from functoriality of postcomposition in the defining fibers of functor categories.

14.5.1 Lax symmetric monoidal right adjoints

A right adjoint to a symmetric monoidal functor inherits a lax symmetric monoidal structure. We construct this structure by lifting the adjunction to the total categories over Span(Fin).

To formulate the compatibility of the lifted adjunction with the maps to Span(Fin), we first record the small amount of relative adjunction theory that we will need. Given two functors $p\colon C \to B$ and $q\colon D \to B$, we write

$$\mathrm{Fun}_{/B}(C, D) := \mathrm{Fun}(C, D) \times_{\mathrm{Fun}(C,B)} \{p\},$$

where the map to $\mathrm{Fun}(C, B)$ is given by postcomposition with q. Thus an object of $\mathrm{Fun}_{/B}(C, D)$ consists of a functor $H\colon C \to D$ together with a specified identification $qH \simeq p$.

Definition 14.5.3 (Relative adjunction)**.** Let L be an object of $\mathrm{Fun}_{/B}(C, D)$. A *right adjoint to L relative to B* consists of an object $R \in \mathrm{Fun}_{/B}(D, C)$ together with a morphism

$$\varepsilon\colon LR \to \mathrm{id}_D$$

in $\mathrm{Fun}_{/B}(D, D)$ whose image in $\mathrm{Fun}(D, D)$ exhibits the underlying functor of R as a right adjoint to the underlying functor of L. We then call $L \dashv R$ an *adjunction relative to B*.

Equivalently, we may specify the unit over B:

Lemma 14.5.4 (Unit formulation of relative adjunctions)**.** *Let $L \in \mathrm{Fun}_{/B}(C, D)$ and $R \in \mathrm{Fun}_{/B}(D, C)$. The following data are equivalent:*

(1) A morphism $\varepsilon\colon LR \to \mathrm{id}_D$ in $\mathrm{Fun}_{/B}(D, D)$ which exhibits R as a right adjoint to L after forgetting to $\mathrm{Fun}(D, D)$.

(2) A morphism $\eta\colon \mathrm{id}_C \to RL$ in $\mathrm{Fun}_{/B}(C, C)$ which exhibits L as a left adjoint to R after forgetting to $\mathrm{Fun}(C, C)$.

Proof. Suppose first that the data in (1) are given. By Proposition 22.1.2, composition with the counit gives an equivalence

$$\mathrm{Nat}(\mathrm{id}_C, RL) \xrightarrow{\sim} \mathrm{Nat}(L, L), \qquad \alpha \longmapsto (\varepsilon L)(L\alpha).$$

Under the specified identifications over B, the fact that ε is a morphism in $\mathrm{Fun}_{/B}(D, D)$ gives a commutative square

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{id}_C, RL) & \xrightarrow{\simeq} & \mathrm{Nat}(L, L) \\ \downarrow & & \downarrow \\ \mathrm{Nat}(p, p) & = & \mathrm{Nat}(p, p). \end{array}$$

It therefore restricts to an equivalence on the fibers over id_p, which are the hom animae

$$\mathrm{Hom}_{\mathrm{Fun}_{/B}(C,C)}(\mathrm{id}_C, RL) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}_{/B}(C,D)}(L, L).$$

The preimage of id_L gives a morphism $\eta\colon \mathrm{id}_C \to RL$ over B, together with an identification $(\varepsilon L)(L\eta) \simeq \mathrm{id}_L$ in $\mathrm{Fun}_{/B}(C, D)$. After forgetting the maps to B, this is the unit corresponding to ε under Proposition 22.1.2, so it gives the data in (2).

The converse follows by duality. These constructions are inverse, since they restrict the usual correspondence between unit and counit data to the fibers over the identity transformations of p and q; the triangle identifications therefore also hold over B. □

Lemma 14.5.5 (Relative adjunctions on functor categories)**.** *Let $L\colon C \rightleftarrows D : R$ be an adjunction relative to B. For every functor $E \to B$, postcomposition induces an adjunction*

$$L_*\colon \mathrm{Fun}_{/B}(E, C) \rightleftarrows \mathrm{Fun}_{/B}(E, D) : R_*.$$

Proof. The counit in Definition 14.5.3 and the corresponding unit of Lemma 14.5.4 may both be whiskered with a functor $E \to C$ or $E \to D$ over B. They therefore give a unit and counit between the displayed functors, with triangle identifications inherited from those of $L \dashv R$. The result follows from Proposition 22.1.2. □

We now produce the promised lax symmetric monoidal structure on the right adjoint of a symmetric monoidal functor.

Proposition 14.5.6. *Let $L\colon C \to D$ be a symmetric monoidal functor whose underlying functor admits a right adjoint R. Then $L^{\otimes}$ admits a right adjoint $R^{\otimes}$ with a canonical structure over* Span(Fin)*, refining R, for which $L^{\otimes} \dashv R^{\otimes}$ is a relative adjunction. The functor $R^{\otimes}$ preserves finite products and hence equips R with a lax symmetric monoidal structure.*

Proof. For $Y = \{y_j\}_{j\in J}$, take $\{Ry_j\}_{j\in J}$ as the candidate right adjoint value. The componentwise counits of $L \dashv R$ give a map

$$\varepsilon_Y \colon \{LRy_j\}_{j\in J} \to \{y_j\}_{j\in J}.$$

By Lemma 22.1.4, it suffices to show that for every tuple $\{x_i\}_{i\in I}$ the map

$$\mathrm{Hom}_{C^\otimes}(\{x_i\}, \{Ry_j\}) \xrightarrow{\varepsilon_Y \circ L^\otimes(-)} \mathrm{Hom}_{D^\otimes}(\{Lx_i\}, \{y_j\}) \tag{14.1}$$

is an isomorphism. Both sides map to $\mathrm{Hom}_{\mathrm{Span(Fin)}}(I, J)$. By Lemmas 14.4.6 and 14.1.10, the induced map on fibers over $\alpha = (I \xleftarrow{f} K \xrightarrow{g} J)$ is the composite

$$\begin{aligned}\prod_{j\in J} \mathrm{Hom}_C\left(\bigotimes_{k\in g^{-1}(j)} x_{f(k)}, Ry_j\right) &\simeq \prod_{j\in J} \mathrm{Hom}_D\left(L\left(\bigotimes_{k\in g^{-1}(j)} x_{f(k)}\right), y_j\right) \\ &\simeq \prod_{j\in J} \mathrm{Hom}_D\left(\bigotimes_{k\in g^{-1}(j)} Lx_{f(k)}, y_j\right).\end{aligned}$$

The first isomorphism is induced by the adjunction and the second by the symmetric monoidal structure of L. This constructs the right adjoint $R^\otimes$ and its counit.

Applying p_D to the counit gives a natural transformation

$$p_C R^\otimes \simeq p_D L^\otimes R^\otimes \xrightarrow{p_D \varepsilon} p_D$$

whose components are identities by construction. It is therefore an isomorphism, equipping $R^\otimes$ with a structure over Span(Fin) for which the counit lies over the base. This gives the relative adjunction. Finally, $R^\otimes$ preserves finite products because it is a right adjoint, and its restriction to the singleton fiber is R by construction. Thus it defines the asserted lax symmetric monoidal structure. □

The lax structure map $Ry_1 \otimes \cdots \otimes Ry_n \to R(y_1 \otimes \cdots \otimes y_n)$ is adjoint to the composite

$$L(Ry_1 \otimes \cdots \otimes Ry_n) \simeq LRy_1 \otimes \cdots \otimes LRy_n \xrightarrow{\bigotimes_i \varepsilon_{y_i}} y_1 \otimes \cdots \otimes y_n.$$

For $n = 0$, this gives the unit map $\mathbb{1}_C \to R(\mathbb{1}_D)$.

Remark 14.5.7. In the previous proposition, it does not suffice for L to be a lax symmetric monoidal functor itself. It would suffice for L to be an *oplax* symmetric monoidal functor, but producing the lax symmetric monoidal right adjoint is more subtle in this case. Dually, the left adjoint to a lax symmetric monoidal functor is oplax symmetric monoidal. We refer to [HHLN23a, Proposition A] for a detailed discussion.

14.5.2 Symmetric monoidal adjunctions

We now use the lax symmetric monoidal structure on a right adjoint to obtain adjunctions on ∞-categories of algebras. Applying this to the constant-diagram functor will show that limits of algebras are computed at each color.

Definition 14.5.8. A *symmetric monoidal adjunction* is an adjunction

$$L\colon C \rightleftarrows D : R$$

between symmetric monoidal ∞-categories in which L is symmetric monoidal and R is equipped with the canonical lax symmetric monoidal structure of Proposition 14.5.6.

Lemma 14.5.9 (Adjunctions on O-algebras)**.** *Every symmetric monoidal adjunction $L \dashv R$ induces, for each ∞-operad O, an adjunction*

$$\mathrm{Alg}_O(L)\colon \mathrm{Alg}_O(C) \rightleftarrows \mathrm{Alg}_O(D) : \mathrm{Alg}_O(R).$$

The underlying functors are obtained by applying L and R, respectively, at every color of O.

Proof. By Proposition 14.5.6, the adjunction $L^\otimes \dashv R^\otimes$ is relative to Span(Fin). Applying Lemma 14.5.5 to $p_O\colon O^\otimes \to \mathrm{Span(Fin)}$ gives an adjunction

$$\mathrm{Fun}_{/\mathrm{Span(Fin)}}(O^\otimes, C^\otimes) \rightleftarrows \mathrm{Fun}_{/\mathrm{Span(Fin)}}(O^\otimes, D^\otimes).$$

Postcomposition with both functors preserves finite-product-preserving functors because $L^\otimes$ and $R^\otimes$ are morphisms of ∞-operads. Restricting the adjunction to these subcategories gives the asserted adjunction on $\mathrm{Alg}_O(C)$ and $\mathrm{Alg}_O(D)$. □

As an application, it follows that limits of O-algebras may be computed pointwise.

Proposition 14.5.10 (Limits of O-algebras)**.** *Let C be a symmetric monoidal ∞-category, let O be an ∞-operad, and let I be a small ∞-category such that C admits I-indexed limits. Then* $\mathrm{Alg}_O(C)$ *admits I-indexed limits, and the evaluation functors*

$$\mathrm{ev}_x\colon \mathrm{Alg}_O(C) \to C, \qquad A \mapsto A_x \qquad (x \in O^\simeq),$$

jointly create them.

Proof. By Corollary 22.2.2, the constant-diagram functor fits into an adjunction

$$\mathrm{const}\colon C \rightleftarrows \mathrm{Fun}(I, C) : \lim\nolimits_I .$$

Being given by restriction along $I \to *$, the functor const is symmetric monoidal for the pointwise symmetric monoidal structure, so Proposition 14.5.6 makes this a symmetric monoidal adjunction. By Lemma 14.5.9, it induces an adjunction

$$\mathrm{Alg}_O(C) \rightleftarrows \mathrm{Alg}_O(\mathrm{Fun}(I, C)) \simeq \mathrm{Fun}(I, \mathrm{Alg}_O(C)).$$

Under the equivalence of Lemma 14.4.13, the left adjoint is the constant-diagram functor. The right adjoint consequently computes limits in $\mathrm{Alg}_O(C)$, and its value at every color x is $\lim_I A_x$. Thus all the evaluation functors preserve these limits. They jointly detect isomorphisms: every object of $O^\otimes$ is a finite product of colors, and algebra maps preserve these products, so a natural transformation that is invertible at every color is invertible at every object. Consequently the evaluation functors jointly reflect limit cones and hence jointly create the limits. □

14.5.3 Symmetric monoidal left adjoints

Let $R\colon D \to C$ be lax symmetric monoidal and suppose its underlying functor admits a left adjoint L. The lax structure maps of R induce canonical maps

$$L\left(\bigotimes_{i\in I} x_i\right) \longrightarrow \bigotimes_{i\in I} Lx_i,$$

adjoint to the composites

$$\bigotimes_{i\in I} x_i \xrightarrow{\bigotimes_i \eta_{x_i}} \bigotimes_{i\in I} RLx_i \xrightarrow{\mathrm{lax}_R} R\left(\bigotimes_{i\in I} Lx_i\right).$$

We show that if these maps are isomorphisms, then L admits a symmetric monoidal structure compatible with the adjunction.

Proposition 14.5.11. *Let $R\colon D \to C$ be a lax symmetric monoidal functor whose underlying functor admits a left adjoint L. Suppose that the canonical maps*

$$L\left(\bigotimes_{i\in I} x_i\right) \longrightarrow \bigotimes_{i\in I} Lx_i$$

are isomorphisms for every finite family. Then $R^\otimes$ admits a left adjoint $L^\otimes$ over Span(Fin)*, refining L, which preserves cocartesian morphisms and makes $L^\otimes \dashv R^\otimes$ a relative adjunction. This equips L with a symmetric monoidal structure for which $L \dashv R$, with the given lax structure on R, is a symmetric monoidal adjunction.*

It suffices to check the hypothesis for empty and two-element families, that is, for the maps

$$L(\mathbb{1}_C) \to \mathbb{1}_D \qquad \text{and} \qquad L(x \otimes y) \to Lx \otimes Ly.$$

Indeed, taking adjoints of the substitution identities for the lax structure of R shows that the canonical maps respect iterated tensor products. The unary map is the identity by the triangle identity, so induction gives the assertion.

Proof. For a tuple $\{x_i\}_{i\in I}$, take $\{Lx_i\}_{i\in I}$ as the candidate left adjoint value, with unit given by the componentwise maps $\eta_{x_i}\colon x_i \to RLx_i$. By Lemma 22.1.4, we must show that applying $R^\otimes$ and precomposing with this unit gives an isomorphism

$$\mathrm{Hom}_{D^\otimes}(\{Lx_i\}, \{y_j\}) \longrightarrow \mathrm{Hom}_{C^\otimes}(\{x_i\}, \{Ry_j\})$$

for every tuple $\{y_j\}_{j\in J}$. By Lemmas 14.4.6 and 14.1.10, its map on fibers over a span $\alpha = (I \xleftarrow{f} K \xrightarrow{g} J)$ is the product over $j \in J$ of

$$\mathrm{Hom}_D\left(\bigotimes_{k\in g^{-1}(j)} Lx_{f(k)}, y_j\right) \longrightarrow \mathrm{Hom}_D\left(L\left(\bigotimes_{k\in g^{-1}(j)} x_{f(k)}\right), y_j\right) \simeq \mathrm{Hom}_C\left(\bigotimes_{k\in g^{-1}(j)} x_{f(k)}, Ry_j\right).$$

The first map is precomposition with the canonical map in the hypothesis and is therefore an isomorphism; the second is adjunction. Their composite is the required map by the definition of the canonical map and naturality of the lax structure of R. This constructs $L^{\otimes}$ and its unit.

Applying p_C to the unit gives a natural isomorphism

$$p_C \xrightarrow{p_C \eta} p_C R^{\otimes} L^{\otimes} \simeq p_D L^{\otimes},$$

since its components are identities by construction. This equips $L^{\otimes}$ with a structure over Span(Fin) for which the unit lies over the base. By Lemma 14.5.4, the adjunction is relative.

Let $\{x_i\} \to \alpha_!\{x_i\}$ be a cocartesian lift of α. Applying $L^{\otimes}$ and factoring through a cocartesian lift in $D^{\otimes}$ gives a map

$$\alpha_!\{Lx_i\} \longrightarrow L^{\otimes}(\alpha_!\{x_i\}).$$

Naturality of the unit, under the adjunction $L \dashv R$, says that its j-th component composed with the canonical map

$$L\left(\bigotimes_{k \in g^{-1}(j)} x_{f(k)}\right) \longrightarrow \bigotimes_{k \in g^{-1}(j)} Lx_{f(k)}$$

is the identity. Thus its components are the inverses of these isomorphisms, so $L^{\otimes}$ preserves cocartesian morphisms. In particular, it preserves inert morphisms and hence finite products by Corollary 14.1.14. It therefore refines L to a symmetric monoidal functor. Uniqueness of the relative right adjoint identifies the given lax structure on R with the one induced by this symmetric monoidal structure. □

14.6 O-monoids and O-monoidal ∞-categories

A commutative monoid in an ∞-category D with finite products can be described by a finite-product-preserving functor Span(Fin) $\to D$. Replacing Span(Fin) by the total category $O^{\otimes}$ of an ∞-operad lets us describe objects of D equipped with the operations specified by O. Such a functor assigns an object A_x to each color x, sends a tuple of colors to the product of the corresponding objects, and sends a multimorphism $\{x_i\}_{i \in I} \to y$ to a map

$$\prod_{i \in I} A_{x_i} \longrightarrow A_y.$$

Functoriality expresses compatibility with identities and substitution. This leads to the following definition.

Definition 14.6.1. Let D be an ∞-category with finite products. An *O-monoid in D* is a finite-product-preserving functor $O^{\otimes} \to D$. We write

$$\mathrm{Mon}_O(D) \quad := \quad \mathrm{Fun}^{\times}(O^{\otimes}, D) \quad \subseteq \quad \mathrm{Fun}(O^{\otimes}, D)$$

for the full subcategory of O-monoids in D.

When D is equipped with its cartesian monoidal structure, O-monoids in D will be identified with O-algebras in Theorem 15.3.11.

Definition 14.6.2. An *$\mathcal{O}$-monoidal ∞-category* is an $\mathcal{O}$-monoid in Cat_∞. For $\mathcal{O} = \mathcal{C}omm$, this recovers the definition of a symmetric monoidal ∞-category. When $\mathcal{O} = \mathcal{A}ssoc$, we refer to $\mathcal{O}$-monoidal ∞-categories simply as *monoidal ∞-categories*.

Concretely, an $\mathcal{O}$-monoidal ∞-category assigns an ∞-category C_x to each color x and a functor $\prod_{i\in I} C_{x_i} \to C_y$ to each multimorphism $\{x_i\}_{i\in I} \to y$, compatibly with identities and substitution. An *$\mathcal{O}$-monoidal functor* is a natural transformation between the corresponding finite-product-preserving functors $\mathcal{O}^\otimes \to \mathrm{Cat}_\infty$.

Example 14.6.3. When $\mathcal{O} = \mathcal{T}riv$, an $\mathcal{O}$-monoidal ∞-category is just an ∞-category.

The multimorphism operad functor $\mathcal{M}\colon \mathrm{Cat}_\infty^\otimes \to \mathrm{Op}_\infty$ admits an analogue $\mathcal{M}_{-/\mathcal{O}}\colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty) \to (\mathrm{Op}_\infty)_{/\mathcal{O}}$ for $\mathcal{O}$-monoidal ∞-categories. This relies on the following relative analogue of Lemma 14.4.4.

Lemma 14.6.4. *Let $\mathcal{O}$ be an ∞-operad and let $q\colon C^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration. Then the composite*

$$C^\otimes \xrightarrow{q} \mathcal{O}^\otimes \xrightarrow{p_{\mathcal{O}}} \mathrm{Span}(\mathrm{Fin})$$

exhibits $C^\otimes$ as an ∞-operad if and only if the functor $\mathrm{Str}^{\mathrm{cc}}(q)\colon \mathcal{O}^\otimes \to \mathrm{Cat}_\infty$ preserves finite products. In this case, the morphism $q\colon C^\otimes \to \mathcal{O}^\otimes$ is automatically a morphism of ∞-operads.

Proof. Write $F := \mathrm{Str}^{\mathrm{cc}}(q)$. Condition (i) of Proposition 14.1.12 holds for $p_{\mathcal{O}}q$: first lift a backwards span along $p_{\mathcal{O}}$ and then lift the resulting morphism along q. Pasting the cocartesian universal properties shows that this gives a $(p_{\mathcal{O}}q)$-cocartesian lift.

For every finite set I, transport along the inert projections gives a square

$$\begin{array}{ccc} C^\otimes_I & \longrightarrow & \displaystyle\prod_{i\in I} C^\otimes_{\{i\}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \prod_{i\in I} q} \\ \mathcal{O}^\otimes_I & \longrightarrow & \displaystyle\prod_{i\in I} \mathcal{O}^\otimes_{\{i\}}. \end{array}$$

The bottom functor is an equivalence, so condition (ii) holds precisely when this square is a pullback. Checking on fibers reduces this to requiring that $F(o) \to \prod_{i\in I} F(o_i)$ be an equivalence for every tuple $o = \{o_i\}_{i\in I}$. Since products in $\mathcal{O}^\otimes$ concatenate tuples, this is equivalent to F preserving finite products.

Assume these conditions hold, and let $X \to X_i$ be the q-cocartesian lifts of the projections $\rho_i\colon o \to o_i$. For any Y over o', the canonical map $\mathrm{Hom}_{C^\otimes}(Y, X) \to \prod_i \mathrm{Hom}_{C^\otimes}(Y, X_i)$ lies over the isomorphism $\mathrm{Hom}_{\mathcal{O}^\otimes}(o', o) \xrightarrow{\sim} \prod_i \mathrm{Hom}_{\mathcal{O}^\otimes}(o', o_i)$. Its fiber at $\beta\colon o' \to o$ identifies with

$$\mathrm{Hom}_{F(o)}(\beta_! Y, X) \longrightarrow \prod_i \mathrm{Hom}_{F(o_i)}((\rho_i\beta)_! Y, X_i),$$

which is an isomorphism because $F(o) \simeq \prod_i F(o_i)$. Thus $X \simeq \prod_i X_i$, proving condition (iii). These products, and hence all finite products obtained by concatenating tuples, are preserved by q. Therefore q is a morphism of ∞-operads. □

Definition 14.6.5. Given an $\mathcal{O}$-monoidal ∞-category C, we let $p_C\colon C^\otimes \to \mathcal{O}^\otimes$ denote its cocartesian unstraightening, which by the lemma defines an ∞-operad $\mathcal{M}_{C/\mathcal{O}} = (C^\otimes, p_\mathcal{O} \circ p_C)$. Since p_C is a morphism of ∞-operads, it turns $\mathcal{M}_{C/\mathcal{O}}$ into an object of the slice category $(\mathrm{Op}_\infty)_{/\mathcal{O}}$, producing a functor

$$\mathcal{M}_{-/\mathcal{O}}\colon \mathrm{Mon}_\mathcal{O}(\mathrm{Cat}_\infty) \to (\mathrm{Op}_\infty)_{/\mathcal{O}}.$$

If $\mathcal{P} \to \mathcal{O}$ and $\mathcal{Q} \to \mathcal{O}$ are two ∞-operads over $\mathcal{O}$, we denote by $\mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{O}}}(\mathcal{P}, \mathcal{Q})$ the ∞-category of operad maps $\mathcal{P} \to \mathcal{Q}$ over $\mathcal{O}$, defined as the following fiber:

$$\begin{array}{ccc} \mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{O}}}(\mathcal{P}, \mathcal{Q}) & \longrightarrow & \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{Q}) \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{O}). \end{array}$$

Definition 14.6.6. Let C be an $\mathcal{O}$-monoidal ∞-category and let $\mathcal{P} \to \mathcal{O}$ be a morphism of ∞-operads. The ∞-category of *$\mathcal{P}/\mathcal{O}$-algebras in C* is

$$\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(C) := \mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{O}}}(\mathcal{P}, \mathcal{M}_{C/\mathcal{O}}).$$

For a fixed map $\mathcal{P} \to \mathcal{A}ssoc$, we usually abbreviate $\mathrm{Alg}_{\mathcal{P}/\mathcal{A}ssoc}(C)$ to $\mathrm{Alg}_\mathcal{P}(C)$. For the identity map of $\mathcal{A}ssoc$, we write $\mathrm{Alg}(C)$.

If C is symmetric monoidal and $\mathcal{P} \to \mathcal{C}omm$ is an ∞-operad over $\mathcal{C}omm$, then this relative definition agrees with the earlier one:

$$\mathrm{Alg}_{\mathcal{P}/\mathcal{C}omm}(C) \simeq \mathrm{Alg}_\mathcal{P}(C).$$

Indeed, $\mathcal{C}omm$ is terminal in Op_∞, so passing to the slice over $\mathcal{C}omm$ imposes no additional condition on operad maps.

Definition 14.6.7. Given $\mathcal{O}$-monoidal ∞-categories C and D, a *lax $\mathcal{O}$-monoidal functor $C \to D$* is an operad morphism $\mathcal{M}_{C/\mathcal{O}} \to \mathcal{M}_{D/\mathcal{O}}$ over $\mathcal{O}^\otimes$.

Under unstraightening, $\mathcal{O}$-monoidal functors correspond to functors over $\mathcal{O}^\otimes$ that preserve cocartesian morphisms. In particular, they are lax $\mathcal{O}$-monoidal. We sometimes call them *strong* to distinguish them from lax functors.

Proposition 14.6.8 (cf. [Lur17, Corollary 7.3.2.7])**.** *Let $L\colon C \to D$ be an $\mathcal{O}$-monoidal functor such that each $L_x\colon C_x \to D_x$ admits a right adjoint R_x. Then the R_x extend to a lax $\mathcal{O}$-monoidal functor R for which $L^\otimes \dashv R^\otimes$ is a relative adjunction over $\mathcal{O}^\otimes$.*

Proof. For a tuple $\{y_i\}_{i\in I}$, the product of the adjunctions $L_{y_i} \dashv R_{y_i}$ gives a right adjoint on the corresponding fiber. For a morphism $\beta\colon \{x_j\}_{j\in J} \to \{y_i\}_{i\in I}$ in $\mathcal{O}^\otimes$, preservation of cocartesian morphisms by $L^\otimes$ and these fiberwise adjunctions identify

$$\mathrm{Hom}^\beta_{D^\otimes}(L^\otimes\{c_j\}, \{d_i\}) \simeq \mathrm{Hom}^\beta_{C^\otimes}(\{c_j\}, \{R_{y_i} d_i\}).$$

Thus the fiberwise counits exhibit $\{R_{y_i} d_i\}$ as the value of a right adjoint to $L^\otimes$. As in Proposition 14.5.6, the counit gives its structure over $\mathcal{O}^\otimes$ and makes the adjunction relative. Since $R^\otimes$ is a right adjoint, it preserves finite products and hence is a morphism of ∞-operads. □

Conversely, let $R\colon D \to C$ be lax $\mathcal{O}$-monoidal, and suppose each R_x admits a left adjoint L_x. For a multimorphism $\alpha\colon \{x_i\}_{i\in I} \to y$, write $\alpha^C_!$ and $\alpha^D_!$ for the corresponding operations in C and D. The units of the adjunctions and the lax structure of R give a transformation

$$\alpha^C_! \longrightarrow \alpha^C_! \circ \prod_i (R_{x_i} L_{x_i}) \longrightarrow R_y \alpha^D_! \circ \prod_i L_{x_i}.$$

Its adjoint is the canonical transformation

$$L_y \alpha^C_! \longrightarrow \alpha^D_! \circ \prod_i L_{x_i}. \tag{14.2}$$

Proposition 14.6.9 (cf. [Lur17, Corollary 7.3.2.12])**.** *Let $R\colon D \to C$ be a lax $\mathcal{O}$-monoidal functor such that each R_x admits a left adjoint L_x. If the map in* (14.2) *is an isomorphism for every multimorphism of $\mathcal{O}$, then the L_x extend to an $\mathcal{O}$-monoidal functor L for which $L^\otimes \dashv R^\otimes$ is a relative adjunction over $\mathcal{O}^\otimes$.*

Proof. For an arbitrary morphism $\beta\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ of $\mathcal{O}^\otimes$, the analogous transformation

$$\Big(\prod_j L_{y_j}\Big)\beta^C_! \longrightarrow \beta^D_!\Big(\prod_i L_{x_i}\Big)$$

is an isomorphism: after projecting to each output, it is the transformation for the corresponding multimorphism, by Lemma 14.1.10. Consequently the fiberwise units induce isomorphisms

$$\mathrm{Hom}^\beta_{D^\otimes}(\{L_{x_i} c_i\}, \{d_j\}) \simeq \mathrm{Hom}^\beta_{C^\otimes}(\{c_i\}, R^\otimes\{d_j\}).$$

They therefore exhibit $\{L_{x_i} c_i\}$ as the value of a left adjoint to $R^\otimes$, with a relative adjunction over $\mathcal{O}^\otimes$. The argument of Proposition 14.5.11 identifies the maps expressing preservation of cocartesian morphisms with the inverses of the displayed transformations. Thus $L^\otimes$ preserves cocartesian morphisms and defines an $\mathcal{O}$-monoidal functor. □

The limit result for algebras also admits the following relative form.

Proposition 14.6.10 (cf. [Lur17, Corollary 3.2.2.5])**.** *Let $q\colon \mathcal{P} \to \mathcal{O}$ be a morphism of ∞-operads, let C be an $\mathcal{O}$-monoidal ∞-category, and let I be a small ∞-category. Assume that each fiber C_x, for $x \in \mathcal{O}^\simeq$, admits I-indexed limits. Then $\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(C)$ admits I-indexed limits, and the evaluation functors*

$$\mathrm{ev}_y\colon \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(C) \to C_{q(y)}, \qquad A \mapsto A_y \qquad (y \in \mathcal{P}^\simeq),$$

jointly create them.

Proof. The proof is analogous to that of Proposition 14.5.10. Applying $\mathrm{Fun}(I, -)$ fiberwise gives an $\mathcal{O}$-monoidal ∞-category whose fiber at x is $\mathrm{Fun}(I, C_x)$, and algebras in it identify with I-diagrams in $\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(C)$. The fiberwise constant-diagram functors form an $\mathcal{O}$-monoidal functor, with lax $\mathcal{O}$-monoidal right adjoint given in each fiber by $\lim_I$. By Proposition 14.6.8, these functors form a relative adjunction over $\mathcal{O}^\otimes$, so Lemma 14.5.5 gives the required right adjoint to the constant-diagram functor after applying $\mathrm{Fun}_{/\mathcal{O}^\otimes}(\mathcal{P}^\otimes, -)$ and restricting to operad maps. Its evaluation at y is $\lim_I$ in $C_{q(y)}$. The evaluation functors jointly detect isomorphisms, so they jointly create the limits. □

Corollary 14.6.11. *Let C be a monoidal ∞-category and let I be a small ∞-category such that C admits I-indexed limits. Then* $\mathrm{Alg}(C)$ *admits I-indexed limits, and the forgetful functor* $\mathrm{Alg}(C) \to C$ *creates and preserves them. The same holds for the forgetful functor* $\mathrm{CAlg}(C) \to C$ *when C is symmetric monoidal.*

Proof. This is the one-color case of Proposition 14.6.10: a monoidal ∞-category is an $\mathcal{A}ssoc$-monoidal ∞-category, with $\mathrm{Alg}(C) = \mathrm{Alg}_{\mathcal{A}ssoc/\mathcal{A}ssoc}(C)$, and a symmetric monoidal ∞-category is a $\mathcal{C}omm$-monoidal ∞-category, with $\mathrm{CAlg}(C) = \mathrm{Alg}_{\mathcal{C}omm/\mathcal{C}omm}(C)$. □

Corollary 14.6.12. *Let C be a monoidal ∞-category with a terminal object T. Then T carries a unique associative algebra structure, and it is terminal in* $\mathrm{Alg}(C)$. *When C is symmetric monoidal, the same holds for the commutative algebra structure and* $\mathrm{CAlg}(C)$.

Proof. By Corollary 14.6.11, the empty diagram has a terminal algebra whose underlying object is T. Any algebra with underlying object T maps isomorphically to this terminal algebra, since the forgetful functor detects isomorphisms. Hence the anima of algebra structures on T is contractible. The commutative case is identical. □

14.7 Monoidal subcategories and monoidal Bousfield localizations

A full subcategory D of a symmetric monoidal ∞-category C determines a full suboperad of $\mathcal{M}_C$ by Construction 14.1.7. If D contains the unit and is closed under tensor products, this suboperad is itself symmetric monoidal. For a reflective subcategory, there is another possibility: tensor the objects in the ambient ∞-category and then apply the reflector. We will show that this defines a symmetric monoidal structure whenever tensoring preserves the morphisms inverted by the reflector.

Lemma 14.7.1. *Let $(C, \otimes_C, \mathbb{1}_C)$ be a symmetric monoidal ∞-category. If $D \subseteq C$ is a full subcategory whose collection of objects $D^{\simeq} \subseteq C^{\simeq}$ contains $\mathbb{1}_C$ and is closed under $\otimes_C$, then D inherits a symmetric monoidal structure $(D, \otimes_D, \mathbb{1}_D)$ from C such that the inclusion $i\colon D \hookrightarrow C$ is a (strong) symmetric monoidal functor.*

Proof. The full suboperad on the colors of D is closed under cocartesian transport in $C^{\otimes}$: by Observation 14.4.7, this transport is computed by finite tensor products, and closure under these follows from closure under the unit and binary tensor products. The cocartesian universal properties restrict by full faithfulness, so $D^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration and its inclusion preserves cocartesian morphisms. □

Recall that a *Bousfield localization* is a functor $L\colon C \to D$ that admits a fully faithful right adjoint $i\colon D \hookrightarrow C$. In this case a morphism $x \to y$ is called an *L-equivalence* if the map $L(x) \to L(y)$ in D is an isomorphism.

Proposition 14.7.2. *Let $(C, \otimes_C, \mathbb{1}_C)$ be a symmetric monoidal ∞-category, and let $i\colon D \hookrightarrow C$ be the inclusion of a full subcategory such that i admits a left adjoint $L\colon C \to D$. Assume that for every L-equivalence $f\colon x \to y$ in C and for every $z \in C$, the morphism $f \otimes_C \mathrm{id}_z\colon x \otimes z \to y \otimes z$ is also an L-equivalence. Then the following hold:*

(1) *The full suboperad $D^{\otimes}$ of $C^{\otimes}$ defines a symmetric monoidal structure on D, with*

$$x \otimes_D y \simeq L(ix \otimes_C iy), \qquad \mathbb{1}_D \simeq L(\mathbb{1}_C).$$

(2) *The inclusion $i^{\otimes}\colon D^{\otimes} \hookrightarrow C^{\otimes}$ admits a left adjoint $L^{\otimes}\colon C^{\otimes} \to D^{\otimes}$ which is a strong symmetric monoidal refinement of the functor L.*

(3) *If C admits small colimits and its tensor product preserves them separately in both variables, then the same holds for D.*

Proof. We regard objects of D as objects of C, omitting i. For every span $\alpha\colon I \leftarrow K \to J$, the transport functor $\alpha_!\colon C^I \to C^J$ preserves componentwise L-equivalences. Indeed, it is computed by restriction and finite tensor products, and a tensor product of L-equivalences is an L-equivalence by changing one factor at a time, using symmetry and the hypothesis.

We first construct a left adjoint to $i^{\otimes}$ by sending $\{x_i\}_{i\in I}$ to $\{Lx_i\}_{i\in I}$. For a tuple $X = \{x_i\}_{i\in I}$ in $C^{\otimes}$ and a tuple Y of local objects, the componentwise unit induces a map

$$\mathrm{Hom}_{D^{\otimes}}(\{Lx_i\}, Y) \longrightarrow \mathrm{Hom}_{C^{\otimes}}(X, Y).$$

On the fibers over a span $\alpha\colon I \leftarrow K \to J$, cocartesian transport identifies this with

$$\prod_{j\in J} \mathrm{Hom}_C((\alpha_!\{Lx_i\})_j, Y_j) \longrightarrow \prod_{j\in J} \mathrm{Hom}_C((\alpha_! X)_j, Y_j).$$

This is an isomorphism: $\alpha_!$ preserves componentwise L-equivalences, and maps into the local objects Y_j invert these morphisms. The pointwise criterion for adjunctions therefore gives $L^{\otimes} \dashv i^{\otimes}$. Its unit lies over identity spans, so the adjunction is relative to $\mathrm{Span}(\mathrm{Fin})$. Moreover, $L^{\otimes}$ preserves inert projections, since it applies L componentwise. Hence it is a morphism of ∞-operads by Corollary 14.1.14.

For $X \in D^{\otimes}$ and a span α, consider the composite

$$X \longrightarrow \alpha_! X \longrightarrow L^{\otimes}\alpha_! X.$$

The first morphism is cocartesian in $C^{\otimes}$ and the second is the unit. The relative adjunction identifies maps from its target to any $Y \in D^{\otimes}$ with maps from $\alpha_! X$ to Y in $C^{\otimes}$, over the same spans. Thus the cocartesian universal property in $C^{\otimes}$ proves that the composite is cocartesian in $D^{\otimes}$. This establishes the symmetric monoidal structure on D and the stated tensor and unit formulas.

For arbitrary $X \in C^{\otimes}$, the unit induces an isomorphism

$$L^{\otimes}\alpha_! X \xrightarrow{\sim} L^{\otimes}\alpha_! L^{\otimes} X,$$

since $\alpha_!$ preserves componentwise L-equivalences. Under this identification, naturality of the unit identifies the image under $L^{\otimes}$ of a cocartesian lift $X \to \alpha_! X$ with the cocartesian lift starting at $L^{\otimes}X$ just constructed. Hence $L^{\otimes}$ is strong symmetric monoidal.

Finally, colimits in D are obtained by applying L to colimits in C. For a small diagram (y_j) in D and $x \in D$, we obtain

$$\begin{aligned} x \otimes_D \mathrm{colim}_j^D\, y_j &\simeq L\big(x \otimes_C iL(\mathrm{colim}_j^C\, iy_j)\big) \\ &\simeq L\big(x \otimes_C \mathrm{colim}_j^C\, iy_j\big) \\ &\simeq \mathrm{colim}_j^D\, L(x \otimes_C iy_j) \simeq \mathrm{colim}_j^D (x \otimes_D y_j). \end{aligned}$$

The second isomorphism uses that tensoring the unit of L with x is an L-equivalence. Symmetry gives preservation of colimits in the other variable. □

When C admits internal homs, the compatibility condition on L can equivalently be expressed as a closure property of the local objects.

Lemma 14.7.3. *Let C be a symmetric monoidal ∞-category in which each functor $- \otimes z$ admits a right adjoint $\underline{\mathrm{Hom}}_C(z, -)$, and let $L\colon C \rightleftarrows D\colon i$ be a Bousfield localization. The following conditions are equivalent:*

(1) *Tensoring with any object of C preserves L-equivalences.*

(2) *For every $z \in C$ and every local object w, the object $\underline{\mathrm{Hom}}_C(z, w)$ is local.*

Under these conditions, D inherits a symmetric monoidal structure for which L is strong symmetric monoidal.

Proof. An object is local precisely when mapping into it inverts all L-equivalences, and a morphism is an L-equivalence precisely when mapping from it into every local object gives an isomorphism. Thus, for $f\colon x \to y$ an L-equivalence, $z \in C$, and w local, the tensor–hom adjunction identifies the conditions in (1) and (2) by identifying the maps

$$\mathrm{Hom}_C(y \otimes z, w) \longrightarrow \mathrm{Hom}_C(x \otimes z, w)$$

with

$$\mathrm{Hom}_C(y, \underline{\mathrm{Hom}}_C(z, w)) \longrightarrow \mathrm{Hom}_C(x, \underline{\mathrm{Hom}}_C(z, w)).$$

The final assertion follows from Proposition 14.7.2. □

A symmetric monoidal localization descends to algebras.

Corollary 14.7.4 (cf. [Lur17, Proposition 2.2.1.9])**.** *Let $L\colon C \rightleftarrows D\colon i$ be as in Proposition 14.7.2, and let O be an ∞-operad. Then postcomposition induces a Bousfield localization*

$$\mathrm{Alg}_O(L)\colon \mathrm{Alg}_O(C) \rightleftarrows \mathrm{Alg}_O(D)\colon \mathrm{Alg}_O(i)$$

Its local objects are precisely the O-algebras whose values on all colors lie in D.

Proof. The symmetric monoidal adjunction $L \dashv i$ induces the asserted adjunction on algebras by Lemma 14.5.9. Since $D^{\otimes}$ is the full suboperad on the local colors, postcomposition with $i^{\otimes}$ is fully faithful, with the stated image. The left adjoint applies L on each color, so it inverts precisely the maps whose underlying morphisms are L-equivalences. □

The reflector applies L on each color. Consequently, a morphism of O-algebras is inverted precisely when its value on every color is an L-equivalence. In particular, a commutative algebra is local precisely when its underlying object is local, and its localization has underlying object $L(A)$.

The preceding results concern reflective localizations inside an ∞-category. The complementary problem of constructing a monoidal structure on a Dwyer–Kan localization at a class of weak equivalences is treated later in Chapter 21.

Exercises

Exercise 14.1. Show that the following functors are examples of ∞-operads:

- $\mathcal{C}omm = (\mathrm{Span}(\mathrm{Fin}), \mathrm{id})$;
- $\mathcal{T}riv = (\mathrm{Fin}^{\mathrm{op}}, \mathrm{inclusion})$;
- $\mathcal{T}riv_{\emptyset} = (*, \mathrm{inclusion})$;
- $\mathcal{E}_0 = (\mathrm{Span}(\mathrm{Fin}, \mathrm{all}, \mathrm{inj}), \mathrm{inclusion})$.

In each case, compute the multimorphism animae and the corresponding composition of multimorphisms, show that these animae are discrete, and identify the resulting ∞-operads with their classical counterparts from Section 12.1, or with the empty-colored operad in the case of $\mathcal{T}riv_{\emptyset}$.

Exercise 14.2. (1) Prove the converse of Lemma 14.4.9: if an ∞-operad $\mathcal{O}$ is the multimorphism operad of a symmetric monoidal ∞-category, then both hypotheses of that lemma are satisfied. Conclude that these two conditions characterize the essential image of $\mathcal{M}\colon \mathrm{Cat}_\infty^{\otimes} \to \mathrm{Op}_\infty$ on objects.

(2) Given symmetric monoidal ∞-categories C and D, show that an operad map $f\colon \mathcal{M}_C \to \mathcal{M}_D$ lies in the image of the functor $\mathcal{M}$ if and only if the map $f^{\otimes}\colon C^{\otimes} \to D^{\otimes}$ preserves the cocartesian lifts $(x_1, \dots, x_n) \to x_1 \otimes \cdots \otimes x_n$ of the span $\langle n\rangle \xleftarrow{=} \langle n\rangle \to \langle 1\rangle$ for every $n \geq 0$. Use Observation 14.4.7 to pass from these one-output lifts to arbitrary active spans.

Exercise 14.3 (Nullary and binary operations suffice)**.** Let $\mathcal{O}$ be an ∞-operad. Suppose that for every family $\{x_i\}_{i\in I}$ with $|I| = 0$ or $|I| = 2$, there is a multimorphism $\varphi\colon \{x_i\}_{i\in I} \to y$ such that substitution induces an isomorphism

$$\mathcal{O}(\{y\} \sqcup \{z_j\}_{j\in J}; w) \longrightarrow \mathcal{O}(\{x_i\}_{i\in I} \sqcup \{z_j\}_{j\in J}; w)$$

for every finite family of colors $\{z_j\}_{j\in J}$ and every color w.

(1) Show that multimorphisms with this property are closed under operadic substitution, and deduce that every finite tuple admits such a multimorphism.

(2) Apply Lemma 14.4.9 to show that $\mathcal{O}$ arises from a symmetric monoidal ∞-category.

Exercise 14.4. Equip Ab with its tensor product and Set with its cartesian monoidal structure. Show that the forgetful functor $U\colon \mathrm{Ab} \to \mathrm{Set}$ is lax symmetric monoidal, with structure maps

$$U(A) \times U(B) \longrightarrow U(A \otimes B), \qquad (a, b) \longmapsto a \otimes b,$$

and unit determined by $1 \in \mathbb{Z}$. Identify the associative and commutative algebra structures induced on the underlying set of a ring.

Exercise 14.5 (Algebras in an arrow category)**.** Let C be a symmetric monoidal ∞-category and equip $\mathrm{Fun}([1], C)$ with the pointwise symmetric monoidal structure. Show that an $\mathcal{O}$-algebra in $\mathrm{Fun}([1], C)$ is precisely a morphism of $\mathcal{O}$-algebras in C. Specialize to $\mathcal{O} = \mathcal{A}ssoc$ and show that sources, targets, and composites of algebra morphisms are computed on the underlying objects of C.

Exercise 14.6 (Arithmetic localization is monoidal)**.** Let P be a set of primes and let $L(X) := X[P^{-1}]$ be the smashing localization of spectra from Section 7.2. Show that the L-equivalences are stable under tensoring with arbitrary spectra. Construct the symmetric monoidal structure on the full subcategory of P-divisible spectra and describe the localization of associative and commutative ring spectra.

15 Cartesian and cocartesian monoidal structures

Every ∞-category C with finite products admits a *cartesian monoidal structure* in which the tensor product of two objects is their cartesian product; the higher coherences required to make this an honest symmetric monoidal structure are uniquely determined by the universal property of the product. Dually, if C admits finite coproducts, it admits a *cocartesian monoidal structure*. The goal of this chapter is to construct these two symmetric monoidal structures.

As a first step, we reformulate the problem in terms of the construction of coherent adjoints. Recall that C admits finite products if and only if for every finite set I the diagonal functor $C \to C^I$ admits a right adjoint; for $I = \emptyset$ this encodes the terminal object, while for $I = \{1, 2\}$ this returns the cartesian product. As a consequence, the restriction functors $C^I \to C^J$ have right adjoints for all morphisms of finite sets $J \to I$. Our task in constructing the cartesian monoidal structure on C is then to assemble both the contravariant restrictions as well as their covariant right adjoints into a single functor

$$\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty, \qquad I \mapsto C^I.$$

The cocartesian monoidal structure should be constructed similarly by extending by left adjoints instead.

The general construction that allows us to carry this out is due to Barwick [Bar17], and is known as *unfurling*. Its input is a contravariant functor whose restriction functors admit left or right adjoints satisfying a certain compatibility known as the *Beck–Chevalley condition*. Its output is a functor on a span category, which on backwards legs restricts to the original functor and on forward legs implements the adjoints. Applying the unfurling construction to $I \mapsto C^I$ then produces the two monoidal structures we are after.

If C is not assumed to admit coproducts, the first steps in the unfurling procedure still make sense and produce an ∞-operad $O^{\amalg}(C)$, which we refer to as the *cocartesian operad of* C. The reason for this is that the universal property of a coproduct $x_1 \sqcup \cdots \sqcup x_n$ tells us how to map out of it: we need to specify maps out of each object x_i. Even if the coproduct does not actually exist, the formula

$$O^{\amalg}(C)(x_1, \ldots, x_n; y) \simeq \prod_{i=1}^{n} \mathrm{Hom}_C(x_i, y)$$

can be used to define the multimorphism animae of an ∞-operad.

The algebras over ∞-operads of the form $O^{\amalg}(C)$ are surprisingly simple: they correspond to C-indexed families of commutative algebras. To see why this is the case at an intuitive level, note

that the cocartesian operad for the terminal ∞-category $*$ is the commutative ∞-operad $Comm$. The assignment $C \mapsto \mathcal{O}^{\amalg}(C)$ is functorial, and so every object of C determines an operad map $Comm \to \mathcal{O}^{\amalg}(C)$, and hence evaluating an $\mathcal{O}^{\amalg}(C)$-algebra at this object produces a commutative algebra. We will show in Section 15.2 that an $\mathcal{O}^{\amalg}(C)$-algebra is completely captured by the resulting C-indexed family of commutative algebras.

The classification of algebras over cocartesian operads directly leads to a classification of cocartesian operads themselves: an ∞-operad $\mathcal{O}$ is of the form $\mathcal{O}^{\amalg}(C)$ for some C if and only if the multimorphism anima in $\mathcal{O}$ from a disjoint union of unordered tuples is the product of the individual multimorphism animae out of each of the constituents. This then gives a direct classification of the cocartesian monoidal structures, resulting in an equivalence of ∞-categories

$$\mathrm{Cat}_\infty^{\mathrm{coprod}} \xrightarrow{\sim} \mathrm{Cat}_\infty^{\otimes,\mathrm{cocart}}$$

between ∞-categories with finite coproducts and cocartesian symmetric monoidal ∞-categories. The analogous result for cartesian monoidal structures follows by duality.

Here and below, $\mathrm{Cat}_\infty^{\mathrm{coprod}}$ denotes the ∞-category of ∞-categories with finite coproducts and finite-coproduct-preserving functors.

For the cartesian monoidal structure, algebras admit a description in terms of products alone: for every ∞-category C with finite products and every ∞-operad $\mathcal{O}$, there is an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(C,\times) \xrightarrow{\sim} \mathrm{Mon}_{\mathcal{O}}(C)$$

between $\mathcal{O}$-algebras and $\mathcal{O}$-monoids, i.e. finite-product-preserving functors $\mathcal{O}^{\otimes} \to C$. In particular, commutative algebras in $(C,\times)$ are nothing but the commutative monoids studied in Chapter 5.

The proofs in Section 15.1.1 are technical and may be skipped on a first reading. Two results from that subsection will be used later: the criterion for cocartesian morphisms in a span category and the adequate triple associated with a cartesian fibration.

15.1 Unfurling

In this section, we explain Barwick's *unfurling construction* [Bar17], which extends a contravariant functor $F\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ to the span category by coherently assembling adjoints. We begin by isolating the required adjointability condition.

Definition 15.1.1. Let (C, C, C_R) be an adequate triple and let $F\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a functor.

(1) The functor F is *left C_R-adjointable* if, for every $r\colon X \to Y$ in C_R, the functor $r^* := F(r)\colon F(Y) \to F(X)$ admits a left adjoint $r_!$, and for every pullback square

$$\begin{array}{ccc} X' & \xrightarrow{r'} & X \\ {\scriptstyle l'}\downarrow & \lrcorner & \downarrow{\scriptstyle l} \\ Y' & \xrightarrow{r} & Y \end{array}$$

with $r, r' \in C_R$, the Beck–Chevalley transformation

$$r'_! l'^* \xrightarrow{r'_! l'^* \eta_r} r'_! l'^* r^* r_! \simeq r'_! r'^* l^* r_! \xrightarrow{\varepsilon_{r'} l^* r_!} l^* r_!$$

is a natural isomorphism.

(2) The functor F is *right C_R-adjointable* if every r^* admits a right adjoint r_* and, for every pullback square as above, the dually defined Beck–Chevalley transformation $l^* r_* \to r'_* l'^*$ is a natural isomorphism.

When $C_R = C$, we simply say that F is *left adjointable* or *right adjointable*.

The trick for extending F to the span category is by reformulating the problem in terms of (co)cartesian fibrations: starting with the unstraightening of F, we wish to construct the unstraightening of its extension. We now provide a formulation of the adjointability conditions on F in terms of its associated fibration over C.

Definition 15.1.2. Let (C, C_L, C_R) be an adequate triple. A functor $p\colon E \to C$ is called a *Beck–Chevalley fibration* with respect to (C_L, C_R) if the following three conditions are satisfied:

(1) Every morphism in C_L admits p-cartesian lifts with given target.

(2) Every morphism in C_R admits p-cocartesian lifts with given source.

(3) In a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{r'} & X \\ {\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle l} \\ Y' & \xrightarrow{r} & Y \end{array}$$

whose image in C is a pullback square with $p(l) \in C_L$ and $p(r) \in C_R$, assume that r is p-cocartesian and l' is p-cartesian. Then r' is p-cocartesian if and only if l is p-cartesian.

Conditions (1) and (2) say that the restrictions $E \times_C C_L \to C_L$ and $E \times_C C_R \to C_R$ are a cartesian and a cocartesian fibration, respectively. They provide contravariant transport l^* along C_L, covariant transport $r_!$ along C_R, and Beck–Chevalley transformations $r'_! l'^* \to l^* r_!$ for pullback squares as above. By choosing cartesian and cocartesian lifts around such a square, one sees that condition (3) amounts to the invertibility of these transformations.

The following result is the key technical input to the unfurling construction.

Theorem 15.1.3. *Let (C, C_L, C_R) be an adequate triple and let $p\colon E \to C$ be a Beck–Chevalley fibration with respect to (C_L, C_R). Set $E_L^{p\text{-cart}} := p^{-1}(C_L) \cap E^{p\text{-cart}}$ and $E_R := p^{-1}(C_R)$. Then $(E, E_L^{p\text{-cart}}, E_R)$ is an adequate triple, and the induced functor*

$$\mathrm{Span}(p)\colon \mathrm{Span}(E, E_L^{p\text{-cart}}, E_R) \longrightarrow \mathrm{Span}(C, C_L, C_R)$$

is a cocartesian fibration with fiber E_X over $X \in C$. Its transport along a span $X \xleftarrow{l} U \xrightarrow{r} Y$ is $r_! l^*$.

For expository purposes, we postpone the proof to Section 15.1.1.

Corollary 15.1.4 (Contravariant left-adjoint unfurling)**.** *Let (C, C, C_R) be an adequate triple. Every left C_R-adjointable functor $F\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ extends canonically to a functor*

$$\mathrm{Unf}^{\mathrm{L}}(F)\colon \mathrm{Span}(C, C, C_R) \longrightarrow \mathrm{Cat}_\infty$$

which sends a span $X \xleftarrow{l} U \xrightarrow{r} Y$ to $r_! l^*$. Its restriction along $C^{\mathrm{op}} \hookrightarrow \mathrm{Span}(C, C, C_R)$ is F itself.

Proof. Let $p\colon \mathrm{Un}^{\mathrm{ct}}(F) \to C$ be the cartesian unstraightening. The dual of Lemma 24.1.20, applied over each $r \in C_R$, identifies the existence of $r_!$ with that of p-cocartesian lifts over r. The Beck–Chevalley condition for F is condition (3) of Definition 15.1.2, so Theorem 15.1.3 gives the extension and its transport formula. The descriptions of the fibers and backwards transport identify its restriction with F. □

Corollary 15.1.5 (Contravariant right-adjoint unfurling)**.** *Let (C, C, C_R) be an adequate triple. Every right C_R-adjointable functor $F\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ extends canonically to a functor*

$$\mathrm{Unf}^{\mathrm{R}}(F)\colon \mathrm{Span}(C, C, C_R) \longrightarrow \mathrm{Cat}_\infty$$

which sends a span $X \xleftarrow{l} U \xrightarrow{r} Y$ to $r_* l^*$. Its restriction along $C^{\mathrm{op}} \hookrightarrow \mathrm{Span}(C, C, C_R)$ is F itself.

Proof. The pointwise opposite $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is left C_R-adjointable: the left adjoint to $(r^*)^{\mathrm{op}}$ is $(r_*)^{\mathrm{op}}$, and its Beck–Chevalley transformations are the opposites of those for F. Apply Corollary 15.1.4 to F^{op} and take opposites pointwise once more; the transport functor becomes $r_* l^*$. □

For completeness, we record the covariant form which appears in Barwick's original construction. Let $F\colon C \to \mathrm{Cat}_\infty$ be a functor and let $C_L \subseteq C$ be a wide subcategory such that (C, C_L, C) is an adequate triple. Write $l_! := F(l)$. We call F *covariantly right C_L-adjointable* if every $l_!$ admits a right adjoint l^* and, for every pullback square as in Definition 15.1.1 with $l, l' \in C_L$, the Beck–Chevalley transformation $r'_! l'^* \to l^* r_!$ is a natural isomorphism.

Proposition 15.1.6 (Covariant right-adjoint unfurling)**.** *The cocartesian unstraightening $p\colon \mathrm{Un}^{\mathrm{cc}}(F) \to C$ of a functor $F\colon C \to \mathrm{Cat}_\infty$ is a Beck–Chevalley fibration with respect to (C_L, C) if and only if F is covariantly right C_L-adjointable. Consequently, such an F extends to* $\mathrm{Span}(C, C_L, C)$ *by sending $X \xleftarrow{l} U \xrightarrow{r} Y$ to $r_! l^*$.*

Proof. Condition (2) of Definition 15.1.2 holds automatically, since p is a cocartesian fibration. For every $l \in C_L$, we may apply Lemma 24.1.20 after pulling p back along $l\colon [1] \to C$. It follows that cartesian lifts over l exist precisely when $l_!$ admits a right adjoint l^*.

Now consider a pullback square as in Definition 15.1.1, with $l, l' \in C_L$. The canonical commutativity isomorphism $r_! l'_! \simeq l_! r'_!$, together with the units and counits of the adjunctions, determines the Beck–Chevalley transformation

$$r'_! l'^* \longrightarrow l^* r_!.$$

Under the correspondence between right adjoints and cartesian lifts, the invertibility of this transformation is exactly condition (3) of Definition 15.1.2. This proves the first claim. The final assertion now follows from Theorem 15.1.3. □

15.1.1 Cocartesian morphisms in span categories

In this subsection we prove Theorem 15.1.3. The main tool is a general criterion for morphisms in span categories to be cocartesian. We will use this criterion again in Section 15.2.1 below to construct cocartesian operads that need not arise from symmetric monoidal structures.

Theorem 15.1.7 (Barwick [Bar17], [HHLN23b, Theorem 3.1]). *Let $p\colon (E, E_L, E_R) \to (C, C_L, C_R)$ be a morphism of adequate triples, and consider a span*

$$X \xleftarrow{\varphi} U \xrightarrow{\psi} Y,$$

with $\varphi \in E_L$ and $\psi \in E_R$. Then this morphism is cocartesian with respect to $\mathrm{Span}(p)\colon \mathrm{Span}(E, E_L, E_R) \to \mathrm{Span}(C, C_L, C_R)$ *whenever the following conditions are satisfied:*

(1) *The morphism φ is p-cartesian and the morphism ψ is p-cocartesian;*

(2) *(Left cancellation for E_L) Given a commutative triangle*

$$\begin{array}{ccc} & U & \\ {\scriptstyle\xi}\nearrow & & \searrow{\scriptstyle\varphi} \\ T & \xrightarrow{\varphi\xi} & X \end{array}$$

such that $p(\xi) \in C_L$ and $\varphi\xi \in E_L$, we have $\xi \in E_L$;

(3) *(Right cancellation for E_R) Given a commutative triangle*

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\psi}\nearrow & & \searrow{\scriptstyle\xi} \\ U & \xrightarrow{\xi\psi} & T \end{array}$$

such that $p(\xi) \in C_R$ and $\xi\psi \in E_R$, we have $\xi \in E_R$;

(4) *(Enough p-cocartesian lifts) For any pullback square in C of the form*

$$\begin{array}{ccc} K & \xrightarrow{g} & J \\ {\scriptstyle l'}\downarrow & \lrcorner & \downarrow{\scriptstyle l} \\ pU & \xrightarrow{p\psi} & pY, \end{array}$$

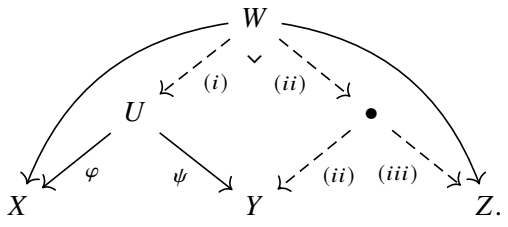
with $l \in C_L$, and for any $W \in E_K$, the morphism g admits a p-cocartesian lift $\psi'\colon W \to V$ which lies in E_R and also satisfies the condition from (3);

(5) *(Beck–Chevalley) Consider a commutative square in E of the form*

$$\begin{array}{ccc} W & \xrightarrow{\psi'} & V \\ {\scriptstyle\xi'}\downarrow & & \downarrow{\scriptstyle\xi} \\ U & \xrightarrow{\psi} & Y \end{array}$$

such that its image in C is a pullback square, $\xi' \in E_L$, $\psi' \in E_R$, and $p(\xi) \in C_L$. Then ψ' is p-cocartesian if and only if $\xi \in E_L$ and the square is a pullback square.

Proof. We consider any other span $X \leftarrow W \to Z$ in E. We have to show that every prescribed filling of the image in C of the following dashed diagram admits a unique lift to a filling in E:



Here all left-pointing maps are in E_L and all right-pointing maps are in E_R. By the description of the hom animae of a span category from Lemma 13.1.13, writing $\mathcal{E} := (E, E_L, E_R)$ and $C := (C, C_L, C_R)$, the anima of such fillings is the fiber of the comparison map

$$\mathrm{Hom}_{\mathrm{Span}(\mathcal{E})}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathrm{Span}(\mathcal{E})}(X, Z) \times_{\mathrm{Hom}_{\mathrm{Span}(C)}(pX, pZ)} \mathrm{Hom}_{\mathrm{Span}(C)}(pY, pZ)$$

over the given point, so that showing all these fibers to be contractible is exactly showing that our span is $\mathrm{Span}(p)$-cocartesian. We produce the filling in three steps, each of which is unique up to a contractible choice.

(i) Since φ is p-cartesian, there exists a unique lift $W \to U$ making the left triangle commute, which lies in E_L by condition (2).

(ii) We claim that this lift $W \to U$ may uniquely be written as the pullback along ψ of some morphism $V \to Y$ in E_L compatible with the given pullback square in C; more precisely, we claim that the following commutative square is a pullback square:

$$\begin{array}{ccc} (E_{/Y})^L & \xrightarrow{\psi^*} & (E_{/U})^L \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (C_{/pY})^L & \xrightarrow{(p\psi)^*} & (C_{/pU})^L. \end{array}$$

Here the superscript $(-)^L$ means that we take the full subcategories spanned by the morphisms in E_L and C_L, respectively.

(a) For essential surjectivity of $(E_{/Y})^L \to (E_{/U})^L \times_{(C_{/pU})^L} (C_{/pY})^L$, we consider a morphism $\xi' \colon W \to U$ in E_L and assume we are given a pullback square in C of the form

$$\begin{array}{ccc} pW & \xrightarrow{g} & J \\ {\scriptstyle p\xi'}\downarrow & \lrcorner & \downarrow{\scriptstyle l} \\ pU & \xrightarrow{p\psi} & pY \end{array}$$

with $l \in C_L$. By assumption (4) we can find a p-cocartesian lift $\psi' \colon W \to V$ of g which lies in E_R and satisfies condition (3). The composite $\psi \circ \xi' \colon W \to Y$ then factors through a map $\xi \colon V \to Y$, giving a commutative square in E of the form considered in (5). By assumption (5), the map ξ then lies in E_L and the resulting square is a pullback square. As ξ lifts l, this shows that the pair (ξ', l) lies in the essential image, as desired.

(b) For full faithfulness, consider two objects $\xi_1 \colon V_1 \to Y$ and $\xi_2 \colon V_2 \to Y$ of $(E_{/Y})^L$. We must show that the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{E_{/Y}}(V_1, V_2) & \xrightarrow{\psi^*} & \mathrm{Hom}_{E_{/U}}(U \times_Y V_1, U \times_Y V_2) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathrm{Hom}_{C_{/pY}}(pV_1, pV_2) & \xrightarrow{(p\psi)^*} & \mathrm{Hom}_{C_{/pU}}(pU \times_{pY} pV_1, pU \times_{pY} pV_2) \end{array}$$

is a pullback square. By the universal property of pullbacks, the right vertical map is isomorphic to the map

$$p \colon \mathrm{Hom}_{E_{/Y}}(U \times_Y V_1, V_2) \to \mathrm{Hom}_{C_{/pY}}(pU \times_{pY} pV_1, pV_2),$$

hence the claim follows from the fact that the projection map $U \times_Y V_1 \to V_1$ is p-cocartesian: this is the implication from right to left in assumption (5), applied to the pullback square of ξ_1 along ψ, whose remaining two sides lie in E_L and E_R by adequacy. This morphism remains cocartesian after passing to the slice over Y, since its cocartesian universal property is unchanged when the target is fixed.

(iii) Finally, given a p-cocartesian morphism $W \to V$, the given map $W \to Z$ uniquely factors through a map $V \to Z$, which lies in E_R by condition (3). □

In practice, we will often be interested in the situation where the subcategories E_L and E_R of E are determined by C_L, C_R and the functor $p\colon E \to C$. To set this up, we start with the following auxiliary lemma:

Lemma 15.1.8. *Let $p\colon E \to C$ be a functor, and consider a commutative square*

$$\begin{array}{ccc} X' & \xrightarrow{r'} & X \\ {\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle l} \\ Y' & \xrightarrow{r} & Y \end{array}$$

in E where l is a p-cartesian morphism. Assume that the image of this square in C is a pullback square. Then this square is a pullback square if and only if the morphism l' is p-cartesian.

Proof. Consider for any object $Z \in E$ the following two diagrams:

$$\begin{array}{ccccc} \mathrm{Hom}_E(Z,X') & \xrightarrow{r'\circ -} & \mathrm{Hom}_E(Z,X) & \xrightarrow{p} & \mathrm{Hom}_C(pZ,pX) \\ {\scriptstyle l'\circ -}\downarrow & & \downarrow{\scriptstyle l\circ -} & \lrcorner & \downarrow{\scriptstyle pl\circ -} \\ \mathrm{Hom}_E(Z,Y') & \xrightarrow{r\circ -} & \mathrm{Hom}_E(Z,Y) & \xrightarrow{p} & \mathrm{Hom}_C(pZ,pY), \end{array}$$

$$\begin{array}{ccccc} \mathrm{Hom}_E(Z,X') & \xrightarrow{p} & \mathrm{Hom}_C(pZ,pX') & \xrightarrow{pr'\circ -} & \mathrm{Hom}_C(pZ,pX) \\ {\scriptstyle l'\circ -}\downarrow & & \downarrow{\scriptstyle pl'\circ -} & \lrcorner & \downarrow{\scriptstyle pl\circ -} \\ \mathrm{Hom}_E(Z,Y') & \xrightarrow{p} & \mathrm{Hom}_C(pZ,pY') & \xrightarrow{pr\circ -} & \mathrm{Hom}_C(pZ,pY). \end{array}$$

The right-hand squares of both diagrams are pullback squares: for the first one since l is p-cartesian, for the second one since the image of the given square in C is a pullback square. As the functor p preserves composition, the outer rectangles of both diagrams agree, so by the pasting law the left-hand square in the first diagram is a pullback square if and only if the left-hand square in the second diagram is a pullback square. Since the former holds for all Z if and only if the given diagram is a pullback square and the latter holds for all Z if and only if l' is p-cartesian, this finishes the proof. □

Proposition 15.1.9. *Let (C, C_L, C_R) be an adequate triple and let $p\colon E \to C$ be a functor such that every morphism in C_L admits p-cartesian lifts with given target (for example: p is a cartesian fibration). Then:*

(1) *The triple $(E, E_L^{p\text{-cart}}, E_R)$ is adequate, where $E_R := p^{-1}(C_R)$ consists of morphisms over C_R and $E_L^{p\text{-cart}} := p^{-1}(C_L) \cap E^{p\text{-cart}}$ consists of p-cartesian morphisms over C_L.*

(2) *The functor $p\colon E \to C$ is a morphism of adequate triples.*

Proof. Both classes are wide subcategories of E: this is clear for $E_R = p^{-1}(C_R)$, while $E_L^{p\text{-cart}}$ contains all identities and is closed under composition since C_L is and since composites of p-cartesian morphisms are again p-cartesian.

We next construct the pullback in E of a morphism $l \in E_L^{p\text{-cart}}$ along some $r \in E_R$. Since $pl \in C_L$ and $pr \in C_R$, we may form the pullback

$$\begin{array}{ccc} pX \times_{pY} pY' & \longrightarrow & pX \\ \downarrow & \lrcorner & \downarrow{\scriptstyle pl} \\ pY' & \xrightarrow{pr} & pY \end{array}$$

in C. By assumption on p, there exists a p-cartesian lift $l'\colon X' \to Y'$ of the morphism $pX \times_{pY} pY' \to pY'$. The composite $rl'\colon X' \to Y$ then uniquely factorizes through some $r'\colon X' \to X$ lifting the projection $pX \times_{pY} pY' \to pX$, using that l is p-cartesian. All in all, we have lifted the pullback square in C to a commutative square in E of the form

$$\begin{array}{ccc} X' & \xrightarrow{r'} & X \\ {\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle l} \\ Y' & \xrightarrow{r} & Y. \end{array} \tag{15.1}$$

Since l and l' are p-cartesian, it follows from Lemma 15.1.8 that this square is a pullback square in E, showing that the required pullbacks exist. Moreover, in this square we have $l' \in E_L^{p\text{-cart}}$ and $r' \in E_R$, so both classes are closed under the relevant base changes. This shows part (1).

For part (2), consider a pullback square in E of the form (15.1). We need to show that applying p to this square gives a pullback square in C. Proceeding as before, we may always form the pullback of pl and pr in C and lift this to a pullback square in E. But by uniqueness of pullback squares of l and r, this new square must then agree with the given one. In particular their images in C agree, hence this image is a pullback square. □

Remark 15.1.10. For the morphism of adequate triples in the previous result, parts (2) and (3) of Theorem 15.1.7 follow from cancellation of p-cartesian morphisms and the definition of E_R, respectively. For a p-cocartesian morphism $\psi\colon U \to Y$ satisfying part (4), part (5) amounts to the invertibility of the Beck–Chevalley transformations

$$g_! l'^* \longrightarrow l^*(p\psi)_!\colon E_{pU} \longrightarrow E_J$$

for the pullback squares in part (4). Here $(p\psi)_!$ exists by taking $J = pY$ and $l = \mathrm{id}$ in that condition.

We are now ready for the main result of this section.

Proof of Theorem 15.1.3. Condition (1) of Definition 15.1.2 and Proposition 15.1.9 show that $(E, E_L^{p\text{-cart}}, E_R)$ is adequate and that

$$\mathrm{Span}(p)\colon\ \mathrm{Span}(E, E_L^{p\text{-cart}}, E_R) \longrightarrow \mathrm{Span}(C, C_L, C_R)$$

is defined. Given a span $X \xleftarrow{l} U \xrightarrow{r} Y$ and an object over X, choose a p-cartesian lift of l followed by a p-cocartesian lift of r. In the criterion of Theorem 15.1.7, parts (2) and (3) hold by the preceding remark, part (4) follows from condition (2) of Definition 15.1.2 because g is a base change of r, and part (5) follows from condition (3) and Lemma 15.1.8. Hence the chosen span is cocartesian, with transport $r_! l^*$.

Finally, by Lemma 13.1.13, a morphism in the fiber over X is represented by a span whose backwards leg is p-cartesian over id_X, hence an isomorphism, and whose forward leg is an arbitrary morphism of E_X. Thus this fiber is E_X. □

15.2 Cocartesian monoidal structures

We now specialize the unfurling construction to construct cocartesian monoidal structures on ∞-categories with finite coproducts. Throughout, we will make use of the ∞-category of finite families from Construction 14.3.1: for an ∞-category C, we write $q\colon \mathrm{Fin}(C) \to \mathrm{Fin}$ for the cartesian unstraightening of the finite-product-preserving functor $C^{(-)}\colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, $I \mapsto C^I$, so that the objects of $\mathrm{Fin}(C)$ are the finite unordered tuples $\{x_i\}_{i\in I}$ of objects of C.

Assume now that C admits finite coproducts. For every map $f\colon I \to J$, the restriction functor $f^*\colon C^J \to C^I$ admits a left adjoint $f_!\colon C^I \to C^J$, which forms coproducts over the fibers of f. These adjunctions satisfy the Beck–Chevalley condition. Indeed, consider a pullback square as in Definition 15.1.1 and a family $A = (A_{y'})_{y'\in Y'}$. For every $x \in X$, the Beck–Chevalley transformation is the natural isomorphism

$$(r'_! l'^* A)_x \simeq \coprod_{x' \in (r')^{-1}(x)} A_{l'(x')} \simeq \coprod_{y' \in r^{-1}(l(x))} A_{y'} \simeq (l^* r_! A)_x,$$

where the middle isomorphism follows from the pullback property. Thus $C^{(-)}$ is left adjointable, and we may apply contravariant unfurling.

Proposition 15.2.1. *Let C be an ∞-category with finite coproducts. The unfurling of $C^{(-)}$ is a finite-product-preserving functor*

$$(C, \amalg) := \mathrm{Unf}^{\mathrm{L}}(C^{(-)})\colon \mathrm{Span}(\mathrm{Fin}) \longrightarrow \mathrm{Cat}_\infty$$

which defines a symmetric monoidal structure on C whose unit is the initial object and whose tensor product is the coproduct.

Proof. By Corollary 15.1.4, the functor $C^{(-)}$ extends to $\mathrm{Span}(\mathrm{Fin})$. To check whether it preserves finite products, recall from part (1) of Lemma 13.3.8 that the inclusion $\mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ preserves finite products. Since this inclusion is the identity on objects, we may test whether the unfurling preserves finite products after restricting along it. By the last statement of Corollary 15.1.4, the resulting functor is the original functor $C^{(-)}$, which preserves finite products by construction. This finishes the construction of the monoidal structure.

Given the description of coproducts in terms of left adjoints, we see that transport along $\emptyset \to \langle 1\rangle$ selects the initial object, while transport along the active span $\langle 2\rangle \xleftarrow{=} \langle 2\rangle \xrightarrow{\nabla} \langle 1\rangle$ is the binary coproduct. □

Definition 15.2.2 (Cocartesian monoidal structure)**.** The symmetric monoidal ∞-category $(C, \amalg)$ from Proposition 15.2.1 is called the *cocartesian monoidal structure on* C.

15.2.1 Cocartesian operads

The construction of the cartesian fibration $q\colon \mathrm{Fin}(C) \to \mathrm{Fin}$ did not use the existence of finite coproducts in C. Those coproducts were needed only to supply cocartesian transport along all morphisms of Fin and thereby make q a Beck–Chevalley fibration. For an arbitrary ∞-category C, we may still apply the span-category construction to q. The resulting map need not be a cocartesian fibration over all of $\mathrm{Span}(\mathrm{Fin})$, but it still has the cocartesian lifts over inert morphisms required of an ∞-operad. We now make this precise.

Recall from Construction 14.3.1 that $\mathrm{Fin}(C)$ admits finite coproducts, given by unordered concatenation of tuples, that q preserves them, and that a morphism $(f, (\varphi_i)_{i\in I})$ of $\mathrm{Fin}(C)$ is q-cartesian if and only if every φ_i is an isomorphism in C; we write $\mathrm{Fin}(C)_{\mathrm{ct}} \subseteq \mathrm{Fin}(C)$ for the wide subcategory spanned by the q-cartesian morphisms.

Lemma 15.2.3. *Let C be an ∞-category. Then the triple $(\mathrm{Fin}(C), \mathrm{Fin}(C)_{\mathrm{ct}}, \mathrm{Fin}(C))$ is an adequate triple. Furthermore, it is extensive in the sense of Definition 13.3.3.*

Proof. The adequate triple is obtained by applying Proposition 15.1.9 to the cartesian fibration $q\colon \mathrm{Fin}(C) \to \mathrm{Fin}$.

The cartesian straightening $C^{(-)}\colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ of q preserves finite products by construction, while Fin is extensive. The final assertion of Proposition 24.2.10 therefore shows that $\mathrm{Fin}(C)$ is extensive.

The q-cartesian morphisms are closed under finite coproducts, and the morphisms $\emptyset \to X$ and $\nabla\colon X \sqcup X \to X$ are q-cartesian, by the componentwise description recalled above. The class of all morphisms has the same closure and containment properties automatically. Hence Lemma 13.3.4 shows that the adequate triple is extensive. □

Construction 15.2.4 (Cocartesian operad)**.** Let C be an ∞-category. We define the ∞-category $C^{\amalg}$ as

$$C^{\amalg} := \mathrm{Span}(\mathrm{Fin}(C), \mathrm{Fin}(C)_{\mathrm{ct}}, \mathrm{Fin}(C)).$$

Combining Lemma 15.2.3 with Lemma 13.3.8, we see that this is a semiadditive ∞-category, with finite products and finite coproducts both induced by the unordered concatenation of tuples in C. We denote by

$$p_C^{\amalg} := \mathrm{Span}(q)\colon C^{\amalg} \to \mathrm{Span}(\mathrm{Fin})$$

the induced functor to $\mathrm{Span}(\mathrm{Fin})$, which preserves finite products.

Proposition 15.2.5. *Let C be an ∞-category. Then the pair $(C^{\amalg}, p_C^{\amalg})$ is an ∞-operad.*

Proof. We have already argued that $C^{\amalg}$ has finite products and that $p_C^{\amalg} = \mathrm{Span}(q)$ preserves them, giving condition (1) of the definition. The objects in the fiber over a finite set I form the fiber $\mathrm{Fin}(C)_I = C^I$. Moreover, by Lemma 13.1.13, a morphism over the identity span of I is represented by a span whose backwards leg is q-cartesian over id_I, hence an isomorphism, and therefore reduces to a morphism in C^I. Thus the fiber of $C^{\amalg}$ over I is C^I, giving condition (2). For

condition (3), consider objects $X_i \in \mathrm{Fin}(C)$ and a morphism $f\colon J \to I$ of finite sets. Then the lift $\widetilde{f}\colon \prod_{i\in I} X_i \to \prod_{j\in J} X_{f(j)}$ in $C^{\amalg}$ is a backwards span of the form

$$\coprod_{i\in I} X_i \xleftarrow{(f,(\mathrm{id})_j)} \coprod_{j\in J} X_{f(j)} \xrightarrow{(\mathrm{id},(\mathrm{id}))} \coprod_{j\in J} X_{f(j)}.$$

The left-pointing morphism is q-cartesian and the right-pointing morphism is an identity. In particular, condition (4) of Theorem 15.1.7 only asks for cocartesian lifts of base changes of an identity, which are again isomorphisms; the remaining conditions follow directly from cartesianness and Lemma 15.1.8. Hence this span is $\mathrm{Span}(q)$-cocartesian, as desired. □

The proof only uses cocartesian lifts over inert morphisms, for which the forward leg of the displayed span is an identity. Thus it does not require the covariant transport $f_!$ used in the full unfurling.

Definition 15.2.6 (Cocartesian operad)**.** For an ∞-category C, we refer to the ∞-operad $O^{\amalg}(C) := (C^{\amalg}, p_C^{\amalg})$ from Proposition 15.2.5 as the *cocartesian operad of* C.

Warning 15.2.7. In various sources, notably in Lurie [Lur17], no notational distinction is made between the ∞-operad $O^{\amalg}(C)$ and its total ∞-category $C^{\amalg}$: both are denoted by $C^{\amalg}$. Although the notation $O^{\amalg}(C)$ is slightly more verbose, we find it useful to distinguish between the ∞-operad and its total category.

The ∞-operad $O^{\amalg}(C)$ models coproducts even when they do not exist: its multimorphisms $(x_1, \dots, x_n) \to y$ are the morphisms $(x_1, \dots, x_n) \to y$ in $\mathrm{Fin}(C)$, which correspond to tuples $(\varphi_i\colon x_i \to y)$ of morphisms in C. If C admits finite coproducts, cocartesian transport along the active fold map $\langle n\rangle \to \langle 1\rangle$ sends $(x_1, \dots, x_n)$ to $x_1 \amalg \cdots \amalg x_n$. Factoring through this cocartesian lift gives the familiar isomorphism

$$\mathrm{Hom}_C(x_1 \amalg \cdots \amalg x_n, y) \simeq \prod_{i=1}^{n} \mathrm{Hom}_C(x_i, y).$$

The construction of $O^{\amalg}(C)$ uses the expression on the right even when the coproduct representing it does not exist.

Proposition 15.2.8. *Let C be an ∞-category. Then the following conditions are equivalent:*

(1) *The ∞-category C admits finite coproducts;*

(2) *The cartesian fibration $q\colon \mathrm{Fin}(C) \to \mathrm{Fin}$ is a cocartesian fibration;*

(3) *The functor $p_C^{\amalg} = \mathrm{Span}(q)\colon C^{\amalg} \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration.*

If these conditions hold, $p_C^{\amalg}$ is the cocartesian unstraightening of the unfurling $(C, \amalg)$ from Proposition 15.2.1.

Proof. If C admits finite coproducts, we showed above that $q\colon \mathrm{Fin}(C) \to \mathrm{Fin}$ is a Beck–Chevalley fibration. The construction in Theorem 15.1.3 identifies the cocartesian unstraightening of the unfurling from Proposition 15.2.1 with $\mathrm{Span}(q)$. This proves that (1) implies (3), as well as the final assertion.

We next show that (3) implies (2). Recall from the dual of Lemma 24.1.15 that every q-cartesian morphism in $\mathrm{Fin}(C)$ whose image in Fin is an isomorphism is itself an isomorphism. Consequently, the square

$$\begin{array}{ccc} \mathrm{Fin}(C) & \hookrightarrow & C^{\amalg} \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{Span}(q)} \\ \mathrm{Fin} & \hookrightarrow & \mathrm{Span}(\mathrm{Fin}) \end{array}$$

is a pullback square, where the horizontal functors are the inclusions of the forward spans, using that $\mathrm{Fin}(C) \simeq \mathrm{Span}(\mathrm{Fin}(C), \mathrm{Fin}(C)^{\simeq}, \mathrm{Fin}(C))$ and $\mathrm{Fin} \simeq \mathrm{Span}(\mathrm{Fin}, \mathrm{Fin}^{\simeq}, \mathrm{Fin})$ by Lemma 13.1.16. Since cocartesian fibrations are closed under pullback, (3) implies (2).

Finally, suppose that q is a cocartesian fibration. By Lemma 24.1.20, its cocartesian transport $f_!\colon C^I \to C^J$ along a map $f\colon I \to J$ is left adjoint to the cartesian transport $f^*\colon C^J \to C^I$. For the map $\emptyset \to \langle 1 \rangle$ this produces an initial object of C, while for the fold map $\langle 2 \rangle \to \langle 1 \rangle$ it produces binary coproducts. Thus C admits finite coproducts, proving that (2) implies (1). □

Thus, when C admits finite coproducts, the symmetric monoidal structure associated to $O^{\amalg}(C)$ is the cocartesian monoidal structure constructed by unfurling at the beginning of the section.

Remark 15.2.9. Note that all the steps in the construction of $O^{\amalg}(C) = (C^{\amalg}, p_C^{\amalg})$ are functorial in C. Using Proposition 15.1.9, we may write it as the composite

$$\mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{Fun}^{\times}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\mathrm{Un}^{\mathrm{ct}}} \mathrm{Cart}(\mathrm{Fin}) \xrightarrow{E \mapsto (E, E^{p\text{-cart}}, E)} \mathrm{AdTrip}_{/\mathrm{Fin}} \xrightarrow{\mathrm{Span}} (\mathrm{Cat}_\infty)_{/\mathrm{Span}(\mathrm{Fin})}.$$

In particular, we obtain a functor

$$O^{\amalg}\colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty .$$

15.2.2 Algebras over cocartesian operads

Algebras over a cocartesian operad $O^{\amalg}(C)$ are simple: as we will show next, they correspond to C-indexed families of commutative algebras.

Recall from Lemma 14.3.2 that $\mathrm{Fin}(C)$ is freely generated by C under finite coproducts; this will be the key input below.

Construction 15.2.10. Let C be an ∞-category. We construct a functor

$$\iota\colon \mathrm{Span}(\mathrm{Fin}) \times C \to C^{\amalg},$$

informally given by sending a pair (I, x) to the 'constant' I-tuple $(x)_{i \in I}$, and by sending a pair $(I \xleftarrow{f} J \xrightarrow{g} K, \varphi\colon x \to y)$ of a span in Fin and a morphism in C to the span

$$(x)_{i \in I} \xleftarrow{(f, (\mathrm{id}_x)_{j \in J})} (x)_{j \in J} \xrightarrow{(g, (\varphi)_{j \in J})} (y)_{k \in K}.$$

(Note that the left-pointing morphism is indeed q-cartesian.) Formally, this map is constructed as follows:

- Consider the natural transformation $\Delta\colon \mathrm{const}_C \to C^{(-)}$ of functors $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which on the one-point set is given by the identity $\mathrm{id}_C\colon C \to C$; this property determines the entire transformation since $C^{(-)}$ is by definition right Kan extended from the point. For a finite set I, it is given by the diagonal functor

$$\Delta\colon C \to C^I, \quad x \mapsto (x)_{i\in I}.$$

- By cartesian unstraightening, we obtain a morphism of cartesian fibrations over Fin:

$$\begin{array}{ccc} \mathrm{Fin}\times C & \xrightarrow{\mathrm{Un}^{\mathrm{ct}}(\Delta)} & \mathrm{Fin}(C) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathrm{Fin}. & \end{array}$$

- By using Proposition 15.1.9, both sides become adequate triples with left class given by the cartesian morphisms. The functor $\mathrm{Un}^{\mathrm{ct}}(\Delta)$ preserves cartesian morphisms and becomes a morphism of adequate triples. Since a morphism on the left-hand side is cartesian if and only if the C-component is an isomorphism, we may use the equivalence $\mathrm{Span}(C, C^{\simeq}, C) \simeq C$ to get a map on spans:

$$\begin{array}{ccc} \mathrm{Span}(\mathrm{Fin})\times C & \xrightarrow{\iota} & C^{\amalg} \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \mathrm{Span}(q)} & \\ & \mathrm{Span}(\mathrm{Fin}). & \end{array}$$

Notation 15.2.11. Let C, D and E be ∞-categories and assume D and E admit finite coproducts. We denote by

$$\mathrm{Fun}^{\amalg,-}(E\times C, D) \quad \subseteq \quad \mathrm{Fun}(E\times C, D)$$

the full subcategory consisting of functors $F\colon E\times C \to D$ that preserve finite coproducts in the first variable, in the sense that $F(-,x)\colon E \to D$ preserves finite coproducts for all $x \in C$.

