Synthetic Category Theory

Volume I: Foundations

Book project, in progress

Denis-Charles Cisinski, Bastiaan Cnossen, and Tashi Walde

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Abstract

This book aims at introducing higher category theory in an axiomatic way: instead of building higher category theory on top of set theory, as is done in the framework of quasi-categories, we will introduce higher category theory synthetically, with the aim of having access to its main features as quickly as possible: the Yoneda embedding, the straightening/unstraightening correspondence, the theory of Kan extensions, the theory of presentable categories, etcetera. Our axiomatic approach will not only provide tools to comprehend the important aspects of higher categories as they are used in practice (derived algebraic geometry, homotopical algebra, etc.) but also in more general contexts (e.g. higher category theory internally in any higher topos) and in logic (dependent type theory).

Preface

This book is under active construction: various parts of it are very much unfinished and will be refined and expanded over the course of 2025 and 2026. The most recent version of this book can be found here.

Earlier versions of the project are available from April 2024 and November 2025. The present volume focuses on synthetic category theory. The former material on tribes is being developed separately in Volume II, *Models for Synthetic Category Theory*.

Kim Nguyen was listed as co-author in the first versions of this book: he was indeed very actively involved in the redaction of the first draft, with many crucial contributions on the right concepts and formulations on which all the theory would be developed. We feel extremely grateful to him and we cherish all the memories of the many lively discussions we had on these topics over the years.

We would like to thank Ulrik Buchholtz, Daniel Gratzer and Jonathan Weinberger for useful discussions on (multimodal) type theory. We further thank the members of a reading group in Université Sorbonne Paris Nord for their useful constructive feedback that improved the exposition of this work.

We thank Christian Sattler for various suggestions, among which the suggestion to define the walking triangle as Fun($[1]$, $[1]$).

We also wish to thank Johannes Gloßner for various useful discussions, and Ishan Dasgupta Samarendra, Rashiqa Dawood, Yang Jiatong, Hayato Nasu, Andrea Panontin, and Jonathan Weinberger for pointing out mistakes and/or typos.

Change log

The following is an overview of the major updates in this book:

- **September 2026** New chapters on presentable categories.
- **July 2026:** New chapter on K-theory: Waldhausen's S-construction and its universal property, and the localization theorem.
- **June 2026:** New chapters on ind-objects and stable categories.
- **May 2026:** New chapter on left exact localizations.
- **March 2026:** Completed part on Peano arithmetic and filtered colimits of animae. Established equivalence between univalent Segal animae and synthetic categories.
- **February 2026:** Completed the parts on straightening/unstraightening, the Yoneda lemma, and functoriality of the opposite category functor. Established full faithfulness of the nerve functor by explicit computations of colimits.
- **November 2025:** Retitled book to *Synthetic Category Theory*; Rewrote Proposition 3.4.5 to emphasize the symmetry between the equivalent conditions from Proposition 3.4.4; Changed conventions on underlining; Fixed various typos; Removed lots of older material.
- **April 2025:** Redefined $[2]$ as $\mathrm{Fun}([1],[1])$, as suggested by Christian Sattler. Various reorganizations from Chapter 12 onwards. Changed terminology "fundamental groupoid" to "geometric realization". Changed convention regarding 'surjective' and 'essentially surjective', see Definition 6.3.6 and Definition 11.2.3;
- **November 2024:** Started a change log. Revisions and cleaner exposition in Chapters 8, 9 and 12. Reformulation of Axiom B.8 for functor categories;
- **October 2024:** Turned the 'lecture notes' into a 'book project'; Started Section 10.1 on naive set theory within synthetic category theory; Started Section 16.1 on the partially ordered set of natural numbers;

- **August 2024:** Removed axioms on diagram categories, replaced with simpler but more ad hoc approach merely using two synthetic categories [1] and [2];
- **July 2024:** Major rewrite of the introduction;
- **May 2024:** Added Chapter 5 on contexts;
- **April - July 2024:** Incorporated the material from Cisinski's lecture course *Formalization of Higher Categories II* during the summer term 2024 in Regensburg: the current Chapters 8, 9, 12 and 10 on universes, cofinality, the Yoneda embedding, peano arithmetic and the simplex category, plus some material which made its way into earlier chapters;
- **April 2024:** Major revision of the lecture notes into its current form: Introduced the language of 'naive category theory' in Chapter 1, rewrote the notes in terms of that language. Removed material on tribes, see Part II here;
- **December 2023:** Expanded Chapter 4 with more material on (co)cartesian fibrations;
- **October 2023 - February 2024:** Lecture notes for Cisinski's lecture course *Formalization of Higher Category* during the winter term 23/24 in Regensburg. Topics included: the theory of clans and tribes; a discussion of homotopy type theory within a tribe; disjoint unions and functor categories; a definition of simplices via the join construction; a discussion of diagram categories, the Segal axiom and the Rezk axiom; a discussion of groupoids; a (very brief) treatment of (co)cartesian fibrations; subcategories and localizations; the Fundamental Theorem of category theory; universes and directed univalence.

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Introduction

The goal of this book is to establish an axiomatic framework for higher category theory. This approach is best thought of as a form of *synthetic* category theory: instead of constructing higher categories from more primitive objects, the axioms impose the intended behavior of higher categories directly. This stands in contrast to *analytic* approaches, such as the theory of quasicategories, in which one first constructs higher categories inside a background foundation, typically set theory, and then proves that the resulting objects have the desired categorical properties.

Mathematical motivation

Category theory has long been a useful unifying language for abstract mathematics. Higher category theory[1] refines ordinary category theory by allowing all constructions and properties to be considered up to coherent homotopy. In recent decades, higher category theory and homotopy theory have become central tools in algebraic topology, algebraic geometry, representation theory, mathematical physics, and many other fields.

The traditional foundations of the field have rested on specific models, most notably the framework of quasicategories developed by Joyal [Joy08] and Lurie [Lur09; Lur17]. As the subject matured, however, it became clear that most of its results and constructions are *model-independent*: they never touch the combinatorial details of quasicategories, or of any other model. One way to say this is that there is a *language* of category theory, of which the classical set-based notion, a category as a set of objects and a set of morphisms, is one interpretation, and the theory of quasicategories another. The working content of higher category theory lives in the language, not in any single interpretation of it. Following this viewpoint, much recent work is written in a deliberately model-agnostic style that avoids the combinatorial scaffolding of quasicategories.

One primary goal of this book is to provide a detailed account of this model-independent language of category theory, and to identify a list of key categorical axioms one needs to assume to build up the rest of the theory. We believe this allows for a more direct and intuitive engagement with the core concepts of higher category theory, and that it leads to proofs carrying insights that may have been obscured by model-specific details in more traditional treatments of the theory.

Our framework has three further advantages:

- *Parametrized ∞-category theory*: Our formalism is designed to serve as the internal language of the ∞-category of sheaves of ∞-categories on any ∞-topos. Thus all results proved in this book are automatically valid in parametrized category theory, as in the work of Barwick et al.

[1] More specifically: $(\infty, 1)$-category theory.

[Bar+16] on genuine equivariant homotopy theory and the work of Martini and Wolf [MW24] on category theory internal to an ∞-topos.

- *A working language for ∞-category theory*: The theory specifies the basic notions, operations, and identifications in a language in which one can teach and use ∞-category theory in the same way one uses naive set theory, making the subject more accessible.

- *Elementary ∞-category theory*: The axioms are constructive, and hence shed light on the scope of "elementary ∞-category theory": Even constructively it is possible to recover substantial parts of standard higher category theory.

An alternative formal approach to higher category theory was developed by Riehl and Verity [RV22]. Their ∞-cosmoi are certain categories enriched in quasicategories and make systematic use of cotensors by simplicial sets. Their work is model-independent in the sense that a common formalism applies to any ∞-cosmos satisfying suitable axioms. Our aim is different: the language developed here is model-agnostic, in that it does not refer to a model of higher categories at all.

Philosophical motivation

There is also a philosophical reason to formalize higher category theory: category theory can serve as a foundation for mathematics. This idea goes back at least to Lawvere, and it has been developed in many forms since then; see for instance Landry and Marquis [LM05] for historical context.

The point of view in this book is that categories, in fact $(\infty, 1)$-categories, should be treated as the building blocks of mathematics, in the role traditionally played by sets or types. One way to study mathematical objects within category theory is through the categories they form; this perspective is already a common practice in fields like algebraic geometry and algebraic topology. We wish to highlight a second way to recover ordinary mathematics as a fragment of the theory of categories: by first restricting the theory to *animae*, those categories in which every morphism is an isomorphism, and then further to *sets*, those animae that admit at most one isomorphism between every two objects. The resulting set theory can then be used to define other mathematical objects in the usual way.

The latter perspective also makes clear that the jump from set-based mathematics to category-based mathematics consists of two conceptual leaps, corresponding to the passage from sets to animae and the passage from animae to categories.

The first leap concerns the notion of *equality*. In set-based mathematics, equality is treated as a property: two elements of a set are either equal or they are not, and there is no further question of *how* they are equal. In anima-based mathematics, equality is instead treated as an intrinsic structure. To illustrate the difference, set-based mathematics treats two abelian groups as the same if they are strictly equal as sets with equal operations, while the anima-based perspective on such equalities would amount to group isomorphisms. The structured perspective on equality is not new; for example:

- Makkai [Mak98] proposed the **Principle of Isomorphism**: All grammatically correct properties of objects of a category are to be invariant under isomorphism.

- Voevodsky's *homotopy type theory (HoTT)* [Uni13] builds in a flexible notion of equality which has the key feature that it is compatible with the **Univalence Axiom**, which roughly states that two isomorphic types (groups, topological spaces, etcetera) are equal to each other.

We take it as our main guiding principle throughout the book:

> **Fundamental Principle of Homotopy Theory:** When expressing that two entities are equal, we must always specify *how* they are equal by providing an isomorphism between them. All valid mathematical constructions should preserve these isomorphisms.

Taking this principle seriously, we are forced to replace sets as our basic notion of 'collections of mathematical objects': equality in sets is too rigid. The new fundamental object in this paradigm is called an *anima*, which may be described as a collection of objects such that for each two objects there is an anima of isomorphisms between them. This gives animae the structure of a groupoid. By considering equality of morphisms, we also have *isomorphisms between isomorphisms*, and repeating this process we see that animae naturally behave like ∞-*groupoids*. Other common names in the literature are *spaces* or *homotopy types*; HoTT for example uses the latter framing. In this book, however, the word *space* is reserved for an actual topological space.

The second leap is to replace animae by categories as the primitive building blocks of the theory. In practice, a class of mathematical objects often naturally comes with more general relations between objects than mere isomorphisms; think of homomorphisms between groups, continuous maps between topological spaces, functors between categories. The concept of a category encapsulates this by encoding morphisms between objects and behavior of their compositions.

The classical perspective on categories is analytic: a category consists of a *set* of objects and a *set* of morphisms, satisfying various axioms. This perspective clashes with Makkai's principle of isomorphism, as it allows one to distinguish two isomorphic objects: categories should have an *anima* of objects and a *anima* of morphisms. In light of this, it may be tempting to try to define categories directly within HoTT. However, we argue that it is better to make a second leap, and directly take categories to be the primitive objects of the theory, with sets and animae arising as special cases.

Just like the fundamental principle of homotopy theory forces animae to behave like ∞-groupoids, it also forces categories to behave like $(\infty, 1)$-*categories*: equality between morphisms must be encoded by some notion of 2-isomorphisms, equality between those by 3-isomorphisms, and so on. In this way, the language of category theory natively speaks about $(\infty, 1)$-categories, and strict 1-categories are merely a fragment of this theory.

For broader discussions of categorical and structural foundations of mathematics, we refer to Tsementzis [Tse17], Marquis [Mar13], Rodin [Rod14], and Awodey [Awo14].

Synthetic category theory eats itself

One of the distinctive features of synthetic category theory is that it can interpret itself. We take this as a key design criterion: There should be a universe Cat whose objects are small synthetic categories, and all the axioms of the theory we develop should ultimately be satisfied in the internal language of Cat. In this sense, it is analogous to the role of Grothendieck universes in set theory:

sets large enough to contain the usual set-theoretic constructions, which can therefore serve as a model of set theory inside set theory. A systematic verification of this design criterion is deferred to Volume II.

The axiom that makes this precise is *directed univalence*, which relates morphisms in Cat to functors between synthetic categories. The main consequence of this axiom is the familiar Grothendieck correspondence: for every small synthetic category C, functors

$$C \to \mathrm{Cat}$$

correspond to small cocartesian fibrations over C. To illustrate why this relates the external theory of synthetic categories to the internal theory of Cat, it is illustrative to consider the terminal case $C = *$. In this case, the Grothendieck correspondence says that objects of Cat correspond to small cocartesian fibrations over the terminal category, hence to small synthetic categories. Applying the principle to more general instances of C then provides the translation between the internal language of Cat and the external language of categories that we are setting up.

Restricting directed univalence to animae recovers Voevodsky's univalence axiom: isomorphisms between animae correspond to equalities in the subuniverse $\mathrm{An} \subseteq \mathrm{Cat}$ of animae. But the directed form is stronger: it also says that maps between animae correspond to morphisms in An. This also marks a crucial difference with homotopy type theory: HoTT has a universe of types, and univalence explains equality in that universe, but this does not yet give a self-interpretation of HoTT inside HoTT. The problem is to internalize the syntax of type theory together with its higher coherence in plain homotopy type theory. Synthetic category theory is designed so that the theory eats itself: the universe Cat should serve not only as one object of the theory, but also as a semantic universe in which the theory can be interpreted. This model theorem will be developed in Volume II.

Relation to type theory

In this book, we present the theory of synthetic categories in ordinary mathematical language, specifying its basic notions, operations, and coherence identifications. The main objective of this book is to provide a model-agnostic presentation of synthetic category theory, and build up standard category theory from a sizable list of foundational axioms. No knowledge of type theory is required to read this book.

Nevertheless, our work is heavily inspired by type-theoretic considerations, and we expect it to be possible to write down rigorous type-theoretic syntax and derivation rules that allow the theory to be formalized in proof assistants. We are exploring several approaches:

- A presentation of the theory à la Martin-Löf is in preparation in collaboration with Michele Riva;
- A presentation of the theory within the type theory CaTT of ω-categories [FM17; BFM24] is in preparation in collaboration with Ivan Kobe.

For readers familiar with dependent type theory, we wish to flag two differences with synthetic category theory.

The first difference concerns the *Curry–Howard correspondence*: the idea that types may be interpreted as mathematical statements, and terms of a type as proofs of those statements. Constructions

of types, like the disjoint union $X \sqcup Y$ or the product $X \times Y$, then admit logical interpretations, like the logical disjunction $P \vee Q$ and logical conjunction $P \wedge Q$. Under this correspondence, propositions appear as the special case of (-1)-truncated types: those types which admit at most one term.

In synthetic category theory, it is no longer the case that arbitrary types correspond to mathematical statements. General types are now interpreted as categories, which allow for non-invertible maps between objects. The types corresponding to statements are the primitive *animae*, which the groupoid recognition axiom identifies with the categories in which every morphism is an isomorphism.

The second difference concerns *type dependency*. In ordinary dependent type theory, for each context Γ, the types in context Γ form a category, and any substitution

$$f : \Delta \to \Gamma$$

induces a pullback operation on types. This pullback operation preserves the logical structure of the theory: products, sums, identity types, and the other type-forming operations are stable under substitution. In this sense, substitution along an arbitrary map of contexts preserves the logic.

In synthetic category theory, this principle breaks in a controlled way. It continues to behave as expected when the contexts involved are animae. But if Γ is a general synthetic category, then a type in context Γ should be thought of as a category over Γ, that is, as a functor

$$p : X \to \Gamma.$$

The resulting theory of categories over Γ is not, in general, again a semantic interpretation of synthetic category theory.

However, there is a better category-theoretic replacement: among functors $X \to \Gamma$, one should restrict to cocartesian fibrations over Γ. The resulting category *does* form a semantic interpretation of synthetic category theory. This theory is not equivalent to the naive theory of all categories over Γ, except when Γ is an anima. A precise form of this result is treated in Volume II.

Still, even when we restrict to cocartesian fibrations, pullback along an arbitrary functor $f : \Delta \to \Gamma$ does not generally preserve all categorical structure: unlike in type theory, substitution does not fully preserve the logic.

We wish to regard this phenomenon as a feature rather than a bug: It is one of the ways in which synthetic category theory is more expressive than ordinary dependent type theory. The comparison between the naive theory of categories over Γ and the theory of cocartesian fibrations over Γ is itself structured: it is governed by adjunctions and by canonical comparison maps, many of which are not invertible. Such non-invertible comparison maps are not pathological in category theory: they often form an integral part of the mathematical discourse of the subject.

Consistency

The consistency of the theory presented in this book is controlled by the usual models of higher category theory. Cisinski's work on higher categories [Cis19], together with the work of Cisinski and Nguyen on the universal cocartesian fibration [CN22], shows that Joyal's theory of quasicategories gives a semantic interpretation of synthetic category theory. Thus the theory is at least as consistent as the usual set-theoretic foundations used to construct quasicategories.

Let us note that the proof of directed univalence, namely the fact that there is a reasonable ∞-category of ∞-categories, subsumes Voevodsky's proof of the univalence axiom for Kan complexes [KL21].

As mentioned before, the work of Buchholtz, Gratzer and Weinberger [GWB26] on simplicial type theory is expected to provide a constructive semantic interpretation of synthetic category theory within the framework of Riehl and Shulman [RS17]. Conversely, the present book proves among other things that the synthetic category of synthetic categories is equivalent to the synthetic category of univalent Segal objects in animae, which can be used as the basis for an internal version of simplicial homotopy type theory. This indicates that synthetic category theory and simplicial type theory are mutually consistent, constructively.

Organization of the book

The present version of the book is organized as follows:

(1) In Chapter 1, we introduce the language of *synthetic category theory*. Categories, animae, functors and isomorphism animae between functors are taken as primitive, and all our axioms are expressed in terms of them. We introduce products, coproducts, pullbacks and functor categories via universal properties, and use mapping animae to define the collection of objects of a category.

(2) In Chapter 2, we study the internal structure of categories. We introduce morphisms and commutative diagrams, and formulate the Segal axiom and the Rezk axiom to ensure that the objects we call synthetic categories do indeed behave like categories. We then relate primitive animae to the structural notion of a *groupoid* and study groupoid cores. Animae play a fundamental role in our approach: an object of a synthetic category C is a functor $x\colon \Gamma \to C$ whose source is allowed to be an arbitrary anima Γ.

(3) In Chapter 3, we introduce various fundamental categorical constructions in the setting of synthetic category theory: full subcategories, localizations, geometric realizations, relative functor categories, joins and slices.

(4) In Chapter 4, we introduce adjunctions, initial and terminal objects, left and right fibrations, and cartesian and cocartesian fibrations.

(5) In Chapter 5, we develop the language of categories in an anima context, weakening, dependent sums, and substitution. We explain how to interpret statements and constructions involving objects of a category using the universal object.

(6) In Chapter 6, we introduce the functoriality of universals axiom: to provide a left adjoint section to a cartesian fibration, it suffices to provide initial objects in its fibers. We show that this is equivalent to the Fundamental Theorem of category theory, which says that a functor is an equivalence if and only if it is fully faithful and strongly surjective, and develop pointwise criteria for adjunctions and fibrations.

(7) In Chapter 7, we introduce limits and colimits, their functoriality and preservation, and their computation in functor categories.

(8) The main goal of Chapter 8 is to axiomatize a synthetic category Cat of small synthetic categories. It comes equipped with a cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ satisfying the property that for every small synthetic category C there is an equivalence between functors from C to Cat and

cocartesian fibrations over C with small fibers. The equivalence is given by pulling back the universal small cocartesian fibration π_{univ} along the given functor $C \to \text{Cat}$.

(9) Chapter 9 studies initial and terminal functors, proper and smooth functors, weak homotopy equivalences, and base change.

(10) Chapter 10 develops sets, natural numbers, posets and simplices.

(11) Chapter 11 introduces covers and truncation, thus opening the possibility of dealing constructively with mere existence statements.

(12) Chapter 12 develops opposite categories, twisted arrow categories, the Yoneda lemma, representable approximation, locally small categories, and descent.

(13) Chapter 13 free completions by small colimits and descent. In particular, Quillen's theorem B is proven.

(14) Chapter 14 studies products in localizations, localizations and adjunctions, and the calculus of fractions.

(15) Chapter 15 explains how univalent Segal animae and the Rezk nerve functor yield a new description of the universe of categories.

(16) Chapter 16 develops filtered categories and filtered colimits of animae, together with the theory of finite categories.

(17) Chapter 17 studies left or right exact localizations as well as presentable categories with respect to a fixed universe (later in the book, this will specialize to finite posets or finite sets, for instance).

(18) Chapter 18 studies compact generation, tensor product of finitely cocomplete categories, and Ind-objects.

(19) Chapter 19 develops finite-colimit completions, stable categories, spectra, and Verdier quotients.

(20) Chapter 20 develops additive and Waldhausen categories, completions by sifted colimits, the S-construction, additivity, the universal property of K-theory (and of the waldhausen S-construction in general), and the localization theorem in algebraic K-theory.

(21) Chapter 21 introduces a new accessibility axiom and develops the theory of presentable categories in full generality - existence of left Bousfield locazitions, presentation as left Bousfield localizations of categories of presheaves, adjoint functor theorem.

Overview of the axioms

The axioms are introduced progressively, since several require preliminary categorical language before they can be stated. This chapter gives a concise overview of the actual axiomatic package and the principal constructions used in its formulation. Precise statements and dependencies appear in the chapters that follow.

The language of synthetic category theory

Synthetic category theory rests upon a language of categories, functors and identifications, together with specified operations and coherences. We will now provide a rough overview of this language, leaving the details to Chapter 1.

Axiom A. There are categories and, among them, animae. Between categories there are functors, with identity functors and composition. For parallel functors $f, g \colon C \to D$, there is an anima of identifications $(f = g)$, with prewhiskering, postwhiskering, and inversion functors. An object $c : C$ is a functor $* \to C$, and an identification $\alpha \colon f = g$ is a functor $* \to (f = g)$. Composition of identifications is a functor, with unit, associativity, and inverse identifications. Composition of functors has unitors and an associator, satisfying the triangle and pentagon identities and the specified whiskering and interchange laws.

Animae serve as the homotopy-coherent collections of the theory, playing the role of sets in ordinary mathematics. The distinguishing feature is the notion of equality in an anima: given two identifications $p, q : x =_X y$, we may talk about identifications between p and q, and this process may be iterated. This also applies to identifications between functors.

A functor $f \colon C \to D$ may be displayed in term notation as $x : C \vdash f(x) : D$. Substitution displays composition of functors; identifications of expressions are identifications of the whole functors.

A functor $f \colon C \to D$ is said to be an *equivalence* if there exists another functor $g \colon D \to C$ and identifications $\mathrm{id}_C = gf$ and $\mathrm{id}_D = fg$.

There are various ways to build new categories out of old ones. In each case, we will specify the behavior of these constructions by prescribing how to define functors into/out of them and how to define identifications between functors into/out of them. We record only the coherence data and relations used explicitly in the development.

Axiom B. We demand the existence of the following categories:

(1) A terminal anima $*$: for every C there is a functor $p_C \colon C \to *$, and for $f, g \colon C \to *$ the anima $(f = g)$ is contractible.

(2) A product $C \times D$ with two projections. Functors into it are formed by pairing, and for $h, k \colon T \to C \times D$ projection induces an equivalence

$$(h = k) \xrightarrow{\sim} (\mathrm{pr}_C\, h = \mathrm{pr}_C\, k) \times (\mathrm{pr}_D\, h = \mathrm{pr}_D\, k).$$

(3) A mapping anima $\mathrm{Map}(C, D)$ with evaluation $\mathrm{Map}(C, D) \times C \to D$. Functors $X \times C \to D$ with anima parameter X can be curried to $X \to \mathrm{Map}(C, D)$, and uncurrying induces equivalences of animae of identifications.

(4) An initial category $\emptyset$, with $\mathrm{Map}(\emptyset, T)$ contractible for every category T.

(5) A coproduct $C \sqcup D$ with inclusions, such that restriction gives an equivalence $\mathrm{Map}(C \sqcup D, T) \xrightarrow{\sim} \mathrm{Map}(C, T) \times \mathrm{Map}(D, T)$ for every T.

(6) A pullback $C \times_E D$ with projections and a matching identification. Maps to the pullback are constructed by mapping into C and D and exhibiting an identification; animae of identifications of functors into the pullback are the corresponding pullbacks of animae of identifications.

(7) A functor category $\mathrm{Fun}(C, D)$ with evaluation inducing equivalences $\mathrm{Map}(T, \mathrm{Fun}(C, D)) \xrightarrow{\sim} \mathrm{Map}(T \times C, D)$ for every category T. This provides currying and uncurrying for arbitrary categorical parameters.

Products of animae are animae, as are pullbacks of animae over animae. We further demand strictness of the initial category and universality of coproducts, in the sense of Axioms B.4' and B.5'.

We define the collection of objects in a category C to be $C^{\simeq} = \mathrm{Map}(*, C)$. We define an *absolute object* of a category C to be a functor $x \colon * \to C$.

The internal structure of categories

We now turn to morphisms and their composition, developed in Chapter 2. To speak of *morphisms*, we need another axiom:

Axiom C. There is a category $[1]$, called the *walking morphism*, which comes equipped with an initial object 0 and a terminal object 1. These induce an equivalence $\langle 0, 1 \rangle \colon * \sqcup * \xrightarrow{\sim} [1]^{\simeq}$.

We may then define a *morphism* in C to be a functor $f \colon [1] \to C$. We refer to the objects $f(0)$ and $f(1)$ of C as the *source* and *target* of f, respectively, and often write $f \colon x \to y$ to indicate that f comes equipped with identifications $x = f(0)$ and $y = f(1)$. Similarly, a *natural transformation* of functors $C \to D$ is defined as a functor $C \times [1] \to D$, or equivalently a functor $C \to \mathrm{Fun}([1], D)$.

The *walking commutative triangle* is the category $[2] := \mathrm{Fun}([1], [1])$. It comes with objects 0, 1 and 2, and with functors $d_0, d_1, d_2 \colon [1] \to [2]$ and $s_0, s_1 \colon [2] \to [1]$ satisfying the simplicial identities. A *commutative triangle* in C is a functor $[2] \to C$, and a *commutative square* is a functor $[1] \times [1] \to C$. Every commutative square determines two commutative triangles by precomposition with the maps $j_0 := (s_0, s_1) \colon [2] \to [1] \times [1]$ and $j_1 := (s_1, s_0) \colon [2] \to [1] \times [1]$, and we demand that the original square can be uniquely recovered from this data:

Axiom D (Commutative square axiom)**.** For every category C, the functor

$$(j_0^*, j_1^*)\colon \operatorname{Fun}([1]\times[1], C) \to \operatorname{Fun}([2], C) \times_{d_1, \operatorname{Fun}([1], C), d_1} \operatorname{Fun}([2], C)$$

is an equivalence.

Every object $x\colon * \to C$ gives rise to an *identity morphism* $\mathrm{id}_x\colon x \to x$, defined as the composite $[1] \xrightarrow{p_{[1]}} * \xrightarrow{x} C$. The definition of *composition* of morphisms in C requires another axiom:

Axiom E (Segal axiom)**.** For every category C, the restriction functor

$$(d_2^*, d_0^*)\colon \operatorname{Fun}([2], C) \to \operatorname{Fun}([1], C) \times_C \operatorname{Fun}([1], C)$$

is an equivalence of categories.

Given morphisms $f\colon x \to y$ and $g\colon y \to z$, the axiom provides a unique commutative triangle $\sigma\colon [2] \to C$ satisfying $d_2^*(\sigma) = f$ and $d_0^*(\sigma) = g$. The resulting morphism $g \circ f := d_1^*(\sigma)\colon [1] \to C$ is called the *composite of f and g*. One can prove that composition of morphisms is automatically unital and associative.

Having both identity morphisms and composition, we may now introduce the notion of *isomorphism* in C: a morphism $f\colon x \to y$ in C is an isomorphism if it admits a left inverse $g\colon y \to x$ (satisfying $gf = \mathrm{id}_x$) as well as a right inverse $h\colon y \to x$ (satisfying $fh = \mathrm{id}_y$). Using diagram categories, we may in fact construct for every C a category $\operatorname{Iso}(C)$ whose objects are precisely the isomorphisms in C (equipped with the data of left and right inverses). The canonical example of an isomorphism in C is the identity morphism $\mathrm{id}_x\colon x \to x$, in which case the left and right inverses are simply id_x itself. This construction provides a functor $i\colon C \to \operatorname{Iso}(C)$, which we demand to be an equivalence:

Axiom F (Rezk axiom)**.** Every isomorphism in a category C is isomorphic to an identity morphism. More precisely, the functor $i\colon C \to \operatorname{Iso}(C)$ is an equivalence.

Axioms E and F are inspired by Rezk's notion of *complete Segal spaces*.

We now introduce the following recognotion principle for animae.

Axiom G (Groupoid recognition axiom)**.** A category is an anima if and only if it is a groupoid.

Constructions of categories

Equipped with a basic language of categories, functors and identifications, we can start formulating our remaining axioms of category theory. We first recall the *groupoid cores* of categories. Using the above notion of isomorphisms in a category C, we say that C is a *groupoid* if all of its morphisms are isomorphisms.

For every category C, there is an anima $C^{\simeq}$ called the *groupoid core of C*, equipped with a functor $\iota_C\colon C^{\simeq} \to C$ simply defined as

$$C^{\simeq} = \operatorname{Map}(*, C)\,.$$

Every functor $f\colon X \to C$ from an anima X factors through ι_C, and for functors $g, h\colon X \to C^{\simeq}$ every identification $\iota_C \circ g = \iota_C \circ h$ may be lifted to an identification $g = h$.