Lemma 15.2.12 ([BH21, Lemma C.4], cf. [CLL25, Proof of Proposition 3.30])**.** *Let C and D be ∞-categories and assume D admits finite coproducts. Then restriction along ι induces an equivalence*

$$\iota^*\colon \mathrm{Fun}^{\amalg}(C^{\amalg}, D) \xrightarrow{\sim} \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin})\times C, D),$$

with inverse given by left Kan extension along ι.

Proof. Step 1: We start with some auxiliary diagrams. Consider the following commutative diagram:

$$\begin{array}{ccccc} C & \xrightarrow{m} & \mathrm{Fin}\times C & \xrightarrow{l} & \mathrm{Span}(\mathrm{Fin})\times C \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} \\ & & \mathrm{Fin}(C) & \xrightarrow{k} & C^{\amalg}. \end{array}$$

In particular, passing to restriction functors induces a commutative diagram as follows:

$$\begin{array}{ccccc} \mathrm{Fun}(C,D) & \xleftarrow[\simeq]{m^*} & \mathrm{Fun}^{\amalg,-}(\mathrm{Fin}\times C, D) & \xleftarrow{l^*} & \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin})\times C, D) \\ & {\scriptstyle i^*}\nwarrow^{\simeq} & \uparrow{\scriptstyle \simeq}{\scriptstyle \iota'^*} & & \uparrow{\scriptstyle \iota^*} \\ & & \mathrm{Fun}^{\amalg}(\mathrm{Fin}(C), D) & \xleftarrow{k^*} & \mathrm{Fun}^{\amalg}(C^{\amalg}, D). \end{array}$$

The functor m^* is an equivalence, since Fin is the free ∞-category with finite coproducts. The functor i^* was shown to be an equivalence in Lemma 14.3.2. It follows by 2-out-of-3 that ι'^* is an equivalence as well. Since the inverses to m^* and i^* are given by left Kan extension along m and i, the inverse to ι'^* is given by left Kan extension along ι'.

Step 2: We now prove the following auxiliary statement: for every object $X = \{x_i\}_{i\in I}$ of $\mathrm{Fin}(C)$, the relative slice category of ι' over X embeds fully faithfully into the relative slice category of ι over X, and this inclusion admits a left adjoint. To see this, let us denote these two relative slice categories as follows:

$$P_X := (C^{\amalg})_{/X} \times_{C^{\amalg}} (\mathrm{Span}(\mathrm{Fin}) \times C);$$

$$Q_X := \mathrm{Fin}(C)_{/X} \times_{\mathrm{Fin}(C)} (\mathrm{Fin} \times C).$$

The inclusions l and k induce a fully faithful inclusion $Q_X \hookrightarrow P_X$. We claim that this inclusion admits a left adjoint $P_X \to Q_X$. To this end, note that an object of P_X consists of a pair (J, y) and a span

$$\{y\}_{j\in J} \xleftarrow{(f,(\mathrm{id}_y))} \{y\}_{k\in K} \xrightarrow{(g,(\varphi_k : y\to x_i))} \{x_i\}_{i\in I} = X$$

in $\mathrm{Fin}(C)$, where the left-pointing map is q-cartesian. We will send this to the object of Q_X given by the morphism

$$\{y\}_{k\in K} \xrightarrow{(g,(\varphi_k : y\to x_i))} \{x_i\}_{i\in I} = X.$$

If we include this back into P_X by taking the left-pointing map to be the identity, the span $J \xleftarrow{f} K \xrightarrow{=} K$ and the identity of y define a morphism from the original object to the resulting object. Let $L : P_X \to Q_X$ denote the construction just described and let $j : Q_X \hookrightarrow P_X$ denote the inclusion. Composition with this morphism induces natural equivalences

$$\mathrm{Hom}_{Q_X}(L(p), q) \simeq \mathrm{Hom}_{P_X}(p, j(q)).$$

Indeed, a morphism on the right is uniquely determined by its map from the middle object of the left leg, because that leg is q-cartesian. Thus L is left adjoint to j.

Step 3: We now return to the question of interest by showing that ι^* admits a left adjoint given by left Kan extension. Consider a functor $F : \mathrm{Span}(\mathrm{Fin}) \times C \to D$ which preserves finite coproducts in the first variable. We need to show that the left Kan extension $\iota_! F : C^{\amalg} \to D$ exists and preserves finite coproducts. For the former, we may use the pointwise criterion for left Kan extensions and show that for each object $\{x_i\}_{i\in I}$ of $C^{\amalg}$ the colimit

$$(\iota_! F)(\{x_i\}_{i\in I}) = \mathrm{colim}_{(J,y)\in P_X} F(J, y)$$

exists in D, where P_X denotes the relative slice category of ι considered in Step 2. Since the inclusion $Q_X \hookrightarrow P_X$ is a right adjoint, it is in particular final, hence the above colimit exists in D if and only if so does $\mathrm{colim}_{(J,y)\in Q_X} F(J, y)$. But this is indeed the case, since Q_X is the relative slice category of ι' and the pointwise left Kan extension of $l^* F : \mathrm{Fin} \times C \to D$ along ι' exists by Step 1.

We conclude that $\iota_! F$ exists, and that the canonical map $\iota'_! l^* F \to k^* \iota_! F$ is a natural isomorphism. It remains to show that $\iota_! F$ preserves finite coproducts, but this can be checked after restricting along the inclusion $k : \mathrm{Fin}(C) \hookrightarrow C^{\amalg}$, and thus follows from the fact that $\iota'_! l^* F$ preserves finite coproducts.

Step 4: Finally, we show that the resulting adjunction

$$\iota_! : \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin}) \times C, D) \rightleftarrows \mathrm{Fun}^{\amalg}(C^{\amalg}, D) : \iota^*$$

is an adjoint equivalence. Since $\mathrm{Fin}(C)$ is generated under coproducts by the image of $\mathrm{Fin} \times C$, the functor ι^* is conservative, hence it suffices to show that for every $F \in \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin}) \times C, D)$ the unit $F \to \iota^*\iota_! F$ is an isomorphism. By essential surjectivity of l, this in turn reduces to $l^*F \xrightarrow{\sim} l^*\iota^*\iota_! F$. Under the isomorphisms $l^*\iota^*\iota_! F = \iota'^* k^* \iota_! F \simeq \iota'^* \iota'_! l^* F$, this corresponds to the unit of the adjunction $\iota'_! \dashv \iota'^*$, hence is an isomorphism by Step 1. □

For an ∞-operad $\mathcal{O}$, write $\mathrm{CAlg}(\mathcal{O}) := \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathit{Comm}, \mathcal{O})$. We may now deduce the characterization of algebras over cocartesian operads we are after:

Proposition 15.2.13. *Let C be an ∞-category.*

(1) *If E is an ∞-category with finite products, restriction along ι induces an equivalence*

$$\mathrm{Fun}^{\times}(C^{\amalg}, E) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{CMon}(E)).$$

(2) *If $\mathcal{O}$ is an ∞-operad, we obtain an equivalence*

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}^{\amalg}(C), \mathcal{O}) \simeq \mathrm{Fun}(C, \mathrm{CAlg}(\mathcal{O})).$$

(3) *If D is a symmetric monoidal ∞-category, we obtain an equivalence*

$$\mathrm{Alg}_{\mathcal{O}^{\amalg}(C)}(D) \simeq \mathrm{Fun}(C, \mathrm{CAlg}(D)).$$

Proof. Part (3) is an instance of (2) by taking $\mathcal{O} = \mathcal{M}_D$ to be the multimorphism operad of D. Part (2) follows by applying Part (1) to $E = \mathcal{O}^{\otimes}$ and $E = \mathrm{Span}(\mathrm{Fin})$, and passing to horizontal fibers:

$$\begin{array}{ccccc}
\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}^{\amalg}(C), \mathcal{O}) & \longrightarrow & \mathrm{Fun}^{\times}(C^{\amalg}, \mathcal{O}^{\otimes}) & \xrightarrow{p_{\mathcal{O}} \circ -} & \mathrm{Fun}^{\times}(C^{\amalg}, \mathrm{Span}(\mathrm{Fin})) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{Fun}(C, \mathrm{CAlg}(\mathcal{O})) & \longrightarrow & \mathrm{Fun}(C, \mathrm{CMon}(\mathcal{O}^{\otimes})) & \xrightarrow{p_{\mathcal{O}} \circ -} & \mathrm{Fun}(C, \mathrm{CMon}(\mathrm{Span}(\mathrm{Fin}))).
\end{array}$$

It thus remains to prove part (1). For this, recall from Construction 15.2.4 that $C^{\amalg}$ is semiadditive. The left vertical map in the following commutative diagram is an equivalence by Corollary 5.3.24. The right vertical map is an equivalence by Proposition 5.3.17, since $\mathrm{CMon}(E)$ is semiadditive:

$$\begin{array}{ccc}
\mathrm{Fun}^{\times}(C^{\amalg}, \mathrm{CMon}(E)) & \longrightarrow & \mathrm{Fun}(C, \mathrm{CMon}(\mathrm{CMon}(E))) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Fun}^{\times}(C^{\amalg}, E) & \longrightarrow & \mathrm{Fun}(C, \mathrm{CMon}(E)).
\end{array}$$

Showing that the bottom functor in this square is an equivalence thus reduces to showing that the top functor is an equivalence. Since a functor between semiadditive ∞-categories preserves finite products if and only if it preserves finite coproducts, this top functor may be rewritten as the restriction functor

$$\iota^*\colon \mathrm{Fun}^{\amalg}(C^{\amalg}, \mathrm{CMon}(E)) \to \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin}) \times C, \mathrm{CMon}(E)) \simeq \mathrm{Fun}(C, \mathrm{Fun}^{\amalg}(\mathrm{Span}(\mathrm{Fin}), \mathrm{CMon}(E))).$$

This is an equivalence by Lemma 15.2.12. □

15.2.3 Classification of cocartesian operads

As a consequence of the results from the previous subsection, we obtain an explicit classification of ∞-operads equivalent to $\mathcal{O}^{\amalg}(C)$ for some ∞-category C.

Definition 15.2.14. An ∞-operad $\mathcal{O}$ is called *cocartesian* if there exists an equivalence $\mathcal{O} \simeq \mathcal{O}^{\amalg}(C)$ for some ∞-category C. We denote by $\mathrm{Op}_\infty^{\mathrm{cocart}} \subseteq \mathrm{Op}_\infty$ the full subcategory spanned by the cocartesian operads.

Example 15.2.15. The commutative ∞-operad $\mathcal{C}omm$ is cocartesian: it is equivalent to $\mathcal{O}^{\amalg}(\mathrm{pt})$.

Proposition 15.2.16. *Let $\mathcal{O}$ be a cocartesian operad and let $\mathcal{P}$ be an ∞-operad such that the ∞-category $\mathcal{P}^{\otimes}$ is semiadditive. Then the underlying category functor induces an equivalence*

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Fun}(U(\mathcal{O}), U(\mathcal{P}))$$

between operad maps $\mathcal{O} \to \mathcal{P}$ and functors $U(\mathcal{O}) \to U(\mathcal{P})$.

Proof. Picking an equivalence $\mathcal{O} \simeq \mathcal{O}^{\amalg}(C)$ for some C, we may identify $U(\mathcal{O})$ with C. The inclusion $U(\mathcal{O}) \hookrightarrow \mathcal{O}^{\otimes}$ then corresponds to the composite $C \hookrightarrow \mathrm{Span}(\mathrm{Fin}) \times C \xrightarrow{\iota} C^{\amalg}$, where ι is the functor from Construction 15.2.10. The functor in question then factors as

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}^{\amalg}(C), \mathcal{P}) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{CAlg}(\mathcal{P})) \to \mathrm{Fun}(C, U(\mathcal{P})),$$

where the first functor is the equivalence from Proposition 15.2.13(2) and the second functor is induced by $\mathrm{CAlg}(\mathcal{P}) \to U(\mathcal{P})$ given by evaluation at $\langle 1 \rangle \in \mathrm{Span}(\mathrm{Fin})$. It thus remains to show that the latter is an equivalence. Unwinding definitions, we see that this map sits as the left vertical map in a diagram of fiber sequences in Cat_∞ as follows:

$$\begin{array}{ccccc}
\mathrm{CAlg}(\mathcal{P}) & \hookrightarrow & \mathrm{CMon}(\mathcal{P}^{\otimes}) & \xrightarrow{(p_{\mathcal{P}})_*} & \mathrm{CMon}(\mathrm{Span}(\mathrm{Fin})) \\
\downarrow{\scriptstyle \mathrm{ev}_{\langle 1 \rangle}} & & \simeq \downarrow{\scriptstyle \mathrm{ev}_{\langle 1 \rangle}} & & \simeq \downarrow{\scriptstyle \mathrm{ev}_{\langle 1 \rangle}} \\
U(\mathcal{P}) & \hookrightarrow & \mathcal{P}^{\otimes} & \xrightarrow{p_{\mathcal{P}}} & \mathrm{Span}(\mathrm{Fin}).
\end{array}$$

Since $\mathcal{P}^{\otimes}$ and $\mathrm{Span}(\mathrm{Fin})$ are semiadditive, Proposition 5.3.17 shows that the middle and right vertical maps are equivalences, and the claim follows. □

Corollary 15.2.17. *The functor $\mathcal{O}^{\amalg} \colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty$ is fully faithful and induces an equivalence*

$$\mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{Op}_\infty^{\mathrm{cocart}}.$$

Proof. Observe that $\mathcal{O}^{\amalg} \colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty$ admits a retraction, given by $U \colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty$. It follows that for two ∞-categories C and D, the composite

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(C, D) \to \mathrm{Hom}_{\mathrm{Op}_\infty}(\mathcal{O}^{\amalg}(C), \mathcal{O}^{\amalg}(D)) \to \mathrm{Hom}_{\mathrm{Cat}_\infty}(C, D)$$

is the identity. Since the second map is an equivalence by Proposition 15.2.16, the first map is an equivalence as well, showing full faithfulness. Since $\mathrm{Op}_\infty^{\mathrm{cocart}}$ is by definition the image of $\mathcal{O}^{\amalg}$, the second claim follows immediately. □

Corollary 15.2.18. *The following conditions are equivalent for an ∞-operad O:*

(1) *The ∞-operad O is cocartesian;*

(2) *The ∞-category $O^{\otimes}$ is semiadditive;*

(3) *The following two conditions are satisfied:*

   (a) For every color $x \in O^{\simeq}$, the anima $O(\emptyset; x)$ of nullary operations is contractible;

   (b) For objects $\{x_i\}_{i\in I}, \{y_j\}_{j\in J}$ of $O^{\otimes}$ and a color $z \in O^{\simeq}$, composition with the morphisms $\{x_i\}_{i\in I} \to \{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J}$ and $\{y_j\}_{j\in J} \to \{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J}$ obtained from condition (a) provides an equivalence

$$O(\{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J}; z) \simeq O(\{x_i\}_{i\in I}; z) \times O(\{y_j\}_{j\in J}; z).$$

Proof. The equivalence between (2) and (3) is a simple unwinding of definitions: condition (a) says that the terminal object of $O^{\otimes}$ is also initial, and condition (b) says that the maps $X \simeq X \times * \to X \times Y$ and $Y \simeq * \times Y \to X \times Y$ in $O^{\otimes}$ exhibit the product also as a coproduct.

Since $C^{\amalg}$ is semiadditive, it is clear that (1) implies (2). To see that (2) implies (1), let O be such an ∞-operad, and write $C := U(O)$ for its underlying ∞-category. By Proposition 15.2.16, the identity $\mathrm{id}_C \colon C \to U(O)$ extends to a morphism of ∞-operads $O^{\amalg}(C) \to O$. It is clear that it induces an equivalence on colors, so it remains to show that it induces equivalences on all multimorphism animae:

$$O^{\amalg}(C)(\{x_i\}_{i\in I}; y) \simeq O(\{x_i\}_{i\in I}; y).$$

But using condition (3), we may inductively reduce this to the case of the one-point set $I = \langle 1 \rangle$, where it is simply the equality $C = U(O)$. The resulting fully faithful operad map is an equivalence by Lemma 14.1.11. □

15.2.4 Classification of cocartesian monoidal structures

We now determine the morphisms between cocartesian monoidal structures and deduce their classification from that of cocartesian operads.

Lemma 15.2.19. *Let C and D be ∞-categories with finite coproducts and let $F \colon C \to D$ be a functor. Then F preserves finite coproducts if and only if the induced map* $\mathrm{Span}(F) \colon C^{\amalg} \to D^{\amalg}$ *preserves cocartesian morphisms over* $\mathrm{Span}(\mathrm{Fin})$.

Proof. The map $\mathrm{Span}(F)$ is always a morphism of ∞-operads, so it remains to check cocartesian morphisms over forward spans. Under the unfurling description from Propositions 15.2.1 and 15.2.8, their restrictions along $\mathrm{Fin} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ are the cocartesian morphisms in $\mathrm{Fin}(C)$ and $\mathrm{Fin}(D)$ which form coproducts over the fibers of a map of finite sets. The functor $\mathrm{Fin}(F)$ preserves these morphisms if and only if F preserves finite coproducts. □

Definition 15.2.20. Let $\mathrm{Cat}_\infty^{\otimes,\mathrm{cocart}} \subseteq \mathrm{Cat}_\infty^{\otimes}$ denote the full subcategory spanned by the symmetric monoidal ∞-categories of the form $(C, \amalg)$ for some ∞-category C. We say that a symmetric monoidal ∞-category $(D, \otimes)$ is *cocartesian monoidal* if it is equivalent to $(C, \amalg)$ for some ∞-category C with finite coproducts.

Corollary 15.2.21. *The equivalence* $O^{\amalg}\colon \mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{Op}_\infty^{\mathrm{cocart}}$ *from Corollary 15.2.17 restricts to an equivalence*

$$\mathrm{Cat}_\infty^{\mathrm{coprod}} \xrightarrow{\sim} \mathrm{Cat}_\infty^{\otimes,\mathrm{cocart}}.$$

Proof. We need to check that the equivalence from Corollary 15.2.17 restricts to objects and morphisms. On objects this is the content of Proposition 15.2.8. On morphisms this is the content of Lemma 15.2.19. □

Lemma 15.2.22. *A symmetric monoidal* ∞*-category* $(D, \otimes)$ *is cocartesian monoidal if and only if the following two conditions are satisfied:*

(1) *The monoidal unit* $\mathbb{1} \in D$ *is an initial object;*

(2) *For any two objects* X *and* Y *of* D*, the two maps* $X \simeq X \otimes \mathbb{1} \to X \otimes Y$ *and* $Y \simeq \mathbb{1} \otimes Y \to X \otimes Y$ *induced by the maps* $\mathbb{1} \to X$ *and* $\mathbb{1} \to Y$ *from (1) exhibit the tensor product* $X \otimes Y$ *as a coproduct of* X *and* Y*.*

Proof. In light of the equivalence

$$\mathcal{M}_D(\{x_i\}_{i\in I}; y) \simeq \mathrm{Hom}_D(\bigotimes_{i\in I} x_i, y),$$

this is an immediate consequence of Corollary 15.2.18, together with Proposition 15.2.8. □

15.3 Cartesian monoidal structures

By dualizing the results of the previous section, every ∞-category C with finite products admits a cartesian monoidal structure: the tensor product is the cartesian product, and the unit is the terminal object. The characterization and uniqueness of this structure follow by duality as well. We will also show that O-algebras in $(C, \times)$ are precisely O-monoids in C. This identification requires a separate argument, for which we construct an explicit model of the cartesian operad.

Let C be an ∞-category with finite products. The functor $C^{(-)}\colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is now right adjointable: restriction along $f\colon I \to J$ admits a right adjoint f_*, computed by taking products over the fibers of f. The Beck–Chevalley isomorphisms again follow by reindexing fiberwise products along pullback squares.

Proposition 15.3.1. *Let* C *be an* ∞*-category with finite products. Right-adjoint unfurling extends* $C^{(-)}$ *to a finite-product-preserving functor*

$$(C, \times) := \mathrm{Unf}^{\mathrm{R}}(C^{(-)})\colon \mathrm{Span}(\mathrm{Fin}) \longrightarrow \mathrm{Cat}_\infty$$

which sends a span $I \xleftarrow{f} K \xrightarrow{g} J$ *to* $g_* f^*$*. It defines a symmetric monoidal structure on* C *whose unit is the terminal object and whose tensor product is the cartesian product.*

Proof. Apply Corollary 15.1.5 to $C^{(-)}$. The resulting transport sends an I-indexed family to

$$(x_i)_{i\in I} \longmapsto \left(\prod_{k\in g^{-1}(j)} x_{f(k)} \right)_{j\in J}.$$

The argument of Proposition 15.2.1 applies verbatim to show that the construction preserves finite products: the inclusion $\mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ preserves finite products and is the identity on objects, so product preservation may be tested on the original functor $C^{(-)}$. The empty product gives the terminal object as unit, and transport along $\langle 2\rangle \xleftarrow{=} \langle 2\rangle \xrightarrow{\nabla} \langle 1\rangle$ gives the binary product. The maps $X \times Y \to X \times * \cong X$ and $X \times Y \to * \times Y \cong Y$ induced by the terminal maps of the factors and the unitors are the product projections. □

Definition 15.3.2 (Cartesian monoidal structure)**.** The symmetric monoidal ∞-category $(C, \times)$ from Proposition 15.3.1 is called the *cartesian monoidal structure on* C. We write $\mathcal{O}^{\times}(C) := \mathcal{M}_{(C,\times)}$ for its underlying ∞-operad and $C^{\times} := (\mathcal{O}^{\times}(C))^{\otimes}$ for its total category.

We first use duality to characterize cartesian monoidal structures and prove their uniqueness. Afterwards we construct a concrete model for $C^{\times}$. That model is not needed for existence, but it gives the explicit control required to prove the universal property of the cartesian operad.

15.3.1 Cartesian monoidal ∞-categories

The preceding construction gives a distinguished symmetric monoidal structure on every ∞-category with finite products. We next compare this construction with the intrinsic notion of a cartesian monoidal ∞-category.

Definition 15.3.3. A symmetric monoidal ∞-category $(D, \otimes)$ is called *cartesian monoidal* if the opposite symmetric monoidal ∞-category $(D, \otimes)^{\mathrm{op}}$ is cocartesian monoidal. We denote by

$$\mathrm{Cat}_{\infty}^{\otimes,\mathrm{cart}} \subseteq \mathrm{Cat}_{\infty}^{\otimes}$$

the full subcategory spanned by the cartesian monoidal ∞-categories.

Lemma 15.3.4. *A symmetric monoidal ∞-category $(D, \otimes)$ is cartesian monoidal if and only if the following two conditions are satisfied:*

(1) *The monoidal unit $\mathbb{1} \in D$ is a terminal object;*

(2) *For any two objects X and Y of D, the two maps $X \otimes Y \to X \otimes \mathbb{1} \simeq X$ and $X \otimes Y \to \mathbb{1} \otimes Y \simeq Y$ induced by the maps $X \to \mathbb{1}$ and $Y \to \mathbb{1}$ from the first condition exhibit the tensor product $X \otimes Y$ as a product of X and Y.*

Proof. This is an instance of Lemma 15.2.22 applied to the opposite symmetric monoidal ∞-category. □

The characterization shows that the underlying ∞-category of a cartesian monoidal ∞-category admits finite products. Moreover, every symmetric monoidal functor between cartesian monoidal ∞-categories preserves these finite products. Hence there is a forgetful functor

$$U\colon \mathrm{Cat}_{\infty}^{\otimes,\mathrm{cart}} \to \mathrm{Cat}_{\infty}^{\mathrm{prod}}.$$

Proposition 15.3.5. *Let C be an ∞-category with finite products. Then the cartesian monoidal structure $(C, \times)$ is cartesian monoidal.*

Proof. The transport formula in Proposition 15.3.1 shows that the monoidal unit is the terminal object of C and the binary tensor product is the product in C. The maps $X \otimes Y \to X \otimes * \cong X$ and $X \otimes Y \to * \otimes Y \cong Y$ induced by the terminal maps of the factors and the unitors are, by this transport formula, the product projections. Thus $(C, \times)$ satisfies the two conditions of Lemma 15.3.4. □

Proposition 15.3.6. *The forgetful functor*

$$U \colon \mathrm{Cat}_\infty^{\otimes,\mathrm{cart}} \to \mathrm{Cat}_\infty^{\mathrm{prod}}$$

is an equivalence, with inverse given by the assignment $C \mapsto (C, \times)$.

Proof. Passing to opposites identifies $\mathrm{Cat}_\infty^{\otimes,\mathrm{cart}}$ with $\mathrm{Cat}_\infty^{\otimes,\mathrm{cocart}}$, and identifies the forgetful functor U with the corresponding forgetful functor

$$\mathrm{Cat}_\infty^{\otimes,\mathrm{cocart}} \to \mathrm{Cat}_\infty^{\mathrm{coprod}}.$$

The latter is an equivalence by Corollary 15.2.21: the functor $C \mapsto (C, \amalg)$ is an equivalence whose composite with the forgetful functor is the identity on $\mathrm{Cat}_\infty^{\mathrm{coprod}}$. Hence U is an equivalence as well.

By Proposition 15.3.5, the assignment $C \mapsto (C, \times)$ lands in $\mathrm{Cat}_\infty^{\otimes,\mathrm{cart}}$. Full faithfulness of U uniquely lifts every finite-product-preserving functor $C \to D$ to a symmetric monoidal functor $(C, \times) \to (D, \times)$. These lifts define a functor whose composite with U is the identity, and hence an inverse to U. □

15.3.2 A concrete model for the cartesian operad

The unfurling construction gives the cartesian monoidal structure directly from the functor $I \mapsto C^I$. To prove its universal property, it is useful to replace an I-indexed family by a functor that also records all products of its subfamilies coherently. This leads to the following model based on the span-functoriality of $I \mapsto \mathrm{Fin}_{/I}$.

Construction 15.3.7. The target projection $\mathrm{ev}_1 \colon \mathrm{Ar}(\mathrm{Fin}) \to \mathrm{Fin}$ is a cartesian fibration whose cartesian straightening sends I to $\mathrm{Fin}_{/I}$. Thus pullback defines a functor $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ with value $\mathrm{Fin}_{/I}$ at I. Each pullback functor $f^* \colon \mathrm{Fin}_{/J} \to \mathrm{Fin}_{/I}$ has a left adjoint $f_!$, given by postcomposition with f. The Beck–Chevalley isomorphisms follow from the pasting law for pullback squares. Its unfurling therefore sends a span $I \xleftarrow{f} K \xrightarrow{g} J$ to $g_! f^* \colon \mathrm{Fin}_{/I} \to \mathrm{Fin}_{/J}$.

Precomposing this unfurling with the leg-swap equivalence $\mathrm{Span}(\mathrm{Fin}) \simeq \mathrm{Span}(\mathrm{Fin})^{\mathrm{op}}$ from Lemma 13.1.15 and then passing to opposites gives a functor $\mathrm{Span}(\mathrm{Fin})^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Let

$$r \colon A \to \mathrm{Span}(\mathrm{Fin})$$

be its cartesian unstraightening. Its fiber over I is $(\mathrm{Fin}_{/I})^{\mathrm{op}}$, and its cartesian transport along a span $I \xleftarrow{f} K \xrightarrow{g} J$ is $(f_! g^*)^{\mathrm{op}}$.

Applying $B \mapsto \mathrm{Fun}(B, C)$ to the straightening of r gives a functor $\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$. It sends I to $\mathrm{Fun}((\mathrm{Fin}_{/I})^{\mathrm{op}}, C)$ and a span $I \xleftarrow{f} K \xrightarrow{g} J$ to restriction along

$$(f_! g^*)^{\mathrm{op}} \colon (\mathrm{Fin}_{/J})^{\mathrm{op}} \longrightarrow (\mathrm{Fin}_{/I})^{\mathrm{op}}.$$

Let $\widetilde{C}^{\times} \to \mathrm{Span}(\mathrm{Fin})$ be its cocartesian unstraightening, and let $C^{\times}_{\mathrm{con}} \subseteq \widetilde{C}^{\times}$ be the fiberwise full subcategory whose fiber over I consists of the finite-product-preserving functors $(\mathrm{Fin}_{/I})^{\mathrm{op}} \to C$. The subscript con indicates that this is the concrete model.

Lemma 15.3.8. *The subcategory $C^{\times}_{\mathrm{con}} \subseteq \widetilde{C}^{\times}$ is closed under cocartesian transport. In particular, $C^{\times}_{\mathrm{con}} \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration.*

Proof. For a span $I \xleftarrow{f} K \xrightarrow{g} J$, the functor $f_! g^* \colon \mathrm{Fin}_{/J} \to \mathrm{Fin}_{/I}$ preserves finite coproducts: pullback along g does so because Fin is extensive, while postcomposition along f plainly does so. Hence $(f_! g^*)^{\mathrm{op}}$ preserves finite products, and restriction along it preserves finite-product-preserving functors. □

Lemma 15.3.9. *For every finite set I, restriction along the functor $I \hookrightarrow (\mathrm{Fin}_{/I})^{\mathrm{op}}$ given by $i \mapsto (\{i\} \hookrightarrow I)$ induces an equivalence*

$$\mathrm{Fun}^{\times}((\mathrm{Fin}_{/I})^{\mathrm{op}}, C) \xrightarrow{\sim} C^I.$$

Proof. Regarding I as a discrete ∞-category, the description of finite families in Construction 14.3.1 gives a finite-coproduct-preserving equivalence

$$\mathrm{Fin}(I) \xrightarrow{\sim} \mathrm{Fin}_{/I}$$

which sends a finite family of elements of I to its indexing map to I. The dual of Lemma 14.3.2 therefore identifies restriction to the singleton families with an equivalence

$$\mathrm{Fun}^{\times}((\mathrm{Fin}_{/I})^{\mathrm{op}}, C) \simeq \mathrm{Fun}(I, C) = C^I.$$

Under this equivalence, a functor F corresponds to its values on the singleton inclusions, and one has

$$F(U \to I) \simeq \prod_{u \in U} F(\{u\} \to I).$$ □

Proposition 15.3.10. *There is a canonical equivalence over* Span(Fin) *between the concrete model $C^{\times}_{\mathrm{con}}$ and the total category $C^{\times}$ of the cartesian monoidal structure.*

Proof. By the uniqueness of cartesian monoidal structures from Proposition 15.3.6, it suffices to show that $C^{\times}_{\mathrm{con}}$ defines a cartesian monoidal structure on C.

By Lemma 15.3.8, the projection $C^{\times}_{\mathrm{con}} \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration. Under the fiber equivalences of Lemma 15.3.9, the canonical maps $F(I \sqcup J) \to F(I) \times F(J)$ for its straightening F are the equivalences

$$C^{I \sqcup J} \simeq C^I \times C^J,$$

and its value at $\emptyset$ is the terminal ∞-category. Thus the straightening preserves finite products, so Lemma 14.4.4 makes $C^{\times}_{\mathrm{con}}$ into a symmetric monoidal ∞-category with underlying ∞-category C.

Transport along a span $I \xleftarrow{f} K \xrightarrow{g} J$ sends an I-indexed family to

$$(x_i)_{i \in I} \longmapsto \left(\prod_{k \in g^{-1}(j)} x_{f(k)} \right)_{j \in J}.$$

Hence its unit is terminal, its binary tensor product is the product in C, and the maps induced by the terminal maps of the factors and the unitors are the product projections. By Lemma 15.3.4, it is therefore cartesian monoidal. □

15.3.3 Algebras in cartesian monoidal structures

Let C be an ∞-category with finite products, and let $\mathcal{O}$ be an ∞-operad. We have introduced two notions of algebraic structure in C: an $\mathcal{O}$-monoid is a finite-product-preserving functor $\mathcal{O}^{\otimes} \to C$, while an $\mathcal{O}$-algebra for the cartesian monoidal structure is an operad map $\mathcal{O} \to \mathcal{O}^{\times}(C)$.

Informally, both specify an object M_x for every color x and an operation

$$P_M \colon M_{x_1} \times \cdots \times M_{x_n} \longrightarrow M_y$$

for every multimorphism $P \in \mathcal{O}((x_1, \ldots, x_n); y)$, compatibly with composition, identities, and permutations. The following theorem identifies these two notions. We will then recover the product functor $C^{\times} \to C$ by naturality.

Theorem 15.3.11 (Universal property of cartesian monoidal structures)**.** *Let C be an ∞-category with finite products. For every ∞-operad $\mathcal{O}$, there is an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(C, \times) \xrightarrow{\sim} \mathrm{Mon}_{\mathcal{O}}(C),$$

natural in $\mathcal{O}$ and in finite-product-preserving functors $C \to C'$.

Proof. We use the concrete model from Construction 15.3.7, identified with $C^{\times}$ by Proposition 15.3.10. Let $r \colon A \to \mathrm{Span}(\mathrm{Fin})$ be the cartesian fibration used in that construction. By Theorem 24.8.3, evaluation gives an equivalence

$$\mathrm{Fun}_{/\mathrm{Span}(\mathrm{Fin})}(\mathcal{O}^{\otimes}, \widetilde{C}^{\times}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathrm{Fin})} A, C). \tag{15.2}$$

The initial objects id_I of the fibers $(\mathrm{Fin}_{/I})^{\mathrm{op}}$ give a section $s \colon \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathrm{Fin})} A$, sending X over I to (X, id_I), by Proposition 24.3.4. Under (15.2), precomposition with s sends an operad map $\Phi \colon \mathcal{O} \to \mathcal{O}^{\times}(C)$ to the functor $X \mapsto \Phi(X)(\mathrm{id}_I)$. This preserves finite products: Φ preserves products, and evaluation at the identity sends concatenation of families in $C^{\times}$ to their product in C, by Lemma 15.3.9.

To construct the inverse, we use inert restriction. By Theorem 15.1.3 and Lemma 13.1.15, A is the span category of $\mathrm{Ar}(\mathrm{Fin})$ with arbitrary squares as left legs and pullback squares as right legs, and r is induced by ev_1. Both evaluation functors preserve pullbacks, and the tautological transformation $\mathrm{ev}_0 \Rightarrow \mathrm{ev}_1$ has pullback naturality squares along the right legs. Thus Remark 13.2.2 gives a natural transformation $\mathrm{Span}(\mathrm{ev}_1) \Rightarrow \mathrm{Span}(\mathrm{ev}_0)$ on A, or equivalently a functor $A \to \mathrm{Ar}(\mathrm{Span}(\mathrm{Fin}))$ sending $u \colon U \to I$ to the backwards span $I \xleftarrow{u} U \xrightarrow{=} U$.

Pulling back along $\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$, applying Proposition 24.1.4, and taking targets gives a functor $\theta \colon \mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathrm{Fin})} A \to \mathcal{O}^{\otimes}$ sending (X, u) to the inert restriction X_U. Applying the same span construction to the natural squares $\mathrm{id}_U \to u$ in $\mathrm{Ar}(\mathrm{Fin})$ gives the cartesian lifts $u \to \mathrm{id}_U$ in A, naturally in u. Together with the inert lifts, these give a natural transformation $\mathrm{id} \to s\theta$ with component $(X, u) \to (X_U, \mathrm{id}_U)$. Restriction along identity maps gives $\theta s \simeq \mathrm{id}$.

For a finite-product-preserving functor $M \colon \mathcal{O}^{\otimes} \to C$, currying $M\theta$ gives a functor $\widehat{M} \colon \mathcal{O}^{\otimes} \to \widetilde{C}^{\times}$ with

$$\widehat{M}(X)(u \colon U \to I) = M(X_U).$$

Inert restriction takes disjoint unions over I to products in $\mathcal{O}^{\otimes}$, so $X_{\coprod_a U_a} \simeq \prod_a X_{U_a}$. Since M preserves finite products, each $\widehat{M}(X)$ preserves finite products. Thus $\widehat{M}$ factors through $C^{\times}_{\mathrm{con}}$.

To check that $\widehat{M}$ preserves inert morphisms, let $V \to U \xrightarrow{u} I$. Transport of $\widehat{M}(X)$ along the backwards span given by u has value $M(X_V)$ at $V \to U$, whereas $\widehat{M}(X_U)$ has value $M((X_U)_V)$. The image of the inert lift $X \to X_U$ has component the canonical isomorphism $M(X_V) \to M((X_U)_V)$ given by composition of inert lifts. Thus $\widehat{M}$ preserves inert morphisms and is a morphism of ∞-operads by Corollary 14.1.14.

Evaluating $\widehat{M}(X)$ at id_I recovers $M(X)$, naturally in M, since $\theta s \simeq \mathrm{id}$. In the other direction, apply the evaluation functor corresponding to Φ under (15.2) to the natural transformation $\mathrm{id} \to s\theta$. This gives maps

$$\Phi(X)(u) \longrightarrow \Phi(X_U)(\mathrm{id}_U),$$

natural in (X, u) and in Φ. They are isomorphisms because Φ preserves inert morphisms and transport along $I \xleftarrow{u} U \xrightarrow{=} U$ sends a functor F to the functor $v \mapsto F(uv)$. Currying gives the required natural isomorphism for the other composite. Thus the two constructions are inverse equivalences.

Naturality in $\mathcal{O}$ follows from pullback and functoriality of inert lifts. Postcomposition with a finite-product-preserving functor $C \to C'$ commutes with currying and preserves the concrete subcategories. Under Proposition 15.3.10, it induces the corresponding map of cartesian operads, proving naturality in C. □

Corollary 15.3.12. *The functor* $\mathrm{Op}_\infty \to \mathrm{Cat}_\infty^{\mathrm{prod}}$ *sending* $\mathcal{O}$ *to* $\mathcal{O}^\otimes$ *is left adjoint to the cartesian operad functor* $\mathcal{O}^\times \colon \mathrm{Cat}_\infty^{\mathrm{prod}} \to \mathrm{Op}_\infty$.

Proof. Pass to groupoid cores in the natural equivalence of Theorem 15.3.11. □

Corollary 15.3.13. *There is a finite-product-preserving functor*

$$\Pi_C \colon C^\times \to C$$

such that the equivalence of Theorem 15.3.11 is given by postcomposition with Π_C.

Proof. Take Π_C to be the counit of Corollary 15.3.12. By naturality in $\mathcal{O}$, the equivalence of the theorem is given by postcomposition with Π_C. In the concrete model it sends $F \colon (\mathrm{Fin}_{/I})^{\mathrm{op}} \to C$ to $F(\mathrm{id}_I)$, so it sends a family $\{x_i\}_{i \in I}$ to $\prod_{i \in I} x_i$ by Lemma 15.3.9. □

Corollary 15.3.14. *Let* C *be an* ∞*-category with finite products. Then associative algebras in the cartesian monoidal structure agree with associative operadic monoids in* C*:*

$$\mathrm{Alg}(C, \times) \xrightarrow{\sim} \mathrm{Mon}_{\mathcal{A}ssoc}(C).$$

Similarly, commutative algebras agree with commutative monoids:

$$\mathrm{CAlg}(C, \times) \xrightarrow{\sim} \mathrm{Mon}_{Comm}(C) = \mathrm{CMon}(C).$$

Proof. This is the special case of Theorem 15.3.11 where $\mathcal{O} = \mathcal{A}ssoc$ and $\mathcal{O} = Comm$, respectively. □

The first equivalence uses the operadic notion $\mathrm{Mon}_{\mathcal{A}ssoc}(C) = \mathrm{Fun}^{\times}(\mathcal{A}ssoc^{\otimes}, C)$. We now relate it to the simplicial definition of $\mathrm{Mon}(C)$ from Chapter 5. The cut functor of Construction 5.3.11 has a refinement

$$\mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{A}ssoc^{\otimes}\,:$$

equip each fiber of a forward map with the order inherited from its source set of cuts. Composition respects these orders, so this defines a functor. A finite-product-preserving functor $A\colon \mathcal{A}ssoc^{\otimes} \to C$ restricts along Cut to a Segal monoid: its value at $[n]$ is $A(\langle 1\rangle)^n$, and the interval inclusions give the product projections. The following result says that every Segal monoid arises in this way.

Theorem 15.3.15 ([Lur17, Proposition 4.1.3.19])**.** *For every ∞-category C with finite products, restriction along* Cut *induces an equivalence*

$$\mathrm{Mon}_{\mathcal{A}ssoc}(C) \xrightarrow{\sim} \mathrm{Mon}(C).$$

Proof. Apply [Lur17, Proposition 4.1.3.19] to the cartesian monoidal structure on C. Its inert-preserving sections over the cut functor identify with Segal monoids in C, while its associative algebras identify with $\mathrm{Mon}_{\mathcal{A}ssoc}(C)$ by Corollary 15.3.14. □

Together with Corollary 15.3.14, this identifies associative algebras in $(C, \times)$ with the monoids of Chapter 5. Their multiplication and unit are given by the middle face map in degree 2 and the degeneracy map from degree 0, respectively.

For $C = \mathrm{Cat}_\infty$, this identifies monoidal ∞-categories with Segal monoids in Cat_∞. More generally, Theorem 15.3.11 identifies $\mathcal{O}$-algebras in $(\mathrm{Cat}_\infty, \times)$ with $\mathcal{O}$-monoidal ∞-categories. Taking $\mathcal{O} = \mathcal{LM}od$, an algebra is a monoidal ∞-category together with an ∞-category left-tensored over it, in the terminology of Definition 17.1.1.

Exercises

Exercise 15.1 (Transport along spans)**.** Let C admit finite products and finite coproducts, and consider a span of finite sets

$$I \xleftarrow{f} K \xrightarrow{g} J.$$

(1) Show that transport along this span in the cocartesian monoidal unfurling is $g_! f^*\colon C^I \to C^J$, given by

$$(X_i)_{i\in I} \longmapsto \left(\coprod_{k\in g^{-1}(j)} X_{f(k)} \right)_{j\in J}.$$

(2) Show that transport in the cartesian monoidal unfurling is $g_* f^*\colon C^I \to C^J$, obtained by replacing the coproducts in the preceding formula by products.

(3) Recover the binary tensor products and tensor units of the cocartesian and cartesian monoidal structures from these formulas.

Exercise 15.2 (A cocartesian monoidal functor)**.** Show that the free-abelian-group functor preserves finite coproducts and hence refines to a symmetric monoidal functor

$$(\mathrm{Set}, \amalg) \longrightarrow (\mathrm{Ab}, \oplus).$$

Identify its unit and binary structure maps explicitly.

16 Day convolution and the tensor product of spectra

Let C and D be symmetric monoidal ∞-categories. It is natural to wonder whether the ∞-category $\mathrm{Fun}(C, D)$ of functors from C to D inherits a symmetric monoidal structure. The *Day convolution*, first introduced by Day [Day71] for classical symmetric monoidal categories, provides an affirmative answer to this question under suitable assumptions on C and D.

Let us first recall the classical construction. Let $(C, \otimes_C, \mathbb{1}_C)$ and $(D, \otimes_D, \mathbb{1}_D)$ be symmetric monoidal 1-categories, and assume that C is small and that D admits all small colimits. For any two functors $F, G\colon C \to D$, their *Day convolution* $F \otimes_{\mathrm{Day}} G$ is a functor $C \to D$ defined as the left Kan extension of the composite functor

$$C \times C \xrightarrow{F \times G} D \times D \xrightarrow{\otimes_D} D$$

along the tensor product functor $\otimes_C\colon C \times C \to C$. More explicitly, for an object $X \in C$, the value of $F \otimes_{\mathrm{Day}} G$ is given by the colimit

$$(F \otimes_{\mathrm{Day}} G)(X) = \mathrm{colim}_{X_0 \otimes_C X_1 \to X} F(X_0) \otimes_D G(X_1),$$

where the colimit is taken over the relative slice category $(\otimes_C)_{/X}$, whose objects are morphisms $X_0 \otimes_C X_1 \to X$ and whose morphisms are compatible maps of pairs (X_0, X_1). Two properties of this construction are relevant for us:

(a) If the tensor product $\otimes_D\colon D \times D \to D$ preserves small colimits in each variable separately, then the Day convolution equips the functor category $\mathrm{Fun}(C, D)$ with a symmetric monoidal structure.

(b) Under this monoidal structure, the ∞-category of commutative algebra objects $\mathrm{CAlg}(\mathrm{Fun}(C, D))$ is equivalent to the ∞-category of lax symmetric monoidal functors from C to D.

The goal of this chapter is to discuss an ∞-categorical analogue of Day convolution. A first ∞-categorical version was constructed by Glasman [Gla16] and subsequently generalized by Lurie [Lur17, Section 2.2.6]. We will prove the form needed here using the Hinich–Winges model for Day convolution between symmetric monoidal ∞-categories. The description of the Hinich–Winges model in Theorem 16.1.6 is the principal input about Day convolution that we take as a black box from the literature. We use the presentability results of Chapter 23 when constructing reflective subcategories.

Our main application is the symmetric monoidal tensor product on the ∞-category Sp of spectra, upgrading the bifunctor $- \otimes -\colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ from Proposition 4.4.16. To this end, we use an equivalent description of Sp as the ∞-category of certain *reduced excisive functors* from $\mathrm{An}_*^{\mathrm{fin}}$ to An. We can then exhibit Sp as a symmetric monoidal Bousfield localization of a Day convolution category.

A second application of Day convolution is that it gives tensor products of commutative monoids and groups in animae. The resulting algebra objects provide homotopy coherent analogues of semirings and rings.

16.1 Day convolution operads

Day convolution is characterized by an internal-hom universal property in the ∞-category of ∞-operads. We take this universal property as our definition, then construct Day convolution operads for symmetric monoidal source and target.

Definition 16.1.1. Let $\mathcal{O}$ and $\mathcal{P}$ be ∞-operads. An ∞-operad $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ is called a *Day convolution operad for $\mathcal{O}$ and $\mathcal{P}$* if it comes equipped with a morphism of ∞-operads

$$\mathrm{ev}\colon \mathcal{D}ay(\mathcal{O},\mathcal{P}) \times \mathcal{O} \to \mathcal{P}$$

satisfying the following universal property: if $\mathcal{Q}$ is another ∞-operad, the composite

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{Q}, \mathcal{D}ay(\mathcal{O},\mathcal{P})) \xrightarrow{-\times\mathcal{O}} \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{Q}\times\mathcal{O}, \mathcal{D}ay(\mathcal{O},\mathcal{P})\times\mathcal{O}) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{Q}\times\mathcal{O},\mathcal{P})$$

is an equivalence. Informally, this means we have the following classification of operad morphisms into $\mathcal{D}ay(\mathcal{O},\mathcal{P})$:

$$\{\text{Operad morphisms } \mathcal{Q} \to \mathcal{D}ay(\mathcal{O},\mathcal{P})\} \quad \leftrightsquigarrow \quad \{\text{Operad morphisms } \mathcal{Q}\times\mathcal{O} \to \mathcal{P}\}.$$

Note that this property uniquely determines $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ if it exists.

Remark 16.1.2. Let $\mathcal{O}$ be an ∞-operad for which $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ exists for all $\mathcal{P}$. Then the functor $-\times\mathcal{O}\colon \mathrm{Op}_\infty \to \mathrm{Op}_\infty$ admits a right adjoint

$$\mathcal{D}ay(\mathcal{O},-)\colon \mathrm{Op}_\infty \to \mathrm{Op}_\infty\,.$$

Whenever the relevant Day convolution operads exist, this internal-hom description is functorial in both variables. An operad map $u\colon \mathcal{O}' \to \mathcal{O}$ induces by precomposition an operad map

$$u^*\colon \mathcal{D}ay(\mathcal{O},\mathcal{P}) \longrightarrow \mathcal{D}ay(\mathcal{O}',\mathcal{P}),$$

and an operad map $v\colon \mathcal{P} \to \mathcal{P}'$ induces by postcomposition an operad map

$$v_*\colon \mathcal{D}ay(\mathcal{O},\mathcal{P}) \longrightarrow \mathcal{D}ay(\mathcal{O},\mathcal{P}').$$

The first map is adjoint to evaluation after precomposition with $\mathrm{id}\times u$, while the second is adjoint to postcomposition of evaluation with v. Their functoriality follows from the universal property. The same universal property gives a natural exponential law

$$\mathcal{D}ay\big(\mathcal{O}, \mathcal{D}ay(\mathcal{P},\mathcal{Q})\big) \simeq \mathcal{D}ay(\mathcal{O}\times\mathcal{P},\mathcal{Q}).$$

Indeed, maps from an ∞-operad $\mathcal{R}$ into either side are naturally equivalent to maps $\mathcal{R}\times\mathcal{O}\times\mathcal{P} \to \mathcal{Q}$.

The universal property of $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ allows us to deduce most of its basic structure. Let us first describe its underlying ∞-category.

Lemma 16.1.3. *Assume $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ exists. Then the underlying ∞-category of $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ is the ∞-category of functors $U(\mathcal{O}) \to U(\mathcal{P})$ between the underlying ∞-categories of $\mathcal{O}$ and $\mathcal{P}$:*

$$U(\mathcal{D}ay(\mathcal{O},\mathcal{P})) \quad \simeq \quad \mathrm{Fun}(U(\mathcal{O}), U(\mathcal{P})).$$

Proof. By Lemma 14.3.5, the universal property of Day convolution, and Lemma 14.3.8, we have natural equivalences

$$\begin{aligned} U(\mathcal{D}ay(\mathcal{O},\mathcal{P})) &\simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv, \mathcal{D}ay(\mathcal{O},\mathcal{P})) \\ &\simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv \times \mathcal{O}, \mathcal{P}) \\ &\simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv_{U(\mathcal{O})}, \mathcal{P}) \\ &\simeq \mathrm{Fun}(U(\mathcal{O}), U(\mathcal{P})), \end{aligned}$$

where the final equivalence is another instance of Lemma 14.3.5. □

As another consequence of the universal property, we may describe the commutative algebras in Day convolution operads:

Lemma 16.1.4. *Assume $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ exists. Then commutative algebras in $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ correspond to operad morphisms $\mathcal{O} \to \mathcal{P}$:*

$$\mathrm{CAlg}(\mathcal{D}ay(\mathcal{O},\mathcal{P})) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O},\mathcal{P}).$$

Proof. Since $\mathcal{C}omm$ is the terminal ∞-operad, we have $\mathcal{O} \simeq \mathcal{C}omm \times \mathcal{O}$, and so

$$\begin{aligned} \mathrm{CAlg}(\mathcal{D}ay(\mathcal{O},\mathcal{P})) &= \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{C}omm, \mathcal{D}ay(\mathcal{O},\mathcal{P})) \\ &\simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{C}omm \times \mathcal{O}, \mathcal{P}) \\ &\simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O},\mathcal{P}). \end{aligned}$$

□

We now construct $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ for symmetric monoidal ∞-categories C and D, using the model of Hinich [Hin20] and its synthetic treatment by Winges [Win26]. By the preceding lemmas, its underlying category will be $\mathrm{Fun}(C,D)$ and its commutative algebras will be the lax symmetric monoidal functors from C to D.

All constructions below are understood in an ambient universe containing the symmetric monoidal ∞-categories under discussion. If C or D is large relative to the basic universe, we enlarge once as in Remark 1.8.1 and carry out the construction there. Further enlargement does not change the resulting Day convolution operad, since its universal property is expressed entirely in terms of the same ∞-operads and their mapping animae.

The main ingredient for this is an explicit description of the unstraightening of the functor

$$(\mathrm{Cat}_\infty^\otimes)^{\mathrm{op}} \times \mathrm{Cat}_\infty^\otimes \to \mathrm{Cat}_\infty, \qquad (C,D) \mapsto \mathrm{Fun}^{\otimes\text{-lax}}(C,D).$$

Definition 16.1.5. We denote by

$$\mathrm{Ar}^{\mathrm{oplax}} \subseteq (\mathrm{Cat}_\infty)_{/[1]}$$

the full subcategory spanned by the cartesian fibrations $E \to [1]$.

Thus $\mathrm{Ar}^{\mathrm{oplax}}$ has the same objects as the ordinary arrow category $\mathrm{Ar}(\mathrm{Cat}_\infty)$, but more morphisms. Under cartesian straightening, an object of $\mathrm{Ar}^{\mathrm{oplax}}$ corresponds to a functor $[1]^{\mathrm{op}} \to \mathrm{Cat}_\infty$. We write such a functor as $F\colon C \to D$, with C as the fiber over 1 and D as the fiber over 0. Since the morphisms in $\mathrm{Ar}^{\mathrm{oplax}}$ do not need to preserve the cartesian edges, they correspond to *laxly* commutative squares of the form

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle F}\downarrow & \swarrow & \downarrow{\scriptstyle F'} \\ D & \longrightarrow & D'. \end{array}$$

Observe that $\mathrm{Ar}^{\mathrm{oplax}}$ is closed under finite products in $(\mathrm{Cat}_\infty)_{/[1]}$, and that pulling back along the inclusions $\{1\} \hookrightarrow [1]$ and $\{0\} \hookrightarrow [1]$ provides a finite-product-preserving functor

$$(s,t)\colon \mathrm{Ar}^{\mathrm{oplax}} \to \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty.$$

Theorem 16.1.6 (Winges [Win26, Theorem 2.7])**.** *Let $\mathcal{O}$ be an ∞-operad, and consider the induced functor*

$$(s,t)\colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{oplax}}) \to \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty) \times \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty).$$

Then:

(1) *The fiber over a pair (C,D) of $\mathcal{O}$-monoidal ∞-categories is the ∞-category $\mathrm{Fun}^{\mathcal{O}\text{-lax}}(C,D)$ of lax $\mathcal{O}$-monoidal functors $C \to D$.*

(2) *For fixed C, the restriction of (s,t) to $\{C\} \times \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty)$ is a cocartesian fibration classifying*

$$\mathrm{Fun}^{\mathcal{O}\text{-lax}}(C,-)\colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty.$$

(3) *For fixed D, the restriction of (s,t) to $\mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty) \times \{D\}$ is a cartesian fibration classifying*

$$\mathrm{Fun}^{\mathcal{O}\text{-lax}}(-,D)\colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty)^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

Remark 16.1.7. Winges in fact shows the stronger claim that this identification of the fibers is natural in both C and D simultaneously. Formulating this precisely requires the notion of *orthofibrations*, which are functors of the form $E \to C \times D$ that correspond under a generalized form of straightening to functors $C^{\mathrm{op}} \times D \to \mathrm{Cat}_\infty$. What Winges shows is that the functor (s,t) from Theorem 16.1.6 is the orthofibration classifying the assignment $(C,D) \mapsto \mathrm{Fun}^{\mathcal{O}\text{-lax}}(C,D)$.

With this result at hand, we may now prove the explicit description of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$:

Theorem 16.1.8 (Hinich [Hin20, Proposition 2.8.9], Winges [Win26, Theorem 1.1])**.** *Let C and D be symmetric monoidal ∞-categories, which we identify with commutative algebras in Cat_∞ with the cartesian monoidal structure. Then the Day convolution operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ exists, and is given by the following pullback square:*

$$\begin{array}{ccc} \mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D) & \longrightarrow & \mathcal{O}^{\times}(\mathrm{Ar}^{\mathrm{oplax}}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ \mathcal{C}omm & \xrightarrow{(C,D)} & \mathcal{O}^{\times}(\mathrm{Cat}_\infty \times \mathrm{Cat}_\infty). \end{array}$$

Proof. Let us write $\mathcal{D}$ for this pullback. We will show directly that $\mathcal{D}$ satisfies the defining universal property of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$. It will suffice to produce for any ∞-operad $\mathcal{P}$ a natural equivalence

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P} \times \mathcal{M}_C, \mathcal{M}_D).$$

For the left-hand side, note that $\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, -)$ commutes with pullbacks. Since $\mathcal{C}omm$ is the terminal ∞-operad, we have $\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{C}omm) \simeq *$. Since operad morphisms from $\mathcal{P}$ into a cartesian operad correspond to $\mathcal{P}$-monoids by Theorem 15.3.11, we obtain equivalences

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{O}^\times(\mathrm{Ar}^{\mathrm{oplax}})) \simeq \mathrm{Mon}_{\mathcal{P}}(\mathrm{Ar}^{\mathrm{oplax}}), \qquad \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{O}^\times(\mathrm{Cat}_\infty{\times}\mathrm{Cat}_\infty)) \simeq \mathrm{Mon}_{\mathcal{P}}(\mathrm{Cat}_\infty{\times}\mathrm{Cat}_\infty).$$

It follows that the ∞-category $\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{D})$ is equivalent to the fiber of the map

$$\mathrm{Mon}_{\mathcal{P}}(\mathrm{Ar}^{\mathrm{oplax}}) \to \mathrm{Mon}_{\mathcal{P}}(\mathrm{Cat}_\infty) \times \mathrm{Mon}_{\mathcal{P}}(\mathrm{Cat}_\infty)$$

over the pair $(C_{\mathcal{P}}, D_{\mathcal{P}})$, where $C_{\mathcal{P}}$ and $D_{\mathcal{P}}$ are obtained from C and D by precomposition with the operad map $p_{\mathcal{P}} : \mathcal{P} \to \mathcal{C}omm$. By Theorem 16.1.6, this fiber is the ∞-category $\mathrm{Fun}^{\mathcal{P}\text{-lax}}(C_{\mathcal{P}}, D_{\mathcal{P}}) = \mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{P}}}(\mathcal{M}_{C_{\mathcal{P}}/\mathcal{P}}, \mathcal{M}_{D_{\mathcal{P}}/\mathcal{P}})$ of lax $\mathcal{P}$-monoidal functors $C_{\mathcal{P}} \to D_{\mathcal{P}}$. By the definition of $C_{\mathcal{P}}$ and $D_{\mathcal{P}}$ we have

$$\mathcal{M}_{C_{\mathcal{P}}/\mathcal{P}} \simeq \mathcal{P} \times_{\mathcal{C}omm} \mathcal{M}_C \qquad \text{and} \qquad \mathcal{M}_{D_{\mathcal{P}}/\mathcal{P}} \simeq \mathcal{P} \times_{\mathcal{C}omm} \mathcal{M}_D,$$

hence

$$\mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{P}}}(\mathcal{M}_{C_{\mathcal{P}}/\mathcal{P}}, \mathcal{M}_{D_{\mathcal{P}}/\mathcal{P}}) \simeq \mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{P}}}(\mathcal{P}{\times}_{\mathcal{C}omm}\mathcal{M}_C, \mathcal{P}{\times}_{\mathcal{C}omm}\mathcal{M}_D) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}{\times}_{\mathcal{C}omm}\mathcal{M}_C, \mathcal{M}_D),$$

where the second equivalence uses the universal property of pullbacks. Since $\mathcal{C}omm$ is terminal, the fiber product $\mathcal{P} \times_{\mathcal{C}omm} \mathcal{M}_C$ is the product $\mathcal{P} \times \mathcal{M}_C$ in Op_∞. This is what we wanted to show. □

Remark 16.1.9. The above description of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ immediately generalizes to a description of the $\mathcal{O}$-monoidal Day convolution operad $\mathcal{D}ay_{/\mathcal{O}}(\mathcal{M}_{C/\mathcal{O}}, \mathcal{M}_{D/\mathcal{O}}) \in (\mathrm{Op}_\infty)_{/\mathcal{O}}$ for $\mathcal{O}$-monoidal ∞-categories C and D. The proof is essentially identical, up to replacing Op_∞ with $(\mathrm{Op}_\infty)_{/\mathcal{O}}$ everywhere. It is this generality in which Hinich and Winges state and prove the result. Note that Theorem 16.1.6, the principal external input about Day convolution, is already stated for an arbitrary ∞-operad $\mathcal{O}$.

Concretely, the universal property reads

$$\mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{O}}}\big(\mathcal{Q}, \mathcal{D}ay_{/\mathcal{O}}(\mathcal{M}_{C/\mathcal{O}}, \mathcal{M}_{D/\mathcal{O}})\big) \quad \simeq \quad \mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{O}}}\big(\mathcal{Q} \times_{\mathcal{O}} \mathcal{M}_{C/\mathcal{O}}, \mathcal{M}_{D/\mathcal{O}}\big)$$

for every ∞-operad $\mathcal{Q}$ over $\mathcal{O}$. If C and D are symmetric monoidal and their $\mathcal{O}$-monoidal structures are obtained by restriction along $\mathcal{O} \to \mathcal{C}omm$, associativity of pullbacks identifies

$$\mathcal{O} \times_{\mathcal{C}omm} \mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D) \quad \simeq \quad \mathcal{D}ay_{/\mathcal{O}}(\mathcal{M}_{C/\mathcal{O}}, \mathcal{M}_{D/\mathcal{O}}).$$

In other words, the underlying $\mathcal{O}$-monoidal ∞-category of the Day convolution is the $\mathcal{O}$-monoidal Day convolution. For $\mathcal{O} = \mathcal{A}ssoc$, this is the relative version underlying the final assertion of Observation 16.6.7, where the relevant functor is lax monoidal but *not* lax symmetric monoidal.

As a consequence of this result, we may explicitly compute the multimorphism animae of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$:

Proposition 16.1.10. *Let C and D be symmetric monoidal ∞-categories, let I be a finite set, and let $\{F_i\colon C \to D\}_{i\in I}$ and $G\colon C \to D$ be functors. Then there is an equivalence*

$$\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)(\{F_i\}_{i\in I}; G) \simeq \mathrm{Nat}\left(\bigotimes_D^I \circ \prod_{i\in I} F_i, G \circ \bigotimes_C^I\right)$$

between the anima of multimorphisms $\{F_i\}_{i\in I} \to G$ in the Day convolution operad and the anima of natural transformations α fitting in the following diagram:

$$\begin{array}{ccc} C^I & \xrightarrow{\prod_{i\in I} F_i} & D^I \\ {\scriptstyle \otimes_C^I}\downarrow & \swarrow & \downarrow{\scriptstyle \otimes_D^I} \\ C & \xrightarrow[G]{} & D. \end{array}$$

Proof. Let us write $O := \mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ for simplicity. By definition, the anima of multimorphisms $O(\{F_i\}_{i\in I}; G)$ is the fiber of the projection map

$$p_O\colon \mathrm{Hom}_{O^\otimes}(\{F_i\}_{i\in I}, G) \to \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, \langle 1\rangle)$$

over the active span $I \xleftarrow{\mathrm{id}} I \to \langle 1\rangle$. By Theorem 16.1.8, the ∞-operad O sits in the following pullback square of ∞-operads:

$$\begin{array}{ccc} O & \longrightarrow & O^\times(\mathrm{Ar}^{\mathrm{oplax}}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ \mathcal{C}omm & \xrightarrow{(C,D)} & O^\times(\mathrm{Cat}_\infty \times \mathrm{Cat}_\infty). \end{array}$$

Since hom animae in a pullback of ∞-categories form a pullback of animae, $\mathrm{Hom}_{O^\otimes}(\{F_i\}_{i\in I}, G)$ is itself the fiber in a pullback square of hom animae. Taking the fibers of these hom animae over the active span $I \to \langle 1\rangle$, we find that $O(\{F_i\}_{i\in I}; G)$ is the fiber in the following pullback square of animae:

$$\begin{array}{ccc} O(\{F_i\}_{i\in I}; G) & \longrightarrow & O^\times(\mathrm{Ar}^{\mathrm{oplax}})(\{\mathrm{Un}(F_i)\}_{i\in I}; \mathrm{Un}(G)) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s_*,t_*)} \\ * & \xrightarrow{(\otimes_C^I, \otimes_D^I)} & O^\times(\mathrm{Cat}_\infty)(\{C\}_{i\in I}; C) \times O^\times(\mathrm{Cat}_\infty)(\{D\}_{i\in I}; D). \end{array}$$

Here the object $\mathrm{Un}(F_i) \in \mathrm{Ar}^{\mathrm{oplax}}$ is the cartesian unstraightening of the functor $[1]^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which takes the arrow $1 \to 0$ to $F_i\colon C \to D$, and similarly for $\mathrm{Un}(G)$. Since Cat_∞ and $\mathrm{Ar}^{\mathrm{oplax}}$ have finite products, we may write the three multimorphism animae in the source and target of the right vertical map as hom animae in $\mathrm{Ar}^{\mathrm{oplax}}$ and Cat_∞ out of the corresponding products, which results in a pullback square as follows:

$$\begin{array}{ccc} O(\{F_i\}_{i\in I}; G) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ar}^{\mathrm{oplax}}}(\mathrm{Un}(\prod_{i\in I} F_i), \mathrm{Un}(G)) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s_*,t_*)} \\ * & \xrightarrow{(\otimes_C^I, \otimes_D^I)} & \mathrm{Hom}_{\mathrm{Cat}_\infty}(C^I, C) \times \mathrm{Hom}_{\mathrm{Cat}_\infty}(D^I, D). \end{array} \tag{16.1}$$

By applying Theorem 16.1.6 to the trivial operad, we get that the functor $(s,t)\colon \mathrm{Ar}^{\mathrm{oplax}} \to \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty$ is a cartesian fibration over $\mathrm{Cat}_\infty \times \{D\}$ and a cocartesian fibration over $\{C\} \times \mathrm{Cat}_\infty$. The fibers of (s,t) are given by the functor categories $\mathrm{Fun}(C,D)$, with functoriality in C given by precomposition and functoriality in D given by postcomposition. The fiber of the upper-right hom anima over a pair (f,g) is therefore the anima of natural transformations $g \circ \prod_i F_i \to G \circ f$. In particular, (16.1) identifies the required fiber as

$$\mathcal{O}(\{F_i\}_{i\in I}; G) \simeq \mathrm{Hom}_{\mathrm{Fun}(C^I,D)}(\bigotimes_D^I \circ \prod_{i\in I} F_i, G \circ \bigotimes_C^I),$$

as claimed. □

16.2 The Day convolution monoidal structure

The preceding section constructed the ∞-operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ and described its multimorphism animae. We now give conditions under which these multimorphisms are represented by tensor products in $\mathrm{Fun}(C,D)$. The resulting Day convolution tensor product is computed by left Kan extension.

Proposition 16.2.1 (Lurie [Lur17, Proposition 2.2.6.16], Winges [Win26, Proposition 4.1])**.** *Let $(C, \otimes_C, \mathbb{1}_C)$ and $(D, \otimes_D, \mathbb{1}_D)$ be symmetric monoidal ∞-categories. Let $\mathcal{K}$ be the collection of all relative slice categories $(\bigotimes_C^n)_{/c}$ defined by the pullback squares*

$$\begin{array}{ccc} (\bigotimes_C^n)_{/c} & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ C^n & \xrightarrow{(\bigotimes_C^n, c)} & C \times C \end{array}$$

for every natural number $n \geq 0$ and every object $c \in C$. Assume that the following two conditions are satisfied:

(1) The ∞-category D admits all $\mathcal{K}$-indexed colimits.

(2) The tensor product functor $\otimes_D\colon D \times D \to D$ preserves $\mathcal{K}$-indexed colimits in each variable separately.

Then the Day convolution operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ defines a symmetric monoidal structure $\otimes_{\mathrm{Day}}$ on $\mathrm{Fun}(C,D)$.

In practice, we will usually apply Proposition 16.2.1 through the following corollary, so we record it before proving the proposition.

Corollary 16.2.2. *Let C and D be symmetric monoidal ∞-categories. Assume that C is small, that D is cocomplete (i.e., admits all small colimits), and that the tensor product of D preserves small colimits in each variable. Then the Day convolution operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ defines a symmetric monoidal structure $\otimes_{\mathrm{Day}}$ on* $\mathrm{Fun}(C,D)$.

Proof. Every relative slice category $(\bigotimes_C^n)_{/c}$ is a pullback of the diagram $C^n \to C \times C \leftarrow \mathrm{Ar}(C)$. Since C is small, so are C^n and $\mathrm{Ar}(C)$, and hence so is $(\bigotimes_C^n)_{/c}$. The collection $\mathcal{K}$ of Proposition 16.2.1 therefore consists of small ∞-categories, so both of its conditions are implied by the assumptions on D. □

Proof of Proposition 16.2.1. We wish to show that the Day convolution operad $O := \mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ defines a symmetric monoidal ∞-category, i.e., that the functor $p_O \colon O^\otimes \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration. Recall from Lemma 14.4.9 that it is enough to show that the following two conditions are satisfied:

(a) For every finite collection of functors $\{F_i\}_{i\in I}$, there exists a functor $G = \bigotimes_{i\in I}^{\mathrm{Day}} F_i$ and a multimorphism $\{F_i\}_{i\in I} \to G$ in O inducing for every other functor $H \colon C \to D$ an equivalence

$$\mathrm{Hom}_{U(O)}(G, H) \xrightarrow{\sim} O(\{F_i\}; H).$$

(b) For a map $f \colon I \to J$ of finite sets, if we let $G_j := \bigotimes_{i\in f^{-1}(j)}^{\mathrm{Day}} F_i$ for all $j \in J$, then the canonical map

$$\bigotimes_{i\in I}^{\mathrm{Day}} F_i \to \bigotimes_{j\in J}^{\mathrm{Day}} G_j$$

is a natural isomorphism of functors $C \to D$.

For part (a), we define G as the left Kan extension of the composite functor

$$\prod_{i\in I} C \xrightarrow{\prod_{i\in I} F_i} \prod_{i\in I} D \xrightarrow{\otimes_D} D$$

along the tensor product functor $\otimes_C \colon C^I \to C$. Explicitly, for an object $c \in C$, its value is given by the colimit:

$$G(c) := \mathrm{colim}_{\{c_i'\}_{i\in I} \in (\bigotimes_C^I)_{/c}} \left(\bigotimes_{i\in I} F_i(c_i') \right).$$

The existence of this left Kan extension is guaranteed by condition (1) of our hypothesis, as the indexing ∞-category $(\bigotimes_C^I)_{/c}$ is in our collection $\mathcal{K}$. The universal property of this left Kan extension provides a natural transformation

$$\eta \colon \bigotimes_D^I \circ \prod_{i\in I} F_i \Longrightarrow G \circ \bigotimes_C^I,$$

which by Proposition 16.1.10 corresponds to a multimorphism $\eta \colon \{F_i\}_{i\in I} \to G$ in O. The universal property of left Kan extension says that for any other functor $H \colon C \to D$, composition with η induces an equivalence

$$\mathrm{Hom}_{U(O)}(G, H) \simeq \mathrm{Nat}(G, H) \xrightarrow{\sim} \mathrm{Nat}\left(\bigotimes_D^I \circ \prod_{i\in I} F_i, H \circ \bigotimes_C^I \right) \simeq O(\{F_i\}_{i\in I}; H),$$

as desired.

We now show that also (b) holds. Let $f\colon I \to J$ be a map in Fin, and write $G_j := \bigotimes^{\mathrm{Day}}_{i \in f^{-1}(j)} F_i$ for all $j \in J$. We need to show that the canonical transformation

$$\bigotimes^{\mathrm{Day}}_{i \in I} F_i \to \bigotimes^{\mathrm{Day}}_{j \in J} G_j$$

is a natural isomorphism of functors $C \to D$. To help visualize this, consider the following diagram:

$$\begin{array}{ccc} C^I & \xrightarrow{\prod_{i\in I} F_i} & D^I \\ \otimes^I_C \Big(\big\downarrow{\scriptstyle (\otimes^f_C)} & & \big\downarrow{\scriptstyle (\otimes^f_D)} \Big) \otimes^I_D \\ C^J & \xrightarrow{\prod_{j\in J} G_j} & D^J \\ \big\downarrow{\scriptstyle \otimes^J_C} & & \big\downarrow{\scriptstyle \otimes^J_D} \\ C & \dashrightarrow & D. \end{array}$$

Since left Kan extensions compose, it will suffice to show that the composite $\bigotimes^J_D \circ \prod_{j\in J} G_j\colon C^J \to D$ is the left Kan extension of the composite $\bigotimes^I_D \circ \prod_{i\in I} F_i\colon C^I \to D$ along $(\bigotimes^f_C)\colon C^I \to C^J$. By the pointwise formula for Kan extensions, this amounts to showing that for every tuple $(c_j)_{j\in J} \in C^J$, the map

$$\mathrm{colim}_{(c'_i)\in(\bigotimes^f_C)_{/c}} \bigotimes^D_{i\in I} F_i(c'_i) \to \bigotimes_{j\in J} G_j(c_j)$$

is a natural isomorphism. Since the functor $\bigotimes^f_C\colon C^I \to C^J$ is a product of the individual functors $\bigotimes^{I_j}_C\colon C^{I_j} \to C$, the indexing ∞-category $(\bigotimes^f_C)_{/c}$ on the left is a product over $j \in J$ of the relative slice categories $(\bigotimes^{I_j}_C)_{/c_j}$. But we also have

$$G_j(c_j) = \mathrm{colim}_{(c''_i)\in(\bigotimes^{I_j}_C)_{/c_j}} \bigotimes^D_{i\in I_j} F_i(c''_i),$$

and since the functor $\bigotimes^D_{j\in J}\colon D^J \to D$ preserves the relevant colimits in each variable separately it follows that the map is a natural isomorphism. This finishes the proof. □

Remark 16.2.3. The proof of the proposition provides an explicit description of the tensor product functor

$$\bigotimes^I_{\mathrm{Day}}\colon \mathrm{Fun}(C,D)^I \to \mathrm{Fun}(C,D)$$

for every finite set I. For $I = \langle 0\rangle$ and $I = \langle 2\rangle$, this specializes as follows:

- The monoidal unit $\mathbb{1}_{\mathrm{Day}}\colon C \to D$ of the Day convolution monoidal structure is the left Kan extension of $\mathbb{1}_D\colon * \to D$ along $\mathbb{1}_C\colon * \to C$.
- The Day convolution $F \otimes_{\mathrm{Day}} G$ of two functors $F, G\colon C \to D$ is the left Kan extension of $C \times C \xrightarrow{F\times G} D \times D \xrightarrow{\otimes_D} D$ along $\otimes_C\colon C \times C \to C$.

Proposition 16.2.4 (Functoriality of monoidal Day convolution)**.** *Assume the relevant Day convolution monoidal structures exist.*

(1) A lax symmetric monoidal functor $u\colon C' \to C$ induces a lax symmetric monoidal precomposition functor

$$u^*\colon \mathrm{Fun}(C, D) \longrightarrow \mathrm{Fun}(C', D).$$

(2) Let $\mathcal{K}$ be the collection of relative slice categories from Proposition 16.2.1. If $v\colon D \to D'$ is symmetric monoidal and preserves $\mathcal{K}$-indexed colimits, then postcomposition induces a symmetric monoidal functor

$$v_*\colon \mathrm{Fun}(C, D) \longrightarrow \mathrm{Fun}(C, D').$$

Proof. Part (1) is induced by precomposition on Day convolution operads. For part (2), postcomposition first gives a lax symmetric monoidal refinement. Since v is symmetric monoidal and preserves the colimits in the pointwise formulas of Remark 16.2.3, it carries each Day convolution product and the Day convolution unit to the corresponding product and unit in $\mathrm{Fun}(C, D')$. Its structure maps are therefore isomorphisms. □

Lemma 16.2.5 (Left Kan extensions and Day convolution)**.** *Let $u\colon A \to B$ be a symmetric monoidal functor, and let D be a symmetric monoidal ∞-category. Assume that the Day convolution monoidal structures on $\mathrm{Fun}(A, D)$ and $\mathrm{Fun}(B, D)$ exist. Suppose that for every object $b \in B$, the ∞-category D admits colimits indexed by the relative slice category $u_{/b}$, and that its tensor product preserves these colimits separately in each variable. Then pointwise left Kan extension along u defines a symmetric monoidal functor*

$$u_!\colon \mathrm{Fun}(A, D) \longrightarrow \mathrm{Fun}(B, D).$$

In particular, this applies when A and B are small, while D is cocomplete and its tensor product preserves small colimits separately in each variable.

Proof. The hypotheses on the relative slice categories ensure that pointwise left Kan extension along u exists. Its right adjoint u^* is lax symmetric monoidal by Proposition 16.2.4. By Proposition 14.5.11, it therefore suffices to show that for every finite collection of functors $F_i\colon A \to D$, indexed by a finite set I, the canonical map

$$u_!\Big(\bigotimes_{i\in I}^{\mathrm{Day}} F_i\Big) \longrightarrow \bigotimes_{i\in I}^{\mathrm{Day}} u_!(F_i)$$

is an isomorphism.

Set

$$H := \bigotimes_{D}^{I} \circ \prod_{i\in I} F_i\colon A^I \to D.$$

By the description of Day convolution in Remark 16.2.3 and the composition law for left Kan extensions, the source of this map is the left Kan extension of H along

$$A^I \xrightarrow{\otimes_A^I} A \xrightarrow{u} B.$$

On the other hand, the pointwise formula for left Kan extensions and the assumptions on the tensor product of D show that

$$\bigotimes_{D}^{I} \circ \prod_{i\in I} u_!(F_i)\colon B^I \to D$$

is the left Kan extension of H along $u^I\colon A^I \to B^I$: the relative slice of u^I over a tuple $(b_i)_{i\in I}$ is the product of the relative slices $u_{/b_i}$, and its colimit may be computed iteratively. Applying Day convolution and composing left Kan extensions therefore identifies its target with the left Kan extension of H along

$$A^I \xrightarrow{u^I} B^I \xrightarrow{\bigotimes_B^I} B.$$

Since u is symmetric monoidal, these two composite functors are naturally equivalent. This map is therefore an isomorphism, as desired. □

As a basic instance of Corollary 16.2.2, we obtain:

Corollary 16.2.6. *Let C be a small symmetric monoidal ∞-category. Then the functor category* $\mathrm{Fun}(C, \mathrm{An})$ *admits a symmetric monoidal structure via Day convolution.*

Proof. The cited corollary applies since An admits small colimits and the functor $-\times-\colon \mathrm{An}\times\mathrm{An} \to \mathrm{An}$ preserves colimits in both variables. □

For later use, we record the following description of the internal hom for Day convolution.

Proposition 16.2.7. *Let C be a small symmetric monoidal ∞-category and let D be a complete closed symmetric monoidal ∞-category for which the Day convolution symmetric monoidal structure on* $\mathrm{Fun}(C, D)$ *exists. Then this symmetric monoidal structure is closed. Its internal hom is pointwise given by the end formula*

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,D)}(F, G)(X) \simeq \int_{Y\in C} \underline{\mathrm{Hom}}_D(F(Y), G(X \otimes_C Y)).$$

Here the integral denotes the end of Definition 24.6.3.

Proof. It suffices to show that for every third functor $H\colon C \to D$ there is a natural equivalence

$$\mathrm{Nat}(H \otimes_{\mathrm{Day}} F, G) \simeq \mathrm{Nat}\left(H, \int_{Y\in C} \underline{\mathrm{Hom}}_D(F(Y), G(- \otimes_C Y))\right).$$

Since the Day convolution is a left Kan extension, the left-hand side is equivalent to the anima

$$\mathrm{Nat}(H(-) \otimes_D F(-), G(- \otimes_C -))$$

of natural transformations of functors $C \times C \to D$. By Proposition 24.6.6, this anima is the end

$$\int_{(X,Y)\in C\times C} \mathrm{Hom}_D(H(X) \otimes_D F(Y), G(X \otimes_C Y)).$$

Using the Fubini rule for ends, Lemma 24.6.5, the adjunction $- \otimes_D F(Y) \dashv \underline{\mathrm{Hom}}_D(F(Y), -)$, and the fact that $\mathrm{Hom}_D(H(X), -)$ preserves limits, we may rewrite this as

$$\begin{aligned}\int_{X\in C}\int_{Y\in C} &\mathrm{Hom}_D(H(X), \underline{\mathrm{Hom}}_D(F(Y), G(X \otimes_C Y)))\\ &\simeq \int_{X\in C} \mathrm{Hom}_D\left(H(X), \int_{Y\in C} \underline{\mathrm{Hom}}_D(F(Y), G(X \otimes_C Y))\right).\end{aligned}$$

Using the end description of animae of natural transformations once more gives the desired result. □

For a presentable target, preservation of colimits by the tensor product has a useful consequence: it guarantees the existence of internal homs. We introduce the standard terminology before applying this to Day convolution.

Definition 16.2.8. A symmetric monoidal ∞-category $(C, \otimes)$ is called *presentably symmetric monoidal* if C is presentable and the tensor product $- \otimes -\colon C \times C \to C$ preserves small colimits separately in both variables.

Proposition 16.2.9. *Every presentably symmetric monoidal ∞-category is closed: for every object $X \in C$, the functor $X \otimes -\colon C \to C$ admits a right adjoint $\underline{\mathrm{Hom}}_C(X, -)$.*

Proof. This follows immediately from Theorem 23.2.5. □

Proposition 16.2.10. *Let C be a small symmetric monoidal ∞-category and let D be presentably symmetric monoidal. Then the Day convolution structure on $\mathrm{Fun}(C, D)$ is presentably symmetric monoidal.*

Proof. The underlying functor category is presentable by Theorem 23.2.2. Since D is complete and closed, Proposition 16.2.7 shows that its Day convolution structure is closed. Thus the tensor product preserves colimits separately in both variables. □

16.3 Examples of Day convolution

This section works out Day convolution for graded and filtered objects and for arrow categories, where it recovers pushout products and smash products.

16.3.1 Graded and filtered objects

Example 16.3.1 (Graded objects)**.** Regard $\mathbb{Z}$ as a discrete symmetric monoidal category under addition, and let C be a cocomplete symmetric monoidal ∞-category whose tensor product preserves colimits separately in each variable. Day convolution equips the ∞-category $\mathrm{Fun}(\mathbb{Z}, C)$ of $\mathbb{Z}$-graded objects with a symmetric monoidal structure. Explicitly,

$$(F \otimes_{\mathrm{Day}} G)_n \simeq \coprod_{i+j=n} F_i \otimes G_j.$$

For $C = \mathrm{Ab}$ this is the usual tensor product of graded abelian groups, and for $C = \mathrm{Sp}$ it is the corresponding tensor product of graded spectra.

Example 16.3.2 (Filtered objects)**.** Under the same hypotheses on C, regard $(\mathbb{Z}, \leq)$ as a poset with its usual ordering and as a symmetric monoidal category under addition. The Day convolution on $\mathrm{Fun}((\mathbb{Z}, \leq), C)$ is the standard tensor product of increasing filtered objects:

$$(F \otimes_{\mathrm{Day}} G)_n \simeq \mathrm{colim}_{i+j \leq n} F_i \otimes G_j.$$

The transition maps are induced by the order on $\mathbb{Z}$. Replacing $(\mathbb{Z}, \leq)$ by $(\mathbb{N}, \leq)$ gives the analogous tensor product of nonnegatively filtered objects.