For categories C and D, the primitive mapping anima $\mathrm{Map}(C, D)$ is related to the functor category by a preferred equivalence

$$\mathrm{Map}(C, D) \simeq \mathrm{Fun}(C, D)^{\simeq}.$$

We also demand the existence of subcategories, localizations, geometric realizations and joins, discussed in detail in Chapter 3:

Axiom H (Subcategory axiom)**.** Consider a category C equipped with a *collection of morphisms* $m\colon M \hookrightarrow \mathrm{Map}([1], C)$ of C which is *closed under composition* (see Definition 3.1.6 for a precise definition of these terms). Then there exists a category $\langle M\rangle_C$ equipped with a functor $i_M\colon \langle M\rangle_C \to C$ satisfying:

- The induced functor $(i_M)_*\colon \mathrm{Map}([1], \langle M\rangle_C) \to \mathrm{Map}([1], C)$ factors through M;
- The functor i_M is *universal* with respect to the previous property: every other functor $f\colon D \to C$ whose induced functor $f_*\colon \mathrm{Map}([1], D) \to \mathrm{Map}([1], C)$ factors through M admits a unique factorization through $\langle M\rangle_C$.

Axiom I (Localization axiom)**.** For every category C and every collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$, there exists a functor $l\colon C \to C[W^{-1}]$ exhibiting $C[W^{-1}]$ as a localization of C at the morphisms in W:

- The functor l sends the morphisms in W to isomorphisms in $C[W^{-1}]$;
- The functor l is *universal* with respect to the previous property: every other functor $f\colon C \to D$ which sends morphisms in W to isomorphisms in D admits a unique factorization through $C[W^{-1}]$.

In the case where W consists of *all* morphisms in C, we denote the localization by $\|C\|$ and call it the *geometric realization of* C. It is automatically an anima.

Axiom J (Join axioms)**.** For every category Γ and functors $C \to \Gamma$ and $D \to \Gamma$, there is a relative join $C \star_\Gamma D \to \Gamma$. If

$$j_\Gamma := \mathrm{id}_\Gamma \times j\colon \Gamma \times \partial[1] \longrightarrow \Gamma \times [1]$$

is the weakening of the boundary inclusion, then the height functor $h_{C,D}\colon C \star_\Gamma D \to \Gamma \times [1]$ is a dependent product of $C \sqcup D$ over $\Gamma \times \partial[1]$ along j_Γ.

When Γ is an anima, the relative join also satisfies the mapping-out exactness property: it is exhibited by a pushout square

$$\begin{array}{ccc} (C \times_\Gamma D) \sqcup (C \times_\Gamma D) & \longrightarrow & \times(C \times_\Gamma D) \\ {\scriptstyle \mathrm{pr}_C \sqcup \mathrm{pr}_D}\downarrow & & \downarrow \\ C \sqcup D & \longrightarrow & C \star_\Gamma D. \end{array}$$

The absolute join $C \star D$ is the special case $\Gamma = *$.

If we write $[0]$ for the terminal category $*$, we have $[1] \xrightarrow{\sim} [0] \star [0]$, and we may inductively define $[n+1] := [n] \star [0]$.

Cartesian and cocartesian fibrations

An important concept in category theory is that of a cocartesian fibration: a functor $E \to C$ whose fibers E_c vary covariantly and functorially in the object $c \in C$. In Chapter 4, we introduce a synthetic analogue of this definition: a functor $f\colon A \to B$ is called a cocartesian fibration if a certain 'directed evaluation map' $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) \to A \mathbin{\vec{\times}_f} B$ admits a left adjoint section $\mathrm{lift}_0^f\colon A \mathbin{\vec{\times}_f} B \to \mathrm{Fun}([1], A)$. Informally, this left adjoint section provides the cocartesian lifts $a \to \beta_!(a)$ in A for morphisms $\beta\colon b \to b'$ in B with specified lift a of b. One may dually define *cartesian fibrations*.

Axiom L (Functoriality of universals axiom)**.** Let $p\colon E \to C$ be a cocartesian (resp. cartesian) fibration. If the fiber E_x of p admits a terminal (resp. initial) object for every object x of C, then there is a right (resp. left) adjoint section $s\colon C \to E$ of p whose value at x is isomorphic to the chosen terminal (resp. initial) object of E_x.

Axiom M (Exponentiability axiom)**.** Cartesian and cocartesian fibrations are *exponentiable*, meaning that pullback along a (co)cartesian fibration admits a right adjoint, and these right adjoints satisfy a Beck-Chevalley condition.

Given a cocartesian fibration $p\colon U_\bullet \to U$, one can formulate what it means for p to satisfy *directed univalence*: roughly speaking, this means that for functors $f_0, f_1\colon C \to U$ with pullback fibrations $E_i := f_i^*(U_\bullet) \to C$, the anima of natural transformations $\alpha\colon f_0 \to f_1$ is equivalent to the anima of cocartesian functors $E_0 \to E_1$ over C. This condition will in particular imply a form of *straightening/unstraightening*. A category U equipped with such a univalent cocartesian fibration p is called a *universe*. We say that U is a *universe of categories* if it is closed under all the categorical operations introduced so far; see Definition 8.4.5 for a complete list.

Axiom N (Directed univalence axiom)**.** For every cocartesian fibration $q\colon E \to B$, there exists a universe of categories U such that B is U-small and q has U-small fibers.

We will usually fix a universe of categories $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ and work with categories that are small with respect to this universe, i.e. those categories C that can be obtained from an object c of Cat via the following pullback square:

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{c} & \mathrm{Cat}. \end{array}$$

A consequence of directed univalence is the straightening/unstraightening equivalence

$$\mathrm{Un}\colon \mathrm{Fun}(C, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{CoCart}(C)$$

for every small category C. Here $\mathrm{CoCart}(C)$ is the subcategory of the slice category $\mathrm{Cat}_{/C}$ whose objects are small cocartesian fibrations over C and whose morphisms are cocartesian functors. This correspondence motivates the intended interpretation of synthetic category theory in the universe Cat; its systematic verification belongs to Volume II.

1 The language of synthetic category theory

At the heart of mathematics lies the concept of a set, a collection of mathematical objects. In modern mathematics, sets serve as the building blocks for defining everything from numbers and relations to more complex structures like groups, rings, and topological spaces. Sets are manipulated using basic operations like forming unions, intersections, and complements, and are compared using functions between sets. It is fair to say that the language of set theory forms an essential toolkit for mathematicians.

Although formal axiomatizations of set theory exist, such as those by Zermelo and Fraenkel, they often diverge from the working mathematician's intuitive understanding of sets. For instance, in formal set theory, every natural number needs to be defined as a set, and in particular one is allowed to speak of the elements of a natural number. Similarly, every function between sets needs to be defined as a set. While this approach is formally valid, it can seem disconnected from everyday mathematical discourse.

In practice, mathematicians often use a more pragmatic approach known as *naive set theory*. In naive set theory, one works with sets based on intuitive understanding, without invoking formal logical or axiomatic implementations. For example, the set of natural numbers is simply the set containing the numbers $0, 1, 2$, etcetera, and functions between sets X and Y are simply 'specifications' that assign to every element of X an element of Y. The simplicity and accessibility of naive set theory make it relatively easy for students to learn, and indeed in many introductory mathematics textbooks sets are used in the naive way without being rigorously introduced.

A similar trend is occurring in the field of higher category theory. In recent decades, higher category theory has seen rapid growth, in large part thanks to contributions from people like Joyal, Lurie, and others. Their work provides solid foundations for the field, and the language of higher categories is now extending beyond its origins in algebraic topology to fields like algebraic geometry and representation theory. But while the original works of Joyal and Lurie are based on explicit set-theoretical models for higher categories called 'quasicategories', more and more research papers are written in a 'model-independent' way in which the usage of details about the specific choice of implementations is avoided. We refer to this as the language of *naive category theory*.

The goal of this chapter is to provide a structured form of this language, with explicit rules and axioms for how to manipulate categories and functors. Our approach is *synthetic* in nature rather than *analytic*: constructions of categories, like products or functor categories, are not *defined* in terms of set-level constructions, but are *axiomatized* to exist and satisfy the expected properties. Our rules are deliberately model-agnostic: we expect them to be satisfied in any reasonable foundational

system for (higher) category theory. For example, we wish to cover the following three examples:

(1) The standard theory of quasicategories;

(2) Type-theoretic approaches like simplicial type theory [RS17; GWB24];

(3) The internal theory of the (synthetic) category of small categories Cat.

While the first two examples satisfy various strictness conditions, like strict unitality or associativity for functor composition, the third example forces us to work in a fully 'homotopical' setting, where no strictness conditions are imposed.

The chapter is organized as follows. The foundational language is introduced in Section 1.1, and its layer of coherences in Section 1.2. These are used in Section 1.3 to study equivalences of categories. We then introduce a variety of category constructors via their universal properties.

Epistemic status: This chapter started off giving a natural language description of 'naive category theory', capturing the model-independent language used in modern papers in higher category theory. Over time, more explicit coherences have been added to the list of axioms when it became clear that some natural language proofs were implicitly using them. We expect the present list to cover most of the required coherences, but it may be the case that we missed some. We hope to eventually settle the complete list of coherences by formalizing everything in a proof assistant.

1.1 The basic vocabulary

We start by introducing the basic vocabulary of synthetic category theory.

Axiom A.1 (Primitives)**.** The following are the primitives of synthetic category theory:

(1) There are *categories*, which we denote by symbols like C, D, and E. Among categories, there are *animae*.

(2) For categories C and D, there is a primitive notion of a *functor* $f : C \to D$. Every category has an *identity functor* $\mathrm{id}_C : C \to C$. Given $f : C \to D$ and $g : D \to E$, there is a *composite functor* $g \circ f : C \to E$, sometimes also denoted gf.

(3) There exists an anima $*$, called the *terminal category*.

Categories and functors play the foundational role in synthetic category theory that sets and functions play in naive set theory. The role of the animae among the categories is to encode the idea of a *homotopy-coherent collection*: where objects in classical mathematics assemble into sets, in synthetic category theory they assemble into animae. A category additionally allows objects to be related by morphisms that need not be invertible.

For now, animae form a primitive distinguished class of categories. We regard every category equivalent to an anima as an anima. Once we have introduced morphisms and their invertibility, we

can define groupoids as those categories in which every morphism is invertible. In the next chapter, we will impose the axiom that the animae are precisely the groupoids.

Definition 1.1.1 (Terms)**.** A functor $f\colon C \to D$ may be displayed in *term notation* as

$$x : C \vdash f(x) : D.$$

The functor and the expression $f(x)$ are two notations for the same datum. We call $x : C$ its *context*. More generally, we may write $C \vdash t : D$ when the name of the variable is left implicit.

Throughout the chapter, we will indicate how the various rules we introduce may be written in their corresponding term notation. Conversely, any term expression obtained by combining these rules corresponds to a well-defined functor. The notation is inspired by type-theoretic approaches to synthetic category theory, such as [RS17; GWB24].

For example, composition of functors is displayed by substitution: for a second functor $g\colon D \to E$, the term notation for the composite $g \circ f$ is the expression $x : C \vdash g(f(x)) : E$. The term notation for the identity functor on C is $x : C \vdash x : C$. We also refer to the latter as the *universal term* of C.

Definition 1.1.2 (Objects)**.** Given a term $x : C$ in context Γ, we say that x is an *object (in context* Γ) if Γ is an anima. If we speak about an *object of* C, we mean an object in an arbitrary anima context.

An *absolute object* is a functor $x\colon * \to C$, i.e. an object in context $*$.

As already mentioned in the introduction of this book, one major difference between classical mathematics and homotopy theory is the idea that *equality is structure*. In classical mathematics, equality is a yes-no-question: two elements of a set are either equal or not. In homotopy theory, we change our perspective and allow there to be potentially several different identifications. The collection of identifications thus needs to assemble into an anima.

We thus introduce a primitive anima of identifications between every pair of functors. All our definitions and constructions should always respect identifications; we choose to implement this via explicit rules rather than via some deeper foundational principle as is for example done in homotopy type theory.

Axiom A.2 (Identifications)**.** Given parallel functors $f, g\colon C \to D$, there is an anima $(f = g)$, called the *anima of identifications of* f *and* g. An *identification* $\alpha : f = g$ is an absolute object of this anima.

For $h\colon D \to E$ and $k\colon B \to C$, there are functors

$$h \star -\colon (f = g) \to (hf = hg), \qquad - \star k\colon (f = g) \to (fk = gk),$$

called *postwhiskering* and *prewhiskering*, respectively.

Notation 1.1.3. We also denote identifications $\alpha : f = g$ in term notation as[1]

$$x : C \vdash \alpha(x) : f(x) = g(x).$$

[1]There is a slight clash between this notation and that of Definition 1.1.1: this does *not* mean a functor $C \to f(x) = g(x)$, which does not even make sense.

Given an anima A and two absolute objects $a, b : A$, we obtain an anima $a = b$ of identifications between them. In particular, given two identifications $\alpha, \beta : f = g$, we may form the anima $\alpha = \beta$. This construction may be iterated.

Definition 1.1.4. A *commutative square of functors*, often displayed by means of a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D', \end{array}$$

consists of four functors f, g, h and f' with sources and targets as displayed in the diagram, together with a specified identification $\alpha\colon h \circ f = f' \circ g$.

We will often refer to the data of just the four functors f, g, h and f' as a *square*, and say that *the square commutes* whenever an identification $\alpha\colon h \circ f = f' \circ g$ can be provided.

A *commutative triangle of functors*, displayed by means of a diagram

$$\begin{array}{ccc} & D & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ C & \xrightarrow[h]{} & E, \end{array}$$

consists of three functors f, g and h with sources and targets as displayed in the diagram, together with a specified identification $\alpha\colon h = g \circ f$.

We will similarly display more general diagrams of functors by drawing directed planar graphs with vertices labeled by categories and edges labeled by functors. We consider diagrams whose faces are bounded by two directed paths with the same source and target, and fix a parenthesization for each composite. We say the diagram *commutes* if every face comes equipped with an identification between these two composites.

Definition 1.1.5 (Equivalence of categories). A functor $f\colon C \to D$ is said to be an *equivalence* if there exists another functor $g\colon D \to C$ equipped with identifications $\mathrm{id}_C = gf$ and $\mathrm{id}_D = fg$. We refer to g as the *inverse* of f. Equivalences will often be introduced by writing $f\colon C \xrightarrow{\sim} D$. We may write $C \simeq D$ to indicate that C and D are equivalent via an unnamed equivalence.

The next two axioms belong to the universal constructions of group B. We introduce them here because they are needed to formulate the coherence axioms of group A. We begin with the universal property of the terminal category.

Axiom B.1 (Terminal category)**.** For every category T, there is a functor $p_T\colon T \to *$. For any functors $f, g\colon T \to *$, the functor

$$p_{(f=g)}\colon (f = g) \longrightarrow *$$

is an equivalence.

In particular, any two functors to $*$ are identified.

Definition 1.1.6 (Contractible category). A category C is called *contractible* if the functor $p_C : C \to *$ is an equivalence.

Definition 1.1.7 (Constant functor). Let C and D be categories and let x be an absolute object of D. We define the *constant functor* $\mathrm{const}_x : C \to D$ as the composite $x \circ p_C : C \to D$:

$$\mathrm{const}_x : C \xrightarrow{p_C} * \xrightarrow{x} D.$$

Remark 1.1.8. Observe that a category C is contractible if and only if there exists an absolute object $x : * \to C$ equipped with an identification $\mathrm{id}_C = \mathrm{const}_x$. Such an identification $H : \mathrm{id}_C = \mathrm{const}_x$ is called a *contraction of C onto x*.

We will postpone most category constructors until later in the chapter, but we need one crucial one right now already in order to state various coherence relations.

Axiom B.2 (Product of categories)**.** The *product* of categories C and D is a category $C \times D$ equipped with *projection functors* $\mathrm{pr}_1 : C \times D \to C$ and $\mathrm{pr}_2 : C \times D \to D$.

Given $f : T \to C$ and $g : T \to D$, there is a functor $(f, g) : T \to C \times D$ equipped with identifications $\beta^1_{f,g} : \mathrm{pr}_1(f, g) = f$ and $\beta^2_{f,g} : \mathrm{pr}_2(f, g) = g$. For any two functors $h, k : T \to C \times D$, the functor

$$(\mathrm{pr}_1 \star -, \mathrm{pr}_2 \star -) : (h = k) \longrightarrow (\mathrm{pr}_1\, h = \mathrm{pr}_1\, k) \times (\mathrm{pr}_2\, h = \mathrm{pr}_2\, k)$$

is an equivalence. If C and D are animae, then $C \times D$ is an anima.

We refer to the functor (f, g) as the *pair construction*. In term notation, it is written

$$x : T \vdash (f(x), g(x)) : C \times D,$$

with identifications $\mathrm{pr}_1(f(x), g(x)) = f(x)$ and $\mathrm{pr}_2(f(x), g(x)) = g(x)$. For arbitrary expressions $h(x), k(x) : C \times D$, two identifications $\alpha(x) : \mathrm{pr}_1\, h(x) = \mathrm{pr}_1\, k(x)$ and $\beta(x) : \mathrm{pr}_2\, h(x) = \mathrm{pr}_2\, k(x)$ give an identification $(\alpha(x), \beta(x)) : h(x) = k(x)$ by applying an inverse of the displayed equivalence. Its two projections are identified with α and β.

Remark 1.1.9. Given categories C and D, we define a *span* from C to D to be a triple (T, f, g) consisting of a category T together with functors $f : T \to C$ and $g : T \to D$. Axiom B.2 may be rephrased as saying that the triple $(C \times D, \mathrm{pr}_1, \mathrm{pr}_2)$ is the *terminal span* from C to D: every span $C \leftarrow T \to D$ admits a factorization through $C \times D$, and an identification of two functors $T \to C \times D$ amounts to identifications between their respective components.

Convention 1.1.10. A term of the form $z : C \times D \vdash t : E$ may also be written as

$$x : C, y : D \vdash t(x, y) : E$$

whenever the term t only depends on the variable z through the two projections $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$. The expression $t(x, y)$ is then obtained from t by replacing $\mathrm{pr}_1(z)$ by x and $\mathrm{pr}_2(z)$ by y. For more than two variables we use a fixed parenthesization of their product.

1.2 Coherences

So far, we have introduced functors and their identifications, but we have essentially no relations between them. We next specify the coherence relations we will use. We first describe composition of identifications and its compatibility with whiskering. We then introduce the unit and associativity identifications for composition of functors, and the relations among them.

For readability, we write most of these coherences in their 'term notation', cf. Notation 1.1.3. In each case, the axiom introduces a specified identification of functors, which we will often leave nameless to avoid cluttering of notation.

1.2.1 Composition of identifications

Identifications can be composed: given identifications $\sigma : f = g$ and $\tau : g = h$, we obtain an identification $\tau \circ \sigma : f = h$. We call this *vertical composition* to distinguish it from the horizontal composition introduced below. Every functor has an identity identification, and every identification has an inverse.

Axiom A.3 (Composition of identifications)**.** For every functor f, there is an identity identification $\mathrm{id}_f \colon f = f$. For parallel functors $f, g, h \colon C \to D$, there is a composition functor

$$- \circ - \colon (g = h) \times (f = g) \longrightarrow (f = h).$$

There is also an *inverse functor*

$$(-)^{-1} \colon (f = g) \to (g = f).$$

This composition is unital and associative, and the inverse functor supplies two-sided inverses, through specified identifications of functors on the corresponding animae of identifications. For $\alpha : f = g$, $\beta : g = h$, and $\gamma : h = k$, where $k \colon C \to D$, these laws read

$$\mathrm{id}_g \circ \alpha = \alpha, \qquad \alpha \circ \mathrm{id}_f = \alpha,$$

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha),$$

$$\alpha^{-1} \circ \alpha = \mathrm{id}_f, \qquad \alpha \circ \alpha^{-1} = \mathrm{id}_g\,.$$

The displayed laws are specified identifications of functors on the relevant animae of inputs. For parallel functors $f, g, h, k \colon C \to D$, the associativity law compares the two functors

$$(h = k) \times \big((g = h) \times (f = g)\big) \longrightarrow (f = k)$$

given by $(\gamma, (\beta, \alpha)) \mapsto (\gamma \circ \beta) \circ \alpha$ and $(\gamma, (\beta, \alpha)) \mapsto \gamma \circ (\beta \circ \alpha)$. Precomposing these comparisons with a functor into the input anima gives the same laws for the resulting expressions.

We can now also speak of commutative squares, triangles, and more general diagrams of identifications, using the convention of Definition 1.1.4 with vertical composition along their boundaries.

1.2.2 Whiskering and horizontal composition

Whiskering composes an identification with a functor on either side. We first require that it respect identity identifications and vertical composition.

Axiom A.4 (Whiskering and vertical composition)**.** Prewhiskering and postwhiskering preserve identity identifications and vertical composition through specified identifications of functors on the input animae of identifications. For functors $f, g, h\colon C \to D$, $k\colon B \to C$ and $u\colon D \to E$, these laws read

$$u \star \mathrm{id}_f = \mathrm{id}_{u\circ f}, \qquad \mathrm{id}_f \star k = \mathrm{id}_{f\circ k},$$

$$u \star (\tau \circ \sigma) = (u \star \tau) \circ (u \star \sigma), \qquad (\tau \circ \sigma) \star k = (\tau \star k) \circ (\sigma \star k),$$

where $\sigma\colon f = g$ and $\tau\colon g = h$. The two vertical-composition comparisons are identifications of functors on $(g = h) \times (f = g)$.

Given identifications $\tau : F = G$ and $\sigma : h = k$, where $F, G\colon D \to E$ and $h, k\colon T \to D$, there are two ways to obtain an identification $Fh = Gk$: one can first change h to k, or first change F to G. The interchange rule compares these two constructions.

Axiom A.5 (Interchange)**.** For $F, G\colon D \to E$ and $h, k : T \to D$, the two functors

$$(F = G) \times (h = k) \longrightarrow (Fh = Gk)$$

given by

$$(\tau, \sigma) \longmapsto (\tau \star k) \circ (F \star \sigma) \quad \text{and} \quad (\tau, \sigma) \longmapsto (G \star \sigma) \circ (\tau \star h)$$

are identified.

Given $\alpha\colon f = f'$ and $\beta\colon g = g'$, we write

$$\beta * \alpha := (\beta \star f') \circ (g \star \alpha) : gf = g'f'$$

for their *horizontal composite*. Interchange compares this with $(g' \star \alpha) \circ (\beta \star f)$. In particular, horizontal composition preserves identities and vertical composites. For fixed functors f, f', g, g', this construction defines a functor

$$- * -\colon (g = g') \times (f = f') \to (gf = g'f').$$

1.2.3 Units and associativity of functor composition

Composition of functors is unital and associative through specified identifications, called the *unitors* and *associator*.

Axiom A.6 (Unitors and associator)**.** For functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$, there are identifications

$$\lambda_f\colon \mathrm{id}_D \circ f = f, \qquad \rho_f\colon f \circ \mathrm{id}_C = f, \qquad \alpha_{f,g,h}\colon (h \circ g) \circ f = h \circ (g \circ f).$$

The unitors and associator are compatible with identifications of the functors to which they are applied, in the following sense.

Axiom A.7 (Compatibility with horizontal composition)**.** Horizontal composition of identifications is compatible with the unitality and associativity identifications from Axiom A.6:

- For an identification $\alpha\colon f = f'$ of functors $C \to D$ there are commutative squares of identifications

$$\begin{array}{ccc} \mathrm{id}_D \circ f & \xrightarrow[=]{\mathrm{id}_{\mathrm{id}_D} * \alpha} & \mathrm{id}_D \circ f' \\ {\scriptstyle \lambda_f}\downarrow & & \downarrow{\scriptstyle \lambda_{f'}} \\ f & \xrightarrow[=]{\alpha} & f' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} f \circ \mathrm{id}_C & \xrightarrow[=]{\alpha * \mathrm{id}_{\mathrm{id}_C}} & f' \circ \mathrm{id}_C \\ {\scriptstyle \rho_f}\downarrow & & \downarrow{\scriptstyle \rho_{f'}} \\ f & \xrightarrow[=]{\alpha} & f'. \end{array}$$

- For identifications $\alpha\colon f = f'$, $\beta\colon g = g'$ and $\gamma\colon h = h'$, the following square of identifications commutes:

$$\begin{array}{ccc} (h \circ g) \circ f & \xrightarrow[=]{(\gamma * \beta) * \alpha} & (h' \circ g') \circ f' \\ {\scriptstyle \alpha_{f,g,h}}\downarrow & & \downarrow{\scriptstyle \alpha_{f',g',h'}} \\ h \circ (g \circ f) & \xrightarrow[=]{\gamma * (\beta * \alpha)} & h' \circ (g' \circ f'). \end{array}$$

These comparisons are identifications of the corresponding functors on products of the input animae of identifications. Their restrictions along functors into those animae give the same formulas for identifications.

Remark 1.2.1. Taking some of the input identifications to be identities gives compatibility with identifications in each individual functor.

It will sometimes be convenient to abbreviate the conditions from Axiom A.7 as follows:

$$\mathrm{id} * \alpha = \alpha = \alpha * \mathrm{id} \qquad \text{and} \qquad (\gamma * \beta) * \alpha = \gamma * (\beta * \alpha).$$

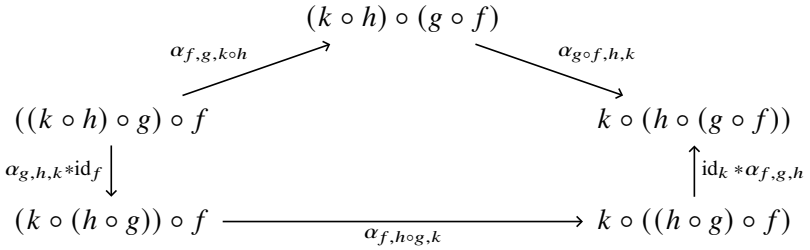
More generally, if we are given identifications $\alpha\colon f = g$ and $\alpha'\colon f' = g'$ of functors $C \to D$, and we happen to be given identifications $f = f'$ and $g = g'$, we will write $\alpha = \alpha'$ to indicate the existence of a commutative square of identifications as follows:

$$\begin{array}{ccc} f & \xrightarrow[=]{\alpha} & g \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ f' & \xrightarrow[=]{\alpha'} & g'. \end{array}$$

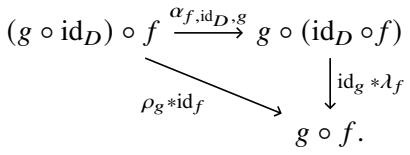
1.2.4 The pentagon and triangle identities

Finally, the unitors and associator must themselves be compatible. The pentagon compares the two ways of reassociating a composite of four functors. The triangle compares removing an identity functor before or after reassociation.

Axiom A.8 (Pentagon and triangle identities)**.** For functors $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ and $k\colon D \to E$, the following pentagon of identifications commutes:



Moreover, for composable functors $f\colon C \to D$ and $g\colon D \to E$, the following triangle of identifications commutes:



1.3 Equivalences of categories

We now develop some basic properties of the equivalences introduced in Definition 1.1.5.

Proposition 1.3.1. *Let $f\colon C \to D$ be a functor. For f to be an equivalence, it suffices that the following properties hold for every category T:*

(1) For $a, b\colon T \to C$, any identification $fa = fb$ gives an identification $a = b$.

(2) For any $d\colon T \to D$, there is a functor $c\colon T \to C$ equipped with an identification $fc = d$.

Proof. Apply the second property to id_D to obtain $g\colon D \to C$ and $fg = \mathrm{id}_D$. The associator, this identification, and the unitors identify $f(gf)$ with $f\,\mathrm{id}_C$. The first property, with $T = C$, gives $gf = \mathrm{id}_C$. □

Definition 1.3.2 (Section/retraction). Consider a functor $f\colon C \to D$. A *section* of f is a functor $s\colon D \to C$ equipped with an identification $fs = \mathrm{id}_D$. A *retraction* of f is a functor $r\colon D \to C$ equipped with an identification $\mathrm{id}_C = rf$.

Lemma 1.3.3. *A functor $f\colon C \to D$ is an equivalence if and only if it admits both a section s and a retraction r. In this case there exists an identification $s = r$ and s is an inverse of f.*

Proof. If f is an equivalence with inverse $g\colon D \to C$, then clearly g is both a section and a retraction of f. Conversely, given a section $s\colon D \to C$, $fs = \mathrm{id}_D$ and a retraction $r\colon D \to C$, $\mathrm{id}_C = rf$, we obtain an identification $s = r$ by composing the following chain of identifications:

$$s = \mathrm{id}_C \circ s = (r \circ f) \circ s = r \circ (f \circ s) = r \circ \mathrm{id}_D = r.$$

It follows that f is an equivalence with inverse s, since there are identifications $fs = \mathrm{id}_D$ and $sf = rf = \mathrm{id}_C$. □

Corollary 1.3.4. *Any two inverses g and h of an equivalence f are identified.* □

Lemma 1.3.5. *If $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.*

Proof. We have $gg'f'f = g\,\mathrm{id}_D\, f = gf = \mathrm{id}_C$ and $f'fgg' = f'\,\mathrm{id}_D\, g' = f'g' = \mathrm{id}_E$. □

Lemma 1.3.6. *Equivalences of categories satisfy the 2-out-of-3 property: given functors $f\colon C \to D$ and $g\colon D \to E$, if two of the three functors f, g and gf are equivalences, then so is the third.*

Proof. If g and f are equivalences, then so is gf by Lemma 1.3.5. If f and gf are equivalences, then we may rewrite g as the composite $g = g \circ \mathrm{id}_D = g \circ f \circ f^{-1} = gf \circ f^{-1}$, so that g is the composite of two equivalences and hence itself an equivalence. A similar argument shows that f is an equivalence whenever g and gf are equivalences. □

Lemma 1.3.7. *Equivalences of categories satisfy the 2-out-of-6 property: given functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.*

Proof. Consider the functors $f' := f \circ (gf)^{-1}\colon E \to D$ and $h' := (hg)^{-1} \circ h\colon E \to D$. Then there are identifications

$$g \circ f' = g \circ f \circ (gf)^{-1} = gf \circ (gf)^{-1} = \mathrm{id}_E$$

and

$$h' \circ g = (hg)^{-1} \circ h \circ g = (hg)^{-1} \circ hg = \mathrm{id}_D\,.$$

Hence it follows from Lemma 1.3.3 that g is an equivalence. In light of Lemma 1.3.6, we then get that also f and h are equivalences, and then so is hgf. □

Definition 1.3.8. We say that a functor $f\colon C \to D$ is a *retract* of a functor $f'\colon C' \to D'$ if there exists a commutative diagram of the form

$$\begin{array}{ccccc} C & \xrightarrow{g} & C' & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{k} & D' & \xrightarrow{j} & D, \end{array}$$

together with identifications $\alpha\colon \mathrm{id}_C = h \circ g$ and $\beta\colon j \circ k = \mathrm{id}_D$, and an identification between $\mathrm{id}_f\colon f = f$ and the composite identification

$$f = f \circ \mathrm{id}_C \overset{\alpha}{=} f \circ h \circ g = j \circ f' \circ g = j \circ k \circ f \overset{\beta}{=} \mathrm{id}_D \circ f = f.$$

Lemma 1.3.9. *Assume that $f\colon C \to D$ is a retract of $f'\colon C' \to D'$. If f' is an equivalence, then so is f.*

Proof. Let $(f')^{-1}\colon D' \to C'$ be an inverse of f'. We claim that the composite $w := h \circ (f')^{-1} \circ k\colon D \to C$ is inverse to f. Indeed, we have identifications

$$f \circ w = f \circ h \circ (f')^{-1} \circ k = j \circ f' \circ (f')^{-1} \circ k = j \circ k = \mathrm{id}_D$$

and

$$w \circ f = h \circ (f')^{-1} \circ k \circ f = h \circ (f')^{-1} \circ f' \circ g = h \circ g = \mathrm{id}_C\,.$$

This finishes the proof. □

Lemma 1.3.10. *Prewhiskering or postwhiskering by an equivalence induces equivalences of animae of identifications.*

Proof. Composition with a fixed identification is an equivalence, with inverse given by composition with its inverse. This follows from the vertical unit, associativity, and inverse laws.