16.3.2 Pushout products and smash products

Lemma 16.3.3 (Pushout product)**.** *Let C be a symmetric monoidal ∞-category that admits finite colimits and such that $X \otimes -\colon C \to C$ preserves finite colimits for every $X \in C$. Then $\mathrm{Ar}(C)$ admits a symmetric monoidal structure via Day convolution. Its unit is $\emptyset \to \mathbb{1}$, and its tensor product is the pushout product*

$$(f\colon X \to Y) \square (f'\colon X' \to Y') \;:=\; (X \otimes Y') \sqcup_{X\otimes X'} (Y \otimes X') \longrightarrow Y \otimes Y'.$$

Proof. Equip $[1] = \{0 \le 1\}$ with the symmetric monoidal structure given by the minimum. Since the relative tensor-product slices of $([1], \min)$ are finite, Proposition 16.2.1 applies to $\mathrm{Fun}([1], C) = \mathrm{Ar}(C)$. By Remark 16.2.3, the unit is the left Kan extension of $\mathbb{1}\colon * \to C$ along $\{1\} \hookrightarrow [1]$, hence is $\emptyset \to \mathbb{1}$. The binary Day convolution is the left Kan extension along $\min\colon [1] \times [1] \to [1]$; the pointwise formula identifies its value at 0 with the displayed pushout and its value at 1 with $Y \otimes Y'$. □

Lemma 16.3.4 (Smash product)**.** *Let C be a symmetric monoidal ∞-category that admits finite colimits and a terminal object, and assume that $X \otimes -\colon C \to C$ preserves finite colimits for every $X \in C$. Equip $\mathrm{Ar}(C)$ with the pushout-product monoidal structure of Lemma 16.3.3. Then $C_* \subseteq \mathrm{Ar}(C)$ is symmetric monoidal, and the localization functor*

$$\mathrm{cofib}\colon \mathrm{Ar}(C) \to C_*, \qquad (f\colon X \to Y) \longmapsto (* \to \mathrm{cofib}(f))$$

is symmetric monoidal. The tensor product of pointed objects is therefore

$$(x\colon * \to X) \otimes (y\colon * \to Y) = \mathrm{cofib}(X \otimes * \sqcup_{*\otimes *} * \otimes Y \to X \otimes Y).$$

Proof. The inclusion $C_* \hookrightarrow \mathrm{Ar}(C)$ admits a left adjoint given by taking cofibers. By Proposition 14.7.2, it thus remains to show that tensoring in $\mathrm{Ar}(C)$ with an object preserves those morphisms (= commutative squares in C) which induce isomorphisms on cofibers. To see this, consider such a morphism

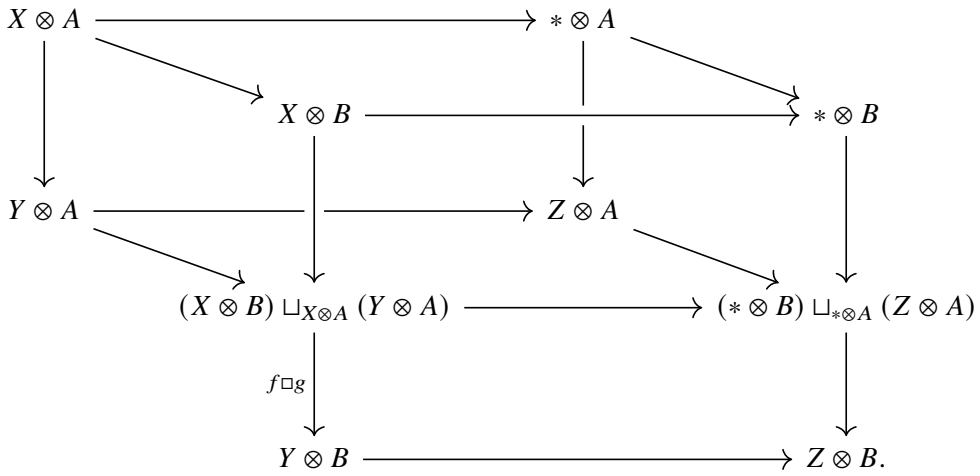
$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y'. \end{array}$$

Let us write $Z := \mathrm{cofib}(f\colon X \to Y)$ and $Z' := \mathrm{cofib}(f'\colon X' \to Y')$, so that the condition on the square is that the induced map $Z \to Z'$ is an isomorphism. Our task is to show that for every other morphism $(g\colon A \to B)$ in C the induced commutative square

$$\begin{array}{ccc} (X \otimes B) \sqcup_{X\otimes A} (Y \otimes A) & \longrightarrow & (X' \otimes B) \sqcup_{X'\otimes A} (Y' \otimes A) \\ {\scriptstyle f\square g}\downarrow & & \downarrow{\scriptstyle f'\square g} \\ Y \otimes B & \longrightarrow & Y' \otimes B \end{array}$$

again induces an isomorphism on (vertical) cofibers. Since $Z \xrightarrow{\sim} Z'$, it will suffice to express the two vertical cofibers purely in terms of g, Z and Z'. We will do so by showing that the bottom front

square in the following commutative diagram is a pushout square:



But this follows from the pasting law for pushout squares: the left and right faces of the diagram are pushouts by definition, while the back square and front rectangle are pushout squares since $- \otimes A$ and $- \otimes B$ preserve the pushout square defining Z as a cofiber of $X \to Y$. □

Definition 16.3.5. Let C be an ∞-category with finite products and finite colimits such that the functor $X \times -\colon C \to C$ preserves finite colimits for all $X \in C$. We may then apply Lemma 16.3.4 to the cartesian monoidal structure on C. We refer to the resulting symmetric monoidal structure $(C_*, \wedge, S^0)$ on the pointed objects in C as the *smash product*. The monoidal unit is $S^0 := * \sqcup *$, while the smash product $X \wedge Y$ is given by

$$X \wedge Y \quad := \quad \mathrm{cofib}(X \vee Y \to X \times Y),$$

where $X \vee Y := X \sqcup_* Y$.

Example 16.3.6. Since the cartesian product in An preserves colimits separately in both variables, this equips An_* with the smash product monoidal structure. Its underlying tensor product is the smash product introduced in Definition 2.4.10.

16.4 Rings and semirings

Recall that an ordinary commutative ring may be equivalently described as a commutative algebra object in the symmetric monoidal category $(\mathrm{Ab}, \otimes)$ of abelian groups. Given two abelian groups A and B, their tensor product $A \otimes B$ is characterized by the property that group homomorphisms $A \otimes B \to C$ correspond to bilinear maps $A \times B \to C$. We now use Day convolution to construct analogous tensor products on $\mathrm{CGrp}(C)$ and $\mathrm{CMon}(C)$ for any presentably symmetric monoidal ∞-category C. This gives general notions of commutative ring and semiring objects.

Proposition 16.4.1. *Let C be a presentably symmetric monoidal ∞-category. The inclusions*

$$\mathrm{CGrp}(C) \hookrightarrow \mathrm{CMon}(C) \hookrightarrow \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$$

admit left adjoints. Moreover, the full subcategories $\mathrm{CMon}(C)$ *and* $\mathrm{CGrp}(C)$ *inherit symmetric monoidal structures for which these left adjoints are symmetric monoidal Bousfield localizations of*

the Day convolution monoidal structure. Their tensor products preserve small colimits separately in both variables.

The free commutative monoid functor

$$F_C^{\mathrm{CMon}} \colon C \longrightarrow \mathrm{CMon}(C),$$

left adjoint to evaluation at $\langle 1 \rangle \in \mathrm{Span}(\mathrm{Fin})$*, is symmetric monoidal.*

Proof. The existence of the two left adjoints is the structured-object case of presentability recorded in Proposition 23.2.7.

The ∞-category $\mathrm{Span}(\mathrm{Fin})$ is small, so Corollary 16.2.2 and Proposition 16.2.10 equip $\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$ with its presentably symmetric monoidal Day convolution structure. It remains to apply the internal-hom criterion of Lemma 14.7.3. By Proposition 16.2.9, the ∞-category C is closed, and the Day convolution internal hom is computed by the end formula

$$\underline{\mathrm{Hom}}_{\mathrm{Day}}(F, G)(S) \simeq \int_{T \in \mathrm{Span}(\mathrm{Fin})} \underline{\mathrm{Hom}}_C\big(F(T), G(S \times T)\big).$$

For every finite set T, the functor $- \times T$ preserves finite coproducts of finite sets, which are the finite products in $\mathrm{Span}(\mathrm{Fin})$. Consequently, if G preserves finite products, then so does the functor $S \mapsto G(S \times T)$. Internal homs and ends preserve limits, so $\underline{\mathrm{Hom}}_{\mathrm{Day}}(F, G)$ again lies in $\mathrm{CMon}(C)$. If G is grouplike, the shear map of $\underline{\mathrm{Hom}}_{\mathrm{Day}}(F, G)$ is obtained by applying the limit-preserving functor $\underline{\mathrm{Hom}}_{\mathrm{Day}}(F, -)$ to the shear isomorphism of G, so $\underline{\mathrm{Hom}}_{\mathrm{Day}}(F, G)$ is grouplike as well. Thus both local subcategories are closed under internal hom with arbitrary F, and Lemma 14.7.3 gives the claimed symmetric monoidal Bousfield localizations.

Part (3) of Proposition 14.7.2 shows that the localized tensor products preserve small colimits separately in both variables.

It remains to consider the free commutative monoid functor. Let $\iota \colon * \to \mathrm{Span}(\mathrm{Fin})$ select the monoidal unit $\langle 1 \rangle$ for the cartesian product of finite sets. Left Kan extension along ι is left adjoint to evaluation at $\langle 1 \rangle$,

$$\iota_! \colon C \simeq \mathrm{Fun}(*, C) \rightleftarrows \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C) \colon \iota^*.$$

The functor ι is symmetric monoidal, and Lemma 16.2.5 shows that $\iota_!$ is symmetric monoidal. The composite of $\iota_!$ with the symmetric monoidal reflection onto $\mathrm{CMon}(C)$ is left adjoint to evaluation at $\langle 1 \rangle$ on commutative monoids, hence is F_C^{CMon}. It is therefore symmetric monoidal. □

Proposition 16.4.2 (Commutative monoids and groups under adjunctions)**.** *Let*

$$L \colon C \rightleftarrows D \colon R$$

be an adjunction between presentably symmetric monoidal ∞-categories, and assume that L is symmetric monoidal. Postcomposition with R induces functors on commutative monoids and groups which admit symmetric monoidal left adjoints:

$$\widetilde{L} \colon \mathrm{CMon}(C) \rightleftarrows \mathrm{CMon}(D) \colon R_* \qquad \textit{and} \qquad \widetilde{L} \colon \mathrm{CGrp}(C) \rightleftarrows \mathrm{CGrp}(D) \colon R_*.$$

If L preserves finite products, then the two functors $\widetilde{L}$ are given by postcomposition with L.

Proof. The right adjoint R preserves finite products, so postcomposition with R preserves commutative monoids. It also preserves grouplike commutative monoids, since it carries shear isomorphisms to shear isomorphisms. This gives the two functors denoted by R_* in the statement.

Let E_C and E_D denote either the commutative-monoid categories or the commutative-group categories of C and D, respectively. Write Q_C and Q_D for the corresponding reflections of Proposition 16.4.1, and i_C and i_D for their fully faithful right adjoints. Postcomposition gives an adjunction

$$L^{\mathrm{Span}} \colon \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C) \rightleftarrows \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), D) : R^{\mathrm{Span}}.$$

The left adjoint of R_* is the composite

$$\widetilde{L} := Q_D L^{\mathrm{Span}} i_C.$$

The adjunctions $Q_D \dashv i_D$ and $L^{\mathrm{Span}} \dashv R^{\mathrm{Span}}$, together with the full faithfulness of i_C, immediately give $\widetilde{L} \dashv R_*$.

Since L is a left adjoint, it preserves small colimits, so Proposition 16.2.4 shows that L^{Span} is symmetric monoidal for the ambient Day convolution structures. Moreover, the two functors

$$\widetilde{L} Q_C \qquad \text{and} \qquad Q_D L^{\mathrm{Span}}$$

are left adjoint to the same functor $R^{\mathrm{Span}} i_D = i_C R_*$. They are therefore naturally isomorphic. Since $Q_D L^{\mathrm{Span}}$ is symmetric monoidal, the universal property of the symmetric monoidal Bousfield localization Q_C supplies $\widetilde{L}$ with a unique symmetric monoidal refinement. Its right adjoint R_* acquires the canonical lax symmetric monoidal structure of Proposition 14.5.6.

If L preserves finite products, postcomposition with L preserves commutative monoids and groups. The object $L^{\mathrm{Span}} i_C(A)$ is then already local for every A, so the localization map

$$L^{\mathrm{Span}} i_C(A) \longrightarrow i_D Q_D L^{\mathrm{Span}} i_C(A)$$

is an isomorphism. Thus $\widetilde{L}$ is given by postcomposition with L, as claimed. □

Lemma 16.4.3 (Bilinear maps)**.** *Let C be a presentably symmetric monoidal ∞-category. For $A, B, M \in \mathrm{CMon}(C)$ there is a natural equivalence*

$$\mathrm{Hom}_{\mathrm{CMon}(C)}(A \otimes B, M) \simeq \mathrm{Nat}_{\mathrm{Span}(\mathrm{Fin})^2}\big(A(-) \otimes_C B(-), M(- \times -)\big).$$

The same formula holds in $\mathrm{CGrp}(C)$ *for commutative group objects. We refer to objects of the anima on the right as* bilinear maps *from A and B to M.*

Proof. The tensor product $A \otimes B$ is the reflection of the ambient Day convolution $A \otimes_{\mathrm{Day}} B$. Since M is local, maps from this reflection to M are the same as maps from $A \otimes_{\mathrm{Day}} B$ to M. The universal property of Day convolution, in the form of Proposition 16.1.10, identifies the latter anima with the displayed anima of natural transformations. The argument for commutative groups is identical. □

Example 16.4.4 (Abelian groups)**.** Take $C = \mathrm{Set}$ with its cartesian monoidal structure. Then $\mathrm{CGrp}(\mathrm{Set}) = \mathrm{Ab}$, and the symmetric monoidal structure supplied by Proposition 16.4.1 is the usual tensor product of abelian groups. Indeed, regard abelian groups A, B and C as product-preserving functors $\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Set}$, so that $A(S) \cong A^S$, and similarly for B and C. By the universal property

of Day convolution and the fact that C is local, maps from the localized Day convolution of A and B to C correspond to families of maps

$$A(S) \times B(T) \longrightarrow C(S \times T)$$

that are functorial in the finite sets S and T and their spans.

Evaluating such a family at $S = T = *$ gives a map $\beta\colon A \times B \to C$. Compatibility with the spans encoding addition shows that β is additive in each variable. Conversely, a bilinear map $\beta\colon A \times B \to C$ determines the family

$$\big((a_s)_{s\in S}, (b_t)_{t\in T}\big) \longmapsto \big(\beta(a_s, b_t)\big)_{(s,t)\in S\times T},$$

and bilinearity is exactly what is needed for compatibility with composition of spans. We therefore obtain natural bijections

$$\mathrm{Hom}_{\mathrm{CGrp}(\mathrm{Set})}(A \otimes B, C) \cong \{\text{Bilinear maps } A \times B \to C\} \cong \mathrm{Hom}_{\mathrm{Ab}}(A \otimes_{\mathbb{Z}} B, C).$$

By Yoneda, the tensor product furnished by the Day convolution localization is the usual tensor product $A \otimes_{\mathbb{Z}} B$.

Proposition 16.4.5 (Path components)**.** *The adjunction*

$$\mathrm{CGrp}(\mathrm{An}) \underset{\longleftarrow}{\overset{\pi_0}{\longrightarrow}} \mathrm{CGrp}(\mathrm{Set}) = \mathrm{Ab}$$

is symmetric monoidal on the left and lax symmetric monoidal on the right, where Ab *is equipped with its usual tensor product.*

Proof. The functor $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ is a symmetric monoidal left adjoint for the cartesian monoidal structures and preserves finite products. Thus Proposition 16.4.2 applies. Finally, Example 16.4.4 identifies the symmetric monoidal structure on $\mathrm{CGrp}(\mathrm{Set}) = \mathrm{Ab}$ with the usual tensor product. □

For a presentably symmetric monoidal ∞-category D, we now set

$$\mathrm{CRig}(D) := \mathrm{CAlg}(\mathrm{CMon}(D)) \qquad \text{and} \qquad \mathrm{CRing}(D) := \mathrm{CAlg}(\mathrm{CGrp}(D)),$$

where $\mathrm{CMon}(D)$ and $\mathrm{CGrp}(D)$ carry the symmetric monoidal structures of Proposition 16.4.1. We refer to their objects as *commutative semiring objects* and *commutative ring objects* in D, respectively. For $D = \mathrm{Set}$, these recover the 1-categories of commutative semirings and commutative rings.

Proposition 16.4.6 (Groupoid cores)**.** *Passing to groupoid cores defines a canonical lax symmetric monoidal functor*

$$(-)^{\simeq}\colon \mathrm{CMon}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{CMon}(\mathrm{An}),$$

where both ∞-categories carry the symmetric monoidal structures of Proposition 16.4.1.

Proof. The inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$ is a symmetric monoidal left adjoint for the cartesian monoidal structures, preserves finite products, and has right adjoint $(-)^{\simeq}$. The result is therefore the commutative-monoid instance of Proposition 16.4.2. □

16.5 Spectra as reduced excisive functors

The construction of Sp in Chapter 4 used prespectra. For the symmetric monoidal structure, a different model is more convenient. We will consider functors from finite pointed animae which send the zero object to a terminal object and pushout squares to pullback squares. We will show that these reduced excisive functors give a second model for spectra, and that their inclusion into the full functor category admits a left adjoint. This realizes Sp as a Bousfield localization of a functor category on which Day convolution is available.

Definition 16.5.1. Let $F\colon C \to D$ be a functor between ∞-categories.

(1) If C admits pushouts, we say that F is *excisive* if it sends pushout squares in C to pullback squares in D.

(2) If C admits a terminal object, we say that F is *reduced* if it sends the terminal object of C to a terminal object of D.

We write $\mathrm{Exc}(C,D) \subseteq \mathrm{Fun}(C,D)$ for the full subcategory of excisive functors, $\mathrm{Fun}_*(C,D) \subseteq \mathrm{Fun}(C,D)$ for the full subcategory of reduced functors, and $\mathrm{Exc}_*(C,D)$ for their intersection.

Notation 16.5.2. We write $\mathrm{An}_*^{\mathrm{fin}}$ for the small ∞-category of finite pointed animae. Its universal property is given in Lemma 4.3.12.

Definition 16.5.3. Let C be an ∞-category with finite limits. We denote by $\mathrm{Sp}^{\mathrm{exc}}(C)$ the full subcategory

$$\mathrm{Sp}^{\mathrm{exc}}(C) := \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \subseteq \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$$

spanned by the reduced, excisive functors from $\mathrm{An}_*^{\mathrm{fin}}$ to C. We denote by $\Omega^\infty\colon \mathrm{Sp}^{\mathrm{exc}}(C) \to C$ the evaluation at $S^0 \in \mathrm{An}_*^{\mathrm{fin}}$.

A reduced functor $\mathrm{An}_*^{\mathrm{fin}} \to C$ has a canonical lift to C_*, obtained from the maps $F(*) \to F(X)$; cf. Lemma 4.1.13. We will therefore freely use the equivalence

$$\mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \simeq \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C_*),$$

and its restriction to reduced excisive functors.

The following criterion is the basic reason that reduced excisive functors behave like spectra.

Construction 16.5.4. Let C be a pointed ∞-category with finite colimits, let D be a pointed ∞-category with finite limits, and let $F\colon C \to D$ be a reduced functor. Applying F to the pushout square defining $\Sigma_C X$ gives a square

$$\begin{array}{ccc} F(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F(\Sigma_C X), \end{array}$$

and hence a canonical map

$$\eta_X\colon F(X) \to \Omega_D F(\Sigma_C X).$$

Proposition 16.5.5 (Excisiveness criterion)**.** *Let C be a pointed ∞-category with finite colimits and let D be a pointed ∞-category with finite limits. A reduced functor $F\colon C \to D$ is excisive if and only if the map $\eta_X\colon F(X) \to \Omega_D F(\Sigma_C X)$ is an isomorphism for every object $X \in C$.*

Proof. If F is excisive, it sends the pushout square defining $\Sigma_C X$ to a pullback square, so the canonical map

$$\eta_X\colon F(X) \longrightarrow \Omega_D F(\Sigma_C X)$$

is an isomorphism.

Conversely, suppose these maps are isomorphisms. Consider a pushout square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ Y & \longrightarrow & W. \end{array}$$

We must show that its comparison map

$$\alpha\colon F(Z) \longrightarrow P := F(X) \times_{F(W)} F(Y)$$

is an isomorphism. Put $A := X \sqcup_Z 0$ and $B := 0 \sqcup_Z Y$, and extend the square by pushouts to the diagram

$$\begin{array}{ccccccc} Z & \to & X & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ Y & \to & W & \longrightarrow & B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \to & \Sigma_C Z & \longrightarrow & \Sigma_C Y \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & \Sigma_C X & \to & \Sigma_C W. \end{array}$$

Applying F, the map $W \to \Sigma_C Z$ and its specified nullhomotopies on X and Y induce a map

$$\beta\colon P \longrightarrow \Omega_D F(\Sigma_C Z).$$

The outer suspension square of Z identifies $\beta\alpha$ with η_Z.

The suspension rectangles of Z, X, Y, W form a coherent square of squares. After applying F and replacing $F(Z)$ by its pullback cone P, they give a square of arrows with vertices $\beta, \eta_X, \eta_Y, \eta_W$. Taking the pullback of the last three arrows therefore identifies the composite

$$P \xrightarrow{\beta} \Omega_D F(\Sigma_C Z) \longrightarrow \Omega_D F(\Sigma_C X) \times_{\Omega_D F(\Sigma_C W)} \Omega_D F(\Sigma_C Y)$$

with the map induced by η_X, η_W, η_Y. This composite is an isomorphism, since all three maps are isomorphisms. Since $\beta\alpha = \eta_Z$ is also an isomorphism, two-out-of-six shows that α is an isomorphism. Thus F is excisive. □

Corollary 16.5.6. *The following conditions are equivalent for a reduced functor $F\colon C \to D$ between stable ∞-categories:*

(1) *F is an exact functor;*

(2) *F commutes with loop objects;*

(3) *F commutes with suspensions.*

Proof. Since C and D are stable, the functor F is exact if and only if it is excisive. By Proposition 16.5.5, this holds if and only if the canonical map $F(X) \to \Omega_D F(\Sigma_C X)$ is an isomorphism for every $X \in C$. By adjunction, this is equivalent to the canonical map $\Sigma_D F(X) \to F(\Sigma_C X)$ being an isomorphism. Thus exactness is equivalent to preservation of suspensions, and preservation of loops is equivalent because suspension and loops are inverse equivalences in both ∞-categories. □

Proposition 16.5.7. *Let C be a pointed ∞-category with finite colimits and let D be an ∞-category with finite limits. Then $\mathrm{Exc}_*(C, D)$ is stable.*

Proof. By replacing D with D_*, we may assume D is pointed. The ∞-category $\mathrm{Exc}_*(C, D)$ has finite limits computed pointwise, and reduced excisive functors are closed under such limits. It is also pointed. By Theorem 4.2.2, it remains to show that its loop functor is an equivalence.

The loop functor is computed pointwise: $\Omega(F) = \Omega_D \circ F$. Define a shift functor by $\mathrm{Shift}(F) := F \circ \Sigma_C$. Since Σ_C preserves zero objects and pushout squares, $\mathrm{Shift}(F)$ is again reduced excisive. Applying an excisive functor F to the pushout square defining suspension gives natural isomorphisms

$$F \simeq \Omega_D F \Sigma_C = \Omega(\mathrm{Shift}\, F) = \mathrm{Shift}(\Omega F),$$

so Ω and Shift are inverse equivalences. □

Lemma 16.5.8. *If C is stable, the evaluation functor $\Omega^\infty \colon \mathrm{Sp}^{\mathrm{exc}}(C) \to C$ is an equivalence.*

Proof. When C is stable, a reduced functor $\mathrm{An}_*^{\mathrm{fin}} \to C$ is excisive if and only if it preserves finite colimits. Hence

$$\mathrm{Sp}^{\mathrm{exc}}(C) = \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \simeq \mathrm{Fun}_*^{\mathrm{rex}}(\mathrm{An}_*^{\mathrm{fin}}, C).$$

Evaluation at S^0 is an equivalence by the universal property of $\mathrm{An}_*^{\mathrm{fin}}$ from Notation 16.5.2. □

Corollary 16.5.9. *The ∞-category $\mathrm{Sp}^{\mathrm{exc}}(C)$ is stable for every ∞-category C with finite limits.*

Proof. Take the source ∞-category in Proposition 16.5.7 to be $\mathrm{An}_*^{\mathrm{fin}}$. □

Proposition 16.5.10. *For every ∞-category C with finite limits, the evaluation functor $\mathrm{ev}_{S^0} \colon \mathrm{Sp}^{\mathrm{exc}}(C) \to C$ exhibits $\mathrm{Sp}^{\mathrm{exc}}(C)$ as a stabilization of C.*

Proof. The functor ev_{S^0} is left exact, since limits in the functor category are computed pointwise and reduced excisive functors are closed under limits. Let D be stable. Since pushout squares and pullback squares agree in D, a functor $D \to C$ is left exact if and only if it is reduced and excisive. Thus it is enough to show that

$$\mathrm{Exc}_*(D, \mathrm{Sp}^{\mathrm{exc}}(C)) \to \mathrm{Exc}_*(D, C)$$

is an equivalence. Since reducedness and excisiveness are pointwise conditions, currying restricts to the relevant full subcategories and identifies the source with

$$\mathrm{Exc}_*(D, \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C)) \simeq \mathrm{Sp}^{\mathrm{exc}}(\mathrm{Exc}_*(D, C)).$$

The target $\mathrm{Exc}_*(D, C)$ is stable by Proposition 16.5.7, so evaluation at S^0 is an equivalence by Lemma 16.5.8. □

Corollary 16.5.11. *There is a unique equivalence*

$$\mathrm{Sp} \simeq \mathrm{Sp}^{\mathrm{exc}}(\mathrm{An}) = \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$$

under which the two functors to An *given by* Ω^∞ *and evaluation at* S^0 *agree.*

Proof. By Propositions 16.5.10 and 4.3.16, both sides, equipped with the displayed functors to An, are stabilizations of An. The result follows from the uniqueness of stabilization. □

Proposition 16.5.12 (Excisive approximation)**.** *Let* C *be an* ∞*-category that admits finite limits, finite colimits, and sequential colimits, and assume that the loop functor* $\Omega\colon C_* \to C_*$ *preserves sequential colimits. Then the two inclusions*

$$\mathrm{Sp}^{\mathrm{exc}}(C) = \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \qquad \textit{and} \qquad \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$$

admit left adjoints

$$P_1\colon \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \to \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \qquad \textit{and} \qquad (-)^{\mathrm{red}}\colon \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C) \to \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C).$$

Proof. For the reduction functor, let $F\colon \mathrm{An}_*^{\mathrm{fin}} \to C$ be arbitrary. Since $*$ is initial in $\mathrm{An}_*^{\mathrm{fin}}$, the unique natural transformation $\mathrm{const}_* \to \mathrm{id}$ induces $\mathrm{const}_{F(*)} \to F$. Define

$$F^{\mathrm{red}}(X) := * \sqcup_{F(*)} F(X).$$

This is reduced. If G is reduced, every natural transformation $F \to G$ vanishes on $F(*)$, and hence factors uniquely through $F \to F^{\mathrm{red}}$. Thus $(-)^{\mathrm{red}}$ is left adjoint to the inclusion of reduced functors.

For P_1, we may replace C by C_*, which inherits finite and sequential colimits from C, so assume C is pointed. Write

$$T(F) := \Omega F \Sigma$$

for the endofunctor of $\mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C)$, equipped with the natural transformation $\eta\colon \mathrm{id} \to T$ from Construction 16.5.4. Apply the final composite identity in the proof of Proposition 16.5.5 to the suspension square of X. In the notation of that proof, $P = T(F)(X)$ and β is the identity, so it gives a natural identification

$$T(\eta_F) \simeq \eta_{T(F)}.$$

Now let $F\colon \mathrm{An}_*^{\mathrm{fin}} \to C$ be reduced. Iterate η to form

$$F \xrightarrow{\eta_F} T(F) \xrightarrow{\eta_{T(F)}} T^2(F) \longrightarrow \cdots$$

and define

$$P_1(F) := \mathrm{colim}_{n \geq 0}\, T^n(F).$$

Thus $T^n(F) \simeq \Omega^n F \Sigma^n$, with the transition maps given by $\eta_{T^n(F)}$.

The functor T preserves sequential colimits by assumption. Consequently,

$$T(P_1(F)) \simeq \mathrm{colim}_{n \geq 0}\, T^{n+1}(F),$$

and under this isomorphism $\eta_{P_1(F)}$ is the map from the defining telescope to its tail. The shift $n \mapsto n+1$ is cofinal, so this map is an isomorphism. The excisiveness criterion now shows that $P_1(F)$ is excisive.

It remains to verify the universal property. The coherence $T(\eta_G) \simeq \eta_{T(G)}$ identifies the sequential diagram defining $P_1(T(G))$ with the tail of the diagram defining $P_1(G)$, compatibly with

$$P_1(\eta_G)\colon P_1(G) \longrightarrow P_1(T(G)).$$

Thus $P_1(\eta_G)$ corresponds to the cofinal tail map and is an isomorphism. Moreover, P_1 preserves sequential colimits, since T does.

Let $u_F\colon F \to P_1(F)$ be the map into the telescope. Since P_1 preserves sequential colimits, the map

$$P_1(u_F)\colon P_1(F) \longrightarrow P_1(P_1(F))$$

is the sequential colimit of the maps $P_1(F) \to P_1(T^n(F))$. Each of these is a composite of the isomorphisms $P_1(\eta_{T^k(F)})$, so $P_1(u_F)$ is an isomorphism. If G is already excisive, the excisiveness criterion shows that every map in the defining telescope for $P_1(G)$ is an isomorphism, so $u_G\colon G \to P_1(G)$ is an isomorphism. In particular, $u_{P_1(F)}$ is an isomorphism. By Proposition 22.7.9, the functor P_1 is left adjoint to the inclusion of reduced excisive functors. □

Corollary 16.5.13. *The conclusion of Proposition 16.5.12 applies when $C = \mathrm{An}$ or $C = \mathrm{An}_*$.*

Proof. Both ∞-categories admit the required limits and colimits. Moreover, filtered colimits of animae commute with finite limits, so their loop functors preserve sequential colimits. □

Lemma 16.5.14. *The evaluation functor*

$$\mathrm{ev}_{S^0}\colon \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}_*) \to \mathrm{An}_*$$

admits a left adjoint. It sends a pointed anima X to the excisive approximation of the smash product functor

$$P_1(X \wedge -)\colon \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}_*.$$

Proof. First consider evaluation at S^0 on the full functor category. Its left adjoint is the left Kan extension along $S^0\colon * \to \mathrm{An}_*^{\mathrm{fin}}$, which sends a pointed anima X to the functor

$$K \longmapsto X \wedge \mathrm{fgt}(K)_+.$$

Its value at the zero object is X. Applying the reduction functor gives $K \mapsto X \wedge K$, since the cofiber of the canonical map $S^0 \to \mathrm{fgt}(K)_+$ is K. Thus $X \wedge -$ is left adjoint to evaluation on reduced functors. Composing with the excisive approximation P_1, which exists by Corollary 16.5.13, gives the asserted left adjoint on reduced excisive functors. □

16.6 The symmetric monoidal structure on Sp

We will now combine the results from the previous sections to upgrade the tensor product $X \otimes Y$ of spectra from Section 4.4.3 to a symmetric monoidal structure $(\mathrm{Sp}, \otimes, \mathbb{S})$ on the ∞-category of spectra.

Let us start by outlining the strategy. By Corollary 16.5.11, we have an equivalence $\mathrm{Sp} \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ between the ∞-category of spectra and the ∞-category of reduced excisive functors

from finite pointed animae to animae. In particular, we may regard Sp as a full subcategory of the functor category $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$. By Proposition 16.2.1, this functor category admits the Day convolution monoidal structure. We will show that Sp is a symmetric monoidal Bousfield localization of it.

As the first step in this plan, we need to equip $\mathrm{An}_*^{\mathrm{fin}}$ with a monoidal structure. The smash product on An_* from Definition 16.3.5 restricts to $\mathrm{An}_*^{\mathrm{fin}}$ by Lemma 14.7.1. Indeed, for fixed $X \in \mathrm{An}_*^{\mathrm{fin}}$, the functor $X \wedge -$ preserves finite colimits, so the fact that $\mathrm{An}_*^{\mathrm{fin}}$ is generated under finite colimits by S^0 reduces closure under smash product to $X \wedge S^0 \xrightarrow{\sim} X$.

Combining everything, we can now construct the tensor product on Sp:

Theorem 16.6.1. *The ∞-category* Sp *admits a unique symmetric monoidal structure such that the reduced excisive approximation functor*

$$P_1^{\mathrm{red}} \colon \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \to \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \simeq \mathrm{Sp}$$

is symmetric monoidal with respect to Day convolution on its source. This structure has the following properties:

(1) Its monoidal unit is the sphere spectrum $\mathbb{S}$, and the unreduced suspension spectrum functor $\mathbb{S}[-] \colon (\mathrm{An}, \times) \to (\mathrm{Sp}, \otimes)$ is strongly monoidal.

(2) Its tensor product preserves small colimits separately in both variables.

(3) Its underlying bifunctor agrees with the tensor product of spectra characterized in Proposition 4.4.16.

Proof. Equip $\mathrm{An}_*^{\mathrm{fin}}$ with the smash product monoidal structure from Definition 16.3.5, and equip An with the cartesian monoidal structure. The ∞-category $\mathrm{An}_*^{\mathrm{fin}}$ is small by Notation 16.5.2. Corollary 16.2.6 therefore gives the Day convolution monoidal structure on $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$.

By Corollary 16.5.13, the inclusion

$$\iota \colon \mathrm{Sp} \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$$

admits the left adjoint

$$P_1^{\mathrm{red}} := P_1 \circ (-)^{\mathrm{red}}.$$

We show that this Bousfield localization satisfies the internal-hom criterion of Lemma 14.7.3. By Proposition 16.2.7, the Day convolution monoidal structure on $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ admits an internal hom given by

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})}(F, G)(X) \simeq \int_{Y \in \mathrm{An}_*^{\mathrm{fin}}} \underline{\mathrm{Hom}}_{\mathrm{An}}(F(Y), G(X \wedge Y)).$$

Here the internal hom in An agrees with the hom anima:

$$\underline{\mathrm{Hom}}_{\mathrm{An}}(A, B) \simeq \mathrm{Hom}_{\mathrm{An}}(A, B).$$

By Lemma 14.7.3, it now remains to show that this internal hom is reduced and excisive whenever G is.

We first check that it is reduced. If X is the zero object of $\mathrm{An}_*^{\mathrm{fin}}$, then so is $X \wedge Y$, hence $G(X \wedge Y)$ is terminal by reducedness of G. Since $\underline{\mathrm{Hom}}_{\mathrm{An}}(F(Y), -)$ preserves terminal animae and an end of terminal animae is terminal, this shows that the internal hom is reduced.

We now check that it is also excisive. Note that $- \wedge Y$ preserves pushouts, and so the functor $G(- \wedge Y)\colon \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}$ sends pushouts to pullbacks for all Y. The same then holds true for $\underline{\mathrm{Hom}}_{\mathrm{An}}(F(Y), G(- \wedge Y))$, and thus also for the end-expression since its defining limit commutes with pullbacks. We conclude that $\underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})}(F, G)$ is excisive.

The internal-hom criterion, together with Proposition 14.7.2, now supplies the symmetric monoidal structure on the full subcategory of reduced excisive functors and the strong symmetric monoidal refinement of P_1^{red}. The ambient Day convolution structure is presentably symmetric monoidal by Proposition 16.2.10, so part (3) of Proposition 14.7.2 shows that the tensor product on Sp preserves small colimits separately in both variables.

To see uniqueness, suppose that another symmetric monoidal structure made P_1^{red} symmetric monoidal. For local objects X_i and Y, adjunction and symmetric monoidality give

$$\mathcal{M}_{\mathrm{Sp}}(\{X_i\}; Y) \simeq \mathcal{D}ay(\mathcal{M}_{\mathrm{An}_*^{\mathrm{fin}}}, \mathcal{M}_{\mathrm{An}})(\{\iota(X_i)\}; \iota(Y)).$$

Its multimorphism operad is therefore the full suboperad of the ambient Day convolution operad on the local objects, which is determined by the localization. The remaining assertions are proved in Proposition 16.6.2 and Corollary 16.6.3. □

Proposition 16.6.2. *The unreduced suspension spectrum functor* $\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}$ *lifts to a strongly monoidal functor* $\mathbb{S}[-]\colon (\mathrm{An}, \times) \to (\mathrm{Sp}, \otimes)$. *In particular, the monoidal unit of* Sp *is the sphere spectrum* $\mathbb{S}$.

Proof. We will show that $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$ is lax symmetric monoidal and that it has a strong monoidal left adjoint.

Under the equivalence $\mathrm{Sp} \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$, we may identify Ω^∞ with the composite

$$\mathrm{Sp} \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \xrightarrow{\mathrm{ev}_{S^0}} \mathrm{An}.$$

The first functor is lax symmetric monoidal by construction of the tensor product of Sp. The second is lax symmetric monoidal by the precomposition statement of Proposition 16.2.4, applied to the symmetric monoidal functor $S^0\colon * \to \mathrm{An}_*^{\mathrm{fin}}$.

The left adjoint of the first functor is symmetric monoidal by construction of the tensor product on Sp. Since $\mathrm{An}_*^{\mathrm{fin}}$ is small and the cartesian product of animae preserves small colimits separately in both variables, Lemma 16.2.5 shows that left Kan extension along $S^0\colon * \to \mathrm{An}_*^{\mathrm{fin}}$ is symmetric monoidal. The left adjoint $\mathbb{S}[-]$ of the displayed composite is therefore symmetric monoidal. □

Corollary 16.6.3. *The tensor product functor* $- \otimes -\colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ *defined by Theorem 16.6.1 agrees with the one characterized by Proposition 4.4.16.*

Proof. By parts (1) and (2) of Theorem 16.6.1, the tensor product preserves colimits in both variables, and the relation $\mathbb{S} \otimes \mathbb{S} \simeq \mathbb{S}$ holds since $\mathbb{S}$ is the monoidal unit. These are the properties that determine the tensor product uniquely in Proposition 4.4.16, so the two agree. □

16.6.1 The multiplicative infinite loop space machine

The Day convolution structures constructed above are compatible with the free functors from animae to commutative monoids, commutative groups, and spectra.

Proposition 16.6.4. *The lax symmetric monoidal functors*

$$\mathrm{Sp} \xrightarrow{\Omega^\infty} \mathrm{CGrp}(\mathrm{An}) \hookrightarrow \mathrm{CMon}(\mathrm{An}) \xrightarrow{\mathrm{fgt}} \mathrm{An}_* \xrightarrow{\mathrm{fgt}} \mathrm{An}$$

admit strongly monoidal left adjoints:

$$(\mathrm{An}, \times) \xrightarrow{(-)_+} (\mathrm{An}_*, \wedge) \xrightarrow{F^{\mathrm{CMon}}} (\mathrm{CMon}(\mathrm{An}), \otimes) \xrightarrow{(-)^{\mathrm{grp}}} (\mathrm{CGrp}(\mathrm{An}), \otimes) \xhookrightarrow{\mathbf{B}^\infty} (\mathrm{Sp}, \otimes).$$

Here, $(-)_+$ *adjoins a disjoint basepoint,* F^{CMon} *is the free commutative monoid functor,* $(-)^{\mathrm{grp}}$ *is group completion, and* $\mathbf{B}^\infty$ *is infinite delooping.*

Proof. The functor

$$(\mathrm{An}, \times) \longrightarrow \big(\mathrm{Ar}(\mathrm{An}), \square\big), \qquad X \longmapsto (\emptyset \to X),$$

is symmetric monoidal: the pushout-product formula gives $(\emptyset \to X) \square (\emptyset \to Y) \simeq (\emptyset \to X \times Y)$, and the unit is $\emptyset \to *$. Its composite with the symmetric monoidal cofiber functor of Lemma 16.3.4 is $(-)_+$, so adjoining a basepoint is strongly monoidal.

The same conclusion for F^{CMon} follows from the bilinear universal property of Lemma 16.4.3. Indeed, for pointed animae X, Y and a commutative monoid M, there are natural equivalences

$$\begin{aligned}\mathrm{Hom}_{\mathrm{CMon}(\mathrm{An})}(F^{\mathrm{CMon}}(X) \otimes F^{\mathrm{CMon}}(Y), M) &\simeq \mathrm{Nat}_{\mathrm{Span}(\mathrm{Fin})^2}\big(F^{\mathrm{CMon}}(X)(-) \times F^{\mathrm{CMon}}(Y)(-), M(- \times -)\big)\\ &\simeq \mathrm{Hom}_{\mathrm{An}_*}(X \wedge Y, \mathrm{fgt}(M))\\ &\simeq \mathrm{Hom}_{\mathrm{CMon}(\mathrm{An})}(F^{\mathrm{CMon}}(X \wedge Y), M).\end{aligned}$$

The middle equivalence is obtained by applying the free commutative monoid adjunction in each variable; the functors $S \mapsto M(S \times T)$ remain commutative monoids because $- \times T$ preserves the finite products of Span(Fin). The analogous nullary argument identifies the monoidal units. The group completion functor is strongly monoidal because it is the symmetric monoidal Bousfield localization of Proposition 16.4.1.

It remains to consider infinite delooping. Since Sp is additive, evaluation $U\colon \mathrm{CGrp}(\mathrm{Sp}) \to \mathrm{Sp}$ is an equivalence by Proposition 5.3.17. Its left adjoint, the free commutative group functor $F_{\mathrm{Sp}}^{\mathrm{CGrp}}$, is symmetric monoidal by Proposition 16.4.1. Hence its inverse U is symmetric monoidal as well.

Applying Proposition 16.4.2 to the symmetric monoidal adjunction $\mathbb{S}[-] \dashv \Omega^\infty$ of Proposition 16.6.2 gives a symmetric monoidal left adjoint $\widetilde{L}$ to postcomposition with Ω^∞. Thus the composite

$$\mathrm{CGrp}(\mathrm{An}) \xrightarrow{\widetilde{L}} \mathrm{CGrp}(\mathrm{Sp}) \xrightarrow{U} \mathrm{Sp}$$

is symmetric monoidal. Its right adjoint $(\Omega^\infty)_* F_{\mathrm{Sp}}^{\mathrm{CGrp}}$ is a finite-product-preserving lift of Ω^∞ to $\mathrm{CGrp}(\mathrm{An})$: evaluation commutes with postcomposition, and $U F_{\mathrm{Sp}}^{\mathrm{CGrp}} \simeq \mathrm{id}_{\mathrm{Sp}}$. By the uniqueness assertion in Theorem 5.4.6, this right adjoint is $\mathbf{\Omega}^\infty$. Uniqueness of left adjoints therefore identifies the displayed composite with $\mathbf{B}^\infty$, giving its symmetric monoidal refinement. Finally, the right adjoints acquire the displayed lax symmetric monoidal structures by Proposition 14.5.6. □

Corollary 16.6.5. *Infinite delooping induces a symmetric monoidal equivalence*

$$\mathbf{B}^{\infty}\colon \big(\mathrm{CGrp}(\mathrm{An}),\otimes\big) \xrightarrow{\simeq} \big(\mathrm{Sp}_{\geq 0},\otimes\big).$$

Proof. The underlying functor is an equivalence by Theorem 5.4.6, and it is symmetric monoidal by Proposition 16.6.4. □

16.6.2 Graded homotopy groups

The ordinary Day convolution tensor product on $\mathrm{Ab}^{\mathbb{Z}} = \mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$ is

$$(A \otimes B)_k = \bigoplus_{n+m=k} A_n \otimes B_m.$$

Its usual symmetry merely interchanges the two factors. For homotopy groups, the natural symmetry is instead the following signed variant.

Definition 16.6.6 (Koszul symmetry)**.** Keeping the same underlying Day convolution monoidal structure on $\mathrm{Ab}^{\mathbb{Z}}$, we replace its ordinary symmetry by the *Koszul symmetry*, which on homogeneous elements is given by

$$A_n \otimes B_m \longrightarrow B_m \otimes A_n, \qquad a \otimes b \longmapsto (-1)^{nm} b \otimes a.$$

We refer to the resulting symmetric monoidal category as the category of graded abelian groups with the Koszul symmetry.

Since $\mathrm{Ab}^{\mathbb{Z}}$ is a 1-category, it is enough to check the ordinary hexagon and involutivity axioms. These follow directly from the fact that $(n, m) \mapsto (-1)^{nm}$ is a symmetric bicharacter: bilinearity gives the two hexagon identities, while $(-1)^{nm}(-1)^{mn} = 1$ gives involutivity.

Observation 16.6.7 (Graded homotopy groups)**.** The functor

$$\pi_*\colon \mathrm{Sp} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

admits a lax symmetric monoidal refinement when the target is equipped with the Koszul symmetry.

Indeed, for all $n, m \in \mathbb{Z}$ the tensor product of representatives defines natural pairings

$$\pi_n(X) \otimes \pi_m(Y) \longrightarrow \pi_{n+m}(X \otimes Y), \qquad [x] \otimes [y] \longmapsto \big[\mathbb{S}^{n+m} \simeq \mathbb{S}^n \otimes \mathbb{S}^m \xrightarrow{x\otimes y} X \otimes Y\big].$$

Associativity and unitality are inherited from the symmetric monoidal structure on Sp. Interchanging x and y introduces the symmetry

$$\mathbb{S}^n \otimes \mathbb{S}^m \longrightarrow \mathbb{S}^m \otimes \mathbb{S}^n,$$

which acts by $(-1)^{nm}$ on $\mathbb{S}^{n+m}$. For $n, m \geq 0$, this is the sign of the block permutation interchanging n and m suspension coordinates; the general case follows by invertibility of suspension. This iterates the basic sign calculation of Warning 4.2.11. Thus the symmetry compatibility is exactly the Koszul rule. Associativity, unitality, and symmetry give all coherence conditions because the target is a 1-category, so the pairings assemble into the claimed lax symmetric monoidal structure.

With the ordinary, unsigned Day symmetry, the same construction is only lax monoidal. Consequently, the homotopy groups of an associative ring spectrum form a graded ring, while those of a commutative ring spectrum form a graded-commutative ring:

$$xy = (-1)^{nm} yx \qquad (x \in \pi_n R,\ y \in \pi_m R).$$

Exercises

Exercise 16.1 (The tensor-product filtration)**.** Let $F, G\colon (\mathbb{Z}, \leq) \to \mathrm{Ab}$ be filtered abelian groups, and put $A := \mathrm{colim}_i F_i$ and $B := \mathrm{colim}_j G_j$. Show that the canonical map

$$(F \otimes_{\mathrm{Day}} G)_n \longrightarrow A \otimes B$$

has image

$$\sum_{i+j \leq n} \mathrm{im}\big(F_i \otimes G_j \longrightarrow A \otimes B\big).$$

Explain why these images define the usual tensor-product filtration on $A \otimes B$.

Exercise 16.2 (A concrete description of the smash product)**.** Let X and Y be pointed animae. Use Lemma 16.3.4 to construct a natural isomorphism

$$X \wedge Y \simeq (X \times Y)/(X \vee Y),$$

where the right-hand side denotes the pushout which sends $X \vee Y$ to the basepoint along the canonical map $X \vee Y \to X \times Y$. Deduce that $S^m \wedge S^n \simeq S^{m+n}$ for $m, n \geq 0$.

Exercise 16.3 (Tensor products of abelian groups)**.** Recover the usual universal property of $A \otimes_{\mathbb{Z}} B$ for abelian groups. Then compute

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n$$

and identify the result as a cyclic group.

Exercise 16.4 (Excisive approximation of the identity)**.** Let $U\colon \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}$ be the forgetful functor, which is reduced. Identify its excisive approximation at a finite pointed anima X with

$$\mathrm{colim}_n\, \Omega^n \Sigma^n X.$$

Relate this anima to $\Omega^\infty \Sigma^\infty X$.

Exercise 16.5 (Monoid animae and ring spectra)**.** Use the symmetric monoidal functor

$$\Sigma_+^\infty\colon (\mathrm{An}, \times) \longrightarrow (\mathrm{Sp}, \otimes)$$

to construct an associative ring spectrum $\Sigma_+^\infty M$ from a monoid anima M, and a commutative ring spectrum when M is commutative. For a discrete monoid M, identify the induced multiplication on $\pi_0(\Sigma_+^\infty M)$ with that of the monoid ring $\mathbb{Z}[M]$.

17 Algebras and modules

The operads $\mathcal{A}ssoc$ and $\mathcal{LM}od$ introduced in Chapter 12 allow us to define associative algebras and their modules in an arbitrary monoidal ∞-category. Most of the familiar structures from ordinary algebra admit anaogues in this generality: We can construct free modules, form relative tensor products over an algebra, and extend scalars along algebra maps. We already encountered these constructions in Chapter 8, where we took them as a black box. We now provide some details on the constructions.

We start with various basic constructions on algebras and modules. The monoidal unit in a symmetric monoidal ∞-category admits a unique commutative algebra structure, making it the initial commutative algebra. The forgetful functor that forgets the left module structure over an associative algebra A admits a left adjoint, given on objects by sending X to $A \otimes X$. Limits and colimits of modules may be computed on the underlying objects.

Next, we construct relative tensor products by the two-sided bar construction. This construction extends to bimodules, and produces an extension of scalars functor along a map of associative algebras. For a commutative algebra, it also provides the symmetric monoidal structure on its module category.

We then turn to endomorphism algebras, whose universal property identifies algebra actions on an object M with algebra maps into $\mathrm{End}(M)$. Applied to the ∞-category of small ∞-categories, this equips the functor category $\mathrm{Fun}(C, C)$ with a monoidal structure given by composition of functors. Moreover, given an adjunction $F\colon C \rightleftarrows D : G$, the composite GF becomes a monad, that is, an algebra in $\mathrm{Fun}(C, C)$. The functor G induces a functor $D \to \mathrm{LMod}_{GF}(C)$, and the Barr–Beck theorem gives conditions under which this functor is an equivalence.

To keep the chapter compact, we quote several results from Lurie [Lur17]. The comparison with Lurie's model of ∞-operads on which this relies is proved in Proposition 19.4.8.

17.1 Algebras and left modules

Recall the left module ∞-operad $\mathcal{LM}od$ from Example 12.2.9 and Corollary 14.2.6. It has two colors $\mathfrak{a}$ and $\mathfrak{m}$, and the full suboperads on these colors are isomorphic to $\mathcal{A}ssoc$ and $\mathcal{T}riv$, respectively. In particular, restriction along these inclusions gives the associative algebra and the underlying object of its module:

$$\begin{array}{ccccc} \mathcal{A}ssoc & \xhookrightarrow{\mathfrak{a}} & \mathcal{LM}od & \xhookleftarrow{\mathfrak{m}} & \mathcal{T}riv \\ & \searrow\!\!\searrow & \downarrow & & \\ & & \mathcal{A}ssoc. & & \end{array}$$

The operations with output $\mathfrak{a}$ are the associative algebra operations. An operation with output $\mathfrak{m}$ has exactly one input of color $\mathfrak{m}$, placed last, and a linearly ordered family of inputs of color $\mathfrak{a}$. The map $\mathcal{LM}od \to \mathcal{A}ssoc$ forgets the colors and preserves the ordering of the inputs. Thus an $\mathcal{LM}od$-algebra in a monoidal ∞-category C consists of objects $A, M \in C$ with multiplication maps $A^{\otimes n} \to A$ and action maps $A^{\otimes n} \otimes M \to M$, compatibly with operadic substitution.

These definitions generalize those from classical algebra. The nullary operation and the binary operation with its standard ordering give the unit and multiplication maps

$$e\colon \mathbb{1} \to A \qquad \text{and} \qquad m\colon A \otimes A \to A.$$

Operadic substitution supplies the unit and associativity identifications in the following diagrams:

$$\begin{array}{ccccc} A \otimes \mathbb{1} & \xrightarrow{\mathrm{id}_A \otimes e} & A \otimes A & \xleftarrow{e \otimes \mathrm{id}_A} & A \\ & \searrow^{\simeq} & \downarrow{\scriptstyle m} & \swarrow_{\simeq} & \\ & & A, & & \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}_A} & A \otimes A \\ {\scriptstyle \mathrm{id}_A \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A. \end{array}$$

This captures the full structure on A whenever C is a 1-category, but for general C the algebra structure further records higher coherences between these identifications.

Similarly, the operation $(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ in $\mathcal{LM}od$ gives the action map

$$\mathrm{act}\colon A \otimes M \to M,$$

which also comes with its unitality and associativity identifications and their higher coherences:

$$\begin{array}{ccc} \mathbb{1} \otimes M & \xrightarrow{e \otimes \mathrm{id}_M} & A \otimes M \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \mathrm{act}} \\ & & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mathrm{id}_A \otimes \mathrm{act}} & A \otimes M \\ {\scriptstyle m \otimes \mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle \mathrm{act}} \\ A \otimes M & \xrightarrow{\mathrm{act}} & M. \end{array}$$

The definition of a module also makes sense when the algebra and its module do not lie in the same ∞-category: the algebra may lie in a monoidal ∞-category C, and the module in an ∞-category D acted on by C.

Definition 17.1.1 (Left tensored ∞-categories)**.** Let C be a monoidal ∞-category. A *left tensoring of an ∞-category D over C* consists of an $\mathcal{LM}od$-monoidal ∞-category, given by a cocartesian fibration of ∞-operads $p_D\colon D^{\otimes} \to \mathcal{LM}od^{\otimes}$, together with the following identifications:

- An equivalence between C and the underlying monoidal ∞-category $\mathfrak{a}^*(p_D)\colon \mathfrak{a}^*(D^{\otimes}) \to \mathcal{A}ssoc^{\otimes}$;
- An equivalence D and the ∞-category corresponding to the underlying $\mathcal{T}riv$-monoidal ∞-category $\mathfrak{m}^*(p_D)\colon \mathfrak{m}^*(D^{\otimes}) \to \mathcal{T}riv^{\otimes}$ under the equivalence $\mathrm{Alg}_{\mathcal{T}riv}(\mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty$.

We say that D is *left-tensored over C* if it is equipped with such a left tensoring.

The operation $(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ induces an action functor $C \times D \to D$, denoted $(X, M) \mapsto X \otimes M$; the operadic structure supplies its associativity and unit identifications.

Example 17.1.2 (Self-tensoring)**.** By pulling back along the operad map $\mathcal{LM}od \to \mathcal{A}ssoc$, every monoidal ∞-category C is canonically left-tensored over itself, with action given by its tensor product.

Example 17.1.3 (Tensoring on terminal ∞-category)**.** Sending a tuple of colors to its subtuple of algebra-colored entries defines a finite-product-preserving functor $r\colon \mathcal{LM}od^{\otimes} \to \mathcal{A}ssoc^{\otimes}$, left adjoint to the inclusion $j\colon \mathcal{A}ssoc^{\otimes} \hookrightarrow \mathcal{LM}od^{\otimes}$: every morphism into an algebra-colored tuple factors uniquely through the projection to this subtuple. Precomposing the functor $\mathcal{A}ssoc^{\otimes} \to \mathrm{Cat}_\infty$ encoding a monoidal ∞-category C with r equips $*$ with a left tensoring over C, since r sends the module color to the empty tuple. For any left tensoring (C, D) encoded by $F\colon \mathcal{LM}od^{\otimes} \to \mathrm{Cat}_\infty$, the adjunction unit $\mathrm{id} \to jr$ induces $F \to Fjr$, giving a morphism $(C, D) \to (C, *)$ whose components are id_C and $D \to *$.

Example 17.1.4 (Restriction of tensorings)**.** Let $f\colon C \to C'$ be a monoidal functor and let D be left-tensored over C'. Then D is left-tensored over C by $X \otimes_f M := f(X) \otimes M$. To construct this coherently, denote the given tensoring by (C', D) and use the terminal tensorings of Example 17.1.3 to form the pullback

$$(C, *) \times_{(C', *)} (C', D)$$

in $\mathrm{Alg}_{\mathcal{LM}od}(\mathrm{Cat}_\infty)$, with Cat_∞ cartesian monoidal. Via the description by finite-product-preserving functors $\mathcal{LM}od^{\otimes} \to \mathrm{Cat}_\infty$, this pullback is computed pointwise. Its values at the algebra and module colors are C and D, respectively, and its action is the one displayed above.

Definition 17.1.5 (Left modules)**.** Let D be left-tensored over a monoidal ∞-category C, and write $\mathrm{LMod}(D) := \mathrm{Alg}_{\mathcal{LM}od}(D)$ and $\mathrm{Alg}(C) := \mathrm{Alg}_{\mathcal{A}ssoc}(C)$. Restriction along $\mathfrak{a}$ defines a functor $\mathfrak{a}^*\colon \mathrm{LMod}(D) \to \mathrm{Alg}(C)$. For an associative algebra $A \in \mathrm{Alg}(C)$, we define the ∞-category $\mathrm{LMod}_A(D)$ of *left A-modules in D* by the pullback square

$$\begin{array}{ccc} \mathrm{LMod}_A(D) & \longrightarrow & \mathrm{LMod}(D) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathfrak{a}^*} \\ * & \xrightarrow{A} & \mathrm{Alg}(C). \end{array}$$

For C left-tensored over itself as in Example 17.1.2, we write $\mathrm{LMod}(C)$ and $\mathrm{LMod}_A(C)$. Note that a left A-module in C amounts to an operad map $\mathcal{LM}od \to \mathcal{M}_C$ whose restriction to $\mathcal{A}ssoc$ is identified with A.

Remark 17.1.6. Suppose that C is symmetric monoidal. To define modules over *commutative* algebras, we replace $\mathcal{LM}od$ by its commutative version $\mathcal{M}od$ from Example 12.2.8. These ∞-operads fit into a commutative diagram:

$$\begin{array}{ccccc} & & \mathcal{A}ssoc & \longrightarrow & \mathcal{C}omm \\ & & \downarrow & & \downarrow \\ \mathcal{T}riv & \hookrightarrow & \mathcal{LM}od & \longrightarrow & \mathcal{M}od \\ & & \downarrow & & \downarrow \\ & & \mathcal{A}ssoc & \longrightarrow & \mathcal{C}omm. \end{array}$$

In particular, restriction along the inclusion $\mathcal{C}omm \hookrightarrow \mathcal{M}od$ defines a forgetful functor $\mathrm{Mod}(C) := \mathrm{Alg}_{\mathcal{M}od}(C) \to \mathrm{CAlg}(C)$, allowing us to define an ∞-category $\mathrm{Mod}_A(C)$ for every commutative

algebra A in C. It turns out this ∞-category only depends on the underlying associative algebra of A:

Proposition 17.1.7. *Let C be a symmetric monoidal ∞-category. Then the following commutative square is a pullback square:*

$$\begin{array}{ccc} \mathrm{Mod}(C) & \xrightarrow{-|_{\mathcal{LM}od}} & \mathrm{LMod}(C) \\ {\scriptstyle -|_{\mathcal{C}omm}}\downarrow & & \downarrow{\scriptstyle -|_{\mathcal{A}ssoc}} \\ \mathrm{CAlg}(C) & \xrightarrow{-|_{\mathcal{A}ssoc}} & \mathrm{Alg}(C). \end{array}$$

In particular, passing to vertical fibers over $R \in \mathrm{CAlg}(C)$ induces an equivalence

$$\mathrm{Mod}_R(C) \xrightarrow{\sim} \mathrm{LMod}_{R'}(C),$$

where we denote the underlying associative algebra of R by R'.

Proof. By Glasman [Gla14, Proposition 7], our commutative module ∞-operad recovers Lurie's ∞-category of modules over commutative algebras. The pullback description is then [Lur17, Corollary 4.5.1.6]. □

17.1.1 Unit algebras

The monoidal unit carries a canonical associative algebra structure, which is commutative when C is symmetric monoidal. Modules over this algebra are simply objects of the original ∞-category, as we will see after constructing free modules. The algebra structure is a special case of the following construction for unital ∞-operads.

Definition 17.1.8. An ∞-operad $\mathcal{O}$ is called *unital* if its total ∞-category $\mathcal{O}^{\otimes}$ is pointed, i.e., if the terminal object (given by the empty tuple $\emptyset_{\mathcal{O}} := \{\}$) is also an initial object. Equivalently, $\mathcal{O}$ is unital if we have

$$\mathcal{O}(\{\}; x) \simeq *$$

for every color $x \in \mathcal{O}^{\simeq}$.

Example 17.1.9. The ∞-operads $\mathcal{C}omm$, $\mathcal{A}ssoc$ and $\mathcal{E}_k$ are unital. The ∞-operads $\mathcal{T}riv$, $\mathcal{M}od$ and $\mathcal{LM}od$ are not unital.

Definition 17.1.10. Let $\mathcal{O}$ be a unital ∞-operad and let $C\colon \mathcal{O}^{\otimes} \to \mathrm{Cat}_\infty$ be an $\mathcal{O}$-monoidal ∞-category. For each color x, the unique nullary operation of $\mathcal{O}$ gives a functor

$$\mathbb{1}_x\colon * \simeq C(\{\}) \longrightarrow C_x.$$

We call the corresponding object $\mathbb{1}_x \in C_x$ the *unit at the color x*.

For an $\mathcal{O}$-algebra A, factoring its nullary operation at x through cocartesian transport gives a unit map $\mathbb{1}_x \to A_x$. The units $\mathbb{1}_x$ themselves assemble into an $\mathcal{O}$-algebra, which we denote by $\mathbb{1}_C$.

Proposition 17.1.11 ([Lur17, Proposition 3.2.1.8 and Lemma 3.2.1.10])**.** *Let $\mathcal{O}$ be a unital ∞-operad and let C be an $\mathcal{O}$-monoidal ∞-category. Then $\mathbb{1}_C$ is initial in $\mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(C)$. An $\mathcal{O}$-algebra A is initial if and only if its unit maps $\mathbb{1}_x \to A_x$ are isomorphisms for all colors x.*

The characterization by initiality uses operadic left Kan extensions, which we do not develop here. The construction of $\mathbb{1}_C$ itself is short; we follow [HW21, Lemma II.45a]. Applying $C\colon \mathcal{O}^{\otimes} \to \mathrm{Cat}_\infty$ to the natural transformation $\mathrm{const}_{\emptyset_{\mathcal{O}}} \to \mathrm{id}_{\mathcal{O}^{\otimes}}$ supplied by the initial object gives a natural transformation $\mathrm{const}_* \to C$, since $C(\emptyset_{\mathcal{O}}) \simeq *$. By straightening, this defines a cocartesian section of p_C. Its value at a tuple $\{x_i\}_{i\in I}$ is $\{\mathbb{1}_{x_i}\}_{i\in I}$, compatibly with the projections, so it preserves finite products and defines the required algebra.

Corollary 17.1.12 ([Lur17, Corollary 3.2.1.9])**.** *Let C be a symmetric monoidal ∞-category. Then a commutative algebra R is initial in $\mathrm{CAlg}(C)$ if and only if the unit map $e\colon \mathbb{1} \to R$ is an isomorphism in C. In particular, the underlying object of the commutative algebra $\mathbb{1}_C \in \mathrm{CAlg}(C)$ from Proposition 17.1.11 is the monoidal unit of C.* □

Remark 17.1.13. The unit-algebra result extends to a morphism $p\colon \mathcal{P} \to \mathcal{O}$ of ∞-operads, with $\mathcal{O}$ unital, provided that for each color x of $\mathcal{O}$ there is a color $\mathbb{1}_x$ of $\mathcal{P}$ over x and a nullary operation $e_x\colon \{\} \to \{\mathbb{1}_x\}$ such that

$$e_x \times \mathrm{id}_Z\colon Z \longrightarrow \{\mathbb{1}_x\} \times Z$$

is $p^{\otimes}$-cocartesian for every tuple $Z \in \mathcal{P}^{\otimes}$. This is the *operadic unit* condition of [Lur17, Definition 3.2.1.1]: the nullary operation must have its universal property also in the presence of additional inputs. For an $\mathcal{O}$-monoidal ∞-category, this follows from the compatibility of cocartesian transport with concatenation.

Under this hypothesis, the units assemble into an initial operad section of p, and a section is initial precisely when its unit maps are isomorphisms; see [Lur17, Proposition 3.2.1.8 and Lemma 3.2.1.10]. Thus the construction only requires these universal nullary operations, even if higher-arity tensor products are not represented. Having such units does not mean that $\mathcal{P}$ is unital: that would require every nullary-operation anima to be contractible.

17.1.2 Free modules

Let A be an ordinary associative ring and let M_0 be an abelian group. Multiplication on the first factor turns the tensor product of abelian groups $A \otimes M_0$ into a left A-module. It is the free A-module on M_0 relative to the forgetful functor to abelian groups: restriction along the map $M_0 \to A \otimes M_0,\ m \mapsto 1 \otimes m$ gives a natural bijection

$$\mathrm{Hom}_A(A \otimes M_0, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ab}}(M_0, N).$$

This admits the following ∞-categorical refinement:

Proposition 17.1.14 ([Lur17, Proposition 4.2.4.2, Corollaries 4.2.4.4 and 4.2.4.8])**.** *Let C be a monoidal ∞-category and let D be an ∞-category left tensored over C.*

(1) *The forgetful functor* $\mathrm{LMod}(D) \xrightarrow{(\mathfrak{a}^*, \mathfrak{m}^*)} \mathrm{Alg}(C) \times D$ *admits a left adjoint*

$$F\colon \mathrm{Alg}(C) \times D \to \mathrm{LMod}(D), \qquad (A, M_0) \mapsto F_A(M_0)$$

satisfying $\mathfrak{a}^* F_A(M_0) \simeq A$ and $\mathfrak{m}^* F_A(M_0) \simeq A \otimes M_0$.

(2) *For an associative algebra $A \in \mathrm{Alg}(C)$, the forgetful functor $\mathfrak{m}^*\colon \mathrm{LMod}_A(D) \to D$ admits a left adjoint*

$$F_A\colon D \to \mathrm{LMod}_A(D),$$

where the underlying object of $F_A(M_0)$ is $A \otimes M_0$.

As a consequence, we may deduce that modules over a trivial algebra are simply objects of D:

Corollary 17.1.15 ([Lur17, Proposition 4.2.4.9])**.** *Let C be a monoidal ∞-category and let $A \in \mathrm{Alg}(C)$ be an algebra object such that the unit map $e\colon \mathbb{1} \to A$ is an isomorphism in C. Then for every left C-tensored ∞-category D the forgetful functor $\mathrm{LMod}_A(D) \to D$ is an equivalence of ∞-categories.*

Proof. By Proposition 17.1.14, the forgetful functor admits a left adjoint F_A. The unit $M_0 \to A \otimes M_0$ is obtained by tensoring e with M_0, so it is an isomorphism. The triangle identity identifies the underlying counit with the inverse of the unit at the underlying object. Since the forgetful functor is conservative, the counit is also an isomorphism. □

17.1.3 Limits and colimits of modules

Limits of modules are computed on their underlying objects. The same holds for colimits whenever the action preserves them.

Proposition 17.1.16 ([Lur17, Corollaries 4.2.3.3 and 4.2.3.5])**.** *Let C be a monoidal ∞-category, let D be left-tensored over C, and let $A \in \mathrm{Alg}(C)$. Let $\mathfrak{m}^*\colon \mathrm{LMod}_A(D) \to D$ be the forgetful functor, and let I be a small ∞-category.*

(1) *If D admits I-indexed limits, then $\mathfrak{m}^*$ creates and preserves I-indexed limits.*

(2) *If D admits I-indexed colimits and the functor $X \otimes -\colon D \to D$ preserves I-indexed colimits for every $X \in C$, then $\mathfrak{m}^*$ creates and preserves I-indexed colimits.*

Proof. For (1), the functor $\mathrm{Fun}(I,-)\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ preserves finite products and is therefore symmetric monoidal for the cartesian structures. It sends the given left tensoring to a left tensoring of $\mathrm{Fun}(I,D)$ over $\mathrm{Fun}(I,C)$; restriction along $\mathrm{const}_I\colon C \to \mathrm{Fun}(I,C)$ as in Example 17.1.4 then gives $\mathrm{Fun}(I,D)$ the pointwise left tensoring over C. The symmetric monoidal natural transformation $\mathrm{const}_I\colon \mathrm{id}_{\mathrm{Cat}_\infty} \to \mathrm{Fun}(I,-)$ induces a morphism of the corresponding $\mathcal{LM}od$-algebras in Cat_∞, which factors through this pullback to give the $\mathcal{LM}od$-monoidal functor $(\mathrm{id}_C, \mathrm{const})\colon (C,D) \to (C, \mathrm{Fun}(I,D))$. Since const has right adjoint $\lim_I$, Proposition 14.6.8 equips the pair

$$(\mathrm{id}_C, \lim_I)\colon (C, \mathrm{Fun}(I,D)) \to (C,D)$$

with the right-adjoint lax $\mathcal{LM}od$-monoidal structure. Passing to $\mathcal{LM}od$-algebras and taking the fiber over A therefore gives an adjunction

$$\mathrm{const}\colon \mathrm{LMod}_A(D) \rightleftarrows \mathrm{LMod}_A(\mathrm{Fun}(I,D)) \simeq \mathrm{Fun}(I, \mathrm{LMod}_A(D)) : \lim_I .$$

The underlying object of the right adjoint is the limit in D. Thus $\mathfrak{m}^*$ preserves these limits. Conversely, a cone of A-modules whose underlying cone is a limit cone in D maps to the limit just

constructed by an underlying isomorphism. Since $\mathfrak{m}^*$ detects isomorphisms, this is an isomorphism of modules. Hence $\mathfrak{m}^*$ also creates these limits.

For (2), we instead use the left adjoint colim_I of const. By assumption, the canonical maps

$$\mathrm{colim}_{i\in I}(X \otimes M_i) \longrightarrow X \otimes \mathrm{colim}_{i\in I} M_i$$

are isomorphisms for every $X \in C$. In particular, the hypothesis of Proposition 14.6.9 is satisfied for $\mathcal{O} = \mathcal{LM}od$: on the algebra color the left adjoint is id_C, and operations with several algebra inputs reduce to the displayed maps by taking their tensor product. Thus $(\mathrm{id}_C, \mathrm{colim}_I)$ is an $\mathcal{LM}od$-monoidal left adjoint to $(\mathrm{id}_C, \mathrm{const})$. As in (1), passing to algebras and taking the fiber over A produces an adjunction

$$\mathrm{colim}_I \colon \mathrm{Fun}(I, \mathrm{LMod}_A(D)) \rightleftarrows \mathrm{LMod}_A(D) \,:\mathrm{const}\,.$$

Its underlying functor takes colimits in D, so $\mathfrak{m}^*$ preserves these colimits. The same argument as in (1), with cocones in place of cones, shows that it also creates them. □

Proposition 17.1.17. *Let D be left-tensored over a monoidal ∞-category C. Suppose that D is presentable and $X \otimes - \colon D \to D$ preserves small colimits for every $X \in C$. Then, for every $A \in \mathrm{Alg}(C)$, the ∞-category $\mathrm{LMod}_A(D)$ is presentable, and its forgetful functor to D creates limits and small colimits.*

Proof. Presentability is [Lur17, Corollary 4.2.3.7]. The assertions about limits and colimits follow from Proposition 17.1.16. □

Corollary 17.1.18. *Let C be a monoidal ∞-category whose underlying ∞-category is stable and whose tensor product preserves finite colimits separately in both variables. For every associative algebra $A \in \mathrm{Alg}(C)$, the ∞-category $\mathrm{LMod}_A(C)$ is stable.*

Proof. By Proposition 17.1.16, the forgetful functor $\mathrm{LMod}_A(C) \to C$ creates and preserves finite limits and colimits. Thus $\mathrm{LMod}_A(C)$ has a zero object, and a commutative square in $\mathrm{LMod}_A(C)$ is a pullback if and only if it is a pushout, since this may be checked in C. Hence $\mathrm{LMod}_A(C)$ is stable. □

Remark 17.1.19. The limit and colimit statements of Proposition 17.1.16 also apply to individual diagrams. A diagram of A-modules admits a limit whenever its underlying diagram in D does, and the forgetful functor creates this limit. Likewise, it creates a colimit whenever the underlying diagram admits a colimit preserved by $X \otimes -$ for every $X \in C$. See [Lur17, Corollaries 3.2.2.3 and 4.2.3.5].

17.2 Relative tensor products and change of algebras

Given associative rings A and B, we may consider (A, B)-*bimodules*: abelian groups equipped with compatible left A-module and right B-module structures. For another associative ring D, the *relative tensor product* $M \otimes_B N$ of an (A, B)-bimodule M and a (B, D)-bimodule N is canonically an (A, D)-bimodule. We now construct relative tensor products in general monoidal ∞-categories using the two-sided bar construction, and explain their associativity and unit isomorphisms. For commutative algebras in a symmetric monoidal ∞-category, this gives a symmetric monoidal structure on modules, compatible with extension of scalars.

17.2.1 Bimodules

We start by extending the notion of bimodules to the ∞-categorical setting.

Definition 17.2.1. We define a colored operad $\mathcal{BM}od$. It has three colors: $\mathfrak{a}_-$, $\mathfrak{a}_+$ and $\mathfrak{m}$. The multimorphism sets $\mathcal{BM}od(\{X_i\}_{i\in I}; Y)$ are then given as follows:

- If $Y = \mathfrak{a}_-$, then this set is empty unless $X_i = \mathfrak{a}_-$ for all $i \in I$, in which case it is the set of linear orders on I;
- If $Y = \mathfrak{a}_+$, then this set is empty unless $X_i = \mathfrak{a}_+$ for all $i \in I$, in which case it is the set of linear orders on I;
- If $Y = \mathfrak{m}$, then this set is the set of linear orders $\{i_1 < i_2 < \cdots < i_n\}$ on I with the following property: there exists precisely one index $i_k \in I$ such that $X_{i_k} = \mathfrak{m}$, $X_{i_j} = \mathfrak{a}_-$ for $j < k$ and $X_{i_j} = \mathfrak{a}_+$ for $j > k$.

Composition in $\mathcal{BM}od$ is given by composition of linear orders.

By Proposition 14.2.4, this colored operad canonically determines the ∞-operad $\mathcal{BM}od$ used below.

Observation 17.2.2. We identify $\mathcal{LM}od$ with the full suboperad of $\mathcal{BM}od$ spanned by the colors $\mathfrak{a}_-$ and $\mathfrak{m}$, and let $\mathcal{RM}od \subseteq \mathcal{BM}od$ denote the full suboperad spanned by $\mathfrak{m}$ and $\mathfrak{a}_+$. Thus $\mathcal{BM}od$ contains two copies of $\mathcal{A}ssoc$, on $\mathfrak{a}_-$ and $\mathfrak{a}_+$, a copy of $\mathcal{T}riv$ on $\mathfrak{m}$, and the full suboperads $\mathcal{LM}od$ and $\mathcal{RM}od$. There is also an operad map

$$\mathcal{BM}od \longrightarrow \mathcal{A}ssoc$$

which sends all three colors to $\mathfrak{a}$ and forgets the restrictions on the linear orders. Its restrictions to $\mathcal{LM}od$ and $\mathcal{RM}od$ are the usual collapse maps to $\mathcal{A}ssoc$.

Notation 17.2.3. For a monoidal ∞-category C we write

$$\mathrm{RMod}(C) := \mathrm{Alg}_{\mathcal{RM}od}(C) \qquad \text{and} \qquad \mathrm{BMod}(C) := \mathrm{Alg}_{\mathcal{BM}od}(C).$$

Similarly, if A and B are associative algebras in C, we write

$$\mathrm{RMod}_B(C) \qquad \text{and} \qquad {}_A\mathrm{BMod}_B(C)$$

for the fibers over B and (A, B) of the two forgetful functors $\mathfrak{a}_+^*\colon \mathrm{RMod}(C) \to \mathrm{Alg}(C)$ and $(\mathfrak{a}_-^*, \mathfrak{a}_+^*)\colon \mathrm{BMod}(C) \to \mathrm{Alg}(C) \times \mathrm{Alg}(C)$, respectively.

The results of the preceding section apply equally to right modules, with right tensorings in place of left tensorings. In particular, Propositions 17.1.16 and 17.1.17 and Corollary 17.1.18 have analogous statements for right modules.

Observation 17.2.4. Suppose that C is symmetric monoidal. There are equivalences of ∞-operads

$$\mathrm{rev}\colon \mathcal{A}ssoc \xrightarrow{\sim} \mathcal{A}ssoc, \qquad \mathrm{rev}\colon \mathcal{LM}od \xrightarrow{\sim} \mathcal{RM}od \qquad \text{and} \qquad \mathrm{rev}\colon \mathcal{BM}od \xrightarrow{\sim} \mathcal{BM}od$$

given on objects by $\mathfrak{a} \mapsto \mathfrak{a}$, $\mathfrak{a}_- \mapsto \mathfrak{a}_+$, $\mathfrak{a}_+ \mapsto \mathfrak{a}_-$ and $\mathfrak{m} \mapsto \mathfrak{m}$, and on multimorphisms by reversing the order on the finite set. The map $\mathcal{A}ssoc \to \mathcal{C}omm$ coequalizes rev and the identity, so the symmetric monoidal structure on C canonically identifies the two induced monoidal structures. Consequently, restriction along rev defines an endofunctor $\mathrm{rev}^*\colon \mathrm{Alg}(C) \to \mathrm{Alg}(C)$. It sends an associative algebra A to its *opposite algebra*, denoted A^{rev}, which has the same underlying object but whose multiplication is given by

$$A \otimes A \xrightarrow{\sim} A \otimes A \xrightarrow{m_A} A,$$

where the first map is the swap map. For associative algebras A and B we thus get equivalences

$$\mathrm{LMod}_A(C) \xrightarrow{\sim} \mathrm{RMod}_{A^{\mathrm{rev}}}(C) \qquad \text{and} \qquad {}_A\mathrm{BMod}_B(C) \xrightarrow{\sim} {}_{B^{\mathrm{rev}}}\mathrm{BMod}_{A^{\mathrm{rev}}}(C).$$

Observation 17.2.5. There is an operad map $\mathcal{BM}od \to \mathcal{M}od$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends both $\mathfrak{a}_-$ and $\mathfrak{a}_+$ to $\mathfrak{a}$. Composing it with the inclusion $\mathcal{LM}od \hookrightarrow \mathcal{BM}od$ gives the canonical map $\mathcal{LM}od \to \mathcal{M}od$, and similarly for $\mathcal{RM}od$. It follows that for a commutative algebra A in a symmetric monoidal ∞-category C, restriction along these operad maps defines forgetful functors

$$\begin{array}{ccccc} & & \mathrm{Mod}_A(C) & & \\ & \swarrow^{\sim} & \downarrow & \searrow^{\sim} & \\ \mathrm{LMod}_A(C) & \longleftarrow & {}_A\mathrm{BMod}_A(C) & \longrightarrow & \mathrm{RMod}_A(C). \end{array}$$

Here the left diagonal equivalence is the one from Proposition 17.1.7. The right one follows by applying Observation 17.2.4, since a commutative algebra is canonically isomorphic to its opposite algebra.

An (A, B)-bimodule M has structure maps

$$A^{\otimes n} \otimes M \otimes B^{\otimes m} \to M$$

for all $n, m \geq 0$, with coherent compatibility with the multiplications and units of A and B. Setting $m = 0$ or $n = 0$ gives the left and right actions. Their compatibility can be expressed by regarding M as a left A-module in $\mathrm{RMod}_B(C)$, or as a right B-module in $\mathrm{LMod}_A(C)$:

Theorem 17.2.6 ([Lur17, Theorem 4.3.2.7])**.** *Let A and B be associative algebras in a monoidal ∞-category C.*

(1) *The ∞-category $\mathrm{RMod}_B(C)$ is left tensored over C, and there is an equivalence*

$${}_A\mathrm{BMod}_B(C) \simeq \mathrm{LMod}_A(\mathrm{RMod}_B(C));$$

(2) *The ∞-category $\mathrm{LMod}_A(C)$ is right tensored over C, and there is an equivalence*

$${}_A\mathrm{BMod}_B(C) \simeq \mathrm{RMod}_B(\mathrm{LMod}_A(C)).$$

Corollary 17.2.7. *For associative algebras A and B in C, there are equivalences*

$${}_A\mathrm{BMod}_{\mathbb{1}}(C) \xrightarrow{\sim} \mathrm{LMod}_A(C) \qquad \textit{and} \qquad {}_{\mathbb{1}}\mathrm{BMod}_B(C) \xrightarrow{\sim} \mathrm{RMod}_B(C).$$

Proof. In light of the equivalences $\mathrm{RMod}_{\mathbb{1}}(C) \xrightarrow{\sim} C$ and $\mathrm{LMod}_{\mathbb{1}}(C) \xrightarrow{\sim} C$ from Corollary 17.1.15, this follows directly from the previous theorem. □

17.2.2 The relative tensor product

Let B be a discrete associative ring. A central construction in the theory of modules is the *relative tensor product*: Given a right B-module M and a left B-module N, their relative tensor product $M \otimes_B N$ is obtained as the quotient of $M \otimes N$ by identifying $mb \otimes n$ with $m \otimes bn$ for all $m \in M$, $n \in N$ and $b \in B$. In other words, $M \otimes_B N$ is the coequalizer of the following diagram:

$$M \otimes B \otimes N \rightrightarrows M \otimes N.$$

In the ∞-categorical setting, we replace this coequalizer by the geometric realization of the *two-sided bar construction*. We first describe its objects, face maps, and degeneracies. In Remark 17.2.10, we explain how to construct this simplicial object using ∞-operads, following [Lur17, Construction 4.4.2.7].

Construction 17.2.8 (Bar construction)**.** Let B be an associative algebra in a monoidal ∞-category C, let M be a right B-module, and let N be a left B-module. The *two-sided bar construction* for M and N over B, denoted $\mathrm{Bar}_B(M,N)_\bullet$, is a simplicial object in C described as follows:

- In degree $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, it is given by $\mathrm{Bar}_B(M,N)_n := M \otimes B^{\otimes n} \otimes N$;
- The face maps $d_i \colon \mathrm{Bar}_B(M,N)_n \to \mathrm{Bar}_B(M,N)_{n-1}$ are given by:
  - $d_0 = \mathrm{act}_M \otimes \mathrm{id}_{B^{\otimes(n-1)}} \otimes \mathrm{id}_N \colon M \otimes B \otimes B^{\otimes(n-1)} \otimes N \to M \otimes B^{\otimes(n-1)} \otimes N$.
  - For $0 < i < n$, d_i is induced by the multiplication map $m \colon B \otimes B \to B$ on the i-th and $(i+1)$-th factors of $B^{\otimes n}$.
  - $d_n = \mathrm{id}_M \otimes \mathrm{id}_{B^{\otimes(n-1)}} \otimes \mathrm{act}_N \colon M \otimes B^{\otimes(n-1)} \otimes B \otimes N \to M \otimes B^{\otimes(n-1)} \otimes N$.
- The degeneracy maps $s_i \colon \mathrm{Bar}_B(M,N)_n \to \mathrm{Bar}_B(M,N)_{n+1}$ are induced by inserting the unit $e \colon \mathbb{1} \to B$ at the $(i+1)$-th position.

Definition 17.2.9 (Relative tensor product)**.** Let C be a monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product $- \otimes - \colon C \times C \to C$ preserves geometric realizations in both variables. For an associative algebra B, a right B-module M and a left B-module N, their *relative tensor product* is the object of C given by the geometric realization of the bar construction:

$$M \otimes_B N := |\mathrm{Bar}_B(M,N)_\bullet| = \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Bar}_B(M,N)_n.$$

More generally, let $A, B, D \in \mathrm{Alg}(C)$. If $M \in {}_A\mathrm{BMod}_B(C)$ and $N \in {}_B\mathrm{BMod}_D(C)$, then the bar construction lifts to a simplicial object of ${}_A\mathrm{BMod}_D(C)$. Geometric realizations of bimodules are created on underlying objects by [Lur17, Proposition 4.3.3.9], so its realization defines an (A,D)-bimodule

$$M \otimes_B N \in {}_A\mathrm{BMod}_D(C).$$

Remark 17.2.10. To formally construct $\mathrm{Bar}_B(M,N)_\bullet$, Lurie defines in [Lur17, Definition 4.4.1.1] an ∞-operad $\mathcal{T}ens_2$ and shows in [Lur17, Proposition 4.4.1.11] that it sits in a pushout square of ∞-operads as follows:

$$\begin{array}{ccc} \mathcal{A}ssoc & \xrightarrow{\mathfrak{a}_+} & \mathcal{B}\mathcal{M}od \\ {\scriptstyle \mathfrak{a}_-}\downarrow & \ulcorner & \downarrow \\ \mathcal{B}\mathcal{M}od & \longrightarrow & \mathcal{T}ens_2. \end{array}$$

In particular, for a monoidal ∞-category C, the ∞-operad $\mathcal{T}ens_2$ encodes the data of algebras $A, B, D \in \mathrm{Alg}(C)$, an (A, B)-bimodule M, and a (B, D)-bimodule N: there is a pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{T}ens_2}(C) & \longrightarrow & \mathrm{BMod}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathfrak{a}^*_-} \\ \mathrm{BMod}(C) & \xrightarrow{\mathfrak{a}^*_+} & \mathrm{Alg}(C). \end{array}$$

For this reason, we denote the five colors of $\mathcal{T}ens_2$ by $\mathfrak{a}$, $\mathfrak{m}$, $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{c}$.

Lurie's construction supplies a functor

$$\mathrm{BCut}\colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{T}ens_2^{\otimes}$$

which sends $[n]$ to the tuple $(\mathfrak{m}, \mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}, \mathfrak{n})$ encoding $M \otimes B^{\otimes n} \otimes N$. It can be described as the pullback, in $\mathcal{T}ens_2^{\otimes}$, of the following cospan:

$$\begin{array}{ccc} & & (\mathfrak{m}, \mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}) \\ & & \downarrow \\ (\mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}, \mathfrak{n}) & \longrightarrow & (\mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}). \end{array}$$

Both maps are inert, and their pullback is the indicated tuple. The compatibility of these pullbacks with the simplicial operators is part of [Lur17, Construction 4.4.2.7].

Now, given an (A, B)-bimodule M and a (B, D)-bimodule N, we may think of this data as defining a $\mathcal{T}ens_2$-algebra in C, which in particular defines a functor $\mathcal{T}ens_2^{\otimes} \to C^{\otimes}$. Precomposing this with BCut gives a functor $\mathbf{\Delta}^{\mathrm{op}} \to C^{\otimes}$. One can check that this actually factors through the active morphisms. Using cocartesian transport, we can then turn this into the desired functor $\mathrm{Bar}_B(M, N)_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to C$.

The construction of Remark 17.2.10 also applies to a left C-tensored ∞-category D: the full suboperad of $\mathcal{T}ens_2$ on $\mathfrak{m}, \mathfrak{b}, \mathfrak{n}$ maps to $\mathcal{LM}od$ by sending these colors to $\mathfrak{a}, \mathfrak{a}, \mathfrak{m}$, respectively, and preserving the order of inputs, so we may use the cocartesian transport of the left tensoring. For $M \in \mathrm{RMod}_B(C)$ and $N \in \mathrm{LMod}_B(D)$, we write $M \otimes_B N$ for the realization in D of the bar construction with terms $M \otimes B^{\otimes n} \otimes N$, whenever it exists. Inserting the unit in the leftmost position gives extra degeneracies on the augmented bar construction $\mathrm{Bar}_B(B, N)_\bullet \to N$, so $B \otimes_B N \simeq N$ by Definition 5.2.3.

Lemma 17.2.11 (Colimits and relative tensor products)**.** *Let C be a cocomplete monoidal ∞-category, and let D be a cocomplete ∞-category left-tensored over C. Suppose that the tensor product on C and the action on D preserve small colimits separately in both variables. For every $B \in \mathrm{Alg}(C)$, the relative tensor product*

$$- \otimes_B -\colon \mathrm{RMod}_B(C) \times \mathrm{LMod}_B(D) \longrightarrow D$$

preserves small colimits separately in both variables. The same holds for relative tensor products of bimodules in C.

Proof. By Proposition 17.1.16 and its right-module version, colimits of modules are computed on underlying objects. Thus each term $M \otimes B^{\otimes n} \otimes N$ preserves small colimits separately in M and N.

Since colimits commute with geometric realizations, the same holds for $M \otimes_B N$. The bimodule case follows by the same argument, since Theorem 17.2.6 identifies bimodules with iterated modules, whose colimits are again computed on underlying objects. □

Proposition 17.2.12. *Let C and D be monoidal ∞-categories which admit geometric realizations and whose tensor products preserve geometric realizations separately in both variables. If $F\colon C \to D$ is a monoidal functor which preserves geometric realizations, then for every associative algebra $B \in \mathrm{Alg}(C)$, right B-module M, and left B-module N, there is a natural isomorphism*

$$F(M) \otimes_{F(B)} F(N) \xrightarrow{\sim} F(M \otimes_B N).$$

Proof. The monoidal structure on F, together with the functorial construction of the coherent bar diagram from Remark 17.2.10, gives an isomorphism of simplicial objects

$$\mathrm{Bar}_{F(B)}(F(M), F(N))_\bullet \xrightarrow{\sim} F(\mathrm{Bar}_B(M, N)_\bullet).$$

Taking geometric realizations and using that F preserves them gives the asserted isomorphism. □

The relative tensor product is functorial in the algebras and bimodules, giving a functor

$$\mathrm{BMod}(C) \times_{\mathrm{Alg}(C)} \mathrm{BMod}(C) \longrightarrow \mathrm{BMod}(C),$$

where the fiber product matches the right algebra of the first bimodule with the left algebra of the second. Relative tensor products are coherently associative and unital. In particular, for bimodules ${}_AM_B$, ${}_BN_C$ and ${}_CP_D$ there is a natural isomorphism

$$(M \otimes_B N) \otimes_C P \simeq M \otimes_B (N \otimes_C P) \quad \in \quad {}_A\mathrm{BMod}_D(C).$$

For a bimodule ${}_AM_B$, the regular bimodules A and B act as units:

$$A \otimes_A M \simeq M \qquad \text{and} \qquad M \otimes_B B \simeq M \quad \in \quad {}_A\mathrm{BMod}_B(C).$$

These statements, including their coherent compatibility, are proved in [Lur17, Propositions 4.4.3.14 and 4.4.3.16]. To describe the resulting extension-of-scalars adjunction, we first introduce restriction of scalars.

Proposition 17.2.13 (Restriction of scalars, [Lur17, Corollary 4.2.3.2])**.** *Let D be left-tensored over a monoidal ∞-category C. The restriction functor*

$$\mathfrak{a}^*\colon \mathrm{LMod}(D) \longrightarrow \mathrm{Alg}(C)$$

is a cartesian fibration. A morphism in $\mathrm{LMod}(D)$ is $\mathfrak{a}^*$-cartesian if and only if its underlying morphism in D is an isomorphism. Consequently, a morphism $f\colon A \to B$ of associative algebras induces a restriction-of-scalars functor

$$f_*\colon \mathrm{LMod}_B(D) \longrightarrow \mathrm{LMod}_A(D)$$

which does not change the underlying object of D.

Under the hypotheses used to construct relative tensor products, restriction of scalars has the following left adjoint.

Proposition 17.2.14 (Extension of scalars)**.** *Let C be a monoidal ∞-category which admits geometric realizations and whose tensor product preserves geometric realizations separately in both variables. For every morphism $f\colon A \to B$ in $\mathrm{Alg}(C)$, the restriction functor from Proposition 17.2.13*

$$f_*\colon \mathrm{LMod}_B(C) \longrightarrow \mathrm{LMod}_A(C)$$

admits a left adjoint given by extension of scalars:

$$f^* := B \otimes_A -\colon \mathrm{LMod}_A(C) \longrightarrow \mathrm{LMod}_B(C).$$

Analogously, restriction from right B-modules to right A-modules admits the left adjoint $- \otimes_A B$.

Proof. The morphism f makes every left B-module into a left A-module and makes B into a (B, A)-bimodule. For a left A-module M and a left B-module N, an A-linear morphism $g\colon M \to f_*N$ induces the B-linear composite

$$B \otimes_A M \xrightarrow{\mathrm{id}_B \otimes_A g} B \otimes_A f_*N \longrightarrow N,$$

where the final map is induced by the B-action on N. Conversely, precomposition with

$$M \xrightarrow{\sim} A \otimes_A M \xrightarrow{f \otimes_A \mathrm{id}_M} B \otimes_A M$$

turns every B-linear morphism $B \otimes_A M \to N$ into an A-linear morphism $M \to f_*N$. These assignments arise from the displayed natural transformations, so they define maps between the corresponding mapping animae. Unitality and associativity of relative tensor products show that these maps are inverse. The argument for right modules is analogous. □

If all the algebras involved are a fixed algebra A, the relative tensor product takes the form of a functor

$$- \otimes_A -\colon {}_A\mathrm{BMod}_A(C) \times {}_A\mathrm{BMod}_A(C) \to {}_A\mathrm{BMod}_A(C)$$

which satisfies associativity and unitality. Lurie shows that this assembles into a monoidal structure on ${}_A\mathrm{BMod}_A(C)$:

Proposition 17.2.15 ([Lur17, Proposition 4.4.3.12])**.** *Let C be a monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product $- \otimes -\colon C \times C \to C$ preserves geometric realizations in both variables. For an associative algebra A, there exists a monoidal structure on ${}_A\mathrm{BMod}_A(C)$ given by the relative tensor product over A, with unit A.*

17.2.3 Modules over commutative algebras

Let R be a discrete commutative ring. Given R-modules M and N, the relative tensor product $M \otimes_R N$ is again an R-module, with the scalar multiplication given by $r(m \otimes n) = rm \otimes n = m \otimes rn$. In fact, the construction $(M, N) \mapsto M \otimes_R N$ equips the category $\mathrm{Mod}_R(\mathrm{Ab})$ of R-modules with a symmetric monoidal structure. The analogous statement also holds in the ∞-categorical setting:

Theorem 17.2.16 ([Lur17, Theorem 4.5.2.1])**.** *Let C be a symmetric monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product of C preserves geometric realizations in both variables. Let R be a commutative algebra in C. Then:*

(1) *The* ∞*-category* $\mathrm{Mod}_R(C)$ *admits a symmetric monoidal structure with unit* R*.*

(2) *The functor* $\mathrm{Mod}_R(C) \to {}_R\mathrm{BMod}_R(C)$ *from Observation 17.2.5 refines to a monoidal functor, where the target has the monoidal structure from Proposition 17.2.15.*

In particular, the tensor product on $\mathrm{Mod}_R(C)$ *is given as the following composite:*

$$\begin{aligned}\mathrm{Mod}_R(C) \times \mathrm{Mod}_R(C) &\longrightarrow {}_R\mathrm{BMod}_R(C) \times {}_R\mathrm{BMod}_R(C) \\ &\xrightarrow{-\otimes_R -} {}_R\mathrm{BMod}_R(C) \longrightarrow \mathrm{LMod}_R(C) \xrightarrow{\sim} \mathrm{Mod}_R(C).\end{aligned}$$

Proposition 17.2.17 ([Lur17, Theorem 5.1.4.10 and Corollary 3.4.1.7])**.** *Let* C *be a symmetric monoidal* ∞*-category. Assume that* C *admits geometric realizations and that the tensor product of* C *preserves geometric realizations in both variables. Let* R *be a commutative algebra in* C*. The forgetful functor* $\mathrm{CAlg}(\mathrm{Mod}_R(C)) \to \mathrm{CAlg}(C)$ *lifts to an equivalence of* ∞*-categories*

$$\mathrm{CAlg}(\mathrm{Mod}_R(C)) \xrightarrow{\sim} \mathrm{CAlg}(C)_{R/}.$$

Warning 17.2.18. There is no analogous equivalence $\mathrm{Alg}(\mathrm{Mod}_R(C)) \simeq \mathrm{Alg}(C)_{R/}$. Already for ordinary rings, the structure map from R to an associative algebra in $\mathrm{Mod}_R(C)$ is central, whereas an arbitrary morphism from R to an associative algebra need not have central image; see also [Lur17, Warning 7.1.3.9].

Corollary 17.2.19 (Localization of a commutative base algebra)**.** *Let* $R \in \mathrm{CAlg}(C)$ *be as in Theorem 17.2.16, and let*

$$L\colon \mathrm{Mod}_R(C) \rightleftarrows D\colon i$$

be a symmetric monoidal Bousfield localization as in Proposition 14.7.2. Then the local R*-module* $L(R)$ *canonically underlies a commutative algebra in* C *equipped with a morphism* $R \to L(R)$*. It is initial among commutative algebras under* R *whose underlying* R*-modules are local.*

Proof. As the monoidal unit of $\mathrm{Mod}_R(C)$, the algebra R is initial in $\mathrm{CAlg}(\mathrm{Mod}_R(C))$. By Corollary 14.7.4, its reflection $L(R)$ is the initial object of $\mathrm{CAlg}(D)$, and the essential image of

$$\mathrm{CAlg}(D) \hookrightarrow \mathrm{CAlg}(\mathrm{Mod}_R(C))$$

consists of the commutative algebras with local underlying R-module. By Proposition 17.2.17, these are precisely the commutative algebras under R whose underlying R-modules are local. □

When C is presentably symmetric monoidal, the symmetric monoidal ∞-category $\mathrm{Mod}_R(C)$ is again presentably symmetric monoidal: its underlying category is presentable by Propositions 17.1.7 and 17.1.17, and its relative tensor product preserves colimits separately in each variable by Lemma 17.2.11. It is therefore closed by Proposition 16.2.9; in particular, each functor $- \otimes_R N$ admits an internal right adjoint $\underline{\mathrm{Hom}}_R(N, -)$.

17.2.4 Restriction and extension of scalars

For a morphism $f\colon R \to S$ of commutative algebras, Proposition 17.2.14 gives extension of scalars $f^* = S \otimes_R -$ left adjoint to *restriction of scalars* f_*.[19] Extension of scalars is also symmetric monoidal. The following theorem constructs this refinement functorially in the algebra.

[19] The notation comes from viewing restriction as pushforward along $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Theorem 17.2.20 ([Lur17, Theorem 4.5.3.1])**.** *Let C be a symmetric monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product $\otimes\colon C\times C\to C$ preserves geometric realizations in both variables separately. Then there exists a cocartesian fibration*

$$p\colon \mathrm{Mod}(C)^{\otimes}\to \mathrm{CAlg}(C)\times \mathrm{Span}(\mathrm{Fin})$$

such that for every commutative algebra R the pullback along $\{R\}\times\mathrm{Span}(\mathrm{Fin})\hookrightarrow \mathrm{CAlg}(C)\times\mathrm{Span}(\mathrm{Fin})$ is the cocartesian fibration

$$p_R\colon \mathrm{Mod}_R(C)^{\otimes}\to \mathrm{Span}(\mathrm{Fin})$$

encoding the symmetric monoidal structure on $\mathrm{Mod}_R(C)$ from Theorem 17.2.16. Cocartesian transport along $(f,\mathrm{id}_{\langle 1\rangle})$ for a morphism $f\colon R\to S$ in $\mathrm{CAlg}(C)$ is the extension-of-scalars functor $S\otimes_R -\colon \mathrm{Mod}_R(C)\to\mathrm{Mod}_S(C)$.

By straightening-unstraightening, the cocartesian fibration p corresponds to a functor

$$\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{CAlg}(C)\times\mathrm{Span}(\mathrm{Fin})\to \mathrm{Cat}_\infty$$

which preserves finite products in the second variable. By currying, this may equivalently be encoded as a functor

$$\mathrm{CAlg}(C)\to \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}),\mathrm{Cat}_\infty)=\mathrm{Cat}_\infty^{\otimes}$$

from commutative algebras in C to symmetric monoidal ∞-categories. We record the resulting functoriality of module categories.

Corollary 17.2.21 (Extension of scalars)**.** *In the situation of Theorem 17.2.20, every morphism $f\colon R\to S$ in $\mathrm{CAlg}(C)$ induces a symmetric monoidal functor*

$$f^*\colon \mathrm{Mod}_R(C)\longrightarrow \mathrm{Mod}_S(C),$$

functorially in f. Its underlying functor is $S\otimes_R -$ from Proposition 17.2.14. For $R=\mathbb{1}$, the resulting symmetric monoidal functor

$$C\simeq \mathrm{Mod}_{\mathbb{1}}(C)\longrightarrow \mathrm{Mod}_S(C)$$

is the free module functor $F_S\colon C\to\mathrm{Mod}_S(C)$ from Proposition 17.1.14.

For morphisms of commutative ring spectra, the symmetric monoidal extension-of-scalars functor of Corollary 17.2.21 fits into a triple of adjoints.

Proposition 17.2.22. *Let $f\colon R\to S$ be a morphism of commutative ring spectra.*

(1) *The functor $f^*\colon \mathrm{Mod}_R\to\mathrm{Mod}_S$ admits a right adjoint*

$$f_*\colon \mathrm{Mod}_S\to\mathrm{Mod}_R.$$

For every S-module M, the underlying spectrum of $f_*(M)$ is naturally isomorphic to the underlying spectrum of M.

(2) *The functor* $f_*\colon \mathrm{Mod}_S \to \mathrm{Mod}_R$ *admits a further right adjoint*

$$\underline{\hom}_R(S,-)\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$$

whose underlying R*-module is the internal hom: it satisfies* $f_*\,\underline{\hom}_R(S,-) \simeq \underline{\mathrm{Hom}}_R(S,-)$ *as functors* $\mathrm{Mod}_R \to \mathrm{Mod}_R$.

Proof. Part (1) is the extension-of-scalars adjunction of Proposition 17.2.14, transported along Proposition 17.1.7. Its right adjoint is restriction of scalars, which leaves the underlying spectrum unchanged.

For part (2), the adjoint functor theorem applies once more because f_* preserves colimits. This can be checked on underlying spectra, since restriction of scalars does not change them and the forgetful functors create colimits. The identification $f_*\,\underline{\hom}_R(S,-) \simeq \underline{\mathrm{Hom}}_R(S,-)$ may be checked after passing to left adjoints, where it is the identity

$$f_*f^* \simeq S \otimes_R -\colon \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R. \qquad \square$$

Remark 17.2.23. Restriction of scalars gives a contravariant functor

$$\mathrm{CAlg}(\mathrm{Sp})^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty, \qquad A \longmapsto \mathrm{Mod}_A, \quad f \longmapsto f_*.$$

Indeed, the cocartesian fibration $\mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})$ of Theorem 17.2.20 is also cartesian: its transport functors admit right adjoints by Proposition 17.2.22(1), so Lemma 24.1.20 applies. Its cartesian transport is restriction of scalars.

17.3 Endomorphism algebras

For an abelian group M, the endomorphisms of M form a ring $\mathrm{End}(M)$ under pointwise addition and composition. An R-module structure on M is the same as a ring homomorphism $R \to \mathrm{End}(M)$. We prove the corresponding statement for an object of an ∞-category left-tensored over a monoidal ∞-category.

Let C be a monoidal ∞-category and let D be left-tensored over C. For an object $M \in D$, an object $E \in C$ equipped with a morphism $a\colon E \otimes M \to M$ is called an *endomorphism object of* M if composition with a induces an isomorphism

$$\mathrm{Hom}_C(X,E) \xrightarrow{\sim} \mathrm{Hom}_D(X \otimes M, M)$$

for every $X \in C$. To construct its algebra structure, we first form an ∞-category of pairs $(A, \eta\colon A \otimes M \to M)$, with A an object of C. Composition of these maps gives a monoidal structure on this ∞-category, in which (E,a) is terminal. We use the coherent construction from Lurie [Lur17, Section 4.7.1] and explain how it gives the required algebra and its universal property.

Definition 17.3.1. Let D be left-tensored over a monoidal ∞-category C, and let $M \in D$. The *endomorphism category of* M is defined by the pullback

$$\begin{array}{ccc} C[M] & \longrightarrow & D_{/M} \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow[-\otimes M]{} & D, \end{array}$$

where the right vertical functor sends a morphism to its source.

An object of $C[M]$ is equivalently a pair (A, η) consisting of an object $A \in C$ and a morphism $\eta\colon A \otimes M \to M$. For two such objects, the mapping anima is

$$\mathrm{Hom}_{C[M]}((X,\zeta),(A,\eta)) \simeq \mathrm{Hom}_C(X,A) \times_{\mathrm{Hom}_D(X\otimes M, M)} \{\zeta\},$$

where the map from $\mathrm{Hom}_C(X,A)$ is induced by η. Thus (A,η) is terminal precisely when composition with η induces an isomorphism

$$\mathrm{Hom}_C(X,A) \xrightarrow{\sim} \mathrm{Hom}_D(X\otimes M, M)$$

for every $X \in C$, which is exactly the endomorphism-object condition from the beginning of the section.

Composition of actions suggests a tensor product on $C[M]$:

$$(A,\eta)\otimes(A',\eta') = \left(A\otimes A', A\otimes A'\otimes M \xrightarrow{\mathrm{id}_A\otimes\eta'} A\otimes M \xrightarrow{\eta} M\right).$$

The unit is $(\mathbb{1}, \mathbb{1}\otimes M \simeq M)$. We quote the following theorem of Lurie, which supplies the coherent monoidal structure with this tensor product and unit, and identifies its algebras with algebra actions on M.

Theorem 17.3.2 ([Lur17, Proposition 4.7.1.30, Theorem 4.7.1.34])**.** *Let D be left-tensored over a monoidal ∞-category C, and let $M \in D$. The endomorphism category $C[M]$ admits a monoidal structure for which the forgetful functor $C[M] \to C$ is monoidal. Moreover, there is an equivalence*

$$\mathrm{Alg}(C[M]) \xrightarrow{\sim} \mathrm{LMod}(D) \times_D \{M\}$$

compatible with the forgetful functors to $\mathrm{Alg}(C)$.

Under this equivalence, an algebra in $C[M]$ is sent to its underlying algebra A together with the action $A \otimes M \to M$.

Corollary 17.3.3 ([Lur17, Corollary 4.7.1.40])**.** *Let D be left-tensored over a monoidal ∞-category C, and let $M \in D$ admit an endomorphism object* $\mathrm{End}(M) \in C$. *Then* $\mathrm{End}(M)$ *admits a preferred structure of an associative algebra, and M admits a preferred structure of a left module over* $\mathrm{End}(M)$. *Moreover, there is an equivalence of ∞-categories making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Alg}(C)_{/\mathrm{End}(M)} & \xrightarrow{\simeq} & \mathrm{LMod}(D)\times_D\{M\} \\ & {\scriptstyle\mathrm{fgt}}\searrow \quad \swarrow{\scriptstyle\mathrm{fgt}} & \\ & \mathrm{Alg}(C). & \end{array}$$

Proof. By definition, $\mathrm{End}(M)$ is terminal in $C[M]$. The monoidal structure from Theorem 17.3.2 and Corollary 14.6.12 therefore give $\mathrm{End}(M)$ a unique associative algebra structure in $C[M]$. Its image in C is an associative algebra, while its image under the equivalence of Theorem 17.3.2 is a left module structure on M.

The forgetful functor $\mathrm{Alg}(C[M]) \to \mathrm{Alg}(C)$ is a right fibration by [Lur17, Proposition 4.7.1.39]. Since its total category has the terminal object just constructed, this right fibration is represented by $\mathrm{End}(M)$. This gives the displayed equivalence. □

17.4 Monads and monadicity

For an adjunction $F\colon C \rightleftarrows D :G$, every object $d \in D$ gives an object $Gd \in C$ with an action $GF(Gd) \to Gd$, induced by the counit. We ask when this action determines d: can D be recovered as the ∞-category of modules over the endofunctor GF? To formulate this question, we first equip endofunctors with the monoidal structure given by composition.

Evaluation exhibits $\mathrm{Fun}(C,C)$ as the endomorphism object of C in $(\mathrm{Cat}_\infty, \times)$, since currying gives

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(X, \mathrm{Fun}(C,C)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}_\infty}(X \times C, C).$$

By Corollary 17.3.3 and Theorem 15.3.11, it therefore carries a monoidal structure with tensor product given by composition, and acts on C by evaluation. When C is not small, we work in a larger universe.

Definition 17.4.1 ([Lur17, Definition 4.7.0.1])**.** A *monad* on an ∞-category C is an associative algebra object

$$T \in \mathrm{Alg}(\mathrm{Fun}(C,C))$$

for the composition monoidal structure. Its *Eilenberg–Moore category* is the ∞-category

$$\mathrm{LMod}_T(C)$$

of left T-module objects in C, formed using the evaluation left tensoring constructed above. It comes with a forgetful functor $U_T\colon \mathrm{LMod}_T(C) \to C$.

Informally, the algebra structure consists of a unit $\mathrm{id}_C \to T$ and a multiplication $T \circ T \to T$, together with all coherent associativity and unitality data. A left T-module is an object $c \in C$ with a coherent action $T(c) \to c$.

Consider an adjunction

$$F\colon C \rightleftarrows D :G,$$

and let η and ε denote its unit and counit. The ∞-category $\mathrm{Fun}(D,C)$ is left-tensored over $\mathrm{Fun}(C,C)$ by postcomposition. For $S \in \mathrm{Fun}(C,C)$, the adjunction gives a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(C,C)}(S, GF) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(D,C)}(SG, G).$$

It sends $\alpha\colon S \to GF$ to $G\varepsilon \circ \alpha G$; its inverse sends $\beta\colon SG \to G$ to $\beta F \circ S\eta$. The triangle identities show that these are inverse. Thus GF, with the action $G\varepsilon\colon GFG \to G$, is an endomorphism object of G; see also [Lur17, Lemma 4.7.3.1]. By Corollary 17.3.3, GF acquires a monad structure, with unit η and multiplication $G\varepsilon F$. Under the equivalence

$$\mathrm{LMod}_{GF}(\mathrm{Fun}(D,C)) \simeq \mathrm{Fun}(D, \mathrm{LMod}_{GF}(C)),$$

the resulting GF-module G in $\mathrm{Fun}(D,C)$ corresponds to a lift of G:

$$D \longrightarrow \mathrm{LMod}_{GF}(C), \qquad d \longmapsto \big(Gd,\ G\varepsilon_d\colon GFGd \to Gd\big).$$

Definition 17.4.2. A right adjoint $G\colon D \to C$, with left adjoint F, is *monadic* if the induced functor $D \to \mathrm{LMod}_{GF}(C)$ is an equivalence.

The Barr–Beck theorem gives a criterion for monadicity in terms of conservativity and geometric realizations. The relevant simplicial objects are those that become split after applying G; we first explain this condition.

Recall the augmented simplicial objects with extra degeneracies from Definition 5.2.3. We use the following terminology.

Definition 17.4.3. An augmented simplicial object is *split* if it admits extra degeneracies in the sense of Definition 5.2.3. A simplicial object is *split* if it extends to a split augmented simplicial object.

Let $G\colon D \to C$ be a functor. A simplicial object $X_\bullet$ of D is *G-split* if $GX_\bullet$ is split.

Our convention differs from [Lur17, Definition 4.7.2.2] by reversing the order on the finite linear orders.

Lemma 17.4.4 (Absolute realization of split simplicial objects)**.** *Every split augmented simplicial object is a colimit diagram. In particular, every split simplicial object admits a geometric realization, and every functor preserves this realization.*

Proof. By Lemma 5.2.4, a split augmented simplicial object exhibits its augmentation as a colimit. Postcomposition with any functor preserves the extra degeneracies, so the same lemma shows that it preserves this colimit. □

The forgetful functor U_T is conservative and admits a left adjoint F_T, whose underlying endofunctor is T. Moreover, every U_T-split simplicial object of $\mathrm{LMod}_T(C)$ admits a geometric realization which is preserved by U_T; see [Lur17, Corollary 4.2.3.2, Corollary 4.2.4.8, Lemma 4.7.3.12].

Theorem 17.4.5 (Barr–Beck monadicity, [Lur17, Theorem 4.7.3.5])**.** *Let $G\colon D \to C$ be a conservative functor admitting a left adjoint. Assume that D admits geometric realizations of G-split simplicial objects and that G preserves them. Then G is monadic.*

Proof sketch. Write $F \dashv G$. The adjunction gives each object $d \in D$ an augmented simplicial resolution whose terms are $(FG)^{n+1}d$. After applying G, the unit of the adjunction supplies an extra degeneracy, so this resolution is G-split. The hypotheses and conservativity of G imply that its realization is d. The corresponding free-module resolution in $\mathrm{LMod}_{GF}(C)$ is also realized by its augmentation. The induced functor $D \to \mathrm{LMod}_{GF}(C)$ identifies free objects and their mapping animae by the two free-forgetful adjunctions. Realizing the free resolutions recovers every GF-module from an object of D; computing maps out of these realizations as limits gives full faithfulness. The coherent construction and the proof are given in [Lur17, Theorem 4.7.3.5]. □

For the free-module adjunction, the associated monad is $A \otimes -$. The forgetful functor is always monadic:

Proposition 17.4.6 (Monadicity of module categories)**.** *Let C be a monoidal ∞-category, let D be left-tensored over C, and let $A \in \mathrm{Alg}(C)$. The forgetful functor*

$$U_A\colon \mathrm{LMod}_A(D) \longrightarrow D$$

is monadic, and its associated monad is $A \otimes -\colon D \to D$, with unit and multiplication induced by those of A.

Proof. The left adjoint exists by Proposition 17.1.14, and its composite with U_A is $A \otimes -$. The forgetful functor is conservative by [Lur17, Corollary 4.2.3.2]. Moreover, every U_A-split simplicial object admits a geometric realization which is preserved by U_A; this is [Lur17, Lemma 4.7.3.12]. The claim follows from Theorem 17.4.5. □

Proposition 17.4.7 (Limits and colimits of Eilenberg–Moore categories)**.** *Let T be a monad on an ∞-category C, let $U_T \colon \mathrm{LMod}_T(C) \to C$ be its forgetful functor, and let I be a small ∞-category.*

(1) *If C admits I-indexed limits, then U_T creates and preserves them.*

(2) *If C admits I-indexed colimits and T preserves them, then U_T creates and preserves them.*

Proof. For (1), apply Proposition 17.1.16(1) to the evaluation left tensoring of C over $\mathrm{Fun}(C, C)$.

For (2), restrict the evaluation left tensoring to the full subcategory of $\mathrm{Fun}(C, C)$ spanned by the endofunctors preserving I-indexed colimits. This subcategory contains the identity and is closed under composition, so it is monoidal. Since T preserves these colimits, its algebra structure belongs to this full monoidal subcategory. The ∞-category of T-modules is unchanged by this restriction, and every object of the acting ∞-category now preserves I-indexed colimits. Thus Proposition 17.1.16(2) applies. □

Applied to the monad $A \otimes -$ of Proposition 17.4.6, part (2) shows that the forgetful functor $\mathrm{LMod}_A(D) \to D$ creates any colimits which exist in D and are preserved by $A \otimes -$. For a fixed algebra this sharpens Proposition 17.1.16(2), whose more uniform hypothesis ensures the same conclusion simultaneously for every algebra in C.

Exercises

Exercise 17.1 (Free modules)**.** Let C be a presentably monoidal ∞-category and let $A \in \mathrm{Alg}(C)$.

(1) Use Proposition 17.1.14 to show that the free-module functor preserves coproducts, and compute the free A-module on a finite coproduct.

(2) Specialize to $C = \mathrm{Sp}$ and identify the free R-module on $\mathbb{S}[n]$ for an associative ring spectrum R. Deduce that $\pi_0 \mathrm{Hom}_{\mathrm{LMod}_R}(R[n], M) \cong \pi_n(M)$ for every left R-module M.

Exercise 17.2 (Transitivity of extension of scalars)**.** Let $A \to B \to C$ be morphisms of associative algebras in a presentably monoidal ∞-category. For a left A-module M, construct a natural isomorphism

$$C \otimes_B (B \otimes_A M) \simeq C \otimes_A M.$$

Check that it identifies the composite of the two extension-of-scalars adjunctions with extension of scalars along $A \to C$.

18 Tensor products of ∞-categories and Morita theory

The tensor product $A \otimes B$ of abelian groups classifies bilinear morphisms out of $A \times B$. Replacing abelian groups by ∞-categories admitting colimits, we similarly obtain a universal functor $C \times D \to C \otimes D$ preserving colimits separately in both variables. Its target is called the *tensor product* of C and D. For presentable ∞-categories, these tensor products give a symmetric monoidal structure on Pr^{L}. This allows us to do *categorical algebra*: the results from Chapter 17 apply to algebras and modules in Pr^{L}.

The commutative algebras in Pr^{L} are precisely the presentably symmetric monoidal ∞-categories. Our main examples, An_*, $\mathrm{CMon}(\mathrm{An})$, $\mathrm{CGrp}(\mathrm{An})$, and Sp, have an additional property: their multiplication $M \otimes M \to M$ is an equivalence. Such a commutative algebra is called a *mode*. Being a module over a mode is a property of a presentable ∞-category; in these four examples, it amounts to pointedness, semiadditivity, additivity, and stability, respectively.

In particular, every presentable stable ∞-category carries a canonical action by spectra. Applying the endomorphism-algebra construction from Section 17.3 gives the endomorphism ring spectrum of an object and makes mapping spectra out of it into right modules. We use this to prove a spectral version of Morita theory: a presentable stable ∞-category with a compact generator is equivalent to the ∞-category of right modules over its endomorphism ring spectrum. Finally, the spectral Eilenberg–Watts theorem identifies colimit-preserving functors between module categories with bimodules.

18.1 Tensor products of presentable ∞-categories

A bifunctor that preserves colimits separately in each variable should correspond to a colimit-preserving functor out of a tensor product. We now construct such tensor products for presentable ∞-categories. In fact, we start more generally with ∞-categories admitting colimits for a prescribed collection $\mathcal{K}$ of ∞-categories; this will also give tensor products for stable ∞-categories or ∞-categories with finite coproducts.

Write

$$\widehat{\mathrm{Cat}}_\infty^{\mathcal{K}\text{-colim}} \quad \subseteq \quad \widehat{\mathrm{Cat}}_\infty$$

for the ∞-category whose objects are the possibly large ∞-categories admitting I-indexed colimits for every $I \in \mathcal{K}$, and whose morphisms preserve all such colimits. More generally, for $C_1, \ldots, C_n, D$

admitting these colimits, we say that a functor

$$F\colon C_1 \times \cdots \times C_n \longrightarrow D$$

is $\mathcal{K}$-*multilinear* if it preserves I-indexed colimits in each variable separately for every $I \in \mathcal{K}$.

The ∞-categories admitting $\mathcal{K}$-indexed colimits and the $\mathcal{K}$-multilinear functors form a suboperad of the cartesian operad $O^{\times}(\widehat{\mathrm{Cat}}_\infty)$. Indeed, identities preserve the prescribed colimits, and substitution preserves multilinearity because each input occurs in a single inner operation, so Construction 14.1.7 applies.

A symmetric monoidal structure with this multimorphism operad amounts to tensor products corepresenting these multilinear functors.

Theorem 18.1.1 (Tensor products of ∞-categories with prescribed colimits, [Lur17, Corollary 4.8.1.4 and Variant 4.8.1.11])**.** *The ∞-operad of $\mathcal{K}$-multilinear functors is the multimorphism operad of a symmetric monoidal ∞-category*

$$\big(\widehat{\mathrm{Cat}}_\infty^{\mathcal{K}\text{-colim}}, \otimes_{\mathcal{K}}\big).$$

Thus, for every finite collection $C_1, \ldots, C_n$ of its objects, there is an object

$$C_1 \otimes_{\mathcal{K}} \cdots \otimes_{\mathcal{K}} C_n$$

and a $\mathcal{K}$-multilinear functor from $C_1 \times \cdots \times C_n$ to this object such that composition induces an equivalence

$$\mathrm{Fun}_{\mathcal{K}}\big(C_1 \otimes_{\mathcal{K}} \cdots \otimes_{\mathcal{K}} C_n, D\big) \xrightarrow{\sim} \mathrm{Fun}_{\mathcal{K},\ldots,\mathcal{K}}(C_1 \times \cdots \times C_n, D).$$

Here $\mathrm{Fun}_{\mathcal{K}}$ denotes the full subcategory of functors preserving the prescribed colimits, and the right-hand side is the full subcategory of $\mathcal{K}$-multilinear functors. The equivalence of ∞-categories follows by applying the universal property with target $\mathrm{Fun}(B, D)$ for every small ∞-category B, since its prescribed colimits are computed pointwise. If $\mathcal{K}$ and each C_i are small, then $C_1 \otimes_{\mathcal{K}} \cdots \otimes_{\mathcal{K}} C_n$ is small as well.

Example 18.1.2 (Finite coproducts)**.** When $\mathcal{K}$ consists of the finite discrete ∞-categories, we obtain the symmetric monoidal ∞-category $(\widehat{\mathrm{Cat}}_\infty^{\mathrm{coprod}}, \otimes_{\mathrm{coprod}})$ of possibly large ∞-categories with finite coproducts and finite-coproduct-preserving functors. Its tensor product classifies functors preserving finite coproducts separately in each variable. The full subcategory $\mathrm{Cat}_\infty^{\mathrm{coprod}}$ on the small ∞-categories is closed under this tensor product. In particular, a commutative algebra in $(\mathrm{Cat}_\infty^{\mathrm{coprod}}, \otimes_{\mathrm{coprod}})$ is precisely a small symmetric monoidal ∞-category whose tensor product preserves finite coproducts separately in each variable.

Example 18.1.3 (Finite colimits and stable ∞-categories)**.** Taking $\mathcal{K}$ to consist of all finite ∞-categories gives a symmetric monoidal structure $(\widehat{\mathrm{Cat}}_\infty^{\mathrm{rex}}, \otimes_{\mathrm{rex}})$ on the ∞-category of possibly large ∞-categories with finite colimits and right exact functors.

If C is stable and D admits finite colimits, then $C \otimes_{\mathrm{rex}} D$ is stable. Indeed, the universal bifunctor sends $(0, d)$ to an initial object. Composing it with $\mathrm{id}_C \to \mathrm{const}_0$ in the first variable therefore induces a natural transformation $\mathrm{id}_{C \otimes_{\mathrm{rex}} D} \to \mathrm{const}_0$ by the functor-category universal property. Since its component at the initial object is the identity, this exhibits the functor selecting that object as right adjoint to the functor to $*$, proving pointedness (Proposition 22.1.2). Since the universal bifunctor

preserves finite colimits separately in each variable, its universal property identifies suspension with $\Sigma_C \otimes_{\mathrm{rex}} \mathrm{id}_D$. This is an equivalence, so Theorem 4.2.2 proves stability.

For stable C, D, the tensor product therefore classifies functors $C \times D \to E$ that are exact separately in each variable, for every stable target E; see Corollary 4.2.27.

Example 18.1.4 (All small colimits)**.** Let $\mathcal{K}$ consist of all small ∞-categories. We denote the resulting ∞-category by $\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}$ and write $\otimes$ for this specialization of $\otimes_{\mathcal{K}}$. Its objects are the cocomplete ∞-categories and its morphisms are the colimit-preserving functors.

For small ∞-categories A and B, the external product of presheaves, defined by $(F \boxtimes G)(a, b) := F(a) \times G(b)$, exhibits an equivalence

$$\mathrm{PSh}(A) \otimes \mathrm{PSh}(B) \quad \simeq \quad \mathrm{PSh}(A \times B);$$

see Exercise 18.1. The monoidal unit is $\mathrm{An} = \mathrm{PSh}(*)$.

We extend the presentability assertion to arbitrary presentable ∞-categories by imposing the relations defining each factor.

Proposition 18.1.5 ([Lur17, Proposition 4.8.1.15])**.** *The tensor product of two presentable ∞-categories is presentable.*

Proof. We may choose presentations

$$C \simeq \mathrm{PSh}(A)^{S\text{-loc}}, \qquad D \simeq \mathrm{PSh}(B)^{T\text{-loc}},$$

where A, B are small ∞-categories and S, T are small collections of morphisms, as in Definition 23.2.1. Let U be the collection of maps

$$s \boxtimes \mathrm{id}_{Y(b)} \quad (s \in S,\ b \in B), \qquad \mathrm{id}_{Y(a)} \boxtimes t \quad (a \in A,\ t \in T)$$

in $\mathrm{PSh}(A \times B)$. By Theorem 23.2.2, the ∞-category of U-local presheaves is presentable. We show that it has the universal property of $C \otimes D$.

Let E be any cocomplete ∞-category. By Theorem 23.2.2(4), colimit-preserving functors $\mathrm{PSh}(A \times B)^{U\text{-loc}} \to E$ correspond to colimit-preserving functors $\mathrm{PSh}(A \times B) \to E$ that invert U.

By the presheaf tensor-product formula, colimit-preserving functors $\mathrm{PSh}(A \times B) \to E$ correspond to functors $\mathrm{PSh}(A) \times \mathrm{PSh}(B) \to E$ preserving colimits in both variables separately. Inverting U means that the latter invert S in the first variable when the second variable is representable, and T in the second variable when the first variable is representable. Since representables generate under colimits, these conditions hold for arbitrary values of the other variable as well. Applying Theorem 23.2.2(4) in each variable gives an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}\big(\mathrm{PSh}(A \times B)^{U\text{-loc}}, E\big) \simeq \mathrm{Fun}^{\mathrm{colim},\mathrm{colim}}(C \times D, E).$$

This identifies $\mathrm{PSh}(A \times B)^{U\text{-loc}}$ with $C \otimes D$, proving the claim. □

The full subcategory $\mathrm{Pr}^{\mathrm{L}} \subseteq \widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}$ on the presentable colors contains the unit An and is closed under tensor products. It is therefore a symmetric monoidal subcategory

$$(\mathrm{Pr}^{\mathrm{L}}, \otimes) \quad \subseteq \quad \big(\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}, \otimes\big),$$

with tensor product characterized by

$$\mathrm{Fun}^{\mathrm{L}}(C \otimes D, E) \simeq \mathrm{Fun}^{\mathrm{L},\mathrm{L}}(C \times D, E).$$

For presentable ∞-categories, the tensor product also has the following description. We write $\mathrm{Fun}^{\mathrm{lim}}$ for the full subcategory of functors preserving small limits.

Proposition 18.1.6 ([Lur17, Proposition 4.8.1.17])**.** *For presentable ∞-categories C and D, there is a canonical equivalence*

$$C \otimes D \simeq \mathrm{Fun}^{\mathrm{lim}}(C^{\mathrm{op}}, D).$$

If $L\colon D' \rightleftarrows D : R$ is an adjunction between presentable ∞-categories, the right adjoint of $\mathrm{id}_C \otimes L$ corresponds to postcomposition with R.

Proof. The tensor product is computed in the ∞-category of all cocomplete ∞-categories. Its universal property therefore applies with target $\mathrm{An}^{\mathrm{op}}$, which admits small colimits. Taking opposites and using Corollary 23.2.6 gives

$$\begin{aligned} C \otimes D &\simeq \mathrm{Fun}^{\mathrm{lim}}((C \otimes D)^{\mathrm{op}}, \mathrm{An}) \\ &\simeq \mathrm{Fun}^{\mathrm{lim},\mathrm{lim}}(C^{\mathrm{op}} \times D^{\mathrm{op}}, \mathrm{An}) \\ &\simeq \mathrm{Fun}^{\mathrm{lim}}\big(C^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{lim}}(D^{\mathrm{op}}, \mathrm{An})\big) \\ &\simeq \mathrm{Fun}^{\mathrm{lim}}(C^{\mathrm{op}}, D). \end{aligned}$$

Here the second line consists of functors preserving small limits in both variables separately, and the third uses pointwise computation of limits. For the last assertion, precomposition with $\mathrm{id}_C \otimes L$ on representable functors corresponds in the second line to precomposition with $\mathrm{id}_{C^{\mathrm{op}}} \times L^{\mathrm{op}}$. Under representability in the D-variable, this is postcomposition with R. □

Remark 18.1.7. The suboperad description above identifies presentably symmetric monoidal ∞-categories with commutative algebra objects of $(\mathrm{Pr}^{\mathrm{L}}, \otimes)$. Indeed, a commutative algebra object is an operad map

$$Comm \longrightarrow \mathcal{M}_{(\mathrm{Pr}^{\mathrm{L}},\otimes)} \subseteq \mathcal{O}^{\times}(\widehat{\mathrm{Cat}}_{\infty}).$$

Such a map is precisely a symmetric monoidal structure on a presentable ∞-category whose tensor operations preserve colimits in each variable separately. It suffices to check this for the binary tensor product. Morphisms are colimit-preserving symmetric monoidal functors.

18.2 Modes and universal properties

Recall from Chapter 16 the sequence of colimit-preserving symmetric monoidal functors

$$(\mathrm{An}, \times) \xrightarrow{(-)_+} (\mathrm{An}_*, \wedge) \xrightarrow{F^{\mathrm{CMon}}} (\mathrm{CMon}(\mathrm{An}), \otimes) \xrightarrow{(-)^{\mathrm{grp}}} (\mathrm{CGrp}(\mathrm{An}), \otimes) \xrightarrow{\mathbf{B}^{\infty}} (\mathrm{Sp}, \otimes).$$

We will show that tensoring a presentable ∞-category C with the four ∞-categories after An gives C_*, $\mathrm{CMon}(C)$, $\mathrm{CGrp}(C)$, and $\mathrm{Sp}(C)$, respectively. These constructions freely impose pointedness, semiadditivity, additivity, and stability. Each leaves C unchanged when it already has the corresponding property, so applying the construction twice has the same effect as applying it once.

This idempotence comes from the commutative algebra structures on these ∞-categories. We first discuss it in an arbitrary symmetric monoidal ∞-category.

Definition 18.2.1 ([Lur17, Definition 4.8.2.8])**.** Let E be a symmetric monoidal ∞-category. A commutative algebra $A \in \mathrm{CAlg}(E)$ is *idempotent* if its multiplication $A \otimes A \to A$ is an isomorphism. An object $X \in E$ is called *A-local* if the unit induces an isomorphism $X \to A \otimes X$. We write $E^{A\text{-loc}} \subseteq E$ for the full subcategory of A-local objects.

Proposition 18.2.2 ([Lur17, Propositions 4.8.2.4, 4.8.2.7 and 4.8.2.10])**.** *Let A be an idempotent commutative algebra in a symmetric monoidal ∞-category E.*

(1) The forgetful functor $\mathrm{Mod}_A(E) \to E$ is fully faithful, with essential image $E^{A\text{-loc}}$.

(2) Tensoring with A defines a symmetric monoidal Bousfield localization

$$A \otimes -\colon E \longrightarrow E^{A\text{-loc}},$$

where the target has its induced symmetric monoidal structure with unit A.

Proof. Both unit insertions $A \to A \otimes A$ are inverse to multiplication. Thus Proposition 22.7.9 identifies $A \otimes -$ as a Bousfield localization onto the A-local objects. If $A \otimes f$ is an isomorphism, then so is $A \otimes (f \otimes Y)$ for every Y. Hence Proposition 14.7.2 supplies the symmetric monoidal structure and refinement in (2).

For an A-module X, the unit and action exhibit its underlying object as a retract of $A \otimes X$. Since locality is closed under retracts, X is A-local, and the action map $A \otimes X \to X$ is inverse to the unit. The counit of the free-module adjunction of Proposition 17.1.14 is therefore an isomorphism, proving full faithfulness in (1). Its image is exactly the local objects, since each local object X is isomorphic to the underlying object of its free module $A \otimes X$. □

A localization given by tensoring with a fixed object is called *smashing*. We apply this construction to ∞-categories.

Definition 18.2.3 (Mode, [CSY21])**.** A *mode* is an idempotent commutative algebra in Pr^{L}.

Since $\mathrm{Pr}^{\mathrm{L}} \hookrightarrow \widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}$ is fully faithful and symmetric monoidal, a mode M is also an idempotent commutative algebra in the latter ∞-category. Thus a cocomplete ∞-category C is M-local if the unit induces an equivalence $C \simeq \mathrm{An} \otimes C \to M \otimes C$. By Proposition 18.2.2, this is equivalent to admitting a unique M-module structure. Tensoring with M defines a symmetric monoidal Bousfield localization of $\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}$, which restricts to the corresponding localization of Pr^{L}.

Corollary 18.2.4. *Let M be a mode and let C, D be cocomplete ∞-categories. If D is M-local, restriction along the unit $C \to M \otimes C$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{colim}}(M \otimes C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{colim}}(C, D).$$

Proof. The localization universal property gives an equivalence on groupoid cores. For every small ∞-category K, the pointwise M-action equips $\mathrm{Fun}(K, D)$ with an M-module structure, so it is again M-local. Applying the universal property to these targets and currying proves the equivalence of functor categories by Yoneda. □

The four constructions at the beginning of this section provide the following examples.

Theorem 18.2.5 (The basic modes, [GGN15, Sections 3–5]; see also [Lur17, Section 4.8.2])**.** *The presentably symmetric monoidal ∞-categories*

$$(\mathrm{An}_*, \wedge), \qquad (\mathrm{CMon}(\mathrm{An}), \otimes), \qquad (\mathrm{CGrp}(\mathrm{An}), \otimes), \qquad (\mathrm{Sp}, \otimes)$$

are modes. For every presentable ∞-category C, there are natural equivalences

$$\mathrm{An}_* \otimes C \simeq C_*, \qquad \mathrm{CMon}(\mathrm{An}) \otimes C \simeq \mathrm{CMon}(C),$$
$$\mathrm{CGrp}(\mathrm{An}) \otimes C \simeq \mathrm{CGrp}(C), \qquad \mathrm{Sp} \otimes C \simeq \mathrm{Sp}(C).$$

Under these equivalences, the localization units are the functors adjoining a basepoint, forming the free commutative monoid or group, and taking the suspension spectrum, respectively.

Proof. Let F be one of $(-)_*$, CMon, CGrp, and Sp, and put $M := F(\mathrm{An})$. These constructions are defined by diagrams satisfying limit conditions, so they commute with forming ∞-categories of limit-preserving functors. Thus Proposition 18.1.6 and Corollary 23.2.6 give

$$C \otimes M \simeq \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, F(\mathrm{An})) \simeq F\big(\mathrm{Fun}^{\lim}(C^{\mathrm{op}}, \mathrm{An})\big) \simeq F(C).$$

Indeed, limits in all four diagram categories are computed pointwise, and the defining conditions can be checked pointwise as well.

By Proposition 16.6.4, the unit $\mathrm{An} \to M$ is the corresponding free functor, whose right adjoint is the forgetful or evaluation functor $M \to \mathrm{An}$. The naturality assertion of Proposition 18.1.6 identifies the right adjoint of $C \to C \otimes M$ with the usual forgetful or evaluation functor $F(C) \to C$. Hence the localization units have the asserted descriptions.

The ∞-category M is pointed, semiadditive, additive, or stable, respectively, and in each case $F(M) \to M$ is an equivalence; see Lemma 4.1.9, Propositions 5.3.17 and 5.3.20, Theorem 4.3.14, and Corollary 4.3.17. Consequently $M \to M \otimes M$ is an equivalence, and its left inverse given by the existing multiplication is an equivalence as well. This proves that M is a mode. □

The functors in the sequence at the beginning of this section are morphisms of modes, meaning morphisms in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$: they are colimit-preserving and symmetric monoidal by Proposition 16.6.4.

We now identify the local cocomplete ∞-categories. Among presentable ∞-categories, this can directly be inferred from the tensor descriptions in Theorem 18.2.5. For the general case, we first compute colimit-preserving functors out of the four basic modes.

Lemma 18.2.6 (Nonmonoidal universal properties of the basic modes)**.** *Let M be one of* An_*, $\mathrm{CMon}(\mathrm{An})$, $\mathrm{CGrp}(\mathrm{An})$, *and* Sp*, and let D be a cocomplete ∞-category that is respectively pointed, semiadditive, additive, or stable. Evaluation at the monoidal unit of M induces an equivalence*

$$\mathrm{ev}_{\mathbb{1}_M} \colon \mathrm{Fun}^{\mathrm{colim}}(M, D) \xrightarrow{\sim} D.$$

Proof. The case $M = \mathrm{Sp}$ is Theorem 4.4.14. For the other three cases, we claim that for every cocomplete D there are natural equivalences

$$\begin{aligned} \mathrm{Fun}^{\mathrm{colim}}(\mathrm{An}_*, D) &\simeq \big((D^{\mathrm{op}})_*\big)^{\mathrm{op}}, \\ \mathrm{Fun}^{\mathrm{colim}}(\mathrm{CMon}(\mathrm{An}), D) &\simeq \mathrm{CMon}(D^{\mathrm{op}})^{\mathrm{op}}, \\ \mathrm{Fun}^{\mathrm{colim}}(\mathrm{CGrp}(\mathrm{An}), D) &\simeq \mathrm{CGrp}(D^{\mathrm{op}})^{\mathrm{op}}. \end{aligned}$$

These follow from the presheaf universal property and the localization universal property of Theorem 23.2.2(4), which allows arbitrary cocomplete targets.

For the first equivalence, regard An_* as the full subcategory of $\mathrm{Fun}([1], \mathrm{An}) = \mathrm{PSh}([1]^{\mathrm{op}})$ consisting of arrows whose source is terminal. If Y_0 denotes the representable corresponding to 0, this is the localization at $\emptyset \to Y_0$. A colimit-preserving functor out of this localization therefore corresponds to a functor $[1]^{\mathrm{op}} \to D$ whose value at 0 is initial. Such a functor is an arrow $d \to \emptyset_D$, where $\emptyset_D$ is initial in D, or equivalently a pointed object of D^{op}, with the direction of natural transformations reversed.

For commutative monoids, we use the presentation in the proof of Proposition 23.2.7: $\mathrm{CMon}(\mathrm{An})$ is the localization of $\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), \mathrm{An}) = \mathrm{PSh}(\mathrm{Span}(\mathrm{Fin})^{\mathrm{op}})$ imposing preservation of finite products. A colimit-preserving functor from this localization to D corresponds to a functor $G\colon \mathrm{Span}(\mathrm{Fin})^{\mathrm{op}} \to D$ carrying the corresponding finite coproduct diagrams to coproduct diagrams. This says exactly that $G^{\mathrm{op}}\colon \mathrm{Span}(\mathrm{Fin}) \to D^{\mathrm{op}}$ preserves finite products. Localizing in addition at the shear maps imposes the grouplike condition, giving the equivalence for commutative groups.

Under these equivalences, evaluation at $\mathbb{1}_M$ corresponds to the underlying-object functor on the right: the unit is the reflection of the representable at $1 \in [1]$ or $\langle 1\rangle \in \mathrm{Span}(\mathrm{Fin})$, respectively. In the pointed case this reflection is S^0; in the other two cases it is the free commutative monoid or group on a point. If D is pointed, semiadditive, or additive, respectively, then so is D^{op}, and the underlying-object functor is an equivalence by Lemma 4.1.9 and Proposition 5.3.17. This proves the claim. □

Proposition 18.2.7. *Let M be one of* An_*, $\mathrm{CMon}(\mathrm{An})$, $\mathrm{CGrp}(\mathrm{An})$, *and* Sp. *A cocomplete ∞-category D is M-local if and only if it is respectively pointed, semiadditive, additive, or stable.*

Proof. Suppose first that D has the corresponding property. The functor category $\mathrm{Fun}^{\mathrm{colim}}(M, D)$ is cocomplete, with colimits computed pointwise. For every other cocomplete ∞-category E, currying identifies restriction along $E \to M \otimes E$ with

$$\mathrm{Fun}^{\mathrm{colim}}\big(E, \mathrm{Fun}^{\mathrm{colim}}(M, D)\big) \xrightarrow{(\mathrm{ev}_{\mathbb{1}_M})_*} \mathrm{Fun}^{\mathrm{colim}}(E, D).$$

This is an equivalence by Lemma 18.2.6, so D is M-local.

Conversely, if D is M-local, then Corollary 18.2.4 with $C = \mathrm{An}$ gives $D \simeq \mathrm{Fun}^{\mathrm{colim}}(M, D)$. For the first three choices of M, the formulas in the preceding proof express the latter as the opposite of a pointed, semiadditive, or additive ∞-category, respectively; see Lemma 4.1.9 and Proposition 5.3.20.

For $M = \mathrm{Sp}$, the functor category $\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Sp}, D)$ is pointed by the dual of Corollary 4.1.12, with zero object the constant functor at the initial object of D. Its colimits are computed pointwise, and the suspension of F is $F \circ \Sigma_{\mathrm{Sp}}$, since F preserves suspension pushouts. Precomposition with Σ_{Sp} is an equivalence, so Theorem 4.2.2 proves stability. □

The local cocomplete ∞-categories are listed in Table 18.1.

Mode	Local cocomplete ∞-categories
An_*	Pointed
$\mathrm{CMon}(\mathrm{An})$	Semiadditive
$\mathrm{CGrp}(\mathrm{An})$	Additive
Sp	Stable

Table 18.1: The four basic modes and their local cocomplete ∞-categories.

We write $\mathrm{Fun}^{\mathrm{colim},\otimes}(C, D)$ for the ∞-category of colimit-preserving symmetric monoidal functors and monoidal natural transformations.

Proposition 18.2.8 (Multiplicative universal property of modes)**.** *Let M be a mode, and let $C, D \in \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}})$. Equip $M \otimes C$ with the symmetric monoidal structure obtained by applying the symmetric monoidal localization $M \otimes -$ to C. If D is M-local, restriction along $C \to M \otimes C$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{colim},\otimes}(M \otimes C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{colim},\otimes}(C, D).$$

Proof. Apply Proposition 18.2.2(2) in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}$. By Corollary 14.7.4, its symmetric monoidal localization induces a localization on commutative algebra objects, giving the asserted equivalence on groupoid cores.

For every small ∞-category K, the pointwise M-action makes $\mathrm{Fun}(K, D)$ M-local. Its colimits and tensor product are computed pointwise, so it satisfies the same hypotheses as D. Applying the same argument with this target and currying gives the equivalence of functor categories by Yoneda. □

Remark 18.2.9. Let C be a presentably symmetric monoidal ∞-category. For $M = \mathrm{CMon}(\mathrm{An})$, the equivalence $M \otimes C \simeq \mathrm{CMon}(C)$ of Theorem 18.2.5 identifies the induced symmetric monoidal structure with the Day convolution structure of Proposition 16.4.1. Indeed, the latter makes the free functor $C \to \mathrm{CMon}(C)$ symmetric monoidal, and its target is semiadditive. By Proposition 18.2.8, this functor factors through a colimit-preserving symmetric monoidal functor

$$\mathrm{CMon}(\mathrm{An}) \otimes C \longrightarrow \mathrm{CMon}(C).$$

Its underlying functor is the equivalence of Theorem 18.2.5, by the uniqueness in Corollary 18.2.4.

Taking $C = \mathrm{An}$ gives the following initiality statement.

Corollary 18.2.10 (Multiplicative universal property)**.** *Every mode M is initial among M-local cocomplete symmetric monoidal ∞-categories whose tensor products preserve colimits separately in each variable.*

Proof. For M-local D, Proposition 18.2.8 gives

$$\mathrm{Fun}^{\mathrm{colim},\otimes}(M, D) \simeq \mathrm{Fun}^{\mathrm{colim},\otimes}(\mathrm{An}, D).$$

Since An is the monoidal unit of $\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}$, it is initial in its ∞-category of commutative algebras. Thus the groupoid core on the right is contractible. Applying this also with target $\mathrm{Fun}(K, D)$ shows that the whole functor category is contractible. □

By Proposition 18.2.7, if M is one of the four basic modes and $D \in \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}})$ has the corresponding property in Table 18.1, then $\mathrm{Fun}^{\mathrm{colim},\otimes}(M, D)$ is contractible. These universal properties characterize the four symmetric monoidal ∞-categories and determine the transition functors.

18.3 Spectral Morita theory and Eilenberg–Watts

A compact generator allows us to recover a presentable stable ∞-category from its endomorphism ring spectrum. We first construct this ring spectrum and the associated module-valued mapping functor. We then prove the Morita equivalence and identify colimit-preserving functors between module categories with bimodules.

Proposition 18.3.1 ([Lur17, Remark 7.1.2.2])**.** *Let D be a stable ∞-category and $M \in D$.*

(1) *The mapping spectrum $A := \hom_D(M, M)$ admits a preferred structure of an associative ring spectrum.*

(2) *The mapping-spectrum functor $\hom_D(M, -) \colon D \to \mathrm{Sp}$ admits a preferred lift*

$$\hom_D(M, -) \colon D \longrightarrow \mathrm{RMod}_A,$$

with right A-action given by precomposition.

If D is presentable, this lift fits into an adjunction

$$- \otimes_A M \colon \mathrm{RMod}_A \rightleftarrows D \colon \hom_D(M, -).$$

Proof. Suppose first that D is presentable. By Theorem 18.2.5 and Proposition 18.2.2, it carries a unique Sp-module structure in Pr^{L}. The action functor $- \otimes M \colon \mathrm{Sp} \to D$ preserves colimits and sends $\mathbb{S}$ to M, so Theorem 4.4.14 identifies it with the tensoring of Proposition 4.4.9. Its right adjoint is $\hom_D(M, -)$. In particular, there are natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Sp}}(E, A) \simeq \mathrm{Hom}_D(E \otimes M, M),$$

so A is an endomorphism object of M. By Corollary 17.3.3, it has a preferred associative algebra structure and acts on M from the left.

By Lemma 17.2.11, the relative tensor product $- \otimes_A M \colon \mathrm{RMod}_A \to D$ preserves small colimits, so it has a right adjoint G by Theorem 23.2.5.

Let $U \colon \mathrm{RMod}_A \to \mathrm{Sp}$ be the forgetful functor. The composite UG is right adjoint to

$$K \longmapsto (K \otimes A) \otimes_A M \simeq K \otimes M,$$

hence $UG \simeq \hom_D(M, -)$. Under this identification, the right A-action is precomposition: its adjoint is the composite of evaluations

$$\hom_D(M, X) \otimes A \otimes M \longrightarrow \hom_D(M, X) \otimes M \longrightarrow X.$$

This proves both assertions and the stated adjunction in the presentable case.

For arbitrary D, enlarge the universe so that D is small and use the fully faithful exact embedding $D \hookrightarrow \mathrm{Ind}(D)$. Its target is presentable and stable by Proposition 23.3.6 and [Lur17, Proposition 1.1.3.6]. The embedding preserves mapping spectra, so applying the presentable case in $\mathrm{Ind}(D)$ and restricting its module-valued functor to D proves the result. □

Theorem 18.3.2 (Spectral Morita theorem, [Lur17, Theorem 7.1.2.1])**.** *Let D be a presentable stable ∞-category with a compact generator M, and let $A := \hom_D(M, M)$. Then the adjunction of Proposition 18.3.1 is an equivalence:*

$$- \otimes_A M \colon \mathrm{RMod}_A \rightleftarrows D \colon \hom_D(M, -).$$

Proof. Since M is compact, $\hom_D(M, -)$ preserves filtered colimits; being exact, it preserves finite colimits, and hence all colimits. The forgetful functor $\mathrm{RMod}_A \to \mathrm{Sp}$ detects and preserves colimits, so the same holds for the module-valued functor.

The counit at M is the canonical isomorphism $A \otimes_A M \to M$. The triangle identity then shows that the unit at $A = \hom_D(M, M)$ is an isomorphism as well. Both adjoints are exact and preserve colimits, so the full subcategories on which the unit, respectively the counit, is an isomorphism are stable and closed under colimits. They contain the generators A of RMod_A and M of D (Example 23.3.10), and are therefore the entire ∞-categories. □

Remark 18.3.3. In the setting of stable model categories, this result is due to Schwede and Shipley [SS03], and it is often referred to as the *Schwede–Shipley theorem*.

Remark 18.3.4. Barr–Beck gives an alternative proof of Theorem 18.3.2. Consider the adjunction

$$- \otimes M \colon \mathrm{Sp} \rightleftarrows D \colon \hom_D(M, -).$$

The right adjoint is conservative because M is a generator, and it preserves all colimits by compactness and exactness. Barr–Beck therefore identifies D with the Eilenberg–Moore category of the associated monad on Sp.

The evaluation action of $A = \hom_D(M, M)$ on M induces a morphism from the monad $- \otimes A$ to the adjunction monad, with underlying natural transformation

$$- \otimes A \longrightarrow \hom_D(M, - \otimes M).$$

The action unit and associativity give compatibility with the monad unit and multiplication. The underlying transformation is an isomorphism because both functors preserve colimits and it is an isomorphism on the sphere spectrum. Hence the Eilenberg–Moore category is RMod_A, and its comparison functor is the module-valued functor $\hom_D(M, -)$.

When D is presentably symmetric monoidal and M is its tensor unit, the equivalence also recovers the tensor product.

Corollary 18.3.5 (Symmetric monoidal Morita equivalence, [Lur17, Proposition 7.1.2.7])**.** *Let C be a presentably symmetric monoidal stable ∞-category whose unit $\mathbb{1}$ is a compact generator. Then $A := \hom_C(\mathbb{1}, \mathbb{1})$ is canonically a commutative ring spectrum, and $\hom_C(\mathbb{1}, -)$ lifts to a symmetric monoidal equivalence*

$$\Phi \colon C \xrightarrow{\sim} \mathrm{Mod}_A.$$

Proof. By Corollary 18.2.10, the functor $- \otimes \mathbb{1} \colon \mathrm{Sp} \to C$ is symmetric monoidal. Its right adjoint $\hom_C(\mathbb{1}, -)$ is therefore lax symmetric monoidal by Corollary 4.4.10 and Proposition 14.5.6, giving A its commutative ring structure and inducing a lax symmetric monoidal functor

$$\Phi \colon C \simeq \mathrm{Mod}_{\mathbb{1}}(C) \longrightarrow \mathrm{Mod}_A.$$

The evaluation action of A on $\mathbb{1}$ identifies its underlying associative algebra with the endomorphism algebra of $\mathbb{1}$ by Corollary 17.3.3, since the induced algebra map is the identity on underlying spectra. The lax structure maps are adjoint to tensor products of evaluation maps; thus the induced right A-action on $\hom_C(\mathbb{1}, X)$ is the precomposition action of Proposition 18.3.1. Identifying Mod_A with RMod_A by Observation 17.2.5, we may therefore apply Theorem 18.3.2 to conclude that Φ is an equivalence and preserves colimits.

It remains to show that Φ is symmetric monoidal. Its unit map $A \to \Phi(\mathbb{1})$ is an isomorphism. For fixed Y, both sides of the lax structure map

$$\Phi(X) \otimes_A \Phi(Y) \longrightarrow \Phi(X \otimes Y)$$

preserve colimits in X (Lemma 17.2.11), and hence are exact. The map is an isomorphism at $X = \mathbb{1}$ by unitality, so it is an isomorphism everywhere because $\mathbb{1}$ generates C. Thus Φ is a symmetric monoidal equivalence. □

We now describe colimit-preserving functors between module categories in terms of bimodules.

For associative ring spectra R and S, the relative tensor product gives a functor

$$- \otimes_R - \colon \mathrm{RMod}_R \times {}_R\mathrm{BMod}_S \longrightarrow \mathrm{RMod}_S$$

which preserves colimits separately in both variables by Lemma 17.2.11. Currying therefore gives a functor

$${}_R\mathrm{BMod}_S \longrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathrm{RMod}_R, \mathrm{RMod}_S).$$

Theorem 18.3.6 (Spectral Eilenberg–Watts theorem, [Lur17, Proposition 7.1.2.4])**.** *For all associative ring spectra R and S, the functor*

$${}_R\mathrm{BMod}_S \longrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathrm{RMod}_R, \mathrm{RMod}_S)$$

is an equivalence.

Proof. We construct an inverse by sending F to $F(R)$. By Theorem 18.2.5 and Proposition 18.2.2, every colimit-preserving functor between presentable stable ∞-categories has a unique Sp-linear structure. This also applies to all natural transformations: apply the same assertion with target $\mathrm{Fun}(K, \mathrm{RMod}_S)$ for every small ∞-category K and curry, as in the proof of Proposition 18.2.8. Applying F to the regular bimodule R, regarded as a left R-module in RMod_R, therefore gives an (R, S)-bimodule $F(R)$ by Theorem 17.2.6. Its left action is

$$R \otimes F(R) \simeq F(R \otimes R) \longrightarrow F(R),$$

where the last map is induced by multiplication. This construction defines a functor $F \mapsto F(R)$.

For a right R-module M, the bar construction $\mathrm{Bar}_R(M, R)_\bullet$ has terms $M \otimes R^{\otimes n} \otimes R$ and realization $M \otimes_R R \simeq M$. In the operadic construction of Remark 17.2.10, we may place the final input in

RMod_R and the preceding inputs in Sp. Since F is Sp-linear, it preserves these operations and their cocartesian transport, identifying $F(\mathrm{Bar}_R(M,R)_\bullet)$ with $\mathrm{Bar}_R(M,F(R))_\bullet$. The last face map uses the left R-action just constructed; the other faces and degeneracies are induced by maps of spectra. Since F preserves geometric realizations, we obtain

$$F(M) \simeq |\mathrm{Bar}_R(M,F(R))_\bullet| \simeq M \otimes_R F(R),$$

naturally in M and F.

Conversely, evaluating $- \otimes_R B$ at R gives $R \otimes_R B \simeq B$ as (R,S)-bimodules. These natural isomorphisms show that the two functors are inverse. □

Associative ring spectra R and S are called *Morita equivalent* if RMod_R and RMod_S are equivalent. By Theorem 18.3.6, this amounts to the existence of an (R,S)-bimodule B and an (S,R)-bimodule C with isomorphisms $B \otimes_S C \simeq R$ and $C \otimes_R B \simeq S$ as bimodules; see Exercise 18.3.

Exercises

Exercise 18.1 (Tensor products of presheaf categories)**.** Let C and D be small ∞-categories. Show that the external-product functor

$$\mathrm{PSh}(C) \times \mathrm{PSh}(D) \longrightarrow \mathrm{PSh}(C \times D), \qquad (F,G) \longmapsto \big((c,d) \longmapsto F(c) \times G(d)\big),$$

exhibits $\mathrm{PSh}(C \times D)$ as the tensor product $\mathrm{PSh}(C) \otimes \mathrm{PSh}(D)$ in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{colim}}$.

Exercise 18.2. Consider the sequence of modes

$$\mathrm{An} \longrightarrow \mathrm{An}_* \longrightarrow \mathrm{CMon}(\mathrm{An}) \longrightarrow \mathrm{CGrp}(\mathrm{An}) \longrightarrow \mathrm{Sp}$$

from Theorem 18.2.5. Show that the tensor product of any two of these modes is the one which occurs further to the right in this sequence.

Exercise 18.3. Let R, S, T be associative ring spectra. For an (R,S)-bimodule B, write $F_B := - \otimes_R B \colon \mathrm{RMod}_R \to \mathrm{RMod}_S$.

(1) Show that natural transformations $F_B \to F_{B'}$ correspond to morphisms of (R,S)-bimodules $B \to B'$.

(2) For an (S,T)-bimodule C, identify the (R,T)-bimodule corresponding to $F_C \circ F_B$.

(3) Identify the bimodule corresponding to the identity functor of RMod_R.

(4) Deduce that F_B is an equivalence if and only if there is an (S,R)-bimodule C with isomorphisms

$$B \otimes_S C \simeq R \qquad \text{and} \qquad C \otimes_R B \simeq S$$

of (R,R)- and (S,S)-bimodules, respectively.

Exercise 18.4 (Matrix Morita equivalence)**.** Let R be an associative ring spectrum and let $n \geq 1$. Put $P := R^{\oplus n} \in \mathrm{RMod}_R$ and let

$$A := \hom_{\mathrm{RMod}_R}(P,P).$$

(1) Show that P is a compact generator of RMod_R. Identify the underlying spectrum of A with $\bigoplus_{i,j=1}^{n} R$ and show that composition gives matrix multiplication. In particular, deduce an isomorphism $\pi_* A \cong M_n(\pi_* R)$ of graded rings. We call A the *matrix ring spectrum* $M_n(R)$.

(2) The construction of Proposition 18.3.1 gives an adjunction

$$- \otimes_A P \colon \mathrm{RMod}_A \rightleftarrows \mathrm{RMod}_R \colon \hom_{\mathrm{RMod}_R}(P, -).$$

Deduce from Theorem 18.3.2 that this adjunction is an equivalence.

19 The envelope of an ∞-operad

An ∞-operad $\mathcal{O}$ describes operations with several inputs without requiring those inputs to have a tensor product. Its *envelope* $\mathrm{Env}(\mathcal{O})$ is a symmetric monoidal ∞-category that formally supplies such tensor products: its objects are finite unordered tuples of colors of $\mathcal{O}$, its tensor product is unordered concatenation, and its morphisms are assembled from operations of $\mathcal{O}$. This extends the construction for classical colored operads from Section 12.3.

An $\mathcal{O}$-algebra in a symmetric monoidal ∞-category C determines a symmetric monoidal functor $\mathrm{Env}(\mathcal{O}) \to C$: it sends a tuple of colors to the tensor product of their values in C, and sends operations to the algebra's structure maps. We prove in Corollary 19.3.12 that this gives a natural equivalence

$$\mathrm{Alg}_{\mathcal{O}}(C) \quad \simeq \quad \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), C).$$

In particular, the envelope construction is left adjoint to the multimorphism operad functor.

The envelope also comes with a symmetric monoidal functor $\mathrm{Env}(\mathcal{O}) \to (\mathrm{Fin}, \amalg)$, which sends each tuple to its indexing set. The envelope, together with this functor, determines $\mathcal{O}$: the colors are the objects over a singleton, and operations are morphisms from tuples to singleton tuples. More precisely, we show in Proposition 19.3.19 that the envelope construction is fully faithful when regarded as taking values in symmetric monoidal ∞-categories over $(\mathrm{Fin}, \amalg)$, and we characterize its essential image.

To construct the envelope, we use the characterization of symmetric monoidal ∞-categories as ∞-operads whose projections to $\mathrm{Span}(\mathrm{Fin})$ are cocartesian fibrations. An ∞-operad already has cocartesian lifts of backwards spans; to construct its envelope, we freely adjoin lifts of forward spans, which encode tensor products. The factorization of spans into forwards and backwards allows us to prove that the resulting projection is a cocartesian fibration. Following Barkan, Haugseng, and Steinebrunner [BHS22], we develop this construction using factorization systems and apply it both to our definition of ∞-operads and to Lurie's definition over Fin_*. The resulting envelope constructions allow us to prove that the two definitions are equivalent (Proposition 19.4.8).

The final two sections give applications of the envelope. The symmetric monoidal Yoneda embedding of $\mathrm{Env}(\mathcal{O})$ induces a fully faithful morphism from $\mathcal{O}$ to the multimorphism operad of its presheaf category, equipped with Day convolution (Section 19.5). We also extend Day convolution to arbitrary operadic targets: viewing $\mathcal{O}$ as a full suboperad of the multimorphism operad of its envelope allows us to restrict the construction already established for symmetric monoidal targets (Section 19.6).

19.1 Factorization systems

A factorization system specifies two classes of morphisms such that every morphism factors uniquely as a morphism in the first class followed by one in the second. The definition expresses this uniqueness through orthogonality.

Definition 19.1.1 (Orthogonality)**.** Let C be an ∞-category. A morphism $l\colon A \to B$ is said to be *left orthogonal* to a morphism $r\colon X \to Y$, written $l \perp r$, if the following square is a pullback square:

$$\begin{array}{ccc} \mathrm{Hom}_C(B,X) & \xrightarrow{-\circ l} & \mathrm{Hom}_C(A,X) \\ {\scriptstyle r\circ -}\downarrow & & \downarrow{\scriptstyle r\circ -} \\ \mathrm{Hom}_C(B,Y) & \xrightarrow{-\circ l} & \mathrm{Hom}_C(A,Y). \end{array}$$

We then also say that r is *right orthogonal* to l. Equivalently, for every solid commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle l}\downarrow & \nearrow & \downarrow{\scriptstyle r} \\ B & \longrightarrow & Y, \end{array}$$

the anima of dashed fillers with compatible commutativity identifications for the two triangles is contractible; here compatibility means that pasting the triangles recovers the given square.

Given a collection of morphisms S, we denote by $S^{\perp}$ the collection of all morphisms that are right orthogonal to every morphism in S, and by ${}^{\perp}S$ the collection of those that are left orthogonal to every morphism in S.

Observation 19.1.2. Every morphism $f\colon X \to Y$ satisfying $f \perp f$ is an isomorphism in C: the unique dashed filler in the following diagram defines an inverse to f:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y. \end{array}$$

Definition 19.1.3 (Factorization system (Joyal), see [Lur09b, Section 5.2.8])**.** A *factorization system* on an ∞-category C consists of a pair of wide subcategories $(\mathcal{L}, \mathcal{R})$ of C satisfying the following conditions:

(1) *Orthogonality*: Every morphism $l \in \mathcal{L}$ is left orthogonal to every morphism $r \in \mathcal{R}$.

(2) *Factorization*: Every morphism f in C admits a factorization $f \simeq r \circ l$, where $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

Example 19.1.4. Surjections and injections form a factorization system on Set. For factorization, we may factor every map through its image. For orthogonality, consider a commutative square with a surjection $l\colon A \to B$ on the left and an injection $r\colon X \to Y$ on the right. The map $A \to X$ is constant on the fibers of l, since its composite with r factors through B, and hence descends uniquely to a filler $B \to X$.

Further examples are developed in Exercise 19.1. To study factorization systems, we first record the closure properties of orthogonal classes.

Proposition 19.1.5. *Let S be a collection of morphisms in an ∞-category C.*

(1) *The classes $S^{\perp}$ and ${}^{\perp}S$ are closed under composition and contain all isomorphisms. In particular, they are encoded by wide subcategories of C.*

(2) *The classes $S^{\perp}$ and ${}^{\perp}S$ are closed under retracts.*

(3) *The class $S^{\perp}$ is closed under base change (pullbacks), and ${}^{\perp}S$ is closed under cobase change (pushouts).*

(4) *The class $S^{\perp}$ has the left cancellation property: if $g \circ f, g \in S^{\perp}$, then $f \in S^{\perp}$. Dually, ${}^{\perp}S$ has the right cancellation property: if $g \circ f, f \in {}^{\perp}S$, then $g \in {}^{\perp}S$.*

(5) *If C admits small limits, $S^{\perp}$ is closed under small limits in $\mathrm{Ar}(C)$. Dually, if C admits small colimits, ${}^{\perp}S$ is closed under small colimits in $\mathrm{Ar}(C)$.*

Proof. These properties all follow from the definition of orthogonality in terms of hom animae. The proofs of (1) and (2) are left to the reader. For (3), consider a pullback square in C where $r \in S^{\perp}$:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle r'}\downarrow & \lrcorner & \downarrow{\scriptstyle r} \\ Y' & \longrightarrow & Y. \end{array}$$

For any $l\colon A \to B$ in S, we must show $l \perp r'$. This means showing that the left square in the diagram below is a pullback square of animae:

$$\begin{array}{ccccc} \mathrm{Hom}_C(B,X') & \longrightarrow & \mathrm{Hom}_C(A,X') & \longrightarrow & \mathrm{Hom}_C(A,X) \\ {\scriptstyle r'\circ -}\downarrow & & \downarrow{\scriptstyle r'\circ -} & & \downarrow{\scriptstyle r\circ -} \\ \mathrm{Hom}_C(B,Y') & \longrightarrow & \mathrm{Hom}_C(A,Y') & \longrightarrow & \mathrm{Hom}_C(A,Y). \end{array}$$

The right square is a pullback because $\mathrm{Hom}_C(A,-)$ preserves pullbacks, so by the pasting lemma for pullback squares it remains to show that the outer rectangle is a pullback. But this holds as we may rewrite it as the pasting of the square obtained from the original pullback square by applying $\mathrm{Hom}_C(B,-)$ and the pullback square exhibiting the orthogonality relation $l \perp r$. The assertion about pushouts is dual.

For (4), consider morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in C, and assume that $g \circ f$ and g lie in $S^{\perp}$. To show that $f \in S^{\perp}$, let $l\colon A \to B$ be a morphism in S, and consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_C(B,X) & \xrightarrow{f\circ -} & \mathrm{Hom}_C(B,Y) & \xrightarrow{g\circ -} & \mathrm{Hom}_C(B,Z) \\ {\scriptstyle -\circ l}\downarrow & & \downarrow{\scriptstyle -\circ l} & & \downarrow{\scriptstyle -\circ l} \\ \mathrm{Hom}_C(A,X) & \xrightarrow{f\circ -} & \mathrm{Hom}_C(A,Y) & \xrightarrow{g\circ -} & \mathrm{Hom}_C(A,Z). \end{array}$$

By assumption, the right and outer squares are pullback squares, hence by the pasting law so is the left square. Right cancellation for ${}^{\perp}S$ follows dually.

For (5), let r be the limit in $\mathrm{Ar}(C)$ of a diagram of morphisms $r_i \in S^{\perp}$. For any $l\colon A \to B$ in S, the square defining $l \perp r$ is the limit of the squares defining $l \perp r_i$, since $\mathrm{Hom}_C(A,-)$ and $\mathrm{Hom}_C(B,-)$ preserve limits. Limits of pullback squares are pullback squares, so $r \in S^{\perp}$. The assertion about colimits is dual. □

Proposition 19.1.6. *If $(\mathcal{L},\mathcal{R})$ is a factorization system on C, then $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\perp}$.*

Proof. By symmetry, it suffices to prove $\mathcal{R} = \mathcal{L}^{\perp}$. The inclusion $\mathcal{R} \subseteq \mathcal{L}^{\perp}$ holds by definition. Given $r \in \mathcal{L}^{\perp}$, factor it as $r'l$ with $l \in \mathcal{L}$ and $r' \in \mathcal{R}$. Since r and r' lie in $\mathcal{L}^{\perp}$, left cancellation gives $l \in \mathcal{L}^{\perp}$. Thus $l \perp l$, so l is an isomorphism by Observation 19.1.2. As $\mathcal{R}$ contains all isomorphisms and is closed under composition, we obtain $r \in \mathcal{R}$. □

Remark 19.1.7. By combining the previous proposition with Proposition 19.1.5, we conclude that the class $\mathcal{R}$ is automatically closed under retracts, base change, and left cancellation. Similarly, $\mathcal{L}$ is closed under retracts, cobase change, and right cancellation.

In particular, closure under retracts need not be imposed as a separate axiom in the definition of a factorization system.

To express uniqueness of factorizations, let $\mathrm{Fact}_{\mathcal{L},\mathcal{R}}(C)$ be the full subcategory of $\mathrm{Fun}([2],C)$ spanned by diagrams $X \to Y \to Z$ with first morphism in $\mathcal{L}$ and second in $\mathcal{R}$. Restriction along $d_1\colon [1] \to [2]$ defines the composition functor $-\circ-\colon \mathrm{Fact}_{\mathcal{L},\mathcal{R}}(C) \to \mathrm{Ar}(C)$.

Proposition 19.1.8 ([Lur09b, Proposition 5.2.8.17])**.** *Let $\mathcal{L}$ and $\mathcal{R}$ be wide subcategories of an ∞-category C. Then the following two conditions are equivalent:*

(1) *The pair $(\mathcal{L},\mathcal{R})$ is a factorization system on C.*

(2) *The composition functor $-\circ-\colon \mathrm{Fact}_{\mathcal{L},\mathcal{R}}(C) \to \mathrm{Ar}(C)$ is an equivalence.*

Proof. Suppose $(\mathcal{L},\mathcal{R})$ is a factorization system. Existence of factorizations gives essential surjectivity. For full faithfulness, consider factorizations $(l\colon X \to Z, r\colon Z \to Y)$ and $(l'\colon X' \to Z', r'\colon Z' \to Y')$. We must show that

$$\mathrm{Hom}_{\mathrm{Fact}_{\mathcal{L},\mathcal{R}}(C)}((l,r),(l',r')) \to \mathrm{Hom}_{\mathrm{Ar}(C)}(rl, r'l')$$

is an isomorphism. A point of the target is a pair of maps $a\colon X \to X'$ and $b\colon Y \to Y'$ together with an identification making the outer rectangle below commute:

$$\begin{array}{ccc} X & \xrightarrow{a} & X' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l'} \\ Z & \dashrightarrow & Z' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ Y & \xrightarrow[b]{} & Y' \end{array}$$

By the commutative-square and Segal axioms, the fiber over this point is the fiber of

$$(l^*, r'_*)\colon \mathrm{Hom}_C(Z,Z') \to \mathrm{Hom}_C(X,Z') \times_{\mathrm{Hom}_C(X,Y')} \mathrm{Hom}_C(Z,Y')$$

over the pair $(l'a, br)$ with the identification inherited from the outer rectangle. This fiber is contractible since $l \perp r'$.

Conversely, suppose composition is an equivalence. Essential surjectivity gives the required factorizations. For $l\colon A \to B$ in $\mathcal{L}$ and $r\colon X \to Y$ in $\mathcal{R}$, we have $\mathrm{Hom}_{\mathrm{Fact}_{\mathcal{L},\mathcal{R}}(C)}((l, \mathrm{id}_B), (\mathrm{id}_X, r)) \simeq \mathrm{Hom}_C(B, X)$. Under this identification, the map on hom animae is precisely the map defining $l \perp r$, so full faithfulness gives orthogonality. □

Corollary 19.1.9. *Let $(\mathcal{L}, \mathcal{R})$ be a factorization system on C. Then for every morphism f of C, the fiber of* $\mathrm{Fact}_{\mathcal{L},\mathcal{R}}(C) \to \mathrm{Ar}(C)$ *over f is contractible.* □

The inert-active factorization system on an ∞-operad

We now apply this theory to inert and active morphisms. Recall from Lemma 13.1.16 that for an adequate triple (C, C_L, C_R), the span category contains C_L^{op} and C_R as the wide subcategories of backwards and forwards maps.

Proposition 19.1.10 (The factorization system on a span category, [HHLN23b, Proposition 4.9])**.** *Let $C = (C, C_L, C_R)$ be an adequate triple. The pair $(\mathcal{L}, \mathcal{R}) = (C_L^{\mathrm{op}}, C_R)$ forms a factorization system on* $\mathrm{Span}(C)$.