Let $F\colon C \to D$ have inverse G, and consider two functors $a, b\colon T \to C$. We wish to show that the first functor $F \star -$ in the following diagram is an equivalence:

$$(a = b) \xrightarrow{F\star -} (Fa = Fb) \xrightarrow{G\star -} (GFa = GFb) \xrightarrow{F\star -} (FGFa = FGFb).$$

Interchange with $GF = \mathrm{id}_C$ identifies the composite of the first two functors with transport along the identifications $GFa = a$ and $b = GFb$, and so that composite is an equivalence. The composite of the last two is an equivalence for the same reason, using $FG = \mathrm{id}_D$, and so the claim follows from the 2-out-of-6 property. Prewhiskering is treated in the same way. □

Consequently an equivalence lifts a specified identification, together with an identification of its image with that identification. Applying the same result to functors into an anima of identifications also lifts identifications between identifications.

From the universal property of products, we may directly deduce that products are commutative, unital, and associative. For commutativity of products, one argues as follows:

Lemma 1.3.11. *For categories C and D, the functor $(\mathrm{pr}_2, \mathrm{pr}_1)\colon C \times D \to D \times C$ is an equivalence, with inverse given by $(\mathrm{pr}_2, \mathrm{pr}_1)\colon D \times C \to C \times D$.*

Proof. We need to provide identifications $(\mathrm{pr}_2, \mathrm{pr}_1) \circ (\mathrm{pr}_2, \mathrm{pr}_1) = \mathrm{id}_{C\times D}$ and $(\mathrm{pr}_2, \mathrm{pr}_1) \circ (\mathrm{pr}_2, \mathrm{pr}_1) = \mathrm{id}_{D\times C}$. By symmetry, it will suffice to produce the first identification. For this, it will in turn suffice to produce identifications after composing with the functors pr_1 and pr_2. For the composition with pr_1 we compute that

$$\mathrm{pr}_1 \circ((\mathrm{pr}_2, \mathrm{pr}_1) \circ (\mathrm{pr}_2, \mathrm{pr}_1)) = (\mathrm{pr}_1 \circ(\mathrm{pr}_2, \mathrm{pr}_1)) \circ (\mathrm{pr}_2, \mathrm{pr}_1) = \mathrm{pr}_2 \circ(\mathrm{pr}_2, \mathrm{pr}_1) = \mathrm{pr}_1,$$

which is indeed identified with $\mathrm{pr}_1 \circ \mathrm{id}_{C\times D}$. The case for composition with pr_2 is analogous. This finishes the proof. □

Unitality may be argued similarly:

Lemma 1.3.12. *For every category C, the projection functor $\mathrm{pr}_1 \colon C \times * \to C$ is an equivalence, with inverse given by the functor $(\mathrm{id}_C, p_C) \colon C \to C \times *$.*

Proof. The composite $\mathrm{pr}_1 \circ (\mathrm{id}_C, p_C) \colon C \to C$ is identified with id_C by the defining property of (id_C, p_C). To show that the composite $(\mathrm{id}_C, p_C) \circ \mathrm{pr}_1 \colon C \times * \to C \times *$ is identified with $\mathrm{id}_{C\times *}$, it will suffice to do so after composing with pr_1 and with pr_2. The case for pr_2 is clear, since any two functors $C \times * \to *$ are identified. In the case of pr_1 we compute

$$\mathrm{pr}_1 \circ ((\mathrm{id}_C, p_C) \circ \mathrm{pr}_1) = (\mathrm{pr}_1 \circ (\mathrm{id}_C, p_C)) \circ \mathrm{pr}_1 = \mathrm{id}_C \circ \mathrm{pr}_1 = \mathrm{pr}_1 = \mathrm{pr}_1 \circ \mathrm{id}_{C\times *},$$

where the first identification is associativity, the second is the relation proved in the first paragraph, and the third and fourth are unitality. This finishes the proof. □

Remark 1.3.13. In the previous proofs, we have been very explicit about the usage of associativity and naturality identifications. In the remainder of this book, we will often leave the usage of such identifications implicit in order to make the proofs more concise.

Exercise 1.3.14 (Exercise 1.10.1). Formulate and prove the associativity of the product of categories.

Construction 1.3.15 (Pairing of identifications). Given functors $f, f' \colon T \to C$ and $g, g' \colon T \to D$, pairing defines a functor

$$(-,-) \colon (f = f') \times (g = g') \longrightarrow ((f,g) = (f',g')).$$

For $\varphi : f = f'$ and $\psi : g = g'$, its value (φ, ψ) is equipped with identifications

$$\beta^1_{f',g'} \circ \big(\mathrm{pr}_1 \star (\varphi,\psi)\big) = \varphi \circ \beta^1_{f,g},$$
$$\beta^2_{f',g'} \circ \big(\mathrm{pr}_2 \star (\varphi,\psi)\big) = \psi \circ \beta^2_{f,g}.$$

There are identifications

$$(\mathrm{id}_f, \mathrm{id}_g) = \mathrm{id}_{(f,g)}, \qquad (\varphi' \circ \varphi, \psi' \circ \psi) = (\varphi', \psi') \circ (\varphi, \psi),$$

where $\varphi' : f' = f''$ and $\psi' : g' = g''$.

Proof. Under the identifications $\beta^1_{f,g} : \mathrm{pr}_1(f,g) = f$ and $\beta^2_{f,g} : \mathrm{pr}_2(f,g) = g$, φ and ψ correspond to two absolute objects of the two factors of the following product:

$$\big(\mathrm{pr}_1(f,g) = \mathrm{pr}_1(f',g')\big) \times \big(\mathrm{pr}_2(f,g) = \mathrm{pr}_2(f',g')\big).$$

Since this product is equivalent to the anima $(f,g) = (f',g')$ by Axiom B.2, this provides the identification (φ, ψ) with the two claimed identification.

We leave the identity law and composition law as exercises for the reader. □

Lemma 1.3.16 (Pairing and precomposition). *Let $f \colon T \to C$, $g \colon T \to D$, and $\sigma \colon S \to T$ be functors. Pairing commutes with precomposition: there is an identification*

$$(f,g) \circ \sigma = (f \circ \sigma, g \circ \sigma).$$

This comparison is compatible with identifications of the input functors, with successive precomposition, and with precomposition by an identity functor, using the associativity and unitality identifications.

Proof. After composition with pr_1 and pr_2, both sides identify with $f \circ \sigma$ and $g \circ \sigma$, respectively. The product axiom therefore supplies the comparison, with these prescribed projections. Its compatibilities can likewise be checked after applying both projections, where they reduce to the coherence laws for composition and identifications. For compatibility with identity functors, one also uses

$$\rho_{ab} = (a \star \rho_b) \circ \alpha_{\mathrm{id},b,a} \tag{1.1}$$

for composable functors b, a. This follows by prewhiskering with the identity, applying the pentagon for a, b, id, id and the triangle identities, and cancelling by Lemma 1.3.10. We leave the remaining checks to the reader. □

Construction 1.3.17. The formation of products of categories is functorial: given $f\colon C \to C'$ and $g\colon D \to D'$, we obtain a functor

$$f \times g := (f\,\mathrm{pr}_1, g\,\mathrm{pr}_2) : C \times D \to C' \times D'.$$

For identifications $\varphi : f = f'$ and $\psi : g = g'$ between functors with these same sources and targets, we define

$$\varphi \times \psi := (\varphi \star \mathrm{pr}_1, \psi \star \mathrm{pr}_2).$$

This gives a functor on the product of the two animae of identifications. It preserves identity identifications and vertical composition by Construction 1.3.15 and the corresponding laws for whiskering.

Exercise 1.3.18 (Exercise 1.10.3). Show that there are identifications

$$\mathrm{id}_C \times \mathrm{id}_D = \mathrm{id}_{C\times D},$$
$$(f' \times g') \circ (f \times g) = (f'f) \times (g'g)$$

for $f\colon C \to C'$, $f'\colon C' \to C''$, $g\colon D \to D'$, and $g'\colon D' \to D''$.

1.4 Mapping animae

We now introduce the mapping anima between two categories C and D: an anima whose objects are functors from C to D. The mapping animae will play an important role in the development of the theory: frequently, we may check conditions at the level of mapping animae. We introduce $\mathrm{Map}(C, D)$ via its universal property in terms of currying/uncurrying functors.

Axiom B.3 (Evaluation and currying for mapping animae)**.** For categories C, D, there is an anima $\mathrm{Map}(C, D)$ equipped with an *evaluation functor*

$$\mathrm{ev}_{\mathrm{Map}}\colon \mathrm{Map}(C, D) \times C \longrightarrow D.$$

If X is an anima and $h : X \times C \to D$ is a functor, there is a functor $h_c\colon X \to \mathrm{Map}(C, D)$ equipped with an identification $(h_c)^u = h$, where

$$a^u := \mathrm{ev}_{\mathrm{Map}} \circ (a \times \mathrm{id}_C) : X \times C \longrightarrow D.$$

For any $a, b\colon X \to \mathrm{Map}(C, D)$, the functor

$$(-)^u\colon (a = b) \longrightarrow (a^u = b^u)$$

induced by product functoriality and postwhiskering with evaluation is an equivalence.

We refer to the functor h_c as the *currying* of h, and to a^u as the *uncurrying* of a. Note that the universal property of Map only holds for anima parameters. A more general form of currying with an arbitrary category as parameter will be provided by functor categories in Section 1.7.

Construction 1.4.1 (Functors as points of mapping animae)**.** A functor $f\colon C \to D$ determines a point $\ulcorner f \urcorner\colon * \to \mathrm{Map}(C, D)$ by currying $f\,\mathrm{pr}_2\colon * \times C \to D$. Conversely, a point $a : * \to \mathrm{Map}(C, D)$ determines the functor

$$C \simeq * \times C \xrightarrow{a \times \mathrm{id}_C} \mathrm{Map}(C, D) \times C \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} D.$$

The functor represented by $\ulcorner f \urcorner$ is identified with f. Moreover, for two points a, b, uncurrying followed by prewhiskering with $C \simeq * \times C$ gives an equivalence from $(a = b)$ to the anima of identifications between their represented functors, where we use Lemma 1.3.10. This equivalence also identifies a point with the representative of its represented functor.

From now on we pass freely back and forth between a functor and its representing point using the comparisons just described; in particular, we will usually denote the point $\ulcorner f \urcorner$ simply by f.

Accordingly, a functor $H\colon \mathrm{Map}(A, B) \to \mathrm{Map}(C, D)$ sends a functor $f\colon A \to B$ to a functor $H(f)\colon C \to D$, and induces a functor

$$(f = g) \longrightarrow (H(f) = H(g)).$$

These operations respect composition and identities. If H is an equivalence, the induced functor on identification animae is an equivalence as well.

Proposition 1.4.2 (Uncurrying). *For every anima X and categories C, D, uncurrying is represented by an equivalence*

$$(-)^u\colon \mathrm{Map}\big(X, \mathrm{Map}(C, D)\big) \xrightarrow{\sim} \mathrm{Map}(X \times C, D).$$

Proof. Put $P := \mathrm{Map}(X, \mathrm{Map}(C, D))$. Iterated evaluation gives a functor

$$P \times X \times C \longrightarrow \mathrm{Map}(C, D) \times C \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} D.$$

Since P is an anima, currying this functor constructs an actual functor $(-)^u\colon P \to \mathrm{Map}(X \times C, D)$.

For the inverse, put $Q := \mathrm{Map}(X \times C, D)$, and consider the evaluation functor $Q \times X \times C \to D$. If we first curry in the variable C, with anima parameter $Q \times X$, and then curry in the variable X, with anima parameter Q, we obtain a functor $Q \to P$. To verify that both composites are the identity, it suffices by Axiom B.3 to uncurry: once in $X \times C$ for the composite on Q, and successively in X and C for the composite on P. In both cases, the resulting identification follows from the defining identifications for currying. □

Construction 1.4.3. The mapping animae are covariantly functorial in the second variable and contravariantly functorial in the first. For $g\colon D \to E$, curry the composite

$$\mathrm{Map}(C,D) \times C \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} D \xrightarrow{g} E$$

to obtain $g_*\colon \mathrm{Map}(C,D) \to \mathrm{Map}(C,E)$. For $f\colon B \to C$, curry evaluation precomposed with $\mathrm{id}_{\mathrm{Map}(C,D)} \times f$ to obtain $f^*\colon \mathrm{Map}(C,D) \to \mathrm{Map}(B,D)$. We call these functors postcomposition with g and precomposition with f, respectively.

An identification $\gamma\colon g = g'$ induces an identification $\gamma_*\colon g_* = g'_*$. Indeed, whisker γ with evaluation and use the uncurrying comparisons for g_* and g'_* to obtain an identification between their uncurried functors. Lift this through the equivalence of animae of identifications in the mapping-anima rule. Similarly, for $\varphi\colon f = f'$, pair φ with the identity on $\mathrm{Map}(C,D)$, postcompose with evaluation, and lift to obtain $\varphi^*\colon f^* = f'^*$.

Lemma 1.4.4. *For every category C, the anima $\mathrm{Map}(C,*)$ is contractible.*

Proof. We must show that the functor $\mathrm{Map}(C,*) \to *$ is an equivalence. An inverse is given by the functor $* \to \mathrm{Map}(C,*)$ obtained by currying the terminal functor $* \times C \to *$. The composite on $*$ is automatically identified with the identity. For the composite on $\mathrm{Map}(C,*)$, it suffices by the mapping-anima axiom to compare the two uncurried functors $\mathrm{Map}(C,*) \times C \to *$. These are again identified by the terminal-category axiom. □

Proposition 1.4.5. *For categories C, D, E, the projections induce an equivalence*

$$(\mathrm{pr}_{1,*}, \mathrm{pr}_{2,*})\colon \mathrm{Map}(E, C \times D) \xrightarrow{\sim} \mathrm{Map}(E,C) \times \mathrm{Map}(E,D).$$

Proof. We define a functor

$$h\colon \mathrm{Map}(E,C) \times \mathrm{Map}(E,D) \longrightarrow \mathrm{Map}(E, C \times D)$$

by pairing the two functors

$$(\mathrm{Map}(E,C) \times \mathrm{Map}(E,D)) \times E \xrightarrow{\mathrm{pr}_1 \times \mathrm{id}_E} \mathrm{Map}(E,C) \times E \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} C,$$
$$(\mathrm{Map}(E,C) \times \mathrm{Map}(E,D)) \times E \xrightarrow{\mathrm{pr}_2 \times \mathrm{id}_E} \mathrm{Map}(E,D) \times E \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} D,$$

and currying the resulting functor to $C \times D$. This is possible since $\mathrm{Map}(E,C) \times \mathrm{Map}(E,D)$ is an anima.

We start by showing that the composite $(\mathrm{pr}_{1,*}, \mathrm{pr}_{2,*}) \circ h$ is identified with the identity. It will suffice to show this after projecting to both factors, and by symmetry we may just do this for $\mathrm{Map}(E,C)$. In other words, we must show that $\mathrm{pr}_{1,*} \circ h$ is identified with pr_1. It will in turn suffice to show this after uncurrying. But the uncurrying of both functors is identified with

$$(\mathrm{Map}(E,C) \times \mathrm{Map}(E,D)) \times E \xrightarrow{\mathrm{pr}_1 \times \mathrm{id}_E} \mathrm{Map}(E,C) \times E \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} C,$$

by the construction of h and the definition of postcomposition.

We will now show that also the composite $h \circ (\mathrm{pr}_{1,*}, \mathrm{pr}_{2,*})$ is identified with the identity. Uncurrying reduces this to comparing two functors

$$\mathrm{Map}(E, C \times D) \times E \longrightarrow C \times D.$$

It suffices to compare their two projections, and again we may just consider the C-component. The projection of the uncurried identity is $\mathrm{pr}_1 \circ \mathrm{ev}_{\mathrm{Map}}$; the projection of the other functor is $\mathrm{ev}_{\mathrm{Map}} \circ (\mathrm{pr}_{1,*} \times \mathrm{id}_E)$. These are identified by the defining commutative square for postcomposition:

$$\begin{array}{ccc} \mathrm{Map}(E, C \times D) \times E & \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} & C \times D \\ {\scriptstyle \mathrm{pr}_{1,*} \times \mathrm{id}_E}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathrm{Map}(E, C) \times E & \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} & C. \end{array}$$

The product axiom and then the mapping-anima axiom give the desired identification. □

We next compare uncurrying with substitution in the anima parameter.

Lemma 1.4.6 (Uncurrying and substitution). *Let C, D be categories, X, Y animae, and $f\colon X \to \mathrm{Map}(C, D)$ and $\sigma\colon Y \to X$ functors. Uncurrying commutes with substitution: there is an identification*

$$(f \circ \sigma)^u = f^u \circ (\sigma \times \mathrm{id}_C).$$

This comparison is compatible with identifications of the input functors, with successive substitution, and with substitution by an identity functor, using the associativity and unitality identifications. Moreover, uncurrying preserves identity identifications and their vertical composition.

Proof. Using the definition of uncurrying, functoriality of products, and associativity, we obtain identifications

$$\begin{aligned} (f \circ \sigma)^u &= \mathrm{ev}_{\mathrm{Map}} \circ ((f \circ \sigma) \times \mathrm{id}_C) \\ &= \mathrm{ev}_{\mathrm{Map}} \circ (f \times \mathrm{id}_C) \circ (\sigma \times \mathrm{id}_C) \\ &= f^u \circ (\sigma \times \mathrm{id}_C). \end{aligned}$$

The compatibilities follow from those for pairing and composition, as in Lemma 1.3.16; the unit comparison also uses equation (1.1). Preservation of identity identifications and vertical composition follows from the corresponding properties of products and whiskering with evaluation. We leave the details to the reader. □

We can now lift composition of functors to an operation on mapping animae.

Construction 1.4.7. For categories C, D and E, consider the double evaluation functor

$$\mathrm{Map}(D, E) \times \mathrm{Map}(C, D) \times C \xrightarrow{\mathrm{id} \times \mathrm{ev}} \mathrm{Map}(D, E) \times D \xrightarrow{\mathrm{ev}} E.$$

Currying in C defines a functor

$$\mu_{C,D,E}\colon \mathrm{Map}(D, E) \times \mathrm{Map}(C, D) \longrightarrow \mathrm{Map}(C, E),$$

which by construction comes with an identification

$$\mathrm{ev}_{\mathrm{Map}}\big(\mu(g, f), c\big) = \mathrm{ev}_{\mathrm{Map}}\big(g, \mathrm{ev}_{\mathrm{Map}}(f, c)\big).$$

The projection $* \times C \to C$ similarly curries to a functor

$$\eta_C\colon * \longrightarrow \mathrm{Map}(C, C),$$

whose evaluation is identified with that projection.

We may use these functors to obtain composition and evaluation for term expressions. Given terms

$$\Gamma \vdash f : \mathrm{Map}(C, D) \qquad \text{and} \qquad \Gamma \vdash g : \mathrm{Map}(D, E),$$

we may form

$$\Gamma \vdash g \circ f := \mu_{C,D,E}(g, f) : \mathrm{Map}(C, E).$$

Likewise, given a term $\Gamma \vdash f : \mathrm{Map}(C, D)$ and $\Gamma \vdash x : C$, we define

$$\Gamma \vdash f(x) := \mathrm{ev}_{\mathrm{Map}}(f, x) : D.$$

Proposition 1.4.8 (Coherence of composition). *Composition in mapping animae has specified left and right unitors and an associator. These satisfy the triangle and pentagon identities through specified identifications.*

Proof. Throughout the proof, we make use of the following notation: Given a 'parameter anima' P and a functor $f\colon P \to \mathrm{Map}(C, D)$, we define

$$\widetilde{f} \;:=\; (\mathrm{pr}_1, f^u)\colon P \times C \to P \times D.$$

If $g\colon P \to \mathrm{Map}(D, E)$ is any other functor, then there is an identification

$$\widetilde{g} \circ \widetilde{f} = \widetilde{g \circ f};$$

indeed, we may construct this identification componentwise, where it is clear for the first component and follows from the defining property of $\mu_{C,D,E}$ in the second component. The compatibility with the primitive coherence identifications and change of parameter can likewise be checked after both projections, using Lemmas 1.4.6 and 1.3.16 and the coherence axioms for composition.

We now use this to construct the associator. Given a third functor $h\colon P \to \mathrm{Map}(E, F)$, we obtain three composable functors

$$P \times C \xrightarrow{\widetilde{f}} P \times D \xrightarrow{\widetilde{g}} P \times E \xrightarrow{\widetilde{h}} P \times F.$$

Using the primitive associator and the previous comparisons, we get identifications

$$\widetilde{(h \circ g) \circ f} = \widetilde{h \circ g} \circ \widetilde{f} = (\widetilde{h} \circ \widetilde{g}) \circ \widetilde{f} = \widetilde{h} \circ (\widetilde{g} \circ \widetilde{f}) = \widetilde{h} \circ \widetilde{g \circ f} = \widetilde{h \circ (g \circ f)}.$$

Passing to the second component of this identification, we obtain an identification between the uncurrying of $(h \circ g) \circ f$ and that of $h \circ (g \circ f)$; we may then use the mapping-anima axiom to lift this to the desired associator.

For the unitors, note that the constant functor $\mathrm{const}_{\mathrm{id}_C}\colon P \to \mathrm{Map}(C, C)$ corresponds under (un)currying to the projection functor $\mathrm{pr}_2\colon P \times C \to C$, and so the associated functor $\widetilde{\mathrm{const}_{\mathrm{id}_C}}\colon P \times C \to P \times C$ is identified with the identity. We then consider the unitors $\lambda_{\widetilde{f}}\colon \mathrm{id}_{P\times D} \circ \widetilde{f} = \widetilde{f}$ and $\rho_{\widetilde{f}}\colon \widetilde{f} \circ \mathrm{id}_{P\times C} = \widetilde{f}$, and with the same arguments as above deduce the desired relations $\mathrm{id}_D \circ f = f$ and $f \circ \mathrm{id}_C = f$.

Observe that both of these constructions are compatible with substitution in the parameter anima P: given a map of animae $\sigma\colon Q \to P$, we can construct identifications

$$\alpha^P_{h,g,f} \star \sigma = \alpha^Q_{h\sigma, g\sigma, f\sigma}, \qquad \lambda^P_f \star \sigma = \lambda^Q_{f\sigma}, \qquad \rho^P_f \star \sigma = \rho^Q_{f\sigma},$$

up to the substitution comparisons relating e.g. $((h \circ g) \circ f)\sigma$ to $((h\sigma) \circ (g\sigma)) \circ f\sigma$. To see this, note that by Lemma 1.4.6 there is a commutative square

$$\begin{array}{ccc} Q \times C & \xrightarrow{\widetilde{f\sigma}} & Q \times D \\ {\scriptstyle \sigma \times \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle \sigma \times \mathrm{id}_D} \\ P \times C & \xrightarrow{\widetilde{f}} & P \times D. \end{array}$$

Using this for f, g and h and applying the interchange law and the various primitive coherence identities, we get the relation on associators after uncurrying, which suffices to establish the relation. The analogous argument applies to the unitors.

For the pentagon, assume given a fourth map $k \colon P \to \mathrm{Map}(F, G)$, and consider the primitive pentagon identification for the four functors

$$P \times C \xrightarrow{\widetilde{f}} P \times D \xrightarrow{\widetilde{g}} P \times E \xrightarrow{\widetilde{h}} P \times F \xrightarrow{\widetilde{k}} P \times G.$$

For each of the five vertices of the resulting pentagon, we may use iterated identifications of the form $\widetilde{g} \circ \widetilde{f} = \widetilde{g \circ f}$ to put them into the form $\widetilde{u}$ for some u. By the construction of the associator, the substitution compatibility, and interchange, projecting the five edges of the resulting pentagon to the second component identifies them with the uncurryings of the five edges of the desired pentagon. The primitive pentagon identification therefore gives an identification between the uncurryings of the two composites around this pentagon. The mapping-anima axiom then provides the desired pentagon identification.

The triangle identification is obtained by similar means. □

We end the section by establishing various further properties about mapping animae.

Proposition 1.4.9. *For functors $f \colon A \to B$ and $g \colon B \to C$ and any category E, there is a canonical identification*

$$g_* \circ f_* = (g \circ f)_*$$

of functors $\mathrm{Map}(E, A) \to \mathrm{Map}(E, C)$.

Proof. After uncurrying, both sides evaluate a functor $E \to A$ and then compose it with f and g. The primitive associator gives the comparison between these uncurried functors, and we may lift it by Axiom B.3. □

Proposition 1.4.10. *For functors $f \colon A \to B$ and $g \colon B \to C$ and any category E, there is a canonical identification*

$$f^* \circ g^* = (g \circ f)^*$$

of functors $\mathrm{Map}(C, E) \to \mathrm{Map}(A, E)$.

Proof. This is proved in the same way, using the primitive associator after uncurrying. □

Proposition 1.4.11. *For a functor $f \colon C \to D$, the following are equivalent:*

(i) The functor f is an equivalence.

(ii) For every category E, the functor $f_* \colon \mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$ is an equivalence.

(iii) For every category E, the functor $f^*\colon \mathrm{Map}(D,E) \to \mathrm{Map}(C,E)$ is an equivalence.

If C and D are animae, it suffices in (ii) or (iii) to let E range over animae.

Proof. We prove (i) $\Leftrightarrow$ (ii); the other case is dual. An inverse of f induces an inverse of every f_*. Conversely, pick an inverse of $f_*\colon \mathrm{Map}(D,C) \to \mathrm{Map}(D,D)$, and let $g\colon D \to C$ be the evaluation of this inverse at id_D. The unit of the equivalence and the beta comparison for postcomposition give an identification $f \circ g = \mathrm{id}_D$. Now, since $f_*\colon \mathrm{Map}(C,C) \to \mathrm{Map}(C,D)$ is an equivalence, it induces an equivalence on animae of identifications. It follows that the chain of identifications

$$f \circ (g \circ f) = (f \circ g) \circ f = \mathrm{id}_D \circ f = f = f \circ \mathrm{id}_C$$

lifts to an identification $g \circ f = \mathrm{id}_C$. Here we use Construction 1.4.1 and the beta comparison to identify the action of f_* on functors with postcomposition by f. If C and D are animae, both choices of E in this argument are animae, showing the final claim. □

1.4.1 Collections of objects

We now define the *groupoid core* of a category, which we may think of as its collection of objects.

Definition 1.4.12. For any category C, we define the *groupoid core* of C to be

$$C^{\simeq} := \mathrm{Map}(*, C).$$

It comes equipped with the canonical functor

$$\iota_C\colon C^{\simeq} \simeq C^{\simeq} \times * \xrightarrow{\mathrm{ev}_{\mathrm{Map}}} C.$$

The functor ι_C displays the objects of C with the anima $C^{\simeq}$ as parameter. It is universal among functors from animae in the following sense.

Corollary 1.4.13 (Universal property of the groupoid core)**.** *For any anima X and any category D, postcomposition induces an equivalence*

$$(\iota_D)_*\colon \mathrm{Map}(X, D^{\simeq}) \xrightarrow{\sim} \mathrm{Map}(X, D).$$

Proof. By Proposition 1.4.2 and the equivalence $X \times * \simeq X$, there are equivalences

$$\mathrm{Map}(X, D^{\simeq}) = \mathrm{Map}(X, \mathrm{Map}(*, D)) \simeq \mathrm{Map}(X \times *, D) \simeq \mathrm{Map}(X, D).$$

By the definition of ι_D, their composite is postcomposition with ι_D. □

Remark 1.4.14. The anima $C^{\simeq}$ also describes all functors of the form $x\colon \Gamma \to C$ where Γ is an anima. By the previous result, it suffices to understand the universal such functor, i.e. $\iota_C\colon C^{\simeq} \to C$.

Proposition 1.4.15. *For every anima A, the canonical functor $\iota_A\colon A^{\simeq} \to A$ is an equivalence. Its inverse is obtained by currying the projection $A \times * \to A$.*

Proof. For every anima X, postcomposition with ι_A induces an equivalence by Corollary 1.4.13. Since both $A^{\simeq}$ and A are animae, Proposition 1.4.11 shows that ι_A is an equivalence. The defining identification for currying identifies its composite with the stated inverse with id_A. □

Construction 1.4.16. A functor $f\colon C \to D$ induces a functor

$$f^{\simeq} := f_*\colon C^{\simeq} = \mathrm{Map}(*, C) \longrightarrow \mathrm{Map}(*, D) = D^{\simeq}.$$

The defining identification for postcomposition makes the square

$$\begin{array}{ccc} C^{\simeq} & \xrightarrow{f^{\simeq}} & D^{\simeq} \\ {\scriptstyle \iota_C}\downarrow & & \downarrow{\scriptstyle \iota_D} \\ C & \xrightarrow[f]{} & D \end{array}$$

commute. There are identifications $(\mathrm{id}_C)^{\simeq} = \mathrm{id}_{C^{\simeq}}$ and $g^{\simeq} \circ f^{\simeq} = (g \circ f)^{\simeq}$, obtained by uncurrying and using the unit and associativity identifications for functors; the latter is the case $E = *$ of Proposition 1.4.9. An identification $\alpha : f = f'$ similarly induces an identification $\alpha^{\simeq} : f^{\simeq} = (f')^{\simeq}$, by whiskering α with evaluation and then currying.