Proof. Any span $X \xleftarrow{l} U \xrightarrow{r} Y$ can be factored as a backwards map followed by a forwards map:

$$X \xleftarrow{l} U \xrightarrow{\mathrm{id}_U} U \qquad \text{followed by} \qquad U \xleftarrow{\mathrm{id}_U} U \xrightarrow{r} Y.$$

For orthogonality, let $l\colon A \to B$ lie in C_L and $r\colon X \to Y$ lie in C_R. Precomposition with the backwards map associated to l and postcomposition with the forwards map associated to r give the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Span}(C)}(A, X) & \longrightarrow & \mathrm{Hom}_{\mathrm{Span}(C)}(A, Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Span}(C)}(B, X) & \longrightarrow & \mathrm{Hom}_{\mathrm{Span}(C)}(B, Y) \end{array}$$

which we must show is a pullback. By Lemma 13.1.13, it identifies with

$$\begin{array}{ccc} ((C_L)_{/A})^{\simeq} \times_{C^{\simeq}} ((C_R)_{/X})^{\simeq} & \xrightarrow{\mathrm{id} \times (r \circ -)} & ((C_L)_{/A})^{\simeq} \times_{C^{\simeq}} ((C_R)_{/Y})^{\simeq} \\ \downarrow{\scriptstyle (l \circ -) \times \mathrm{id}} & & \downarrow{\scriptstyle (l \circ -) \times \mathrm{id}} \\ ((C_L)_{/B})^{\simeq} \times_{C^{\simeq}} ((C_R)_{/X})^{\simeq} & \xrightarrow[\mathrm{id} \times (r \circ -)]{} & ((C_L)_{/B})^{\simeq} \times_{C^{\simeq}} ((C_R)_{/Y})^{\simeq}. \end{array}$$

This is the fiber product, over the constant square $C^{\simeq}$, of a square with identity horizontal maps and a square with identity vertical maps. Both are pullback squares, so their fiber product is a pullback square as well. □

Remark 19.1.11. The unique filler in a lifting problem in $\mathrm{Span}(C, C_L, C_R)$ can be visualized directly. A commutative square with a backwards map on the left and a forwards map on the right corresponds

to a diagram of spans in C as follows:

$$\begin{array}{ccccc} A & \longleftarrow & Z & \longrightarrow & X \\ {\scriptstyle l}\uparrow & & \uparrow & \lrcorner & \| \\ B & \longleftarrow & Z' & \longrightarrow & X \\ \| & \llcorner & \downarrow & & \downarrow{\scriptstyle r} \\ B & \longleftarrow & Z'' & \longrightarrow & Y. \end{array}$$

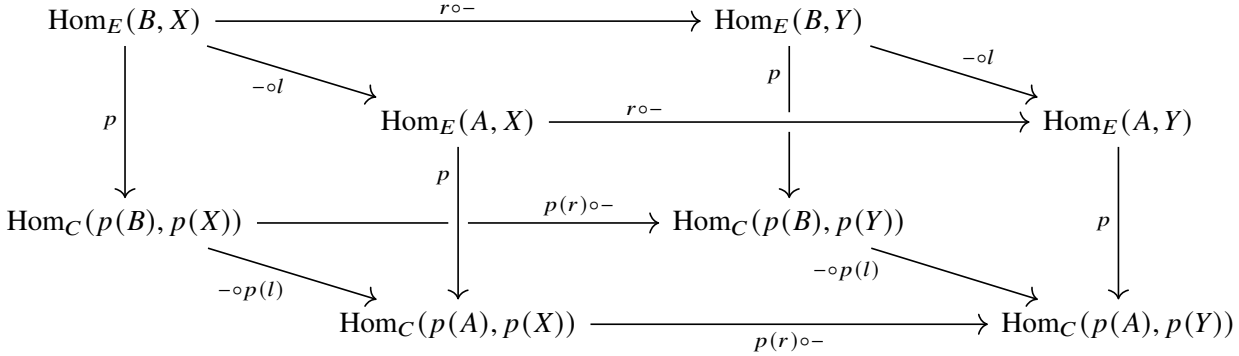
Here the composite span is $A \leftarrow Z' \rightarrow Y$, computed both as the composite of the spans $A \xleftarrow{l} B \xrightarrow{\mathrm{id}_B} B$ and $B \leftarrow Z'' \rightarrow Y$, as well as the composite of the spans $A \leftarrow Z \rightarrow X$ and $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{r} Y$. The unique filler span is then given by the span $B \leftarrow Z' \rightarrow X$.

The factorization system on Span(Fin) lifts to $\mathcal{O}^{\otimes}$ by the following lemma.

Lemma 19.1.12. *Let $(\mathcal{L}, \mathcal{R})$ be a factorization system on an ∞-category C, and let $p\colon E \to C$ admit cocartesian lifts of morphisms in $\mathcal{L}$. Then E has a factorization system $(\mathcal{L}_E, \mathcal{R}_E)$, where $\mathcal{L}_E$ consists of the p-cocartesian morphisms over $\mathcal{L}$ and $\mathcal{R}_E$ consists of all morphisms over $\mathcal{R}$.*

Proof. Both classes define wide subcategories of E: they contain all isomorphisms, and they are closed under composition because $\mathcal{L}$ and $\mathcal{R}$ are, and because composites of p-cocartesian morphisms are again p-cocartesian.

(1) For orthogonality, consider $l\colon A \to B$ in $\mathcal{L}_E$ and $r\colon X \to Y$ in $\mathcal{R}_E$. We need to show that the top square in the following commutative cube is a pullback square:



Since l is p-cocartesian, the left and right squares are pullback squares, and since $p(l) \perp p(r)$ the bottom square is a pullback square. The claim thus follows from the pasting law of pullback squares.

(2) For factorization, consider a morphism $f\colon X \to Z$ in E. Factor $p(f)$ as $p(X) \xrightarrow{l} y \xrightarrow{r} p(Z)$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$, and choose a p-cocartesian lift $\widehat{l}\colon X \to Y$ of l. The cocartesian property gives a unique morphism $\widehat{r}\colon Y \to Z$ over r with $f \simeq \widehat{r} \circ \widehat{l}$. Since $\widehat{l} \in \mathcal{L}_E$ and $\widehat{r} \in \mathcal{R}_E$, this is the required factorization. □

Corollary 19.1.13. *Let $\mathcal{O}$ be an ∞-operad. Then the inert and active morphisms define a factorization system on $\mathcal{O}^{\otimes}$.*

Proof. This is an instance of Lemma 19.1.12 applied to the functor $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$, where we equip the target with the factorization system from Proposition 19.1.10. □

By Corollary 19.1.9, the anima of inert-active factorizations of each morphism in $O^{\otimes}$ is therefore contractible, as asserted in Remark 14.1.15.

19.2 Freely adjoining cocartesian lifts

Fix an ∞-category S and a wide subcategory $\mathcal{R} \subset S$. To freely adjoin a cocartesian lift of a morphism $r\colon p(e) \to y$ in $\mathcal{R}$ to a functor $p\colon E \to S$, we introduce a formal target (e, r) over y. Composition with a morphism $r'\colon y \to y'$ in $\mathcal{R}$ will send (e, r) to $(e, r'r)$. We will show that this construction has the expected universal property. When $(\mathcal{L}, \mathcal{R})$ is a factorization system and p already has cocartesian lifts over $\mathcal{L}$, it produces a cocartesian fibration over all of S.

Our exposition follows Barkan–Haugseng–Steinebrunner [BHS22, Section 2]. The free cocartesian completion previously appeared in work of Gepner–Haugseng–Nikolaus [GHN17, Section 4], and the variant for factorization systems in work of Ayala–Francis–Mazel-Gee–Rozenblyum [AFMR17, Proposition A.0.1] and Shah [Sha21, Theorem 3.6].

Definition 19.2.1 ($\mathcal{R}$-cocartesian fibrations)**.** Let S be an ∞-category and $\mathcal{R} \subset S$ a wide subcategory. A functor $p\colon E \to S$ is called $\mathcal{R}$-*cocartesian* if for every $e \in E$ and every $r\colon p(e) \to y'$ in $\mathcal{R}$ there exists a p-cocartesian morphism $\hat{r}_e\colon e \to e'$ lifting r. A functor $f\colon E \to E'$ over S between $\mathcal{R}$-cocartesian fibrations is $\mathcal{R}$-*cocartesian* if it preserves p-cocartesian morphisms over morphisms in $\mathcal{R}$. We denote by

$$(\mathrm{Cat}_\infty)_{/S}^{\mathcal{R}-\mathrm{cocart}} \quad \subset \quad (\mathrm{Cat}_\infty)_{/S}$$

the (non-full) subcategory spanned by the $\mathcal{R}$-cocartesian fibrations and $\mathcal{R}$-cocartesian functors.

Construction 19.2.2 (Freely adjoining $\mathcal{R}$-cocartesian morphisms)**.** Let $\mathrm{Ar}_{\mathcal{R}}(S) \subseteq \mathrm{Ar}(S)$ be the full subcategory spanned by the morphisms in $\mathcal{R}$, with source and target functors $s, t\colon \mathrm{Ar}_{\mathcal{R}}(S) \to S$. For any $p\colon E \to S$, we define

$$E_{\mathcal{R}}(p) \quad := \quad E \times_{S,s} \mathrm{Ar}_{\mathcal{R}}(S) \xrightarrow{t \circ \mathrm{pr}_2} S.$$

This defines an endofunctor $E_{\mathcal{R}}\colon (\mathrm{Cat}_\infty)_{/S} \to (\mathrm{Cat}_\infty)_{/S}$. For $p'\colon E' \to S$, a natural transformation $E \times [1] \to E'$ over S pulls back to $E_{\mathcal{R}}(p) \times [1] \to E_{\mathcal{R}}(p')$. Thus the construction also acts on natural transformations, compatibly with identities and composition.

Lemma 19.2.3. *For every $p\colon E \to S$, the functor $E_{\mathcal{R}}(p)$ is an $\mathcal{R}$-cocartesian fibration. For every functor f over S, the functor $E_{\mathcal{R}}(f)$ preserves $\mathcal{R}$-cocartesian morphisms.*

Proof. For $r'\colon y \to y'$ in $\mathcal{R}$, the morphism $(e, r) \to (e, r'r)$ is a cocartesian lift of r'. Indeed, for any target (e'', r'') and $g\colon y' \to y''$, a map $(e, r'r) \to (e'', r'')$ over g consists of a map $h\colon e \to e''$ and an identification $r''p(h) \simeq gr'r$. This is precisely a map $(e, r) \to (e'', r'')$ over gr', naturally in g, which proves the cocartesian property. The functor $E_{\mathcal{R}}(f)$ carries these lifts to the corresponding lifts for $f(e)$. □

The identity arrows define a fully faithful functor $i\colon S \to \mathrm{Ar}_{\mathcal{R}}(S)$. Pulling it back along p gives a fully faithful functor over S

$$i_p\colon E \longrightarrow E_{\mathcal{R}}(p), \qquad e \longmapsto (e, \mathrm{id}_{p(e)}).$$

This construction is natural in $p \in (\mathrm{Cat}_\infty)_{/S}$.

Lemma 19.2.4. *If $p\colon E \to S$ is an $\mathcal{R}$-cocartesian fibration, the functor $i_p\colon E \to E_{\mathcal{R}}(p)$ admits a relative left adjoint $\pi_p\colon E_{\mathcal{R}}(p) \to E$ over S, given on objects by sending a pair $(e, r\colon x \to y)$ to the target e' of the p-cocartesian lift $\hat{r}_e\colon e \to e'$ of r.*

Proof. Let $e \to e'$ be a p-cocartesian lift of $r\colon p(e) \to y$. For every $e'' \in E$, the pullback description and cocartesianness give

$$\begin{aligned}\mathrm{Hom}_{E_{\mathcal{R}}(p)}((e,r), i_p(e'')) &\simeq \mathrm{Hom}_E(e, e'') \times_{\mathrm{Hom}_S(p(e),p(e''))} \mathrm{Hom}_S(y, p(e'')) \\ &\simeq \mathrm{Hom}_E(e', e'').\end{aligned}$$

These identifications are natural in e'', so the pointwise criterion for adjoints gives $\pi_p \dashv i_p$. Its unit at (e, r) is induced by $e \to e'$ and projects to id_y, giving an isomorphism $t\,\mathrm{pr}_2 \simeq p\pi_p$. With this identification, the unit and counit lie over the identity of S. □

Lemma 19.2.5. *If $q\colon F \to S$ is $\mathcal{R}$-cocartesian, then π_q preserves $\mathcal{R}$-cocartesian morphisms. For an $\mathcal{R}$-cocartesian functor $g\colon F \to F'$ over S, with $q'\colon F' \to S$, the mate of the naturality square $E_{\mathcal{R}}(g)i_q \simeq i_{q'}g$ is an isomorphism*

$$\pi_{q'} E_{\mathcal{R}}(g) \xrightarrow{\sim} g\pi_q,$$

natural in g.

Proof. The functor π_q sends the canonical lift $(e, r) \to (e, r'r)$ to $r_!e \to (r'r)_!e$. This is q-cocartesian by cancellation, since both its composite with $e \to r_!e$ and that first morphism are q-cocartesian (Lemma 24.1.14).

The mate is the composite

$$\pi_{q'} E_{\mathcal{R}}(g) \xrightarrow{\eta} \pi_{q'} E_{\mathcal{R}}(g) i_q \pi_q \simeq \pi_{q'} i_{q'} g \pi_q \xrightarrow{\varepsilon} g\pi_q.$$

At (e, r) it is the map $r_!g(e) \to g(r_!e)$ induced by the image under g of the cocartesian lift $e \to r_!e$. Since g preserves these lifts, this map is an isomorphism. The construction uses the naturality of i and the adjunction units and counits, so it is natural also with respect to noninvertible natural transformations over S between such functors g. □

Given functors $p\colon E \to S$ and $q\colon F \to S$, we write $\mathrm{Fun}_{/S}(E, F)$ for the fiber of $\mathrm{Fun}(E, F) \to \mathrm{Fun}(E, S)$ over p. If both functors are $\mathcal{R}$-cocartesian, we write $\mathrm{Fun}^{\mathcal{R}-\mathrm{cocart}}_{/S}(E, F)$ for the full subcategory spanned by the functors which preserve cocartesian morphisms over $\mathcal{R}$.

Proposition 19.2.6 (Free adjunction of $\mathcal{R}$-cocartesian lifts)**.** *For $p\colon E \to S$ and an $\mathcal{R}$-cocartesian fibration $q\colon F \to S$, precomposition with i_p induces an equivalence*

$$\mathrm{Fun}^{\mathcal{R}-\mathrm{cocart}}_{/S}\big(E_{\mathcal{R}}(p), F\big) \xrightarrow{\sim} \mathrm{Fun}_{/S}(E, F),$$

with inverse sending $f\colon E \to F$ to $\pi_q \circ E_{\mathcal{R}}(f)$.

Proof. The preceding lemmas give the proposed inverse as a functor. For $f\colon E \to F$ over S, naturality of i and the invertible counit of $\pi_q \dashv i_q$ give

$$\pi_q E_{\mathcal{R}}(f) i_p \simeq \pi_q i_q f \simeq f.$$

For an $\mathcal{R}$-cocartesian functor $g\colon E_{\mathcal{R}}(p) \to F$, Lemma 19.2.5 gives

$$\pi_q E_{\mathcal{R}}(g i_p) \simeq g \pi_{E_{\mathcal{R}}(p)} E_{\mathcal{R}}(i_p) \simeq g.$$

Here $\pi_{E_{\mathcal{R}}(p)}$ is induced by composition of $\mathcal{R}$-arrows: it sends $((e,r),r')$ to $(e,r'r)$. Its composite with $E_{\mathcal{R}}(i_p)$ is therefore the identity by the unit law for composition. Both displayed identifications are natural in functors and natural transformations over S, proving the equivalence of functor categories. □

Passing to groupoid cores gives an adjunction

$$E_{\mathcal{R}}\colon (\mathrm{Cat}_\infty)_{/S} \rightleftarrows (\mathrm{Cat}_\infty)_{/S}^{\mathcal{R}-\mathrm{cocart}} : \mathrm{forget},$$

with unit $i_p\colon E \to E_{\mathcal{R}}(p)$ and counit $\pi_q\colon E_{\mathcal{R}}(q) \to F$.

In case the wide subcategory $\mathcal{R}$ is part of a factorization system $(\mathcal{L},\mathcal{R})$ on S, the functor $E_{\mathcal{R}}$ turns $\mathcal{L}$-cocartesian fibrations into cocartesian fibrations:

Proposition 19.2.7. *Let $(\mathcal{L},\mathcal{R})$ be a factorization system on S. If $p\colon E \to S$ is $\mathcal{L}$-cocartesian, then $E_{\mathcal{R}}(p) \to S$ is a cocartesian fibration, with cocartesian morphisms given by those diagrams*

$$\begin{array}{ccc} e & \xrightarrow{\hat{f}} & e' \\ \downarrow & & \downarrow \\ x & \xrightarrow{f} & x' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ y & \xrightarrow{g} & y' \end{array}$$

such that f lies in $\mathcal{L}$ and $\hat{f}$ is a p-cocartesian morphism in E. Similarly, if $g\colon E \to E'$ is an $\mathcal{L}$-cocartesian functor over S, the induced map $E_{\mathcal{R}}(g)$ is a cocartesian functor over S.

Proof. Given an object $(e, r\colon x \to y)$ of $E_{\mathcal{R}}(p)$ and a morphism $g\colon y \to y'$ in S, factor gr as $r'f$, with $f\colon x \to x'$ in $\mathcal{L}$ and $r'\colon x' \to y'$ in $\mathcal{R}$, and choose a p-cocartesian lift $\hat{f}\colon e \to e'$ of f. We claim that the resulting morphism $(e,r) \to (e',r')$ is cocartesian. By the hom-anima criterion, this amounts to showing that the anima of dashed fillers in the following diagram is contractible for every (e'',r'') and $g'\colon y' \to y''$:

$$\begin{array}{ccccc} e & \xrightarrow{\hat{f}} & e' & \dashrightarrow & e'' \\ \downarrow & & \downarrow & & \downarrow \\ x & \xrightarrow{f} & x' & \dashrightarrow & x'' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r''} \\ y & \xrightarrow{g} & y' & \xrightarrow{g'} & y''. \end{array}$$

First, orthogonality of $f \in \mathcal{L}$ and $r'' \in \mathcal{R}$ gives the unique filler $x' \to x''$. The p-cocartesianness of $\hat{f}$ then gives the unique filler $e' \to e''$. Thus $E_{\mathcal{R}}(p)$ is a cocartesian fibration. Uniqueness of cocartesian lifts gives the stated description of all its cocartesian morphisms, from which the assertion about $E_{\mathcal{R}}(g)$ follows. □

Taking $p = \mathrm{id}_S$ shows that $t\colon \mathrm{Ar}_{\mathcal{R}}(S) \to S$ is a cocartesian fibration, with cocartesian morphisms precisely those whose image under s lies in $\mathcal{L}$.

Corollary 19.2.8. *The functor $E_{\mathcal{R}}$ restricts to a functor*

$$E_{\mathcal{R}}\colon (\mathrm{Cat}_\infty)^{\mathcal{L}-\mathrm{cocart}}_{/S} \longrightarrow \mathrm{Cocart}(S)$$

which is left adjoint to the inclusion.

Proof. By Proposition 19.2.7, $E_{\mathcal{R}}$ restricts as stated and its unit i_p preserves $\mathcal{L}$-cocartesian morphisms. For a cocartesian fibration q, the counit sends a cocartesian morphism displayed there to $r_!e \to r'_!e'$, which is cocartesian by applying the cancellation property to $e \to r_!e \to r'_!e'$. Thus the unit and counit restrict as required, giving the adjunction. □

The construction $E_{\mathcal{R}}(p)$ comes with a projection to $\mathrm{Ar}_{\mathcal{R}}(S)$. Pulling back this projection along $i\colon S \to \mathrm{Ar}_{\mathcal{R}}(S)$, which sends an object to its identity, recovers E. We will show that remembering the projection makes $E_{\mathcal{R}}$ fully faithful. Since id_S is terminal in $(\mathrm{Cat}_\infty)^{\mathcal{L}-\mathrm{cocart}}_{/S}$ and $E_{\mathcal{R}}(\mathrm{id}_S) = \mathrm{Ar}_{\mathcal{R}}(S)$, these projections define a functor

$$E_{\mathcal{R}}\colon (\mathrm{Cat}_\infty)^{\mathcal{L}-\mathrm{cocart}}_{/S} \longrightarrow \mathrm{Cocart}(S)_{/\mathrm{Ar}_{\mathcal{R}}(S)}. \tag{19.1}$$

Lemma 19.2.9. *The functor in* (19.1) *is fully faithful.*

Proof. Let $p\colon E \to S$ and $p'\colon E' \to S$ be $\mathcal{L}$-cocartesian. The definition of $E_{\mathcal{R}}(p')$ gives a pullback square

$$\begin{array}{ccc} E' & \xhookrightarrow{i_{p'}} & E_{\mathcal{R}}(p') \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_2} \\ S & \xhookrightarrow{i} & \mathrm{Ar}_{\mathcal{R}}(S). \end{array}$$

Moreover, $i_{p'}$ preserves and reflects cocartesian morphisms over $\mathcal{L}$: for a morphism h over $\mathcal{L}$, Proposition 19.2.7 says precisely that $i_{p'}(h)$ is cocartesian if and only if h is.

By Corollary 19.2.8, the anima of cocartesian functors $E_{\mathcal{R}}(p) \to E_{\mathcal{R}}(p')$ over $\mathrm{Ar}_{\mathcal{R}}(S)$ gets identified with the anima of $\mathcal{L}$-cocartesian functors $E \to E_{\mathcal{R}}(p')$ whose projection is ip. The pullback square and the preservation and reflection just noted identify the latter anima with that of $\mathcal{L}$-cocartesian functors $E \to E'$ over S. This proves full faithfulness. □

19.3 Operadic envelopes

The envelope construction will also let us relate our definition of an ∞-operad to Lurie's. To treat the two definitions together, we work over span categories that include both Span(Fin) and $\mathrm{Fin}_* \simeq \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$. This is a special case of the envelope construction for algebraic patterns in [BHS22].

Convention 19.3.1. Throughout this section, we fix an adequate triple $\mathcal{F} = (F, F_L, F_R)$ which satisfies the following three conditions:

(1) The ∞-category F is extensive in the sense of Definition 13.3.1.

(2) The classes F_L and F_R are closed under finite coproducts.

(3) For every $I \in F$, the morphism $\emptyset \to I$ is contained in F_L.

In particular, F_L contains every coproduct inclusion $I_i \hookrightarrow \bigsqcup_{j=1}^n I_j$, since this is the coproduct of id_{I_i} with the morphisms $\emptyset \to I_j$ for $j \neq i$.

Example 19.3.2. The main two examples are the adequate triples (Fin, Fin, Fin) and $(\mathrm{Fin}, \mathrm{Fin}_{\mathrm{inj}}, \mathrm{Fin})$, where $\mathrm{Fin}_{\mathrm{inj}} \subseteq \mathrm{Fin}$ is the wide subcategory of injections.

The hypotheses also apply to the following examples from equivariant and global homotopy theory.

Example 19.3.3. More generally, let G be a finite group and let Fin_G be the category of finite G-sets. Then the adequate triples $(\mathrm{Fin}_G, \mathrm{Fin}_G, \mathrm{Fin}_G)$ and $(\mathrm{Fin}_G, (\mathrm{Fin}_G)_{\mathrm{inj}}, \mathrm{Fin}_G)$ are examples.

Example 19.3.4. Let $\mathrm{FinGrpd} \subseteq \mathrm{Grpd} \subseteq \mathrm{Cat}$ denote the full (2,1)-category of finite groupoids. Under the fully faithful inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{An}$, we may identify this with the full subcategory of animae spanned by objects of the form $\bigsqcup_{i=1}^n BG_i$, the finite disjoint unions of classifying animae of finite groups G_i. Then (FinGrpd, FinGrpd, FinGrpd) satisfies the conditions of Convention 19.3.1.

Definition 19.3.5 ($\mathcal{F}$-monoid)**.** Let $\mathcal{F} = (F, F_L, F_R)$ be an adequate triple satisfying the conditions of Convention 19.3.1.

(1) Given an ∞-category C with finite products, we say that a functor $M\colon F_L^{\mathrm{op}} \to C$ satisfies the *Segal condition* if for every $n \geq 0$ and all objects $I_1, \dots, I_n \in F$, the maps $e_i\colon I_i \hookrightarrow \bigsqcup_{j=1}^n I_j$ induce an isomorphism

$$(e_i^*)_{i=1}^n\colon M\Big(\bigsqcup_{j=1}^n I_j\Big) \xrightarrow{\sim} \prod_{i=1}^n M(I_i).$$

(2) For C with finite products, we define an *$\mathcal{F}$-monoid in C* to be a functor

$$M\colon \mathrm{Span}(\mathcal{F}) \to C$$

such that $M|_{F_L^{\mathrm{op}}}$ satisfies the Segal condition. We denote by

$$\mathrm{Mon}_{\mathcal{F}}(C) \quad \subseteq \quad \mathrm{Fun}(\mathrm{Span}(\mathcal{F}), C)$$

the full subcategory spanned by the $\mathcal{F}$-monoids.

Remark 19.3.6. If F_L also contains the fold maps $I \sqcup I \to I$, then (F, F_L, F_R) is weakly extensive by Lemma 13.3.4. In this case F_L admits finite coproducts, so the Segal condition on a functor $F_L^{\mathrm{op}} \to C$ is equivalent to preservation of finite products. By Lemma 13.3.8, the $\mathcal{F}$-monoids in C are likewise precisely the finite-product-preserving functors $\mathrm{Span}(\mathcal{F}) \to C$.

Example 19.3.7. For $\mathcal{F} = (\mathrm{Fin}, \mathrm{Fin}, \mathrm{Fin})$, the $\mathcal{F}$-monoids are precisely the commutative monoids from Definition 5.3.4.

For $\mathcal{F} = (\mathrm{Fin}, \mathrm{Fin}_{\mathrm{inj}}, \mathrm{Fin})$, the ∞-category $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ is equivalent to the (classical) category Fin_* of finite pointed sets, and the resulting $\mathcal{F}$-monoids are precisely the commutative monoids as defined by Lurie [Lur17]. By Proposition 5.3.9, restriction along the inclusion induces an equivalence between these two definitions.

An $\mathcal{F}$*-monoidal* ∞*-category* is an $\mathcal{F}$-monoid in Cat_∞. By straightening-unstraightening, these form a full subcategory of $\mathrm{Cocart}(\mathrm{Span}(\mathcal{F}))$; we write $p_C\colon C^\otimes \to \mathrm{Span}(\mathcal{F})$ for the unstraightening of C. Recall from Proposition 19.1.10 that $\mathrm{Span}(\mathcal{F})$ has a factorization system $(\mathcal{L},\mathcal{R})$, where $\mathcal{L} = F_L^{\mathrm{op}}$ consists of the backwards spans and $\mathcal{R} = F_R$ of the forward spans.

Definition 19.3.8 ($\mathcal{F}$-∞-operad)**.** Let $\mathcal{F} = (F, F_L, F_R)$ be an adequate triple satisfying the conditions of Convention 19.3.1. An $\mathcal{F}$*-*∞*-operad* is a pair $O = (O^\otimes, p_O)$ consisting of an ∞-category $O^\otimes$ and a functor $p_O\colon O^\otimes \to \mathrm{Span}(\mathcal{F})$ satisfying the following three conditions:

(1) The functor p_O is an $\mathcal{L}$-cocartesian fibration;

(2) The straightening $F_L^{\mathrm{op}} \to \mathrm{Cat}_\infty$ of the resulting cocartesian fibration $O^\otimes|_{F_L^{\mathrm{op}}} \to F_L^{\mathrm{op}}$ satisfies the Segal condition: for all $J_1,\dots,J_n \in F$ we have

$$O^\otimes_{\bigsqcup_{i=1}^n J_i} \xrightarrow{\sim} \prod_{i=1}^n O^\otimes_{J_i}.$$

(3) Consider objects $I, J_1,\dots,J_n \in F$, $X \in O_I^\otimes$ and $Y \in O_J^\otimes$ with $J := \bigsqcup_{i=1}^n J_i$. Let $Y \to Y_i$ be cocartesian lifts of the backwards maps $J \hookleftarrow J_i \xrightarrow{=} J_i$. Then the induced commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{O^\otimes}(X,Y) & \longrightarrow & \prod_{i=1}^n \mathrm{Hom}_{O^\otimes}(X,Y_i) \\ {\scriptstyle p_O}\downarrow & & \downarrow{\scriptstyle p_O} \\ \mathrm{Hom}_{\mathrm{Span}(\mathcal{F})}(I,J) & \longrightarrow & \prod_{i=1}^n \mathrm{Hom}_{\mathrm{Span}(\mathcal{F})}(I,J_i) \end{array}$$

is a pullback square.

Given another $\mathcal{F}$-∞-operad $\mathcal{P} = (\mathcal{P}^\otimes, p_{\mathcal{P}})$, a *morphism of* $\mathcal{F}$*-*∞*-operads* is a $\mathcal{L}$-cocartesian functor $f\colon O^\otimes \to \mathcal{P}^\otimes$ over $\mathrm{Span}(\mathcal{F})$. We denote by

$$\mathrm{Op}_{\mathcal{F}} \quad \subseteq \quad (\mathrm{Cat}_\infty)^{\mathcal{L}-\mathrm{cocart}}_{/\mathrm{Span}(\mathcal{F})}$$

the full subcategory spanned by the $\mathcal{F}$-∞-operads.

Construction 19.3.9. We construct a functor

$$\mathcal{M}\colon \mathrm{Mon}_{\mathcal{F}}(\mathrm{Cat}_\infty) \to \mathrm{Op}_{\mathcal{F}}.$$

Given an $\mathcal{F}$-monoidal ∞-category C, its unstraightening $p_C\colon C^\otimes \to \mathrm{Span}(\mathcal{F})$ satisfies conditions (1) and (2) of Definition 19.3.8 by definition. For (3), take $X \in C_I^\otimes$, $Y \in C_J^\otimes$, and a span $\alpha\colon I \to J$. Cocartesian transport identifies $\mathrm{Hom}^\alpha_{C^\otimes}(X,Y)$ with $\mathrm{Hom}_{C_J^\otimes}(\alpha_! X, Y)$, which splits as a product by (2), just as in the proof of Lemma 14.4.4. A morphism of $\mathcal{F}$-monoids unstraightens to a cocartesian functor, so in particular preserves the lifts over $\mathcal{L}$. Thus unstraightening defines $\mathcal{M}$.

Proposition 19.3.10. *The functor* $\mathcal{M}\colon \mathrm{Mon}_{\mathcal{F}}(\mathrm{Cat}_\infty) \to \mathrm{Op}_{\mathcal{F}}$ *admits a left adjoint*

$$\mathrm{Env}_{\mathcal{F}}\colon \mathrm{Op}_{\mathcal{F}} \to \mathrm{Mon}_{\mathcal{F}}(\mathrm{Cat}_\infty),$$

called the $\mathcal{F}$-operadic envelope construction.

Proof. By Corollary 19.2.8, the free $\mathcal{R}$-cocartesian completion is left adjoint to the forgetful functor from cocartesian fibrations to $\mathcal{L}$-cocartesian fibrations over $\mathrm{Span}(\mathcal{F})$. The functor $\mathcal{M}$ is the restriction of this forgetful functor to the full subcategories of $\mathcal{F}$-monoidal ∞-categories and $\mathcal{F}$-∞-operads. It therefore suffices to show that, for an $\mathcal{F}$-∞-operad O, the straightening of $E_{\mathcal{R}}(p_O)$ satisfies the Segal condition.

By the characterization of cocartesian morphisms in Proposition 19.2.7, its straightening takes the following form:

- On objects, it sends $I \in F$ to the pullback ∞-category $(F_R)_{/I} \times_{\mathrm{Span}(\mathcal{F})} O^{\otimes}$.
- Given $g\colon I \to J$ in F_L, transport along the backwards span $J \xleftarrow{g} I \xrightarrow{=} I$ sends $(X, r\colon J' \to J)$ to $(Y, r'\colon I' \to I)$, where
$$\begin{array}{ccc} I' & \xrightarrow{f} & J' \\ {\scriptstyle r'}\downarrow & \lrcorner & \downarrow{\scriptstyle r} \\ I & \xrightarrow{g} & J, \end{array}$$
and $X \to Y$ is a p_O-cocartesian lift of the backwards span $J' \xleftarrow{f} I' \xrightarrow{=} I'$.
- Given $g\colon I \to J$ in F_R, transport along the forward span sends $(X, r\colon K \to I)$ to $(X, gr\colon K \to J)$.

In the backwards case, the pullback computes the composite span whose factorization is used in Proposition 19.2.7. In the forward case, the composite gr already belongs to F_R, so its factorization has left factor id_K.

Let $J = \bigsqcup_{i=1}^{n} J_i$. We must show that the Segal map

$$(F_R)_{/J} \times_{\mathrm{Span}(\mathcal{F})} O^{\otimes} \longrightarrow \prod_{i=1}^{n} \big((F_R)_{/J_i} \times_{\mathrm{Span}(\mathcal{F})} O^{\otimes}\big)$$

is an equivalence. Extensivity gives $F_{/J} \simeq \prod_i F_{/J_i}$, and this restricts to

$$(F_R)_{/J} \simeq \prod_{i=1}^{n} (F_R)_{/J_i}.$$

Indeed, pullback along coproduct inclusions preserves membership in F_R by adequacy, since these inclusions lie in F_L. Closure under finite coproducts gives the converse. This applies both to objects and to morphisms of the slices.

Given objects $(X_i, r_i\colon I_i \to J_i)$, take $r = \bigsqcup_i r_i$ and choose $X \in O^{\otimes}_{\bigsqcup_i I_i}$ corresponding to $(X_i)_i$ under condition (2) of Definition 19.3.8. Then (X, r) maps to the given tuple, proving essential surjectivity.

For full faithfulness, consider $(X, r\colon I \to J)$ and $(X', r'\colon I' \to J)$, and denote their images by $(X_i, r_i\colon I_i \to J_i)$ and $(X'_i, r'_i\colon I'_i \to J_i)$. Since the map between the base slices is an equivalence, it suffices to check the map on hom animae over each morphism $f\colon (I, r) \to (I', r')$ in $(F_R)_{/J}$. Writing $f_i\colon I_i \to I'_i$ for its components, we have

$$\mathrm{Hom}^{f}_{O^{\otimes}}(X, X') \simeq \prod_{i=1}^{n} \mathrm{Hom}^{f_i}_{O^{\otimes}}(X_i, X'_i).$$

Here condition (3) first gives the fibers of $\mathrm{Hom}_{\mathcal{O}^\otimes}(X, X_i')$ over the spans $I \leftarrow I_i \xrightarrow{f_i} I_i'$, and cocartesian transport along $I \leftarrow I_i \xrightarrow{=} I_i$ gives the displayed product. These are precisely the fibers of the hom animae on the two sides, proving full faithfulness. □

Let us explicitly record the main example of the $\mathcal{F}$-operadic envelope construction:

Proposition 19.3.11. *The multimorphism operad functor $\mathcal{M}\colon \mathrm{Cat}_\infty^\otimes \to \mathrm{Op}_\infty$ admits a left adjoint*

$$\mathrm{Env}\colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty^\otimes.$$

Proof. This is a special case of the previous proposition. For $\mathcal{F} = (\mathrm{Fin}, \mathrm{Fin}, \mathrm{Fin})$, Proposition 14.1.12 identifies the objects of $\mathrm{Op}_\mathcal{F}$ with the ∞-operads from Definition 14.1.1, while Corollary 14.1.14 identifies the two notions of morphism. □

Corollary 19.3.12. *Let $\mathcal{O}$ be an ∞-operad and let C be a symmetric monoidal ∞-category. There is a natural equivalence*

$$\mathrm{Alg}_\mathcal{O}(C) \simeq \mathrm{Fun}^\otimes(\mathrm{Env}(\mathcal{O}), C).$$

In other words, an $\mathcal{O}$-algebra in C is equivalently a symmetric monoidal functor from the envelope of $\mathcal{O}$ to C.

Proof. Let E be an ∞-category. Using Lemma 14.4.13, the adjunction of Proposition 19.3.11, and currying, we obtain natural equivalences

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}_\infty}(E, \mathrm{Alg}_\mathcal{O}(C)) &\simeq \mathrm{Hom}_{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{M}_{\mathrm{Fun}(E,C)}) \\ &\simeq \mathrm{Map}^\otimes(\mathrm{Env}(\mathcal{O}), \mathrm{Fun}(E, C)) \\ &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(E, \mathrm{Fun}^\otimes(\mathrm{Env}(\mathcal{O}), C)).\end{aligned}$$

Here $\mathrm{Fun}(E, C)$ carries the pointwise symmetric monoidal structure. Since these equivalences are natural in E, the Yoneda lemma gives the claimed equivalence of ∞-categories. □

We refer to the symmetric monoidal ∞-category $\mathrm{Env}(\mathcal{O})$ as the *envelope* of the ∞-operad $\mathcal{O}$. By construction, its total category is $E_\mathcal{R}(p_\mathcal{O})$, and its underlying ∞-category is the fiber over $\langle 1 \rangle$. This gives the pullback square

$$\begin{array}{ccc}\mathrm{Env}(\mathcal{O}) & \longrightarrow & \mathcal{O}^\otimes \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_\mathcal{O}} \\ \mathrm{Fin} & \hookrightarrow & \mathrm{Span}(\mathrm{Fin}).\end{array}$$

In particular, the objects of $\mathrm{Env}(\mathcal{O})$ are unordered tuples $\{x_i\}_{i\in I}$ of colors of $\mathcal{O}$.

Lemma 19.3.13. *The monoidal unit of* $\mathrm{Env}(\mathcal{O})$ *is the empty tuple, and its tensor product is given by concatenation of unordered tuples:*

$$\{x_i\}_{i\in I} \otimes \{y_j\}_{j\in J} \simeq \{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J}.$$

Proof. Under the Segal equivalence, the two tuples on the left determine the object of the fiber over $\langle 2\rangle$ indexed by their disjoint union $I \sqcup J$. The tensor product is cocartesian transport of this object along the forward fold span $\langle 2\rangle \xleftarrow{=} \langle 2\rangle \to \langle 1\rangle$. By the forward-transport calculation in the proof of Proposition 19.3.10, this transport leaves the object of $\mathcal{O}^{\otimes}$ unchanged and postcomposes its indexing map with the fold map. The result is the concatenated tuple. The same calculation for the unique forward map $\emptyset \to \langle 1\rangle$ gives the empty tuple as the unit. □

Lemma 19.3.14. *Let $\mathcal{O}$ be an ∞-operad, and let $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ be objects of* $\mathrm{Env}(\mathcal{O})$. *Then there is a natural equivalence*

$$\mathrm{Hom}_{\mathrm{Env}(\mathcal{O})}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) \simeq \coprod_{f\colon I\to J} \prod_{j\in J} \mathcal{O}(\{x_i\}_{i\in f^{-1}(j)}; y_j),$$

where the coproduct ranges over all maps of finite sets $f\colon I \to J$. In particular, for a single output color $y \in \mathcal{O}^{\simeq}$ there is a natural equivalence

$$\mathrm{Hom}_{\mathrm{Env}(\mathcal{O})}(\{x_i\}_{i\in I}, y) \simeq \mathcal{O}(\{x_i\}_{i\in I}; y).$$

Proof. The pullback description of $\mathrm{Env}(\mathcal{O})$ identifies its hom anima with the pullback of

$$\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) \longrightarrow \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I, J)$$

along the inclusion $\mathrm{Hom}_{\mathrm{Fin}}(I,J) \hookrightarrow \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J)$. It is therefore the coproduct of the fibers over the forward spans $I \xleftarrow{=} I \xrightarrow{f} J$. Applying Lemma 14.1.10 to each such span gives the formula. The final statement is the case $J = \langle 1\rangle$. □

Lemma 19.3.15. *The envelope* $\mathrm{Env}(\mathcal{C}omm)$ *of the commutative ∞-operad is the category of finite sets, with monoidal structure given by disjoint union.*

Proof. The total category of the terminal ∞-operad $\mathcal{C}omm$ is Span(Fin) itself. The pullback description of the underlying category of the envelope therefore gives $\mathrm{Env}(\mathcal{C}omm) \simeq \mathrm{Fin}$, and Lemma 19.3.13 identifies the tensor product with disjoint union. □

Corollary 19.3.16. *For every symmetric monoidal ∞-category $\mathcal{C}$, there is a natural equivalence*

$$\mathrm{CAlg}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{C}omm}(\mathcal{C}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Fin}, \mathcal{C}),$$

where Fin *carries the monoidal structure given by disjoint union.*

Proof. Apply Corollary 19.3.12 with $\mathcal{O} = \mathcal{C}omm$ and use Lemma 19.3.15. □

Thus the operadic definition of commutative algebras in Example 14.4.11 agrees with Definition 8.1.3.

Corollary 19.3.17. *For a colored operad $\mathcal{O}$, the ∞-categorical envelope of the associated ∞-operad is equivalent to the category underlying its classical envelope from Definition 12.3.3, compatibly with the symmetric monoidal structures and the maps to* $(\mathrm{Fin}, \amalg)$.

Proof. Both envelopes have unordered tuples of colors as objects, and Lemma 19.3.14 identifies their hom animae with the classical morphism sets of Definition 12.3.3. By Construction 14.2.3, composition of labeled forward spans is given by operadic substitution, as in the classical envelope. Both tensor products are given by concatenation, with unit, associativity, and symmetry maps induced by the corresponding bijections of indexing sets. These identifications therefore give a symmetric monoidal equivalence over $(\mathrm{Fin}, \amalg)$. □

The envelopes of the ∞-operads governing associative algebras and modules are also 1-categories. We record them explicitly.

Example 19.3.18 (Envelopes for algebras and modules)**.** All the following categories are symmetric monoidal under disjoint union.

(1) The envelope $\mathrm{Env}(\mathcal{A}ssoc)$ has finite sets as objects. Its morphisms $S \to T$ are maps of finite sets together with a linear order on every fiber over $t \in T$.

(2) The envelope $\mathrm{Env}(\mathcal{LM}od)$ has pairs (S_A, S_M) of finite sets as objects. A morphism $(S_A, S_M) \to (T_A, T_M)$ is a map
$$f\colon S_A \sqcup S_M \longrightarrow T_A \sqcup T_M$$
which restricts to a bijection $S_M \to T_M$, together with a linear order on $f^{-1}(t) \cap S_A$ for every $t \in T_A \sqcup T_M$.

(3) The envelope $\mathrm{Env}(\mathcal{M}od)$ has the same objects and morphisms as in (2), but without the choices of linear orders.

These descriptions follow immediately from Lemma 19.3.14 and the multimorphism sets of $\mathcal{A}ssoc$, $\mathcal{LM}od$, and $\mathcal{M}od$.

By functoriality, any morphism of ∞-operads induces a symmetric monoidal functor on envelopes. In particular, the terminal morphism $\mathcal{O} \to \mathcal{C}omm$ induces a symmetric monoidal functor $\mathrm{Env}(\mathcal{O}) \to \mathrm{Env}(\mathcal{C}omm) \simeq (\mathrm{Fin}, \amalg)$. This functor sends a tuple to its indexing set. Its fiber over a singleton recovers $U(\mathcal{O})$, and Lemma 19.3.14 recovers the multimorphisms as maps from tuples to singleton tuples. The following proposition shows that the envelope, together with this functor, determines $\mathcal{O}$.

Proposition 19.3.19 ([BHS22, Example 4.3.1 and Observation 4.3.2])**.** *The induced functor*
$$\mathrm{Env}\colon \mathrm{Op}_\infty \to (\mathrm{Cat}_\infty^\otimes)_{/(\mathrm{Fin},\amalg)}$$
is fully faithful. A symmetric monoidal functor $p\colon C \to (\mathrm{Fin}, \amalg)$ lies in the essential image if and only if it satisfies the following two conditions:

(1) *For every finite set I, the tensor product in C defines an equivalence*
$$\prod_{i \in I} C_{\{i\}} \xrightarrow{\sim} C_I,$$
where C_I is the fiber of p over $I \in \mathrm{Fin}$;

(2) *For every finite collection $(I_j)_{j\in J}$ of finite sets and objects $X_j \in C_{I_j}$ and $y_j \in C_{\{j\}}$, the commutative square*

$$\begin{array}{ccc} \prod_{j\in J} \mathrm{Hom}_C(X_j, y_j) & \xrightarrow{\bigotimes_{j\in J}} & \mathrm{Hom}_C(\bigotimes_{j\in J} X_j, \bigotimes_{j\in J} y_j) \\ \downarrow & & \downarrow p \\ * & \longrightarrow & \mathrm{Hom}_{\mathrm{Fin}}(\bigsqcup_{j\in J} I_j, J) \end{array}$$

is a pullback square, where the bottom map selects the function $\bigsqcup_{j\in J} I_j \to J$ whose restriction to I_j has value j.

Proof. Under straightening, full faithfulness follows from Lemma 19.2.9. For $C = \mathrm{Env}(\mathcal{O})$, condition (1) follows from the Segal condition on $\mathcal{O}$ and Lemma 19.3.13. Condition (2) follows by taking the fiber in Lemma 19.3.14 over the function $\bigsqcup_j I_j \to J$ with value j on I_j.

Conversely, suppose that $p\colon C \to (\mathrm{Fin}, \sqcup)$ satisfies (1) and (2). Let $\mathcal{O} \subseteq \mathcal{M}_C$ be the full suboperad on objects whose image under p is a singleton, as in Construction 14.1.7. The unit $\mathit{Comm} \to \mathcal{M}_{(\mathrm{Fin},\sqcup)}$ identifies Comm with the full suboperad on the singleton: its multimorphism animae are the singleton sets $\mathrm{Hom}_{\mathrm{Fin}}(I, \langle 1\rangle)$. Thus the morphism of ∞-operads $\mathcal{M}_p\colon \mathcal{M}_C \to \mathcal{M}_{(\mathrm{Fin},\sqcup)}$ induced by p factors through this unit after restriction to $\mathcal{O}$. By naturality of the adjunction $\mathrm{Env} \dashv \mathcal{M}$, the symmetric monoidal functor $\mathrm{Env}(\mathcal{O}) \to C$ adjoint to the inclusion $\mathcal{O} \hookrightarrow \mathcal{M}_C$ is therefore a functor over $(\mathrm{Fin}, \sqcup)$.

The colors of $\mathcal{O}$ identify with the objects of $C_{\langle 1\rangle}$, so condition (1) makes this functor essentially surjective. Since $\mathcal{O}$ is a full suboperad of $\mathcal{M}_C$, we have $\mathcal{O}(\{x_i\}; y) \simeq \mathrm{Hom}_C(\bigotimes_i x_i, y)$. Consequently, Lemma 19.3.14 identifies the fiber over $f\colon I \to J$ of the hom anima from $\{x_i\}_{i\in I}$ to $\{y_j\}_{j\in J}$ in $\mathrm{Env}(\mathcal{O})$ with

$$\prod_{j\in J} \mathrm{Hom}_C\left(\bigotimes_{i\in f^{-1}(j)} x_i, y_j\right).$$

The map induced by $\mathrm{Env}(\mathcal{O}) \to C$ on this fiber tensors the displayed morphisms. By condition (2), it is an isomorphism onto the fiber over f of

$$\mathrm{Hom}_C\left(\bigotimes_{i\in I} x_i, \bigotimes_{j\in J} y_j\right) \longrightarrow \mathrm{Hom}_{\mathrm{Fin}}(I, J).$$

Taking the coproduct over all f proves full faithfulness. □

19.4 Comparison to Lurie's ∞-operads

Lurie's definition of ∞-operads uses finite pointed sets, or equivalently spans with injective backwards leg. We show that restriction to these spans induces an equivalence with the definition used in this book. The proof uses envelopes: both definitions identify ∞-operads with the same full subcategory of symmetric monoidal ∞-categories over $(\mathrm{Fin}, \sqcup)$.

Definition 19.4.1. Given a finite set I, we write $I_+ := I \sqcup *$ for the resulting finite pointed set. A morphism $f\colon I_+ \to J_+$ of finite pointed sets is called *inert* if on the preimage of J it restricts to a bijection $f^{-1}(J) \xrightarrow{\sim} J$. We say that f is *active* if $f^{-1}(*) = \{*\}$.

Given a finite set I, the *Segal map* $\rho_i \colon I_+ \to \{i\}_+$ for $i \in I$ is defined as the map that is the identity on $\{i\}$ and sends all other $j \in I \setminus \{i\}$ to the basepoint.

Remark 19.4.2. Recall from Lemma 5.3.8 that there is an equivalence

$$\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \simeq \mathrm{Fin}_*,$$

given on objects by sending I to I_+ and on morphisms by sending a span $I \xleftarrow{f} K \xrightarrow{g} J$ to the pointed map $h \colon I_+ \to J_+$ given by $h(i) = g(k)$ whenever $i = f(k)$ is in the image of f and $h(i) = *$ otherwise. Under this equivalence, the inert maps in Fin_* correspond to the backwards spans $I \xleftarrow{f} K \xrightarrow{=} K$, and the active maps in Fin_* correspond to the forward spans $I \xleftarrow{=} I \xrightarrow{g} J$. The Segal maps ρ_i correspond to the backwards spans

$$I \hookleftarrow \{i\} \xrightarrow{=} \{i\}.$$

We will write $\mathrm{Fin}_* \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ for the inclusion of the subcategory $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$.

Definition 19.4.3 ([Lur17, Definition 2.1.1.10])**.** An *∞-operad in the sense of Lurie* is a pair $(O^\otimes, p_O)$ consisting of an ∞-category $O^\otimes$ equipped with a functor $p_O \colon O^\otimes \to \mathrm{Fin}_*$. Writing $O^\otimes_I$ for the fiber over I_+, we require the following conditions:

(1) For any inert morphism $f \colon I_+ \to J_+$ and any object $X \in O^\otimes_I$ there is a p_O-cocartesian lift $X \to Y$ of f in $O^\otimes$.

(2) Let X and Y be objects of $O^\otimes$ and let $I_+ := p_O(X)$ and $J_+ := p_O(Y)$. For every $j \in J$, let $Y \to Y_j$ denote a p_O-cocartesian lift of the Segal map $\rho_j \colon J_+ \to \{j\}_+$. Then the induced commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{O^\otimes}(X, Y) & \longrightarrow & \prod_{j \in J} \mathrm{Hom}_{O^\otimes}(X, Y_j) \\ \downarrow{\scriptstyle p_O} & & \downarrow{\scriptstyle p_O} \\ \mathrm{Hom}_{\mathrm{Fin}_*}(I_+, J_+) & \longrightarrow & \prod_{j \in J} \mathrm{Hom}_{\mathrm{Fin}_*}(I_+, \{j\}_+) \end{array}$$

is a pullback square.

(3) For every finite collection $\{X_j\}_{j \in J}$ of objects $X_j \in O^\otimes_{\{j\}}$, there exists an object $X \in O^\otimes_J$ and a collection of p_O-cocartesian morphisms $X \to X_j$ covering $\rho_j \colon J_+ \to \{j\}_+$.

We say a functor $f \colon O^\otimes \to \mathcal{P}^\otimes$ over Fin_* is a *morphism of Lurie-∞-operads* if it preserves cocartesian morphisms over inerts. We let $\mathrm{Op}^{\mathrm{Lurie}}_\infty \subseteq (\mathrm{Cat}_\infty)_{/\mathrm{Fin}_*}$ denote the ∞-category of the Lurie ∞-operads and their morphisms.

Let $\mathcal{F}_{\mathrm{Lurie}} := (\mathrm{Fin}, \mathrm{Fin}_{\mathrm{inj}}, \mathrm{Fin})$, so that $\mathrm{Span}(\mathcal{F}_{\mathrm{Lurie}}) \simeq \mathrm{Fin}_*$.

Lemma 19.4.4. *Under the equivalence* $\mathrm{Span}(\mathcal{F}_{\mathrm{Lurie}}) \simeq \mathrm{Fin}_*$*, the $\mathcal{F}_{\mathrm{Lurie}}$-$\infty$-operads of Definition 19.3.8 are precisely the ∞-operads in the sense of Lurie, and the two notions of morphism agree. In particular, there is a canonical equivalence*

$$\mathrm{Op}_{\mathcal{F}_{\mathrm{Lurie}}} \simeq \mathrm{Op}^{\mathrm{Lurie}}_\infty .$$

Proof. Condition (1) in the two definitions is the same. Suppose first that $p_O\colon O^\otimes \to \mathrm{Fin}_*$ is an ∞-operad in the sense of Lurie. The functor

$$O^\otimes_J \longrightarrow \prod_{j\in J} O^\otimes_{\{j\}}$$

induced by cocartesian transport along the Segal maps is essentially surjective by condition (3) of Definition 19.4.3. It is fully faithful as well: condition (2), restricted to the fiber over id_{J_+}, identifies the hom anima in $O^\otimes_J$ with the product of the hom animae in the fibers over the singletons, using cocartesian transport along the Segal maps on the sources. Thus condition (2) holds for the decomposition of J into singletons, and hence for every finite decomposition by grouping the singleton factors. Condition (3) follows similarly from condition (2): apply it to the singleton decomposition of J and to those of the summands J_i, and group the resulting factors according to $J = \bigsqcup_i J_i$.

Conversely, if O is an $\mathcal{F}_{\mathrm{Lurie}}$-$\infty$-operad, condition (1) gives the required cocartesian lifts of inert maps. Specializing condition (3) to the decomposition of J into singletons gives condition (2) of Definition 19.4.3, while condition (2) gives condition (3). Finally, in both cases a morphism is a functor preserving cocartesian morphisms over inert maps. □

Construction 19.4.5. Using Lemma 19.4.4 and Proposition 5.3.9, we define the *Lurie operadic envelope*

$$\mathrm{Env}^{\mathrm{Lurie}}\colon \mathrm{Op}_\infty^{\mathrm{Lurie}} \longrightarrow \mathrm{Cat}_\infty^\otimes$$

as the functor $\mathrm{Env}_{\mathcal{F}_{\mathrm{Lurie}}}$ of Proposition 19.3.10, followed by the equivalence from $\mathcal{F}_{\mathrm{Lurie}}$-monoidal ∞-categories to symmetric monoidal ∞-categories.

Lemma 19.4.6. *If $O = (O^\otimes, p_O)$ is an ∞-operad in the sense of Definition 14.1.1, then its pullback $p_\mathcal{P}\colon \mathcal{P}^\otimes := O^\otimes \times_{\mathrm{Span(Fin)}} \mathrm{Fin}_* \to \mathrm{Fin}_*$ is an ∞-operad in the sense of Lurie. Moreover, forming pullbacks along the inclusion $\mathrm{Fin}_* \hookrightarrow \mathrm{Span(Fin)}$ induces a restriction functor*

$$\mathrm{Op}_\infty \to \mathrm{Op}_\infty^{\mathrm{Lurie}}.$$

Proof. Any p_O-cocartesian lift of a backwards span $I \hookleftarrow J \xrightarrow{=} J$ will still be $p_\mathcal{P}$-cocartesian, verifying condition (1). Condition (2) is immediate from Lemma 14.1.13. Condition (3) is immediate from condition (ii) in Proposition 14.1.12. The claim on morphisms is checked similarly. □

Lemma 19.4.7. *Let*

$$u\colon \mathrm{Fin}_* \simeq \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \hookrightarrow \mathrm{Span(Fin)}$$

be the inclusion, and let $\mathcal{R}_*$ and $\mathcal{R}$ denote the forward morphisms in Fin_* *and* $\mathrm{Span(Fin)}$*, respectively. For every functor $p\colon E \to \mathrm{Span(Fin)}$, there is a natural equivalence of functors over* Fin_*

$$u^* E_\mathcal{R}(p) \simeq E_{\mathcal{R}_*}(u^* p).$$

Proof. By the pullback formula in Construction 19.2.2, it suffices to construct a natural equivalence

$$\mathrm{Fin}_* \times_{\mathrm{Span(Fin)},t} \mathrm{Ar}_\mathcal{R}(\mathrm{Span(Fin)}) \simeq \mathrm{Ar}_{\mathcal{R}_*}(\mathrm{Fin}_*)$$

compatible with the target functors to Fin_* and the source functors to $\mathrm{Span}(\mathrm{Fin})$, using $u \circ s$ on the right. The objects on both sides are forward spans, equivalently maps of finite sets $r\colon I \to J$. A morphism on the left is a commutative square in $\mathrm{Span}(\mathrm{Fin})$

$$\begin{array}{ccc} I & \xrightarrow{f} & I' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ J & \xrightarrow{g} & J' \end{array}$$

in which g belongs to Fin_*. The composite gr has injective backwards leg because this leg is obtained by pulling back the injective backwards leg of g along r. Since r' is a forward map, the backwards leg of $r'f$ is the backwards leg of f. The equivalence $r'f \simeq gr$ therefore shows that f also belongs to Fin_*. As $\mathrm{Fin}_* \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ is a wide subcategory inclusion, its hom animae are unions of components, so the identification making the square commute also lies in the corresponding hom anima of Fin_*. Thus the displayed square is a morphism in $\mathrm{Ar}_{\mathcal{R}_*}(\mathrm{Fin}_*)$. This identifies the two arrow categories compatibly with source and target, and pulling back along p proves the claim. □

Proposition 19.4.8 ([BHS22, Corollary 5.1.15])**.** *The restriction functor* $\mathrm{Op}_\infty \to \mathrm{Op}_\infty^{\mathrm{Lurie}}$ *is an equivalence.*

Proof. By Lemma 19.4.4, we may work with $\mathcal{F}_{\mathrm{Lurie}}$-$\infty$-operads. The envelope of the terminal such ∞-operad is $(\mathrm{Fin}, \amalg)$, by the same calculation as in Lemma 19.3.15. Thus Lemma 19.2.9, followed by the Segal equivalence of Proposition 5.3.9, gives a fully faithful functor

$$\mathrm{Env}^{\mathrm{Lurie}}\colon \mathrm{Op}_\infty^{\mathrm{Lurie}} \longrightarrow (\mathrm{Cat}_\infty^{\otimes})_{/(\mathrm{Fin},\amalg)}.$$

In these envelopes, objects are again tuples of colors, with tensor product given by concatenation, as in Lemma 19.3.13. The Segal condition therefore gives condition (1) of Proposition 19.3.19. The hom-anima calculation of Lemma 19.3.14 also applies, using condition (2) of Definition 19.4.3 over forward maps, and gives condition (2) of that proposition. Hence the essential image of $\mathrm{Env}^{\mathrm{Lurie}}$ is contained in that of Env.

Conversely, let $u\colon \mathrm{Fin}_* \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ be the inclusion and let $\mathcal{O}$ be an ∞-operad over $\mathrm{Span}(\mathrm{Fin})$. Applying Lemma 19.4.7 to $p_{\mathcal{O}}$ identifies the cocartesian fibration defining $\mathrm{Env}^{\mathrm{Lurie}}(u^*\mathcal{O})$ with the restriction of the one defining $\mathrm{Env}(\mathcal{O})$. Under the Segal equivalence, this gives a natural symmetric monoidal equivalence

$$\mathrm{Env}^{\mathrm{Lurie}}(u^*\mathcal{O}) \simeq \mathrm{Env}(\mathcal{O}).$$

It is compatible with the maps to $(\mathrm{Fin}, \amalg)$ by naturality, so the two envelope functors have the same essential image. We obtain a naturally commutative diagram

$$\begin{array}{ccc} \mathrm{Op}_\infty & \overset{\mathrm{Env}}{\hookrightarrow} & \\ {\scriptstyle -\times_{\mathrm{Span}(\mathrm{Fin})}\mathrm{Fin}_*}\downarrow & & (\mathrm{Cat}_\infty^{\otimes})_{/(\mathrm{Fin},\amalg)}. \\ \mathrm{Op}_\infty^{\mathrm{Lurie}} & \overset{\mathrm{Env}^{\mathrm{Lurie}}}{\hookrightarrow} & \end{array}$$

Both diagonal functors are equivalences onto the same essential image. It follows by 2-out-of-3 that the left vertical functor is an equivalence, as desired. □

Remark 19.4.9. For a classical colored operad O, restricting the ∞-operad of Proposition 14.2.4 along $\mathrm{Fin}_* \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ recovers Lurie's category of operators [Lur17, Construction 2.1.1.7]. Indeed, spans with injective left leg are partial maps, and their operation labels compose by substitution.

19.5 The multiplicative Yoneda embedding

The envelope embeds O into the multimorphism operad of a symmetric monoidal ∞-category. The Yoneda embedding of $\mathrm{Env}(O)$ is symmetric monoidal for Day convolution, so composing gives a fully faithful morphism

$$O \longrightarrow \mathcal{M}_{\mathrm{PSh}(\mathrm{Env}(O))},$$

as in Nikolaus [Nik16, Proposition 2.7]. A color y is sent to the presheaf represented by its singleton tuple, whose value on a tuple $\{x_i\}_{i\in I}$ is $O(\{x_i\}_{i\in I}; y)$ by Lemma 19.3.14. We equip the Yoneda functor with a symmetric monoidal structure by using the lax symmetric monoidal structure on the Hom-functor. Whenever necessary, we enlarge the universe so that the envelope is small before forming its presheaf category.

Lemma 19.5.1. *Let D be a symmetric monoidal ∞-category. The Hom-functor* $\mathrm{Hom}_D(-,-)\colon D^{\mathrm{op}} \times D \to \mathrm{An}$ *admits a canonical lax symmetric monoidal structure, where* An *carries the cartesian monoidal structure. Its structure maps are given by tensoring morphisms:*

$$\prod_{i\in I} \mathrm{Hom}_D(x_i, y_i) \longrightarrow \mathrm{Hom}_D\left(\bigotimes_{i\in I} x_i, \bigotimes_{i\in I} y_i\right).$$

The unit is the point $\mathrm{id}_{\mathbb{1}_D} \in \mathrm{Hom}_D(\mathbb{1}_D, \mathbb{1}_D)$.

Proof. The twisted arrow construction preserves finite products, by the product formula for hom animae and straightening. It therefore preserves commutative monoid objects of Cat_∞, giving $\mathrm{Tw}(D)$ a symmetric monoidal structure. Naturality of the source and target projections makes the left fibration $(s,t)\colon \mathrm{Tw}(D) \to D^{\mathrm{op}} \times D$ a symmetric monoidal functor.

Under straightening over Span(Fin), the induced functor on total categories

$$(s,t)^{\otimes}\colon \mathrm{Tw}(D)^{\otimes} \longrightarrow (D^{\mathrm{op}} \times D)^{\otimes}$$

corresponds to a natural transformation with components $(s,t)^I\colon \mathrm{Tw}(D)^I \to (D^{\mathrm{op}} \times D)^I$. These are left fibrations, so Lemma 24.2.2 shows that $(s,t)^{\otimes}$ is a left fibration as well. Its straightening $H^{\otimes}\colon (D^{\mathrm{op}} \times D)^{\otimes} \to \mathrm{An}$ sends a tuple $\{(x_i, y_i)\}_{i\in I}$ to $\prod_{i\in I} \mathrm{Hom}_D(x_i, y_i)$.

Products in $(D^{\mathrm{op}} \times D)^{\otimes}$ are given by concatenation. Since $H^{\otimes}$ sends the projections from a concatenated tuple to the corresponding projections between products of hom animae, it preserves finite products. Its restriction to the singleton fiber is $\mathrm{Hom}_D(-,-)$, so Theorem 15.3.11 gives the required lax symmetric monoidal structure. The tensor product in $\mathrm{Tw}(D)$ tensors morphisms of D, and its unit is $\mathrm{id}_{\mathbb{1}_D}$, giving the stated structure maps and unit. □

Proposition 19.5.2 (Nikolaus [Nik16, Proposition 3.9])**.** *Let D be a small symmetric monoidal ∞-category. Then the Yoneda embedding*

$$Y\colon D \to \mathrm{PSh}(D) = \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{An})$$

admits a canonical strong symmetric monoidal structure, where $\mathrm{PSh}(D)$ *is equipped with the Day convolution monoidal structure.*

Proof. By the universal property of Day convolution (Definition 16.1.1), a lax symmetric monoidal structure on Y corresponds to one on the associated functor $\mathrm{Hom}_D(-,-)\colon D^{\mathrm{op}} \times D \to \mathrm{An}$. This is provided by Lemma 19.5.1.

It remains to show that this lax structure is strong. Let $\{d_i\}_{i\in I}$ be a finite collection of objects of D. We must show that the induced map

$$\bigotimes_{i\in I}^{\mathrm{Day}} Y(d_i) \to Y\left(\bigotimes_{i\in I} d_i\right)$$

is an isomorphism in $\mathrm{PSh}(D)$. For any presheaf $H\colon D^{\mathrm{op}} \to \mathrm{An}$, Proposition 16.1.10 identifies the anima of maps from $\bigotimes_{i\in I}^{\mathrm{Day}} Y(d_i)$ to H with

$$\mathrm{Nat}\left(\prod_{i\in I} Y(d_i)(-), H\left(\bigotimes_{i\in I} -\right)\right).$$

The first functor is the presheaf on D^I represented by the tuple $\{d_i\}_{i\in I}$. The Yoneda lemma therefore identifies this anima with $H(\bigotimes_{i\in I} d_i)$, which is also $\mathrm{Nat}(Y(\bigotimes_{i\in I} d_i), H)$. Under these identifications, precomposition with the lax structure map is the identity: the tensor product of the identity morphisms of the d_i is the identity of $\bigotimes_{i\in I} d_i$. Thus the structure map is an isomorphism. □

Proposition 19.5.3 (Multiplicative Yoneda embedding)**.** *Let* $\mathcal{O}$ *be a small* ∞*-operad and put* $E := \mathrm{Env}(\mathcal{O})$. *The composite*

$$\mathcal{O} \xrightarrow{\eta_{\mathcal{O}}} \mathcal{M}_E \xrightarrow{\mathcal{M}_Y} \mathcal{M}_{\mathrm{PSh}(E)}$$

is fully faithful.

Proof. The map $\eta_{\mathcal{O}}$ is fully faithful by Lemma 19.3.14. Since Y is fully faithful and symmetric monoidal by Proposition 19.5.2, the induced map $\mathcal{M}_Y$ is fully faithful as well, and hence so is their composite. □

19.6 Day convolution into arbitrary ∞-operads

Let C be a symmetric monoidal ∞-category and $\mathcal{O}$ an ∞-operad. We construct a Day convolution operad whose underlying ∞-category is $\mathrm{Fun}(C, U(\mathcal{O}))$, as in Lemma 16.1.3. To do so, we embed $\mathcal{O}$ into the multimorphism operad of its envelope and restrict Day convolution to the $\mathcal{O}$-valued functors. We then describe its multimorphisms as coherently natural families of operations in $\mathcal{O}$.

Lemma 19.6.1. *Let* $\mathcal{O} \subseteq \mathcal{Q}$ *be a full suboperad, let* C *be a symmetric monoidal* ∞*-category, and assume that* $\mathcal{D}ay(\mathcal{M}_C, \mathcal{Q})$ *exists. Then* $\mathcal{D}ay(\mathcal{M}_C, \mathcal{O})$ *is the full suboperad of* $\mathcal{D}ay(\mathcal{M}_C, \mathcal{Q})$ *spanned by the functors* $C \to U(\mathcal{Q})$ *valued in* $U(\mathcal{O})$.

Proof. The total category of a full suboperad is the full subcategory on tuples of its colors (Construction 14.1.7). Thus maps into a full suboperad form a full subcategory of the operad functor category, consisting of those maps whose colors lie in that suboperad.

For every ∞-operad $\mathcal{R}$, the equivalence

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{R}, \mathcal{D}ay(\mathcal{M}_C, \mathcal{Q})) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{R} \times \mathcal{M}_C, \mathcal{Q})$$

restricts to the full subcategories where, respectively, all color functors are $\mathcal{O}$-valued and all colors lie in $\mathcal{O}$. These conditions correspond because evaluation sends (F, c) to $F(c)$. The restricted evaluation map therefore gives the stated full suboperad the universal property of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{O})$. □

Theorem 19.6.2. *Let C be a symmetric monoidal ∞-category. For every ∞-operad $\mathcal{O}$, the Day convolution operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{O})$ exists.*

Proof. Put $E := \mathrm{Env}(\mathcal{O})$. By Lemma 19.3.14, the unit identifies $\mathcal{O}$ with the full suboperad of $\mathcal{M}_E$ on singleton tuples. Since $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_E)$ exists by Theorem 16.1.8, the result follows from Lemma 19.6.1. □

Theorem 19.6.3 (Multimorphism formula for Day convolution)**.** *Let C be a small symmetric monoidal ∞-category and let $\mathcal{O}$ be an ∞-operad. For every finite set I and functors $F_i, G\colon C \to U(\mathcal{O})$, there is a natural equivalence*

$$\mathcal{D}ay(\mathcal{M}_C, \mathcal{O})(\{F_i\}_{i\in I}; G) \simeq \int_{\{c_i\}_{i\in I}\in C^I} \mathcal{O}(\{F_i(c_i)\}_{i\in I}; G(\textstyle\bigotimes_C^I c_i)).$$

Here the end is taken over the bifunctor on $(C^I)^{\mathrm{op}}\times C^I$ sending $(\{c_i\}, \{d_i\})$ to $\mathcal{O}(\{F_i(c_i)\}; G(\bigotimes_i d_i))$.

Proof. Put $E := \mathrm{Env}(\mathcal{O})$. By Lemma 19.6.1, we may compute the multimorphism anima in $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_E)$. The Day convolution formula of Proposition 16.1.10 and the end formula for natural transformations (Proposition 24.6.6) give

$$\begin{aligned}\mathcal{D}ay(\mathcal{M}_C, \mathcal{O})(\{F_i\}_{i\in I}; G) &\simeq \mathrm{Nat}\left(\textstyle\bigotimes_E^I \circ \prod_{i\in I} F_i, G \circ \bigotimes_C^I\right)\\ &\simeq \int_{\{c_i\}\in C^I} \mathrm{Hom}_E\left(\bigotimes_{i\in I} F_i(c_i), G(\bigotimes_{i\in I} c_i)\right).\end{aligned}$$

By Lemma 19.3.14, the integrand is naturally isomorphic to $\mathcal{O}(\{F_i(c_i)\}_{i\in I}; G(\bigotimes_{i\in I} c_i))$, as required. □

Thus a multimorphism is a coherently natural family of operations $\{F_i(c_i)\}_{i\in I} \to G(\bigotimes_i c_i)$. Substitution evaluates the outer family on the tensor products of the objects in each inner family and composes the resulting operations in $\mathcal{O}$, as in the ambient Day convolution operad. For $I = \emptyset$, the formula gives $\mathcal{O}(; G(\mathbb{1}_C))$.

Evaluation at the monoidal unit defines a morphism of ∞-operads

$$\mathrm{ev}_{\mathbb{1}_C}\colon \mathcal{D}ay(\mathcal{M}_C, \mathcal{O}) \longrightarrow \mathcal{O}, \qquad F \longmapsto F(\mathbb{1}_C).$$

On multimorphisms, it evaluates a natural family at $c_i = \mathbb{1}_C$ for every $i \in I$, using $\bigotimes_{i\in I} \mathbb{1}_C \simeq \mathbb{1}_C$. Its map on multimorphism animae is therefore the projection from the end to this tuple. Indeed,

the symmetric monoidal functor $* \to C$ selecting $\mathbb{1}_C$ induces $Comm \to \mathcal{M}_C$. Precomposition then gives the required map by Remark 16.1.2, since $\mathcal{D}ay(Comm, O) \simeq O$.

Remark 19.6.4. If $f\colon O' \to \mathcal{P}$ is a cocartesian fibration of ∞-operads, then the product functor $O' \times_{\mathcal{P}} -\colon (\mathrm{Op}_\infty)_{/\mathcal{P}} \to (\mathrm{Op}_\infty)_{/\mathcal{P}}$ admits a right adjoint

$$\mathcal{D}ay_{/\mathcal{P}}(O', -)\colon (\mathrm{Op}_\infty)_{/\mathcal{P}} \to (\mathrm{Op}_\infty)_{/\mathcal{P}},$$

by Lurie [Lur17, Theorem 2.2.6.2 and Construction 2.2.6.7]. For $\mathcal{P} = Comm$ and $O' = \mathcal{M}_C$, this specializes to Theorem 19.6.2.

Exercises

Exercise 19.1 (Basic factorization systems)**.** (1) Let C be an ∞-category. Show that the pairs $(C, C^{\simeq})$ and $(C^{\simeq}, C)$ are factorization systems on C.

(2) In An, show that the essentially surjective morphisms and the monomorphisms form a factorization system.

(3) In Cat_∞, show that the essentially surjective functors and the fully faithful functors form a factorization system.

Exercise 19.2 (Free cocartesian lifts and Yoneda)**.** Let c be an object of an ∞-category C. Identify the free cocartesian completion of $c\colon * \to C$ from Construction 19.2.2, with $\mathcal{R} = C$, with $C_{c/} \to C$. Deduce that, for every cocartesian fibration $D \to C$, evaluation at id_c induces an equivalence

$$\mathrm{ev}_{\mathrm{id}_c}\colon \mathrm{Fun}^{\mathrm{cocart}}_{/C}(C_{c/}, D) \xrightarrow{\sim} D_c,$$

and describe its inverse.

Exercise 19.3. Show that initial functors and left fibrations form a factorization system on Cat_∞. Deduce the dual factorization system consisting of final functors and right fibrations. *Hint:* In the first case, obtain a factorization by freely adjoining cocartesian lifts and geometrically realizing the fibers, and use the description of limits as sections to prove orthogonality.

Exercise 19.4 (The envelope of a trivial operad)**.** Let C be an ∞-category and consider its trivial operad $\mathcal{T}riv_C$. Show that objects of $\mathrm{Env}(\mathcal{T}riv_C)$ are finite tuples of objects of C, with hom animae

$$\mathrm{Hom}_{\mathrm{Env}(\mathcal{T}riv_C)}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) \simeq \coprod_{\sigma\colon I \xrightarrow{\sim} J} \prod_{i\in I} \mathrm{Hom}_C(x_i, y_{\sigma(i)}).$$

Deduce that $\mathrm{Env}(\mathcal{T}riv_C)$ is the free symmetric monoidal ∞-category generated by C: restriction to singleton tuples induces an equivalence $\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{T}riv_C), D) \simeq \mathrm{Fun}(C, D)$ for every symmetric monoidal ∞-category D.

Exercise 19.5 (The envelope of $\mathcal{E}_0$)**.** Identify the morphisms in $\mathrm{Env}(\mathcal{E}_0)$ with injections between the indexing sets and construct a symmetric monoidal equivalence

$$\mathrm{Env}(\mathcal{E}_0) \simeq (\mathrm{Fin}_{\mathrm{inj}}, \amalg).$$

Deduce that $(\mathrm{Fin}_{\mathrm{inj}}, \amalg)$ is the free symmetric monoidal ∞-category generated by an object equipped with a map from the unit.

Exercise 19.6 (Day convolution on arrows)**.** Equip $[1]$ with the symmetric monoidal structure given by the maximum. Let $\mathcal{O}$ be an ∞-operad, let I be a finite set, and let $F_i = (x_i \to y_i)$ and $G = (z \to w)$ be arrows in $U(\mathcal{O})$. Show that

$$\mathcal{D}ay(\mathcal{M}_{([1],\max)}, \mathcal{O})(\{F_i\}_{i\in I}; G) \simeq \mathcal{O}(\{x_i\}_{i\in I}; z) \times_{\mathcal{O}(\{x_i\}_{i\in I}; w)} \mathcal{O}(\{y_i\}_{i\in I}; w),$$

where the maps to the middle anima are postcomposition with $z \to w$ and precomposition with the $x_i \to y_i$. For $\mathcal{O} = \mathcal{M}_D$, deduce that this Day convolution operad is the multimorphism operad of $\mathrm{Ar}(D)$ with its pointwise monoidal structure. No colimit hypotheses on D are needed.

20 Operadic stabilization

For an ∞-category C with finite limits, stabilization gives a stable ∞-category $\mathrm{Sp}(C)$ and a functor $\Omega^\infty\colon \mathrm{Sp}(C) \to C$. Every finite-limit-preserving functor from a stable ∞-category to C factors uniquely through Ω^∞. When C is symmetric monoidal, we would like this factorization to respect multiplicative structures as well.

Following Nikolaus [Nik16], we formulate this extension for ∞-operads $\mathcal{O}$. The required hypothesis is that finite limits in $U(\mathcal{O})$ are *operadic*: their universal property holds for arbitrary multimorphism animae. Day convolution on reduced excisive functors then gives a stable ∞-operad $\mathcal{S}p(\mathcal{O})$ and a map $\Omega^\infty\colon \mathcal{S}p(\mathcal{O}) \to \mathcal{O}$ through which maps from stable ∞-operads factor uniquely, provided they preserve finite operadic limits. For presentably symmetric monoidal C, this recovers the monoidal structure on $\mathrm{Sp}(C) \simeq \mathrm{Sp} \otimes C$ from Section 18.2; in particular, it acquires a universal property among lax symmetric monoidal functors preserving finite limits.

Pointed objects, commutative monoids, and commutative groups admit analogous operadic constructions. We use the monoid construction to show that a tensor product distributing over finite coproducts defines a commutative semiring object in Cat_∞. The proofs for pointing and additivization appear in the final online-only section.

20.1 Operadic (co)limits

Limits and colimits in the underlying category of an ∞-operad need not interact with its multimorphisms. We call them operadic when they satisfy the corresponding universal properties for multimorphism animae.

Definition 20.1.1. Let $\mathcal{O}$ be an ∞-operad and let $F\colon I \to U(\mathcal{O})$ be a functor.

(1) A cone $\eta\colon \mathrm{const}_y \to F$ in $U(\mathcal{O})$ is called an *operadic limit* if for every $n \geq 0$ and all colors $x_1, \ldots, x_n \in \mathcal{O}^\simeq$ the induced map

$$\mathcal{O}((x_1, \ldots, x_n); y) \to \lim_{i \in I} \mathcal{O}((x_1, \ldots, x_n); F(i))$$

is an equivalence of animae.

(2) A cocone $\varepsilon\colon F \to \mathrm{const}_x$ in $U(\mathcal{O})$ is called an *operadic colimit* if for every $n \geq 1$ and all colors $x_2, \ldots, x_n, y \in \mathcal{O}^\simeq$ the induced map

$$\mathcal{O}((x, x_2, \ldots, x_n); y) \to \lim_{i \in I^{\mathrm{op}}} \mathcal{O}((F(i), x_2, \ldots, x_n); y)$$

is an equivalence of animae.

Remark 20.1.2. Taking $n = 1$ shows that operadic limits and colimits are limits and colimits in $U(\mathcal{O})$. We will usually assume that these exist and ask whether they are operadic.

Every object of $\mathrm{Env}(\mathcal{O})$ is a tensor $x_1 \otimes \cdots \otimes x_n$ of colors. By Lemma 19.3.14, maps from this tensor to a color y form the multimorphism anima $\mathcal{O}((x_1, \ldots, x_n); y)$. Thus a cone in $U(\mathcal{O})$ is an operadic limit if and only if it becomes a limit cone under the inclusion $U(\mathcal{O}) \hookrightarrow \mathrm{Env}(\mathcal{O})$.

The colimit condition tests maps into colors after tensoring the cocone with any fixed colors $x_2, \ldots, x_n$. It does not test maps into arbitrary objects of the envelope. In particular, an operadic initial object need not be initial in $\mathrm{Env}(\mathcal{O})$.

Example 20.1.3. If $\mathcal{O} = \mathcal{M}_C$ for a symmetric monoidal ∞-category C, then any limit in C is automatically operadic. This is because the tensor product provides a natural equivalence

$$\mathcal{M}_C((x_1, \ldots, x_n); -) \simeq \mathrm{Hom}_C(x_1 \otimes \cdots \otimes x_n, -),$$

and the functor $\mathrm{Hom}_C(X, -)$ preserves limits for any object X.

Example 20.1.4. The unique color of the trivial operad $\mathcal{T}riv$ is terminal in $U(\mathcal{T}riv) \simeq *$, but it is not operadic terminal: the anima of nullary operations into it is empty.

Example 20.1.5. If $\mathcal{O} = \mathcal{M}_C$ for a symmetric monoidal ∞-category C, then a colimit in C is operadic if and only if it is preserved by the functor $x \otimes -\colon C \to C$ for every object $x \in C$; the argument is similar to Example 20.1.3.

Corollary 20.1.6. *Let $\mathcal{O}$ be a small ∞-operad admitting finite operadic limits. The multiplicative Yoneda embedding identifies $\mathcal{O}$ with a full suboperad of $\mathcal{M}_{\mathrm{PSh}(\mathrm{Env}(\mathcal{O}))}$ whose colors are closed under finite limits in* $\mathrm{PSh}(\mathrm{Env}(\mathcal{O}))$.

Proof. An operadic limit in $\mathcal{O}$ becomes a limit in $\mathrm{Env}(\mathcal{O})$ by Remark 20.1.2. The Yoneda embedding preserves limits, so the multiplicative Yoneda embedding preserves these limits. Full faithfulness follows from Proposition 19.5.3. □

We can now extend pointedness, semiadditivity, additivity, and stability to ∞-operads.

Definition 20.1.7. An ∞-operad $\mathcal{O}$ is called:

(1) *Pointed* if $U(\mathcal{O})$ has a zero object which is both operadic initial and operadic terminal.

(2) *Semiadditive* if $U(\mathcal{O})$ is semiadditive and all finite products and coproducts are operadic.

(3) *Additive* if $U(\mathcal{O})$ is additive and all finite products and coproducts are operadic.

(4) *Stable* if $U(\mathcal{O})$ is stable and all finite limits and colimits are operadic.

This allows us to define various subcategories of the ∞-category Op_∞ of ∞-operads:

- We define $\mathrm{Op}_\infty^* \subseteq \mathrm{Op}_\infty$ as the (non-full) subcategory spanned by ∞-operads with an operadic terminal object and maps preserving them. We denote by $\mathrm{Op}_\infty^{\mathrm{pt}} \subseteq \mathrm{Op}_\infty^*$ the full subcategory spanned by pointed ∞-operads.

- We define $\mathrm{Op}_\infty^{\mathrm{prod}} \subseteq \mathrm{Op}_\infty$ as the (non-full) subcategory spanned by ∞-operads with finite operadic products and maps preserving them. We denote by $\mathrm{Op}_\infty^{\mathrm{sadd}}, \mathrm{Op}_\infty^{\mathrm{add}} \subseteq \mathrm{Op}_\infty^{\mathrm{prod}}$ the full subcategories spanned by semiadditive and additive ∞-operads, respectively.
- We define $\mathrm{Op}_\infty^{\mathrm{lex}} \subseteq \mathrm{Op}_\infty$ as the (non-full) subcategory spanned by ∞-operads with finite operadic limits and maps preserving them. We denote by $\mathrm{Op}_\infty^{\mathrm{st}} \subseteq \mathrm{Op}_\infty^{\mathrm{lex}}$ the full subcategory spanned by stable ∞-operads.

For example, the ∞-operad $\mathcal{M}_{\mathrm{Sp}}$ associated with the symmetric monoidal structure constructed in Section 16.6 is stable.

Example 20.1.8 (Pointed objects)**.** Let C satisfy the hypotheses of Lemma 16.3.4, and equip C_* with its smash product monoidal structure. Its zero object is both operadic terminal and operadic initial, so $\mathcal{M}_{C_*}$ is a pointed ∞-operad. Indeed, operadic terminality is automatic by Example 20.1.3, while operadic initiality says that tensoring with the zero object gives the zero object.

Lemma 20.1.9 (Properties inherited by full suboperads)**.** *Let $\mathcal{O} \subseteq \mathcal{P}$ be a full suboperad.*

(1) If $\mathcal{P}$ is pointed and $U(\mathcal{O})$ contains its zero object, then $\mathcal{O}$ is pointed.

(2) If $\mathcal{P}$ is semiadditive or additive and $U(\mathcal{O})$ is closed under finite biproducts in $U(\mathcal{P})$, then $\mathcal{O}$ is semiadditive or additive, respectively.

(3) If $\mathcal{P}$ is stable and $U(\mathcal{O})$ is closed under finite limits and finite colimits in $U(\mathcal{P})$, then $\mathcal{O}$ is stable.

Proof. In each case the relevant (co)limits in $U(\mathcal{O})$ are inherited from $U(\mathcal{P})$. This gives the asserted property of the underlying ∞-category. Since $\mathcal{P}$ has the corresponding operadic (co)limits and $\mathcal{O} \subseteq \mathcal{P}$ is full, the same multimorphism animae compute the operadic (co)limit conditions in $\mathcal{O}$. □

The inclusions of pointed, semiadditive, additive, and stable ∞-operads into the respective ambient ∞-categories above all admit right adjoints. We begin with stabilization.

20.2 Stabilization of ∞-operads

Given an ∞-operad $\mathcal{O}$ with finite operadic limits, the reduced excisive functors $\mathrm{An}_*^{\mathrm{fin}} \to U(\mathcal{O})$ form the underlying ∞-category of its stabilization, and its multimorphisms are supplied by Day convolution. To prove its stability, we embed $\mathcal{O}$ into a presheaf category, using the multiplicative Yoneda embedding. The universal property will then follow by showing that evaluation at S^0 recovers $\mathcal{O}$ when it is already stable.