Lemma 1.4.17. *For categories C and D, the functor $(\mathrm{pr}_C^{\simeq}, \mathrm{pr}_D^{\simeq})\colon (C \times D)^{\simeq} \to C^{\simeq} \times D^{\simeq}$ is an equivalence.*

Proof. This is a special case of Proposition 1.4.5. □

1.5 Initial categories and coproducts

Mapping animae allow us to express universal properties for functors out of a category. We use them to introduce the initial category and coproducts.

Axiom B.4 (Initial category)**.** There is a category $\emptyset$, called the *initial category*, such that $\mathrm{Map}(\emptyset, T)$ is contractible for every category T.

Thus there is a functor $\emptyset \to T$ for every T, and for any two such functors their anima of identifications is contractible. Indeed, their representing points lie in the contractible anima $\mathrm{Map}(\emptyset, T)$, and Lemma 1.3.10 and Construction 1.4.1 give the assertion about animae of identifications.

We further ask the initial category to be *strictly* initial:

Axiom B.4' (Strictness of the initial category). Given a category C, any functor $C \to \emptyset$ is an equivalence.

Lemma 1.5.1. *For a category C, the projection map $C \times \emptyset \to \emptyset$ is an equivalence.*

Proof. This is immediate from Axiom B.4'. □

Axiom B.5 (Coproduct of categories)**.** The *coproduct* of categories C and D is a category $C \sqcup D$ equipped with *inclusion functors* $i_C \colon C \to C \sqcup D$ and $i_D \colon D \to C \sqcup D$, such that for every category T the restriction functor

$$(i_C^*, i_D^*) \colon \mathrm{Map}(C \sqcup D, T) \longrightarrow \mathrm{Map}(C, T) \times \mathrm{Map}(D, T)$$

is an equivalence. We also call $C \sqcup D$ the *disjoint union* of C and D.

Construction 1.5.2 (Copairing)**.** For $f \colon C \to T$ and $g \colon D \to T$, applying an inverse to the displayed equivalence (i_C^*, i_D^*) to the pair (f, g) provides a functor

$$\langle f, g \rangle \colon C \sqcup D \to T$$

with identifications $\langle f, g \rangle i_C = f$ and $\langle f, g \rangle i_D = g$. For any $h, k : C \sqcup D \to T$, restriction also induces an equivalence

$$(h = k) \xrightarrow{\sim} (hi_C = ki_C) \times (hi_D = ki_D).$$

To see this, use Lemma 1.3.10 for the equivalence in the coproduct axiom, then the product rule and the comparison of Construction 1.4.1. In particular, a pair of identifications $\alpha : hi_C = ki_C$ and $\beta : hi_D = ki_D$ results in an identification $\langle \alpha, \beta \rangle : h = k$, with the prescribed restrictions to C and D.

In term notation, the axiom allows us to write expressions of the form

$$z : C \sqcup D \vdash \langle f, g \rangle(z) : T,$$

whenever given two expressions $x : C \vdash f(x) : T$ and $y : D \vdash g(y) : T$; the relations can be phrased as $\langle f, g \rangle(i_C(x)) = f(x)$ and $\langle f, g \rangle(i_D(y)) = g(y)$.

Once we have distributivity available (see Section 1.6.3) we can also use term notation for expressions with more additional parameters.

Exercise 1.5.3 (Exercise 1.10.2). Show that the coproduct of categories is associative, commutative and unital:

$$(C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Construction 1.5.4. The formation of coproducts of categories is functorial: given two functors $f \colon C \to C'$ and $g \colon D \to D'$, we obtain a new functor

$$f \sqcup g := \langle i_{C'} \circ f, i_{D'} \circ g \rangle \colon C \sqcup D \to C' \sqcup D'.$$

We leave it to the reader to verify that there are identifications

$$\mathrm{id}_C \sqcup \mathrm{id}_D = \mathrm{id}_{C \sqcup D} \quad \text{and} \quad (f' \sqcup g') \circ (f \sqcup g) = (f' \circ f) \sqcup (g' \circ g).$$

1.6 Pullbacks of categories

We will now axiomatize the *pullback* of two functors $f\colon C \to E$ and $g\colon D \to E$ of categories. This will be similar to the axiom for the product $C \times D$, but it is more involved due to the fact that we need to record compatibilities with the structure maps to E.

Definition 1.6.1. Consider two functors $f\colon C \to E$ and $g\colon D \to E$. Given a category T, we define a *cone diagram on* T to be a triple $c = (t_C, t_D, \tau)$ consisting of functors $t_C\colon T \to C$ and $t_D\colon T \to D$ together with an identification $\tau\colon f \circ t_C = g \circ t_D$. In other words, a cone diagram consists of a commutative square of the form

$$\begin{array}{ccc} T & \xrightarrow{t_C} & C \\ {\scriptstyle t_D}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{g} & E. \end{array}$$

We call T the *cone point* of the diagram. Given another cone diagram $c' = (t'_C, t'_D, \tau')$, an *identification of cone diagrams* $c = c'$ is a triple $\Phi\colon (\varphi_C, \varphi_D, H)$, where $\varphi_C\colon t_C = t'_C$ and $\varphi_D\colon t_D = t'_D$ are identifications, and H is an identification

$$\tau' \circ (f \star \varphi_C) = (g \star \varphi_D) \circ \tau.$$

In particular, a comparison of cones needs to be compatible with the identifications forming the commutative square. In turn, if $\Psi\colon (\psi_C, \psi_D, G)$ is another identification of cone diagrams $c = c'$, we say that Φ and Ψ are identified if there are identifications $\varphi_C = \psi_C$ and $\varphi_D = \psi_D$ under which H becomes identified with G.

Given a functor $s\colon S \to T$ and a cone diagram $c = (t_C, t_D, \tau)$ on T we obtain another cone diagram $s^*(c) := (t_C \circ s, t_D \circ s, \tau * \mathrm{id}_s)$ on S. Similarly, every identification $s_1 = s_2$ of functors $S \to T$ induces an identification $s_1^*(c) = s_2^*(c)$ of cone diagrams.

Axiom B.6 (Pullback of categories)**.** For functors $f\colon C \to E$ and $g\colon D \to E$, there is a category $C \times_E D$, called their *pullback* or *fiber product*, with projections $\mathrm{pr}_1\colon C \times_E D \to C$ and $\mathrm{pr}_2\colon C \times_E D \to D$, and an identification $\tau_{\mathrm{univ}}\colon f\,\mathrm{pr}_1 = g\,\mathrm{pr}_2$. Given $a\colon T \to C$, $b\colon T \to D$ and $\tau\colon fa = gb$, there is a functor $(a, b)_\tau\colon T \to C \times_E D$ equipped with projection comparisons $c\colon \mathrm{pr}_1(a, b)_\tau = a$ and $d\colon \mathrm{pr}_2(a, b)_\tau = b$ and an identification

$$\tau \circ (f \star c) = (g \star d) \circ (\tau_{\mathrm{univ}} \star (a, b)_\tau).$$

If C, D, E are animae, then $C \times_E D$ is an anima.

For any $h, k\colon T \to C \times_E D$, the projection-induced functor

$$(h = k) \longrightarrow (\mathrm{pr}_1\, h = \mathrm{pr}_1\, k) \times_{(f\,\mathrm{pr}_1\, h = g\,\mathrm{pr}_2\, k)} (\mathrm{pr}_2\, h = \mathrm{pr}_2\, k)$$

is an equivalence; here the two functors defining the pullback on the right are

$$\alpha \longmapsto (\tau_{\mathrm{univ}} \star k) \circ (f \star \alpha), \qquad \beta \longmapsto (g \star \beta) \circ (\tau_{\mathrm{univ}} \star h).$$

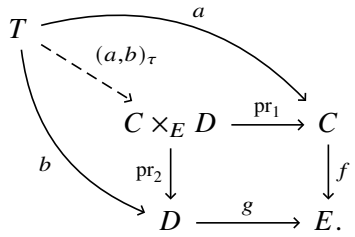
Interchange with τ_{univ} identifies their composites with the two projections from $(h = k)$ and thereby defines the displayed functor.

The final clause uses the pullback construction, now applied to three animae of identifications. In term notation the factorization is written

$$x : T \vdash (a(x), b(x))_{\tau(x)} : C \times_E D.$$

The subscript may be omitted when it is clear from context, or left implicit to simplify notation. An identification between two such expressions is written $(\alpha(x), \beta(x))_{\eta(x)}$, where η is the identification between the two composites in the middle anima of identifications of the axiom.

The resulting map may be displayed as follows:



Definition 1.6.2 (Pullback square)**.** A commutative square

$$\begin{array}{ccc} P & \xrightarrow{p} & C \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{g} & E \end{array}$$

with matching identification $\sigma\colon fp = gq$ is a *pullback square*, or is *cartesian*, if its induced functor $(p, q)_\sigma\colon P \to C \times_E D$ is an equivalence. We indicate this by putting the symbol ⌟ in the square.

A pullback square has the same universal property as the chosen pullback: every cone factors through it with an identification of cone diagrams, and its animae of identifications are given by the displayed pullbacks of animae of identifications. Indeed, transfer the two clauses of the axiom through the equivalence $P \to C \times_E D$, using Lemma 1.3.10. Conversely, these clauses construct inverse functors between P and the chosen pullback.

Note that any functor between two pullbacks of the same cospan that respects the projections and the specified identifications making the squares is an equivalence. Indeed, the induced functors to the chosen pullback are equivalences and form a commutative triangle with this functor, so the claim follows by two-out-of-three.

Proposition 1.6.3. *A commutative square of categories is a pullback square if and only if applying* $\mathrm{Map}(T, -)$ *gives a pullback square for every category* T.

Proof. We first show that applying $\mathrm{Map}(T, -)$ to the chosen pullback $C \times_E D$ results in a pullback square. Postcomposition with the two projections and whiskering with τ_{univ} gives a comparison functor

$$\mathrm{Map}(T, C \times_E D) \longrightarrow \mathrm{Map}(T, C) \times_{\mathrm{Map}(T,E)} \mathrm{Map}(T, D).$$

For an inverse, let P be the anima on the right. Uncurrying the projections to $\mathrm{Map}(T, C)$ and $\mathrm{Map}(T, D)$ gives maps from $P \times T$ to C and D. Moreover, the identification in the commutative square for $C \times_E D$ results in an identification of two functors $P \times T \to E$. The pullback axiom thus

supplies a functor from $P \times T$ to $C \times_E D$, with the projection comparisons and their compatibility with the squares. Currying gives a candidate inverse functor $P \to \mathrm{Map}(T, C \times_D E)$.

To compare the composite on the right to the identity, we first do so on its two projections, where it follows by currying the identifications we just constructed. To show that the resulting two commutative squares are identified, we may again do so after uncurrying, where it follows from unwinding definitions.

For the composite on the left, we need to show its uncurrying agrees with the evaluation map. This can in turn be tested after both projections and comparing the commutativity identifications.

The claim now also follows for an arbitrary pullback square by considering the equivalence from its cone point to $C \times_E D$. □

Lemma 1.6.4. *For functors $C \to E$ and $D \to E$, the functor $(\mathrm{pr}_C^{\simeq}, \mathrm{pr}_D^{\simeq}) \colon (C \times_E D)^{\simeq} \to C^{\simeq} \times_{E^{\simeq}} D^{\simeq}$ is an equivalence.*

Proof. This is a special case of Proposition 1.6.3. □

Exercise 1.6.5 (Exercise 1.10.4). Show that the fiber product is commutative, associative and unital: for functors $C \to \Gamma$, $D \to \Gamma$ and $B \to \Gamma$, there are preferred equivalences

$$C \times_\Gamma D \xrightarrow{\sim} D \times_\Gamma C, \qquad B \times_\Gamma (C \times_\Gamma D) \xrightarrow{\sim} (B \times_\Gamma C) \times_\Gamma D, \qquad C \times_\Gamma \Gamma \xrightarrow{\sim} C.$$

Exercise 1.6.6 (Exercise 1.10.5). Consider categories C and D, and let $p_C \colon C \to *$ and $p_D \colon D \to *$ be their functors to the terminal category. Show that the functor $(\mathrm{pr}_1, \mathrm{pr}_2) \colon C \times_* D \to C \times D$ is an equivalence.

Construction 1.6.7. The construction of pullbacks of categories is functorial: given a commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{f} & \Gamma & \xleftarrow{g} & D \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\psi} \\ C' & \xrightarrow[f']{} & \Gamma' & \xleftarrow[g']{} & D' \end{array}$$

there is an induced functor

$$\varphi \times_\chi \psi := (\varphi \circ \mathrm{pr}_1, \psi \circ \mathrm{pr}_2) \colon C \times_\Gamma D \to C' \times_{\Gamma'} D'.$$

Exercise 1.6.8. In the above situation, show that if each of the functors φ, ψ and χ is an equivalence, then so is $\varphi \times_\chi \psi$.

Corollary 1.6.9. *Equivalences are closed under pullback: if $g \colon D \to E$ is an equivalence and $f \colon C \to E$ is an arbitrary functor, then also $C \times_E D \to C$ is an equivalence.*

Proof. This is a special case of Exercise 1.6.8 by taking $C' = C$, $\Gamma = \Gamma' = E$, $D' = E$, $\varphi = \mathrm{id}_C$, $\chi = \mathrm{id}_E$ and $\psi = g$: this exhibits the projection $C \times_E D \to C$ as the composite of the equivalence $\varphi \times_\chi \psi \colon C \times_E D \to C \times_E E$ with the unitality equivalence $C \times_E E \xrightarrow{\sim} C$ from Exercise 1.6.5. □

1.6.1 Fibers of functors

Pullbacks of categories lead to the notion of a *fiber* of a functor:

Definition 1.6.10 (Fiber). Let $f : C \to D$ be a functor and let $x : * \to D$ be an absolute object. We define the *fiber* of f over x as the pullback

$$\begin{array}{ccc} C_x & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ * & \xrightarrow{x} & D. \end{array}$$

One particular case is the fiber of a map $b : * \to C$ over some object $a : * \to C$. The absolute terms of this fiber can be identified with identifications $\alpha : a = b$. We impose as an axiom that the same holds in families:

Axiom B.7 (Representability of identifications)**.** For every category C and absolute objects $a, b \in C$, there is an identification

$$\varepsilon : \mathrm{const}_a = \mathrm{const}_b$$

of functors $(a = b) \to C$ such that the square

$$\begin{array}{ccc} (a = b) & \xrightarrow{p_{(a=b)}} & * \\ {\scriptstyle p_{(a=b)}}\downarrow & \lrcorner & \downarrow{\scriptstyle b} \\ * & \xrightarrow[a]{} & C \end{array}$$

is a pullback, with commutativity identification given by ε. For every identification $\alpha : a = b$, we require a specified identification

$$\varepsilon(\alpha) = \alpha.$$

Here $\varepsilon(\alpha)$ denotes the identification obtained by prewhiskering ε with $\alpha : * \to (a = b)$ and using the terminal-category comparisons to identify $\mathrm{const}_a \circ \alpha$ with a and $\mathrm{const}_b \circ \alpha$ with b.

For a functor $u : T \to (a = b)$, the corresponding identification between the constant functors $T \to C$ is obtained by prewhiskering ε with u, using the terminal-category comparisons to identify its source and target.

1.6.2 Properties of pullback squares

Lemma 1.6.11 (Pasting lemma for pullback squares). *Consider a commutative diagram*

$$\begin{array}{ccccc} C_1 & \xrightarrow{g_1} & C_2 & \xrightarrow{g_2} & C_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ D_1 & \xrightarrow[h_1]{} & D_2 & \xrightarrow[h_2]{} & D_3. \end{array}$$

(1) There is a preferred equivalence $D_1 \times_{D_3} C_3 \xrightarrow{\sim} D_1 \times_{D_2} (D_2 \times_{D_3} C_3)$.

(2) If the right-hand square is a pullback square, then the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

Proof. For part (1), we define the functor as $(\mathrm{pr}_1, (h_1 \circ \mathrm{pr}_1, \mathrm{pr}_2))$. We claim that an inverse is given by the functor

$$(\mathrm{pr}_1, \mathrm{pr}_2 \circ \mathrm{pr}_2) \colon D_1 \times_{D_2} (D_2 \times_{D_3} C_3) \to D_1 \times_{D_3} C_3.$$

To show that the two composites are identified with the respective identities, it suffices to construct these identifications after projecting to the two components and verifying that these identifications agree after composing with the functors to D_3. But after unwinding the definitions, this holds tautologically by the very construction of the two functors in both directions.

Part (2) is an immediate consequence of the 2-out-of-3 property from Lemma 1.3.6 applied to the following commutative square:

$$\begin{array}{ccc} C_1 & \longrightarrow & D_1 \times_{D_2} C_2 \\ \downarrow & & \downarrow{\scriptstyle \simeq} \\ D_1 \times_{D_3} C_3 & \xrightarrow{\simeq} & D_1 \times_{D_2} (D_2 \times_{D_3} C_3), \end{array}$$

where the right vertical map is an equivalence by applying Exercise 1.6.8 to the equivalence $C_2 \xrightarrow{\sim} D_2 \times_{D_3} C_3$. □

Lemma 1.6.12. *Consider a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ E & \xrightarrow{k} & F \end{array}$$

and assume that h is an equivalence. Then g is an equivalence if and only if the square is a pullback square.

Proof. Since h is an equivalence, also the projection map $\mathrm{pr}_2 \colon D \times_F E \to E$ is an equivalence by Corollary 1.6.9. Since the composite of the map $(f, g) \colon C \to D \times_F E$ with pr_2 is identified with g, the claim follows from the 2-out-of-3 property of equivalences, Lemma 1.3.6. □

Lemma 1.6.13. *Consider functors $f \colon C \to E$ and $g \colon D \to E$. Then the commutative square*

$$\begin{array}{ccc} C \times_E D & \longrightarrow & E \\ {\scriptstyle (\mathrm{pr}_1, \mathrm{pr}_2)}\downarrow & & \downarrow{\scriptstyle \Delta} \\ C \times D & \xrightarrow{f \times g} & E \times E \end{array}$$

is a pullback square.

Proof. This follows immediately from the pasting lemma for pullback squares, Lemma 1.6.11, applied to the following commutative diagram:

$$\begin{array}{ccccc}
C \times_E D & \xrightarrow{\mathrm{pr}_2} & D & \xrightarrow{g} & E \\
{\scriptstyle (\mathrm{pr}_1,\mathrm{pr}_2)}\downarrow & & \downarrow{\scriptstyle (g,\mathrm{id}_D)} & & \downarrow{\scriptstyle (\mathrm{id}_E,\mathrm{id}_E)} \\
C \times D & \xrightarrow{f\times \mathrm{id}_D} & E \times D & \xrightarrow{\mathrm{id}_E \times g} & E \times E \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \\
C & \xrightarrow{f} & E. & &
\end{array}$$

□

1.6.3 Universality of coproducts

The strictness of the initial category immediately implies its universality:

Lemma 1.6.14. *For every functor $g\colon D \to E$, the commutative square*

$$\begin{array}{ccc}
\emptyset & \longrightarrow & \emptyset \\
\downarrow & & \downarrow \\
D & \xrightarrow{g} & E
\end{array}$$

is a pullback square.

Proof. We need to show that the induced map $\emptyset \to \emptyset \times_E D$ is an equivalence. But this follows directly from Axiom B.4' by applying 2-out-of-3 to the maps $\emptyset \to \emptyset \times_E D \xrightarrow{\mathrm{pr}_1} \emptyset$. □

We now postulate a analogous universality property on coproducts of categories:

Axiom B.5' (Universality of coproducts). Consider categories Γ_0 and Γ_1, define $\Gamma := \Gamma_0 \sqcup \Gamma_1$, and let $i_0\colon \Gamma_0 \to \Gamma$ and $i_1\colon \Gamma_1 \to \Gamma$ denote the two inclusions.

(1) For functors $f\colon C \to \Gamma_0$ and $g\colon D \to \Gamma_1$, the commutative squares

$$\begin{array}{ccc}
C & \xrightarrow{i_C} & C \sqcup D \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\sqcup g} \\
\Gamma_0 & \xrightarrow{i_0} & \Gamma_0 \sqcup \Gamma_1
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
D & \xrightarrow{i_D} & C \sqcup D \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f\sqcup g} \\
\Gamma_1 & \xrightarrow{i_1} & \Gamma_0 \sqcup \Gamma_1
\end{array}$$

are pullback squares.

(2) For a functor $h\colon E \to \Gamma$, the functor

$$\langle \mathrm{pr}_1, \mathrm{pr}_1 \rangle\colon (E \times_\Gamma \Gamma_0) \sqcup (E \times_\Gamma \Gamma_1) \to E$$

is an equivalence.

Corollary 1.6.15 (Disjointness of coproducts). *Coproducts are disjoint: for categories C and D, the commutative square*

$$\begin{array}{ccc} \emptyset & \longrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle i_C} \\ D & \xrightarrow{i_D} & C \sqcup D \end{array}$$

is a pullback square. □

Lemma 1.6.16. *For functors $f\colon C \to \Gamma$, $g\colon D \to \Gamma$ and $\varphi\colon \Gamma' \to \Gamma$, the commutative square*

$$\begin{array}{ccc} (C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') & \longrightarrow & C \sqcup D \\ \downarrow & & \downarrow{\scriptstyle \langle f,g\rangle} \\ \Gamma' & \xrightarrow{\varphi} & \Gamma \end{array}$$

is a pullback square.

Proof. We need to show that the induced functor

$$(C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') \to (C \sqcup D) \times_\Gamma \Gamma'$$

is an equivalence. But this is a special case of Axiom B.5' applied to the functor $f = \mathrm{pr}_1\colon (C \sqcup D) \times_\Gamma \Gamma' \to C \sqcup D$: by the pasting law of pullback squares there are equivalences

$$C \times_{C\sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq C \times_\Gamma \Gamma' \qquad \text{and} \qquad D \times_{C\sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq D \times_\Gamma \Gamma'.$$

This finishes the proof. □

Corollary 1.6.17 (Distributivity). *For categories C, D and E, the functor*

$$\langle E \times i_C, E \times i_D\rangle\colon E \times C \sqcup E \times D \to E \times (C \sqcup D)$$

is an equivalence.

Proof. This is an instance of the previous lemma by taking $\Gamma = *$, $\Gamma' = E$, $f = p_C\colon C \to *$, $g = p_D\colon D \to *$, and $\varphi = p_E\colon E \to *$. □

Consequently, expressions with an additional parameter $t : T$ may be defined by cases: combine the functors $T \times C \to E$ and $T \times D \to E$ by copairing, then precompose with the inverse of distributivity. Their restriction comparisons are the comparisons of the resulting expression on the two summands.

Lemma 1.6.18. *The following two conditions are equivalent:*

(1) The category $* \sqcup *$ is contractible;

(2) Every category is contractible.

Proof. It is clear that (2) implies (1). Conversely, assume that $* \sqcup *$ is contractible, i.e. that the map $* \sqcup * \to *$ is an equivalence. Since this map admits a section given by the left inclusion $l\colon * = * \sqcup \emptyset \hookrightarrow * \sqcup *$, it follows that l is an equivalence. In particular, its fiber over the right point must be contractible. Since this fiber is the empty category $\emptyset$ by Corollary 1.6.15, we see that the map $\emptyset \to *$ is an equivalence, and it follows from taking the product with any category C that also $\emptyset \to C$ is an equivalence. By 2-out-of-3 this implies that C is contractible. □

1.6.4 Embeddings of categories

We introduce the notion of *embeddings* of categories and prove basic properties about them. We will eventually see that embeddings precisely correspond to *subcategories*, see Proposition 6.4.5.

Definition 1.6.19 (Embedding). A functor $f\colon C \to D$ is called an *embedding* if the diagonal $\Delta_f := (\mathrm{id}_C, \mathrm{id}_C)\colon C \to C \times_D C$ is an equivalence, or equivalently if the commutative diagram

$$\begin{array}{ccc} C & = & C \\ \| & & \downarrow f \\ C & \xrightarrow[f]{} & D \end{array}$$

is a pullback square.

Proposition 1.6.20. *A functor $f\colon C \to D$ is an embedding if and only if for every category E the induced map $f_*\colon \mathrm{Map}(E,C) \to \mathrm{Map}(E,D)$ is an embedding.*

Proof. This follows immediately from Proposition 1.6.3 by considering the commutative diagram

$$\begin{array}{ccc} C & = & C \\ \| & & \downarrow f \\ C & \xrightarrow{f} & D. \end{array}$$

□

Corollary 1.6.21. *Given an embedding $f\colon C \to D$, the functor $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is also an embedding.*

Proof. In light of the definition of embeddings, this is immediate from Lemma 1.6.4. □

Lemma 1.6.22. *A functor $f\colon C \to D$ is an embedding if and only if the commutative square*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle (\mathrm{id}_C,\mathrm{id}_C)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_D,\mathrm{id}_D)} \\ C \times C & \xrightarrow{f\times f} & D \times D \end{array}$$

is a pullback square.

Proof. By Lemma 1.6.13 the pullback of this diagram is $C \times_D C$. Since the induced map $C \to C \times_D C$ is precisely the diagonal $\Delta_f\colon C \to C \times_D C$ of f, the claim follows. □

Lemma 1.6.23. *Consider a pullback square*

$$\begin{array}{ccc} C' & \xrightarrow{g} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ D' & \xrightarrow{f} & D \end{array}$$

of categories. If v is an embedding, then also u is an embedding.

Proof. This is left as an exercise for the reader, see Exercise 1.10.10 □

Lemma 1.6.24. *Let $f\colon C \to D$ be an embedding which admits a section $s\colon D \to C$. Then f is an equivalence.*

Proof. As $fs = \mathrm{id}_D$, it remains to show that $sf = \mathrm{id}_C$. Consider the following commutative diagram:

$$\begin{array}{ccccc} C & \overset{h}{\dashrightarrow} & C & = & C \\ {\scriptstyle f}\downarrow & & \| & \lrcorner & \downarrow{\scriptstyle f} \\ D & \xrightarrow{s} & C & \xrightarrow{f} & D. \end{array}$$

Since the right square is a pullback square by assumption on f, there exists a map $h\colon C \to C$ making the diagram commute. It follows that $sf = h = \mathrm{id}_C$, finishing the proof. □

We will now prove some more intricate properties about embeddings, which may be skipped on first reading and referred back to whenever needed.

Lemma 1.6.25. *Let $f\colon C \to D$ and $g\colon D \to E$ be functors and assume that g is an embedding. Then the square*

$$\begin{array}{ccc} C & = & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle gf} \\ D & \xrightarrow{g} & E \end{array}$$

is a pullback square.

Proof. This is immediate from the pasting law of pullback squares applied to the following commutative diagram:

$$\begin{array}{ccc} C & = & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ D & = & D \\ \| & & \downarrow{\scriptstyle g} \\ D & \xrightarrow{g} & E. \end{array}$$

Indeed, the bottom square is a pullback square by assumption on g, while the top square is clearly a pullback square. □

Lemma 1.6.26. *Consider a commutative triangle*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle gf}\searrow \quad \swarrow{\scriptstyle g} & \\ & E & \end{array}$$

of functors. If g is an embedding, then f is an embedding if and only if gf is an embedding.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} C & = & C & = & C \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle gf} \\ C & \xrightarrow{f} & D & \xrightarrow{g} & E. \end{array}$$

Since g is an embedding, the right square is a pullback square by Lemma 1.6.25. It follows from the pasting law of pullback squares, Lemma 1.6.11, that the left square is cartesian if and only if the outer rectangle is cartesian, proving the claim. □

Corollary 1.6.27. *Consider a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{v} & D \end{array}$$

of functors such that v is an embedding. Then u is an embedding if and only if the induced map $(f,u)\colon A \to B \times_D C$ is an embedding.

Proof. The map $u\colon A \to C$ factors as the composite $A \xrightarrow{(f,u)} B \times_D C \xrightarrow{\mathrm{pr}_2} C$. The map pr_2 is an embedding by Lemma 1.6.23, and hence the statement follows from Lemma 1.6.26. □

Lemma 1.6.28. *Consider a commutative diagram*

$$\begin{array}{ccc} C' & \xrightarrow{u} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{v} & D \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ E' & \xrightarrow{w} & E. \end{array}$$

If v and w are embeddings and the composite square is a pullback square, then also the top square is a pullback square.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} C' & = & C' & \xrightarrow{u} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle (g'f',fu)} & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{(g',v)} & E' \times_E D & \longrightarrow & D \\ & & \downarrow & & \downarrow \\ & & E' & \xrightarrow{w} & E. \end{array}$$

Since w is an embedding, the map $E' \times_E D \to D$ is an embedding by Lemma 1.6.23. Since $v\colon D' \to D$ is also an embedding, it thus follows from Lemma 1.6.26 that the map $(g', v)\colon D' \to E' \times_E D$ is an embedding, and thus by Lemma 1.6.25 the top left square is a pullback square. Since the right bottom square and right outer rectangle are pullback squares, so is the top right square by the pasting law Lemma 1.6.11. Another instance of Lemma 1.6.11 shows that the outer rectangle is a pullback square, as desired. □

1.7 Functor categories

Given two categories C and D, we are interested in forming the *functor category* $\mathrm{Fun}(C, D)$ whose objects are precisely the functors from C to D.

Given any functor $e\colon F \times C \to D$ and any category T, we can form a functor

$$u_T\colon \mathrm{Map}(T, F) \to \mathrm{Map}(T \times C, D)$$

by currying the composite

$$\mathrm{Map}(T, F) \times T \times C \xrightarrow{\mathrm{ev}_{\mathrm{Map}} \times \mathrm{id}_C} F \times C \xrightarrow{e} D.$$

It satisfies the property that it evaluations on an absolute object of $\mathrm{Map}(T, F)$, corresponding to a functor $g\colon T \to F$, as

$$u_T(g) = e \circ (g \times \mathrm{id}_C).$$

Definition 1.7.1. A *functor category* from C to D is a category F equipped with a functor $e\colon F \times C \to D$ such that, for any category T, the induced map

$$u_T\colon \mathrm{Map}(T, F) \xrightarrow{\sim} \mathrm{Map}(T \times C, D)$$

is an equivalence. We then call F the category of functors from C to D and e the *evaluation functor*.

Proposition 1.7.2. *If F and F' both are categories of functors from C to D with evaluation functors e and e' respectively, then there is an equivalence $\varphi\colon F' \xrightarrow{\sim} F$ as well as $\sigma\colon e \circ (\varphi \times \mathrm{id}_C) = e'$.*

Proof. Let v be an inverse of the equivalence $u_{F'}\colon \mathrm{Map}(F', F) \xrightarrow{\sim} \mathrm{Map}(F' \times C, D)$ and define $\varphi := v(e')\colon F' \to F$. By Construction 1.4.1, the identification $u_{F'} \circ v = \mathrm{id}$ and the beta comparison for $u_{F'}$ provide

$$\sigma\colon e \circ (\varphi \times \mathrm{id}_C) = e'.$$

Exchanging the roles of (F, e) and (F', e'), we similarly obtain a functor $\psi\colon F \to F'$ together with $e' \circ (\psi \times \mathrm{id}_C) = e$. Combining the two, the uncurried functors associated to $\varphi \circ \psi$ and id_F are identified:

$$e \circ ((\varphi \circ \psi) \times \mathrm{id}_C) = e' \circ (\psi \times \mathrm{id}_C) = e = e \circ (\mathrm{id}_F \times \mathrm{id}_C).$$

Since u_F is an equivalence, it reflects this identification and gives $\varphi \circ \psi = \mathrm{id}_F$. The same argument shows $\psi \circ \varphi = \mathrm{id}_{F'}$, so φ is an equivalence with inverse ψ. □

Axiom B.8 (Functor category)**.** For any two categories C and D, there is a category $\mathrm{Fun}(C, D)$ equipped with an *evaluation functor* $\mathrm{ev}\colon \mathrm{Fun}(C, D) \times C \to D$ that exhibits it as a functor category from C to D.

Definition 1.7.3. Given another category T, choose an inverse v_T of the equivalence u_T. For a functor $f : T \times C \to D$, define its *currying* by applying v_T:

$$f_c := v_T(f) : T \longrightarrow \mathrm{Fun}(C, D).$$

It comes equipped with a commutative diagram as follows:

$$\begin{array}{ccc} T \times C & \xrightarrow{f_c \times \mathrm{id}_C} & \mathrm{Fun}(C,D) \times C \\ & \searrow^{f} & \downarrow{\scriptstyle \mathrm{ev}} \\ & & D. \end{array}$$

Conversely, given a functor $g : T \to \mathrm{Fun}(C, D)$, we call the composite

$$T \times C \xrightarrow{g \times \mathrm{id}_C} \mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D$$

the *uncurrying* of g and denote it by g^u. The discussion before shows it is identified with $u_T(g)$. Similarly, an identification $\beta : g = h$ induces an identification $\beta^u : g^u = h^u$. At the level of mapping animae, uncurrying is the equivalence

$$(-)^u : \mathrm{Map}(T, \mathrm{Fun}(C, D)) \xrightarrow{\sim} \mathrm{Map}(T \times C, D)$$

with inverse defined by currying.

For parallel functors $g, h : T \to \mathrm{Fun}(C, D)$, the induced functor

$$(g = h) \longrightarrow (g^u = h^u)$$

is an equivalence. Indeed, represent g, h by points of $\mathrm{Map}(T, \mathrm{Fun}(C, D))$, apply Lemma 1.3.10 to u_T, and use Construction 1.4.1 at both ends.

Notation 1.7.4. In term notation, an expression $x : T, y : C \vdash f(x, y) : D$ may therefore be curried to $x : T \vdash (y \mapsto f(x, y)) : \mathrm{Fun}(C, D)$. Conversely, evaluation of $x : T \vdash F(x) : \mathrm{Fun}(C, D)$ and $x : T \vdash c(x) : C$ is formed by pairing, giving $x : T \vdash F(x)(c(x)) : D$. These conventions also apply to identifications of the whole functors, with the beta and eta comparisons above.

Remark 1.7.5. By considering the terminal category, we observe that absolute objects $* \to \mathrm{Fun}(C, D)$ correspond to functors $C \simeq * \times C \to D$, justifying the notation and terminology of $\mathrm{Fun}(C, D)$.

Construction 1.7.6 (Evaluation functor)**.** Given categories C and D, and $x : * \to C$, the *evaluation functor at x* is defined as the composite

$$\mathrm{ev}_x : \mathrm{Fun}(C, D) \simeq \mathrm{Fun}(C, D) \times * \xrightarrow{\mathrm{id} \times x} \mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D.$$

The functoriality of the product naturally provides functoriality for the functor categories:

Construction 1.7.7 (Functoriality of functor categories)**.** Given a functor $g\colon D \to D'$, we define the functor $g \circ -\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, D')$ as the currying of the composite

$$\mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D \xrightarrow{g} D'.$$

By assumption we obtain a commutative square as follows:

$$\begin{array}{ccc} \mathrm{Fun}(C, D) \times C & \xrightarrow{\mathrm{ev}} & D \\ {\scriptstyle (g\circ -)\times \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Fun}(C, D') \times C & \xrightarrow{\mathrm{ev}} & D'. \end{array}$$

Similarly, given a functor $f\colon C' \to C$, we define the functor $- \circ f\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C', D)$ as the currying of the composite

$$\mathrm{Fun}(C, D) \times C' \xrightarrow{\mathrm{id}\times f} \mathrm{Fun}(C, D) \times C \xrightarrow{\mathrm{ev}} D.$$

It satisfies the property that it makes the following square commute:

$$\begin{array}{ccc} \mathrm{Fun}(C, D) \times C' & \xrightarrow{\mathrm{id}\times f} & \mathrm{Fun}(C, D) \times C \\ {\scriptstyle (-\circ f)\times \mathrm{id}_{C'}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ \mathrm{Fun}(C', D) \times C' & \xrightarrow{\mathrm{ev}} & D. \end{array}$$

In a completely analogous way, every identification $\beta\colon g = g'$ of functors $D \to D'$ induces an identification $(\beta \circ -)\colon (g \circ -) = (g' \circ -)$ of functors $\mathrm{Fun}(C, D) \to \mathrm{Fun}(C, D')$, and similarly for the construction $- \circ f$.

Exercise 1.7.8 (Exercise 1.10.6)**.** Formulate and prove that the assignments $g \mapsto (g \circ -)$ and $f \mapsto (- \circ f)$ are functorial, in the sense that they respect identity functors and composition of functors.

1.7.1 Properties of functor categories

The universal property of functor categories allows us to deduce their compatibilities with the terminal category, the initial category, products, coproducts and pullbacks from the corresponding properties of mapping animae. We will use Proposition 1.4.11 to test the comparison functors after applying $\mathrm{Map}(T, -)$.

Lemma 1.7.9. *For every category C the functor category* $\mathrm{Fun}(C, *)$ *is contractible.*

Proof. For every category T, the universal property of the functor category gives

$$\mathrm{Map}(T, \mathrm{Fun}(C, *)) \simeq \mathrm{Map}(T \times C, *) \simeq *,$$

where the last equivalence is Lemma 1.4.4. Thus $\mathrm{Fun}(C, *) \to *$ induces an equivalence on mapping animae for every T, and is an equivalence by Proposition 1.4.11. □

Lemma 1.7.10. *Let C, D and E be categories. Then the functor*

$$(\mathrm{pr}_1 \circ -, \mathrm{pr}_2 \circ -)\colon \mathrm{Fun}(C, D \times E) \to \mathrm{Fun}(C, D) \times \mathrm{Fun}(C, E)$$

is an equivalence.

Proof. For every category T, the universal property of functor categories and Proposition 1.4.5 give equivalences

$$\begin{aligned}\mathrm{Map}(T, \mathrm{Fun}(C, D \times E)) &\simeq \mathrm{Map}(T \times C, D \times E)\\ &\simeq \mathrm{Map}(T \times C, D) \times \mathrm{Map}(T \times C, E)\\ &\simeq \mathrm{Map}(T, \mathrm{Fun}(C, D)) \times \mathrm{Map}(T, \mathrm{Fun}(C, E))\\ &\simeq \mathrm{Map}(T, \mathrm{Fun}(C, D) \times \mathrm{Fun}(C, E)).\end{aligned}$$

This composite is induced by the functor in the statement: after uncurrying, its two components are postcomposition with pr_1 and pr_2, by Lemma 1.7.7. The result follows from Proposition 1.4.11. □

Lemma 1.7.11. *Let T be a category and let $f\colon C \to E$ and $g\colon D \to E$ be functors. Then the functor*

$$(\mathrm{pr}_1 \circ -, \mathrm{pr}_2 \circ -)\colon \mathrm{Fun}(T, C \times_E D) \to \mathrm{Fun}(T, C) \times_{\mathrm{Fun}(T,E)} \mathrm{Fun}(T, D)$$

induced by the identification $(f \circ -) \circ (\mathrm{pr}_1 \circ -) = (g \circ -) \circ (\mathrm{pr}_2 \circ -)$ is an equivalence.

Proof. By Proposition 1.6.3, it suffices to show that the pullback square defining $C \times_E D$ remains a pullback square after applying $\mathrm{Map}(S, \mathrm{Fun}(T, -))$ for every category S. Under uncurrying, its four corners are identified with

$$\mathrm{Map}(S \times T, C \times_E D), \quad \mathrm{Map}(S \times T, C), \quad \mathrm{Map}(S \times T, D), \quad \mathrm{Map}(S \times T, E).$$

By Lemma 1.7.7, the four functors become postcomposition with the original projections, f, and g. These comparisons respect the specified commutativity identification, as follows by uncurrying its defining postcomposition identification. We conclude that this square agrees with the one obtained by applying $\mathrm{Map}(S \times T, -)$ to the original pullback square, including its commutativity data. So the resulting square is a pullback by Proposition 1.6.3. □

Lemma 1.7.12. *For categories C, D and E, there is a preferred equivalence*

$$\mathrm{Fun}(C, \mathrm{Fun}(D, E)) \xrightarrow{\sim} \mathrm{Fun}(C \times D, E)\,.$$

Proof. We define the preferred functor as the curried functor of the following composite:

$$\mathrm{Fun}(C, \mathrm{Fun}(D, E)) \times C \times D \xrightarrow{\mathrm{ev} \times \mathrm{id}_D} \mathrm{Fun}(D, E) \times D \xrightarrow{\mathrm{ev}} E.$$

To construct an inverse, it will by currying suffice to construct a functor $\mathrm{Fun}(C \times D, E) \times C \to \mathrm{Fun}(D, E)$, which we may take to be the curried functor of the following composite

$$\mathrm{Fun}(C \times D, E) \times C \times D \xrightarrow{\mathrm{ev}} E.$$

By construction, these two functors are compatible with the chain of equivalences

$$\begin{aligned}\mathrm{Map}(T, \mathrm{Fun}(C, \mathrm{Fun}(D, E))) &\simeq \mathrm{Map}(T \times C, \mathrm{Fun}(D, E)) \\ &\simeq \mathrm{Map}((T \times C) \times D, E) \\ &\simeq \mathrm{Map}(T \times (C \times D), E) \\ &\simeq \mathrm{Map}(T, \mathrm{Fun}(C \times D, E)) \,.\end{aligned}$$

This shows that applying $\mathrm{Map}(T, -)$ to the preferred functor is an equivalence for all T, hence that the preferred functor is an equivalence. □

Lemma 1.7.13. *For a category C, the functor* $\mathrm{ev}_* \colon \mathrm{Fun}(*, C) \to C$ *is an equivalence.*

Proof. We leave it to the reader to verify that an inverse is given by currying the equivalence $C \times * \xrightarrow{\sim} C$. □

Lemma 1.7.14. *For a category C, the functor category* $\mathrm{Fun}(\emptyset, C)$ *is contractible, i.e.* $\mathrm{Fun}(\emptyset, C) \xrightarrow{\sim} *$.

Proof. For every category T, there are equivalences

$$\mathrm{Map}(T, \mathrm{Fun}(\emptyset, C)) \simeq \mathrm{Map}(T \times \emptyset, C) \simeq \mathrm{Map}(\emptyset, C) \simeq *,$$

using Lemma 1.5.1 and the initial-category axiom. Hence $\mathrm{Fun}(\emptyset, C) \to *$ induces an equivalence on mapping animae for every T, and Proposition 1.4.11 applies. □

Lemma 1.7.15. *For categories C, D and E, restriction along the inclusions $i_C \colon C \hookrightarrow C \sqcup D$ and $i_D \colon D \hookrightarrow C \sqcup D$ induces an equivalence*

$$(i_C^*, i_D^*) \colon \mathrm{Fun}(C \sqcup D, E) \xrightarrow{\sim} \mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E).$$

Proof. For every category T, there are equivalences

$$\begin{aligned}\mathrm{Map}(T, \mathrm{Fun}(C \sqcup D, E)) &\simeq \mathrm{Map}(T \times (C \sqcup D), E) \\ &\simeq \mathrm{Map}((T \times C) \sqcup (T \times D), E) \\ &\simeq \mathrm{Map}(T \times C, E) \times \mathrm{Map}(T \times D, E) \\ &\simeq \mathrm{Map}(T, \mathrm{Fun}(C, E)) \times \mathrm{Map}(T, \mathrm{Fun}(D, E)) \\ &\simeq \mathrm{Map}(T, \mathrm{Fun}(C, E) \times \mathrm{Fun}(D, E)),\end{aligned}$$

using distributivity, the coproduct axiom, and Proposition 1.4.5. This composite is induced by (i_C^*, i_D^*): after uncurrying, its components are restriction along $T \times i_C$ and $T \times i_D$, which correspond under Corollary 1.6.17 to the two coproduct inclusions. The result follows from Proposition 1.4.11. □

1.7.2 Mapping animae and functor categories

We now compare the mapping anima with the functor category. Currying mapping-anima evaluation defines a canonical functor

$$\iota_{C,D} := (\mathrm{ev}_{\mathrm{Map}})_c \colon \mathrm{Map}(C, D) \longrightarrow \mathrm{Fun}(C, D).$$

The following result identifies $\mathrm{Map}(C, D)$ with the collection of objects of $\mathrm{Fun}(C, D)$.

Proposition 1.7.16. *For any categories C and D, there is a preferred equivalence*

$$\mathrm{Map}(C, D) \xrightarrow{\sim} \mathrm{Fun}(C, D)^{\simeq}.$$

Under this equivalence, the canonical functor $\iota_{\mathrm{Fun}(C,D)} \colon \mathrm{Fun}(C, D)^{\simeq} \to \mathrm{Fun}(C, D)$ is identified with $\iota_{C,D}$.

Proof. The universal property of functor categories gives equivalences

$$\mathrm{Fun}(C, D)^{\simeq} = \mathrm{Map}(*, \mathrm{Fun}(C, D)) \simeq \mathrm{Map}(* \times C, D) \simeq \mathrm{Map}(C, D).$$

The inverse of their composite is the desired equivalence. To check compatibility with the canonical functors, it suffices to uncurry. Both resulting functors $\mathrm{Map}(C, D) \times C \to D$ are identified with mapping-anima evaluation. □

Proposition 1.7.17. *For any anima X, postcomposition with $\iota_{C,D}$ induces an equivalence*

$$(\iota_{C,D})_* \colon \mathrm{Map}(X, \mathrm{Map}(C, D)) \xrightarrow{\sim} \mathrm{Map}(X, \mathrm{Fun}(C, D)).$$

Proof. By the preceding proposition, this is the universal property of the groupoid core of $\mathrm{Fun}(C, D)$ from Corollary 1.4.13. □

1.8 Pushout squares

Using mapping animae, we may dualize the definition of a pullback square of categories:

Definition 1.8.1 (Pushout square)**.** A commutative square of categories

$$\begin{array}{ccc} C' & \xrightarrow{u} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{v} & D \end{array}$$

is called a *pushout square* (or *cocartesian*) if, for every category E, the induced square

$$\begin{array}{ccc} \mathrm{Map}(D, E) & \xrightarrow{v^*} & \mathrm{Map}(D', E) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \mathrm{Map}(C, E) & \xrightarrow{u^*} & \mathrm{Map}(C', E) \end{array}$$

is a pullback square.

Remark 1.8.2. We do not assume in the present metatheory that arbitrary pushouts of categories exist. We do not need this as a metatheoretic axiom, although such pushouts exist in all intended models of our theory. We will eventually prove that the synthetic category Cat of small synthetic categories admits small colimits, which in particular provides pushouts of small categories.

Proposition 1.8.3. *A commutative square of categories*

$$\begin{array}{ccc} C' & \xrightarrow{u} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{v} & D \end{array}$$

is a pushout square if and only if, for every category E, the induced square

$$\begin{array}{ccc} \mathrm{Fun}(D,E) & \xrightarrow{v^*} & \mathrm{Fun}(D',E) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \mathrm{Fun}(C,E) & \xrightarrow{u^*} & \mathrm{Fun}(C',E) \end{array}$$

is a pullback square.

Proof. For any category T, the commutative square

$$\begin{array}{ccc} \mathrm{Map}(T,\mathrm{Fun}(D,E)) & \xrightarrow{v^*} & \mathrm{Map}(T,\mathrm{Fun}(D',E)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \mathrm{Map}(T,\mathrm{Fun}(C,E)) & \xrightarrow{u^*} & \mathrm{Map}(T,\mathrm{Fun}(C',E)) \end{array}$$

is equivalent to the commutative square

$$\begin{array}{ccc} \mathrm{Map}(T\times D,E) & \xrightarrow{v^*} & \mathrm{Map}(T\times D',E) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \mathrm{Map}(T\times C,E) & \xrightarrow{u^*} & \mathrm{Map}(T\times C',E) \end{array}$$

which in turn is equivalent to the square

$$\begin{array}{ccc} \mathrm{Map}(D,\mathrm{Fun}(T,E)) & \xrightarrow{v^*} & \mathrm{Map}(D',\mathrm{Fun}(T,E)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \mathrm{Map}(C,\mathrm{Fun}(T,E)) & \xrightarrow{u^*} & \mathrm{Map}(C',\mathrm{Fun}(T,E)). \end{array}$$

The latter is a pullback square for all T and E if and only if it is so for all E in the special case where $T = *$ since there is a canonical equivalence $E \xrightarrow{\sim} \mathrm{Fun}(*,E)$. On the other hand, the first square is a pullback square for all T if and only if

$$\begin{array}{ccc} \mathrm{Fun}(D,E) & \xrightarrow{v^*} & \mathrm{Fun}(D',E) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f')^*} \\ \mathrm{Fun}(C,E) & \xrightarrow{u^*} & \mathrm{Fun}(C',E) \end{array}$$

is a pullback square. This proves the proposition. □

Lemma 1.8.4 (Pasting lemma for pushout squares)**.** *Consider a commutative diagram*

$$\begin{array}{ccccc} C_1 & \xrightarrow{g_1} & C_2 & \xrightarrow{g_2} & C_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ D_1 & \xrightarrow[h_1]{} & D_2 & \xrightarrow[h_2]{} & D_3. \end{array}$$

If the left-hand square is a pushout square, then the right-hand square is a pushout square if and only if the outer rectangle is a pushout square.

Proof. This follows immediately by applying Lemma 1.6.11 to the commutative diagrams

$$\begin{array}{ccccc} \mathrm{Map}(D_3, E) & \xrightarrow{h_2^*} & \mathrm{Map}(D_2, E) & \xrightarrow{h_1^*} & \mathrm{Map}(D_1, E) \\ {\scriptstyle f_3^*}\downarrow & & \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle f_1^*} \\ \mathrm{Map}(C_3, E) & \xrightarrow[g_2^*]{} & \mathrm{Map}(C_2, E) & \xrightarrow[g_1^*]{} & \mathrm{Map}(C_1, E). \end{array}$$

for each E. □

1.9 Categories over a base

We will frequently consider categories equipped with a functor to a fixed category S. We introduce the corresponding notions of functors and natural transformations over S.

Definition 1.9.1 (Categories and functors over a base)**.** A *category over* S is a category C equipped with a functor $p\colon C \to S$. A *functor over* S from $p\colon C \to S$ to $q\colon D \to S$ consists of a functor $f\colon C \to D$ and a specified identification $qf = p$. We may sometimes also refer to such a functor as a *relative functor* if we wish to leave the base implicit. We will generally display such relative functors as commutative triangles

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S. & \end{array}$$

Definition 1.9.2. Let S be a category and let $p\colon C \to S$ and $q\colon D \to S$ be categories over S. We define the category $\mathrm{Fun}_S(C, D)$ of relative functors over S as the following pullback:

$$\begin{array}{ccc} \mathrm{Fun}_S(C, D) & \longrightarrow & \mathrm{Fun}(C, D) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle g_*} \\ * & \xrightarrow{f} & \mathrm{Fun}(C, S). \end{array}$$

The objects of $\mathrm{Fun}_S(C, D)$ are precisely the relative functors from C to D over S.

We write $\mathrm{Map}_S(C, D)$ for the groupoid core of $\mathrm{Fun}_S(C, D)$, which fits into a pullback square of the form

$$\begin{array}{ccc} \mathrm{Map}_S(C, D) & \longrightarrow & \mathrm{Map}(C, D) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle g_*} \\ * & \xrightarrow{f} & \mathrm{Map}(C, S). \end{array}$$

We may sometimes also write $\mathrm{Map}_{/S}(C, D)$ for this anima.

A *natural transformation over* S is a morphism in $\mathrm{Fun}_S(C, D)$. Given two relative functors $f, g\colon C \to D$, with structural identifications $\alpha : qf = p$ and $\beta : qg = p$, such a relative natural transformation consists of a natural transformation $\eta\colon f \to g$ together with the identification

$$\beta \circ (q\eta) = \alpha.$$

Similarly, *identifications over* S are identifications in the anima $\mathrm{Map}_S(C, D)$.

Definition 1.9.3. A functor over S is an *equivalence over* S if it admits an inverse over S, with identifications of the two composites with the identity functors over S.

Identity functors over S use the unitality of composition, while composition of functors over S uses the associativity of composition. Whiskering and composition of natural transformations likewise preserve their identifications over S. Unitality and associativity of functor composition of S is a consequence of the triangle and pentagon laws for composition of absolute functors. In particular, note that by passing to categories over S, we loose some amount of coherences.

Given a functor $p\colon S \to T$ and a category $C \to T$, its *base change* along p is the pullback $p^*C = C \times_T S \to S$. This operation can be extended to functors and natural transformations over T:

Construction 1.9.4. Let $p\colon S \to T$ be a functor. For categories C and D over T, we obtain an induced functor

$$p^*\colon \mathrm{Fun}_T(C, D) \to \mathrm{Fun}_S(C \times_T S, D \times_T S)$$

by applying pullback universality to a functor $h\colon C \to D$ together with its specified identification over T. This gives a functor $h \times_T S\colon C \times_T S \to D \times_T S$ over S. Applying the same construction to the universal functor parameterized by $\mathrm{Fun}_T(C, D)$ and currying gives the displayed functor, including its action on natural transformations and their identifications.

The category $S \xrightarrow{\mathrm{id}_S} S$ is terminal among categories over S, in the sense that we have

$$\mathrm{Fun}_S(C, S) \xrightarrow{\sim} *$$

for every category C over S. Products over S are computed by the corresponding absolute pullbacks: for categories C, D over S and any $X \to S$, the universal property of pullbacks implies that the two projections induce an equivalence

$$\mathrm{Fun}_S(X, C \times_S D) \xrightarrow{\sim} \mathrm{Fun}_S(X, C) \times \mathrm{Fun}_S(X, D).$$

Indeed, a term on either side consists of two functors out of X into C and D with their specified identifications to its structural functor $X \to S$; the same pullback description applies to natural transformations and their identifications.

1.10 Exercises

Exercise 1.10.1. Show that the product of categories is associative: for categories C, D and E there is a preferred equivalence $C \times (D \times E) \xrightarrow{\sim} (C \times D) \times E$.

Exercise 1.10.2. Show that the coproduct of categories is associative, commutative and unital: for categories C, D and E there are preferred equivalences

$$(C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Exercise 1.10.3. Given functors $f\colon C \to C'$, $f'\colon C' \to C''$, $g\colon D \to D'$ and $g'\colon D' \to D''$, show that there are identifications

$$\mathrm{id}_C \times \mathrm{id}_D = \mathrm{id}_{C\times D} \quad \text{and} \quad (f' \times g') \circ (f \times g) = (f' \circ f) \times (g' \circ g).$$

Exercise 1.10.4. Show that the fiber product of categories is commutative, associative and unital: for functors $C \to E$, $D \to E$ and $B \to E$, there are preferred equivalences

$$C \times_E D \xrightarrow{\sim} D \times_E C, \qquad B \times_E (C \times_E D) \xrightarrow{\sim} (B \times_E C) \times_E D, \qquad C \times_E E \xrightarrow{\sim} C.$$

Exercise 1.10.5. Consider categories C and D, and let $p_C\colon C \to *$ and $p_D\colon D \to *$ be their functors to the terminal category. Show that the functor $(\mathrm{pr}_1, \mathrm{pr}_2)\colon C \times_* D \to C \times D$ is an equivalence.

Exercise 1.10.6. Prove that the functors

$$g \circ -\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, D') \qquad \text{and} \qquad - \circ f\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C', D)$$

constructed in Construction 1.7.7 respect identity functors and composition of functors:

$$\begin{aligned} (\mathrm{id}_D \circ -) &= \mathrm{id}_{\mathrm{Fun}(C,D)}, & g' \circ (g \circ -) &= (g' \circ g) \circ - \\ (- \circ \mathrm{id}_C) &= \mathrm{id}_{\mathrm{Fun}(C,D)}, & (- \circ f) \circ f' &= - \circ (f \circ f'). \end{aligned}$$

Exercise 1.10.7. Show that also the operation of currying is functorial in E: given functors $f\colon E \times C \to D$ and $h\colon E' \to E$, there is a preferred identification between the currying of the functor $E' \times C \xrightarrow{h\times \mathrm{id}_C} E \times C \xrightarrow{f} D$ and the composite $E' \xrightarrow{h} E \xrightarrow{f_c} \mathrm{Fun}(C, D)$.

Exercise 1.10.8. Verify that the uncurrying equivalences in the proof of Lemma 1.7.11 respect the commutativity identifications.

Exercise 1.10.9. Let $f\colon C \to D$ and $g\colon D \to E$ be two embeddings of categories. Show that the composite $gf\colon C \to E$ is also an embedding.

Exercise 1.10.10. Consider a pullback square

$$\begin{array}{ccc} C' & \xrightarrow{g} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ D' & \xrightarrow{f} & D \end{array}$$

of categories and assume that v is an embedding. Show that also u is an embedding.

2 The internal structure of categories

In the previous chapter, our discussion of categories focused on their external theory: relating them via functors and equivalences, and constructing them via products or functor categories. Something we have mostly ignored so far is the *internal* structure that categories are expected to have: beyond objects, they should also have morphisms, and there should be a unital and associative composition law.

Rather than directly axiomatizing such internal structure, we will instead gain access to it by 'probing' a category under consideration with various test categories. We begin with morphisms and commutative diagrams in Section 2.1. The Segal axiom in Section 2.2 then provides composition of morphisms, and the Rezk axiom in Section 2.3 relates isomorphisms to identifications of objects.

Finally, we relate the primitive animae of our theory to the structural notion of a *groupoid*, a category in which every morphism is an isomorphism. The groupoid recognition axiom identifies these two points of entry. This also explains why the collection of objects $C^{\simeq}$ introduced in the previous chapter is called the groupoid core of C: it remembers the objects of C and their isomorphisms, while forgetting the noninvertible morphisms.

2.1 Morphisms and diagrams

In the case of morphisms, we use the following test category.

Axiom C.1 (Walking morphism). There is a category $[1]$ equipped with objects $0, 1 \colon * \to [1]$. Moreover, the functors $0, 1 \colon * \to [1]^{\simeq}$ induce an equivalence

$$\langle 0, 1 \rangle \colon \ * \sqcup * \xrightarrow{\sim} [1]^{\simeq}.$$

In particular, $* \sqcup *$ is an anima, since it is equivalent to the anima $[1]^{\simeq}$.

By substitution along $p_\Gamma \colon \Gamma \to *$, the two displayed objects also give terms

$$\Gamma \vdash 0, 1 : [1]$$

in every categorical context Γ.

The category $[1]$ is called the *walking morphism* and should be thought of as the poset $\{0 \leq 1\}$.

Let us introduce some standard terminology regarding functors out of $[1]$.

Definition 2.1.1 (Morphisms). A *morphism* in a category C is defined as a functor $f\colon [1] \to C$, or equivalently as an object of the arrow category $\mathrm{Fun}([1], C)$.

Every morphism $f\colon [1] \to C$ has a *source/domain* $f(0)\colon * \to C$ and a *target/codomain* $f(1)\colon * \to C$, obtained by precomposition with $0, 1\colon * \to [1]$. If f is a morphism in C, and x and y are objects of C, we will often write $f\colon x \to y$ to indicate that f comes equipped with identifications $x = f(0)$ and $y = f(1)$, which we will mostly leave implicit in the notation. In this case we say that *f is a morphism from x to y*.

We write $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$ for the source and target functors, given by evaluation at 0 and 1, cf. Construction 1.7.6.

Definition 2.1.2 (Natural transformations). If C and D are categories, we define a *natural transformation of functors $C \to D$* to be a morphism in $\mathrm{Fun}(C, D)$. By (un)currying, this may equivalently be encoded as a functor $[1] \times C \to D$.