Construction 20.2.1. Let $\mathcal{O}$ be an ∞-operad that admits finite operadic limits. Consider the ∞-category $\mathrm{An}_*^{\mathrm{fin}}$ equipped with the smash product monoidal structure from Definition 16.3.5. We define

$$\mathcal{S}p(\mathcal{O}) \quad \subseteq \quad \mathcal{D}ay(\mathcal{M}_{\mathrm{An}_*^{\mathrm{fin}}}, \mathcal{O})$$

to be the full suboperad whose colors are those functors $F\colon \mathrm{An}_*^{\mathrm{fin}} \to U(\mathcal{O})$ that are reduced and excisive. Note that by construction we have

$$U(\mathcal{S}p(\mathcal{O})) \simeq \mathrm{Sp}^{\mathrm{exc}}(U(\mathcal{O})),$$

and so by Proposition 16.5.10 we may identify this underlying ∞-category with the abstract stabilization $\mathrm{Sp}(U(\mathcal{O}))$. The monoidal unit S^0 defines an operad map $\mathcal{C}omm \to \mathcal{M}_{\mathrm{An}^{\mathrm{fin}}_*}$, and restricting the evaluation map along it defines a morphism of ∞-operads

$$\Omega^\infty \colon \mathcal{S}p(\mathcal{O}) \to \mathcal{O}.$$

Lemma 20.2.2. *Let D be a small symmetric monoidal ∞-category and equip $\mathrm{PSh}(D)$ with the Day convolution symmetric monoidal structure. Then $\mathcal{S}p(\mathcal{M}_{\mathrm{PSh}(D)})$ is a stable ∞-operad.*

Proof. By construction of the Day convolution structure on $\mathrm{PSh}(D)$, its multimorphism operad $\mathcal{M}_{\mathrm{PSh}(D)}$ is identified with $\mathcal{D}ay(\mathcal{M}_{D^{\mathrm{op}}}, \mathcal{M}_{\mathrm{An}})$. Thus, using the universal property of Day convolution twice, $\mathcal{D}ay(\mathcal{M}_{\mathrm{An}^{\mathrm{fin}}_*}, \mathcal{M}_{\mathrm{PSh}(D)})$ identifies with

$$\mathcal{D}ay(\mathcal{M}_{D^{\mathrm{op}}}, \mathcal{D}ay(\mathcal{M}_{\mathrm{An}^{\mathrm{fin}}_*}, \mathcal{M}_{\mathrm{An}})).$$

Under this identification, $\mathcal{S}p(\mathcal{M}_{\mathrm{PSh}(D)})$ is obtained by restricting the inner $\mathcal{D}ay(\mathcal{M}_{\mathrm{An}^{\mathrm{fin}}_*}, \mathcal{M}_{\mathrm{An}})$ to reduced excisive functors, using Lemma 19.6.1. By Theorem 16.6.1 and the construction of the symmetric monoidal localization there, this restriction is the multimorphism operad $\mathcal{M}_{\mathrm{Sp}}$. Hence

$$\mathcal{S}p(\mathcal{M}_{\mathrm{PSh}(D)}) \simeq \mathcal{D}ay(\mathcal{M}_{D^{\mathrm{op}}}, \mathcal{M}_{\mathrm{Sp}}).$$

The latter is the multimorphism operad of $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Sp})$ with Day convolution. This functor category is stable, with finite limits and colimits computed pointwise. Its Day tensor product preserves colimits separately in each variable, since it is obtained from the pointwise tensor product of spectra by a colimit-preserving left Kan extension. Thus its finite limits and colimits are operadic, proving the claim. □

Proposition 20.2.3. *Let $\mathcal{O}$ be an ∞-operad that admits finite operadic limits. Then $\mathcal{S}p(\mathcal{O})$ is stable.*

Proof. Put $E := \mathrm{Env}(\mathcal{O})$. By Corollary 20.1.6, the multiplicative Yoneda embedding realizes $\mathcal{O}$ as a full suboperad of $\mathcal{M}_{\mathrm{PSh}(E)}$ and preserves finite operadic limits. It therefore induces a fully faithful functor

$$\mathrm{Sp}^{\mathrm{exc}}(U(\mathcal{O})) \longrightarrow \mathrm{Sp}^{\mathrm{exc}}(\mathrm{PSh}(E))$$

preserving finite limits, since these are computed pointwise. Both ∞-categories are stable by Proposition 16.5.7, so this functor is exact and its essential image is closed under finite limits and colimits. By Lemma 19.6.1, the corresponding full suboperad of $\mathcal{S}p(\mathcal{M}_{\mathrm{PSh}(E)})$ is $\mathcal{S}p(\mathcal{O})$. The ambient ∞-operad is stable by Lemma 20.2.2, so Lemma 20.1.9 proves the claim. □

For a stable ∞-operad $\mathcal{O}$, evaluation at S^0 already gives an equivalence on underlying categories (Lemma 16.5.8). We must show that it also induces equivalences on multimorphism animae. By Theorem 19.6.3, these consist of coherent families of operations

$$F_1(K_1), \dots, F_n(K_n) \longrightarrow G(K_1 \wedge \cdots \wedge K_n).$$

We will show that such a family is determined by its value at $K_1 = \cdots = K_n = S^0$. The following lemma allows us to evaluate one variable at a time.

Lemma 20.2.4. *Let $T\colon (\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}} \times \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}$ be a functor such that $T(-, L)$ sends finite colimits to limits for every L, and $T(K, -)$ is reduced and excisive for every K. Then the projection*

$$\int_{K \in \mathrm{An}_*^{\mathrm{fin}}} T(K, K) \to T(S^0, S^0),$$

is an equivalence.

Proof. By Lemma 23.3.9, the condition on the first variable gives a functor $F\colon \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}_*$ and natural identifications

$$T(K, L) \simeq \mathrm{Hom}_{\mathrm{An}_*}(K, F(L)).$$

Since the restricted Yoneda embedding preserves and reflects limits, the condition on the second variable says that F is reduced and excisive. Let $j\colon \mathrm{An}_*^{\mathrm{fin}} \hookrightarrow \mathrm{An}_*$ be the inclusion. The end formula for natural transformations and Lemma 16.5.14 give

$$\begin{aligned}\int_{K \in \mathrm{An}_*^{\mathrm{fin}}} T(K, K) &\simeq \mathrm{Nat}(j, F) \simeq \mathrm{Nat}(P_1 j, F) \\ &\simeq \mathrm{Hom}_{\mathrm{An}_*}(S^0, F(S^0)) \simeq T(S^0, S^0).\end{aligned}$$

The composite sends a transformation $j \to F$ to its component at S^0, so it is the stated projection. □

Proposition 20.2.5. *If $\mathcal{O}$ is stable, then the evaluation map*

$$\Omega^\infty\colon \mathcal{S}p(\mathcal{O}) \to \mathcal{O}$$

is an equivalence of ∞-operads.

Proof. On underlying ∞-categories this is Lemma 16.5.8. By Lemma 14.1.11, it remains to prove that Ω^∞ is fully faithful.

Let I be a finite set, and let $\{F_i\colon \mathrm{An}_*^{\mathrm{fin}} \to U(\mathcal{O})\}_{i \in I}$ and $G\colon \mathrm{An}_*^{\mathrm{fin}} \to U(\mathcal{O})$ be reduced excisive functors. By Theorem 19.6.3, the source of the induced map on multimorphism animae is

$$\int_{\{K_i\}_{i \in I} \in (\mathrm{An}_*^{\mathrm{fin}})^I} \mathcal{O}(\{F_i(K_i)\}_{i \in I}; G(\bigwedge\nolimits_{i \in I} K_i)).$$

By naturality of Theorem 19.6.3 in the evaluation operad map and the identification $\bigwedge_{i \in I} S^0 \simeq S^0$, the map induced by Ω^∞ is the projection from this end to the component where all K_i are equal to S^0.

We evaluate the variables at S^0 one at a time. In order to iterate the argument, we allow additional fixed colors $z_1, \dots, z_m$ among the inputs and prove the assertion by induction on $|I|$. The case $I = \emptyset$ is immediate. For $I \neq \emptyset$, choose $i_0 \in I$. By Fubini, the end is $\int_K T(K, K)$ for the bifunctor

$$T(K, L) := \int_{\{K_i\}_{i \neq i_0} \in (\mathrm{An}_*^{\mathrm{fin}})^{I \setminus \{i_0\}}} \mathcal{O}(z_1, \dots, z_m, F_{i_0}(K), \{F_i(K_i)\}_{i \neq i_0}; G(L \wedge \bigwedge_{i \neq i_0} K_i)).$$

We check the two hypotheses of Lemma 20.2.4. Since $U(\mathcal{O})$ is stable, the reduced excisive functor F_{i_0} preserves finite colimits. These colimits are operadic, so $T(-, L)$ sends them to limits.

For the other variable, smashing with the fixed $\bigwedge_{i \neq i_0} K_i$ preserves finite colimits. Applying G gives a reduced excisive functor, and its finite limits are operadic. Thus $T(K, -)$ is reduced and excisive.

In both verifications, taking the end preserves the relevant limits. These arguments also apply with the fixed inputs $z_1, \dots, z_m$, since operadic limits and colimits are tested on multimorphism animae with arbitrary other inputs.

By the lemma, the projection $\int_K T(K,K) \to T(S^0, S^0)$ is an equivalence. The latter is the end over the remaining variables, with $F_{i_0}(S^0)$ now a fixed input. The induction hypothesis identifies this end with the multimorphism anima obtained by setting all remaining $K_i = S^0$. Taking $m = 0$, we obtain

$$\mathcal{O}(\{F_i(S^0)\}_{i \in I}; G(S^0)).$$

Each step is the projection to the indicated component, so the resulting equivalence is the map induced by Ω^∞. This proves full faithfulness. □

Theorem 20.2.6 (Stabilization of ∞-operads)**.** *Let $\mathcal{O}$ be an ∞-operad that admits finite operadic limits. Then the ∞-operad $\mathcal{S}p(\mathcal{O})$ is stable, and the resulting functor*

$$\mathcal{S}p \colon \mathrm{Op}_\infty^{\mathrm{lex}} \to \mathrm{Op}_\infty^{\mathrm{st}}$$

is right adjoint to the inclusion functor.

Proof. Stability is Proposition 20.2.3. Postcomposition with a map in $\mathrm{Op}_\infty^{\mathrm{lex}}$ preserves reduced excisive functors. Finite limits of these functors are computed pointwise, so postcomposition and evaluation at S^0 preserve finite operadic limits. Thus $\mathcal{S}p$ is a functor on $\mathrm{Op}_\infty^{\mathrm{lex}}$, with a natural transformation $\Omega^\infty \colon \mathcal{S}p \to \mathrm{id}$.

By Proposition 20.2.5, this transformation is an equivalence on stable ∞-operads. For the right adjoint criterion dual to Proposition 22.7.9, it remains to show that $\mathcal{S}p(\Omega^\infty_{\mathcal{O}})$ is an equivalence for any $\mathcal{O} \in \mathrm{Op}_\infty^{\mathrm{lex}}$. The universal property of Day convolution identifies $\mathcal{S}p(\mathcal{S}p(\mathcal{O}))$ with the full suboperad of $\mathcal{D}ay(\mathcal{M}_{\mathrm{An}_*^{\mathrm{fin}} \times \mathrm{An}_*^{\mathrm{fin}}}, \mathcal{O})$ on bifunctors that are reduced and excisive separately. The maps $\mathcal{S}p(\Omega^\infty_{\mathcal{O}})$ and $\Omega^\infty_{\mathcal{S}p(\mathcal{O})}$ evaluate at S^0 in the two respective variables, so interchanging those variables exchanges the maps. The latter is an equivalence because $\mathcal{S}p(\mathcal{O})$ is stable; hence so is the former. □

For a presentably symmetric monoidal ∞-category C, Theorem 18.2.5 equips $\mathrm{Sp}(C) \simeq \mathrm{Sp} \otimes C$ with a symmetric monoidal structure. Its multimorphism operad is the operadic stabilization constructed above.

Corollary 20.2.7 (Presentable operadic stabilization, [Nik16, Proposition 4.9])**.** *Let C be a presentably symmetric monoidal ∞-category. For the presentably symmetric monoidal structure on $\mathrm{Sp}(C) \simeq \mathrm{Sp} \otimes C$, there is a natural equivalence*

$$\mathcal{S}p(\mathcal{M}_C) \quad \simeq \quad \mathcal{M}_{\mathrm{Sp}(C)}.$$

Proof. By Proposition 23.2.7(3), the reduced excisive functors form a presentable reflective subcategory of $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$. Equip the latter with its presentably symmetric monoidal Day convolution structure. For an arbitrary functor F and a reduced excisive functor G, Proposition 16.2.7 gives

$$\underline{\mathrm{Hom}}_{\mathrm{Day}}(F,G)(K) \simeq \int_{L \in \mathrm{An}_*^{\mathrm{fin}}} \underline{\mathrm{Hom}}_C(F(L), G(K \wedge L)).$$

This is reduced excisive: smashing with L preserves finite colimits, while internal homs and ends preserve limits. Thus Lemma 14.7.3 and Proposition 14.7.2 make the reflection symmetric monoidal, with a presentably symmetric monoidal target whose multimorphism operad is $\mathcal{S}p(\mathcal{M}_C)$.

Left Kan extension along $S^0 \colon * \to \mathrm{An}_*^{\mathrm{fin}}$ is symmetric monoidal by Lemma 16.2.5. Its composite with the reflection is left adjoint to evaluation at S^0 on reduced excisive functors, hence is the stabilization functor $C \to \mathrm{Sp}(C)$. This functor is therefore symmetric monoidal.

Since $\mathrm{Sp}(C)$ is stable, the localization associated to the mode Sp factors the stabilization functor through a colimit-preserving symmetric monoidal functor from $\mathrm{Sp} \otimes C$ (Proposition 18.2.8). On underlying categories this is the stabilization equivalence $\mathrm{Sp} \otimes C \simeq \mathrm{Sp}(C)$ of Theorem 18.2.5. Hence it is an equivalence of symmetric monoidal ∞-categories, as claimed. □

Remark 20.2.8. For a general symmetric monoidal ∞-category C with finite limits, $\mathcal{S}p(\mathcal{M}_C)$ need not be represented by a symmetric monoidal structure on $\mathrm{Sp}(C)$ (see Exercise 20.2).

Recall from Definition 11.1.16 that a symmetric monoidal ∞-category D is *stably symmetric monoidal* if it is stable and its tensor product is exact in each variable. Equivalently, its multimorphism operad $\mathcal{M}_D$ is stable. The operadic stabilization adjunction therefore gives the following universal property whenever $\mathcal{S}p(\mathcal{M}_C)$ is represented by a symmetric monoidal structure.

Corollary 20.2.9. *Let C be a symmetric monoidal ∞-category with finite limits whose operadic stabilization $\mathcal{S}p(\mathcal{M}_C)$ is represented by a symmetric monoidal structure on* $\mathrm{Sp}(C)$*. For every stably symmetric monoidal ∞-category D, evaluation induces an equivalence*

$$\Omega^\infty \circ - \colon \mathrm{Fun}^{\otimes\text{-lax},\mathrm{lex}}(D, \mathrm{Sp}(C)) \xrightarrow{\sim} \mathrm{Fun}^{\otimes\text{-lax},\mathrm{lex}}(D, C),$$

where lex *means that the underlying functor preserves finite limits.*

Proof. Since $\mathcal{M}_D$ is stable, Theorem 20.2.6 shows that postcomposition with the counit Ω^∞ induces an equivalence on groupoid cores. To include the monoidal natural transformations, we apply the same argument with $\mathrm{Fun}(E, C)$ in place of C, for any small ∞-category E.

A spectrum object in $\mathrm{Fun}(E, C)$ is an E-diagram of spectrum objects in C, since reducedness and excisiveness are checked pointwise in E. Under this identification, the pointwise tensor product on $\mathrm{Fun}(E, \mathrm{Sp}(C))$ represents $\mathcal{S}p(\mathcal{M}_{\mathrm{Fun}(E,C)})$. Indeed, the Day formula of Theorem 19.6.3 computes the I-ary multimorphisms on the two sides by taking the ends over E and $(\mathrm{An}_*^{\mathrm{fin}})^I$ in opposite orders. Interchanging these ends respects substitution, since the operations compose pointwise in E. Evaluation at S^0 likewise applies Ω^∞ to each object of the diagram.

By Lemma 14.4.13, a lax symmetric monoidal functor $D \to \mathrm{Fun}(E, C)$ is an E-diagram of lax symmetric monoidal functors $D \to C$, and preservation of finite limits is again checked pointwise. Thus postcomposition with Ω^∞ induces an equivalence between the groupoid cores of the two categories of E-diagrams. As this holds for every E, the result follows from the Yoneda lemma. For the case $C = \mathrm{An}$, see also [Nik16, Corollary 4.13]. □

The other right adjoints from Section 20.1 are obtained by adapting the construction of stabilization.

Theorem 20.2.10 (Additivization of ∞-operads, [Nik16, Theorem 5.7])**.** *Let $\mathcal{O}$ be an ∞-operad with an operadic terminal object, and assume finite operadic products for the monoid and group constructions. The ∞-categories of pointed objects, commutative monoids, and commutative groups in $U(\mathcal{O})$ admit natural operad structures $\mathcal{O}_*$, $\mathcal{CM}on(\mathcal{O})$, and $\mathcal{CG}rp(\mathcal{O})$, respectively. The resulting functors*

$$(-)_* \colon \mathrm{Op}_\infty^* \to \mathrm{Op}_\infty^{\mathrm{pt}}, \qquad \mathcal{CM}on \colon \mathrm{Op}_\infty^{\mathrm{prod}} \to \mathrm{Op}_\infty^{\mathrm{sadd}} \qquad \textit{and} \qquad \mathcal{CG}rp \colon \mathrm{Op}_\infty^{\mathrm{prod}} \to \mathrm{Op}_\infty^{\mathrm{add}}$$

are right adjoint to the respective inclusion functors.

For pointed objects, replace $\mathrm{An}_*^{\mathrm{fin}}$ by $[1]$ with its minimum tensor product and use diagrams whose value at 0 is terminal. For commutative monoids and groups, use finite-product-preserving diagrams on Span(Fin), with its tensor product induced by cartesian products of finite sets, imposing the grouplike condition for groups. Restricting Day convolution to these diagrams gives the three ∞-operads; evaluation at the monoidal unit gives the counits. For a presentable ∞-category C whose cartesian product preserves colimits separately in each variable, these constructions on $\mathcal{M}_{(C,\times)}$ recover the smash product on C_* and the monoidal structures on $\mathrm{CMon}(C)$ and $\mathrm{CGrp}(C)$ from Chapter 16. The constructions and proof of Theorem 20.2.10 are given in Section 20.4.

20.3 Semiring objects from distributivity

A symmetric monoidal ∞-category whose tensor product distributes over finite coproducts has two compatible operations: addition by coproduct and multiplication by tensor product. We show that these operations define a coherent commutative semiring object in Cat_∞. To do so, we use the tensor product $\otimes_{\mathrm{coprod}}$ from Example 18.1.2, which represents functors preserving finite coproducts separately in each variable.

Lemma 20.3.1. *The ∞-category $\mathrm{Cat}_\infty^{\mathrm{coprod}}$ is presentable and semiadditive. Its zero object is the terminal ∞-category, and the cartesian product $C \times D$ is also the coproduct of C and D. Moreover, the tensor product $\otimes_{\mathrm{coprod}}$ makes $\mathrm{Cat}_\infty^{\mathrm{coprod}}$ presentably symmetric monoidal.*

Proof. Presentability is the finite-coproduct case of [Lur17, Lemma 4.8.4.2]. Finite products are computed in Cat_∞ and therefore given by cartesian products of ∞-categories. The terminal ∞-category is also initial in $\mathrm{Cat}_\infty^{\mathrm{coprod}}$, since a finite-coproduct-preserving functor from it to C must select an initial object of C.

Let 0_C and 0_D denote initial objects. For every $E \in \mathrm{Cat}_\infty^{\mathrm{coprod}}$, restriction along $c \mapsto (c, 0_D)$ and $d \mapsto (0_C, d)$ induces an equivalence

$$\mathrm{Fun}^{\amalg}(C \times D, E) \xrightarrow{\sim} \mathrm{Fun}^{\amalg}(C, E) \times \mathrm{Fun}^{\amalg}(D, E).$$

An inverse sends (F, G) to the functor $(c, d) \mapsto F(c) \amalg G(d)$. This preserves finite coproducts because the coproduct functor $E \times E \to E$ is left adjoint to the diagonal. The two composites are naturally isomorphic to the identity, using $(c, d) \simeq (c, 0_D) \amalg (0_C, d)$ and preservation of initial objects. Thus $C \times D$ is also a coproduct in $\mathrm{Cat}_\infty^{\mathrm{coprod}}$, proving semiadditivity.

Finally, the universal property of $\otimes_{\mathrm{coprod}}$ gives a right adjoint $\mathrm{Fun}^{\amalg}(D, -)$ to $- \otimes_{\mathrm{coprod}} D$, with finite coproducts in this functor category computed pointwise. Hence tensoring preserves colimits, proving the final assertion. □

Proposition 20.3.2 (Cocartesian semiring construction)**.** *The functor*

$$\mathrm{Cat}_\infty^{\mathrm{coprod}} \longrightarrow \mathrm{CMon}(\mathrm{Cat}_\infty), \qquad C \longmapsto (C, \amalg),$$

has a canonical lax symmetric monoidal refinement, where the source carries $\otimes_{\mathrm{coprod}}$ and the target carries the symmetric monoidal structure of Proposition 16.4.1.

Proof. The forgetful functor $U\colon \mathrm{Cat}_\infty^{\mathrm{coprod}} \to \mathrm{Cat}_\infty$ is canonically lax symmetric monoidal: by Example 18.1.2, the multimorphisms in the source are functors preserving finite coproducts separately in each variable, and their inclusion into all functors is an operad map.

By Lemma 20.3.1, the multimorphism operad of $\mathrm{Cat}_\infty^{\mathrm{coprod}}$ is semiadditive. Moreover, U preserves finite products, since they are computed by cartesian products of ∞-categories on both sides. The adjunction of Theorem 20.2.10 therefore gives a unique lift

$$\begin{array}{ccc} & & \mathcal{CM}on\big(\mathcal{M}_{(\mathrm{Cat}_\infty,\times)}\big) \\ & \nearrow & \downarrow \\ \mathcal{M}_{\mathrm{Cat}_\infty^{\mathrm{coprod}}} & \longrightarrow & \mathcal{M}_{(\mathrm{Cat}_\infty,\times)}. \end{array}$$

The upper-right ∞-operad is $\mathcal{M}_{\mathrm{CMon}(\mathrm{Cat}_\infty)}$ by Corollary 20.4.5. The lift therefore gives a lax symmetric monoidal functor to $\mathrm{CMon}(\mathrm{Cat}_\infty)$. On C, its commutative monoid structure is induced by the biproducts in $\mathrm{Cat}_\infty^{\mathrm{coprod}}$: the unit selects 0_C, and addition is the codiagonal $C \times C \simeq C \oplus C \to C$. By the preceding lemma, this is the coproduct functor $\amalg\colon C \times C \to C$, as required. □

Applying this functor to commutative algebras gives the following result.

Proposition 20.3.3. *Let C be a small symmetric monoidal ∞-category with finite coproducts, and assume that its tensor product preserves finite coproducts separately in each variable. Then C canonically defines an object of*

$$\mathrm{CRig}(\mathrm{Cat}_\infty) = \mathrm{CAlg}(\mathrm{CMon}(\mathrm{Cat}_\infty)).$$

Its addition is the coproduct, with zero the initial object, and its multiplication is the tensor product, with unit $\mathbb{1}_C$.

Proof. By Example 18.1.2, the given symmetric monoidal structure makes C a commutative algebra in $(\mathrm{Cat}_\infty^{\mathrm{coprod}}, \otimes_{\mathrm{coprod}})$. Applying the lax symmetric monoidal functor of Proposition 20.3.2 produces a commutative algebra in $\mathrm{CMon}(\mathrm{Cat}_\infty)$, whose underlying commutative monoid is the cocartesian monoidal structure $(C, \amalg)$. □

Corollary 20.3.4. *Under the hypotheses of Proposition 20.3.3, the groupoid core $C^{\simeq}$ is canonically an object of* $\mathrm{CRig}(\mathrm{An})$.

Proof. Apply the lax symmetric monoidal groupoid core functor of Proposition 16.4.6 to the commutative algebra supplied by Proposition 20.3.3. □

20.4 Pointing and additivization (online only)

Pointed objects, commutative monoids, and commutative groups are obtained by restricting Day convolution to suitable diagrams. In place of $\mathrm{An}_*^{\mathrm{fin}}$, we use $[1]$ for pointed objects and $\mathrm{Span}(\mathrm{Fin})$ for monoids and groups. Evaluation at their monoidal units gives the counits of the adjunctions in Theorem 20.2.10.

Definition 20.4.1. If $\mathcal{O}$ is an ∞-operad with an operadic terminal object, equip $[1]$ with the symmetric monoidal structure given by the minimum, as used in the proof of Lemma 16.3.3, and denote by

$$\mathcal{O}_* \quad \subseteq \quad \mathcal{D}ay(\mathcal{M}_{[1]}, \mathcal{O})$$

the full suboperad whose colors are those functors $F\colon [1] \to U(\mathcal{O})$ satisfying $F(0) \simeq *$. If in addition $\mathcal{O}$ has finite operadic products, equip Span(Fin) with the symmetric monoidal structure induced by the cartesian product of finite sets, and denote by

$$\mathcal{CG}rp(\mathcal{O}) \quad \subseteq \quad \mathcal{CM}on(\mathcal{O}) \quad \subseteq \quad \mathcal{D}ay(\mathcal{M}_{\mathrm{Span(Fin)}}, \mathcal{O})$$

the full suboperads whose colors are those functors $F\colon \mathrm{Span(Fin)} \to U(\mathcal{O})$ that are commutative groups or commutative monoids in $U(\mathcal{O})$, respectively.

The tensor product on Span(Fin) is the cartesian product of finite sets, as in Proposition 16.4.1. Its categorical products, which the functors defining commutative monoids preserve, are disjoint unions by Lemma 13.3.8.

By construction, their underlying ∞-categories are $U(\mathcal{O})_*$, $\mathrm{CMon}(U(\mathcal{O}))$, and $\mathrm{CGrp}(U(\mathcal{O}))$, respectively.

Lemma 20.4.2 (Day convolution on pointed objects)**.** *Let C be a symmetric monoidal ∞-category satisfying the hypotheses of Lemma 16.3.4, and denote the resulting tensor on C_* by $\wedge$. Then there is a natural equivalence*

$$(\mathcal{M}_C)_* \simeq \mathcal{M}_{(C_*,\wedge)}.$$

Proof. The ∞-operad $(\mathcal{M}_C)_*$ is the full suboperad of the Day convolution operad on $\mathrm{Ar}(C)$ spanned by the arrows with source the terminal object. The monoidal localization

$$\mathrm{cofib}\colon \mathrm{Ar}(C)^{\square} \longrightarrow (C_*, \wedge)$$

of Lemma 16.3.4 identifies this full suboperad with the multimorphism operad of its local objects. □

Lemma 20.4.3. *If $\mathcal{O}$ is pointed, then evaluation at the target defines an equivalence of ∞-operads*

$$\mathrm{ev}_1\colon \mathcal{O}_* \to \mathcal{O}.$$

Proof. On underlying ∞-categories, evaluation identifies the ∞-category of arrows from the zero object with $U(\mathcal{O})$. It remains to prove full faithfulness on multimorphism animae.

Let $\{F_i\}_{i\in I}$ and G be colors of $\mathcal{O}_*$. By Theorem 19.6.3, their multimorphism anima is an end over $[1]^I$. We evaluate its variables at 1 one at a time. By Fubini, each step is the end of a bifunctor $T\colon [1]^{\mathrm{op}} \times [1] \to \mathrm{An}$, allowing additional fixed colors among the inputs. This end is the pullback

$$T(0,0) \times_{T(0,1)} T(1,1).$$

Both $T(0,0)$ and $T(0,1)$ are contractible because $F_i(0)$ is the zero object and hence operadic initial. Thus this pullback is equivalent to $T(1,1)$. Fubini for ends and induction, with the values $F_i(1)$ already obtained treated as fixed inputs, identify the original end with

$$\mathcal{O}(\{F_i(1)\}_{i\in I}; G(1)).$$

By naturality of the end formula, this equivalence is the map induced by evaluation at 1. □

For commutative monoids, we use the following finite-product analogue of Lemma 20.2.4.

Lemma 20.4.4. *Let* $T\colon \mathrm{Span}(\mathrm{Fin})^{\mathrm{op}} \times \mathrm{Span}(\mathrm{Fin}) \to \mathrm{An}$ *be a functor such that* $T(-, L)$ *sends finite coproducts to products for every* L*, and* $T(K, -)$ *preserves finite products for every* K*. Then the projection*

$$\int_{K \in \mathrm{Span}(\mathrm{Fin})} T(K, K) \longrightarrow T(1, 1)$$

is an equivalence, where 1 *denotes a singleton.*

Proof. Put $B := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin})^{\mathrm{op}}, \mathrm{An})$. Reversal of spans identifies B with $\mathrm{CMon}(\mathrm{An})$ by Lemma 13.1.15, so B is semiadditive by Proposition 5.3.20. Since finite products and coproducts agree in $\mathrm{Span}(\mathrm{Fin})$, the Yoneda embedding factors through a finite-product-preserving functor $j\colon \mathrm{Span}(\mathrm{Fin}) \to B$. The hypotheses on T give another such functor $F\colon \mathrm{Span}(\mathrm{Fin}) \to B$, defined by $F(L) = T(-, L)$. Yoneda and Proposition 5.3.17 give

$$\int_K T(K, K) \simeq \mathrm{Nat}(j, F) \simeq \mathrm{Hom}_B(j(1), F(1)) \simeq T(1, 1).$$

The last two maps evaluate a natural transformation at 1 and then its component on id_1. Thus the composite is the stated projection. □

Proof of Theorem 20.2.10. We first prove the pointed case. Put $E := \mathrm{Env}(\mathcal{O})$. The multiplicative Yoneda embedding and Lemma 19.6.1 give a fully faithful map

$$\mathcal{O}_* \hookrightarrow (\mathcal{M}_{\mathrm{PSh}(E)})_*.$$

By Lemma 20.4.2, the target is the multimorphism operad of the pointed presheaf category and is therefore pointed by Example 20.1.8. The suboperad $\mathcal{O}_*$ contains its zero object, since the multiplicative Yoneda embedding preserves the operadic terminal object of $\mathcal{O}$. It follows from Lemma 20.1.9(1) that $\mathcal{O}_*$ is pointed.

Evaluation at the target gives a natural operad map $\mathcal{O}_* \to \mathcal{O}$. If $\mathcal{O}$ is pointed, this map is an equivalence by Lemma 20.4.3. Applying the pointed-object construction a second time gives the same conclusion because $\mathcal{O}_*$ is pointed. Interchanging the two coordinates of $[1] \times [1]$ exchanges the two ways of evaluating a doubly pointed object at 1, so both whiskerings of the evaluation transformation are equivalences. The dual of Proposition 22.7.9, applied in $(\mathrm{Op}_\infty^*)^{\mathrm{op}}$, therefore identifies $(-)_*$ as right adjoint to the inclusion of pointed ∞-operads.

For commutative monoids and groups, apply the same embedding into $\mathcal{M}_{\mathrm{PSh}(E)}$. Proposition 16.4.1 constructs their Day convolution monoidal structures in $\mathrm{PSh}(E)$. Their underlying categories are semiadditive and additive by Proposition 5.3.20, and their tensor products preserve colimits separately in both variables. Hence their finite biproducts are operadic. These biproducts are computed pointwise. Since the multiplicative Yoneda image is closed under finite products, the full suboperads consisting of objects valued in that image are closed under these biproducts. They are $\mathcal{CM}on(\mathcal{O})$ and $\mathcal{CG}rp(\mathcal{O})$, so Lemma 20.1.9(2) shows that the former is semiadditive and the latter additive.

Suppose now that $\mathcal{O}$ is semiadditive. Evaluation at 1 gives an equivalence on underlying categories by Proposition 5.3.17. By Theorem 19.6.3, its map on multimorphism animae is the projection

$$\int_{\{K_i\} \in \mathrm{Span}(\mathrm{Fin})^I} \mathcal{O}(\{F_i(K_i)\}_{i \in I}; G(\prod_{i \in I} K_i)) \longrightarrow \mathcal{O}(\{F_i(1)\}_{i \in I}; G(1)).$$

We apply Lemma 20.4.4 one variable at a time, as in the proof of Proposition 20.2.5. Each F_i preserves finite coproducts: a finite-product-preserving functor between semiadditive ∞-categories preserves zero morphisms and hence the biproduct injections. Since these coproducts in $\mathcal{O}$ are operadic, the contravariant variable sends them to products. For the covariant variable, $(-) \times R$ preserves disjoint unions in Span(Fin) for every finite set R, hence preserves its finite products. Applying G and using operadic products gives the other hypothesis of the lemma. Taking ends preserves these products. The same arguments allow additional fixed inputs, so induction identifies the displayed projection with an equivalence. For $I = \emptyset$, it is already the identity of $\mathcal{O}(;G(1))$. Thus $\mathcal{CM}on(\mathcal{O}) \to \mathcal{O}$ is an equivalence of ∞-operads.

If $\mathcal{O}$ is additive, every commutative monoid in $U(\mathcal{O})$ is grouplike by Proposition 5.3.17, so the same conclusion holds for $\mathcal{CG}rp(\mathcal{O}) \to \mathcal{O}$.

Finally, postcomposition with maps preserving finite operadic products preserves commutative monoids and groups, and their finite products are computed pointwise. Thus both constructions and their evaluation maps belong to $\mathrm{Op}_\infty^{\mathrm{prod}}$. Applying either construction twice gives functors on Span(Fin) $\times$ Span(Fin) satisfying the respective monoid or group conditions in each variable. Interchanging the variables exchanges the two evaluation maps. Since the once-constructed ∞-operads are semiadditive or additive, the preceding paragraph shows that both whiskerings of evaluation are equivalences. The dual of Proposition 22.7.9 therefore gives the two remaining right adjoints. □

Corollary 20.4.5. *Let C be a presentably symmetric monoidal ∞-category. With the smash product on C_*, the localized Day convolution structures on* $\mathrm{CMon}(C)$ *and* $\mathrm{CGrp}(C)$*, and the symmetric monoidal structure on* $\mathrm{Sp}(C) \simeq \mathrm{Sp} \otimes C$*, there are natural equivalences*

$$(\mathcal{M}_C)_* \simeq \mathcal{M}_{(C_*,\wedge)}, \qquad \mathcal{CM}on(\mathcal{M}_C) \simeq \mathcal{M}_{\mathrm{CMon}(C)},$$

$$\mathcal{CG}rp(\mathcal{M}_C) \simeq \mathcal{M}_{\mathrm{CGrp}(C)}, \qquad \mathcal{S}p(\mathcal{M}_C) \simeq \mathcal{M}_{\mathrm{Sp}(C)}.$$

Taking $C = (\mathrm{An}, \times)$ recovers the four symmetric monoidal structures of Chapter 16.

Proof. The pointed case is Lemma 20.4.2. By Definition 20.4.1, $\mathcal{CM}on(\mathcal{M}_C)$ is the full Day convolution suboperad on product-preserving functors Span(Fin) $\to C$. These are precisely the local objects of the symmetric monoidal localization onto $\mathrm{CMon}(C)$ in Proposition 16.4.1, so this suboperad is $\mathcal{M}_{\mathrm{CMon}(C)}$. The same argument with grouplike objects gives the assertion for $\mathrm{CGrp}(C)$. The stable case is Corollary 20.2.7. □

Exercises

Exercise 20.1 (A stable suboperad of dualizable objects)**.** Let C be a stably symmetric monoidal ∞-category. Use Exercise 11.2 to show that the full suboperad of $\mathcal{M}_C$ on the dualizable objects is stable. Deduce that the full suboperad of $\mathcal{M}_{\mathrm{Sp}}$ on the finite spectra is stable.

Exercise 20.2 (A stabilization that is not represented)**.** Let C be a nonzero stable ∞-category, and consider its cartesian operad $\mathcal{O}^\times(C)$. Show that evaluation identifies the underlying category of $\mathcal{S}p(\mathcal{O}^\times(C))$ with C, but every multimorphism anima of arity other than one is contractible. Deduce that $\mathcal{S}p(\mathcal{O}^\times(C))$ is not represented by a symmetric monoidal structure on C. *Hint:* Use finite

biproducts to split the multimorphism end, then evaluate each factor in a variable absent from its input at the initial pointed anima.

21 Monoidal Dwyer–Kan localizations and derived categories

Let R be a commutative ring and let A be a differential graded R-algebra with multiplication μ. How does its multiplication give a multiplication in the derived category? Writing $\gamma\colon \mathrm{Ch}(R) \to \mathrm{D}(R)$ for the localization functor, we would like to form the composite

$$\gamma A \otimes^{\mathbf{L}}_{R} \gamma A \longrightarrow \gamma(A \otimes_R A) \xrightarrow{\gamma(\mu)} \gamma A.$$

For A bounded below, we can describe the first map using Definition 6.6.25. Choose a projective resolution $P \to A$. Then $P \otimes_R P$ represents the derived tensor product, and the resolution map induces $P \otimes_R P \to A \otimes_R A$. We will show that these maps, for arbitrary pairs of complexes, equip γ with a lax symmetric monoidal structure. It then follows that the displayed multiplication makes γA an associative algebra in $\mathrm{D}(R)$, and a commutative algebra if A is commutative (Corollary 21.2.3).

More generally, suppose we wish to invert a class of morphisms W in a symmetric monoidal ∞-category C. If tensoring with any object preserves W, the tensor product descends to $C[W^{-1}]$ and the localization functor is symmetric monoidal (Proposition 21.1.1). For chain complexes this hypothesis can fail even when tensoring with a module concentrated in degree zero, as we saw for $\mathbb{Z}/2$ in Example 6.6.24. Following Nikolaus and Scholze [NS18, Appendix A], we instead invert the tuples of weak equivalences in each fiber of $C^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$. When the tensor products admit absolute left derived functors compatible with composition, these derived functors give the transport along spans in the localized ∞-category. In this way we obtain both the symmetric monoidal structure on $C[W^{-1}]$ and the lax symmetric monoidal localization functor to it (Theorem 21.1.6).

We apply this construction to unbounded chain complexes. The resulting tensor product on $\mathrm{D}(R)$ extends the bounded below derived tensor product of Chapter 6 and preserves colimits separately in each variable (Propositions 21.2.2 and 21.2.4). Its unit $R[0]$ is a compact generator. The compact objects are the perfect complexes, and we will see that their Ind-completion recovers $\mathrm{D}(R)$ together with its tensor product (Corollary 21.2.6). These are the facts about $\mathrm{D}(R)$ used in Corollary 8.2.4 to identify it, by Morita theory, with the ∞-category of modules over the Eilenberg–MacLane ring spectrum of R.

21.1 Monoidal Dwyer–Kan localizations

We first consider the case in which tensoring with any object of C preserves the morphisms to be inverted.

Proposition 21.1.1 (Tensor-stable monoidal localization)**.** *Let C be a symmetric monoidal ∞-category, and let W be a class of morphisms in C such that, for every object $X \in C$, the functor $X \otimes -$ sends morphisms in W to morphisms in W.*

(1) *The localization $C[W^{-1}]$ inherits a unique symmetric monoidal structure turning the localization functor $\gamma\colon C \to C[W^{-1}]$ into a symmetric monoidal functor.*

(2) *For any other symmetric monoidal ∞-category D, restriction along γ induces a fully faithful functor*

$$\mathrm{Fun}^{\otimes}(C[W^{-1}], D) \hookrightarrow \mathrm{Fun}^{\otimes}(C, D)$$

whose essential image consists of those symmetric monoidal functors $C \to D$ which invert the morphisms in W.

Proof. We may replace W by the wide subcategory it generates. This leaves the localization unchanged, and tensoring with any object still preserves W. Let $\mathrm{RelCat}_\infty \subseteq \mathrm{Ar}(\mathrm{Cat}_\infty)$ be the full subcategory of *relative ∞-categories*, that is, inclusions of wide subcategories $W \hookrightarrow C$. By the universal property of localization, the functor $(C, W) \mapsto C[W^{-1}]$ is left adjoint to $D \mapsto (D, D^{\simeq})$.

Both functors preserve finite products (Exercise 1.5.24), so this adjunction induces an adjunction on commutative monoids. Tensoring one variable at a time shows that the tensor product carries pairs of morphisms in W to morphisms in W. Thus (C, W) is a commutative monoid in RelCat_∞. Applying the left adjoint gives a symmetric monoidal structure on $C[W^{-1}]$, and the adjunction unit gives the symmetric monoidal functor γ.

The adjunction identifies the anima of symmetric monoidal functors $C[W^{-1}] \to D$ with the subanima of symmetric monoidal functors $C \to D$ inverting W. Apply this also to $\mathrm{Fun}([n], D)$ with its pointwise tensor product. A functor $C \to \mathrm{Fun}([n], D)$ inverts W precisely when its evaluation at every vertex does. We thus obtain the same identification on $[n]$-diagrams for every n, proving (2) at the level of functor categories (Definition 14.4.3). The universal property gives uniqueness in (1). □

Remark 21.1.2. In the situation of Proposition 14.7.2, we may consider the class W of L-equivalences in C. The assumption on L guarantees that also the condition of Proposition 21.1.1 is satisfied. In particular, Proposition 21.1.1 provides an alternative way of equipping the subcategory $D \subseteq C$ with a symmetric monoidal structure.

For chain complexes, we can compute the derived tensor product by first choosing projective resolutions (Definition 6.6.25). More generally, consider a tuple of cofibrant objects. After transport along a span, its entries are finite tensor products of cofibrant objects. To derive transport along a second span, we may have to replace these entries by cofibrant objects again. The following condition ensures that applying the second tensor product to these replacements gives a weak equivalence.

Definition 21.1.3 (Left derivable monoidal structure)**.** Let (C, W, I) be an ∞-category with weak equivalences and cofibrations, and suppose C is symmetric monoidal. We call an object *tensor-cofibrant* if it is a finite tensor product of cofibrant objects, allowing the empty tensor product. The monoidal structure is *left derivable* if for every finite family $(f_j\colon X_j \to Y_j)_{j\in J}$ of weak equivalences with each X_j cofibrant and each Y_j tensor-cofibrant, the map

$$\bigotimes_{j\in J} f_j\colon \bigotimes_{j\in J} X_j \to \bigotimes_{j\in J} Y_j$$

is again a weak equivalence.

Remark 21.1.4. In the model category literature, there exists a notion of *symmetric monoidal model category*. The underlying category with weak equivalences and cofibrations of such a symmetric monoidal model category is always left derivable. However, the condition of being left derivable is much weaker: we require only that tensoring preserve the relevant weak equivalences, with no general compatibility between tensor products and cofibrations.

A convenient sufficient criterion is the following. Assume that the monoidal unit is cofibrant, that the tensor product of two cofibrant objects is again cofibrant, and that for every cofibrant object Z the functor $- \otimes Z$ sends weak equivalences between cofibrant objects to weak equivalences. Then every tensor-cofibrant object is again cofibrant, and an induction on the cardinality of J shows that the monoidal structure is left derivable in the sense of the definition.

Construction 21.1.5. Let (C, W, I) be an ∞-category with weak equivalences and cofibrations which comes equipped with a left derivable symmetric monoidal structure. For a finite set J, let W^J denote the collection of morphisms in $C^J \simeq C^{\otimes}_J$ given by J-tuples of weak equivalences, and let $W^{\otimes}$ be the collection of morphisms in $C^{\otimes}$ given by the union of the W^J. By construction, the cocartesian fibration $p_C \colon C^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ inverts all morphisms in $W^{\otimes}$, so the universal property of localizations produces a functor

$$p_{C[W^{-1}]} \colon C^{\otimes}[(W^{\otimes})^{-1}] \to \mathrm{Span}(\mathrm{Fin}).$$

The localization functor $\gamma^{\otimes} \colon C^{\otimes} \to C^{\otimes}[(W^{\otimes})^{-1}]$ then becomes a functor over $\mathrm{Span}(\mathrm{Fin})$.

Theorem 21.1.6 (cf. [NS18, Theorem A.7])**.** *Let (C, W, I) be an ∞-category with weak equivalences and cofibrations, and assume that C comes equipped with a left derivable symmetric monoidal structure. Then the following statements hold:*

(1) *The functor $p_{C[W^{-1}]} \colon C^{\otimes}[(W^{\otimes})^{-1}] \to \mathrm{Span}(\mathrm{Fin})$ defines a symmetric monoidal ∞-category.*

(2) *The localization functor $\gamma^{\otimes} \colon C^{\otimes} \to C^{\otimes}[(W^{\otimes})^{-1}]$ encodes a lax symmetric monoidal functor.*

(3) *The underlying functor of $\gamma^{\otimes}$ exhibits its target as the localization of C at W, thus equipping $C[W^{-1}]$ with the structure of a symmetric monoidal ∞-category.*

(4) *For every symmetric monoidal ∞-category D, the functor $\gamma^{\otimes}$ induces a fully faithful functor*

$$\mathrm{Fun}^{\otimes\text{-lax}}(C[W^{-1}], D) \hookrightarrow \mathrm{Fun}^{\otimes\text{-lax}}(C, D)$$

whose essential image consists of those lax symmetric monoidal functors $C \to D$ that invert the morphisms in W.

For chain complexes, the binary lax structure map $\gamma X \otimes^{\mathbf{L}} \gamma Y \to \gamma(X \otimes Y)$ is the structure transformation of the derived tensor product. When X and Y are bounded below, it is represented by the map $P \otimes Q \to X \otimes Y$ induced by a pair of projective resolutions.

Example 21.1.7 (Chain complexes)**.** Let $\mathcal{A}$ be a symmetric monoidal abelian category with enough projectives whose tensor product is additive in each variable. Assume its unit is projective and tensor products of projectives are projective. In the projective cofibration structure on $\mathrm{Ch}^{-}(\mathcal{A})$, the

cofibrant objects are the bounded below complexes of projectives (Proposition 6.6.15). They include the unit and are closed under tensor products, since only finite direct sums occur in each degree. Quasi-isomorphisms between them are chain homotopy equivalences, which tensoring preserves in each variable (Proposition 6.6.11 and Definition 6.6.23). Hence the monoidal structure is left derivable.

By Theorem 21.1.6, the localization $\mathrm{Ch}^-(\mathcal{A}) \to \mathrm{D}^-(\mathcal{A})$ is lax symmetric monoidal. On bounded below complexes of projectives its structure maps are isomorphisms, since these complexes serve as their own resolutions. Thus localizing the symmetric monoidal subcategory $\mathrm{Ch}^-_{\mathrm{proj}}(\mathcal{A})$ as in Proposition 21.1.1 gives the same symmetric monoidal structure on $\mathrm{D}^-(\mathcal{A})$ by Proposition 6.6.16. Its tensor product is the one of Definition 6.6.25.

To prove Theorem 21.1.6, we describe what happens to cocartesian transport when we localize the fibers of a cocartesian fibration.

Recall from Definition 6.6.1 that a total left derived functor of $F\colon C \to D$ is a right Kan extension of F along the localization $\gamma\colon C \to C[W^{-1}]$. Such a derived functor $\mathbf{L}F$ is called *absolute* if for every functor $H\colon D \to E$, the composite $H \circ \mathbf{L}F$ is again a total left derived functor of $H \circ F$.

Example 21.1.8. Assume that $F\colon C \to D$ inverts morphisms in W, so that it induces a functor $\mathbf{L}F\colon C[W^{-1}] \to D$ by the universal property of localization. Then the natural isomorphism $\mathbf{L}F \circ \gamma \cong F$ exhibits $\mathbf{L}F$ as an absolute left derived functor of F.

Example 21.1.9. More generally, let (C, W, I) be an ∞-category with weak equivalences and cofibrations, in the sense of Definition 2.2.4, and consider a functor $F\colon C \to D$ for which the restriction $F|_{C_c}\colon C_c \to D$ to the cofibrant objects inverts weak equivalences. It follows from Proposition 6.6.3 that F admits an absolute left derived functor $\mathbf{L}F\colon C[W^{-1}] \to D$, uniquely characterized by the existence of an isomorphism $\mathbf{L}F \circ \gamma \circ i \simeq F \circ i$ on the cofibrant objects.

Definition 21.1.10. Let $p\colon C \to S$ be a cocartesian fibration. Assume that for every object $s \in S$ we are given a class W_s of morphisms in the fiber $C_s := p^{-1}(s)$, and let $\gamma_s\colon C_s \to C_s[W_s^{-1}]$ denote the localization functor. We say that p is *left derivable with respect to* $\{W_s\}_{s\in S}$ if the following two conditions are satisfied:

(1) For every morphism $f\colon s \to t$ in S, consider the cocartesian transport functor $f_!\colon C_s \to C_t$. Then $\gamma_t \circ f_!\colon C_s \to C_t[W_t^{-1}]$ admits an absolute left derived functor

$$\mathbf{L}f_!\colon C_s[W_s^{-1}] \to C_t[W_t^{-1}].$$

(2) For each composable pair $s \xrightarrow{f} t \xrightarrow{g} u$ in S, the canonical map

$$\mathbf{L}g_! \circ \mathbf{L}f_! \to \mathbf{L}(g \circ f)_!$$

is a natural isomorphism.

To obtain the map in (2), compose the structure transformations to get $\mathbf{L}g_! \circ \mathbf{L}f_! \circ \gamma_s \Rightarrow \mathbf{L}g_! \circ \gamma_t \circ f_! \Rightarrow \gamma_u \circ (g \circ f)_!$. The universal property of the right Kan extension $\mathbf{L}(g \circ f)_!$ then gives the stated map.

Example 21.1.11. In the situation of Definition 21.1.10, assume that every cocartesian transport functor $f_! : C_s \to C_t$ satisfies $f_!(W_s) \subseteq W_t$. Then p is left derivable with respect to $\{W_s\}_{s \in S}$. Indeed, the condition on $f_!$ shows that $\gamma_t \circ f_!$ uniquely factors through a functor $\mathbf{L}f_! : C_s[W_s^{-1}] \to C_t[W_t^{-1}]$, which by Example 21.1.8 is an absolute left derived functor. The second condition is then an immediate consequence of the universal property.

Example 21.1.12. If $p : C \to S$ is left derivable with respect to $\{W_s\}$, then for any functor $f : T \to S$ the pullback functor $C \times_S T \to T$ is left derivable with respect to $\{W_{f(t)}\}_{t \in T}$.

Proposition 21.1.13 ([NS18, Proposition A.14])**.** *Let $p : C \to S$ be a cocartesian fibration that is left derivable with respect to $\{W_s\}_{s \in S}$. Write W for the union of these classes and $q : C[W^{-1}] \to S$ for the functor induced by p.*

(1) *The functor q is a cocartesian fibration.*

(2) *For each $s \in S$, the localization functor induces an equivalence $C_s[W_s^{-1}] \xrightarrow{\sim} C[W^{-1}]_s$.*

(3) *For $f : s \to t$ in S, write $f'_!$ for transport along f in q. Under the identifications in (2), the natural transformation*

$$f'_! \circ \gamma_s \Rightarrow \gamma_t \circ f_!$$

induced by $\gamma : C \to C[W^{-1}]$ over S exhibits $f'_!$ as the absolute right Kan extension of $\gamma_t \circ f_!$ along γ_s.

Proof sketch. When $S = [0]$, the assertion follows from the definition of localization. When $S = [1]$, the cocartesian fibration p is classified by a functor $F : C_0 \to C_1$. Unstraightening the absolute left derived functor of $\gamma_1 \circ F$ gives a cocartesian fibration over $[1]$ with the desired fibers. The description of its total category in Example 24.2.6, together with the absolute right Kan extension property, gives the universal property of the localization of C.

Suppose now that $S = \operatorname{colim}_i S_i$ and the result holds over each S_i. Pullback preserves left derivability (Example 21.1.12). For each transition functor $S_i \to S_j$, the map from the localized fibration over S_i to the pullback of the one over S_j is an equivalence on fibers by (2) and preserves cocartesian lifts by (3). The transition squares are therefore pullbacks. By descent for cocartesian fibrations, the colimit of their total categories is cocartesian over S, and by the universal property of localization this colimit is $C[W^{-1}]$. Part (3) extends to composites of arrows from the S_i by the composition condition in Definition 21.1.10. Since $[0]$ and $[1]$ generate Cat_∞ under colimits (Proposition 25.1.14), this gives the result. See Nikolaus and Scholze [NS18, Lemma A.16 and proof of Proposition A.14]. □

Proposition 21.1.14. *In the situation of Theorem 21.1.6, the cocartesian fibration $p_C : C^\otimes \to$ Span(Fin) is left derivable with respect to the classes $\{W^I\}_{I \in \mathrm{Fin}}$ on its fibers $C^\otimes_I \simeq C^I$.*

Proof. Fix a span $\alpha : I \xleftarrow{f} K \xrightarrow{g} J$ in Span(Fin). Under the identification $C^\otimes_I \simeq C^I$, the cocartesian transport functor along α is given by

$$\alpha_! : C^I \xrightarrow{f^*} C^K \xrightarrow{g_\otimes} C^J, \qquad (x_i)_{i \in I} \mapsto \left(\bigotimes_{k \in g^{-1}(j)} x_{f(k)} \right)_{j \in J}.$$

The product category C^I again carries weak equivalences and cofibrations componentwise, by Lemma 6.6.7. It therefore suffices, by Proposition 6.6.3, to show that $\alpha_!$ preserves componentwise weak equivalences between componentwise cofibrant objects.

This is clear for the restriction functor f^*. For the functor $g_\otimes$, let $\{u_k : x_k \to y_k\}_{k\in K}$ be a componentwise weak equivalence between componentwise cofibrant tuples. Then every object y_k is in particular tensor-cofibrant, and the j-th component of $g_\otimes(u)$ is the map

$$\bigotimes_{k\in g^{-1}(j)} x_k \xrightarrow{\bigotimes_{k\in g^{-1}(j)} u_k} \bigotimes_{k\in g^{-1}(j)} y_k,$$

which is a weak equivalence by the left derivability assumption. Thus $\alpha_!$ admits an absolute left derived functor

$$\mathbf{L}\alpha_! : C^I[(W^I)^{-1}] \to C^J[(W^J)^{-1}].$$

It remains to verify compatibility with composition. Consider composable spans

$$\alpha : I \xleftarrow{f} K \xrightarrow{g} J \qquad \text{and} \qquad \beta : J \xleftarrow{h} L \xrightarrow{k} M.$$

We show that the canonical map

$$\mathbf{L}\beta_! \circ \mathbf{L}\alpha_! \to \mathbf{L}(\beta\circ\alpha)_!$$

is an isomorphism. Let $X = (X_i)_{i\in I}$ be componentwise cofibrant. Every component of $\alpha_!(X)$ is a finite tensor product of cofibrant objects, and hence is tensor-cofibrant. Choose any componentwise weak equivalence

$$q : P \xrightarrow{\sim} \alpha_!(X)$$

with P componentwise cofibrant. Such a map is obtained by factoring the morphism from the initial object to $\alpha_!(X)$ in C^J, and no functorial choice is required. By the defining property of total left derived functors, we have

$$(\mathbf{L}\beta_! \circ \mathbf{L}\alpha_!)(\gamma_I X) \simeq \mathbf{L}\beta_!(\gamma_J \alpha_!(X)) \simeq \gamma_M \beta_!(P).$$

The structure transformation for $\mathbf{L}\beta_!$ at $\alpha_!(X)$ is represented under this identification by the map

$$\gamma_M\beta_!(q) : \gamma_M\beta_!(P) \longrightarrow \gamma_M\beta_!\alpha_!(X).$$

Each component of $\beta_!(q)$ is a tensor product of weak equivalences from cofibrant to tensor-cofibrant objects, so it is a weak equivalence by Definition 21.1.3. Thus the composite structure transformation of Definition 21.1.10 is invertible at X, as is the structure transformation for $\mathbf{L}(\beta\circ\alpha)_!$ since X is componentwise cofibrant. It follows that $\mathbf{L}\beta_! \circ \mathbf{L}\alpha_! \to \mathbf{L}(\beta\circ\alpha)_!$ is invertible at $\gamma_I X$. Every object of the localized fiber is represented by such an X by part (1) of Corollary 2.2.9. □

Proof of Theorem 21.1.6. Let us write

$$q : C^\otimes[(W^\otimes)^{-1}] \to \mathrm{Span}(\mathrm{Fin})$$

for the functor from Construction 21.1.5. By Proposition 21.1.14, the cocartesian fibration $p_C : C^\otimes \to \mathrm{Span}(\mathrm{Fin})$ is left derivable with respect to the classes $\{W^I\}_{I\in\mathrm{Fin}}$ on its fibers. In particular, it follows

from Proposition 21.1.13 that q is a cocartesian fibration. Moreover, for every finite set I, the induced map on fibers

$$C^{\otimes}_I = C^I \to C^{\otimes}[(W^{\otimes})^{-1}]_I$$

exhibits the target as the localization $C^I[(W^I)^{-1}]$. Using Exercise 1.5.24, we obtain natural equivalences

$$C^{\otimes}[(W^{\otimes})^{-1}]_I \simeq C^I[(W^I)^{-1}] \simeq (C[W^{-1}])^I.$$

For $i \in I$, let

$$\rho_i \colon I \hookleftarrow \{i\} \xrightarrow{=} \{i\}$$

be the corresponding inert span. Cocartesian transport along ρ_i in $C^{\otimes}$ is the projection $\mathrm{pr}_i \colon C^I \to C$, which preserves componentwise weak equivalences. Its total left derived functor is therefore the induced projection

$$C^I[(W^I)^{-1}] \longrightarrow C[W^{-1}].$$

Part (3) of Proposition 21.1.13 identifies this with cocartesian transport along ρ_i for q. Consequently, under the equivalences above, the Segal map for the straightening of q is the equivalence

$$C^I[(W^I)^{-1}] \xrightarrow{\sim} (C[W^{-1}])^I.$$

Thus the cocartesian straightening of q preserves finite products, and Lemma 14.4.4 shows that q defines a symmetric monoidal ∞-category. This proves (1).

For (2), part (3) of Proposition 21.1.13 identifies the transformation induced by $\gamma^{\otimes}$ along ρ_i with the structure transformation of the derived projection. It is invertible since the projection preserves weak equivalences. Thus $\gamma^{\otimes}$ carries cocartesian lifts of ρ_i to cocartesian lifts. These lifts exhibit the product decompositions in both total categories by part (iii) of Proposition 14.1.12. Hence $\gamma^{\otimes}$ preserves finite products and defines a lax symmetric monoidal functor.

For (3), we evaluate part (2) of Proposition 21.1.13 on the one-point set $\langle 1 \rangle$. This identifies the underlying ∞-category of the symmetric monoidal ∞-category from (1) with the localization $C[W^{-1}]$, and under this equivalence the restriction of $\gamma^{\otimes}$ to the fiber over $\langle 1 \rangle$ is the localization functor $\gamma \colon C \to C[W^{-1}]$. Thus the symmetric monoidal structure from (1) may be regarded as a symmetric monoidal structure on $C[W^{-1}]$ itself.

Finally, let D be a symmetric monoidal ∞-category. Since $\gamma^{\otimes} \colon C^{\otimes} \to C^{\otimes}[(W^{\otimes})^{-1}]$ is the localization of the total category at $W^{\otimes}$, precomposition with $\gamma^{\otimes}$ induces a fully faithful functor

$$\mathrm{Fun}_{/\mathrm{Span(Fin)}}(C^{\otimes}[(W^{\otimes})^{-1}], D^{\otimes}) \hookrightarrow \mathrm{Fun}_{/\mathrm{Span(Fin)}}(C^{\otimes}, D^{\otimes})$$

whose essential image consists of those functors over $\mathrm{Span(Fin)}$ that invert the morphisms in $W^{\otimes}$. In the sense of Definition 14.1.1, the lax symmetric monoidal functors are precisely the morphisms of ∞-operads, i.e. the functors over $\mathrm{Span(Fin)}$ that preserve finite products. Since $\gamma^{\otimes}$ preserves finite products, precomposition with $\gamma^{\otimes}$ certainly sends lax symmetric monoidal functors to lax symmetric monoidal functors. Conversely, let $F \colon C^{\otimes}[(W^{\otimes})^{-1}] \to D^{\otimes}$ be a functor over $\mathrm{Span(Fin)}$ such that $F \circ \gamma^{\otimes}$ preserves finite products. By part (2) of Proposition 21.1.13, the functor $\gamma^{\otimes}$ is essentially surjective on each fiber, hence essentially surjective on the total category. Since it also preserves finite products, it follows that F preserves finite products as well. We conclude that the displayed fully faithful functor restricts to a fully faithful functor

$$\mathrm{Fun}^{\otimes\text{-lax}}(C[W^{-1}], D) \hookrightarrow \mathrm{Fun}^{\otimes\text{-lax}}(C, D)$$

whose essential image consists of those lax symmetric monoidal functors that invert the morphisms in W. Indeed, for a lax symmetric monoidal functor, inverting W on the fiber over $\langle 1 \rangle$ is equivalent to inverting $W^{\otimes}$ on every fiber, since the induced functor on the fiber over I is the product of the underlying functor on the fiber over $\langle 1 \rangle$. This proves (4). □

21.2 The symmetric monoidal derived category

Let R be a commutative ring. For bounded below complexes of R-modules, we computed the derived tensor product using projective resolutions. For unbounded complexes we use the projective cofibrations on $\mathrm{Ch}(R)$. The unit and tensor products of cofibrant complexes are again cofibrant, and tensoring with a cofibrant complex preserves quasi-isomorphisms.

Proposition 21.2.1 (Unbounded projective cofibrations, [Lur17, Propositions 7.1.2.8 and 7.1.2.11])**.** *Let W be the class of quasi-isomorphisms in $\mathrm{Ch}(R)$. There is a class I_{proj} of projective cofibrations with the following properties.*

(1) The triple $(\mathrm{Ch}(R), W, I_{\mathrm{proj}})$ is a homotopy cocomplete ∞-category with weak equivalences and cofibrations.

(2) The unit $R[0]$ is cofibrant, and a finite tensor product of cofibrant complexes is again cofibrant.

(3) If P is cofibrant, then $P \otimes_R -$ is homotopy cocontinuous and preserves quasi-isomorphisms.

To see why $P \otimes_R -$ preserves all quasi-isomorphisms in (3), recall from the proof of Lurie [Lur17, Proposition 7.1.2.8] that every cofibrant complex is a retract of one built from zero by successively adjoining free generators. Tensor this construction with an acyclic complex K. Each inclusion at a successor stage is degreewise split with quotient a shift of K, and at limit stages filtered colimits of R-modules are exact. It follows that $P \otimes_R K$ is acyclic. Since tensoring commutes with mapping cones, the claim follows from Lemma 6.1.15.

Proposition 21.2.2 (The symmetric monoidal derived category)**.** *The derived ∞-category $\mathrm{D}(R)$ admits a canonical symmetric monoidal structure with tensor product $\otimes^{\mathbf{L}}_R$ and unit $R[0]$. The derived tensor product preserves small colimits separately in each variable, and the localization functor $\gamma\colon \mathrm{Ch}(R) \to \mathrm{D}(R)$ admits a canonical lax symmetric monoidal refinement.*

Proof. By Proposition 21.2.1 and Remark 21.1.4, the tensor product on $\mathrm{Ch}(R)$ is left derivable. Hence Theorem 21.1.6 equips $\mathrm{D}(R)$ with a symmetric monoidal structure and γ with a lax symmetric monoidal refinement. Applying part (3) of Proposition 21.1.13 to the binary tensor product identifies the tensor product on $\mathrm{D}(R)$ with the total left derived functor of $\otimes_R$. Applied to the empty tensor product, it identifies the unit with $\gamma(R[0])$.

For a cofibrant complex P, the binary lax structure map gives a transformation

$$\gamma P \otimes^{\mathbf{L}}_R \gamma Y \longrightarrow \gamma(P \otimes_R Y)$$

natural in Y. Choose a cofibrant replacement $Q \to Y$. By the description of derived transport above, the structure map is represented by $P \otimes_R Q \to P \otimes_R Y$, a quasi-isomorphism by part (3) of Proposition 21.2.1. Thus $\gamma P \otimes^{\mathbf{L}}_R -$ is the functor on localizations induced by $P \otimes_R -$. Since $P \otimes_R -$ is

homotopy cocontinuous, its induced functor preserves small colimits by the dual of Theorem 2.2.11. Every object of $\mathrm{D}(R)$ is represented by a cofibrant complex by part (1) of Corollary 2.2.9, and symmetry gives the assertion in the other variable. □

Corollary 21.2.3 (DGAs as derived algebras)**.** *The localization γ induces functors*

$$\mathrm{Alg}(\mathrm{Ch}(R)) \longrightarrow \mathrm{Alg}(\mathrm{D}(R)) \qquad \textit{and} \qquad \mathrm{CAlg}(\mathrm{Ch}(R)) \longrightarrow \mathrm{CAlg}(\mathrm{D}(R)).$$

For an associative or commutative DGA A with multiplication μ, the multiplication on γA is the composite $\gamma A \otimes^{\mathbf{L}}_R \gamma A \to \gamma(A \otimes_R A) \xrightarrow{\gamma(\mu)} \gamma A$.

Proof. Apply Proposition 21.2.2 and the fact that lax symmetric monoidal functors preserve associative and commutative algebras. □

Proposition 21.2.4. *The full subcategory $\mathrm{D}^-(R) \subseteq \mathrm{D}(R)$ contains the unit and is closed under the derived tensor product. Its induced symmetric monoidal structure agrees with the bounded below construction of Example 21.1.7.*

Proof. A complex is cofibrant in the unbounded projective structure if maps out of it lift against every degreewise surjective quasi-isomorphism [Lur17, Proposition 7.1.2.8]. For a bounded below complex of projectives, the induction in Lemma 6.6.14 constructs these lifts starting at the lower bound of the source, regardless of bounds on the other two complexes. Thus if P and Q are bounded below projective resolutions, their ordinary tensor product $P \otimes_R Q$ computes both derived tensor products. It is again bounded below, as is the unit $R[0]$.

By part (4) of Theorem 21.1.6, the composite $\mathrm{Ch}^-(R) \hookrightarrow \mathrm{Ch}(R) \xrightarrow{\gamma} \mathrm{D}(R)$ induces a lax symmetric monoidal functor $\mathrm{D}^-(R) \hookrightarrow \mathrm{D}(R)$. On P, Q its binary structure map is represented by the identity of $P \otimes_R Q$, and its unit map is the identity of $R[0]$. Hence the inclusion is symmetric monoidal, giving the claimed identification by Lemma 14.7.1. □

We next describe the derived category in terms of perfect complexes. To include the tensor product in this description, we first explain how a tensor product on a small stable ∞-category extends to its Ind-completion.

Theorem 21.2.5 (Symmetric monoidal Ind-completion, [Lur17, Corollary 4.8.1.14])**.** *Let C be a small stable symmetric monoidal ∞-category, and assume that the tensor product on C is exact separately in both variables. Then $\mathrm{Ind}(C)$ admits a unique symmetric monoidal structure with the following properties:*

(1) The Yoneda embedding $C \hookrightarrow \mathrm{Ind}(C)$ is symmetric monoidal.

(2) The tensor product on $\mathrm{Ind}(C)$ preserves colimits separately in both variables.

For filtered-colimit presentations $X \simeq \mathrm{colim}_i X_i$ and $Y \simeq \mathrm{colim}_j Y_j$ with $X_i, Y_j \in C$, its underlying tensor product is characterized by

$$X \otimes Y \simeq \mathrm{colim}_{(i,j)}(X_i \otimes Y_j).$$

Proof. Equip $\mathrm{PSh}(C)$ with Day convolution. Its tensor product preserves colimits separately in each variable, and the Yoneda embedding $Y\colon C \to \mathrm{PSh}(C)$ is symmetric monoidal by Propositions 19.5.2 and 16.2.10. Thus

$$\left(\operatorname{colim}_i Y(c_i)\right) \otimes_{\mathrm{Day}} \left(\operatorname{colim}_j Y(d_j)\right) \simeq \operatorname{colim}_{(i,j)} Y(c_i \otimes d_j).$$

Products of filtered ∞-categories are filtered, so $\mathrm{Ind}(C)$ contains the unit and is closed under Day convolution. By Lemma 14.7.1, it inherits a symmetric monoidal structure with the asserted formula.

For $c \in C$, the right adjoint to $Y(c) \otimes_{\mathrm{Day}} -$ on $\mathrm{PSh}(C)$ sends a presheaf F to $F(c \otimes -)$, by Proposition 16.2.7 and Yoneda. This functor preserves $\mathrm{Ind}(C)$: by Proposition 23.3.6, its objects are precisely the presheaves sending finite colimits in C to limits, and $c \otimes -$ preserves finite colimits. The adjunction therefore restricts to $\mathrm{Ind}(C)$, so $Y(c) \otimes -$ preserves all colimits there. Every object of $\mathrm{Ind}(C)$ is a filtered colimit of representables, and filtered colimits are computed in $\mathrm{PSh}(C)$. Tensoring with an arbitrary object is consequently a filtered colimit of these colimit-preserving functors, and hence preserves all colimits. Symmetry gives the assertion in the other variable.

Finally, repeated application of the universal property of $\mathrm{Ind}(C)$ from Proposition 23.1.2 shows that restriction to C^n is fully faithful on functors $\mathrm{Ind}(C)^n \to \mathrm{Ind}(C)$ preserving filtered colimits separately in each variable. Thus the finite tensor products and all their coherence data are uniquely determined by their restrictions to C. This proves uniqueness. □

We now return to the derived category of R.

Corollary 21.2.6 (Perfect complexes)**.** *The unit $R[0]$ is a compact generator of $\mathrm{D}(R)$, and the inclusion of compact objects induces a symmetric monoidal equivalence*

$$\mathrm{Ind}(\mathrm{D}(R)^{\omega}) \xrightarrow{\sim} \mathrm{D}(R).$$

Proof. Since R is projective as an R-module, Corollary 6.4.5 gives

$$\pi_n \hom_{\mathrm{D}(R)}(R[0], X) \cong H_n(X)$$

for every $X \in \mathrm{D}(R)$. The canonical t-structure on $\mathrm{D}(R)$ is compatible with filtered colimits by [Lur17, Proposition 1.3.5.21], so each homology functor preserves filtered colimits. Since homotopy groups of spectra preserve filtered colimits by Lemma 4.4.28, the displayed formula shows that $R[0]$ is compact. Its shifts jointly detect zero objects, since a complex is zero in $\mathrm{D}(R)$ precisely when all its homology groups vanish. It follows from Proposition 23.3.4 that $R[0]$ is a compact generator.

By the same proposition, $\mathrm{D}(R)^{\omega}$ is small and is the smallest thick subcategory containing $R[0]$. For each compact P, the exact functor $P \otimes^{\mathbf{L}}_R -$ sends $R[0]$ to P, so it preserves this subcategory. By Proposition 23.3.7, the inclusion induces an equivalence $\mathrm{Ind}(\mathrm{D}(R)^{\omega}) \xrightarrow{\sim} \mathrm{D}(R)$. Since the derived tensor product preserves colimits separately in each variable, the uniqueness in Theorem 21.2.5 makes this equivalence symmetric monoidal. □

The compact objects of $\mathrm{D}(R)$ are precisely the retracts of bounded complexes of finitely generated free R-modules. Indeed, filtering such a complex by degree builds it from shifts of finite sums of $R[0]$, so it is compact. Conversely, the bounded complexes of finitely generated free modules are closed under shifts and cofibers. Any map between them in $\mathrm{D}(R)$ is represented by a chain map by Lemma 6.6.32, and its mapping cone is again bounded and finite free in each degree. A map

between retracts extends to a map between the original complexes by composing with the retraction from the source and the inclusion into the target. The original map is a retract of its extension, and the same holds for their cofibers. Thus the retracts form a thick subcategory containing $R[0]$, so they include every compact object.

By Proposition 23.3.6, $\mathrm{D}(R)$ is presentable. In Corollary 8.2.4 we apply symmetric monoidal Morita theory to its compact generator $R[0]$ and identify its endomorphism ring spectrum with the Eilenberg–MacLane ring spectrum of R. The resulting equivalence $\mathrm{D}(R) \simeq \mathrm{Mod}_R(\mathrm{Sp})$ sends $R[0]$ to R, so it identifies the subcategory $\mathrm{D}(R)^\omega$ of perfect complexes with the perfect R-modules $\mathrm{Perf}(R)$ of Definition 8.1.10.

Exercises

Exercise 21.1 (A tensor-stable localization)**.** Let k be a field and let X be any (possibly unbounded) complex of k-vector spaces. Show that $X \otimes_k -$ preserves quasi-isomorphisms and deduce that the localization functor $\mathrm{Ch}(k) \to \mathrm{D}(k)$ is symmetric monoidal.

Exercise 21.2 (Derived transport over $[1]$)**.** Let $E \to [1]$ be the cocartesian fibration classified by $F\colon C_0 \to C_1$. Suppose it is left derivable with respect to classes W_0 and W_1 in its fibers. Write $\gamma\colon E \to E[(W_0 \cup W_1)^{-1}]$ for the localization, γ_i for its restriction to C_i, and $\mathbf{L}F$ for the total left derived functor of $\gamma_1 \circ F$.

(1) For $X \in C_0$ and $Y \in C_1$, identify the anima of maps from $\gamma_0 X$ to $\gamma_1 Y$ over $0 \to 1$ with $\mathrm{Hom}_{C_1[W_1^{-1}]}((\mathbf{L}F)(\gamma_0 X), \gamma_1 Y)$.

(2) Under this identification, which morphism corresponds to the image of a cocartesian lift $X \to F(X)$? Deduce a criterion for this lift to remain cocartesian after localization.

Exercise 21.3 (The lax structure map and self-intersection)**.** Let R be a commutative ring and let $\gamma\colon \mathrm{Ch}(R) \to \mathrm{D}(R)$ be the localization functor.

(1) For $C, D \in \mathrm{Ch}(R)$, describe the lax structure map $\gamma C \otimes^{\mathbf{L}}_R \gamma D \to \gamma(C \otimes_R D)$ using cofibrant replacements. Show that it is an isomorphism if either $C \otimes_R -$ or $- \otimes_R D$ preserves quasi-isomorphisms.

(2) Let $x \in R$ be a non-zero-divisor and put $A = R/(x)$. Use the Koszul resolution of Example 8.2.7 to compute $A \otimes^{\mathbf{L}}_R A$ and identify the lax structure map $A \otimes^{\mathbf{L}}_R A \to A \otimes_R A \simeq A$. What does it do on H_0 and H_1? Specialize to $R = \mathbb{Z}$ and $x = p$ prime to obtain an example where the map fails to be an isomorphism.

Exercise 21.4 (Duality for a perfect complex)**.** Let P be a bounded complex of finitely generated projective modules over a commutative ring R. Show that $P^\vee := \underline{\mathrm{Hom}}_R(P, R)$ is perfect and that evaluation induces an isomorphism of complexes $P^\vee \otimes_R M \to \underline{\mathrm{Hom}}_R(P, M)$ for every complex M. Deduce that $P^\vee \otimes^{\mathbf{L}}_R M \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(P, M)$ is an isomorphism for every $M \in \mathrm{D}(R)$ and that $P^\vee$ is a dual of P.

Part III

Background on ∞-category theory

Introduction to Part III

This part collects the ∞-categorical background used throughout the book. The chapters can be consulted as needed: references in the main text point to the relevant statements.

Chapters 22 and 23 develop the results on adjunctions, colimits, and presentability used in Parts I and II. Chapter 24 explains how cartesian and cocartesian fibrations encode functors with values in Cat_∞. Chapter 25 gives a way to construct ∞-categories from simplicial animae, used for the span categories of Part II.

22 Adjunctions

Free algebras, localizations, Kan extensions and (co)limits are all specified by universal properties. Adjunctions express how these constructions depend functorially on their input. For example, the universal property of a free algebra says that the free algebra functor is left adjoint to the forgetful functor.

22.1 Definition and characterizations

Recall from Definition 1.8.14 that an adjunction is given by a natural isomorphism of hom animae:

Definition 22.1.1. Let $F\colon C \to D$ and $G\colon D \to C$ be two functors. An *adjunction* between C and D consists of a natural isomorphism

$$\mathrm{Hom}_D(F(-),-) \cong \mathrm{Hom}_C(-,G(-))$$

of functors $C^{\mathrm{op}} \times D \to \mathrm{An}$. We often write $F \dashv G$ to indicate that such an isomorphism has been given.

Proposition 22.1.2. *Consider two functors $F\colon C \to D$ and $G\colon D \to C$. The following data are equivalent:*

(1) The data of an adjunction $F \dashv G$;

(2) The data of a natural transformation $\eta\colon \mathrm{id}_C \to GF$, called the unit*, such that for all objects X of C and Y of D the induced map*

$$\mathrm{Hom}_D(FX,Y) \xrightarrow{G} \mathrm{Hom}_C(GFX,GY) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_C(X,GY)$$

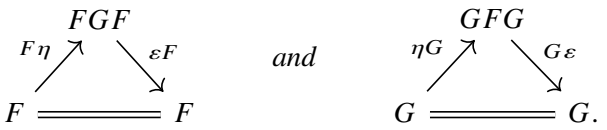
is an equivalence;

(3) The data of a natural transformation $\varepsilon\colon FG \to \mathrm{id}_D$, called the counit*, such that for all objects X of C and Y of D the induced map*

$$\mathrm{Hom}_C(X,GY) \xrightarrow{F} \mathrm{Hom}_D(FX,FGY) \xrightarrow{\varepsilon_Y\circ-} \mathrm{Hom}_D(FX,Y)$$

is an equivalence;

(4) *The data of both a unit* $\eta\colon \mathrm{id} \to GF$ *and a counit* $\varepsilon\colon FG \to \mathrm{id}$ *together with commutative triangles as follows, called the* triangle identities*:*[20]



Proof sketch. First assume (1), so that there is a natural equivalence $\mathrm{Hom}_D(FX, Y) \simeq \mathrm{Hom}_C(X, GY)$. If we fix $X \in C$, then by the Yoneda lemma the natural transformation $\mathrm{Hom}_D(FX, -) \xrightarrow{\sim} \mathrm{Hom}_D(X, G(-))$ corresponds to a map $\eta_X\colon X \to GFX$: we obtain this map by taking $Y = FX$ and letting η_X be the map that under this equivalence corresponds to the identity map $\mathrm{id}_{FX}\colon FX \to FX$. Conversely, the map η_X determines the transformation as the composite

$$\mathrm{Hom}_D(FX, -) \xrightarrow{G} \mathrm{Hom}_C(GFX, G(-)) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_C(X, -).$$

The naturality of the equivalence in X corresponds to the naturality of $\eta\colon \mathrm{id}_C \to GF$, showing the equivalence between (1) and (2).

The equivalence between (1) and (3) is proved similarly, where now the counit map $\varepsilon_Y\colon FGY \to Y$ corresponds to the identity map $\mathrm{id}_{GY}\colon GY \to GY$.

To show that these conditions are also equivalent to (4), assume first that (1)-(3) are satisfied. For the first triangle identity, observe that by definition the map $\eta_X\colon X \to GFX$ is a preimage of $\mathrm{id}_{FX}\colon FX \to FX$ under the equivalence

$$\mathrm{Hom}_C(X, GFX) \xrightarrow{F} \mathrm{Hom}_D(FX, FGFX) \xrightarrow{\varepsilon_{FX}\circ -} \mathrm{Hom}_D(FX, FX).$$

This says that the composite $\varepsilon_{FX} \circ F\eta_X$ is equivalent to id_{FX}, naturally in X, which is the first triangle identity. The second triangle identity similarly follows from the fact that the morphism $\varepsilon_Y\colon FGY \to Y$ is a preimage of $\mathrm{id}_{GY}\colon GY \to GY$ under the equivalence

$$\mathrm{Hom}_D(FGY, Y) \xrightarrow{G} \mathrm{Hom}_C(GFGY, GY) \xrightarrow{-\circ\eta_{GY}} \mathrm{Hom}_C(GY, GY).$$

Finally, assume that (4) is satisfied. An easy check shows that in this case the map defined in (2) is inverse to the map defined in (3), producing the desired natural equivalence $\mathrm{Hom}_D(FX, Y) \simeq \mathrm{Hom}_C(X, GY)$. □

To construct an adjoint, it suffices to construct its value at each object by the appropriate universal property:

Definition 22.1.3. Let $F\colon C \to D$ be a functor and let Y be an object of D.

(1) An object Z of C is called a *right adjoint object to Y under F* if it comes equipped with a map $\varepsilon_Y\colon FZ \to Y$ such that for every other object X of C the induced map

$$\mathrm{Hom}_C(X, Z) \xrightarrow{F} \mathrm{Hom}_D(FX, FZ) \xrightarrow{\varepsilon_Y\circ -} \mathrm{Hom}_D(FX, Y)$$

is an equivalence of animae.

[20] The details aren't super relevant here, but strictly speaking the data of an adjunction should only contain either the first or the second triangle identity, while for the other triangle we only demand the *existence*; see e.g. [Lur26, Tag 02F4].

(2) Dually, Z is called a *left adjoint object to Y under F* if it comes equipped with a map $\eta_Y : Y \to FZ$ such that for every other object X of C the induced map

$$\mathrm{Hom}_C(Z, X) \xrightarrow{F} \mathrm{Hom}_D(FZ, FX) \xrightarrow{-\circ\eta_Y} \mathrm{Hom}_D(Y, FX)$$

is an equivalence of animae.

Lemma 22.1.4. *Consider a functor $F\colon C \to D$ and let $D_R \subseteq D$ be a full subcategory. Assume that for every $Y \in D_R$ there exists a right adjoint object Z to Y under F. Then these right adjoint objects assemble into a functor $G\colon D_R \to C$ which comes equipped with a natural isomorphism*

$$\mathrm{Hom}_D(F(-), -) \cong \mathrm{Hom}_C(-, G(-))$$

of functors $C^{\mathrm{op}} \times D_R \to \mathrm{An}$.

Dually, if $D_L \subseteq D$ is a subcategory such that every $Y \in D_L$ admits a left adjoint object under F, then these left adjoint objects assemble into a functor $L\colon D_L \to C$ that comes equipped with a natural isomorphism

$$\mathrm{Hom}_C(L(-), -) \cong \mathrm{Hom}_D(-, F(-))$$

of functors $D_L^{\mathrm{op}} \times C \to \mathrm{An}$.

Proof. We will prove the claim for right adjoints; the claim for left adjoints is dual. The assumption on F is that for every Y in D_R the functor $\mathrm{Hom}_D(F(-), Y)\colon C^{\mathrm{op}} \to \mathrm{An}$ is representable, in the sense that it lies in the image of the Yoneda embedding $Y\colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$. Since the Yoneda embedding is fully faithful, this means there exists a functor

$$G\colon D_R \to C$$

making the following diagram commute:

$$\begin{array}{ccc} & & C \\ & \overset{G}{\nearrow} & \downarrow{\scriptstyle Y} \\ D_R & \xrightarrow[Y \mapsto \mathrm{Hom}_D(F(-), Y)]{} & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}). \end{array}$$

Here the bottom functor is the one obtained from currying the functor $\mathrm{Hom}_D(F(-), -)\colon C^{\mathrm{op}} \times D_R \to \mathrm{An}$. Since $Y(X) = \mathrm{Hom}_C(-, X)$, this commutative triangle encodes a natural isomorphism

$$\mathrm{Hom}_D(F(-), -) \cong \mathrm{Hom}_C(-, G(-))$$

of functors $C^{\mathrm{op}} \times D_R \to \mathrm{An}$, as desired. □

Corollary 22.1.5. *A functor $F\colon C \to D$ admits a right adjoint if and only if every object of D admits a right adjoint object under F. Dually, F admits a left adjoint if and only if every object of D admits a left adjoint object under F.*

Proof. We again only prove the claim about right adjoints. If a right adjoint $G\colon D \to C$ exists, then by Proposition 22.1.2 we see that the object $Z := GY$ together with the counit map $\varepsilon_Y : FGY \to Y$ provides a right adjoint object to Y under F. Conversely, if all the right adjoint objects exists, then we may apply Lemma 22.1.4 with $D = D_R$ to obtain a functor $G\colon D \to C$ together with a natural isomorphism

$$\mathrm{Hom}_D(F(-), -) \cong \mathrm{Hom}_C(-, G(-)),$$

which means that G is a right adjoint to F. □

Precomposition and postcomposition with adjoint functors induce adjunctions between functor categories:

Lemma 22.1.6. *Let $F \dashv G$ be an adjunction of ∞-categories. Then:*

(1) For every ∞-category E the functors

$$G^*\colon \operatorname{Fun}(C,E) \rightleftarrows \operatorname{Fun}(D,E) \colon F^*$$

form an adjunction, with counit and unit given by

$$G^*F^* = (FG)^* \xrightarrow{\varepsilon^*} \mathrm{id}_D^* = \mathrm{id}_{\operatorname{Fun}(D,E)} \qquad \text{and} \qquad \mathrm{id}_{\operatorname{Fun}(C,E)} = \mathrm{id}_C^* \xrightarrow{\eta^*} (GF)^* = F^*G^*.$$

(2) For every ∞-category E the functors

$$F_*\colon \operatorname{Fun}(E,C) \rightleftarrows \operatorname{Fun}(E,D) \colon G_*$$

form an adjunction, with counit and unit given by

$$F_*G_* = (FG)_* \xrightarrow{\varepsilon_*} (\mathrm{id}_D)_* = \mathrm{id}_{\operatorname{Fun}(E,D)} \qquad \text{and} \qquad \mathrm{id}_{\operatorname{Fun}(E,C)} = (\mathrm{id}_C)_* \xrightarrow{\eta_*} (GF)_* = G_*F_*.$$

Proof. The triangle identities for ε^* and η^*, resp. ε_* and η_*, follow immediately from the triangle identities for ε and η. We leave the details to the reader. □

Every adjunction induces an equivalence on geometric realizations:

Lemma 22.1.7. *Let $F\colon C \rightleftarrows D \colon G$ be an adjunction of ∞-categories. Then the induced maps $\|F\|\colon \|C\| \to \|D\|$ and $\|G\|\colon \|D\| \to \|C\|$ are inverse equivalences.*

Proof. Thinking of the counit $\varepsilon\colon FG \to \mathrm{id}_D$ as a functor $[1] \times D \to D$, it induces another functor

$$[1] \times \|D\| \to \|[1]\| \times \|D\| \overset{1.5.24}{\simeq} \|[1] \times D\| \xrightarrow{\|\varepsilon\|} \|D\|,$$

i.e. a natural transformation $\|F\|\|G\| = \|FG\| \to \|\,\mathrm{id}_D\,\| = \mathrm{id}_{\|D\|}$. Since $\|D\|$ is an anima, this natural transformation is automatically a natural isomorphism by Axiom K. Similarly, the unit $\eta\colon \mathrm{id}_C \to GF$ induces a natural transformation $\|\eta\|\colon \mathrm{id}_{\|C\|} \to \|G\|\|F\|$. This shows that $\|G\|$ is inverse to $\|F\|$, as desired. □

22.2 (Co)limits and adjunctions

Taking colimits is left adjoint to the constant-diagram functor, while taking limits is right adjoint to it. We first express this for an individual diagram:

Lemma 22.2.1. *Let I and C be ∞-categories and let $F\colon I \to C$ be a functor.*

(1) An object W in C equipped with a cone $\varepsilon_F\colon \mathrm{const}_W \to F$ is a limit of F in C, in the sense of Definition 1.7.1, if and only if ε_F exhibits W as a right adjoint object to F under the functor $\mathrm{const}\colon C \to \operatorname{Fun}(I,C)$*, in the sense of Definition 22.1.3.*

(2) Dually, an object W equipped with a cocone $\eta_F \colon F \to \mathrm{const}_W$ is a colimit of F in C if and only if it exhibits W as a left adjoint object to F under $\mathrm{const} \colon C \to \mathrm{Fun}(I, C)$.

Proof. This is simply a matter of unwinding definitions: the condition in (1) that W is a right adjoint object to F under const means that for every other object X of C the composite

$$\mathrm{Hom}_C(X, W) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_X, \mathrm{const}_W) \xrightarrow{\varepsilon} \mathrm{Nat}(\mathrm{const}_X, F)$$

is an equivalence, where we recall that we write $\mathrm{Nat}(-,-)$ for the hom anima in $\mathrm{Fun}(I, C)$. But this is the universal property of the limit from Definition 1.7.1. □

Corollary 22.2.2. *Let I and C be ∞-categories. Then C admits all I-indexed limits if and only if the functor*

$$\mathrm{const} \colon C \to \mathrm{Fun}(I, C), \qquad W \mapsto \mathrm{const}_W$$

admits a right adjoint

$$\lim_I \colon \mathrm{Fun}(I, C) \to C.$$

Dually, C admits all I-indexed colimits if and only if $\mathrm{const} \colon C \to \mathrm{Fun}(I, C)$ *admits a left adjoint*

$$\mathrm{colim}_I \colon \mathrm{Fun}(I, C) \to C.$$

Proof. In light of Lemma 22.2.1 this is an immediate consequence of the pointwise criterion for adjoints from Corollary 22.1.5. □

As a consequence, we see that limit cones and colimit cocones are natural in the diagram $F \colon I \to C$: the transformations

$$\mathrm{const}_{\lim F} \to F \qquad \text{and} \qquad F \to \mathrm{const}_{\mathrm{colim}\, F}$$

are given by the counit and unit, respectively, of the adjunctions $\mathrm{const} \dashv \lim_I$ and $\mathrm{colim}_I \dashv \mathrm{const}$.

Remark 22.2.3. For an object x of C, it follows from Corollary 22.2.2 that X is terminal if and only if the functor

$$x \colon * \to C$$

is right adjoint to the functor $p_C \colon C \to *$. Dually, x is initial if and only if it is *left* adjoint to $p_C \colon C \to *$.

Lemma 22.2.4. *Let i be an initial object of an ∞-category I. Then every ∞-category C admits I-indexed limits, and the limit functor is given by evaluation at i:*

$$\lim_I = \mathrm{ev}_i \colon \mathrm{Fun}(I, C) \to C.$$

Dually, if i is terminal in I then C admits I-indexed colimits which are given by evaluating at i.

Proof. By Corollary 22.2.2 it suffices to show that ev_i defines a right adjoint to $\mathrm{const} \colon C \to \mathrm{Fun}(I, C)$. This adjunction may be obtained by considering the adjunction $i \dashv p_I$ and passing to functor categories using Lemma 22.1.6.

(An alternative proof would be to cite Lemma 22.5.5.) □

Lemma 22.2.5. *Let $F\colon C \to D$ be a functor and let I be an ∞-category such that both C and D admit I-indexed limits. Then F preserves I-indexed limits if and only if the 'canonical' natural transformation*

$$F(\lim_I(-)) \to \lim_I(F_*(-))$$

of functors $\mathrm{Fun}(I, C) \to D$ *is a natural isomorphism.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{F} & D \\ {\scriptstyle \mathrm{const}}\downarrow & & \downarrow{\scriptstyle \mathrm{const}} \\ \mathrm{Fun}(I, C) & \xrightarrow{F_*} & \mathrm{Fun}(I, D). \end{array}$$

The 'canonical' transformation above is the one that under the adjunction $\mathrm{const} \dashv \lim_I$ corresponds to the transformation

$$\mathrm{const}_{F(\lim_I(-))} \cong F_*(\mathrm{const}_{\lim_I(-)}) \to F_*(-)$$

induced by the counit $\mathrm{const}_{\lim_I(X)} \to X$. By Axiom K we may check that this map is an isomorphism for every fixed diagram $X\colon I \to C$. But now we observe that the map $F(\lim_I(X)) \to \lim_I(F_*(-))$ is an isomorphism if and only if the associated cone $\mathrm{const}_{F(\lim_I(X))} \to F_*(X) = F \circ X$ is a limit cone, giving the claim. □

Lemma 22.2.6. *Let $F\colon C \rightleftarrows D :G$ be an adjunction of ∞-categories. Then F preserves all colimits that exist in C and G preserves all limits that exist in D.*

Proof. We prove that F preserves colimits; the proof for G preserving limits is dual. Let $X\colon I \to C$ be a diagram that admits a colimit in C, and let $\eta\colon X \to \mathrm{const}_{\mathrm{colim}_I X}$ denote the associated colimit cone. We need to show that the cocone

$$F(\eta)\colon F \circ X \to F \circ \mathrm{const}_{\mathrm{colim}_I X} = \mathrm{const}_{F(\mathrm{colim}_I X)}$$

is a colimit cone in D. By definition, this means that for every other object W in D the map

$$\mathrm{Hom}_D(F(\mathrm{colim}_I X), W) \to \mathrm{Nat}(F \circ X, \mathrm{const}_W)$$

is an equivalence of animae. But under the adjunction $F \dashv G$ and the induced adjunction $F\circ - \dashv G\circ -$ from Lemma 22.1.6, this map is equivalent to the map

$$\mathrm{Hom}_C(\mathrm{colim}_I X, G(W)) \to \mathrm{Nat}(X, G \circ \mathrm{const}_W) = \mathrm{Nat}(X, \mathrm{const}_{G(W)}).$$

Since the latter map is an equivalence by the universal property of the colimit $\mathrm{colim}_I X$, this finishes the proof. □

Thus limits commute with limits, and colimits commute with colimits:

Corollary 22.2.7. *If C admits I-indexed limits, then* $\lim_I\colon \mathrm{Fun}(I, C) \to C$ *preserves all limits that exist in* $\mathrm{Fun}(I, C)$. *If C admits I-indexed colimits, then* $\mathrm{colim}_I\colon \mathrm{Fun}(I, C) \to C$ *preserves all colimits that exist in* $\mathrm{Fun}(I, C)$. □

Lemma 22.2.8. *Consider ∞-categories C and I, and assume that C admits all I-indexed limits (resp. colimits).*

(1) For every ∞-category E, the functor category $\mathrm{Fun}(E, C)$ admits I-indexed limits (resp. colimits).

(2) For every functor $E' \to E$, the restriction functor $\mathrm{Fun}(E, C) \to \mathrm{Fun}(E', C)$ preserves I-indexed limits (resp. colimits).

(3) In particular, for every object X of E the functor

$$\mathrm{ev}_X \colon \mathrm{Fun}(E, C) \to C$$

preserves I-indexed limits (resp. colimits).

Proof. We treat the case for limits; the case for colimits is dual. By Corollary 22.2.2 there is an adjunction

$$\mathrm{const} \colon C \rightleftarrows \mathrm{Fun}(I, C) \colon \lim_I .$$

By Lemma 22.1.6 this adjunction induces an adjunction on functor categories of the form

$$\mathrm{Fun}(E, C) \rightleftarrows \mathrm{Fun}(E, \mathrm{Fun}(I, C)) \simeq \mathrm{Fun}(I, \mathrm{Fun}(E, C)).$$

By applying Corollary 22.2.2 another time this says that $\mathrm{Fun}(E, C)$ admits I-indexed limits.

Observe that the limit-functor for $\mathrm{Fun}(E, C)$ is given as the composite

$$\mathrm{Fun}(I, \mathrm{Fun}(E, C)) \simeq \mathrm{Fun}(E, \mathrm{Fun}(I, C)) \xrightarrow{(\lim_I)_*} \mathrm{Fun}(E, C).$$

This description makes it clear that precomposition with any functor $E' \to E$ preserves I-limits. □

In particular, limits and colimits of presheaves are computed pointwise:

Corollary 22.2.9. *Let C be an ∞-category. The presheaf category $\mathrm{PSh}(C)$ admits all small limits and colimits. The evaluation functors $\mathrm{ev}_x \colon \mathrm{PSh}(C) \to \mathrm{An}$ preserve all small limits and colimits.* □

A colimit indexed by a product category can be computed one variable at a time:

Lemma 22.2.10. *Let I, J and C be ∞-categories such that C admits all I-indexed and J-indexed colimits. Then C admits all $(I \times J)$-indexed colimits, and for a functor $F \colon I \times J \to C$ we have*

$$\mathrm{colim}_{(i,j) \in I \times J} F(i, j) \cong \mathrm{colim}_{i \in I} \mathrm{colim}_{j \in J} F(i, j).$$

Proof. The adjunction $\mathrm{colim}_J \colon \mathrm{Fun}(J, C) \rightleftarrows C \colon \mathrm{const}$ induces an adjunction

$$\mathrm{Fun}(I \times J, C) \simeq \mathrm{Fun}(I, \mathrm{Fun}(J, C)) \rightleftarrows \mathrm{Fun}(I, C)$$

by applying Lemma 22.1.6. Composing this with the adjunction $\mathrm{colim}_I \colon \mathrm{Fun}(I, C) \rightleftarrows C \colon \mathrm{const}$ then gives the result. □

More generally, if the indexing ∞-category is itself a colimit, we can first take colimits over its constituent ∞-categories:

Theorem 22.2.11. *Let J be a small ∞-category and consider a J-indexed family of small ∞-categories $I_\bullet \colon J \to \mathrm{Cat}_\infty$. Define $I := \mathrm{colim}_j I_j$, and let $\varepsilon \colon I_\bullet \to \mathrm{const}_I$ denote the colimit cone, giving functors $\varepsilon_j \colon I_j \to I$. Let C be an ∞-category and let $F \colon I \to C$ be a functor.*

(1) Assume that the restricted diagram

$$F|_{I_j} \colon I_j \xrightarrow{\varepsilon_j} I \xrightarrow{F} C$$

admits a colimit for every $j \in J$. Then these colimits assemble into a functor

$$F' \colon J \to C, \qquad j \mapsto \mathrm{colim}(F|_{I_j} \colon I_j \to C).$$

(2) In this case, the functor F admits a colimit in C if and only if the functor F' admits a colimit in C, and then we have

$$\mathrm{colim}_{i \in I} F(i) \cong \mathrm{colim}_{j \in J} \mathrm{colim}(F|_{I_j}).$$

Dually, assume that $F|_{I_j}$ admits a limit for every $j \in J$. Then these limits assemble into a functor

$$F'' \colon J^{\mathrm{op}} \to C, \qquad j \mapsto \lim(F|_{I_j} \colon I_j \to C).$$

The functor F admits a limit if and only if F'' admits a limit, and then

$$\lim_{i \in I} F(i) \cong \lim_{j \in J^{\mathrm{op}}} \lim_{i \in I_j} F(i).$$

Proof. See Lemma 13.3.1 of [CCNW26]. □

22.3 Slice categories

The slice categories $C_{/x}$ and $C_{x/}$ consist of objects of C equipped with a morphism to or from x. Their relative versions will index the pointwise formulas for Kan extensions.

Definition 22.3.1 (Slice category)**.** Let C be an ∞-category and let x be an object in C. We define the *slice categories* $C_{/x}$ and $C_{x/}$, also known as the *over category* and the *under category*, respectively, via the following two pullback squares:

$$\begin{array}{ccc} C_{/x} & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ * & \xrightarrow{x} & C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C_{x/} & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ * & \xrightarrow{x} & C. \end{array}$$

Note that an object of $C_{/x}$ is a pair $(y, f \colon y \to x)$ of an object y equipped with a morphism to x, and dually objects of $C_{x/}$ are pairs $(y, f \colon x \to y)$. As a result of Proposition 1.4.4 morphisms $(y, f) \to (y', f')$ in $C_{/x}$ may equivalently be encoded by commutative triangles of the form

$$\begin{array}{ccc} y & \xrightarrow{g} & y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & x, & \end{array}$$

and dually for morphisms in $C_{x/}$.

We denote by

$$s\colon C_{/x} \to C \qquad \text{and} \qquad t\colon C_{x/} \to C$$

the assignments $s(y, f\colon y \to x) = y$ and $t(y, f\colon x \to y) = y$, i.e. we first include the slice into the arrow category $\mathrm{Ar}(C)$ and then apply the source/target functor.

Lemma 22.3.2. *Consider an object x of C and consider two objects (y, f) and (y', f') of the slice category $C_{/x}$. Then the hom anima in $C_{/x}$ sits in a pullback square of the form*

$$\begin{array}{ccc} \mathrm{Hom}_{C_{/x}}((y,f),(y',f')) & \longrightarrow & \mathrm{Hom}_C(y,y') \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f'\circ -} \\ * & \xrightarrow{\quad f \quad} & \mathrm{Hom}_C(y,x). \end{array}$$

A dual formula holds for the hom animae of $C_{x/}$.

Proof. By Exercise 1.4, the hom anima in $C_{/x}$ may be computed as the following pullback:

$$\begin{array}{ccc} \mathrm{Hom}_{C_{/x}}((y,f),(y',f')) & \longrightarrow & \mathrm{Nat}(f\colon y \to x, f'\colon y' \to x) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{\quad \mathrm{id}_x \quad} & \mathrm{Hom}_C(x,x). \end{array}$$

Using the equivalence $\mathrm{Fun}([1]\times[1], C) \simeq \mathrm{Fun}([2], C) \times_{\mathrm{Ar}(C)} \mathrm{Fun}([2], C)$ from Axiom D, the top right hom anima in $\mathrm{Ar}(C)$ sits in a pullback square as follows:

$$\begin{array}{ccc} \mathrm{Nat}(f\colon y \to x, f'\colon y' \to x) & \longrightarrow & \mathrm{Hom}_C(y,y') \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f'\circ -} \\ \mathrm{Hom}_C(x,x) & \xrightarrow{\quad -\circ f \quad} & \mathrm{Hom}_C(y,x). \end{array}$$

The claim thus follows by pasting pullback squares. □

Corollary 22.3.3. *Let x be an object of C. Then the slice category $C_{/x}$ admits a terminal object (x, id_x), while $C_{x/}$ admits an initial object (x, id_x).*

Proof. The claim for $C_{/x}$ is immediate from Lemma 22.3.2, since for every $(y, f) \in C_{/x}$ the induced map $\mathrm{id}_x \circ -\colon \mathrm{Hom}_C(y,x) \to \mathrm{Hom}_C(y,x)$ is an equivalence, hence its fiber over f is contractible. The claim for $C_{x/}$ is dual. □

Lemma 22.3.4. *Let x be an object in an ∞-category C.*

(1) The object x is terminal in C if and only if the forgetful functor $s\colon C_{/x} \to C$ is an equivalence;

(2) The object x is initial in C if and only if the forgetful functor $t\colon C_{x/} \to C$ is an equivalence.

Proof. We prove (1); the proof of (2) is dual. If $s\colon C_{/x} \to C$ is an equivalence, then x is terminal in C by Corollary 22.3.3, since equivalences preserve terminal objects. Conversely, suppose that

x is terminal. Every object y of C admits a morphism $y \to x$, so s is essentially surjective. By Lemma 22.3.2, the map

$$\mathrm{Hom}_{C_{/x}}((y,f),(y',f')) \longrightarrow \mathrm{Hom}_C(y,y')$$

is a pullback of $\{f\} \to \mathrm{Hom}_C(y,x)$. Since x is terminal, this last map is an isomorphism, and so s is fully faithful. We deduce s is an equivalence by Axiom J. □

Definition 22.3.5 (Relative slice category)**.** For a functor $F\colon C \to D$ and an object d of D, we define the *relative slice categories* $C_{/d}$ and $C_{d/}$ via the following two pullback squares:

$$\begin{array}{ccc} C_{/d} & \longrightarrow & D_{/d} \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ C & \xrightarrow{F} & D \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C_{d/} & \longrightarrow & D_{d/} \\ {\scriptstyle t}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ C & \xrightarrow{F} & D \end{array}$$

Objects of $C_{/d}$ consist of pairs $(y, f\colon F(y) \to d)$, and objects of $C_{d/}$ consist of pairs $(y, f\colon d \to F(y))$.

Remark 22.3.6. These relative slice categories are sometimes also denoted by F/d and d/F, which makes their dependencies on F more transparent.

22.4 Kan extensions

Given a functor $\varphi\colon I \to J$, we would like to extend a functor $F\colon I \to C$ to J. Left and right Kan extensions formulate this as a universal problem, using adjoint objects for the restriction functor $\varphi^*\colon \mathrm{Fun}(J,C) \to \mathrm{Fun}(I,C)$.

Definition 22.4.1. Let I, J and C be ∞-categories and let $\varphi\colon I \to J$ be a functor.

(1) Given a functor $F\colon I \to C$, a *left Kan extension of F along φ* is a left adjoint object to F under the restriction functor $\varphi^*\colon \mathrm{Fun}(J,C) \to \mathrm{Fun}(I,C)$. More explicitly, it is a functor $\varphi_!(F)\colon J \to C$ equipped with a natural transformation $\eta_F\colon F \to \varphi^*\varphi_!(F)$ such that for every other functor $G\colon J \to C$ the composite

$$\mathrm{Nat}(\varphi_!(F),G) \xrightarrow{\varphi^*} \mathrm{Nat}(\varphi^*\varphi_!(F),\varphi^*G) \xrightarrow{-\circ\eta_F} \mathrm{Nat}(F,\varphi^*G)$$

is an equivalence. Diagrammatically:



(2) Dually, a *right Kan extension of F along φ* is a right adjoint object to F under φ^*, i.e. a functor $\varphi_*(F)\colon J \to C$ equipped with a transformation $\varepsilon_F\colon \varphi^*\varphi_*(F) \to F$ inducing equivalences

$$\mathrm{Nat}(G,\varphi_*(F)) \xrightarrow{\sim} \mathrm{Nat}(\varphi^*(G),F)$$

for all $G\colon J \to C$.

If every functor $F\colon I \to C$ admits a left Kan extension along φ, then by Corollary 22.1.5 these left Kan extensions assemble into a left adjoint

$$\varphi_!\colon \mathrm{Fun}(I, C) \to \mathrm{Fun}(J, C)$$

to the restriction functor $\varphi^*\colon \mathrm{Fun}(J, C) \to \mathrm{Fun}(I, C)$, called the functor of *left Kan extension along* φ. Dually, if every functor F admits a *right* Kan extension along φ then φ^* admits a right adjoint

$$\varphi_*\colon \mathrm{Fun}(I, C) \to \mathrm{Fun}(J, C)$$

called *right Kan extension along* φ.

Observation 22.4.2. Note that the left Kan extension of a functor $F\colon I \to C$ along the functor $p_I\colon I \to *$ is a colimit of F. Dually, a right Kan extension of F along p_I is a limit of F.

The Kan extensions used in this book are computed objectwise by colimits or limits over relative slice categories. Such Kan extensions are called *pointwise Kan extensions*. The following theorem gives the formulas and the hypotheses under which they apply.

Theorem 22.4.3 (Pointwise formula Kan extensions)**.** *Let $F\colon I \to C$ and $\varphi\colon I \to J$ be functors.*

(1) Assume that for every object j of J the composite functor

$$I_{/j} \xrightarrow{s} I \xrightarrow{F} C$$

admits a colimit in C. Then these colimits assemble into a functor

$$\varphi_!(F)\colon J \to C.$$

Furthermore, the maps

$$F(i) \simeq \mathrm{colim}_{i' \in I_{/i}} F(i') \to \mathrm{colim}_{i' \in I_{/\varphi(i)}} F(i') = \varphi_!(F)(\varphi(i)) = \varphi^*\varphi_!(F)(i)$$

assemble into a natural transformation $\eta_F\colon F \to \varphi^*\varphi_!(F)$ exhibiting $\varphi_!(F)$ as a left Kan extension of F along φ.

(2) Dually, if for every j in J the composite

$$I_{j/} \xrightarrow{t} I \xrightarrow{F} C$$

admits a limit in C, then these limits assemble into a functor

$$\varphi_*(F)\colon J \to C$$

which is a right Kan extension of F.

In (1), $I_{/i}$ denotes the ordinary slice in I, while $I_{/\varphi(i)}$ is the relative slice for φ. The functor $(i', f) \mapsto (i', \varphi(f))$ induces the displayed map between colimits. The first isomorphism follows from Corollary 22.3.3 and Lemma 22.2.4.

Proof. See Theorems 13.1.8 and 13.1.9 of [CCNW26]. □

Definition 22.4.4. If the condition in (1) is satisfied, we say that *F admits a pointwise left Kan extension along φ*, and if the condition in (2) is satisfied we say that *F admits a pointwise right Kan extension along φ*.

In general, a Kan extension need not restrict back to the original functor: the unit $F \to \varphi^*(\varphi_! F)$ or counit $\varphi^*(\varphi_* F) \to F$ need not be invertible. However, these maps *are* invertible for pointwise Kan extensions along a fully faithful functor φ. For the proof we first record a characterization of full faithfulness in terms of relative slices.

Lemma 22.4.5. *Let $F\colon C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor F is fully faithful;

(2) For every object y of C the commutative square

$$\begin{array}{ccc} C_{/y} & \xrightarrow{F} & D_{/Fy} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{F} & D \end{array}$$

is a pullback square;

(3) For every object x of C the commutative square

$$\begin{array}{ccc} C_{x/} & \xrightarrow{F} & D_{Fx/} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ C & \xrightarrow{F} & D \end{array}$$

is a pullback square.

Proof. We prove the equivalence between (1) and (2); the equivalence with (3) is dual. Suppose that F is fully faithful. For every $y \in C$, the induced functor $C_{/y} \to C \times_D D_{/Fy}$ is essentially surjective: every object $(x, u\colon Fx \to Fy)$ of the target comes from a morphism $f\colon x \to y$. For $f\colon x \to y$ and $g\colon x' \to y$, the hom anima from f to g and the hom anima between their images are, respectively,

$$\mathrm{Hom}_C(x,x') \times_{\mathrm{Hom}_C(x,y)} \{f\} \qquad \text{and} \qquad \mathrm{Hom}_C(x,x') \times_{\mathrm{Hom}_D(Fx,Fy)} \{Ff\},$$

by Lemma 22.3.2 and Exercise 1.4. The maps from $\mathrm{Hom}_C(x,x')$ are postcomposition with g and its composite with F, respectively. Full faithfulness of F identifies these pullbacks, so the induced functor is fully faithful and hence an equivalence by Axiom J.

Conversely, taking the fibers of the square in (2) over x and Fx gives an isomorphism $\mathrm{Hom}_C(x,y) \xrightarrow{\sim} \mathrm{Hom}_D(Fx,Fy)$ induced by F. Thus F is fully faithful. □

Lemma 22.4.6. *Assume that $\varphi\colon I \hookrightarrow J$ is a fully faithful functor, and let $F\colon I \to C$ be a functor.*

(1) Assume that F admits a pointwise left Kan extension $\varphi_!(F)$ along φ. Then the unit map $\eta_F\colon F \to \varphi^*\varphi_!(F)$ is a natural isomorphism.

(2) Dually, if F admits a pointwise right Kan extension $\varphi_*(F)$ along φ then the counit $\varepsilon_F\colon \varphi^*\varphi_*(F) \to F$ is a natural isomorphism.

Proof. We prove (1); the proof of (2) is dual. By Axiom K we may check that η_F is a pointwise isomorphism. For an object i of I, the pointwise formula of Theorem 22.4.3 identifies the unit component $\eta_F(i)\colon F(i) \to (\varphi_! F)(\varphi(i))$ with the map on colimits induced by the functor

$$I_{/i} \to I_{/\varphi(i)}, \quad (i', f\colon i' \to i) \mapsto (i', \varphi(f)\colon \varphi(i') \to \varphi(i)).$$

But since φ is fully faithful, this functor is an equivalence by Lemma 22.4.5, and it follows that $\eta_F(i)$ is an isomorphism. □

22.5 Cofinal functors

A final functor allows us to change the indexing ∞-category of a diagram without changing its colimit. Initial functors have the analogous property for limits. The following form of Quillen's Theorem A characterizes these functors in terms of relative slices; we take it as a black box:

Theorem 22.5.1 (Joyal's version of Quillen's Theorem A, [Lur09b, Theorem 4.1.3.1])**.** *For a functor $\alpha\colon I \to J$, the following two conditions are equivalent:*

(1) For every ∞-category C and every functor $F\colon J \to C$, the functor F has a colimit if and only if $F \circ \alpha\colon I \to C$ has a colimit, and in this case the canonical map

$$\operatorname{colim}_I(F \circ \alpha) \to \operatorname{colim}_J F$$

is an isomorphism in C;

(2) For every object j of J, the relative slice category $I_{j/} = I \times_J J_{j/}$ is weakly contractible, meaning that its geometric realization $\|I_{j/}\|$ is contractible.

The dual result is also true, where in (1) we use limits and in (2) we use the over categories $I_{/j}$.

Definition 22.5.2. A functor α satisfying the conditions of the theorem is called *final*. We say that α is *initial* if $\alpha^{\mathrm{op}}\colon I^{\mathrm{op}} \to J^{\mathrm{op}}$ is final (i.e. each relative slice $I_{/j}$ is weakly contractible, or equivalently limits of functors $F\colon J \to C$ may be computed as limits of $F \circ \alpha\colon I \to C$).

Lemma 22.5.3. *Every localization functor $q\colon C \to C[W^{-1}]$ is final.*

Proof. Let $F\colon C[W^{-1}] \to D$ be a functor. By the universal property of the localization, precomposition with q is fully faithful. Hence for every object $X \in D$ it induces an equivalence

$$\operatorname{Nat}(F, \operatorname{const}_X) \xrightarrow{\sim} \operatorname{Nat}(F \circ q, \operatorname{const}_X).$$

Thus the cocone functors of F and $F \circ q$ are naturally equivalent, so either is representable if and only if the other is, by the same object. Condition (1) of Theorem 22.5.1 now shows that q is final. □

Warning 22.5.4. The terminology regarding final functors is extremely inconsistent in the literature. Because of historic reasons, final functors are also called *co*final functors, where the 'co' roughly means 'jointly final' and has nothing to do with the usual meaning of 'co' in category theory. Nevertheless, some authors use the words 'final' and 'cofinal' for the two notions from Definition 22.5.2, and there is no consensus as to which of the two words should refer to which of the two notions.

We hope that the convention in Definition 22.5.2 is easy to remember in light of the following lemma:

Lemma 22.5.5. *Consider an object j of an ∞-category J. Then $j\colon * \to J$ is an initial functor if and only if j is an initial object, and it is a final functor if and only if j is a terminal object of J.*

Proof. For an object x of J, the relative slice $*_{/x} = * \times_J J_{/x}$ for the functor $j\colon * \to J$ is the hom anima $\mathrm{Hom}_J(j,x)$. Since an anima agrees with its geometric realization, this slice is weakly contractible precisely when $\mathrm{Hom}_J(j,x)$ is contractible. Thus j is an initial functor if and only if it is an initial object. The assertion about final functors is dual. □

A common way of showing that the relative slice categories $I_{j/}$ or $I_{/j}$ are weakly contractible is by showing they admit an initial or terminal object:

Lemma 22.5.6. *Let C be an ∞-category that admits an initial (resp. terminal) object $x\colon * \to C$. Then C is weakly contractible: $\|C\| \simeq *$.*

Proof. If x is initial, Remark 22.2.3 gives an adjunction $x\colon * \rightleftarrows C : p_C$, which by Lemma 22.1.7 induces an equivalence $* \simeq \| * \| \simeq \|C\|$. The terminal case is dual. □

Corollary 22.5.7. *If a functor $\alpha\colon I \to J$ admits a left adjoint, then α is a final functor. Dually, if it admits a right adjoint it is an initial functor.*

Proof. Let $\beta\colon J \to I$ be a left adjoint to α, with unit $\eta\colon \mathrm{id}_J \to \alpha\beta$. For every $j \in J$, the object $(\beta(j), \eta_j)$ is initial in the relative slice $I_{j/}$. Indeed, by Lemma 22.3.2 and Exercise 1.4, its hom anima to $(i, u\colon j \to \alpha(i))$ is the fiber over u of

$$\mathrm{Hom}_I(\beta(j), i) \longrightarrow \mathrm{Hom}_J(j, \alpha(i)), \qquad f \longmapsto \alpha(f) \circ \eta_j,$$

which is an isomorphism by the adjunction. Thus $I_{j/}$ is weakly contractible by Lemma 22.5.6, proving finality. The assertion about initial functors is dual. □

22.6 Filtered and sifted ∞-categories

In An, filtered colimits commute with finite limits, and sifted colimits commute with finite products. We begin with the conditions on the indexing ∞-categories that characterize these two classes of colimits.

Definition 22.6.1. An ∞-category K is called *filtered* if for every finite diagram $F\colon I \to K$ in K there exists an object $W \in K$ with a cocone $F \to \mathrm{const}_W$; equivalently, F extends to a functor $\widetilde{F}\colon I^{\triangleright} \to K$.

Example 22.6.2. Every directed poset is a filtered ∞-category. In particular, the poset $(\mathbb{N}, \leq)$ of natural numbers is filtered.

More generally, any category which is filtered in the classical sense is filtered as an ∞-category.

The condition on finite diagrams is equivalent to the commutation of K-indexed colimits with finite limits in An. We take this characterization as a black box:

Theorem 22.6.3 ([Lur09b, Proposition 5.3.3.3])**.** *An ∞-category K is filtered if and only if the functor* $\operatorname{colim}_K \colon \operatorname{Fun}(K, \mathrm{An}) \to \mathrm{An}$ *preserves finite limits.*

Here are some consequences:

Corollary 22.6.4. *A filtered colimit* $\operatorname{colim}_K *$ *of terminal animae is again terminal.*

Proof. The constant terminal diagram is terminal in $\operatorname{Fun}(K, \mathrm{An})$, and colim_K preserves terminal objects by Theorem 22.6.3. □

Corollary 22.6.5. *The functor* $\Omega \colon \mathrm{An}_* \to \mathrm{An}_*$ *preserves filtered colimits.*

Proof. View An_* as the full subcategory of $\mathrm{Ar}(\mathrm{An})$ on arrows with terminal source. Colimits of arrows are computed pointwise by Lemma 22.2.8, and filtered colimits of terminal animae are terminal by Corollary 22.6.4. Thus the forgetful functor $\mathrm{An}_* \to \mathrm{An}$ creates filtered colimits. For a filtered diagram $X_\bullet$ of pointed animae, Theorem 22.6.3 gives

$$\operatorname{colim}_i \Omega X_i \simeq \operatorname{colim}_i (* \times_{X_i} *) \simeq * \times_{\operatorname{colim}_i X_i} * \simeq \Omega \operatorname{colim}_i X_i.$$

These isomorphisms respect the basepoints, proving the claim. □

Corollary 22.6.6. *For every $n \geq 0$, the functor* $\pi_n \colon \mathrm{An}_* \to \mathrm{Set}$ *preserves filtered colimits.*

Proof. We may factor this functor as follows:

$$\mathrm{An}_* \xrightarrow{\Omega^n} \mathrm{An}_* \xrightarrow{\mathrm{fgt}} \mathrm{An} \xrightarrow{\pi_0} \mathrm{Set}.$$

Since each of these functors preserves filtered colimits, the claim follows. □

Lemma 22.6.7. *Every filtered ∞-category K is weakly contractible.*

Proof. For every ∞-category I, its geometric realization may be computed as the colimit in An of the constant I-indexed diagram with value $*$. Indeed, for every anima X there are natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathrm{An}}(\operatorname{colim}_{i \in I} *, X) &\simeq \operatorname{Nat}(\mathrm{const}_*, \mathrm{const}_X) \\ &\simeq \operatorname{Fun}(I, X) \\ &\simeq \operatorname{Hom}_{\mathrm{An}}(\|I\|, X). \end{aligned}$$

Indeed, such a natural transformation chooses a point of X for every object of I, a path between the corresponding points for every morphism of I, and all the higher coherence data; this is the data

of a functor $I \to X$. The last isomorphism follows from the universal property of the localization $I \to \|I\|$. The Yoneda lemma therefore gives

$$\|I\| \simeq \mathrm{colim}_{i \in I} *.$$

Applying this to the filtered ∞-category K, we conclude from Corollary 22.6.4 that

$$\|K\| \simeq \mathrm{colim}_{k \in K} * \simeq *.$$

□

Definition 22.6.8. An ∞-category I is called *sifted* if it is nonempty and the diagonal functor $\Delta\colon I \to I \times I$ is final.

Proposition 22.6.9 (The simplex category is sifted)**.** *The ∞-category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted.*

Proof. The ∞-category $\mathbf{\Delta}^{\mathrm{op}}$ is nonempty, and the diagonal $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$ is initial by Proposition 10.4.6 of [CCNW26]. Passing to opposite categories shows that the diagonal of $\mathbf{\Delta}^{\mathrm{op}}$ is final. □

Lemma 22.6.10. *Every filtered ∞-category is sifted.*

Proof. Let K be a filtered ∞-category. Applying filteredness to the empty diagram shows that K is nonempty. To show that its diagonal is final, we will show that for every $(k, l) \in K \times K$ the relative slice category $K \times_{K \times K} (K \times K)_{(k,l)/}$ is filtered, hence weakly contractible by Lemma 22.6.7.

Consider a finite ∞-category I and a functor $F\colon I \to K \times_{K \times K} (K \times K)_{(k,l)/}$. This amounts to a diagram $I \to K$ together with two cones with vertices k and l, respectively. Equivalently, it is encoded by a functor

$$I^{\triangleleft} \sqcup_I I^{\triangleleft} \longrightarrow K$$

taking the two cone vertices to k and l. Its source is finite: finite ∞-categories are closed under pushouts and cones. For the latter assertion, the cone construction preserves pushouts and takes $\emptyset$, $[0]$, and $[1]$ to $[0]$, $[1]$, and $[2]$, respectively, which are finite.

Filteredness of K therefore supplies a cocone on this functor, with some vertex w. Its components at the two cone vertices give maps $k \to w$ and $l \to w$. The restriction to I, together with its compatibility with both cones, gives a cocone on F with vertex $(w, k \to w, l \to w)$ in the relative slice. Thus the relative slice is filtered, as desired. □

Lemma 22.6.11. *Let C be an ∞-category that admits finite products, and assume that for every object $X \in C$ the functor $X \times -\colon C \to C$ preserves colimits. Then the product functor $- \times -\colon C \times C \to C$ preserves sifted colimits.*

Proof. Let I be a sifted ∞-category and let $X, Y\colon I \to C$ be two functors whose colimits exist. We need to show that the map

$$\mathrm{colim}_{i \in I}(X_i \times Y_i) \to (\mathrm{colim}_{i \in I} X_i) \times (\mathrm{colim}_{i \in I} Y_i)$$

is an isomorphism. The assumptions ensure that the iterated colimits below exist, and finality of the diagonal then ensures that the colimit on the left exists:

$$\begin{aligned}\operatorname{colim}_{i\in I}(X_i \times Y_i) &\overset{(1)}{\simeq} \operatorname{colim}_{(i,j)\in I\times I}(X_i \times Y_j)\\ &\overset{(2)}{\simeq} \operatorname{colim}_{i\in I}\operatorname{colim}_{j\in I}(X_i \times Y_j)\\ &\overset{(3)}{\simeq} \operatorname{colim}_{i\in I}(X_i \times \operatorname{colim}_{j\in I} Y_j)\\ &\overset{(4)}{\simeq} (\operatorname{colim}_{i\in I} X_i) \times (\operatorname{colim}_{j\in I} Y_j).\end{aligned}$$

Here isomorphism (1) holds because the diagonal functor $I \to I\times I, i \mapsto (i,i)$ is final. Isomorphism (2) follows by applying the universal properties of the iterated colimits, as in Lemma 22.2.10. Isomorphisms (3) and (4) hold because the product in C preserves colimits in each variable. □

Lemma 22.6.12. *The geometric realization functor* $|-|\colon \mathrm{sAn} \to \mathrm{An}$ *preserves finite products.*

Proof. By definition, this functor is $\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}}\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An}) \to \mathrm{An}$. Since $\mathbf{\Delta}^{\mathrm{op}}$ is sifted by Proposition 22.6.9, Lemma 22.6.11 shows that it preserves binary products: for every anima X, the functor $X \times -$ preserves colimits because it has right adjoint $\mathrm{Hom}_{\mathrm{An}}(X, -)$. The realization of the constant terminal simplicial anima is $\|\mathbf{\Delta}^{\mathrm{op}}\|$. This is contractible by Lemma 22.5.6, since $[0]$ is initial in $\mathbf{\Delta}^{\mathrm{op}}$. Thus geometric realization also preserves the terminal object. □

22.7 Bousfield localizations

A reflective subcategory is a full subcategory such that each object of the ambient ∞-category admits a universal map to an object of the subcategory; the resulting adjoint is then called a *Bousfield localization*.

Definition 22.7.1. A functor $L\colon C \to D$ is called a *Bousfield localization* if it admits a fully faithful right adjoint $R\colon D \hookrightarrow C$. In this case, we also call D a *reflective subcategory of* C.

A *Bousfield colocalization* is instead a functor $C \to D$ that admits a fully faithful *left* adjoint.

Example 22.7.2 (Rationalization)**.** The inclusion $\mathrm{Vect}_{\mathbb{Q}} \hookrightarrow \mathrm{Ab}$ of rational vector spaces into abelian groups admits a left adjoint

$$\mathbb{Q}\otimes_{\mathbb{Z}} -\colon \mathrm{Ab} \to \mathrm{Vect}_{\mathbb{Q}},$$

called the *rationalization functor*. It is a Bousfield localization.

Example 22.7.3 (Abelianization)**.** The inclusion $\mathrm{Ab} \hookrightarrow \mathrm{Grp}$ of abelian groups into all groups admits a left adjoint

$$(-)^{\mathrm{ab}}\colon \mathrm{Grp} \to \mathrm{Ab},$$

given by the *abelianization functor*.

Example 22.7.4 (Group completion)**.** The inclusion $\mathrm{Grp} \hookrightarrow \mathrm{Mon}$ of groups into monoids admits a left adjoint

$$(-)^{\mathrm{grp}}\colon \mathrm{Mon} \to \mathrm{Grp}$$

called the (discrete) *group completion functor*.

Example 22.7.5 (Path components)**.** The inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ admits a left adjoint

$$\pi_0 \colon \mathrm{An} \to \mathrm{Set}$$

given by the set of path components.

Example 22.7.6 (Geometric realization)**.** The inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$ has a left adjoint

$$\| - \| \colon \mathrm{Cat}_\infty \to \mathrm{An}$$

given by the geometric realization functor. It also admits a right adjoint

$$(-)^{\simeq} \colon \mathrm{Cat}_\infty \to \mathrm{An}$$

given by the groupoid core functor.

Example 22.7.7 (Homotopy category)**.** The inclusion $\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$ has a left adjoint

$$\mathrm{Ho}(-) \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_1$$

given by the homotopy category functor.

The terminology is justified by the fact that every Bousfield (co)localization is a localization in the sense of Definition 1.5.19:

Lemma 22.7.8. *Let $L \colon E \to C$ be a Bousfield localization, and let $W := L^{-1}(C^{\simeq})$ be the class of morphisms in E inverted by L. Then L exhibits C as a localization of E at W. The same conclusion holds if L is a Bousfield colocalization.*

Proof. See Proposition 4.4.16 of [CCNW26]. □

In practice, reflective subcategories are often detected via the existence of an idempotent functor with a compatible unit. The following is a precise formulation of this criterion:

Proposition 22.7.9 (cf. [Lur09b, Proposition 5.2.7.4])**.** *Let $L \colon C \to C$ be a functor equipped with a natural transformation $\eta \colon \mathrm{id}_C \Rightarrow L$ such that both maps*

$$\eta_{Lx} \colon Lx \to LLx \qquad \text{and} \qquad L\eta_x \colon Lx \to LLx$$

are isomorphisms for all $x \in C$. Then the functor $L \colon C \to \mathrm{im}(L)$ is left adjoint to the inclusion $\mathrm{im}(L) \hookrightarrow C$, with unit η. In particular, $L \colon C \to \mathrm{im}(L)$ is a Bousfield localization.

Proof. See Proposition 4.4.18 of [CCNW26]. □

Exercises

Exercise 22.1 (Composition of adjunctions)**.** Suppose that $F\colon C \rightleftarrows D : G$ and $F'\colon D \rightleftarrows E : G'$ are adjunctions. Construct the unit and counit of the composite adjunction

$$F'F\colon C \rightleftarrows E : GG'$$

from the units and counits of the given adjunctions, and verify the triangle identities.

Exercise 22.2 (Kan extension along an endpoint)**.** Let $i\colon \{0\} \hookrightarrow [1]$ and let $x \in C$, where C has an initial and a terminal object. Compute both $i_!(x)$ and $i_*(x)$ as arrows in C. Identify the unit and counit maps, and repeat the calculation for the inclusion $\{1\} \hookrightarrow [1]$.

Exercise 22.3 (Cofinal subsequences)**.** Let $u\colon \mathbb{N} \to \mathbb{N}$ be an increasing and unbounded function, regarded as a functor between posets. Show that u is final. Deduce that for every sequence

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots$$

in an ∞-category, passing to a cofinal subsequence does not change its colimit. In particular, compare the colimit of this sequence with the colimit of the subsequence $(X_{2n})_{n\geq 0}$.

Exercise 22.4 (A Bousfield localization of abelian groups)**.** Fix a prime p and consider

$$L\colon \mathrm{Ab} \longrightarrow \mathrm{Ab}, \qquad A \longmapsto A \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].$$

Show that the unit $A \to L(A)$ exhibits L as a Bousfield localization. Identify the local abelian groups and the L-equivalences.

23 Presentable ∞-categories

The large ∞-categories that occur in stable homotopy theory are usually built from small data. Presheaf categories are the basic examples: they are large enough to admit all small colimits, but every presheaf is assembled from the small collection of representable presheaves. Presentable ∞-categories are obtained from presheaf categories by imposing a small collection of relations.

The usual treatment of presentability uses regular cardinals, accessible ∞-categories, and accessible functors. We will not develop that language here. Instead, we take the presentation by generators and relations as our definition. We prove the elementary localization and compactness arguments that fit naturally with this presentation and record the remaining consequences needed elsewhere in the book as black boxes. This is equivalent to the standard definition; see [Lur09b, Theorem 5.5.1.1 and Proposition 5.5.4.15]. We freely use the Yoneda lemma and the elementary presheaf formalism from Section 1.8.

23.1 Presheaves and free cocompletions

The free-cocompletion theorem from Theorem 1.8.10 has a useful variant in which one freely adds only a prescribed collection of colimits.

Definition 23.1.1. Let $\mathcal{K}$ be a collection of small ∞-categories and let C be a small ∞-category. We denote by

$$\mathrm{PSh}^{\mathcal{K}}(C) \subseteq \mathrm{PSh}(C)$$

the smallest full subcategory of $\mathrm{PSh}(C)$ that contains the representable presheaves and is closed under I-indexed colimits for all $I \in \mathcal{K}$.

Proposition 23.1.2 (Lurie [Lur09b, Proposition 5.3.6.2])**.** *Let $\mathcal{K}$ be a collection of small ∞-categories. For every small ∞-category C and every (possibly large) ∞-category D with I-indexed colimits for all $I \in \mathcal{K}$, restriction along the Yoneda embedding $Y\colon C \hookrightarrow \mathrm{PSh}^{\mathcal{K}}(C)$ induces an equivalence*

$$\mathrm{Fun}_{\mathcal{K}}(\mathrm{PSh}^{\mathcal{K}}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D),$$

where the left-hand side denotes the full subcategory spanned by the functors $\mathrm{PSh}^{\mathcal{K}}(C) \to D$ that preserve I-indexed colimits for all $I \in \mathcal{K}$.

23.2 Presentable ∞-categories and adjoint functors

We first make precise what it means to impose a small collection of relations on a presheaf category. Let E be an ∞-category and let S be a collection of morphisms in E. An object $X \in E$ is called *S-local* if for every morphism $f\colon A \to B$ in S, precomposition with f induces an equivalence

$$\mathrm{Hom}_E(B, X) \xrightarrow{\sim} \mathrm{Hom}_E(A, X).$$

We write $E^{S\text{-loc}} \subseteq E$ for the full subcategory of S-local objects.

Definition 23.2.1. An ∞-category D is called *presentable* if there exist a small ∞-category C and a small collection S of morphisms in $\mathrm{PSh}(C)$ such that the inclusion

$$\mathrm{PSh}(C)^{S\text{-loc}} \hookrightarrow \mathrm{PSh}(C)$$

admits a left adjoint and D is equivalent to $\mathrm{PSh}(C)^{S\text{-loc}}$.

Thus a presentation of D consists of a small collection of generators, encoded by C, and a small collection of relations, encoded by S. The presentation is not part of the structure on D, and is very far from unique. The existence of the reflector may in fact be omitted from the definition: any small collection of morphisms in a presheaf category generates a Bousfield localization. We include it to keep the definition directly tied to the localization formalism of Section 22.7.

Theorem 23.2.2 (Basic constructions on presentable ∞-categories, [Lur09b, Theorem 5.5.1.1 and Propositions 5.5.3.6, 5.5.3.10–5.5.3.11, 5.5.4.15 and 5.5.4.20])**.** *The following statements hold.*

(1) For every small ∞-category C, the presheaf category $\mathrm{PSh}(C)$ is presentable. In particular, $\mathrm{An} \simeq \mathrm{PSh}(*)$ is presentable.

(2) If C is presentable and K is small, then the functor category $\mathrm{Fun}(K, C)$ is presentable.

(3) If C is presentable and $X \in C$, then the slice categories $C_{/X}$ and $C_{X/}$ are presentable.

(4) If C is presentable and S is a small collection of morphisms in C, then the full subcategory $C^{S\text{-loc}}$ is presentable and its inclusion into C admits a left adjoint L. For every cocomplete ∞-category E, precomposition with L identifies $\mathrm{Fun}^{\mathrm{L}}(C^{S\text{-loc}}, E)$ with the full subcategory of $\mathrm{Fun}^{\mathrm{L}}(C, E)$ on functors that invert S.

We will also use the following immediate consequence of the definition: every presentable ∞-category C admits a small collection $\mathcal{G}$ of objects such that the functors $\mathrm{Hom}_C(G, -)$, for $G \in \mathcal{G}$, jointly detect isomorphisms. Indeed, in any presentation of C as a localization of a presheaf category, one may take the images under the reflector of the representable presheaves. The adjunction and the Yoneda lemma then reduce detection to the fact that isomorphisms of presheaves are detected objectwise.

We write Pr^{L} for the ∞-category whose objects are presentable ∞-categories and whose morphisms are the colimit-preserving functors. By the adjoint functor theorem below, these are precisely the left adjoint functors. Similarly, we write Pr^{R} for the ∞-category of presentable ∞-categories and right adjoint functors. Passing between left and right adjoints gives an equivalence

$$(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}.$$

Their objects are large ∞-categories, so all statements about Pr^{L} and Pr^{R} are interpreted one universe higher, in the enlarged universe $\widehat{\mathrm{Cat}}_\infty$ from Remark 1.8.1.

Theorem 23.2.3 (Limits of presentable ∞-categories, [Lur09b, Proposition 5.5.3.13 and Theorem 5.5.3.18])**.** *The ∞-categories Pr^{L} and Pr^{R} admit small limits. Moreover, the inclusion functors*

$$\mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Cat}}_\infty \qquad \textit{and} \qquad \mathrm{Pr}^{\mathrm{R}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

preserve small limits.

Corollary 23.2.4 (Presentability of kernels)**.** *Let $F\colon C \to D$ be a colimit-preserving functor between pointed presentable ∞-categories. Then the full subcategory*

$$\ker(F) := \{X \in C \mid F(X) \simeq 0\} \subseteq C$$

is presentable, and its inclusion into C preserves small colimits.

Proof. The functor $* \to D$ selecting the zero object is a left adjoint, since the zero object is initial. The pullback

$$\begin{array}{ccc} \ker(F) & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle F} \\ * & \xrightarrow{0} & D \end{array}$$

may therefore be formed in Pr^{L}. By Theorem 23.2.3, its underlying ∞-category is the pullback in $\widehat{\mathrm{Cat}}_\infty$, which identifies it with the displayed full subcategory. The projection $\ker(F) \to C$ is a morphism in Pr^{L}, so it preserves small colimits. □

One major feature of the presentable setting is that it admits a converse to the statement that every left adjoint preserves colimits:

Theorem 23.2.5 (Adjoint functor theorem, [Lur09b, Corollary 5.5.2.9])**.** *Let C and D be presentable ∞-categories. A functor $F\colon C \to D$ admits a right adjoint if and only if it preserves small colimits.*

There is a dual-looking criterion for the existence of a left adjoint, but preservation of limits by itself is not enough: one must also impose a size condition on the functor. We will not need its general formulation. The special cases involving structured objects that occur later are included in Proposition 23.2.7 below.

Corollary 23.2.6 (Representability, [Lur09b, Proposition 5.5.2.2])**.** *Let C be a presentable ∞-category. A functor $F\colon C^{\mathrm{op}} \to \mathrm{An}$ is representable if and only if it preserves small limits. Equivalently, the Yoneda embedding induces an equivalence*

$$C \xrightarrow{\sim} \mathrm{Fun}^{\lim}(C^{\mathrm{op}}, \mathrm{An}),$$

where $\mathrm{Fun}^{\lim}$ denotes the full subcategory of functors preserving small limits.

Proposition 23.2.7 (Presentability of structured objects)**.** *Let C be a presentable ∞-category.*

(1) The ∞-categories $\mathrm{CMon}(C)$ *and* $\mathrm{CGrp}(C)$ *are presentable, and the inclusions*

$$\mathrm{CGrp}(C) \hookrightarrow \mathrm{CMon}(C) \hookrightarrow \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$$

admit left adjoints.

(2) If C is stable and $\mathcal{G}$ is a small collection of objects of C, then the smallest full stable subcategory $\mathrm{Loc}(\mathcal{G}) \subseteq C$ *that contains $\mathcal{G}$ and is closed under small colimits is presentable. Its inclusion into C preserves small colimits and consequently admits a right adjoint.*

(3) The full subcategory $\mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \subseteq \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$ *of reduced excisive functors is presentable, and its inclusion admits a left adjoint.*

Proof. For (1), choose a small collection $\mathcal{G}$ of objects of C whose corepresentable functors jointly detect isomorphisms. Evaluation at a finite set S has a left adjoint

$$\gamma_S \colon C \longrightarrow \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C).$$

Localizing the presentable functor category $\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$ at the maps

$$\emptyset \longrightarrow \gamma_\emptyset(G) \qquad \text{and} \qquad \gamma_S(G) \sqcup \gamma_T(G) \longrightarrow \gamma_{S \sqcup T}(G)$$

for $G \in \mathcal{G}$ cuts out precisely the product-preserving functors, hence $\mathrm{CMon}(C)$. This is an instance of the accessible-localization theorem [Lur09b, Proposition 5.5.4.15]. Localizing in addition at the endomorphisms induced by the shear $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ cuts out precisely the grouplike objects. This proves the assertion for $\mathrm{CGrp}(C)$ and gives both reflectors.

For (2), apply [Lur17, Proposition 1.4.4.11(2)] to the small collection of all shifts $G[n]$ with $G \in \mathcal{G}$ and $n \in \mathbb{Z}$. The resulting presentable subcategory is $\mathrm{Loc}(\mathcal{G})$. Its inclusion preserves small colimits by definition, so it admits a right adjoint by Theorem 23.2.5.

For (3), choose $\mathcal{G}$ as in (1), and let $\gamma_K \colon C \to \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$ be the left adjoint of evaluation at K. Apply Theorem 23.2.2(4) to the maps

$$\emptyset \longrightarrow \gamma_*(G) \qquad \text{and} \qquad \gamma_Y(G) \sqcup_{\gamma_W(G)} \gamma_Z(G) \longrightarrow \gamma_X(G),$$

where $G \in \mathcal{G}$ and $W \simeq Y \sqcup_X Z$ ranges over pushout squares in $\mathrm{An}_*^{\mathrm{fin}}$. This is a small collection of morphisms. By adjunction and the choice of $\mathcal{G}$, its local objects are precisely the functors taking $*$ to a terminal object and pushout squares to pullback squares, as required. □

23.3 Compact objects and Ind-categories

A compact object behaves like a finite object with respect to filtered colimits: mapping out of it commutes with taking such colimits. This notion suffices for the applications in this book; we will not need the more general notion of κ-compactness.

Definition 23.3.1 (Compact object)**.** Let C be an ∞-category with filtered colimits. An object $X \in C$ is called *compact* if the functor $\mathrm{Hom}_C(X, -) \colon C \to \mathrm{An}$ preserves filtered colimits. We write $C^\omega \subseteq C$ for the full subcategory of compact objects.

Corollary 23.3.2. *If C is a stable ∞-category with filtered colimits, then C^ω is a thick subcategory of C.*

Definition 23.3.3. An ∞-category C is called *compactly generated* if it admits small colimits, the subcategory C^ω is small, and every object of C is a filtered colimit of compact objects.

If C is stable, a small collection $\mathcal{G}$ of compact objects is called a *collection of compact generators* if the smallest localizing subcategory of C containing $\mathcal{G}$ is C itself. Here a localizing subcategory is a full stable subcategory closed under small colimits.

Proposition 23.3.4 (Compact generation criterion, cf. Hovey, Palmieri, and Strickland [HPS97, Corollary 2.3.12])**.** *Let C be a stable ∞-category with small colimits and let $\mathcal{G}$ be a small collection of compact objects. The following conditions are equivalent:*

(1) The collection $\mathcal{G}$ generates C.

(2) The functors

$$\mathrm{Hom}_C(G[n], -) \colon C \to \mathrm{An}, \qquad G \in \mathcal{G},\ n \in \mathbb{Z},$$

are jointly conservative.

(3) Every object of C is a filtered colimit of objects in the smallest thick subcategory of C containing $\mathcal{G}$.

If these conditions hold, then C is compactly generated and its compact objects form the smallest thick subcategory of C containing $\mathcal{G}$.

Proof. Apply the argument of the proof of [Lur17, Corollary 1.4.4.2] to the full subcategory generated by $\mathcal{G}$ under shifts and finite colimits. The description of compact objects then follows from [Lur09b, Lemma 5.4.2.4]. □

The preceding statement is often the most convenient way to prove compact generation. Notice the distinction between the two closure operations: the generators produce all objects under small colimits, but the compact objects are obtained from the generators using only finite colimits, shifts, and retracts.

Definition 23.3.5 (Ind-completion)**.** Let C be a small ∞-category. Its *Ind-completion* is the full subcategory

$$\mathrm{Ind}(C) \subseteq \mathrm{PSh}(C)$$

spanned by the filtered colimits of representable presheaves. The Yoneda embedding factors through a fully faithful functor $C \hookrightarrow \mathrm{Ind}(C)$.

Thus $\mathrm{Ind}(C)$ is the partial free cocompletion from Proposition 23.1.2 obtained by freely adjoining filtered colimits. In particular, if D admits filtered colimits, restriction along the Yoneda embedding induces an equivalence

$$\mathrm{Fun}_\omega(\mathrm{Ind}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D),$$

where the left-hand side denotes the full subcategory of functors preserving filtered colimits.

Proposition 23.3.6 ([Lur09b, Corollary 4.2.3.11, Propositions 5.3.2.9, 5.3.5.10 and 5.3.5.14, and Theorem 5.5.1.1])**.** *If C is a small ∞-category with finite colimits, then* $\mathrm{Ind}(C)$ *admits small colimits and is presentable. Moreover, the inclusion $C \hookrightarrow \mathrm{Ind}(C)$ preserves finite colimits, and the inclusion*

$$\mathrm{Ind}(C) \hookrightarrow \mathrm{PSh}(C)$$

identifies $\mathrm{Ind}(C)$ *with the full subcategory spanned by the presheaves $C^{\mathrm{op}} \to \mathrm{An}$ which send finite colimits in C to limits in* An*.*

Proposition 23.3.7 (Compactly generated ∞-categories as Ind-completions, [Lur09b, Propositions 5.5.7.8 and 5.5.7.10])**.** *Let D be an ∞-category with small colimits and assume that D^{ω} is small. The following are equivalent:*

(1) Every object of D is a filtered colimit of compact objects.

(2) The restricted Yoneda functor induces an equivalence

$$D \xrightarrow{\sim} \mathrm{Ind}(D^{\omega}).$$

Under these conditions, a colimit-preserving functor out of D is uniquely determined by its restriction to D^{ω}.

Remark 23.3.8 (Compact presentations)**.** A presentable ∞-category D is compactly generated if and only if it admits a presentation

$$D \simeq \mathrm{PSh}(C)^{S\text{-loc}}$$

in which C is small and S is a small collection of morphisms between compact objects of $\mathrm{PSh}(C)$. We will not use this characterization; see [Lur09b, Corollary 5.5.7.3 and Proposition 5.5.7.8].

Lemma 23.3.9 (Animae as Ind-completions)**.** *The restricted Yoneda functors induce equivalences*

$$\mathrm{An} \xrightarrow{\sim} \mathrm{Ind}(\mathrm{An}^{\mathrm{fin}}) \qquad \textit{and} \qquad \mathrm{An}_* \xrightarrow{\sim} \mathrm{Ind}(\mathrm{An}_*^{\mathrm{fin}}).$$

Proof. The first equivalence follows immediately from Propositions 23.3.6 and 4.3.11. For the second, Lemma 4.1.13 gives

$$\mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, \mathrm{An}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, \mathrm{An}_*),$$

and the right-hand side is equivalent to An_* by Lemma 4.3.12. In both cases the inverse to evaluation is the restricted Yoneda functor, so the result follows from Proposition 23.3.6. □

Example 23.3.10. The ∞-categories An, An_*, and Sp are compactly generated. For animae and pointed animae this follows from Lemma 23.3.9. The sphere spectrum is a compact generator of Sp: compactness follows because $\mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, -) \simeq \Omega^{\infty}$ preserves filtered colimits by Lemma 4.3.21, while its shifts jointly detect isomorphisms by Corollary 4.4.27; now apply Proposition 23.3.4.

For every associative ring spectrum A, the free module A is a compact generator of LMod_A by Lemma 8.1.11; for every commutative ring R, the complex $R[0]$ is a compact generator of $\mathrm{D}(R)$ by Corollary 21.2.6. Consequently,

$$\mathrm{Sp} \simeq \mathrm{Ind}(\mathrm{Sp}^{\omega}), \qquad \mathrm{LMod}_A \simeq \mathrm{Ind}(\mathrm{Perf}(A)), \qquad \mathrm{D}(R) \simeq \mathrm{Ind}(\mathrm{Perf}(R)).$$

The equivalence $\mathrm{D}(R) \simeq \mathrm{Mod}_R(\mathrm{Sp})$ of Corollary 8.2.4 identifies the last presentation with the corresponding presentation of $\mathrm{Mod}_R(\mathrm{Sp})$.

Exercises

Exercise 23.1 (Representables as compact generators)**.** Let C be a small ∞-category.

(1) Show that every representable presheaf $Y(c) \in \mathrm{PSh}(C)$ is compact.

(2) Show that the representables jointly detect isomorphisms.

(3) Explain how the density presentation of a presheaf exhibits $\mathrm{PSh}(C)$ as generated under colimits by the representables.

Exercise 23.2 (A presheaf localization)**.** Let C be small and let S be a small collection of morphisms in C. Regard each $s\colon c \to d$ as a morphism $Y(c) \to Y(d)$ of representable presheaves.

(1) Show that a presheaf F is local with respect to these maps if and only if $F(s)\colon F(d) \to F(c)$ is an equivalence for every $s \in S$.

(2) Identify the local presheaves with presheaves on the localization $C[S^{-1}]$.

(3) Conclude that this full subcategory is presentable.

Exercise 23.3 (An Ind-completion)**.** Show that every set is a filtered colimit of its finite subsets and that finite sets are compact in Set. Prove that

$$\mathrm{Ind}(\mathrm{Fin}) \simeq \mathrm{Set}.$$

Explain why the analogous statement for animae requires finite animae rather than finite sets.

24 Cocartesian fibrations and (un)straightening

An important notion in ∞-category theory is that of a *cocartesian fibration*. Roughly speaking, a cocartesian fibration is a functor $p\colon E \to C$ whose fibers $E_x := E \times_C \{x\}$ vary covariantly in x, in the sense that for every morphism $f\colon x \to y$ in C we obtain a *cocartesian transport functor* $f_!\colon E_x \to E_y$. These functors are fully coherent in f, in the sense that they assemble into a functor $\mathrm{Str}(p)\colon C \to \mathrm{Cat}_\infty$, given on objects by sending x to E_x and on morphisms by sending f to $f_!$. This provides a powerful way to construct functors into Cat_∞, known as the *straightening/unstraightening correspondence*. For example, it will allow us to formally define the hom functor $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$ of an ∞-category C.

24.1 Cartesian and cocartesian fibrations

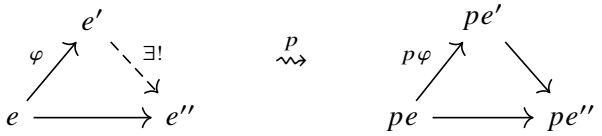
To define transport along $f\colon x \to y$, we consider lifts of f to a morphism $e \to e'$ in E with a prescribed source $e \in E_x$. The cocartesian morphisms are the ones that satisfy the following universal property:

Definition 24.1.1. Consider a functor $p\colon E \to C$.

(1) A morphism $\varphi\colon e \to e'$ in E is called *p-cocartesian* if, for every other object e'' in E, the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_E(e', e'') & \xrightarrow{-\circ\varphi} & \mathrm{Hom}_E(e, e'') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Hom}_C(pe', pe'') & \xrightarrow{-\circ p(\varphi)} & \mathrm{Hom}_C(pe, pe'') \end{array}$$

is a pullback square. Informally, this means that for every solid diagram of the form



in E for which its image in C has been completed to a triangle, there exists a unique map $e' \to e''$ that forms a commutative triangle in E and projects to the given one in C.

(2) We say that p is a *cocartesian fibration* if for every morphism $f\colon x \to y$ in C and every object $e \in E_x$ there exists a p-cocartesian morphism $\varphi\colon e \to e'$ satisfying $p(\varphi) = f$. We refer to such a morphism φ as a *p-cocartesian lift of f*.

(3) Let $p\colon E \to C$ and $p'\colon E' \to C$ be two cocartesian fibrations. A functor $F\colon E \to E'$ over C, i.e., a commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & C, & \end{array}$$

is called *cocartesian over C* if it sends p-cocartesian morphisms in E to p'-cocartesian morphisms in E'.

(4) If C is a small ∞-category, we denote by

$$\mathrm{Cocart}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$$

the non-full subcategory spanned by the cocartesian fibrations $p\colon E \to C$ and the cocartesian functors between them.

The cocartesian lift $\varphi\colon e \to e'$ is functorial in the pair (f, e). To show this, the following reformulation of p-cocartesian morphisms is useful:

Lemma 24.1.2. *Let $p\colon E \to C$ be a functor. Consider the functor*

$$Q\colon \mathrm{Ar}(E) \longrightarrow \mathrm{Ar}(C) \times_{s,C,p} E, \qquad (\varphi\colon e \to e') \longmapsto (p\varphi, e).$$

An arrow $\varphi\colon e \to e'$ satisfying $Q(\varphi) = (f, e)$ is p-cocartesian if and only if, equipped with the identity map of (f, e), it is a left adjoint object to (f, e) under Q.

Proof. Set $B := \mathrm{Ar}(C) \times_{s,C,p} E$. Since $Q(\varphi) = (f, e)$, the identity of (f, e) exhibits φ as a left adjoint object under Q if and only if, for every arrow $\psi\colon a \to b$ in E, the induced map

$$Q_*\colon \mathrm{Hom}_{\mathrm{Ar}(E)}(\varphi, \psi) \longrightarrow \mathrm{Hom}_B((f, e), Q(\psi))$$

is an equivalence. By Axiom D and Exercise 1.4, its source and target admit canonical descriptions

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ar}(E)}(\varphi, \psi) &\xrightarrow{\sim} \mathrm{Hom}_E(e, a) \times_{\mathrm{Hom}_E(e,b)} \mathrm{Hom}_E(e', b), \\ \mathrm{Hom}_B((f, e), Q(\psi)) &\xrightarrow{\sim} \mathrm{Hom}_E(e, a) \times_{\mathrm{Hom}_C(pe,pb)} \mathrm{Hom}_C(pe', pb). \end{aligned}$$

Here the maps out of $\mathrm{Hom}_E(e, a)$ are given by postcomposition with ψ in the first line and by postcomposition with $p(\psi)$ after applying p in the second line. Under these identifications, Q_* is induced by $p\colon \mathrm{Hom}_E(e', b) \to \mathrm{Hom}_C(pe', pb)$ on the second factor.

If φ is p-cocartesian, then for every b the defining pullback square identifies

$$\mathrm{Hom}_E(e', b) \longrightarrow \mathrm{Hom}_E(e, b) \times_{\mathrm{Hom}_C(pe,pb)} \mathrm{Hom}_C(pe', pb)$$

as an equivalence. Pulling this equivalence back along the postcomposition map $\psi \circ -\colon \mathrm{Hom}_E(e, a) \to \mathrm{Hom}_E(e, b)$ shows that Q_* is an equivalence for every ψ. Thus φ is a left adjoint object under Q.

Conversely, suppose that φ is a left adjoint object under Q. Taking $\psi = \mathrm{id}_b$ identifies Q_* with the preceding comparison map. It is therefore an equivalence for every object b of E, which says that the squares defining p-cocartesianness of φ are pullback squares. □

Notation 24.1.3. Given an object $e \in E_x$, we may sometimes denote a cocartesian lift of $f : x \to y$ by $\varphi : e \to f_! e$, and refer to $f_! e$ as the *cocartesian transport of e along f*. This is well-defined, since left adjoint objects are unique whenever they exist.

Proposition 24.1.4 (Functoriality of partial cocartesian lifts)**.** *Let $p : E \to C$ be a functor, and let $W \subseteq \mathrm{Ar}(C)$ be a subcategory. Suppose that for every $(f, e) \in W \times_{s,C,p} E$ there exists a p-cocartesian lift of f starting in e. Then these lifts assemble into a functor*

$$\mathrm{lift}_W : W \times_{s,C,p} E \longrightarrow \mathrm{Ar}(E)$$

More precisely, this functor factors through

$$\mathrm{Ar}(E) \times_{\mathrm{Ar}(C) \times_{s,C,p} E} \big(W \times_{s,C,p} E\big),$$

and the resulting functor is left adjoint to the projection from this pullback to $W \times_{s,C,p} E$.

Proof. By Lemma 24.1.2, the required cocartesian lifts are precisely the left adjoint objects under the restricted functor

$$\mathrm{Ar}(E) \times_{\mathrm{Ar}(C) \times_{s,C,p} E} \big(W \times_{s,C,p} E\big) \longrightarrow W \times_{s,C,p} E.$$

The claim is therefore an immediate consequence of the pointwise criterion for adjunctions from Lemma 22.1.4. □

Corollary 24.1.5. *Let $p : E \to C$ be a cocartesian fibration. Then the cocartesian lifts assemble into a left adjoint*

$$\mathrm{lift} : \mathrm{Ar}(C) \times_{s,C,p} E \to \mathrm{Ar}(E), \qquad (f, e) \mapsto (\varphi : e \to f_! e)$$

to the functor from Lemma 24.1.2.

Proof. Apply Proposition 24.1.4 with $W = \mathrm{Ar}(C)$. □

Corollary 24.1.6. *Let $p : E \to C$ be a cocartesian fibration, and let $f : x \to y$ be a morphism in C. Then cocartesian transport assembles into a functor $f_! : E_x \to E_y$.*

Proof. We may obtain $f_!$ by restricting the functor lift from the previous corollary. □

We may also deduce the uniqueness of cocartesian lifts:

Lemma 24.1.7 (Uniqueness of cocartesian lifts)**.** *For a functor $p : E \to C$, consider the full subcategory*

$$\mathrm{Ar}^{\mathrm{cocart}}(E) \subseteq \mathrm{Ar}(E)$$

spanned by the p-cocartesian morphisms. The functor

$$\mathrm{Ar}^{\mathrm{cocart}}(E) \to \mathrm{Ar}(C) \times_{s,C,p} E, \quad (\varphi : e \to e') \mapsto (p\varphi, e)$$

is fully faithful. It is an equivalence if and only if p is a cocartesian fibration.

Proof. Set $B := \mathrm{Ar}(C) \times_{s,C,p} E$, and let $B^{\mathrm{lift}} \subseteq B$ be the full subcategory spanned by the pairs (f, e) that admit a cocartesian lift. By Lemmas 24.1.2 and 22.1.4, these lifts assemble into a left adjoint

$$L \colon B^{\mathrm{lift}} \longrightarrow \mathrm{Ar}(E) \times_B B^{\mathrm{lift}}$$

to the projection. Its unit is an equivalence, so L is fully faithful. Its essential image is precisely $\mathrm{Ar}^{\mathrm{cocart}}(E)$, since the cocartesian arrows are exactly the left adjoint objects described in Lemma 24.1.2. Thus the displayed functor identifies with the inclusion $B^{\mathrm{lift}} \hookrightarrow B$. It is therefore fully faithful, and it is an equivalence precisely when every pair (f, e) admits a cocartesian lift. □

The definition of cocartesian fibrations may be dualized:

Definition 24.1.8. For a functor $p \colon E \to C$, we say that a morphism $\varphi \colon e \to e'$ in E is *p-cartesian* if for every third object e'' in E the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_E(e'', e) & \xrightarrow{\varphi \circ -} & \mathrm{Hom}_E(e'', e') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Hom}_C(pe'', pe) & \xrightarrow{p\varphi \circ -} & \mathrm{Hom}_C(pe'', pe') \end{array}$$

is a pullback square. We say p is a *cartesian fibration* if for every morphism $f \colon x \to y$ and every $e' \in E_y$ there exists a p-cartesian lift $\varphi \colon e \to e'$ of f. We similarly obtain a notion of a *cartesian functor over C*, and this leads to a (non-full) subcategory

$$\mathrm{Cart}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}.$$

Remark 24.1.9. Note that a functor $p \colon E \to C$ is a cartesian fibration if and only if its opposite $p^{\mathrm{op}} \colon E^{\mathrm{op}} \to C^{\mathrm{op}}$ is a cocartesian fibration. All properties about cocartesian fibrations have dual analogues for cartesian fibrations; we will generally not spell these out explicitly.

For example, the target functor $\mathrm{Ar}(C) \to C$ is a cocartesian fibration:

Example 24.1.10. Let C be an ∞-category, let $E := \mathrm{Ar}(C)$ be its arrow category, and consider the target functor $p := t \colon \mathrm{Ar}(C) \to C$. An object of E is given by a morphism in C, which for emphasis we will write vertically: $\binom{X}{\downarrow}_Y$. A morphism $\varphi \colon \binom{X}{\downarrow}_Y \to \binom{X'}{\downarrow}_{Y'}$ in E is given by a commutative square in C of the form

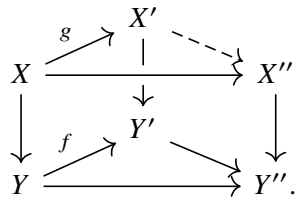
$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & Y'. \end{array}$$

We claim:

(1) The morphism φ in E is p-cocartesian if and only if g is an isomorphism. In particular, t is *always* a cocartesian fibration: given $e = \binom{X}{\downarrow}_Y$ and $f \colon p(\binom{X}{\downarrow}_Y) = Y \to Y'$ we may always form the square with $X' := X$ and $g = \mathrm{id}_X$.

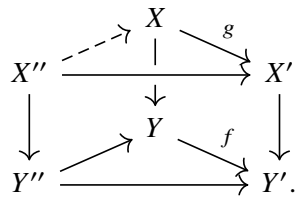
(2) The morphism φ in E is p-cartesian if and only if the square is a pullback square (a.k.a. a *cartesian* square). In particular, t is a cartesian fibration if and only if C admits pullbacks.

Indeed, claim (1) translates the defining universal property of a cocartesian morphism into the following lifting property: for every third object $\binom{X''}{\downarrow}{Y''}$ of E and every solid diagram below, there exists a unique dashed arrow making the diagram commute:



If g is an isomorphism, we may invert it and the dashed filler is unique. Conversely, if φ is p-cocartesian, then by taking $X'' = X$ we obtain a morphism $g' \colon X' \to X$ satisfying $g'g \cong \mathrm{id}_X$, and by taking $X'' = X'$ we use uniqueness to obtain $gg' \cong \mathrm{id}_{X'}$.

Similarly, claim (2) translates the defining universal property of a cartesian morphism into the existence of a unique dashed arrow in every solid diagram of the following form:



This is the universal property of the original square being a pullback square. Passing to opposite categories shows dually that the source functor $s \colon \mathrm{Ar}(C) \to C$ is always a cartesian fibration, and that it is also a cocartesian fibration if and only if C admits pushouts.

Example 24.1.11. The forgetful functor VectBund $\to$ Top, $(X, E) \mapsto X$ is a cartesian fibration. See Exercise 24.1 for details.

Lemma 24.1.12. *The following closure properties hold for cocartesian fibrations.*

(1) If $q \colon E \to D$ and $p \colon D \to C$ are cocartesian fibrations, then $pq \colon E \to C$ is a cocartesian fibration, and q is a cocartesian functor over C.

(2) Consider a pullback square

$$\begin{array}{ccc} E' & \xrightarrow{F} & E \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C' & \longrightarrow & C. \end{array}$$

If p is a cocartesian fibration, then so is p', and a morphism in E' is p'-cocartesian if and only if its image under F is p-cocartesian.

Proof. For part (1), pick a cocartesian lift along p of a morphism in C, and then a further cocartesian lift along q. The two defining pullback squares paste to show that the resulting morphism in E is pq-cocartesian. This proves that pq is a cocartesian fibration. By Lemma 24.1.7, every pq-cocartesian lift is isomorphic to one constructed in this way, so q carries it to a p-cocartesian morphism.

For part (2), a p-cocartesian lift of the image in C of a morphism in C' determines a morphism in the pullback E'. The universal property of the pullback identifies its defining cocartesian square with the base change of the corresponding square for p, so this morphism is p'-cocartesian. Hence p' is a cocartesian fibration. The same argument proves that every morphism whose image under F is p-cocartesian is p'-cocartesian; the converse follows from uniqueness of cocartesian lifts. □

Lemma 24.1.13. *The functor $C \to *$ is both a cartesian and cocartesian fibration. Consequently, every projection functor $\mathrm{pr}_D \colon C \times D \to D$ is both a cartesian and cocartesian fibration, and a morphism $(f \colon c \to c', g \colon d \to d')$ is pr_D-cocartesian or pr_D-cartesian if and only if f is an isomorphism.*

Proof. The unique morphism in $*$ has identity lifts. The assertion for pr_D follows by pulling this fibration back along $D \to *$. Using the product formula for hom animae in the defining pullback square, the cocartesian or cartesian condition reduces to requiring that precomposition or postcomposition with f induce equivalences on all hom animae of C. By the Yoneda lemma, this holds if and only if f is invertible. □

Lemma 24.1.14 (Right cancellation of cocartesian morphisms)**.** *Let $p \colon E \to C$ be a functor and let $f \colon x \to y$ and $g \colon y \to z$ be morphisms in E such that f is p-cocartesian. Then g is p-cocartesian if and only if gf is p-cocartesian.*

Proof. For every other object $w \in E$, we consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_E(z,w) & \xrightarrow{-\circ g} & \mathrm{Hom}_E(y,w) & \xrightarrow{-\circ f} & \mathrm{Hom}_E(x,w) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathrm{Hom}_C(pz,pw) & \xrightarrow{-\circ p(g)} & \mathrm{Hom}_C(py,pw) & \xrightarrow{-\circ p(f)} & \mathrm{Hom}_C(px,pw).
\end{array}$$

Since f is p-cocartesian, the right square is a pullback square. It follows from the pasting law for pullback squares that the left square is a pullback if and only if the outer square is a pullback. This proves the claim. □

Lemma 24.1.15. *Let $p \colon E \to C$ be a functor and let $\varphi \colon e \to e'$ be a p-cocartesian morphism. Then φ is an isomorphism if and only if $p(\varphi)$ is an isomorphism.*

Proof. Only the converse requires proof. Suppose that $p(\varphi)$ is invertible. Cocartesianness provides a morphism $\psi \colon e' \to e$ over $p(\varphi)^{-1}$ satisfying $\psi\varphi \cong \mathrm{id}_e$. The morphisms $\varphi\psi$ and $\mathrm{id}_{e'}$ have the same image in C and agree after precomposition with φ. The defining universal property of φ therefore gives $\varphi\psi \cong \mathrm{id}_{e'}$, so φ is invertible. □

24.1.1 Left and right fibrations

Proposition 24.1.16. *Let $p \colon E \to C$ be a cocartesian fibration. Then the following three conditions are equivalent:*

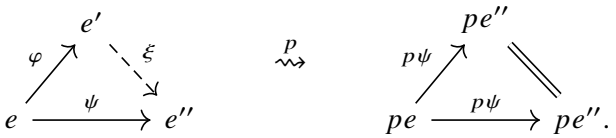
(1) The functor p is conservative, i.e., a morphism f in E is an isomorphism as soon as $p(f)$ is an isomorphism in C;

(2) For each $x \in C$, the fiber $E_x := \{x\} \times_C E$ is an anima;

(3) Every morphism in E is p-cocartesian.

Proof. It is clear that (1) implies (2): every morphism in the fiber E_x is sent to the identity morphism id_x, hence is an isomorphism by conservativity of p.

Assuming (2), consider a morphism $\psi\colon e \to e''$ in E. We need to show that ψ is p-cocartesian. Since p is a cocartesian fibration, there exists a p-cocartesian morphism $\varphi\colon e \to e'$ lifting $p(\psi)$. Since φ is p-cocartesian, there then exists a unique morphism $\xi\colon e' \to e''$ fitting in the following commutative diagram lifting the degenerate one in C:



But then the morphism ξ lies in the fiber $E_{pe''}$, which is an anima, so ξ is an isomorphism. In particular ξ is p-cocartesian, and hence so is $\psi = \xi \circ \varphi$.

Finally, assume that (3) holds, and consider a morphism $\varphi\colon e \to e'$ in E such that $p\varphi$ is invertible. We need to show that φ is invertible. By assumption, there is an inverse $g\colon pe' \to pe$ of $p\varphi$. We may then find a p-cocartesian lift $\psi\colon e' \to e''$ of g. But then $\varphi \circ \psi$ and $\psi \circ \varphi$ are cocartesian lifts of the respective identity maps on pe and pe'. By uniqueness of cocartesian lifts (Lemma 24.1.7) these composites are isomorphisms. It follows that φ is an isomorphism with inverse ψ. □

Definition 24.1.17. A functor $p\colon E \to C$ is called a *left fibration* if it is a cocartesian fibration satisfying the equivalent conditions (1), (2) and (3) from the proposition. This gives rise to a full subcategory

$$\mathrm{LFib}(C) \subseteq \mathrm{Cocart}(C).$$

Note that $\mathrm{LFib}(C)$ is also a full subcategory of $(\mathrm{Cat}_\infty)_{/C}$ (in contrast to $\mathrm{Cocart}(C)$).

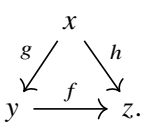
Dually, p is called a *right fibration* if it is a cartesian fibration which satisfies the equivalent conditions (1), (2) and (3'): Every morphism in E is cartesian. We write $\mathrm{RFib}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$ for the resulting full subcategory.