Definition 2.1.3 (Arrow category/hom anima). For a category C, we define its *arrow category* as

$$\mathrm{Ar}(C) \quad := \quad \mathrm{Fun}([1], C).$$

Given absolute objects x and y of C, we define the *hom anima* $C(x, y)$ via the following pullback square:

$$\begin{array}{ccc} C(x,y) & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ * & \xrightarrow{(x,y)} & C \times C. \end{array}$$

At this point, the pullback defines a category. We call it the hom anima in anticipation of Corollary 4.3.5, where it will be shown to be an anima. Observe that the absolute objects of $C(x, y)$ are triples (f, α, β), where $f\colon x' \to y'$ is a morphism in C and $\alpha\colon x = x'$ and $\beta\colon y = y'$ are identifications in C. We will sometimes use the alternative notation $\mathrm{Hom}_C(x, y)$ for $C(x, y)$.

Convention 2.1.4 (Morphism expressions)**.** For functors $f, g\colon C \to D$, the notation

$$x : C \vdash \alpha(x) : f(x) \to g(x)$$

means a functor $H : C \times [1] \to D$ together with identifications $H(-, 0) = f$ and $H(-, 1) = g$. Currying gives its equivalent description as a functor $C \to \mathrm{Ar}(D)$ with the same endpoint comparisons. In the absolute case, we use $[1] \to D$ directly.

An identification between two morphism expressions with the same displayed source and target is an identification of their interval diagrams together with compatibility with the two endpoint comparisons. For absolute morphisms between fixed objects $a, b : D$, it is an identification in $D(a, b)$. In context $x : C$, with the same fixed endpoints, it is an identification between functors $C \to D(a, b)$. For general f and g that vary with x, it is simply a functor into D with the two endpoint compatibilities.

Construction 2.1.5 (From identifications to morphism expressions)**.** A primitive identification $\alpha\colon f = g$ determines a morphism expression from $f(x)$ to $g(x)$. Take the constant interval

$$C \times [1] \xrightarrow{\mathrm{pr}_1} C \xrightarrow{f} D,$$

with its canonical source identification, and with target identification given by α.

Construction 2.1.6. Let $f\colon C \to D$ be a functor and consider two absolute objects x and y of C. Then the induced functor $f_*\colon \mathrm{Ar}(C) \to \mathrm{Ar}(D)$ induces a functor

$$f\colon C(x,y) \to D(f(x), f(y))$$

on fibers.

Example 2.1.7 (Identity morphism). In a common categorical context Γ, a term $\Gamma \vdash x : C$ determines the *identity morphism* $\Gamma \vdash \mathrm{id}_x\colon x \to x$, defined by currying the composite

$$\Gamma \times [1] \xrightarrow{\mathrm{pr}_1} \Gamma \xrightarrow{x} C.$$

It comes with a specified identification from its uncurrying to this composite. For every functor $f\colon C \to D$, there is a preferred identification $\Gamma \vdash f(\mathrm{id}_x) = \mathrm{id}_{f(x)}$. In the absolute case $\Gamma = *$, this recovers the usual identity morphism $\mathrm{id}_x \in C(x,x)$.

Definition 2.1.8 (Slice categories). Let C be a category and let x be an absolute object of C. We define the *slice categories* $C_{/x}$ and $C_{x/}$ via the following two pullback diagrams:

$$\begin{array}{ccc} C_{/x} & \longrightarrow & \mathrm{Ar}(C) \\ {\scriptstyle p_x}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ C & \xrightarrow{\mathrm{id}_C \times x} & C \times C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_2} \\ * & \xrightarrow{x} & C. \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C_{x/} & \longrightarrow & \mathrm{Ar}(C) \\ {\scriptstyle q_x}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ C & \xrightarrow{x \times \mathrm{id}_C} & C \times C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_1} \\ * & \xrightarrow{x} & C. \end{array}$$

In particular, an object of the category $C_{/x}$ consists of a pair (y,u) where y is an object of C and $u\colon y \to x$ is a morphism in C. Similarly the objects of $C_{x/}$ are pairs (y,v) where now v is a morphism in C of the form $v\colon x \to y$.

Definition 2.1.9 (Initial/terminal object). An absolute object x in a category C is called *terminal* if the functor $p_x\colon C_{/x} \to C$ is an equivalence. Dually, x is called *initial* if $q_x\colon C_{x/} \to C$ is an equivalence.

Remark 2.1.10. Observe that the fiber of p_x over an absolute object y is the hom anima $C(y,x)$, using the pasting law for pullback squares. In particular, if x is terminal then $C(y,x)$ is contractible for every y. Dually, if x is initial then $C(x,y)$ is contractible for all y. The converse will be established later, once additional axioms have been introduced.

> **Axiom C.2.** The absolute object $0\colon * \to [1]$ is initial. The absolute object $1\colon * \to [1]$ is terminal.

Construction 2.1.11. As a consequence of Axiom C.2, we obtain the following two pullback squares:

$$\begin{array}{ccc} [1] & \xrightarrow{\overline{\min}} & \mathrm{Ar}([1]) \\ {\scriptstyle \sim}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ [1] & \xrightarrow{(\mathrm{const}_0,\mathrm{id})} & [1] \times [1] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} [1] & \xrightarrow{\overline{\max}} & \mathrm{Ar}([1]) \\ {\scriptstyle \sim}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ [1] & \xrightarrow{(\mathrm{id},\mathrm{const}_1)} & [1] \times [1]. \end{array}$$

Using $\mathrm{Ar}(C) = \mathrm{Fun}([1], C)$, the two top horizontal maps in these diagrams correspond under currying/uncurrying to functors

$$\min\colon [1] \times [1] \to [1] \qquad \text{and} \qquad \max\colon [1] \times [1] \to [1].$$

Lemma 2.1.12. *The maps* max *and* min *satisfy the identifications*

$$\begin{aligned} \max(x, 0) &= x = \max(0, x), & \max(x, 1) &= 1 = \max(1, x), \\ \min(x, 0) &= 0 = \min(0, x), & \min(x, 1) &= x = \min(1, x), \end{aligned}$$

in the sense that the following diagrams commute:

$$\begin{array}{ccccc} [1] & \xrightarrow{(\mathrm{id},0)} & [1] \times [1] & \xleftarrow{(0,\mathrm{id})} & [1] \\ & \searrow^{\mathrm{const}_0} & \downarrow{\scriptstyle \min} & \swarrow_{\mathrm{const}_0} & \\ & & [1] & & \end{array} \qquad \begin{array}{ccccc} [1] & \xrightarrow{(\mathrm{id},1)} & [1] \times [1] & \xleftarrow{(1,\mathrm{id})} & [1] \\ & \searrow & \downarrow{\scriptstyle \min} & \swarrow & \\ & & [1] & & \end{array}$$

$$\begin{array}{ccccc} [1] & \xrightarrow{(\mathrm{id},0)} & [1] \times [1] & \xleftarrow{(0,\mathrm{id})} & [1] \\ & \searrow & \downarrow{\scriptstyle \max} & \swarrow & \\ & & [1] & & \end{array} \qquad \begin{array}{ccccc} [1] & \xrightarrow{(\mathrm{id},1)} & [1] \times [1] & \xleftarrow{(1,\mathrm{id})} & [1] \\ & \searrow^{\mathrm{const}_1} & \downarrow{\scriptstyle \max} & \swarrow_{\mathrm{const}_1} & \\ & & [1]. & & \end{array}$$

Proof. The identifications $\min(x, 0) = 0$, $\min(x, 1) = x$, $\max(x, 0) = x$ and $\max(x, 1) = 1$ are immediate from the two pullback squares stated before the lemma. For the other four relations we use that $q_0^{-1}\colon [1] \xrightarrow{\sim} [1]_{0/}$ sends 0 to $\mathrm{id}_0\colon 0 \to 0$ and 1 to $\mathrm{can}\colon 0 \to 1$, while $p_1^{-1}\colon [1] \xrightarrow{\sim} [1]_{/1}$ sends 0 to $\mathrm{can}\colon 0 \to 1$ and 1 to $\mathrm{id}_1\colon 1 \to 1$. □

2.1.1 Commutative triangles

Apart from the poset $[1] = \{0 \leq 1\}$, which classifies morphisms, there should also be a poset $[2] = \{\overline{0} \leq \overline{1} \leq \overline{2}\}$ classifying *commutative triangles*.

Definition 2.1.13 (Walking commutative triangle). We define the *walking commutative triangle* as

$$[2] \quad := \quad \mathrm{Ar}([1]) \quad = \quad \mathrm{Fun}([1], [1]).$$

We define objects $\overline{0}, \overline{1}, \overline{2} : [2]$ by:

$$\overline{0} := \mathrm{id}_0, \qquad \overline{1} := \mathrm{can}, \qquad \overline{2} := \mathrm{id}_1,$$

where $\mathrm{can}\colon [1] \to [1]$ is the identity of $[1]$, regarded as a morphism $\mathrm{can}\colon 0 \to 1$ in $[1]$.

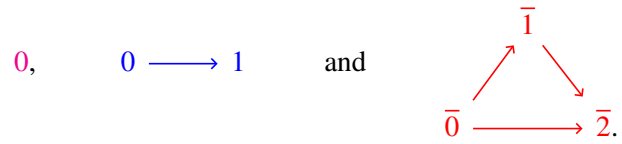
Construction 2.1.14. We construct three *face maps* $d_i\colon [1] \to [2]$ for $i = 0, 1, 2$:

- We set $d_0 := \overline{\max}\colon [1] \to \mathrm{Fun}([1], [1])$;
- We set $d_1 := p^*_{[1]}\colon [1] \to \mathrm{Fun}([1], [1])$;
- We set $d_2 := \overline{\min}\colon [1] \to \mathrm{Fun}([1], [1])$.

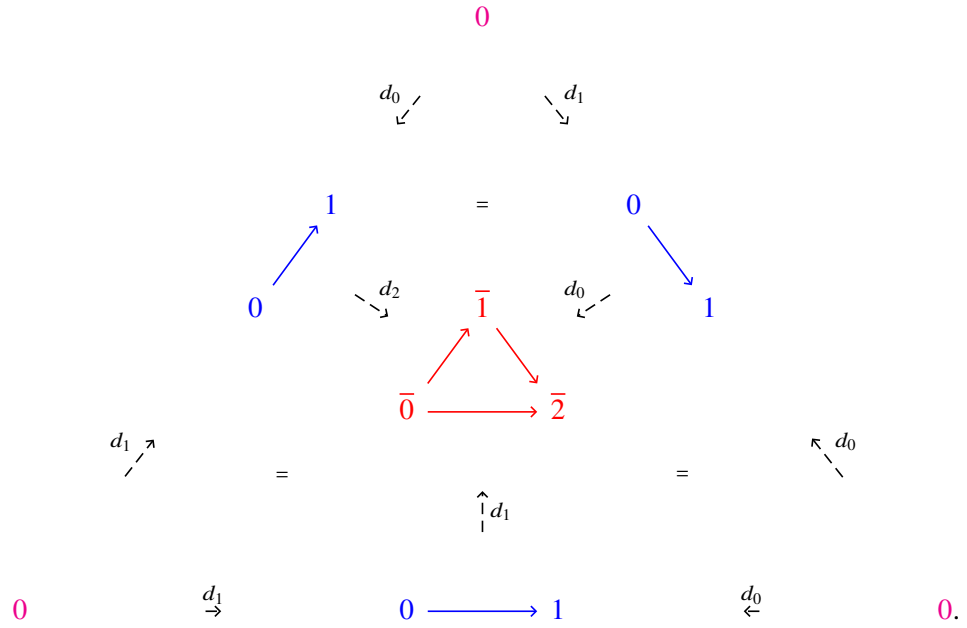
We also construct two *degeneracy maps* $s_i\colon [2] \to [1]$ for $i = 0, 1$:

- We set $s_0 := \mathrm{ev}_0 \colon \mathrm{Fun}([1],[1]) \to [1]$;
- We set $s_1 := \mathrm{ev}_1 \colon \mathrm{Fun}([1],[1]) \to [1]$.

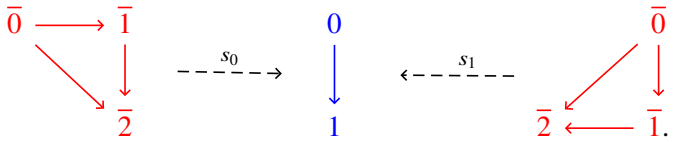
Convention 2.1.15. It will sometimes be useful to denote the terminal category by $[0]$, and to denote the maps $0 : * \to [1]$ and $1 : * \to [1]$ by $d_1 \colon [0] \to [1]$ and $d_0 \colon [0] \to [1]$, respectively. We will visualize the three categories $[0]$, $[1]$ and $[2]$ as follows:



The five face maps $d_0, d_1 \colon [0] \to [1]$ and $d_0, d_1, d_2 \colon [1] \to [2]$ and the relations between them may be displayed as follows:



The two degeneracy maps $s_0, s_1 \colon [2] \to [1]$ may be interpreted as the following 'projections':



Let us establish some of the relations implicit in this figure.

Lemma 2.1.16 (Simplicial identities). *There are vertex identifications*

$$d_2(0) = \overline{0} = d_1(0), \qquad d_2(1) = \overline{1} = d_0(0), \qquad d_0(1) = \overline{2} = d_1(1),$$
$$s_0(\overline{0}) = 0 = s_0(\overline{1}), \qquad s_0(\overline{2}) = 1, \quad s_1(\overline{0}) = 0, \qquad s_1(\overline{1}) = 1 = s_1(\overline{2}).$$

There are identifications of morphisms

$$\begin{aligned} s_0d_0 &= \mathrm{id}_{[1]}, \qquad & s_0d_1 &= \mathrm{id}_{[1]}, \qquad & s_0d_2 &= \mathrm{const}_0, \\ s_1d_0 &= \mathrm{const}_1, \qquad & s_1d_1 &= \mathrm{id}_{[1]}, \qquad & s_1d_2 &= \mathrm{id}_{[1]}\,. \end{aligned}$$

These identifications may be chosen so that the morphism identifications are compatible with the vertex identifications.

The last statement means that if we evaluate one of the six morphism identifications at one of the two vertices 0 and 1, the resulting identification agrees with the one obtained by combining the twelve vertex identifications. For example, in the case of evaluating $s_0d_0 = \mathrm{id}_{[1]}$ at 0 it asserts that the following square of identifications commutes:

$$\begin{array}{ccc} s_0(d_0(0)) & \xrightarrow{=} & s_0(\overline{1}) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathrm{id}_{[1]}(0) & \xrightarrow{=} & 0. \end{array}$$

Proof. Recall that s_0 and s_1 are evaluation at the source and target. The defining pullback squares for $d_2 = \overline{\min}$ and $d_0 = \overline{\max}$ therefore give

$$s_0d_2 = \mathrm{const}_0, \qquad s_1d_2 = \mathrm{id}_{[1]}, \qquad s_0d_0 = \mathrm{id}_{[1]}, \qquad s_1d_0 = \mathrm{const}_1\,.$$

The two remaining identifications follow from the definition of d_1 as the constant-arrow functor.

The vertex identifications for s_0 and s_1 are the specified source and target identifications of

$$\overline{0} = \mathrm{id}_0, \qquad \overline{1} = \mathrm{can}, \qquad \overline{2} = \mathrm{id}_1\,.$$

At the vertices, these constructions give identifications

$$\begin{aligned} d_2(0) &= \overline{0} = d_1(0), \\ d_2(1) &= \overline{1} = d_0(0), \\ d_0(1) &= \overline{2} = d_1(1). \end{aligned}$$

We choose them with the source and target comparisons just specified. This is possible because the categories of morphisms $0 \to 0$, $0 \to 1$, and $1 \to 1$ in $[1]$ are contractible, by initiality of 0 and terminality of 1.

Each of these six vertex comparisons is chosen with its specified source and target identifications. Applying s_0 or s_1 therefore gives the corresponding commuting square of identifications, as required. □

Definition 2.1.17 (Commutative triangle). Given a category C, we define a *commutative triangle in* C to be a functor $[2] \to C$. We define the category of commutative triangles in C as

$$\mathrm{Triang}(C) \quad := \quad \mathrm{Fun}([2], C).$$

The three functors $d_0, d_1, d_2 \colon [1] \to [2]$ induce three restriction functors

$$d_0^*, d_1^*, d_2^* \colon \mathrm{Triang}(C) \to \mathrm{Ar}(C).$$

We will frequently denote commutative triangles by

$$\sigma \quad = \quad \begin{array}{ccccc} & & y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ x & & \xrightarrow[h]{} & & z, \end{array}$$

where $f = d_2^*(\sigma)$, $g = d_0^*(\sigma)$ and $h = d_1^*(\sigma)$. We use the identifications of Lemma 2.1.16 whenever we restrict a triangle to its edges and vertices. Thus the three edges have specified common vertices, and the simplicial identities respect those identifications. For families of triangles, these comparisons are obtained by uncurrying and restricting the evaluation functor; compatibility follows by applying evaluation to the comparisons above. We will often leave commutative triangles unnamed, and only label their three edges. We may sometimes denote in the same manner triples of morphisms (f, g, h) that do *not* (or not necessarily) come from a commutative triangle, which will then be mentioned explicitly.

Definition 2.1.18. Given morphisms $f\colon x \to y$ and $g\colon y \to z$ in C, a *composite* of f and g is a morphism $h\colon x \to z$ that comes equipped with a commutative triangle σ of the above form:

$$\begin{array}{ccccc} & & y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ x & & \xrightarrow[h]{} & & z. \end{array}$$

We define the *category of composites* $\mathrm{Comp}_{f,g}(C)$ as the following pullback:

$$\begin{array}{ccc} \mathrm{Comp}_{f,g}(C) & \longrightarrow & \mathrm{Triang}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d_2^*, d_0^*)} \\ * & \xrightarrow{(f,g)} & \mathrm{Ar}(C) \times_C \mathrm{Ar}(C). \end{array}$$

Here the right vertical map is the map induced by the following commutative diagram of restriction maps:

$$\begin{array}{ccc} \mathrm{Triang}(C) & \xrightarrow{d_0^*} & \mathrm{Ar}(C) \\ {\scriptstyle d_2^*}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathrm{Ar}(C) & \xrightarrow[\mathrm{ev}_1]{} & C. \end{array}$$

Example 2.1.19. The two functors $s_0, s_1\colon [2] \to [1]$ induce functors

$$s_0^*, s_1^*\colon \mathrm{Ar}(C) \to \mathrm{Triang}(C),$$

so that every morphism $f\colon x \to y$ in C gives rise to two commutative diagrams $s_0^*(f)$ and $s_1^*(f)$ in C. As a consequence of the simplicial identities from Lemma 2.1.16, these commutative triangles have the following form:

$$\begin{array}{ccccc} & & x & & \\ & {\scriptstyle \mathrm{id}_x}\nearrow & & \searrow{\scriptstyle f} & \\ x & & \xrightarrow[f]{} & & y \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle \mathrm{id}_y} & \\ x & & \xrightarrow[f]{} & & y. \end{array}$$

In particular, these triangles exhibit f as a composite of id_x and f, as well as a composite of f and id_y.

2.1.2 Commutative squares

Throughout, C denotes a category.

Definition 2.1.20 (Commutative square). We define a *commutative square in C* to be a functor $[1] \times [1] \to C$. We define the category of commutative squares in C as

$$\mathrm{Sq}(C) \quad := \quad \mathrm{Fun}([1] \times [1], C).$$

We will often display commutative squares as follows:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ z & \xrightarrow[k]{} & w. \end{array}$$

In ordinary category theory, a commutative square consists of morphisms f, g, h and k as above that satisfy the condition that $h \circ f = k \circ g$. We will now formulate a synthetic analogue of this.

Definition 2.1.21. We define two functors $j_0, j_1 \colon [2] \to [1] \times [1]$ as

$$j_0 := (s_0, s_1) \qquad \text{and} \qquad j_1 := (s_1, s_0).$$

These two commutative triangles in $[1] \times [1]$ may be visualized as follows:

$$j_0 \quad = \quad \begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ & \searrow & \downarrow \\ & & (1,1), \end{array} \qquad\qquad j_1 \quad = \quad \begin{array}{ccc} (0,0) & & \\ \downarrow & \searrow & \\ (1,0) & \longrightarrow & (1,1). \end{array}$$

The simplicial relations $s_0 \circ d_1 = \mathrm{id}_{[1]} = s_1 \circ d_1$ provide an identification making the following square commute:

$$\begin{array}{ccc} [1] & \xrightarrow{d_1} & [2] \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle j_0} \\ [2] & \xrightarrow[j_1]{} & [1] \times [1]. \end{array} \tag{2.1}$$

Axiom D (Commutative square axiom). The commutative square (2.1) is a pushout square: For every category C, the commutative square

$$\begin{array}{ccc} \mathrm{Sq}(C) & \xrightarrow{j_1^*} & \mathrm{Triang}(C) \\ {\scriptstyle j_0^*}\downarrow & & \downarrow{\scriptstyle d_1^*} \\ \mathrm{Triang}(C) & \xrightarrow{d_1^*} & \mathrm{Ar}(C) \end{array}$$

is a pullback square.

Construction 2.1.22. In a common categorical context Γ, consider terms $\Gamma \vdash \sigma, \tau : \mathrm{Triang}(C)$ equipped with an identification $\Gamma \vdash \varphi\colon d_1^*(\sigma) = d_1^*(\tau)$. By Axiom D, these determine a term

$$\Gamma \vdash \square(\sigma, \tau, \varphi) : \mathrm{Sq}(C)$$

with specified identifications $\sigma = j_0^*(\square(\sigma, \tau, \varphi))$ and $\tau = j_1^*(\square(\sigma, \tau, \varphi))$. In the absolute case it is the commutative square

$$\square(\sigma, \tau, \varphi) \quad = \quad \begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & \overset{\tau}{\searrow} & \downarrow{\scriptstyle k} \\ z & \xrightarrow[h]{} & w \end{array}$$

We will sometimes just write $\square(\sigma, \tau)$ instead of $\square(\sigma, \tau, \varphi)$ when the identification φ is left implicit.

Construction 2.1.23. We construct functors $p_0, p_2\colon [1] \times [1] \to [2]$, pictorially represented by the following diagrams:

$$p_0 \quad = \quad \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \| & \searrow & \downarrow \\ 0 & \longrightarrow & 2, \end{array} \qquad\qquad p_2 \quad = \quad \begin{array}{ccc} 0 & \longrightarrow & 2 \\ \downarrow & \searrow & \| \\ 1 & \longrightarrow & 2. \end{array}$$

More precisely, $p_0 := \square(\mathrm{id}_{[2]}, d_1 s_0)$ is the commutative square determined by the two commutative triangles $\sigma = \mathrm{id}_{[2]}$ and $\tau = d_1 \circ s_0$, with identification

$$d_1^*(\sigma) = \mathrm{id}_{[2]} \circ d_1 = d_1 = d_1 \circ \mathrm{id}_{[1]} = d_1 \circ s_0 \circ d_1 = d_1^*(\tau).$$

Similarly, $p_2 := \square(d_1 s_1, \mathrm{id}_{[2]})$ is the square determined by $\sigma = d_1 \circ s_1$ and $\tau = \mathrm{id}_{[2]}$, with analogous identification between $d_1^*(\sigma)$ and $d_1^*(\tau)$.

By construction these two maps satisfy the following two relations:

$$p_0 \circ j_0 = \mathrm{id}_{[2]}, \qquad p_0 \circ j_1 = d_1 \circ s_0, \qquad p_2 \circ j_0 = d_1 \circ s_1, \qquad p_2 \circ j_1 = \mathrm{id}_{[2]}\,.$$

Lemma 2.1.24. *We have the relations*

$$\square(s_0, s_1) = \mathrm{pr}_1 \qquad \textit{and} \qquad \square(s_1, s_0) = \mathrm{pr}_2$$

of functors $[1] \times [1] \to [1]$.

Proof. It suffices to observe that these two projection maps have the correct restrictions to the triangles j_0 and j_1. □

Lemma 2.1.25. *We have*

$$\max = \square(s_1, s_1) \qquad \textit{and} \qquad \min = \square(s_0, s_0).$$

Proof. It suffices to check that max and min restrict correctly along j_0 and j_1, and this follows directly by unwinding the definitions of $\square(s_1, s_1)$ and $\square(s_0, s_0)$ from Construction 2.1.22. □

Lemma 2.1.26. *We have relations*

$$j_0 p_0(x, y) = (\min(x, y), y), \qquad j_1 p_0(x, y) = (y, \min(x, y)),$$
$$j_0 p_2(x, y) = (y, \max(x, y)), \qquad j_1 p_2(x, y) = (\max(x, y), y).$$

Proof. Since $p_0 := \square(\mathrm{id}_{[2]}, d_1 s_0)$ and $p_2 := \square(d_1 s_1, \mathrm{id}_{[2]})$, we have

$$s_0 p_0 = \square(s_0, s_0 d_1 s_0) = \min, \qquad s_1 p_0 = \square(s_1, s_1 d_1 s_0) = \square(s_1, s_0) = \mathrm{pr}_2$$
$$s_0 p_2 = \square(s_0 d_1 s_1, s_0) = \square(s_1, s_0) = \mathrm{pr}_2, \qquad s_1 p_2 = \square(s_1 d_1 s_1, s_1) = \max,$$

where we used Lemma 2.1.24. Using $j_0 = (s_0, s_1)$ and $j_1 = (s_1, s_0)$ we get the desired four relations. □

Lemma 2.1.27. *The map* $\langle 0, 1, 2\rangle \colon * \sqcup * \sqcup * \to [2]^{\simeq}$ *is an equivalence.*

Proof. Recall the functors $j_0 = (s_0, s_1) \colon [2] \to [1] \times [1]$ and $p_0 \colon [1] \times [1] \to [2]$ from Definition 2.1.21 and Construction 2.1.23. They satisfy $p_0 j_0 = \mathrm{id}_{[2]}$. Choose an inverse to the preferred equivalence

$$\kappa := (\mathrm{pr}_1^{\simeq}, \mathrm{pr}_2^{\simeq}) \colon ([1] \times [1])^{\simeq} \xrightarrow{\sim} ([1]^{\simeq})^2$$

from Lemma 1.4.17, and put $\bar{j}_0 := \kappa \circ j_0^{\simeq}$ and $\bar{p}_0 := p_0^{\simeq} \circ \kappa^{-1}$. Let

$$a \colon * \sqcup * \sqcup * \longrightarrow (* \sqcup *)^2$$

send $0, 1, 2$ to $(0, 0)$, $(0, 1)$, $(1, 1)$, respectively, and let

$$b \colon (* \sqcup *)^2 \longrightarrow * \sqcup * \sqcup *$$

send $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ to $0, 1, 0, 2$, respectively. Then $b \circ a = \mathrm{id}$. The defining identifications for groupoid-core functoriality and the displayed descriptions of j_0 and p_0 show that the following squares commute:

$$\begin{array}{ccc} * \sqcup * \sqcup * & \xrightarrow{a} & (* \sqcup *)^2 \\ \downarrow{\scriptstyle \langle 0,1,2\rangle} & & \downarrow{\scriptstyle \langle 0,1\rangle^2} \\ \simeq & \xrightarrow{\bar{j}_0} & ([1]^{\simeq})^2 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (* \sqcup *)^2 & \xrightarrow{b} & * \sqcup * \sqcup * \\ \downarrow{\scriptstyle \langle 0,1\rangle^2} & & \downarrow{\scriptstyle \langle 0,1,2\rangle} \\ ([1]^{\simeq})^2 & \xrightarrow{\bar{p}_0} & \simeq. \end{array}$$

The comparison for composition of groupoid-core functors, together with $p_0 j_0 = \mathrm{id}_{[2]}$, gives $\bar{p}_0 \bar{j}_0 = \mathrm{id}$. Thus this exhibits the map

$$\langle 0, 1, 2\rangle \colon * \sqcup * \sqcup * \longrightarrow [2]^{\simeq}$$

as a retract of

$$\langle 0, 1\rangle^2 \colon (* \sqcup *)^2 \longrightarrow ([1]^{\simeq})^2.$$

The latter map is an equivalence because it is the product of the equivalence in Axiom C.1 with itself. The claim follows from closure of equivalences under retracts (Lemma 1.3.9). □

Lemma 2.1.28. *For categories C and D, the maps $C^{\simeq} \to (C \sqcup D)^{\simeq}$ and $D^{\simeq} \to (C \sqcup D)^{\simeq}$ induce an equivalence $C^{\simeq} \sqcup D^{\simeq} \xrightarrow{\sim} (C \sqcup D)^{\simeq}$.*

Proof. Let $i_0 \colon * \to * \sqcup *$ and $i_1 \colon * \to * \sqcup *$ denote the inclusions of the first and second point, respectively. By combining Lemma 1.6.4 and Axiom B.5', there are pullback squares

$$\begin{array}{ccc} C^{\simeq} & \longrightarrow & (C \sqcup D)^{\simeq} \\ \downarrow & \lrcorner & \downarrow \\ (*)^{\simeq} & \xrightarrow{i_0^{\simeq}} & (* \sqcup *)^{\simeq} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} D^{\simeq} & \longrightarrow & (C \sqcup D)^{\simeq} \\ \downarrow & \lrcorner & \downarrow \\ (*)^{\simeq} & \xrightarrow{i_1^{\simeq}} & (* \sqcup *)^{\simeq}. \end{array}$$

It remains to show that the two maps $i_0^{\simeq}$ and $i_1^{\simeq}$ induce an equivalence

$$q\colon (*)^{\simeq} \sqcup (*)^{\simeq} \longrightarrow (* \sqcup *)^{\simeq}.$$

The defining identifications for groupoid-core functoriality give a commutative square

$$\begin{array}{ccc} (*)^{\simeq} \sqcup (*)^{\simeq} & \xrightarrow{q} & (* \sqcup *)^{\simeq} \\ {\scriptstyle \iota_* \sqcup \iota_*}\downarrow & & \downarrow{\scriptstyle \iota_{* \sqcup *}} \\ * \sqcup * & \xrightarrow[\mathrm{id}]{} & * \sqcup *. \end{array}$$

The left vertical functor is an equivalence by Proposition 1.4.15, and the right vertical functor is an equivalence by Proposition 1.4.15, since $* \sqcup *$ is an anima. Hence q is an equivalence by 2-out-of-3. The two pullback squares above, descent for coproducts, and base change along q now identify $C^{\simeq} \sqcup D^{\simeq}$ with $(C \sqcup D)^{\simeq}$. □

Remark 2.1.29. By Proposition 1.7.16 and Lemma 2.1.27, there are preferred equivalences

$$\mathrm{Map}([1],[1]) \simeq \mathrm{Fun}([1],[1])^{\simeq} \simeq [2]^{\simeq} \simeq * \sqcup * \sqcup *.$$

We may think of these three points as picking out the three morphisms $\mathrm{id}_0\colon 0 \to 0$, $\mathrm{id}_1\colon 1 \to 1$ and $\mathrm{can}\colon 0 \to 1$ of $[1]$. In particular, we see that $[1]$ only has three morphisms.

2.2 The Segal axiom

In this section, we will introduce the *Segal axiom*, which will provide unique compositions of morphisms in categories.

Axiom E (Segal axiom). For any category C, the functor

$$(d_2^*, d_0^*)\colon \mathrm{Triang}(C) \to \mathrm{Ar}(C) \times_C \mathrm{Ar}(C)$$

is an equivalence.

In particular, for every pair of composable morphisms $f\colon x \to y$ and $g\colon y \to z$ in C the category $\mathrm{Comp}_{f,g}(C)$ of composites of f and g is contractible, being defined as the fiber of the equivalence (d_2^*, d_0^*).

Notation 2.2.1. In a common categorical context Γ, composable morphisms $\Gamma \vdash f\colon x \to y$ and $\Gamma \vdash g\colon y \to z$ determine a term

$$\Gamma \vdash g \circ f\colon x \to z,$$

which we call their unique composite. It comes with a chosen 2-simplex σ and specified identifications $d_2^*(\sigma) = f$, $d_0^*(\sigma) = g$, and $d_1^*(\sigma) = g \circ f$.

Observation 2.2.2 (Exercise 2.6.1). Let C_0 and C_1 be categories and consider morphisms $f_0\colon x_0 \to y_0$ and $g_0\colon y_0 \to z_0$ in C_0 and morphisms $f_1\colon x_1 \to y_1$ and $g_1\colon y_1 \to z_1$ in C_1. Then the composite $(g_0, g_1) \circ (f_0, f_1)\colon (x_0, x_1) \to (z_0, z_1)$ in $C_0 \times C_1$ is given by $(g_0 \circ f_0, g_1 \circ f_1)$.

For fixed objects $x, y, z : C$, the definition of $g \circ f$ works for expressions with values in $C(x, y)$ and $C(y, z)$ and a common source. Thus composition is functorial in f and g. We may make this functoriality more explicit as follows:

Construction 2.2.3 (Composition functors). The Segal axiom gives an equivalence

$$q_C = (d_2^*, d_0^*) \colon \mathrm{Triang}(C) \longrightarrow \mathrm{Ar}(C) \times_C \mathrm{Ar}(C).$$

Here the identification joining the two short edges is induced by $d_2(1) = \overline{1} = d_0(0)$ from Lemma 2.1.16. The vertex identifications of that lemma make q_C a functor over C^3: on triangles we evaluate at $\overline{0}, \overline{1}, \overline{2}$, and on composable pairs we take the source and target of the first edge and the target of the second.

Choose an inverse j_C over C^3, together with an identification $\varepsilon_C \colon q_C j_C = \mathrm{id}$ over C^3. Composition is the functor

$$\mathrm{Ar}(C) \times_C \mathrm{Ar}(C) \xrightarrow{j_C} \mathrm{Triang}(C) \xrightarrow{d_1^*} \mathrm{Ar}(C).$$

Thus we first complete a composable pair (f, g) to a triangle, and then take its long edge. We denote this edge by $g \circ f$.

The vertex identifications of Lemma 2.1.16 also identify the endpoints of the long edge with the first and last vertices. Hence composition is a functor over $C \times C$. Passing to fibers over prescribed vertices x, y, z gives a functor

$$- \circ - \colon C(y, z) \times C(x, y) \longrightarrow C(x, z).$$

The source and target identifications of $g \circ f$ are obtained from these vertex comparisons followed by the corresponding components of ε_C. Different choices of the inverse give identified composition functors over $C \times C$.

Convention 2.2.4 (Composition of morphism expressions)**.** For functors $f, g, h \colon C \to D$, consider morphism expressions $\alpha(x) : f(x) \to g(x)$ and $\beta(x) : g(x) \to h(x)$ in context $x : C$. Their endpoint identifications define a functor $C \to \mathrm{Ar}(D) \times_D \mathrm{Ar}(D)$. Composing with the composition functor gives a morphism expression

$$\beta(x) \circ \alpha(x) : f(x) \to h(x).$$

This construction also acts on identifications of morphism expressions that retain the endpoints.

Lemma 2.2.5. *For a functor $F \colon C \to D$ and morphisms $f \colon x \to y$ and $g \colon y \to z$ in C, there is an identification*

$$F(g \circ f) = F(g) \circ F(f)$$

in $D(F(x), F(z))$. This identification is natural in the pair (g, f): it forms an identification over $D \times D$ between two functors $\mathrm{Ar}(C) \times_C \mathrm{Ar}(C) \to \mathrm{Ar}(D)$.

Proof. Consider the chosen triangle in C expressing $g \circ f$ as a composite. Applying F to this triangle produces a triangle in D with short edges $F(f), F(g)$. By the Segal axiom, this triangle is identified with the chosen triangle in D expressing $F(g) \circ F(f)$ as a composite of $F(f)$ and $F(g)$, in a way compatible with the two short-edge comparison identifications. The relation $F(g \circ f) = F(g) \circ F(f)$ now follows by passing to the long edges of these identified triangles. On source and target, this

identification is the restriction of the two a priori identifications on the short edges, and so it is compatible with the vertex identifications.

For naturality, we apply this argument to terms in the categorical context $\mathrm{Ar}(C) \times_C \mathrm{Ar}(C)$. □

Lemma 2.2.6 (Unitality). *For every morphism $f : x \to y$, there are identifications*

$$\mathrm{id}_y \circ f = f = f \circ \mathrm{id}_x$$

in $C(x, y)$, with its given endpoint identifications. They are natural in f: they define identifications of functors $\mathrm{Ar}(C) \to \mathrm{Ar}(C)$ over $C \times C$.

Proof. We use the degenerate triangle

$$s_0^*(f) \quad = \quad \begin{array}{ccc} & & x \\ & \overset{\mathrm{id}_x}{\nearrow} & \\ x & & \underset{f}{\searrow} \\ & \underset{f}{\longrightarrow} & y, \end{array}$$

obtained by restricting f along s_0. Its short edges are (id_x, f) and its long edge is f. These identifications respect the three vertices: after uncurrying, their compatibility is given by the vertex identifications of Lemma 2.1.16.

The chosen triangle $j_C(\mathrm{id}_x, f)$ has the same short edges, with comparison given by ε_C. The Segal equivalence therefore identifies these two triangles, with this comparison on their short edges. Restriction to the long edge gives an identification

$$f \circ \mathrm{id}_x = d_1^* j_C(\mathrm{id}_x, f) = d_1^* s_0^*(f) = f.$$

This identification also respects the prescribed source and target: the interchange law allows us to compare endpoints before or after comparing the triangles. The short-edge identifications of the degenerate triangle then cancel with their inverses, leaving the vertex comparisons followed by the components of ε_C. These are exactly the source and target identifications used to define composition.

The same argument with s_1 gives $\mathrm{id}_y \circ f = f$.

For naturality in f, we apply both constructions to the universal arrow on $\mathrm{Ar}(C)$, working with term expressions in context $\mathrm{Ar}(C)$. (In terms of actual functors, this translates to restriction and lifting through the Segal equivalence.) This results in the required identifications of functors over $C \times C$. □

Proposition 2.2.7 (Associativity, cf. [RS17, Proposition 5.9]). *For composable morphisms $f : x \to y$, $g : y \to z$, and $h : z \to w$, there is an identification*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

in $C(x, w)$. These identifications are natural in the composable triple: they form an identification over $C \times C$ between the two composition functors from $\mathrm{Ar}(C) \times_C \mathrm{Ar}(C) \times_C \mathrm{Ar}(C)$ to $\mathrm{Ar}(C)$.

Proof. As in the proof of unitality, we establish the desired identification (including the source and target compatibilities) for term expressions $f : x \to y$, $g : y \to z$ and $h : z \to w$ in an arbitrary categorical context. Taking this context to be $\mathrm{Ar}(C) \times_C \mathrm{Ar}(C) \times_C \mathrm{Ar}(C)$ then gives the final claim.

Consider the commutative triangle presenting the composite of f and g, which has $g \circ f$ as its long edge. Consider also the commutative triangle presenting the composite of id_x and $g \circ f$. Using the unitality of composition from Lemma 2.2.6, we may identify its long edge with $g \circ f$ in $C(x, z)$. Using the commutative square axiom, we may glue these two triangles into a commutative square

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle \mathrm{id}_x}\downarrow & \searrow^{g\circ f} & \downarrow{\scriptstyle g} \\ x & \xrightarrow[g\circ f]{} & z. \end{array}$$

The pullback property in the commutative square axiom gives identifications of its two triangle restrictions with the chosen triangles, compatible with the specified identification of their long edges. Since this last identification lies in $C(x, z)$, the endpoint comparisons agree at the two corners on the diagonal. At the other two corners, compatibility follows from the corresponding triangle. To identify these with the corners of the displayed square, recall that $j_0 = (s_0, s_1)$ and $j_1 = (s_1, s_0)$. The simplicial identities identify their boundary edges with the coordinate edges of $[1] \times [1]$; for example, $j_0 d_2 = (0, \mathrm{id}_{[1]})$ and $j_1 d_2 = (\mathrm{id}_{[1]}, 0)$. The vertex compatibilities of Lemma 2.1.16 give the specified comparisons under both product projections, so the product universal property identifies the corner comparisons as well. Thus all four edges have their prescribed endpoints, including the identification of the left side with id_x.

In a similar fashion, gluing the triangles for (g, h) and $(h \circ g, \mathrm{id}_w)$ along the unitality identification in $C(y, w)$ gives a commutative square

$$\begin{array}{ccc} y & \xrightarrow{h\circ g} & w \\ {\scriptstyle g}\downarrow & \searrow^{h\circ g} & \downarrow{\scriptstyle \mathrm{id}_w} \\ z & \xrightarrow[h]{} & w, \end{array}$$

again with the displayed endpoint identifications.

Under the equivalence $\mathrm{Sq}(C) \simeq \mathrm{Ar}(\mathrm{Ar}(C))$, we may regard these commutative squares as morphisms $\mathrm{id}_x \to g$ and $g \to \mathrm{id}_w$ in $\mathrm{Ar}(C)$. Their composition in $\mathrm{Ar}(C)$ corresponds to a commutative square in C of the form

$$\begin{array}{ccc} x & \xrightarrow{(h\circ g)\circ f} & w \\ {\scriptstyle \mathrm{id}_x}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_w} \\ x & \xrightarrow[h\circ(g\circ f)]{} & w. \end{array}$$

Here the top and bottom edges are identified with $(h \circ g) \circ f$ and $h \circ (g \circ f)$ by applying Lemma 2.2.5 to the two evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Ar}(C) \to C$. That lemma also ensures that these identifications retain the prescribed endpoints.

The two triangle restrictions of this square identify its diagonal with the composites along the top and right, and along the left and bottom, respectively. Indeed, the Segal equivalence compares each triangle with the chosen triangle for its two short edges. By Lemma 2.2.6, these composites are identified with the top and bottom edges themselves in $C(x, w)$. We obtain the required identification

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

All constructions use restriction, the two axioms' universal properties, and composition in $\mathrm{Ar}(C)$, so they apply to the universal composable triple as asserted at the start. □

Observe that another way of formulating the Segal axiom, Axiom E, is that the commutative square

$$\begin{array}{ccc} \mathrm{Triang}(C) & \xrightarrow{d_2^*} & \mathrm{Ar}(C) \\ {\scriptstyle d_0^*}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathrm{Ar}(C) & \xrightarrow{\mathrm{ev}_0} & C \end{array}$$

is a pullback square. This has the following corollary:

Corollary 2.2.8. *For every category C, the commutative squares*

$$\begin{array}{ccc} \mathrm{Triang}(C) & \xrightarrow{p_0^*} & \mathrm{Sq}(C) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ C & \xrightarrow{(p_{[1]})^*} & \mathrm{Fun}([1]\times\{0\}, C) \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \mathrm{Triang}(C) & \xrightarrow{p_2^*} & \mathrm{Sq}(C) \\ {\scriptstyle \mathrm{ev}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ C & \xrightarrow{(p_{[1]})^*} & \mathrm{Fun}([1]\times\{1\}, C) \end{array}$$

are pullback squares.

Proof. We will prove the claim for the left square and leave the analogous proof of the right square to the reader. Note that this square can be written as the outer square in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Triang}(C) & \xrightarrow{p_0^*} & \mathrm{Sq}(C) \\ {\scriptstyle d_1^*}\downarrow & & \downarrow{\scriptstyle j_1^*} \\ \mathrm{Ar}(C) & \xrightarrow{s_0^*} & \mathrm{Triang}(C) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & {\scriptstyle d_2^*}\downarrow \\ C & \xrightarrow{(p_{[1]})^*} & \mathrm{Ar}(C), \end{array}$$

and so it will suffice that each of these two squares is a pullback square. The first one follows from applying the pasting law of pullback squares to the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Triang}(C) & \xrightarrow{p_0^*} & \mathrm{Sq}(C) & \xrightarrow{j_0^*} & \mathrm{Triang}(C) \\ {\scriptstyle d_1^*}\downarrow & \lrcorner & \downarrow{\scriptstyle j_1^*} & \lrcorner & \downarrow{\scriptstyle d_1^*} \\ \mathrm{Ar}(C) & \xrightarrow{s_0^*} & \mathrm{Triang}(C) & \xrightarrow{d_1^*} & \mathrm{Ar}(C). \end{array}$$

Here the outer square is evidently a pullback, and the right square is a pullback by the commutative square axiom, Axiom D. The second one similarly follows from applying the pasting law of pullback squares to the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Ar}(C) & \xrightarrow{s_0^*} & \mathrm{Triang}(C) & \xrightarrow{d_0^*} & \mathrm{Ar}(C) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & \lrcorner & {\scriptstyle d_2^*}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_0} \\ C & \xrightarrow{(p_{[1]})^*} & \mathrm{Ar}(C) & \xrightarrow{\mathrm{ev}_1} & C. \end{array}$$

Here the outer rectangle is again evidently a pullback square, and the right square is a pullback square by the Segal axiom, Axiom E. □

Remark 2.2.9. Informally, the previous corollary says that commutative triangles in C may equivalently be encoded in the following three ways:

(1) One might consider commutative triangles $[2] \to C$, which we will display as

$$\begin{array}{ccc} & y & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} \\ x & \xrightarrow{gf} & z. \end{array}$$

(2) One might consider commutative squares $[1] \times [1] \to C$ whose restriction to $[1] \times \{0\}$ is constant, which we will display as

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \| & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{gf} & z. \end{array}$$

(3) One might consider commutative squares $[1] \times [1] \to C$ whose restriction to $[1] \times \{1\}$ is constant, which we will display as

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle gf}\downarrow & & \downarrow{\scriptstyle g} \\ z & = & z. \end{array}$$

We now explain how the usual naturality relation follows from our definition of natural transformations. Let $\alpha\colon f \to g$ be a natural transformation between functors $f, g\colon C \to D$. For a morphism $u\colon a \to b$ in C, precomposition of the interval diagram of α $C \times [1] \to D$ with $u \times \mathrm{id}_{[1]}$ gives a commutative square

$$\begin{array}{ccc} f(a) & \xrightarrow{\alpha(a)} & g(a) \\ {\scriptstyle f(u)}\downarrow & & \downarrow{\scriptstyle g(u)} \\ f(b) & \xrightarrow{\alpha(b)} & g(b). \end{array}$$

Its two triangles have the same long edge. After taking composites, we write

$$\alpha(u) : g(u) \circ \alpha(a) = \alpha(b) \circ f(u)$$

for the resulting comparison in $D(f(a), g(b))$. Its endpoint identifications are those supplied by Lemma 2.1.16 and the comparison with the chosen composites in Construction 2.2.3.

The same construction gives the following interchange law.

Lemma 2.2.10 (Interchange for natural transformations)**.** *Let $F, G\colon C \to D$ and $u, v\colon B \to C$ be functors, and let $\alpha\colon F \to G$ and $\beta\colon u \to v$ be natural transformations. There is a commutative square in* $\mathrm{Fun}(B, D)$

$$\begin{array}{ccc} Fu & \xrightarrow{\alpha u} & Gu \\ {\scriptstyle F\beta}\downarrow & & \downarrow{\scriptstyle G\beta} \\ Fv & \xrightarrow[\alpha v]{} & Gv. \end{array}$$

This gives an identification in $\mathrm{Hom}_{\mathrm{Fun}(B,D)}(Fu, Gv)$ *of the form*

$$(G\beta) \circ (\alpha u) = (\alpha v) \circ (F\beta).$$

Proof. Let $A\colon [1]\times C \to D$ and $H\colon [1]\times B \to C$ encode α and β, respectively. The composite

$$[1]\times[1]\times B \xrightarrow{\mathrm{id}_{[1]}\times H} [1]\times C \xrightarrow{A} D$$

curries to a functor $[1]\times[1] \to \mathrm{Fun}(B, D)$. Its four edges, with the endpoint identifications supplied by A and H, are precisely the four transformations displayed above. Restricting to the two triangles, using the restriction identifications fixed above, gives the two composites with a common long edge and the prescribed source/target. This gives the displayed identification. □

Finally, the conversion of Construction 2.1.5 is compatible with composition of primitive identifications. For $\alpha\colon f = g$ and $\beta\colon g = h$, consider the constant triangle at f, with vertex identifications id_f, α, and $\beta \circ \alpha$. Its short edges represent α and β, using α to identify the second short edge with the constant interval at g. Its long edge represents $\beta \circ \alpha$. The Segal equivalence identifies this triangle with the chosen composite triangle, retaining the short-edge comparisons. Passing to the long edge therefore gives the required compatibility with the prescribed source/target, as in Construction 2.2.3.

2.3 The Rezk axiom

The notions of identity morphisms and composition of morphisms in categories, provided by the Segal axiom, lead to the notion of an *isomorphism* in a category:

Definition 2.3.1 (Isomorphisms). Consider a morphism $f\colon x \to y$ in a category C. We say that f is *invertible*, or that it is an *isomorphism*, if there are commutative triangles in C of the form

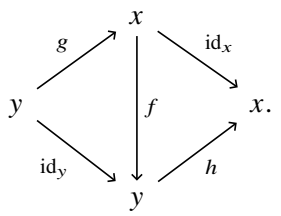
$$\begin{array}{ccc} y & \xrightarrow{g} & x \\ & \searrow^{\mathrm{id}_y} & \downarrow f \\ & & y \end{array} \qquad\qquad \begin{array}{ccc} x & \xrightarrow{f} & y \\ & \searrow^{\mathrm{id}_x} & \downarrow h \\ & & x. \end{array}$$

An invertible natural transformation is called a *natural isomorphism*. The *Rezk axiom* will relate invertible morphisms to the primitive identifications of objects from Definition 1.1.2. We start by defining the category $\mathrm{Iso}(C)$ of isomorphisms in C.

Definition 2.3.2. Given a category C, we define the category $\mathrm{Iso}(C)$ via the following pullback diagram:

$$\begin{array}{ccccc} \mathrm{Iso}(C) & \xrightarrow{\quad\pi_{\mathrm{Iso}}\quad} & \mathrm{Triang}(C)\times_{d_0,\mathrm{Ar}(C),d_2}\mathrm{Triang}(C) & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\Delta} \\ & & \mathrm{Triang}(C)\times\mathrm{Triang}(C) & \xrightarrow{d_0^*\times d_2^*} & \mathrm{Ar}(C)\times\mathrm{Ar}(C) \\ & & \downarrow{\scriptstyle d_1^*\times d_1^*} & & \\ C\times C & \xrightarrow{(x,y)\mapsto(\mathrm{id}_y,\mathrm{id}_x)} & \mathrm{Ar}(C)\times\mathrm{Ar}(C). & & \end{array}$$

Explicitly, an absolute object of $\mathrm{Iso}(C)$ is a commutative diagram in C of the form



We denote the composite functor at the top of the diagram by

$$\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Ar}(C).$$

Notice that by definition a morphism $f \in \mathrm{Ar}(C)$ is invertible in C if and only if it lifts to an absolute object of $\mathrm{Iso}(C)$. We may therefore think of the fiber of π_{Iso} over f as the category of 'proofs that f is invertible'. The following lemma shows that there is at most one such proof:

Lemma 2.3.3. *For a category C, the map $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Ar}(C)$ is an embedding.*

Proof. We use the unit and associativity comparisons over prescribed vertices established in Lemma 2.2.6 and Proposition 2.2.7. We need to show that the diagonal

$$\Delta \colon \mathrm{Iso}(C) \to \mathrm{Iso}(C) \times_{\mathrm{Ar}(C)} \mathrm{Iso}(C)$$

is an equivalence, or equivalently that the first projection

$$\mathrm{pr}_1 \colon \mathrm{Iso}(C) \times_{\mathrm{Ar}(C)} \mathrm{Iso}(C) \to \mathrm{Iso}(C)$$

is an equivalence. To this end, consider the two functors

$$\begin{aligned} \mathrm{Ar}(C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_0} \mathrm{Iso}(C) &\to \mathrm{Ar}(C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_1} \mathrm{Iso}(C), \qquad (g, f) \mapsto (f \circ g, f) \\ \mathrm{Iso}(C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_0} \mathrm{Ar}(C) &\to \mathrm{Iso}(C) \times_{\mathrm{ev}_0, C, \mathrm{ev}_0} \mathrm{Ar}(C), \qquad (f, h) \mapsto (f, h \circ f). \end{aligned}$$

We claim that these functors are equivalences. For the first functor, note that for $f \in \mathrm{Iso}(C)$ we may functorially extract both a left inverse f_l with an identification $f_l \circ f = \mathrm{id}_{s(f)}$ as well as a right inverse f_r with an identification $f \circ f_r = \mathrm{id}_{t(f)}$. Using associativity of composition, a left inverse to the first functor is then given by $(g', f) \mapsto (f_l \circ g', f)$, while a right inverse is given by $(g', f) \mapsto (f_r \circ g', f)$. One argues similarly for the second functor.

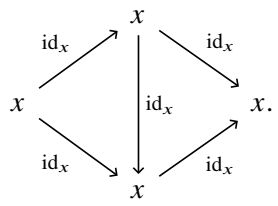
The claim now follows from the observation that the projection pr_1 is the left vertical map in the following pullback square, where the right vertical map is the fiber product over $\mathrm{Iso}(C)$ of the two equivalences just constructed, hence an equivalence itself:

$$\begin{array}{ccc} \mathrm{Iso}(C) \times_{\mathrm{Ar}(C)} \mathrm{Iso}(C) & \longrightarrow & \mathrm{Ar}(C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_0} \mathrm{Iso}(C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_0} \mathrm{Ar}(C) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & \lrcorner & \downarrow{\scriptstyle (g,f,h) \mapsto (f \circ g, f, h \circ f)} \\ \mathrm{Iso}(C) & \xrightarrow{f \mapsto (\mathrm{id}_y, f, \mathrm{id}_x)} & \mathrm{Ar}(C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_1} \mathrm{Iso}(C) \times_{\mathrm{ev}_0, C, \mathrm{ev}_0} \mathrm{Ar}(C). \end{array}$$

□

As one expects, the identity morphism of any term $\Gamma \vdash x : C$ is invertible in the same context:

Construction 2.3.4. We construct a functor $i\colon C \to \mathrm{Iso}(C)$ lifting the functor $1\colon C \to \mathrm{Ar}(C)$ given by $x \mapsto \mathrm{id}_x$. Informally, i sends x to the following commutative diagram in C:



More formally, restriction along the map $p_{[2]}\colon [2] \to [0]$ induces a functor $p_{[2]}^*\colon C \to \mathrm{Triang}(C)$, giving a functor $(p_{[2]}^*, p_{[2]}^*)\colon C \to \mathrm{Triang}(C) \times_{\mathrm{Ar}(C)} \mathrm{Triang}(C)$. Since the two relevant edges of this diagram are both the identity on x, this functor factors through $\mathrm{Iso}(C)$, providing the desired functor $i\colon C \to \mathrm{Iso}(C)$.

By construction the following diagram commutes:

$$\begin{array}{ccccc} C & \xrightarrow{i} & \mathrm{Iso}(C) & \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} & C \times C \\ & {\scriptstyle x\mapsto \mathrm{id}_x}\searrow & \downarrow{\scriptstyle \pi_{\mathrm{Iso}}} & \nearrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} & \\ & & \mathrm{Ar}(C). & & \end{array}$$

Axiom F (Rezk axiom). For any category C, the functor $i\colon C \to \mathrm{Iso}(C)$ is an equivalence.

Together with representability of identifications, the Rezk axiom identifies the anima of identifications between two objects with the category of isomorphisms between them:

Corollary 2.3.5. *For every category C and absolute objects x, y of C, there is an equivalence*

$$(x = y) \simeq * \times_{C\times C} \mathrm{Iso}(C),$$

where the maps defining the fiber product are $(x, y)\colon * \to C \times C$ and $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Iso}(C) \to C \times C$.

Proof. By Axiom B.7, the anima $(x = y)$ is the pullback of $x, y\colon * \to C$. Applying Lemma 1.6.13, we obtain a pullback square

$$\begin{array}{ccc} (x = y) & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ * & \xrightarrow[(x,y)]{} & C \times C. \end{array}$$

The Rezk axiom identifies C with $\mathrm{Iso}(C)$ through the equivalence i, and $(\mathrm{ev}_0, \mathrm{ev}_1)i = \Delta$. Replacing C by $\mathrm{Iso}(C)$ therefore identifies this pullback with the required fiber. □

Corollary 2.3.6. *For functors $f, g\colon C \to D$, there is an equivalence*

$$(f = g) \simeq * \times_{\mathrm{Fun}(C,D)\times\mathrm{Fun}(C,D)} \mathrm{Iso}(\mathrm{Fun}(C, D)),$$

where the maps defining the fiber product are (f, g) and $(\mathrm{ev}_0, \mathrm{ev}_1)$. Thus identifications of functors are equivalently natural isomorphisms of functors.

Proof. By Definition 1.7.3, currying identifies the anima $(f = g)$ with the anima of identifications between the corresponding absolute objects of $\mathrm{Fun}(C, D)$. Applying Corollary 2.3.5 to these objects gives the asserted equivalence. □

Corollary 2.3.7. *For a category C, the functor $p^*_{[1]} \colon C \to \mathrm{Ar}(C)$ is an embedding.*

Proof. This functor factors as $C \xrightarrow{i} \mathrm{Iso}(C) \xrightarrow{\pi_{\mathrm{Iso}}} \mathrm{Ar}(C)$. The first functor is an equivalence by the Rezk axiom, while the second one is an embedding by Lemma 2.3.3. The claim follows. □

2.4 Groupoids

A category is called a *groupoid* if all its morphisms are invertible. In the synthetic setting, we need to phrase this condition for arbitrary terms of the arrow category, as follows.

Definition 2.4.1 (Groupoid). We say that a category C is a groupoid if in an arbitrary categorical context, every morphism expression $f \colon x \to y$ in C is an isomorphism. Explicitly: for every category Γ and every functor $f \colon \Gamma \to \mathrm{Ar}(C)$, there exists a dashed arrow $\tilde{f}$ making the following diagram commute:

$$\begin{array}{ccc} & & \mathrm{Iso}(C) \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow {\scriptstyle \pi_{\mathrm{Iso}}} \\ \Gamma & \xrightarrow[f]{} & \mathrm{Ar}(C). \end{array}$$

The following result provides various alternative formulations for a category to be a groupoid:

Proposition 2.4.2. *Let X be a category. Then the following conditions are equivalent:*

(1) The category X is a groupoid;

(2) The functor $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(X) \to \mathrm{Ar}(X)$ admits a section, in the sense of Definition 1.3.2;

(3) The functor $p^*_{[1]} \colon X \to \mathrm{Ar}(X)$, induced by the map $p_{[1]} \colon [1] \to *$, admits a section;

(4) The functor $p^*_{[1]} \colon X \to \mathrm{Ar}(X)$ is an equivalence.

Proof. To see that (1) implies (2), take $\Gamma = \mathrm{Ar}(X)$ and let $f = \mathrm{id}_{\mathrm{Ar}(X)}$, so that $s = \tilde{f}$ provides a section. Conversely, if $s \colon \mathrm{Ar}(X) \to \mathrm{Iso}(X)$ is a section of π_{Iso}, a lift $\tilde{f}$ of f may be given as $\tilde{f} := s \circ f$.

The fact that (2) and (3) are equivalent is immediate from the following commutative diagram:

$$\begin{array}{ccccc} & & X & & \\ & \overset{i}{\swarrow}_{\sim} & & \overset{(p_{[1]})^*}{\searrow} & \\ \mathrm{Iso}(X) & & \xrightarrow{\pi_{\mathrm{Iso}}} & & \mathrm{Ar}(X), \end{array}$$

using that the map i is an equivalence by Axiom F. Finally, note that the functor $(p_{[1]})^*$ always admits a retraction given by $\mathrm{ev}_0 \colon \mathrm{Ar}(X) \to X$, and thus the equivalence between (3) and (4) is an instance of Lemma 1.3.3. □

Convention 2.4.3. From now on we will mostly take (4) as our working definition of groupoids.

Example 2.4.4. The terminal category $*$ is a groupoid: there is a commutative triangle

$$\begin{array}{ccc} & \mathrm{Ar}(*) & \\ {\scriptstyle (p_{[1]})^*}\nearrow & & \searrow \\ * & \longrightarrow & *, \end{array}$$

and since the bottom and right diagonal maps are equivalences (see Lemma 1.7.9) the claim follows by 2-out-of-3.

Example 2.4.5. If X and Y are groupoids, then their product $X \times Y$ is a groupoid. This is an immediate consequence of Lemma 1.7.10.