Lemma 24.1.18. *Left fibrations are closed under composition and pullback.*

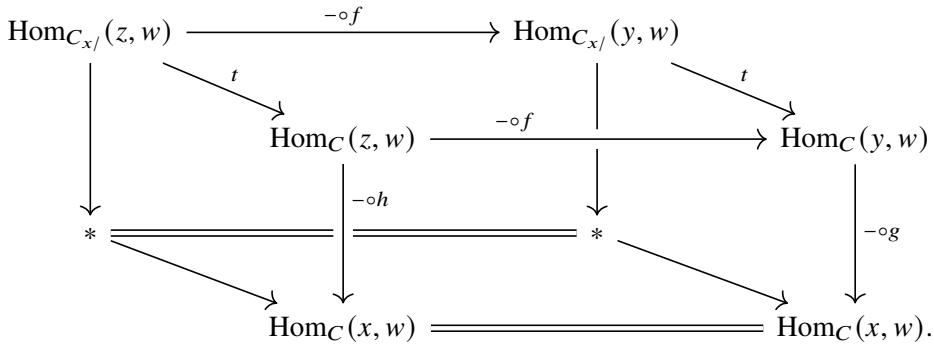
Proof. Follows from Lemma 24.1.12. □

Proposition 24.1.19. *Let C be an ∞-category. For every object $x \in C$, the functor $t\colon C_{x/} \to C$ sending a morphism $x \to y$ to its target y is a left fibration. Dually, the source functor $s\colon C_{/x} \to C$ is a right fibration.*

Proof. We prove the case for $t\colon C_{x/} \to C$; the other case is dual. Consider a morphism in $C_{x/}$, which takes the form of a commutative triangle



To show that this morphism is cocartesian with respect to the target functor $t\colon C_{x/} \to C$, we have to show that for every third object $x \to w$, the top square in the following commutative diagram is a pullback square:



But this follows from the pasting law for pullback squares, since the bottom square and left/right faces are pullback squares. □

24.1.2 Adjunctions

Lemma 24.1.20. *Let $p\colon E \to C$ be a cocartesian fibration. Then the following conditions are equivalent:*

(1) *The functor p is also a cartesian fibration.*

(2) *For every morphism $f\colon X \to Y$ in C, the cocartesian transport functor $f_!\colon E_X \to E_Y$ admits a right adjoint $f^*\colon E_Y \to E_X$.*

Proof. The cartesian lift of $f\colon X \to Y$ ending at $e \in E_Y$ is obtained by composing the cocartesian lift $f^*e \to f_!f^*e$ with the counit $f_!f^*e \to e$. The universal property, as well as the converse construction of the adjunction from cartesian lifts, is proved in Proposition 6.5.11 of [CCNW26]. □

Corollary 24.1.21. *Let $F\colon C \to D$ be a functor between small ∞-categories, encoded by a morphism $[1] \to \mathrm{Cat}_\infty$. Let $E \to [1]$ be the cocartesian unstraightening of this functor. Then F admits a right adjoint if and only if the cocartesian fibration $E \to [1]$ is also a cartesian fibration.* □

24.2 Straightening/unstraightening

Let $p\colon E \to C$ be a cocartesian fibration. For every object x of C, we may consider the fiber E_x of p over x. For a morphism $f\colon x \to y$ in C, there is a *cocartesian transport* functor $f_!\colon E_x \to E_y$. Furthermore, given another morphism $g\colon y \to z$, there is a natural isomorphism

$$g_! \circ f_! \cong (g \circ f)_!\colon E_x \to E_z.$$

The transport functors and their composition isomorphisms thus suggest a functor

$$\mathrm{Str}^{\mathrm{cc}}(p)\colon C \to \mathrm{Cat}_\infty, \quad x \mapsto E_x, \quad f \mapsto f_!.$$

Straightening/unstraightening supplies this functor, including its higher coherences:

Theorem 24.2.1 (Straightening/Unstraightening, Lurie [Lur09b], Hebestreit, Heuts, and Ruit [HHR21])**.** *Let C be a small ∞-category. Then there exist equivalences of ∞-categories*

$$\mathrm{Str}^{\mathrm{cc}}\colon\ \mathrm{Cocart}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Cat}_\infty)$$

and

$$\mathrm{Str}^{\mathrm{ct}}\colon\ \mathrm{Cart}(C) \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty).$$

Furthermore, these equivalences are natural[21] *in C, in the sense that for every functor $F\colon C \to D$ the following two squares commute:*

$$\begin{array}{ccc} \mathrm{Cocart}(D) & \xrightarrow{\mathrm{Str}^{\mathrm{cc}}} & \mathrm{Fun}(D, \mathrm{Cat}_\infty) \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle -\circ F} \\ \mathrm{Cocart}(C) & \xrightarrow{\mathrm{Str}^{\mathrm{cc}}} & \mathrm{Fun}(C, \mathrm{Cat}_\infty) \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \mathrm{Cart}(D) & \xrightarrow{\mathrm{Str}^{\mathrm{ct}}} & \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Cat}_\infty) \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle -\circ F^{\mathrm{op}}} \\ \mathrm{Cart}(C) & \xrightarrow{\mathrm{Str}^{\mathrm{ct}}} & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty). \end{array}$$

Finally, the composites

$$\mathrm{Cat}_\infty \simeq \mathrm{Cocart}(*) \xrightarrow{\mathrm{Str}^{\mathrm{cc}}} \mathrm{Fun}(*, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty$$
$$\mathrm{Cat}_\infty \simeq \mathrm{Cart}(*) \xrightarrow{\mathrm{Str}^{\mathrm{ct}}} \mathrm{Fun}(*, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty$$

are (isomorphic to) the identity functors.

We take straightening/unstraightening as an axiom. It was first proved in the model of quasicategories by Jacob Lurie; a substantially simpler proof is given by Hebestreit, Heuts, and Ruit [HHR21]. It is arguably one of the most fundamental results in the theory of ∞-categories.

Given a cocartesian fibration $p\colon E \to C$, we refer to the functor $\mathrm{Str}^{\mathrm{cc}}(p)\colon C \to \mathrm{Cat}_\infty$ as its *(cocartesian) straightening*, and similarly for $\mathrm{Str}^{\mathrm{ct}}(p)$. The inverses of $\mathrm{Str}^{\mathrm{cc}}$ and $\mathrm{Str}^{\mathrm{ct}}$ are denoted by

$$\mathrm{Un}^{\mathrm{cc}}\colon\ \mathrm{Fun}(C, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cocart}(C), \qquad \text{and} \qquad \mathrm{Un}^{\mathrm{ct}}\colon\ \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cart}(C).$$

Given a functor $F\colon C \to \mathrm{Cat}_\infty$, we refer to the cocartesian fibration $\mathrm{Un}^{\mathrm{cc}}(F) \to C$ as its *cocartesian unstraightening*, and to the cartesian fibration $\mathrm{Un}^{\mathrm{ct}}(F)$ as the *cartesian unstraightening*.

Lemma 24.2.2. *Let S be a small ∞-category, and let $\alpha\colon F \Rightarrow G$ be a natural transformation between functors $F, G\colon S \to \mathrm{Cat}_\infty$. If every functor $\alpha_s\colon F(s) \to G(s)$ is a left fibration, then the induced functor*

$$\mathrm{Un}^{\mathrm{cc}}(\alpha)\colon\ \mathrm{Un}^{\mathrm{cc}}(F) \longrightarrow \mathrm{Un}^{\mathrm{cc}}(G)$$

is a left fibration. Its fiber over $(s, g) \in \mathrm{Un}^{\mathrm{cc}}(G)$ is the fiber of α_s over g.

Proof. By Theorem 24.2.1, the functor $\mathrm{Un}^{\mathrm{cc}}(\alpha)$ is a morphism in $\mathrm{Cocart}(S)$, that is, a cocartesian functor over S. Applying the naturality of straightening to the functor $s\colon * \to S$, and using that straightening over a point is the identity, we see that its restriction to the fibers over $s \in S$ is identified with $\alpha_s\colon F(s) \to G(s)$. Since every α_s is a left fibration, $\mathrm{Un}^{\mathrm{cc}}(\alpha)$ is a left fibration by Corollary 6.5.14 of [CCNW26]. For the description of the fibers, note that $g \in G(s)$ lies over s, so the fiber of $\mathrm{Un}^{\mathrm{cc}}(\alpha)$ over g can be computed after first passing to the fibers over s; it is therefore the fiber of α_s over g. □

[21] See Remark 24.2.11 for a more refined version of the naturality.

Lemma 24.2.3. *Let $p\colon E \to C$ be a cocartesian fibration and let I be a small ∞-category. Then*

$$p_*\colon \mathrm{Fun}(I, E) \longrightarrow \mathrm{Fun}(I, C)$$

is a cocartesian fibration. A natural transformation in $\mathrm{Fun}(I, E)$ is p_*-cocartesian if and only if all its components are p-cocartesian.

Proof. See Corollary 6.5.8 of [CCNW26]. □

We will use the following compatibility of unstraightening with functor categories. Here $\mathrm{const}\colon C \to \mathrm{Fun}(I, C)$ denotes the constant-diagram functor.

Proposition 24.2.4. *Let $p\colon E \to C$ be a cocartesian fibration classified by a functor $F\colon C \to \mathrm{Cat}_\infty$, and let I be a small ∞-category. The composite*

$$C \xrightarrow{F} \mathrm{Cat}_\infty \xrightarrow{\mathrm{Fun}(I,-)} \mathrm{Cat}_\infty$$

is classified by the cocartesian fibration

$$C \times_{\mathrm{Fun}(I,C)} \mathrm{Fun}(I, E) \longrightarrow C,$$

where the map $\mathrm{Fun}(I, E) \to \mathrm{Fun}(I, C)$ is induced by p and the map from C sends an object to its constant I-diagram.

Proof. See Proposition 8.7.13 of [CCNW26]. □

Example 24.2.5. When $C = [0]$, the last statement of the theorem implies that for a cocartesian fibration $p\colon E \to C$ the ∞-categories $\mathrm{Str}^{\mathrm{cc}}(p)(x)$ for $x \in C$ are the fibers of p.

Example 24.2.6. An illustrative example is (un)straightening over the walking morphism $C = [1]$.

Consider first a cocartesian fibration $p\colon E \to [1]$, with E small, and let E_0 and E_1 denote its fibers over 0 and 1, respectively. Cocartesian transport along the canonical morphism in $[1]$ results in a functor $F\colon E_0 \to E_1$, which corresponds to a morphism in Cat_∞, i.e. a functor $[1] \to \mathrm{Cat}_\infty$. This is $\mathrm{Str}^{\mathrm{cc}}(p)$.

Conversely, consider a functor $F\colon E_0 \to E_1$ between small ∞-categories. Regarding this as a morphism in Cat_∞, its unstraightening takes the form of a cocartesian fibration $E \to [1]$. Since the fibers over 0 and 1 are E_0 and E_1, there are inclusions $i_0\colon E_0 \hookrightarrow E$ and $i_1\colon E_1 \hookrightarrow E$. Moreover, by choosing coherent choices of cocartesian lifts, there is a natural transformation $\alpha\colon i_0 \Rightarrow i_1 \circ F$ of functors $E_0 \to E$. All in all, we have constructed a commutative square in Cat_∞ as follows:

$$\begin{array}{ccc} E_0 \times \{1\} & \hookrightarrow & E_0 \times [1] \\ & & \downarrow{\scriptstyle \alpha} \\ E_1 & \xrightarrow{i_1} & E. \end{array}$$

It can be shown that this square is a pushout square in Cat_∞, see for example [Lur09b, Section 3.2.2].

Let us now state some immediate consequences of the theorem.

Corollary 24.2.7. *The straightening equivalences* $\mathrm{Str}^{\mathrm{cc}}$ *and* $\mathrm{Str}^{\mathrm{ct}}$ *restrict to equivalences*

$$\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{LFib}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{An}) \qquad \textit{and} \qquad \mathrm{Str}^{\mathrm{ct}}\colon \mathrm{RFib}(C) \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

Proof. Given a cocartesian fibration $p\colon E \to C$, the individual values of $\mathrm{Str}^{\mathrm{cc}}(p)$ are the fibers of p. In particular, $\mathrm{Str}^{\mathrm{cc}}(p)$ lands in the subcategory $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ if and only if all fibers are animae. By Proposition 24.1.16, this is equivalent to p being a left fibration, proving the first equivalence. The dual discussion provides the right equivalence. □

Corollary 24.2.8. *Let X be an anima. Then there is an equivalence of ∞-categories*

$$\mathrm{Str}\colon \mathrm{An}_{/X} \xrightarrow{\sim} \mathrm{Fun}(X, \mathrm{An}).$$

Proof. If C is an anima, then every functor $E \to C$ is both a cocartesian and a cartesian fibration. Moreover, it is a left/right fibration if and only if also E is an anima. The claim now becomes an instance of Corollary 24.2.7. □

Notation 24.2.9. We will denote the inverses to the straightening functors $\mathrm{Str}^{\mathrm{cc}}$ and $\mathrm{Str}^{\mathrm{ct}}$ by

$$\mathrm{Un}^{\mathrm{cc}}\colon \mathrm{Fun}(C, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cocart}(C) \qquad \text{and} \qquad \mathrm{Un}^{\mathrm{ct}}\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cart}(C),$$

and refer to these as (cocartesian/cartesian) *unstraightening*. Given a functor $F\colon C \to \mathrm{Cat}_\infty$, we may describe its unstraightening $\mathrm{Un}^{\mathrm{cc}}(F)$ informally as follows:

- An object in $\mathrm{Un}^{\mathrm{cc}}(F)$ is a pair (x, a), where x is an object of C and a is an object of $F(x)$;
- A morphism in $\mathrm{Un}^{\mathrm{cc}}(F)$ from (x, a) to (y, b) is a pair (f, φ), where $f\colon x \to y$ is a morphism in C and $\varphi\colon f_!(a) \to b$ is a morphism in $F(y)$. Here we write $f_!$ for the functor $F(f)\colon F(x) \to F(y)$ induced by f using the functoriality of F.
- Composition of $(f, \varphi)\colon (x, a) \to (y, b)$ and $(g, \psi)\colon (y, b) \to (z, c)$ is defined as $(g \circ f, \psi \circ g_!(\varphi))$.

The functor $\mathrm{Un}^{\mathrm{cc}}(F) \to C$ is given by $(x, a) \mapsto x$ and $(f, \varphi) \mapsto f$. A dual description holds for $\mathrm{Un}^{\mathrm{ct}}(F)$.

Proposition 24.2.10 (Coproducts and extensivity in cartesian unstraightenings)**.** *Let $q\colon E \to B$ be a cartesian fibration, where B admits finite coproducts. If the straightening*

$$\mathrm{Str}^{\mathrm{ct}}(q)\colon B^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

preserves finite products, then E admits finite coproducts and q preserves them. More precisely, if $X_a \in E_{b_a}$ and $X \in E_{\coprod_a b_a}$ corresponds to $(X_a)_a$ under the equivalence

$$E_{\coprod_a b_a} \simeq \prod_a E_{b_a},$$

then the q-cartesian lifts $X_a \to X$ of the coproduct inclusions exhibit X as the coproduct $\coprod_a X_a$ in E.

If, in addition, B is extensive, then E is extensive.

Proof. The first assertion is dual to Proposition 8.7.15 of [CCNW26].

For the final assertion, write $F := \mathrm{Str}^{\mathrm{ct}}(q)$ and let $Y := \coprod_a Y_a$, where $Y_a \in E_{b_a}$. Since F preserves finite products, the coproduct of a finite family of q-cartesian morphisms is again q-cartesian: under the product equivalence between the relevant fibers, it corresponds to a tuple of isomorphisms. Hence the coproduct functor

$$\prod_a E_{/Y_a} \longrightarrow E_{/Y}$$

is a cartesian functor over the extensivity equivalence

$$\prod_a B_{/b_a} \xrightarrow{\sim} B_{/\coprod_a b_a}.$$

By the dual of Proposition 8.7.14 of [CCNW26], the induced functor on fibers over a tuple $(f_a\colon c_a \to b_a)_a$ is

$$\prod_a F(c_a)_{/f_a^* Y_a} \longrightarrow F\Big(\coprod_a c_a\Big)_{/(\coprod_a f_a)^* Y}.$$

This is an equivalence because F preserves finite products. Hence the coproduct functor on slices is an equivalence, so E is extensive. □

Remark 24.2.11. In Theorem 24.2.1 we only formulated a 'morphismwise' version of functoriality of the straightening equivalences. Let us remark on how to formulate a fully functorial version.

For this, we will need to assume the existence of an even bigger universe $\widehat{\mathrm{Cat}}_\infty$ of *large* ∞-categories, and assume that Cat_∞ is an object of $\widehat{\mathrm{Cat}}_\infty$, cf. Remark 1.8.1. Applying cartesian straightening to the cartesian fibration $t\colon \mathrm{Ar}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$ results in a contravariant functor

$$\mathrm{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty, \quad C \mapsto (\mathrm{Cat}_\infty)_{/C}, \quad (F\colon C \to D) \mapsto (F^*\colon (\mathrm{Cat}_\infty)_{/D} \to (\mathrm{Cat}_\infty)_{/C}).$$

Since cocartesian fibrations are closed under pullback, this restricts to a functor

$$\mathrm{Cocart}(-)\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty.$$

Similarly, the assignment $C \mapsto \mathrm{Fun}(C, \mathrm{Cat}_\infty)$ defines a functor

$$\mathrm{Fun}(-, \mathrm{Cat}_\infty)\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty,$$

where functoriality is given by precomposition. The fully coherent version of the straightening equivalence would now ask for a natural equivalence

$$\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{Cocart}(-) \xrightarrow{\sim} \mathrm{Fun}(-, \mathrm{Cat}_\infty)$$

of functors $\mathrm{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$.

Remark 24.2.12. Recall that if two functors $F\colon C \to D$ and $G\colon D \to E$ admit right adjoints F^R and G^R, then GF admits a right adjoint $F^R G^R$. Using straightening/unstraightening, we can define a highly coherent version of this functoriality of passing to right adjoints. Consider the wide subcategories

$$\mathrm{Cat}_\infty^{\mathrm{L}} \subseteq \mathrm{Cat}_\infty \qquad \text{and} \qquad \mathrm{Cat}_\infty^{\mathrm{R}} \subseteq \mathrm{Cat}_\infty$$

denote the wide subcategories spanned by the left adjoints and right adjoints, respectively. We claim that passing to right adjoints defines a functor

$$(-)^{\mathbb{R}} \colon (\mathrm{Cat}_\infty^{\mathrm{L}})^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{R}}.$$

To this end, choose a bigger universe $\widehat{\mathrm{Cat}}_\infty$ containing Cat_∞ as an object. The inclusion $\mathrm{Cat}_\infty^{\mathrm{L}} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ can then be straightened to a cocartesian fibration $E \to \mathrm{Cat}_\infty^{\mathrm{L}}$. Given a morphism $F \colon C \to D$ in $\mathrm{Cat}_\infty^{\mathrm{L}}$, the induced cocartesian transport map $E_C \to E_D$ on fibers is the functor F itself. Since by assumption F admits a right adjoint, it follows from Lemma 24.1.20 that this cocartesian fibration is also a cartesian fibration, hence straightens to a functor $(-)^{\mathrm{R}} \colon (\mathrm{Cat}_\infty^{\mathrm{L}})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. By checking on objects and morphisms, this factors through $\mathrm{Cat}_\infty^{\mathrm{R}}$.

24.3 Fiberwise criteria

A natural transformation between functors with values in Cat_∞ is invertible if and only if each of its components is an equivalence. Under straightening, this gives a fiberwise criterion for a cocartesian functor to be an equivalence:

Theorem 24.3.1. *Consider a cocartesian functor*

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & C, & \end{array}$$

where p and p' are cocartesian fibrations between small ∞-categories. Then F is an equivalence of ∞-categories if and only if for every object x of C, the induced functor on fibers $F_x \colon E_x \to E'_x$ is an equivalence.

Proof. Under the equivalence $\mathrm{Str}^{\mathrm{cc}} \colon \mathrm{Cocart}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Cat}_\infty)$, F corresponds to a natural transformation $\mathrm{Str}^{\mathrm{cc}}(F) \colon \mathrm{Str}^{\mathrm{cc}}(p) \to \mathrm{Str}^{\mathrm{cc}}(p')$, and this natural transformation is a natural equivalence if and only if it is a pointwise equivalence. □

Corollary 24.3.2. *A cocartesian fibration $p \colon E \to C$ is an equivalence if and only if all of its fibers are contractible.*

Proof. Apply the previous theorem to $E' = C$. □

Corollary 24.3.3. *A morphism of animae $p \colon X \to Y$ is an equivalence if and only if all its fibers are contractible.*

Proof. Since X is an anima, every morphism in X is an isomorphism, and in particular p-cocartesian. It follows that p is a left fibration, and thus the claim holds by the previous corollary. □

Proposition 24.3.4 (Fiberwise initial objects in cartesian fibrations, [Lur09b, Proposition 2.4.4.9])**.** *Let $p \colon E \to C$ be a cartesian fibration whose fibers admit initial objects. Then the fiberwise initial objects assemble uniquely into a fully faithful left-adjoint section*

$$s \colon C \to E$$

of p. Dually, the fiberwise terminal objects of a cocartesian fibration assemble uniquely into a fully faithful right-adjoint section.

Proof. For every $x \in C$, choose an initial object $e_x \in E_x$. We claim that e_x, together with the identity morphism $x \to p(e_x) = x$, is a left adjoint object to x under p. Indeed, for any $e \in E$ consider the map

$$\mathrm{Hom}_E(e_x, e) \longrightarrow \mathrm{Hom}_C(x, p(e)).$$

For a morphism $f : x \to p(e)$, choose a p-cartesian lift $f^*e \to e$. The defining property of this lift identifies the fiber of the displayed map over f with

$$\mathrm{Hom}_{E_x}(e_x, f^*e),$$

which is contractible because e_x is initial. It follows from Corollary 24.3.3 that $\mathrm{Hom}_E(e_x, e) \to \mathrm{Hom}_C(x, p(e))$ is an equivalence. This proves the claim.

The pointwise criterion for adjunctions from Lemma 22.1.4 now assembles the objects e_x and the identity morphisms $x \to p(e_x)$ into a left adjoint $s : C \to E$ whose unit $\mathrm{id}_C \to ps$ is a natural isomorphism. Hence s is a fully faithful section of p. Any other section selecting initial objects in the fibers is a left adjoint to p by the same argument, and is therefore unique. The final statement follows by duality. □

Lemma 24.3.5. *Let $p : E \to C$ be a functor. Then a morphism $\varphi : e \to e'$ in E is p-cocartesian if and only if the commutative square*

$$\begin{array}{ccc} E_{e'/} & \xrightarrow{-\circ\varphi} & E_{e/} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ C_{pe'/} & \xrightarrow{-\circ p\varphi} & C_{pe/} \end{array}$$

is a pullback square. A dual criterion holds for p-cartesian morphisms.

Proof. The horizontal maps are left fibrations, hence in particular cocartesian fibrations. It follows from closure under base change that also the map from the pullback to $E_{e/}$ is a cocartesian fibration. By Theorem 24.3.1, the functor $E_{e'/} \to E_{e/} \times_{C_{pe/}} C_{pe'/}$ is an equivalence if and only if this holds fiberwise over $E_{e/}$, which means that for every morphism $\psi : e \to e''$ the map

$$\mathrm{Hom}_{E_{e/}}(\varphi, \psi) \to \mathrm{Hom}_{C_{pe/}}(p\varphi, p\psi)$$

is an equivalence. Since this is a morphism of animae, this may be checked fiberwise, where it becomes the definition of cocartesianness for φ. □

Lemma 24.3.6. *Let $q : E \to D$ and $p : D \to C$ be functors and assume that p is a left fibration. Then q is a left fibration if and only if pq is a left fibration.*

Proof. As a consequence of Lemma 24.3.5, we see that a morphism $\varphi\colon e \to e'$ in E is q-cocartesian if and only if it is pq-cocartesian: in the commutative diagram

$$\begin{array}{ccc} E_{e'/} & \xrightarrow{-\circ\varphi} & E_{e/} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ D_{qe'/} & \xrightarrow{-\circ q\varphi} & D_{qe/} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ C_{pqe'/} & \xrightarrow{-\circ pq\varphi} & C_{pqe/}, \end{array}$$

the bottom square is a pullback square, hence the top square is a pullback square if and only if the bottom one is. It follows that every morphism in E is q-cocartesian if and only if every morphism is pq-cocartesian. The claim follows. □

Proposition 24.3.7. *For a functor $p\colon E \to C$, the following conditions are equivalent:*

(1) *The functor p is a left fibration;*

(2) *For every object e of E, the induced functor $p\colon E_{e/} \to C_{pe/}$ is an equivalence;*

(3) *The commutative square*

$$\begin{array}{ccc} \mathrm{Ar}(E) & \xrightarrow{p_*} & \mathrm{Ar}(C) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ E & \xrightarrow{p} & C \end{array}$$

is a pullback square.

Proof. For (1) $\Rightarrow$ (2), assume that p is a left fibration. Since $E_{e/} \to E$ and $C_{pe/} \to C$ are left fibrations as well, it follows from Lemma 24.3.6 that also $E_{e/} \to C_{pe/}$ is a left fibration, hence it is an equivalence if and only if each of its fibers is contractible. But the fiber over $f \in C_{pe/}$ is the anima $L(e, f)$ of p-cocartesian lifts of f starting in e, hence this is contractible by Lemma 24.1.7.

For (2) $\Rightarrow$ (1), first note that it follows from (2) that every morphism $pe \to c'$ has a lift to some morphism $e \to e'$. It remains to show that every morphism $\varphi\colon e \to e'$ in E is p-cocartesian. But this is a direct consequence of Lemma 24.3.5: the two vertical maps in the square are equivalences, hence the square is a pullback square.

For (2) $\iff$ (3), note that the square is a pullback square if and only if the horizontal map in the following diagram is an equivalence:

$$\begin{array}{ccc} \mathrm{Ar}(E) & \xrightarrow{(s,p_*)} & E \times_C \mathrm{Ar}(C) \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & E. & \end{array}$$

By the dual of Example 24.1.10, the two vertical functors are cartesian fibrations. In light of Theorem 24.3.1, the horizontal functor is an equivalence if and only if it induces equivalences on fibers over each object $e \in E$. Unwinding definitions, these functors on fibers are the functors $E_{e/} \to C_{pe/}$ from condition (2). □

24.4 The Hom-functor

The hom animae of an ∞-category C depend contravariantly on the source and covariantly on the target. We will now use straightening/unstraightening to construct the resulting functor

$$\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}.$$

Lemma 24.4.1. *Let C be an ∞-category. Then the functor*

$$\begin{array}{ccc} \mathrm{Ar}(C) & \xrightarrow{(s,t)} & C \times C \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & C & \end{array}$$

is a cocartesian functor over C.

Proof. Recall from Example 24.1.10 that the target functor $t \colon \mathrm{Ar}(C) \to C$ is a cocartesian fibration whose cocartesian morphisms are those squares inverted by $s \colon \mathrm{Ar}(C) \to C$. The projection functor pr_2 is cocartesian by Lemma 24.1.13, and its cocartesian morphisms are those pairs $(g \colon X \to X', f \colon Y \to Y')$ such that g is an isomorphism. It follows immediately from these descriptions that (s,t) is a cocartesian functor. □

Construction 24.4.2 (The Hom-functor)**.** Let C be a small ∞-category. We construct a functor $\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}$.

By Lemma 24.4.1, the functor $(s,t) \colon \mathrm{Ar}(C) \to C \times C$ defines a morphism in $\mathrm{Cocart}(C)$. Applying the cocartesian straightening functor, we thus obtain a morphism

$$\mathrm{Str}^{\mathrm{cc}}(t) \to \mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)$$

in $\mathrm{Fun}(C, \mathrm{Cat}_\infty)$. Now, note that pr_2 is the pullback along $p_C \colon C \to *$ of the map $p_C \colon C \to *$, and it follows from the naturality of straightening that $\mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)$ is given by the composite

$$C \xrightarrow{p_C} * \xrightarrow{\mathrm{Str}(p_C)} \mathrm{Cat}_\infty.$$

Since $\mathrm{Str}(p_C)$ is simply the map picking out the object $C \in \mathrm{Cat}_\infty$, this means that $\mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)$ is the constant functor $\mathrm{const}_C \colon C \to \mathrm{Cat}_\infty$. In particular, the map $\mathrm{Str}^{\mathrm{cc}}(t) \to \mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2) \cong \mathrm{const}_C$ gives rise to an object

$$\mathrm{Str}^{\mathrm{cc}}(t) \quad \in \quad \mathrm{Fun}(C, \mathrm{Cat}_\infty)_{/\mathrm{const}_C} \simeq \mathrm{Fun}(C, (\mathrm{Cat}_\infty)_{/C}),$$

i.e., we may treat it as a functor $C \to (\mathrm{Cat}_\infty)_{/C}$. We now claim that this functor factors through the full subcategory $\mathrm{RFib}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$. We may check this on objects: we have to show that for every object $x \in C$, the functor $\mathrm{Str}^{\mathrm{cc}}(t)(x) \to \mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)(x)$ is a right fibration. By naturality of $\mathrm{Str}^{\mathrm{cc}}(-)$, this evaluation is computed by forming the induced map on fibers over x:

$$\begin{array}{ccccc} & & C & \longrightarrow & C \times C \\ & {\scriptstyle s}\nearrow & / & {\scriptstyle (s,t)}\nearrow & \\ C_{/x} & \longrightarrow & \mathrm{Ar}(C) & & \\ & \searrow \quad \downarrow & & \searrow & \downarrow \\ & * & \xrightarrow{x} & & C. \end{array}$$

But this functor $s\colon C_{/x} \to C$ is indeed a right fibration by Proposition 24.1.19. We thus obtain a functor

$$C \to \mathrm{RFib}(C) \xrightarrow[\sim]{\mathrm{Str}^{\mathrm{ct}}} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

Uncurrying then gives the desired functor $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$.

Remark 24.4.3. There is an asymmetry in our definition of Hom_C: we could also have considered the *cartesian* functor

$$\begin{array}{ccc} \mathrm{Ar}(C) & \xrightarrow{(s,t)} & C \times C \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & C & \end{array}$$

to get a functor $C^{\mathrm{op}} \to \mathrm{LFib}(C) \simeq \mathrm{Fun}(C, \mathrm{An})$. The resulting two functors can be shown to agree with each other: more generally, for *bifibrations* $E \to C \times D$ the two ways of constructing a functor $C^{\mathrm{op}} \times D \to \mathrm{An}$ agree, see [HHLN23b, Proposition 6.18].

24.5 Limits and colimits of ∞-categories

Let $E \to I$ be the cocartesian unstraightening of a diagram $I \to \mathrm{Cat}_\infty$. Our next goal is to give explicit descriptions of the limit and colimit of this functor: Its colimit is obtained from E by inverting the cocartesian morphisms, while its limit is the ∞-category of cocartesian sections. We first define these two constructions.

Construction 24.5.1. Let $p\colon E \to I$ be a cocartesian fibration.

- We write $E[\mathrm{cc}^{-1}]$ for the localization of E at the collection of p-cocartesian morphisms.
- We write $\Gamma_I(E)$ for the ∞-category of *sections of* p, defined via the following pullback square:

$$\begin{array}{ccc} \Gamma_I(E) & \longrightarrow & \mathrm{Fun}(I, E) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_*} \\ * & \xrightarrow{\mathrm{id}_I} & \mathrm{Fun}(I, I). \end{array}$$

- We write $\Gamma_I^{\mathrm{cc}}(E)$ for the full subcategory of $\Gamma_I(E)$ spanned by the *cocartesian sections*: those sections

$$\begin{array}{ccc} I & \xrightarrow{s} & E \\ & \searrow \quad \swarrow{\scriptstyle p} & \\ & I & \end{array}$$

  that are cocartesian functors over I (i.e., that send arbitary morphisms in I to p-cocartesian morphisms in E).

Proposition 24.5.2. *Let I be a small ∞-category and let $F\colon I \to \mathrm{Cat}_\infty$ be a functor.*

(1) The functor F admits a colimit in Cat_∞, which can be computed by the formula

$$\mathrm{colim}_I F(i) \simeq \mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}].$$

(2) The functor F admits a limit in Cat_∞*, which can be computed by the formula*

$$\lim_I F(i) \simeq \Gamma_I^{\mathrm{cc}}(\mathrm{Un}^{\mathrm{cc}}(F)).$$

Proof. See Theorems 13.2.1 and 13.2.4 of [CCNW26]. □

Proposition 24.5.3 (Fiberwise colimits over cocartesian fibrations)**.** *Let $p\colon E \to J$ be a cocartesian fibration between small ∞-categories and let $G\colon E \to C$ be a functor. Assume that the restriction $G|_{E_j}\colon E_j \to C$ admits a colimit for every $j \in J$. Then the left Kan extension $p_!G\colon J \to C$ exists and is given pointwise by*

$$(p_!G)(j) \simeq \mathrm{colim}_{E_j} G|_{E_j}.$$

Moreover, G admits a colimit if and only if $p_!G$ admits a colimit, and in this case there is an isomorphism

$$\mathrm{colim}_E G \cong \mathrm{colim}_J p_!G.$$

Dually, fiberwise limits over a cartesian fibration assemble into a right Kan extension and may be computed by first taking the fiberwise limits and then the limit over the base.

Proof. See Proposition 13.2.14 of [CCNW26]. □

Corollary 24.5.4. *Let I be a small ∞-category and let $F\colon I \to \mathrm{An}$ be functor.*

(1) The functor F admits a colimit in An*, which can be computed by the formula*

$$\mathrm{colim}_I F(i) \simeq \| \mathrm{Un}^{\mathrm{cc}}(F)\|.$$

(2) The functor F admits a limit in An*, which can be computed by the formula*

$$\lim_I F(i) \simeq \Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F)).$$

Proof. We compute the limit and colimit in Cat_∞ and observe that it is still in An. For (1) this is clear: all morphisms in $\mathrm{Un}^{\mathrm{cc}}(F)$ are cocartesian, see Proposition 24.1.16, hence inverting all cocartesian morphisms gives an anima. For (2), any section $s\colon I \to \mathrm{Un}^{\mathrm{cc}}(F)$ of p_F is cocartesian, again since all morphisms are cocartesian, so $\Gamma_I^{\mathrm{cc}}(\mathrm{Un}^{\mathrm{cc}}(F)) = \Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F))$. Note that this is already an anima: any morphism of sections is invertible since all fibers of $p_F\colon \mathrm{Un}^{\mathrm{cc}}(F) \to I$ are animae. □

Adjunctions between the values of two diagrams induce an adjunction between their limits, provided the right adjoints commute with the transition functors. The Beck–Chevalley condition expresses this compatibility:

Proposition 24.5.5 (Limits of adjunctions)**.** *Let K be a small ∞-category and consider functors $C_\bullet, D_\bullet\colon K \to \mathrm{Cat}_\infty$. Let $F_\bullet\colon C_\bullet \to D_\bullet$ be a natural transformation satisfying the following two conditions:*

(1) *(Pointwise adjoints) The functor $F_k\colon C_k \to D_k$ admits a right adjoint $G_k\colon D_k \to C_k$ for every $k \in K$;*

(2) *(Beck-Chevalley condition) For every morphism $f\colon k \to k'$ in K, the Beck-Chevalley transformation* $\mathrm{BC}\colon f_* G_k \to G_{k'} f_*$*, defined as the composite*

$$f_* G_k \xrightarrow{\text{unit}} G_{k'} F_{k'} f_* G_k \cong G_{k'} f_* F_k G_k \xrightarrow{\text{counit}} G_{k'} f_*,$$

is a natural isomorphism.

Then the functors G_k assemble into a natural transformation $G_\bullet\colon D_\bullet \to C_\bullet$, and the units and counits assemble into natural transformations $\varepsilon_\bullet\colon F_\bullet \circ G_\bullet \to \mathrm{id}_\bullet$ and $\eta_\bullet\colon \mathrm{id}_\bullet \to G_\bullet \circ F_\bullet$ that satisfy the triangle identities.

As a result, the induced functors

$$F := \lim_k F_k\colon \lim_k C_k \to \lim_k D_k \qquad \textit{and} \qquad G := \lim_k G_k\colon \lim_k D_k \to \lim_k C_k$$

are adjoint to each other.

Proof. By the dual of [Lur17, Proposition 7.3.2.6], the cocartesian functor obtained by unstraightening $F_\bullet$ admits a right adjoint over K, with unit and counit over K. Its restriction to the fiber over k is G_k. For $f\colon k \to k'$ and $d \in D_k$, it sends a cocartesian arrow $d \to f_* d$ to the composite

$$G_k d \longrightarrow f_* G_k d \xrightarrow{\mathrm{BC}_d} G_{k'} f_* d,$$

where the first arrow is cocartesian. Since BC_d is an isomorphism, the right adjoint preserves cocartesian morphisms. Straightening therefore gives $G_\bullet$ and the asserted unit and counit.

Both functors restrict to the ∞-categories of cocartesian sections, and their unit and counit induce transformations on these ∞-categories satisfying the triangle identities. By Proposition 24.5.2, these section categories compute $\lim_k C_k$ and $\lim_k D_k$, giving the required adjunction. □

Corollary 24.5.6. *Let I and K be small ∞-categories and let $C_\bullet\colon K \to \mathrm{Cat}_\infty$ be a K-indexed diagram of small ∞-categories. Assume that C_k admits I-indexed limits (resp. colimits) for all $k \in K$, and that the functor $C_k \to C_{k'}$ preserves I-indexed limits (resp. colimits) for every morphism $k \to k'$ in K. Then the limit $C := \lim_{k \in K} C_k$ in Cat_∞ also admits I-indexed limits (resp. colimits) and each functor $C \to C_k$ preserves them.*

Proof. See Proposition 13.2.10 of [CCNW26]. □

24.6 Hom animae in functor categories

A natural transformation $F \to G$ between functors $C \to D$ heuristically consists of component morphisms coherently compatible with the morphisms of C. To make this heuristic precise, we will express the anima of natural transformations as a limit of hom animae in D, called an *end*.

Definition 24.6.1 (Twisted arrow category)**.** Let C be an ∞-category. We define its *twisted arrow category* $\mathrm{Tw}(C)$ as the source of the left fibration whose cocartesian straightening is the Hom-functor $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$:

$$\mathrm{Tw}(C) := \mathrm{Un}^{\mathrm{cc}}(\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}).$$

In particular, it comes equipped with a left fibration $(s,t)\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ whose fibers are the hom animae of C:

$$\begin{array}{ccc} \mathrm{Hom}_C(x,y) & \longrightarrow & \mathrm{Tw}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ * & \xrightarrow{(x,y)} & C^{\mathrm{op}} \times C. \end{array}$$

Remark 24.6.2. Under the complete Segal presentation of ∞-categories from Chapter 25, the twisted arrow category has the edgewise-subdivision model

$$[n] \longmapsto \mathrm{Hom}_{\mathrm{Cat}_\infty}([n]^{\mathrm{op}} \star [n], C).$$

The inclusions of the two copies of $[n]$ induce the source and target functors, and the fiber over (x,y) is $\mathrm{Hom}_C(x,y)$. This gives a model-specific description of the left fibration in the preceding definition.

Unwinding the description of the unstraightening construction, we see that $\mathrm{Tw}(C)$ may be described as follows:

- The objects of $\mathrm{Tw}(C)$ are morphisms in C, which we will display as $\left(\begin{smallmatrix} X \\ \downarrow \varphi \\ Y \end{smallmatrix}\right)$. We have $s\left(\begin{smallmatrix} X \\ \downarrow \varphi \\ Y \end{smallmatrix}\right) = X$ and $t\left(\begin{smallmatrix} X \\ \downarrow \varphi \\ Y \end{smallmatrix}\right) = Y$.
- A morphism $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\colon \left(\begin{smallmatrix} X \\ \downarrow \varphi \\ Y \end{smallmatrix}\right) \to \left(\begin{smallmatrix} X' \\ \downarrow \varphi' \\ Y' \end{smallmatrix}\right)$ in $\mathrm{Tw}(C)$ is a triple (f,g,α) consisting of a morphism $f\colon X \to X'$ in C^{op} (i.e., $f\colon X' \to X$ in C), a morphism $g\colon Y \to Y'$ in C, and a morphism $\alpha\colon (f,g)_!\left(\begin{smallmatrix} X \\ \downarrow \varphi \\ Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} X' \\ \downarrow g\varphi f \\ Y' \end{smallmatrix}\right) \to \left(\begin{smallmatrix} X' \\ \downarrow \varphi' \\ Y' \end{smallmatrix}\right)$ in the fiber $\mathrm{Tw}(C)_{(X',Y')} = \mathrm{Hom}_C(X',Y')$. We will display this data by means of diagrams of the form

$$\begin{array}{ccc} X & \xleftarrow{f} & X' \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ Y & \xrightarrow{g} & Y'. \end{array}$$

  Note that this is *not* a morphism in the usual arrow category $\mathrm{Ar}(C)$, since f points in the opposite direction.

Definition 24.6.3 (End)**.** Let C be an ∞-category and consider a functor $F\colon C^{\mathrm{op}} \times C \to D$. The *end of* F, denoted by $\int_{c \in C} F(c,c)$ if it exists, is the limit of the composite functor

$$\mathrm{Tw}(C) \xrightarrow{(s,t)} C^{\mathrm{op}} \times C \xrightarrow{F} D.$$

Remark 24.6.4. If C is small and D admits small limits, the end of F always exists. Furthermore, if $D \to D'$ is a functor preserving small limits, then it also preserves ends.

Lemma 24.6.5 (Fubini rule for ends)**.** *Let C and C' be ∞-categories and let $F\colon C^{\mathrm{op}} \times C \times C'^{\mathrm{op}} \times C' \to D$ be a functor. Then there is an equivalence*

$$\int_{c \in C} \int_{c' \in C'} F(c,c,c',c') \simeq \int_{(c,c') \in C \times C'} F(c,c,c',c').$$

Proof. Since a limit over a product category may be computed as an iterated limit, it will suffice to produce an equivalence

$$\mathrm{Tw}(C \times C') \simeq \mathrm{Tw}(C) \times \mathrm{Tw}(C')$$

of left fibrations over $C^{\mathrm{op}} \times C \times C'^{\mathrm{op}} \times C'$. By straightening-unstraightening, it will suffice to produce a natural isomorphism between their unstraightenings:

$$(x, y, x', y') \quad \mapsto \quad \mathrm{Hom}_{C \times C'}((x, x'), (y, y')) \simeq \mathrm{Hom}_C(x, y) \times \mathrm{Hom}_{C'}(x', y').$$

Unwinding the definition of the hom functors, this boils down to the canonical equivalence

$$\mathrm{Ar}(C \times C') \xrightarrow{\sim} \mathrm{Ar}(C) \times \mathrm{Ar}(C')$$

over $C \times C \times C' \times C'$, coming from the fact that $\mathrm{Ar}(-) = \mathrm{Fun}([1], -)$ preserves products. □

Proposition 24.6.6 (Hom animae in functor categories)**.** *Let $F, G \colon C \to D$ be functors. Then we have the following end-formula for the anima of natural transformations from F to G:*

$$\mathrm{Nat}(F, G) \quad \simeq \quad \int_{x \in C} \mathrm{Hom}_D(F(x), G(x)).$$

Here the right-hand side is the end of the composite functor $C^{\mathrm{op}} \times C \xrightarrow{F^{\mathrm{op}} \times G} D^{\mathrm{op}} \times D \xrightarrow{\mathrm{Hom}_D} \mathrm{An}$.

Proof. The end imposes precisely the naturality conditions on a pointwise family of morphisms $F(x) \to G(x)$. The rigorous identification, including all higher coherence data, is proved in Theorem 12.5.5 of [CCNW26]. □

Proposition 24.6.7. *Let $F \colon I \to C$ be a functor of ∞-categories. A natural transformation $\eta \colon F \to \mathrm{const}_x$ exhibits $x \in C$ as the colimit of F if and only if the natural map*

$$\eta^* \colon \mathrm{Hom}_C(x, y) \xrightarrow{\sim} \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_C(F(i), y)$$

obtained by applying $\mathrm{Hom}_C(-, y)$ to η is an isomorphism for all $y \in C$. A similar assertion holds for limits.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_C(x, y) & \xrightarrow{\mathrm{const}} & \mathrm{Nat}(\mathrm{const}_x, \mathrm{const}_y) & \xrightarrow{-\circ\eta} & \mathrm{Nat}(F, \mathrm{const}_y) \\ & \searrow^{\eta^*} & & & \downarrow \\ & & & & \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_C(F(i), y); \end{array}$$

here the right vertical map is induced by the evaluation functors $\mathrm{ev}_i \colon \mathrm{Fun}(I, C) \to C$. By definition, x is a colimit of F if and only if the top map is an isomorphism for all y, hence it will suffice to show that the right vertical map is an isomorphism. By Proposition 24.6.6, the top right is isomorphic to $\lim_{(i \to i') \in \mathrm{Tw}(I)} \mathrm{Hom}_C(F(i), y)$. The source projection $s \colon \mathrm{Tw}(I) \to I^{\mathrm{op}}$ is initial by Lemma 12.2.7 of [CCNW26], applied to the unique functor $I \to *$. Hence restriction along s induces an isomorphism

$$\lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_C(F(i), y) \xrightarrow{\sim} \lim_{(i \to i') \in \mathrm{Tw}(I)} \mathrm{Hom}_C(F(i), y).$$

Under the end formula, this is the inverse of the right vertical map, completing the proof. □

Corollary 24.6.8. *The functors* $Y_C \colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ *and* $Y_{C^{\mathrm{op}}} \colon C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{An})$ *preserve limits.*

Proof. The statement for $Y_{C^{\mathrm{op}}}$ is immediate from Proposition 24.6.7. The other one is proved dually, or may be recovered by replacing C by C^{op}. □

24.7 Descent and ∞-topoi

An important property that distinguishes the ∞-category An from other ∞-categories is a property called *descent*: a map to a colimit of animae can be recovered from its pullbacks to the objects of the diagram, together with their compatibility under base change. We already used this property in the proof of the recognition principle (Theorem 5.2.5); we now give a self-contained treatment.

Definition 24.7.1 (Descent)**.** Let I be a small ∞-category, and let T be an ∞-category with pullbacks and I-indexed colimits. We say that T *satisfies descent for* I*-indexed colimits* if the functor

$$T^{\mathrm{op}} \to \mathrm{Cat}, \qquad X \mapsto T_{/X}$$

preserves I-indexed limits.

Definition 24.7.2 (∞-Topos)**.** An ∞*-topos* is a presentable ∞-category satisfying descent for all small colimits.

Let us unwind Definition 24.7.1. Given a small ∞-category I and an I-indexed diagram $X_\bullet \colon I \to T$, set $X := \mathrm{colim}_i X_i$. The descent condition demands that the canonical functor $T_{/X} \to \lim_i T_{/X_i}$, obtained from the base change functors along the maps $X_i \to X$, is an equivalence.

Theorem 24.7.3. *The* ∞*-category* An *is an* ∞*-topos.*

Proof. We need to show that for any functor $X_\bullet \colon I \to \mathrm{An}$, with colimit $X_\infty := \mathrm{colim}_i X_i$, the functor $\mathrm{An}_{/X_\infty} \to \lim_i \mathrm{An}_{/X_i}$ is an equivalence. By Corollary 24.2.8, we have an equivalence $\mathrm{An}_{/Y} \xrightarrow{\sim} \mathrm{Fun}(Y, \mathrm{An})$ which is natural in Y. So, the claim is equivalent to showing that the map

$$\mathrm{Fun}(X_\infty, \mathrm{An}) \to \lim_i \mathrm{Fun}(X_i, \mathrm{An})$$

is an equivalence. But this follows from the fact that $\mathrm{Fun}(-, \mathrm{An}) \colon \mathrm{An}^{\mathrm{op}} \to \mathrm{Cat}$ sends colimits in An to limits in Cat. □

The limit of the slice categories appearing in the definition of descent can be described using diagrams over $X_\bullet$ whose naturality squares are pullbacks:

Definition 24.7.4 (Cartesian natural transformation)**.** Given two functors $F, G \colon I \to T$, a natural transformation $\alpha \colon F \to G$ is called *cartesian* if for every morphism $i \to j$ in I the naturality square

$$\begin{array}{ccc} F(i) & \xrightarrow{\alpha(i)} & G(i) \\ \downarrow & \lrcorner & \downarrow \\ F(j) & \xrightarrow{\alpha(j)} & G(j) \end{array}$$

is a pullback square. We denote by $\mathrm{Fun}^{\mathrm{cart}}(I, T) \subseteq \mathrm{Fun}(I, T)$ the wide subcategory spanned by the cartesian natural transformations.

Given a diagram $X_\bullet \colon I \to T$, evaluation at $i \in I$ determines a functor $\mathrm{ev}_i \colon \mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet} \to T_{/X_i}$. Furthermore, by definition of cartesianness these evaluation functors are compatible with base change, resulting in a comparison functor $\mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet} \to \lim_{i \in I^{\mathrm{op}}} T_{/X_i}$.

Lemma 24.7.5. *Let C be an ∞-category with pullbacks and let $X_\bullet \colon I \to C$ be a functor. Then the functor $\mathrm{Fun}^{\mathrm{cart}}(I,C)_{/X_\bullet} \to \lim_{i \in I^{\mathrm{op}}} C_{/X_i}$ is an equivalence.*

More informally, objects of the limit are given by I-indexed diagrams $Y_\bullet$ in C equipped with a cartesian natural transformation $Y_\bullet \to X_\bullet$.

Proof. Recall that a limit of a diagram $I^{\mathrm{op}} \to \mathrm{Cat}$ of ∞-categories may be computed as the ∞-category of cartesian sections of the cartesian unstraightening of the diagram. By definition, the functor $C_{/-} \colon C^{\mathrm{op}} \to \mathrm{Cat}$ is the cartesian straightening of the target map $t \colon \mathrm{Ar}(C) \to C$, so the unstraightening of $C_{/-}$ is t. It follows that the unstraightening of the functor $i \mapsto C_{/X_i}$ is the base change of t along $X_\bullet$:

$$\begin{array}{ccc} C/X_\bullet & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ I & \xrightarrow{X_\bullet} & C. \end{array}$$

The ∞-category of sections $I \to C/X_\bullet$ of this map is equivalent to the ∞-category of maps $I \to \mathrm{Ar}(C)$ whose target projection is $X_\bullet \colon I \to C$, which in turn is equivalent to the slice-∞-category $\mathrm{Fun}(I,C)_{/X_\bullet}$. It thus remains to show that a section $s \colon I \to C/X_\bullet$ is cartesian precisely if the corresponding natural transformation $Y_\bullet \to X_\bullet$ is a cartesian natural transformation. But this is an immediate consequence of the fact that the t-cartesian morphisms in $\mathrm{Ar}(C)$ are precisely the pullback squares in C. □

In terms of cartesian natural transformations, the left adjoint to $T_{/X} \to \lim_i T_{/X_i}$ is given by taking colimits:

Lemma 24.7.6. *Let T be an ∞-category with pullbacks and I-indexed colimits, and let $X = \mathrm{colim}_i X_i$ be a colimit in T. Then the functor $T_{/X} \to \lim_i T_{/X_i}$ admits a left adjoint*

$$\mathrm{colim} \colon \lim_i T_{/X_i} \to T_{/X}$$

sending a cartesian transformation $Y_\bullet \to X_\bullet$ to its colimit $\mathrm{colim}_i Y_i \to \mathrm{colim}_i X_i$.

Proof. The colimit functor $\mathrm{colim} \colon \mathrm{Fun}(I,T) \to T$ admits a right adjoint $\mathrm{const} \colon T \to \mathrm{Fun}(I,T)$ sending an object Y to the constant functor on Y. By slicing this adjunction over an arbitrary diagram $X_\bullet \in \mathrm{Fun}(I,T)$, with colimit denoted X, we obtain another adjunction

$$\mathrm{colim} \colon \mathrm{Fun}(I,T)_{/X_\bullet} \rightleftarrows T_{/X},$$

see [Lur09b, Proposition 5.2.5.1]. The right adjoint in this adjunction is given by the composite $T_{/X} \xrightarrow{\mathrm{const}} \mathrm{Fun}(I,T)_{/\mathrm{const}\, X} \to \mathrm{Fun}(I,T)_{/X_\bullet}$, with the second map given by base change along the colimit cocone $X_\bullet \to \mathrm{const}_X$. We now make the following two observations:

- The ∞-category $\mathrm{Fun}(I,T)_{/X_\bullet}$ contains the limit $\lim_{i \in I^{\mathrm{op}}} C_{/X_i} \simeq \mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet}$ as a full subcategory: given transformations $Y_\bullet \to Z_\bullet \to X_\bullet$, if both $Z_\bullet \to X_\bullet$ and $Y_\bullet \to X_\bullet$ are cartesian transformations, then so is $Y_\bullet \to Z_\bullet$ by the pasting law for pullback squares.

- Since any transformation $\operatorname{const} Y \to \operatorname{const} X$ between constant diagrams is cartesian, and cartesian transformations are closed under base change, the right adjoint $T_{/X} \to \operatorname{Fun}(I, T)_{/X_\bullet}$ lands in this full subcategory.

It follows that the adjunction restricts to an adjunction $\lim_i T_{/X_i} \rightleftarrows T_{/X}$, as desired. □

We can now characterize when the functor $T_{/X} \to \lim_i T_{/X_i}$ is an equivalence. By Lemma 24.7.6, this happens precisely when the unit and counit of the adjunction are natural isomorphisms.

Lemma 24.7.7. *Let T be an ∞-category with pullbacks and I-indexed colimits, and let $X_\bullet \colon I \to T$ be a diagram with colimit $X := \operatorname{colim}_i X_i$. Then T satisfies descent for I-indexed colimits if and only if the following two conditions hold:*

(1) *For any map $Y \to X$, the canonical map $\operatorname{colim}_i (X_i \times_X Y) \to Y$ is an isomorphism.*

(2) *Given a cartesian transformation $Y_\bullet \to X_\bullet$, each map $Y_i \to (\operatorname{colim}_j Y_j) \times_X X_i$ is an isomorphism.*

Proof. By definition, T satisfies descent for I-indexed colimits if and only if the canonical functor $T_{/X} \to \lim_i T_{/X_i}$ is an equivalence. By Lemma 24.7.6, this functor admits a left adjoint given by sending a cartesian transformation $Y_\bullet \to X_\bullet$ to its colimit. The functor is an equivalence if and only if both the unit and counit of this adjunction are isomorphisms. The counit at $Y \to X$ is the map in (1), while the unit at a cartesian transformation $Y_\bullet \to X_\bullet$ consists of the maps in (2). □

24.7.1 Examples of descent

For initial objects and coproducts, descent has the following concrete meaning:

Example 24.7.8 (Strictly initial object)**.** Let T be an ∞-category with an initial object. Then the initial object $\emptyset$ satisfies descent if and only if it is *strictly initial*, meaning that any morphism $X \to \emptyset$ is an isomorphism.

Indeed, by definition $\emptyset$ satisfies descent if and only if the functor $T_{/\emptyset} \to *$ is an equivalence. This functor admits a fully faithful left adjoint $* \to T_{/\emptyset}$ hitting the identity $\mathrm{id}_\emptyset$. For the functor to be an equivalence, the counit $\emptyset \to X$ must be an isomorphism for every $X \in T_{/\emptyset}$, which is the strict initiality condition.

Example 24.7.9 (Disjoint coproducts)**.** For an ∞-category T with arbitrary coproducts, we say that *coproducts are disjoint* if for all $X, Y \in T$ the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \sqcup Y \end{array}$$

is a pullback square. We claim that arbitrary coproducts in T satisfy descent if and only if they are universal and T has disjoint coproducts.

By Lemma 24.7.7, coproducts satisfy descent if and only if for every collection of maps $\{Y_i \to X_i\}_{i \in I}$ the commutative square

$$\begin{array}{ccc} Y_j & \longrightarrow & Y := \coprod_{i \in I} Y_i \\ \downarrow & & \downarrow \\ X_j & \longrightarrow & X := \coprod_{i \in I} X_i \end{array}$$

is a pullback square for every $j \in I$. To see that this reduces to disjointness, consider $X' := \coprod_{i \in I \setminus \{j\}} X_i$ and $Y' := \coprod_{i \in I \setminus \{j\}} Y_i$. We may write $X = X_j \sqcup X'$ and $Y = Y_j \sqcup Y'$. Invoking universality of coproducts once more shows that the condition reduces to disjointness of binary coproducts.

Example 24.7.10 (Grothendieck construction)**.** Let T be an ∞-topos. Given an anima $A \in \mathrm{An}$, we obtain an equivalence

$$\mathrm{Fun}(A, T) \quad = \quad \lim_A T_{/*} \quad \simeq \quad T_{/\operatorname{colim}_A *}.$$

We refer to this equivalence as the *Grothendieck construction*.

24.8 Exponentiability of cartesian fibrations

Cartesian fibrations are *exponentiable*: for a cartesian fibration $p \colon E \to B$, the product functor $- \times_B E$ on $(\mathrm{Cat}_\infty)_{/B}$ admits a right adjoint. We will use the consequence that this product functor preserves colimits.

Proposition 24.8.1 (Exponentiability of cartesian fibrations, [Lur17, Appendix B.3])**.** *Let $p \colon E \to B$ be a cartesian fibration. Then the functor*

$$- \times_B E \colon (\mathrm{Cat}_\infty)_{/B} \to (\mathrm{Cat}_\infty)_{/B}$$

preserves colimits. Dually, if p is a cocartesian fibration, then $- \times_B E$ preserves colimits as well.

Proof. This is Lurie's flatness theorem for cartesian fibrations; see also [AF20]. □

Remark 24.8.2. The proposition fails for arbitrary functors $E \to B$. For example, let $\ell \colon [1] \to [2]$ be the long edge. The ∞-category $[2]$ is the pushout of the spine

$$[1] \sqcup_{[0]} [1],$$

where the two copies of $[1]$ meet in the middle vertex. Pulling this pushout back along ℓ gives two discrete vertices, whereas the pullback of $[2]$ along ℓ is $[1]$. Thus $- \times_{[2]} [1]$ does not preserve this pushout.

Theorem 24.8.3 (Currying for cartesian fibrations)**.** *Let $p \colon E \to B$ be a cartesian fibration and let C be an ∞-category. Write*

$$A := \mathrm{Str}^{\mathrm{ct}}(p) \colon B^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

Thus, for a morphism $\beta \colon b \to b'$ in B, the induced functor is the cartesian transport functor $\beta^* \colon E_{b'} \to E_b$. Let

$$H \colon B \to \mathrm{Cat}_\infty, \qquad b \mapsto \mathrm{Fun}(E_b, C), \qquad \beta \mapsto (\beta^*)^*$$

and let $q\colon Q := \mathrm{Un}^{\mathrm{cc}}(H) \to B$ *be its cocartesian unstraightening. Then there exists a functor*

$$\mathrm{ev}\colon Q \times_B E \to C$$

which restricts on the fiber over $b \in B$ *to the usual evaluation functor*

$$\mathrm{Fun}(E_b, C) \times E_b \to C$$

and which has the following universal property: for every object $D \to B$ *of* $(\mathrm{Cat}_\infty)_{/B}$*, the composite*

$$\mathrm{Fun}_{/B}(D, Q) \xrightarrow{-\times_B E} \mathrm{Fun}_{/B}(D \times_B E, Q \times_B E) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Fun}(D \times_B E, C)$$

is an equivalence of ∞*-categories.*

Proof. This is the dual of Gepner, Haugseng, and Nikolaus [GHN17, Proposition 7.3]. That result identifies the internal hom obtained by exponentiating along a cocartesian fibration as the cartesian unstraightening of the fiberwise functor categories. Passing to opposite categories gives the present statement for a cartesian fibration: the right adjoint to $- \times_B E$ is the cocartesian unstraightening of

$$b \longmapsto \mathrm{Fun}(E_b, C),$$

with transport given by precomposition with cartesian transport in E. Its counit is the displayed evaluation functor, and the adjunction gives the asserted equivalence. □

Exercises

Exercise 24.1 (Vector bundles)**.** Consider the category VectBund whose objects are pairs (X, E) consisting of a topological space X and a real vector bundle $p\colon E \to X$, and whose morphisms are commutative squares

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & X' \end{array}$$

of topological spaces such that F induces $\mathbb{R}$-linear maps $E_x \to E'_{f(x)}$ on fibers. Show that the forgetful functor $\mathrm{VectBund} \to \mathrm{Top}$, $(X, E) \mapsto X$ is a cartesian fibration.

Exercise 24.2 (Coverings of an anima)**.** A morphism $f\colon Y \to X$ of animae is called a *covering* if all its fibers are sets, i.e., if all their higher homotopy groups vanish. Letting $\mathrm{Cov}(X) \subseteq \mathrm{An}_{/X}$ denote the full subcategory spanned by the coverings, deduce that the straightening equivalence induces an equivalence

$$\mathrm{Cov}(X) \xrightarrow{\sim} \mathrm{Fun}(\Pi_1(X), \mathrm{Set})$$

between the category of coverings of the anima X and the category of functors $\Pi_1(X) \to \mathrm{Set}$. Explain how this is the anima-level analogue of the classical classification of covering spaces.

Exercise 24.3 (Constant diagrams of ∞-categories)**.** Let I and C be ∞-categories, and let $F = \mathrm{const}_C\colon I \to \mathrm{Cat}_\infty$ be the constant functor with value C.

(1) Show that the colimit of F is the product $C \times \|I\|$, where $\|I\| := I[\mathrm{all}^{-1}]$ is the geometric realization of I (obtained by inverting arbitrary morphisms).

(2) Show that the limit of F is $\mathrm{Fun}(\|I\|, C)$.

(3) Conclude that for an anima X we have $\mathrm{colim}_X * \simeq X$.

Exercise 24.4 (A pushout of ∞-categories)**.** Use the formula for colimits in Cat_∞ to compute the pushout in Cat_∞ of the following diagram:

$$\begin{array}{ccc} * & \xrightarrow{0} & [1] \\ {\scriptstyle 1}\downarrow & & \\ [1]. & & \end{array}$$

(You may take for granted that for a 1-category I the unstraightening of a functor $I \to \mathrm{Cat}_\infty$ that happens to land in the subcategory $\mathrm{Cat}_1 \subseteq \mathrm{Cat}_\infty$ of 1-categories is again a 1-category, computed by the Grothendieck construction.)

Exercise 24.5 (Descent for a coproduct)**.** Let X and Y be animae. Construct an equivalence

$$\mathrm{An}_{/(X \sqcup Y)} \simeq \mathrm{An}_{/X} \times \mathrm{An}_{/Y}.$$

Describe both directions explicitly using pullback along the coproduct inclusions and formation of coproducts over $X \sqcup Y$. Deduce that every map $Z \to X \sqcup Y$ decomposes uniquely as the coproduct of its restrictions over X and Y.

25 Complete Segal animae

To construct an ∞-category, it is not enough to specify composition and check unitality and associativity: one must also account for the higher coherences. Products, pullbacks, and functor categories inherit these from ∞-categories already constructed. For constructions such as the span categories of Chapter 13, we need another approach.

Complete Segal animae, introduced by Rezk [Rez01], provide such an approach. To every ∞-category C we can assign its *nerve* $N(C)$, the simplicial anima defined by $N(C)_n := \mathrm{Map}([n], C)$ which records composable strings of morphisms and their coherences. It turns out that, conversely, every simplicial anima satisfying the Segal and Rezk conditions is the nerve of an ∞-category, and the nerve determines that ∞-category. Thus we can construct ∞-categories by constructing simplicial animae and checking these two conditions.

25.1 ∞-Categories as complete Segal animae

We first define the nerve and then characterize which simplicial animae arise in this way.

Definition 25.1.1 (Nerve)**.** The *nerve* of an ∞-category C is the simplicial anima $N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ defined by $N(C)_n := \mathrm{Map}([n], C)$, i.e. as the composite

$$\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}_\infty}(-,C)} \mathrm{An}.$$

By Remark 1.8.4, this construction is functorial in C, giving a functor $N\colon \mathrm{Cat}_\infty \to \mathrm{sAn}$.

We may think of points of $\mathrm{Map}([n], C)$ as 'strings of n composable morphisms in C':

- For $n = 0$, $\mathrm{Map}([0], C) = \mathrm{Fun}(*, C)^{\simeq} \simeq C^{\simeq}$ is just the groupoid core of C;
- For $n = 1$, $\mathrm{Map}([1], C)$ is the anima of morphisms in C;
- For $n = 2$, $\mathrm{Map}([2], C)$ is the anima of commutative triangles in C, which by the Segal axiom, Proposition 1.4.5, is equivalent to $\mathrm{Map}([1], C) \times_{C^{\simeq}} \mathrm{Map}([1], C)$;
- More generally, the 'higher version' of the Segal condition inductively implies that for every $n \geq 0$ we have

  $$\mathrm{Map}([n], C) \xrightarrow{\sim} \mathrm{Map}([1], C) \times_{C^{\simeq}} \mathrm{Map}([1], C) \times_{C^{\simeq}} \cdots \times_{C^{\simeq}} \mathrm{Map}([1], C)$$

  for all n.

The nerve records the objects of C in $N(C)_0 = C^{\simeq}$ and its hom animae as the fibers of $(d_1, d_0)\colon N(C)_1 \to N(C)_0 \times N(C)_0$. Its 2-simplices record composition, and the higher simplices record the coherences. Theorem 25.1.10 will show that the nerve determines C.

The Segal condition allows us to interpret a simplicial anima in the same way, with objects, morphisms, and composition:

Definition 25.1.2 (Segal anima)**.** A simplicial anima $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ is called a *Segal anima* if the Segal condition is satisfied: For every $n \geq 1$, the inclusion maps $e_i\colon [1] \cong \{i-1 \leq i\} \hookrightarrow [n]$ induce an isomorphism of animae

$$(e_1^*, \ldots, e_n^*)\colon X_n \xrightarrow{\sim} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$

We write $\mathrm{Seg}(\mathrm{An}) \subseteq \mathrm{sAn} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An})$ for the full subcategory of Segal animae.

The nerve $N(C)$ of an ∞-category C is always a Segal anima, see Exercise 25.2.

One may show, just like we did for ∞-categories in Section 1.4.2, that a Segal anima X comes equipped with a composition map

$$- \circ -\colon X_1 \times_{X_0} X_1 \simeq X_2 \xrightarrow{d_1} X_1$$

and that composition is unital and associative.

Convention 25.1.3. Given a Segal anima X, we refer to 0-simplices $x \in X_0$ as *objects* of X, and to 1-simplices $f \in X_1$ as *morphisms* of X. The notation $f\colon x \to y$ includes specified identifications $d_1(f) = x$ and $d_0(f) = y$ in X_0. Given objects $x, y \in X_0$, we define the *hom anima* in X as the following pullback:

$$\begin{array}{ccc} \mathrm{Hom}_X(x,y) & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d_1,d_0)} \\ * & \xrightarrow{(x,y)} & X_0 \times X_0. \end{array}$$

A 2-simplex with boundary $f\colon x \to y$, $g\colon y \to z$, and $h\colon x \to z$ supplies an identification $g \circ f = h$ in $\mathrm{Hom}_X(x,z)$.

Definition 25.1.4. Let X be a Segal anima. A morphism $f\colon x \to y$ in X is called an *isomorphism* if there exist morphisms $g, h\colon y \to x$ together with identifications

$$g \circ f = \mathrm{id}_x \quad \text{in } \mathrm{Hom}_X(x,x), \qquad f \circ h = \mathrm{id}_y \quad \text{in } \mathrm{Hom}_X(y,y).$$

We write $X_1^{\sim} \subseteq X_1$ for the full subanima spanned by the isomorphisms.

An identification between points $x, y \in X_0$ induces an isomorphism from x to y in X, by transporting the identity morphism. An isomorphism in X need not arise this way: for example, the ordinary simplicial nerve of a 1-category, viewed as a simplicial anima with discrete terms, may have distinct isomorphic objects. The completeness condition below will ensure that identifications in X_0 account for all isomorphisms in X.

Definition 25.1.5 (Dwyer-Kan equivalences)**.** A morphism $f\colon X \to Y$ of Segal animae is called a *Dwyer-Kan equivalence* if the following two conditions are satisfied:

(1) (Essential surjectivity): For every object $y \in Y_0$ there exists an object $x \in X_0$ and an isomorphism $f(x) \cong y$ in the Segal anima Y.

(2) (Full faithfulness): For objects $x, x' \in X_0$, the induced map $f\colon \mathrm{Hom}_X(x, x') \to \mathrm{Hom}_Y(fx, fx')$ is an equivalence of animae.

Remark 25.1.6. Note that a functor $F\colon C \to D$ between ∞-categories is essentially surjective (resp. fully faithful) precisely if the induced map on nerves $N(F)\colon N(C) \to N(D)$ is essentially surjective (resp. fully faithful). In particular, F is an equivalence if and only if $N(F)$ is a Dwyer-Kan equivalence.

Definition 25.1.7 (Complete Segal anima, Rezk [Rez01])**.** A Segal anima $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ is called *complete*, or is said to satisfy the *Rezk condition*, if the map

$$X_0 \to X_1^{\sim}, \quad x \mapsto \mathrm{id}_x$$

is an equivalence of animae. We denote the full subcategory of complete Segal animae by $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{Seg}(\mathrm{An})$.

Taking fibers of the completeness equivalence over $(x, y) \in X_0 \times X_0$ identifies the anima $(x =_{X_0} y)$ with the anima of isomorphisms from x to y in X. Thus, for complete Segal animae, the two notions agree.

The nerve $N(C)$ of an ∞-category is complete. In particular, the nerve functor $N\colon \mathrm{Cat}_\infty \to \mathrm{sAn}$ factors through $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{sAn}$. This assertion is also included in Exercise 25.2.

Lemma 25.1.8 ([HS25, Lemma 2.3])**.** *Let $\varphi\colon X \to Y$ be a morphism of complete Segal animae. Then φ is an equivalence if and only if it is a Dwyer-Kan equivalence.*

Proof. See Corollary 15.2.13 of [CCNW26]. □

Recall from Construction 1.8.12 the associated ∞-category functor

$$\mathrm{ac}\colon \mathrm{sAn} \to \mathrm{Cat}_\infty,$$

which is the colimit-preserving extension of the inclusion $\mathbf{\Delta} \hookrightarrow \mathrm{Cat}_\infty$. By Theorem 12.8.1 of [CCNW26], this functor is left adjoint to the nerve functor:

$$\mathrm{ac}\colon \mathrm{sAn} \rightleftarrows \mathrm{Cat}_\infty : N.$$

The pointwise formula for left Kan extensions (Theorem 22.4.3) expresses $\mathrm{ac}(X)$ as a colimit in Cat_∞. Using Proposition 24.5.2, this gives a description of $\mathrm{ac}(X)$ as a localization; see [HS25, Corollary 3.8]. That description is used to prove the following result:

Proposition 25.1.9 (Hebestreit and Steinebrunner [HS25, Corollary 3.15, Lemma 4.1])**.** *For every Segal anima X, the unit map $\eta\colon X \to N(\mathrm{ac}(X))$ is a Dwyer-Kan equivalence.*

We refer to [HS25] for the proof. For complete Segal animae, the unit is therefore an isomorphism, which will give the following theorem:

Theorem 25.1.10 ([JT07, Section 4], [Lur09a, Corollary 4.3.16], [HS25])**.** *The nerve functor*

$$N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$$

is fully faithful. Its image consists of the complete Segal animae. In particular, the nerve functor induces an equivalence

$$N\colon \mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{CSeg}(\mathrm{An})$$

between Cat_∞ *and the* ∞*-category of complete Segal animae.*

Proof. Since the nerve functor lands in $\mathrm{CSeg}(\mathrm{An})$, the adjunction $\mathrm{ac} \dashv N$ induces an adjunction

$$\mathrm{ac}\colon \mathrm{CSeg}(\mathrm{An}) \rightleftarrows \mathrm{Cat}_\infty : N.$$

We have to show that both the unit and the counit are equivalences. For the unit, let X be a complete Segal anima. The unit map $\eta\colon X \to N(\mathrm{ac}(X))$ is a Dwyer-Kan equivalence by Proposition 25.1.9 and thus an equivalence by Lemma 25.1.8, since both sides are complete Segal animae. For the counit, let C be an ∞-category. To show that $\varepsilon\colon \mathrm{ac}(N(C)) \to C$ is an equivalence, it suffices by Remark 25.1.6 to show that $N(\varepsilon)\colon N(\mathrm{ac}(N(C))) \to N(C)$ is an equivalence. But this is a consequence of the triangle identity and the fact that $\eta_{N(C)}\colon N(C) \to N(\mathrm{ac}(N(C)))$ is an equivalence. □

Corollary 25.1.11. *The associated category functor* $\mathrm{ac}\colon \mathrm{Seg}(\mathrm{An}) \to \mathrm{Cat}_\infty$ *is a Bousfield localization. A morphism* $\varphi\colon X \to Y$ *of Segal animae is inverted by* $\mathrm{ac}(-)$ *if and only if it is a Dwyer-Kan equivalence.*

Proof. The first statement is a consequence of the full faithfulness of the nerve functor $N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{Seg}(\mathrm{An})$ from Theorem 25.1.10. For the second statement, consider the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ N(\mathrm{ac}(X)) & \xrightarrow{N(\mathrm{ac}(\varphi))} & N(\mathrm{ac}(Y)). \end{array}$$

The vertical morphisms are Dwyer-Kan equivalences by Proposition 25.1.9. Since Dwyer-Kan equivalences satisfy the two-out-of-three property by Lemma 15.4.5, it follows that φ is a Dwyer-Kan equivalence if and only if $N(\mathrm{ac}(\varphi))$ is. By Remark 25.1.6, the latter holds if and only if $\mathrm{ac}(\varphi)\colon \mathrm{ac}(X) \to \mathrm{ac}(Y)$ is an equivalence. □

Corollary 25.1.12. *The inclusions* $\mathrm{CSeg}(\mathrm{An}) \hookrightarrow \mathrm{sAn}$ *and* $\mathrm{CSeg}(\mathrm{An}) \hookrightarrow \mathrm{Seg}(\mathrm{An})$ *admit left adjoints given by* $X \mapsto N(\mathrm{ac}(X))$.

Proof. Given a complete Segal anima Y, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CSeg}(\mathrm{An})}(N(\mathrm{ac}(X)), Y) &\simeq \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{ac}(X), \mathrm{ac}(Y)) \\ &\simeq \mathrm{Hom}_{\mathrm{sAn}}(X, N(\mathrm{ac}(Y))) \\ &\simeq \mathrm{Hom}_{\mathrm{sAn}}(X, Y), \end{aligned}$$

where the first isomorphism holds by the equivalence $\mathrm{Cat}_\infty \simeq \mathrm{CSeg}(\mathrm{An})$, the second comes from the adjunction $\mathrm{ac} \dashv N$, and the third is induced by $Y \xrightarrow{\sim} N(\mathrm{ac}(Y))$. This proves the assertion for the inclusion into sAn; restricting to the full subcategory $\mathrm{Seg}(\mathrm{An}) \subseteq \mathrm{sAn}$ proves it for the second inclusion as well. □

Notation 25.1.13. The left adjoint to $\mathrm{CSeg}(\mathrm{An}) \hookrightarrow \mathrm{Seg}(\mathrm{An})$ is often called the *completion functor* or *Rezk completion* and is denoted by

$$\widehat{(-)}\colon \mathrm{Seg}(\mathrm{An}) \to \mathrm{CSeg}(\mathrm{An}).$$

As a consequence of the theorem, every small ∞-category can be built from points and arrows by iterated colimit constructions:

Proposition 25.1.14. *The ∞-category Cat_∞ is generated under colimits by $[0]$ and $[1]$.*

Proof. The presheaf category $\mathrm{sAn} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An})$ is generated by the representable presheaves under colimits by Theorem 1.8.10. It follows that Cat_∞ is generated under colimits by their images under the localization functor $\mathrm{ac}\colon \mathrm{sAn} \to \mathrm{Cat}_\infty$, which are the posets $[n]$. The claim now follows, since the Segal condition shows that each $[n]$ is an iterated pushout of copies of $[1]$ along $[0]$. □

25.2 Filtered colimits and geometric realizations of ∞-categories

Limits and filtered colimits of complete Segal animae can be computed degreewise. This lets us deduce that finite limits commute with filtered colimits in Cat_∞.

Lemma 25.2.1. *The subcategory $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{sAn}$ is closed under limits and filtered colimits.*

Proof. Recall that limits and colimits in sAn are computed pointwise. The conditions defining the subcategory $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{sAn}$ are expressed in terms of finite limits. Since finite limits commute with both limits and filtered colimits[22] in An, the claim follows. □

Corollary 25.2.2. *The nerve functor $N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$ preserves filtered colimits. In particular, finite limits commute with filtered colimits in Cat_∞.*

Proof. The first claim is immediate from Lemma 25.2.1 and the equivalence $N\colon \mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{CSeg}(\mathrm{An})$. The second claim follows since limits commute with filtered colimits in An and hence in sAn. □

Corollary 25.2.3. *For every finite ∞-category I, the functor $\mathrm{Hom}_{\mathrm{Cat}_\infty}(I, -)\colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves filtered colimits. In particular the core functor $(-)^{\simeq}\colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves filtered colimits.*

Proof. By definition, the subcategory $\mathrm{Cat}_\infty^{\mathrm{fin}} \subseteq \mathrm{Cat}_\infty$ is the smallest subcategory closed under pushouts which contains $\emptyset$, $[0]$ and $[1]$. Since filtered colimits commute with pullbacks in An, the class I of ∞-categories for which the claim holds is closed under pushouts. It thus remains to show the claim for $I = \emptyset, [0], [1]$. The claim for $\emptyset$ is clear, since $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\emptyset, C)$ is terminal for each C and filtered colimits of terminal objects are terminal. The claim for $[0]$ and $[1]$ follows from the previous lemma, since $\mathrm{Hom}_{\mathrm{Cat}_\infty}([n], -)$ agrees with the composite $\mathrm{Cat}_\infty \xrightarrow{N} \mathrm{sAn} \xrightarrow{\mathrm{ev}_n} \mathrm{An}$.

The last claim is the special case $I = [0]$. □

Corollary 25.2.4. *The ∞-category Cat_∞ is compactly generated. Its compact objects are precisely the retracts of finite ∞-categories.*

[22] See e.g. [Hau25, Section 9.9] for a 'synthetic' proof of the fact that finite limits commute with filtered colimits in An.

Proof. Every finite ∞-category is compact by Corollary 25.2.3. By Proposition 25.1.14, every small ∞-category is a small colimit of copies of $[0]$ and $[1]$. Writing such a colimit as a filtered colimit of its finite subcolimits expresses every small ∞-category as a filtered colimit of finite ∞-categories.

It remains to identify all compact objects. If C is compact, then applying compactness to a filtered-colimit presentation of C by finite ∞-categories shows that id_C factors through one of its finite stages. Thus C is a retract of a finite ∞-category. Conversely, every such retract is compact. The subcategory of compact objects is therefore the idempotent completion of the small ∞-category $\mathrm{Cat}_\infty^{\mathrm{fin}}$, and is consequently small. □

We have used the term *geometric realization* for two constructions:

- For an ∞-category C, the geometric realization $\|C\|$ is obtained from C by inverting all the morphisms in C:
$$\|C\| := C[W^{-1}], \qquad W := \mathrm{Map}([1], C);$$
- For a simplicial anima $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$, its geometric realization is its colimit:
$$|X| := \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n.$$

These agree when we pass from an ∞-category to its nerve:

Lemma 25.2.5. *The geometric realization $\|C\|$ of an ∞-category C is naturally equivalent to the geometric realization $|N(C)|$ of its nerve.*

Proof. The colimit-functor is by definition a left adjoint to the constant simplicial anima functor:

$$|-|\colon \mathrm{sAn} \rightleftarrows \mathrm{An} \colon \mathrm{const}.$$

For any anima X, the constant simplicial anima $\mathrm{const}_X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ is a complete Segal anima (equivalent to the nerve $N(X)$ of X), and hence this adjunction restricts to an adjunction

$$|-|\colon \mathrm{CS}(\mathrm{An}) \rightleftarrows \mathrm{An} \colon \mathrm{const}.$$

Under the equivalence $\mathrm{Cat}_\infty \simeq \mathrm{CS}(\mathrm{An})$ from Theorem 25.1.10, the constant functor corresponds to the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$, and by uniqueness of adjoints this implies the claim. □

25.3 Homotopy coherent nerves

In topology, the morphisms in various categories under consideration often form a topological space, and composition is continuous. Such a topologically enriched category should give an ∞-category, called its *homotopy coherent nerve*, whose hom animae are obtained from the hom spaces by applying the functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$. One way to make this precise is via the theory of *enriched ∞-categories*: one may identify ∞-categories with An-enriched ∞-categories [Hei23, Lemma 6.18] and then obtain a functor from topologically enriched categories by changing enrichments along Π_∞. Here we bypass this technology and construct the ∞-category directly from the topological enrichment using colimits in Cat_∞; its relation to the quasicategorical construction from Example 1.6.14 is explained in Remark 25.3.3.

Definition 25.3.1. A *topologically enriched category* is a classical 1-category D together with a chosen topology on each of the hom sets $\mathrm{Hom}_D(X,Y)$ for $X, Y \in D$, satisfying the property that the composition maps $\mathrm{Hom}_D(Y,Z) \times \mathrm{Hom}_D(X,Y) \to \mathrm{Hom}_D(X,Z)$ are continuous with respect to these topologies. We denote by $\mathrm{Hom}_D^{\mathrm{top}}(X,Y) \in \mathrm{Top}$ the hom set in C equipped with the chosen topology.

Construction 25.3.2 (Homotopy coherent nerve)**.** Consider a topologically enriched category D. We will define an ∞-category $N^{\mathrm{top}}(D)$, called the *homotopy coherent nerve of* D.

For every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, consider the (classical) 1-category D_n defined as follows:

- The objects of D_n are the objects of D;
- The morphisms $X \to Y$ of D_n are the continuous maps $|\Delta^n| \to \mathrm{Hom}_D^{\mathrm{top}}(X,Y)$;
- The identity in D_n is the constant map $|\Delta^n| \to \mathrm{Hom}_D^{\mathrm{top}}(X,X)$ with value id_X;
- The composition in D_n of morphisms $f\colon |\Delta^n| \to \mathrm{Hom}_D^{\mathrm{top}}(X,Y)$ and $g\colon |\Delta^n| \to \mathrm{Hom}_D^{\mathrm{top}}(Y,Z)$ is the composite

$$|\Delta^n| \xrightarrow{\Delta} |\Delta^n| \times |\Delta^n| \xrightarrow{g \times f} \mathrm{Hom}_D^{\mathrm{top}}(Y,Z) \times \mathrm{Hom}_D^{\mathrm{top}}(X,Y) \xrightarrow{\circ} \mathrm{Hom}_D^{\mathrm{top}}(X,Z).$$

Since $|\Delta^n|$ is functorial in $[n] \in \mathbf{\Delta}$ (see Example 1.6.8) we see that the assignment $[n] \mapsto D_n$ determines a functor

$$D_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}_\infty.$$

We then define the ∞-category $N^{\mathrm{top}}(D)$ as the colimit of this functor:

$$N^{\mathrm{top}}(D) \quad := \quad \mathrm{colim}(D_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty).$$

Note that D_0 is the underlying category of D. Since it admits a canonical functor to $N^{\mathrm{top}}(D)$, we see that every object of D gives rise to an object of $N^{\mathrm{top}}(D)$, and in fact all objects are of this form.

Remark 25.3.3. The fact that we refer to $N^{\mathrm{top}}(D)$ as the 'homotopy coherent nerve' is somewhat abusive: this terminology is often used instead for the simplicial set $N^{\Delta}(\mathbf{C})$ associated to a simplicially enriched category, cf. Example 1.6.14. The two constructions are closely related: given a topologically enriched category D, we may obtain a simplicially enriched category $\mathbf{C}$ by taking $\mathrm{Hom}_{\mathbf{C}}^{\Delta}(X,Y) := \mathrm{Sing}(\mathrm{Hom}_D^{\mathrm{top}}(X,Y))$, i.e. we apply the singular simplicial complex construction from Example 1.6.8 to each of the hom spaces of D. Then the underlying ∞-category of the quasi-category $N^{\Delta}(\mathbf{C})$ is equivalent to $N^{\mathrm{top}}(D)$. The proof of this fact is quite technical: it relies on the fact that there is a model structure (due to Bergner [Ber07]) on the category of (small) simplicially enriched categories whose localization at the weak equivalences is Cat_∞. A reference for this claim in the literature may be found in Gepner and Meier [GM23, Corollary B.4].

Proposition 25.3.4. *For objects X and Y of D, the hom anima between X and Y in $N^{\mathrm{top}}(D)$ is equivalent to the underlying anima of the topological space* $\mathrm{Hom}_D^{\mathrm{top}}(X,Y)$*:*

$$\mathrm{Hom}_{N^{\mathrm{top}}(D)}(X,Y) \simeq \Pi_\infty(\mathrm{Hom}_D^{\mathrm{top}}(X,Y)).$$

Proof. Let B_n be the ordinary simplicial nerve of D_n, regarded as a simplicial anima, and put $B := \operatorname{colim}_n B_n$. Since geometric realizations commute with finite products in animae, we have

$$B_k \simeq \coprod_{x_0,\dots,x_k \in \mathrm{Ob}(D)} \prod_{j=1}^{k} \Pi_\infty \operatorname{Hom}_D^{\mathrm{top}}(x_{j-1}, x_j).$$

Consequently, B is a Segal anima with object anima the discrete set $\mathrm{Ob}(D)$ and the asserted hom animae. By Remark 1.8.13, the associated category of B_n is D_n. Since the associated-category functor preserves colimits, its value on B is therefore $N^{\mathrm{top}}(D)$. The result follows from Proposition 25.1.9, since the completion map preserves hom animae. □

Exercises

Exercise 25.1 (Composition in a Segal anima)**.** Let X be a Segal anima. Show that the degeneracy map $s_0 \colon X_0 \to X_1$ defines identity morphisms and that the composition map

$$X_1 \times_{X_0} X_1 \simeq X_2 \xrightarrow{d_1} X_1$$

is unital and associative.

Exercise 25.2 (Nerves and completeness)**.** (1) Let C be an ∞-category. Show that its ∞-categorical nerve $N(C)_n = \mathrm{Map}([n], C)$ is a complete Segal anima.

(2) Let C be a classical 1-category. Regard its nerve $N(C) \in \mathrm{sSet}$ from Example 1.6.11 as a simplicial anima. Show that it is a Segal anima. Is it complete? Determine precisely when this holds.

Exercise 25.3 (Discrete topological enrichment)**.** Let C be a classical 1-category and regard it as a topologically enriched category C^{disc} by giving each hom set the discrete topology. Show that the canonical functor

$$C \longrightarrow N^{\mathrm{top}}(C^{\mathrm{disc}})$$

is an equivalence of ∞-categories.

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