Lemma 2.4.6. *Let $C \to D$ be an equivalence of categories. Then C is a groupoid if and only if D is a groupoid.*

Proof. This follows from the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\sim} & D \\ {\scriptstyle (p_{[1]})^*}\downarrow & & \downarrow{\scriptstyle (p_{[1]})^*} \\ \mathrm{Ar}(C) & \xrightarrow{\sim} & \mathrm{Ar}(D). \end{array}$$

Here the bottom functor is again an equivalence by the functoriality of functor categories, Construction 1.7.7. □

Lemma 2.4.7. *Let X be a groupoid. Then for every category C, the functor category* $\mathrm{Fun}(C, X)$ *is a groupoid.*

Proof. This follows from the fact that the map $\mathrm{Fun}(C, X) \to \mathrm{Ar}(\mathrm{Fun}(C, X))$ is equivalent to the map $\mathrm{Fun}(C, X) \to \mathrm{Fun}(C, \mathrm{Ar}(X))$ induced by the equivalence $X \xrightarrow{\sim} \mathrm{Ar}(X)$. □

Lemma 2.4.8. *Groupoids are closed under retracts: if X is a groupoid and Y is a retract of X, then Y is also a groupoid.*

Proof. If Y is a retract of X, then the functor $Y \to \mathrm{Ar}(Y)$ is a retract of the functor $X \to \mathrm{Ar}(X)$. In particular, if the latter is an equivalence then so is the former by Lemma 1.3.9. □

Lemma 2.4.9. *Consider functors $f : X \to C$ and $g : Y \to C$ and assume that X and Y are groupoids. Then the pullback $X \times_C Y$ is a groupoid.*

Proof. The functor $X \times_C Y \to \mathrm{Ar}(X \times_C Y)$ factors as

$$X \times_C Y \xrightarrow{\sim} X \times_{\mathrm{Ar}(C)} Y \xrightarrow{\sim} \mathrm{Ar}(X) \times_{\mathrm{Ar}(C)} \mathrm{Ar}(Y) \simeq \mathrm{Ar}(X \times_C Y).$$

Here the second functor is an equivalence since X and Y are groupoids. The last equivalence is Lemma 1.7.11. Finally, we claim that the first functor is a pullback of the diagonal of the embedding

$i\colon C \to \mathrm{Ar}(C)$ from Corollary 2.3.7 and hence is an equivalence. To see this, consider the following commutative diagram:

$$\begin{array}{ccccc}
X \times_C Y & \longrightarrow & X \times_{\mathrm{Ar}(C)} Y & \longrightarrow & X \times Y \\
\downarrow & & \downarrow & & \downarrow \\
C & \xrightarrow[\sim]{\Delta_i} & C \times_{\mathrm{Ar}(C)} C & \longrightarrow & C \times C \\
 & & \downarrow & \lrcorner & \downarrow{\scriptstyle i\times i} \\
 & & \mathrm{Ar}(C) & \xrightarrow{\Delta} & \mathrm{Ar}(C) \times \mathrm{Ar}(C).
\end{array}$$

The right bottom square, the right outer rectangle and the top outer rectangle are pullback squares by Lemma 1.6.13. It thus follows from Lemma 1.6.11 that also the top left square is a pullback square. □

Lemma 2.4.10. *Consider a pullback square*

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
{\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
Y' & \longrightarrow & Y
\end{array}$$

of categories. If X, Y and Y' are groupoids, then so is X'.

Proof. This is a special case of Lemma 2.4.9. □

Lemma 2.4.11. *The empty category $\emptyset$ is a groupoid.*

Proof. We need to show that the functor $\emptyset \to \mathrm{Ar}(\emptyset)$ is an equivalence, or equivalently that the map $\mathrm{ev}_0\colon \mathrm{Ar}(\emptyset) \to \emptyset$ is an equivalence. But this is immediate from Axiom B.4'. □

2.5 Recognizing animae

Animae enter the foundational language primitively as homotopy-coherent collections, whereas the groupoid condition is a property of a category expressed by invertibility of its morphisms. The two notions enter the theory independently. The following axiom asserts that they determine the same class of categories, allowing us to pass between collection-oriented anima language and structural groupoid arguments.

Axiom G (Groupoid recognition axiom)**.** A category is an anima if and only if it is a groupoid.

Proposition 2.5.1. *For any anima X and any category A, the canonical functor*

$$\iota_{A,X}\colon \mathrm{Map}(A, X) \xrightarrow{\sim} \mathrm{Fun}(A, X)$$

is an equivalence.

Proof. By the groupoid recognition axiom and Lemma 2.4.7, the category $\mathrm{Fun}(A, X)$ is an anima. The claim follows from Propositions 1.4.15 and 1.7.16. □

Lemma 2.5.2. *For functors $X \to C$ and $Y \to C$ with X and Y animae, the pullback $X \times_C Y$ is an anima.*

Proof. This follows from Lemma 2.4.9 and the groupoid recognition axiom. □

Corollary 2.5.3. *Let C be a category. Then the functor $\iota_C \colon C^{\simeq} \to C$ is an embedding.*

Proof. We need to show that the diagonal $C^{\simeq} \to C^{\simeq} \times_C C^{\simeq}$ is an equivalence. Its source is an anima, and its target is an anima by Lemma 2.5.2. By Proposition 1.4.11, it therefore suffices to show that for every anima Z the induced map

$$\mathrm{Map}(Z, C^{\simeq}) \to \mathrm{Map}(Z, C^{\simeq} \times_C C^{\simeq}) \simeq \mathrm{Map}(Z, C^{\simeq}) \times_{\mathrm{Map}(Z,C)} \mathrm{Map}(Z, C^{\simeq})$$

is an equivalence, i.e. we need to show that the map $\mathrm{Map}(Z, C^{\simeq}) \to \mathrm{Map}(Z, C)$ is an embedding. But this map is even an equivalence by Corollary 1.4.13. □

Corollary 2.5.4. *The empty category $\emptyset$ is an anima. For animae X and Y, the coproduct $X \sqcup Y$ is an anima.*

Proof. The empty category is an anima by Lemma 2.4.11 and the groupoid recognition axiom.

Put $S := X \sqcup Y$, with inclusions $i_X \colon X \to S$ and $i_Y \colon Y \to S$. By Corollary 1.4.13, these factor through the core inclusion $\iota_S \colon S^{\simeq} \to S$: there are functors $\widetilde{i_X} \colon X \to S^{\simeq}$ and $\widetilde{i_Y} \colon Y \to S^{\simeq}$ with identifications $\iota_S \circ \widetilde{i_X} = i_X$ and $\iota_S \circ \widetilde{i_Y} = i_Y$. Their copairing $s := \langle \widetilde{i_X}, \widetilde{i_Y} \rangle \colon S \to S^{\simeq}$ satisfies

$$\iota_S \circ s = \langle \iota_S \circ \widetilde{i_X}, \iota_S \circ \widetilde{i_Y} \rangle = \langle i_X, i_Y \rangle = \mathrm{id}_S \,.$$

Thus S is a retract of the anima $S^{\simeq}$. By the recognition axiom, $S^{\simeq}$ is a groupoid, so S is a groupoid by Lemma 2.4.8. Recognition then shows that S is an anima. □

Corollary 2.5.5. *Let X and Y be groupoids. Then their disjoint union $X \sqcup Y$ is a groupoid.*

Proof. By the recognition axiom, X and Y are animae. Their coproduct is an anima by Corollary 2.5.4, hence a groupoid by recognition. □

Lemma 2.5.6. *For any two categories C and D, the canonical map*

$$\mathrm{Ar}(C) \sqcup \mathrm{Ar}(D) \to \mathrm{Ar}(C \sqcup D)$$

is an equivalence.

Proof. When $C = * = D$, we have $\mathrm{Ar}(*) \simeq *$ and the statement says that the map $* \sqcup * \to \mathrm{Ar}(* \sqcup *)$ is an equivalence. By Corollary 2.5.4, $* \sqcup *$ is an anima, hence a groupoid by the recognition axiom. The claim follows, since for a groupoid X the functor $X \to \mathrm{Ar}(X)$ is an equivalence. For arbitrary C and D, it follows from Lemma 1.7.11 that there are pullback squares

$$\begin{array}{ccccc} \mathrm{Ar}(C) & \longrightarrow & \mathrm{Ar}(C \sqcup D) & \longleftarrow & \mathrm{Ar}(D) \\ \downarrow & \lrcorner & \downarrow & \llcorner & \downarrow \\ \mathrm{Ar}(*) & \longrightarrow & \mathrm{Ar}(* \sqcup *) & \longleftarrow & \mathrm{Ar}(*). \end{array}$$

The claim thus follows from Lemma 1.6.16. □

2.6 Exercises

Exercise 2.6.1. Let C_0 and C_1 be categories and consider morphisms $f_0 \colon x_0 \to y_0$ and $g_0 \colon y_0 \to z_0$ in C_0 and morphisms $f_1 \colon x_1 \to y_1$ and $g_1 \colon y_1 \to z_1$ in C_1. Show that the composite $(g_0, g_1) \circ (f_0, f_1) \colon (x_0, x_1) \to (z_0, z_1)$ in $C_0 \times C_1$ is given by $(g_0 \circ f_0, g_1 \circ f_1)$.

Exercise 2.6.2. Let $f \colon x \to y$ be a morphism in a category C. Prove that the following assertions are equivalent:

(i) f is an isomorphism;

(ii) for any object c of C, f induces an equivalence of the form

$$f \circ - \colon C(c, x) \xrightarrow{\sim} C(c, y) \, ;$$

(iii) for any object c of C, f induces an equivalence of the form

$$- \circ f \colon C(y, c) \xrightarrow{\sim} C(x, c) \, .$$

3 Constructions of categories

In this chapter, we introduce various fundamental categorical constructions in the setting of synthetic category theory: subcategories and full subcategories, localizations, geometric realizations, exponentiable functors, joins, and slices.

3.1 Subcategories

In ordinary category theory, if one is given a collection M of morphisms in a category C which is closed under composition, then one may form the subcategory of C whose morphisms are precisely the morphisms in M (and the objects in the subcategory are precisely the objects whose identity morphism is in M). In this section, we discuss the analogous situation in the context of synthetic category theory.

We start by defining the notion of a subcategory. For this, we need the following auxiliary construction:

Construction 3.1.1. For categories C, D and E, we construct a map

$$\mathrm{Map}(C, D) \to \mathrm{Map}(\mathrm{Map}(E, C), \mathrm{Map}(E, D))$$

by currying the internal composition functor from Proposition 1.4.8:

$$\mu_{E,C,D} \colon \mathrm{Map}(C, D) \times \mathrm{Map}(E, C) \longrightarrow \mathrm{Map}(E, D).$$

For an absolute functor $f \colon C \to D$, the specialization of the resulting map at f is identified with the postcomposition functor $f_* \colon \mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$ constructed in Construction 1.4.3. This follows by uncurrying and comparing the two double-evaluation functors. Here we use the convention from Construction 1.4.1 of passing back and forth between functors and their representing points in the mapping anima.

Definition 3.1.2 (Subcategory)**.** A functor $f \colon A \to C$ is called a *subcategory* if the following two conditions are satisfied:

(1) The map $f_* \colon \mathrm{Map}([1], A) \to \mathrm{Map}([1], C)$ is an embedding;

(2) For every category D, the induced square

$$\begin{array}{ccc} \mathrm{Map}(D, A) & \xrightarrow{\quad f_* \quad} & \mathrm{Map}(D, C) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], A)) & \xrightarrow{(f_*)_*} & \mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], C)) \end{array}$$

is a pullback square. Here the two vertical maps are given by Construction 3.1.1.

In other words, if $f\colon A \to C$ is a subcategory, then a given functor $D \to C$ factors through A if and only if the associated map on morphisms $\mathrm{Map}([1], D) \to \mathrm{Map}([1], C)$ factors through $\mathrm{Map}([1], A)$.

Remark 3.1.3. The above notion of subcategory corresponds to what is known as a *replete subcategory* in the higher category theory literature. There is no sensible notion of non-replete subcategories in the synthetic setting.

Example 3.1.4. The identity functor $\mathrm{id}_C\colon C \to C$ and the empty category $\emptyset \to C$ are both seen to be subcategories.

Lemma 3.1.5. *Every subcategory $f\colon A \to C$ is an embedding.*

Proof. By Proposition 1.6.20, it will suffice to show that the induced map $\mathrm{Map}(D, A) \to \mathrm{Map}(D, C)$ is an embedding for every category D. But this follows from its defining property, as it is the base change of the map $(f_*)_*$ induced by the map $f_*\colon \mathrm{Map}([1], A) \to \mathrm{Map}([1], C)$ which was assumed to be an embedding. □

We would now like to construct subcategories of C out of the data of a collection of morphisms in C closed under composition. Let us start by defining this notion.

Definition 3.1.6 (Collection of morphisms)**.** Let C be a category. A *collection of morphisms in C* is an anima M equipped with an embedding

$$m\colon M \hookrightarrow \mathrm{Map}([1], C).$$

We say that *M is closed under composition* if the following two conditions are satisfied:

- If a morphism $f\colon x \to y$ in C is in M, then so are id_x and id_y. More concretely, there are commutative diagrams of the following form:

$$\begin{array}{ccccc} M & \dashleftarrow & M & \dashrightarrow & M \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ \mathrm{Map}([1], C) & \xleftarrow{f \mapsto \mathrm{id}_x} & \mathrm{Map}([1], C) & \xrightarrow{f \mapsto \mathrm{id}_y} & \mathrm{Map}([1], C). \end{array}$$

- For maps $f\colon x \to y$ and $g\colon y \to z$ in M, also the composition of f and g is in M. More concretely, there exists a dotted arrow in the following diagram:

$$\begin{array}{ccc} M \times_{\mathrm{ev}_1, C^{\simeq}, \mathrm{ev}_0} M & \dashrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle m} \\ \mathrm{Map}([1], C) \times_{\mathrm{ev}_1, C^{\simeq}, \mathrm{ev}_0} \mathrm{Map}([1], C) & \xleftarrow{\sim} \mathrm{Map}([2], C) \xrightarrow{d_1^*} & \mathrm{Map}([1], C). \end{array}$$

Convention 3.1.7. Let $m\colon M \hookrightarrow \mathrm{Map}([1], C)$ be a collection of morphisms in C. We say that a morphism $f\colon x \to y$ in C *is contained in M*, or simply *is in M*, if the corresponding absolute object of $\mathrm{Map}([1], C)$ factors through m. Since m is an embedding, such a factorization, if it exists, is unique, so being contained in M is a property. Thus, in phrases such as "a morphism in M", the word "morphism" refers to a morphism in C contained in the subanima $M \hookrightarrow \mathrm{Map}([1], C)$, not to a morphism of the anima M.

Example 3.1.8. Taking $M = \mathrm{Map}([1], C)$, it is clear that the identity map $M \to \mathrm{Map}([1], C)$ is a collection of morphisms in C closed under composition.

Example 3.1.9. Let $A \to C$ be a subcategory. Then the map $M_A := \mathrm{Map}([1], A) \to \mathrm{Map}([1], C)$ is a collection of morphisms in C closed under composition. This is immediate from the fact that the functor $A \to C$ preserves compositions and identities.

Also the collection of isomorphisms in C determines a collection of morphisms closed under composition:

Lemma 3.1.10. *For a category C, the map $\pi_{\mathrm{Iso}}^{\simeq}\colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Map}([1], C)$ defines a collection of morphisms in C closed under composition.*

Proof. We saw in Lemma 2.3.3 that the functor π_{Iso} is an embedding, hence $\pi_{\mathrm{Iso}}^{\simeq}$ is an embedding by Corollary 1.6.21. The identity-isomorphism construction shows that the source and target identities of every isomorphism again lie in $\mathrm{Iso}(C)^{\simeq}$. Given composable isomorphisms f and g, choose their functorially specified left and right inverses. Associativity and unitality exhibit $f^{-1} \circ g^{-1}$ as a two-sided inverse of $g \circ f$. This construction is functorial in (f, g) and supplies the dotted composition map in Definition 3.1.6. Thus $\mathrm{Iso}(C)^{\simeq}$ is closed under composition. □

We may now state the subcategory axiom:

Axiom H (Subcategory axiom)**.** Let C be a category and let $M \hookrightarrow \mathrm{Map}([1], C)$ be a collection of morphisms in C closed under composition. Then there exists a category $\langle M \rangle_C$ equipped with a functor $i_M\colon \langle M \rangle_C \to C$ satisfying the following two conditions:

(1) The functor i_M sends morphisms in $\langle M \rangle_C$ into M: the first projection pr_1 in the following pullback square is an equivalence:

$$\begin{array}{ccc} \mathrm{Map}([1], \langle M \rangle_C) \times_{\mathrm{Map}([1], C)} M & \xrightarrow{\mathrm{pr}_2} & M \\ {\scriptstyle \mathrm{pr}_1}\downarrow{\scriptstyle \simeq} & \lrcorner & \downarrow{\scriptstyle m} \\ \mathrm{Map}([1], \langle M \rangle_C) & \xrightarrow{(i_M)_*} & \mathrm{Map}([1], C). \end{array}$$

Equivalently: the map $(i_M)_*\colon \mathrm{Map}([1], \langle M \rangle_C) \to \mathrm{Map}([1], C)$ factors through the map $m\colon M \to \mathrm{Map}([1], C)$.

(2) The functor i_M is universal with this property: for any other category D, the following commutative square is a pullback square:

$$\begin{array}{ccc} \mathrm{Map}(D, \langle M \rangle_C) & \xrightarrow{(i_M)_*} & \mathrm{Map}(D, C) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}(\mathrm{Map}([1], D), M) & \xhookrightarrow{m_*} & \mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], C)). \end{array}$$

Here the vertical maps use Construction 3.1.1 and the map from (1).

Corollary 3.1.11. *In the situation of Axiom H, the functor* $\mathrm{Map}([1], \langle M \rangle_C) \to M$ *is an equivalence.*

Proof. Applying the universal property of $\langle M \rangle_C$ to $D = [1]$, and using the equivalence $\mathrm{Map}([1], [1]) \simeq \{\mathrm{id}_0, \mathrm{id}_1, \mathrm{can}\}$ from Remark 2.1.29, we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Map}([1], \langle M \rangle_C) & \longrightarrow & \mathrm{Map}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \substack{(f\colon x \to y) \\ \updownarrow \\ (\mathrm{id}_x, \mathrm{id}_y, f)}} \\ M^3 & \hookrightarrow & \mathrm{Map}([1], C)^3. \end{array}$$

The claim now follows from the fact that the map $M \hookrightarrow \mathrm{Map}([1], C)$ is an embedding and that M is closed under composition, so that f lies in M if and only if each of $\mathrm{id}_x, \mathrm{id}_y, f$ lie in M. □

Corollary 3.1.12. *In the situation of Axiom H, the functor* $i_M \colon \langle M \rangle_C \to C$ *exhibits* $\langle M \rangle_C$ *as a subcategory of* C*. In particular,* i_M *is an embedding.*

Proof. By Corollary 3.1.11, the map $\mathrm{Map}([1], \langle M \rangle_C) \to \mathrm{Map}([1], C)$ agrees with the embedding $M \hookrightarrow \mathrm{Map}([1], C)$, giving condition (1) of Definition 3.1.2. Condition (2) then follows immediately from the universal property of $\langle M \rangle_C$. □

Lemma 3.1.13. *Let* $f \colon A \to C$ *be a subcategory and let* $M_A := \mathrm{Map}([1], A) \hookrightarrow \mathrm{Map}([1], C)$ *denote the induced collection of morphisms in* C*. Then the map* $i_{M_A} \colon \langle M_A \rangle_C \to C$ *factors through an equivalence* $\langle M_A \rangle_C \xrightarrow{\sim} A$*.*

Proof. Since the functor $\langle M_A \rangle_C \to C$ induces the inclusion $\mathrm{Map}([1], A) \hookrightarrow \mathrm{Map}([1], C)$, the fact that A is a subcategory of C implies that this functor factors through A. It further follows that the induced map $\mathrm{Map}(D, \langle M_A \rangle_C) \to \mathrm{Map}(D, A)$ is an equivalence for every category D, since both sides sit in the same pullback square. The claim thus follows from Proposition 1.4.11. □

In particular, this shows:

Corollary 3.1.14. *When* $M = \mathrm{Map}([1], C)$ *is the collection of all morphisms, the map* $i_M \colon \langle M \rangle_C \to C$ *is an equivalence.* □

We will now explain the functoriality of the construction of the subcategory $\langle M \rangle_C$ in the pair (C, M).

Definition 3.1.15. Let C and D be categories, and assume that they come equipped with collections of morphisms $m \colon M \hookrightarrow \mathrm{Map}([1], C)$ and $n \colon N \hookrightarrow \mathrm{Map}([1], D)$. A functor $f \colon C \to D$ is said to *send morphisms in* M *to morphisms in* N if there exists a map $\tilde{f} \colon M \to N$ making the following diagram commute:

$$\begin{array}{ccc} M & \overset{\tilde{f}}{\dashrightarrow} & N \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ \mathrm{Map}([1], C) & \xrightarrow{f_*} & \mathrm{Map}([1], D). \end{array}$$

Since the map n is an embedding, the map $\tilde{f} \colon M \to N$ is unique if it exists.

Construction 3.1.16. Let C and D be categories equipped with collections of morphisms M and N closed under composition, and let $f\colon C \to D$ be a functor which sends morphisms in M to morphisms in N. We construct a functor $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ making the following diagram commute:

$$\begin{array}{ccc} \langle M\rangle_C & \xrightarrow{f|} & \langle N\rangle_D \\ {\scriptstyle i_M}\downarrow & & \downarrow{\scriptstyle i_N} \\ C & \xrightarrow{f} & D. \end{array}$$

By the defining property of $\langle N\rangle_D$, it suffices to show that the map $f \circ i_M\colon \langle M\rangle_C \to D$ sends all morphisms to morphisms in N. But this follows from the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Map}([1], \langle M\rangle_C) & \xrightarrow{\simeq} & M & \xrightarrow{\tilde{f}} & N \\ & {\scriptstyle (i_M)_*}\searrow & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\ & & \mathrm{Map}([1], C) & \xrightarrow{f_*} & \mathrm{Map}([1], D). \end{array}$$

Remark 3.1.17 (Exercise 3.8.3)**.** The construction $f \mapsto f|$ satisfies the following properties:

(1) The map $f|$ is uniquely determined by f.

(2) Given an identification $f = f'$, we obtain an induced identification $f| = f'|$.

(3) The construction is compatible with identities and composition through identifications

$$(\mathrm{id}_C)| = \mathrm{id}_{\langle M\rangle_C} \qquad \text{and} \qquad (g \circ f)| = g| \circ f|.$$

(4) If the maps f and $\tilde{f}$ are equivalences, then also $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ is an equivalence.

3.2 Full subcategories

As a consequence of the subcategory axiom, we may in particular form *full* subcategories.

Construction 3.2.1. Taking $E = *$ in Construction 3.1.1, and using $C^{\simeq} = \mathrm{Map}(*, C)$ and $D^{\simeq} = \mathrm{Map}(*, D)$, gives a map

$$\mathrm{Map}(C, D) \longrightarrow \mathrm{Map}(C^{\simeq}, D^{\simeq}).$$

Its specialization at $f\colon C \to D$ is the functor $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ from Construction 1.4.16.

Definition 3.2.2 (Full subcategory)**.** A functor $f\colon A \to C$ is called a *full subcategory* if the induced map $f^{\simeq}\colon A^{\simeq} \to C^{\simeq}$ is an embedding and for every category D, the commutative square

$$\begin{array}{ccc} \mathrm{Map}(D, A) & \xrightarrow{f_*} & \mathrm{Map}(D, C) \\ \downarrow & & \downarrow \\ \mathrm{Map}(D^{\simeq}, A^{\simeq}) & \xrightarrow{(f^{\simeq})_*} & \mathrm{Map}(D^{\simeq}, C^{\simeq}) \end{array}$$

is a pullback square.

More informally, this says that a functor $D \to C$ factors through A if and only if the map $D^{\simeq} \to C^{\simeq}$ on objects factors through the embedding $A^{\simeq} \to C^{\simeq}$.

Corollary 3.2.3. *Every full subcategory is an embedding.*

Proof. For every category D, the map $\mathrm{Map}(D, A) \to \mathrm{Map}(D, C)$ is the base change of the embedding

$$(f^{\simeq})_* \colon \mathrm{Map}(D^{\simeq}, A^{\simeq}) \longrightarrow \mathrm{Map}(D^{\simeq}, C^{\simeq}).$$

It is therefore an embedding. The claim follows from Proposition 1.6.20. □

Definition 3.2.4 (Collection of objects)**.** Consider a category C. A *collection of objects* is a pair (Γ, j) consisting of an anima Γ equipped with an embedding $j \colon \Gamma \hookrightarrow C^{\simeq}$. In this case, we define an anima M_Γ via the following pullback:

$$\begin{array}{ccc} M_\Gamma & \xhookrightarrow{m} & \mathrm{Map}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \Gamma \times \Gamma & \xhookrightarrow{j \times j} & C^{\simeq} \times C^{\simeq}. \end{array}$$

The map m is an embedding, as it is a base change of the embedding $j \times j$. The resulting collection M_Γ is closed under composition: the identity of an object represented by Γ again has both endpoints in Γ, and the composite of two composable morphisms with endpoints represented by Γ has the same property. This allows us to make the following definition:

Definition 3.2.5. Given an embedding $j \colon \Gamma \hookrightarrow C^{\simeq}$, we define

$$\langle \Gamma \subseteq C \rangle := \langle M_\Gamma \rangle_C,$$

and refer to it as the *full subcategory of C spanned by* Γ.

Lemma 3.2.6. *There is a preferred equivalence $\langle \Gamma \subseteq C \rangle^{\simeq} \xrightarrow{\sim} \Gamma$ fitting in a commutative triangle*

$$\begin{array}{ccc} \langle \Gamma \subseteq C \rangle^{\simeq} & \xrightarrow{\simeq} & \Gamma \\ & \searrow \quad \swarrow{\scriptstyle j} & \\ & C^{\simeq}. & \end{array}$$

Proof. Given a category D, consider the following two commutative squares:

$$\begin{array}{ccc} \mathrm{Map}(D, \langle \Gamma \subseteq C \rangle) & \longrightarrow & \mathrm{Map}(D, C) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}(\mathrm{Map}([1], D), M_\Gamma) & \longrightarrow & \mathrm{Map}(\mathrm{Map}([1], D), \mathrm{Map}([1], C)) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)_*} \\ \mathrm{Map}(\mathrm{Map}([1], D), \Gamma \times \Gamma) & \longrightarrow & \mathrm{Map}(\mathrm{Map}([1], D), C^{\simeq} \times C^{\simeq}). \end{array}$$

The top square is a pullback by definition of $\langle \Gamma \subseteq C \rangle$, and the bottom square is a pullback by definition of M_Γ. Taking $D = *$ we thus obtain a pullback square of the form

$$\begin{array}{ccc} \langle \Gamma \subseteq C \rangle^{\simeq} & \longrightarrow & C^{\simeq} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ \Gamma \times \Gamma & \xrightarrow{j \times j} & C^{\simeq} \times C^{\simeq}. \end{array}$$

Since j is an embedding, this implies that $\langle \Gamma \subseteq C\rangle^{\simeq} \simeq \Gamma$, cf. Lemma 1.6.22. It is immediate from the construction of this equivalence that it lives over $C^{\simeq}$. □

Proposition 3.2.7. *The inclusion $\tilde{j}\colon \langle \Gamma \subseteq C\rangle \to C$ is a full subcategory, in the sense of Definition 3.2.2.*

Proof. We showed in Lemma 3.2.6 that the induced functor $\tilde{j}^{\simeq}\colon \langle \Gamma \subseteq C\rangle^{\simeq} \to C^{\simeq}$ agrees, under the preferred equivalence $\langle \Gamma \subseteq C\rangle^{\simeq} \xrightarrow{\sim} \Gamma$, with $j\colon \Gamma \to C^{\simeq}$, hence is an embedding by assumption. We thus need to show that for every category D, the top square in the following commutative diagram is a pullback square:

$$\begin{array}{ccc}
\mathrm{Map}(D, \langle \Gamma \subseteq C\rangle) & \longrightarrow & \mathrm{Map}(D, C) \\
\downarrow & & \downarrow \\
\mathrm{Map}(D^{\simeq}, \langle \Gamma \subseteq C\rangle^{\simeq}) & \longrightarrow & \mathrm{Map}(D^{\simeq}, C^{\simeq}) \\
\downarrow & & \downarrow \\
\mathrm{Map}(D^{\simeq} \times D^{\simeq}, \langle \Gamma \subseteq C\rangle^{\simeq} \times \langle \Gamma \subseteq C\rangle^{\simeq}) & & \mathrm{Map}(D^{\simeq} \times D^{\simeq}, C^{\simeq} \times C^{\simeq}) \\
\downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)^*} & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)^*} \\
\mathrm{Map}(\mathrm{Map}([1], D), \langle \Gamma \subseteq C\rangle^{\simeq} \times \langle \Gamma \subseteq C\rangle^{\simeq}) & \longrightarrow & \mathrm{Map}(\mathrm{Map}([1], D), C^{\simeq} \times C^{\simeq}).
\end{array}$$

The bottom two horizontal maps are embeddings since they are induced by the embedding $\langle \Gamma \subseteq C\rangle^{\simeq} \to C^{\simeq}$. The outer rectangle agrees with the outer pullback square in the proof of Lemma 3.2.6, hence is a pullback square itself. It then follows from Lemma 1.6.28 that the top square is also a pullback square. □

Corollary 3.2.8. *For any category D, there is a preferred pullback*

$$\begin{array}{ccc}
\mathrm{Map}(D, \langle \Gamma \subseteq C\rangle) & \longrightarrow & \mathrm{Map}(D, C) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{Map}(D^{\simeq}, \Gamma) & \xrightarrow{j_*} & \mathrm{Map}(D^{\simeq}, C^{\simeq}).
\end{array}$$

Proof. This is immediate from Proposition 3.2.7 and Lemma 3.2.6. □

Lemma 3.2.9. *The inclusion $\langle C^{\simeq} \subseteq C\rangle \hookrightarrow C$ is an equivalence.*

Proof. Since the map $j\colon C^{\simeq} \to C^{\simeq}$ is the identity, it follows from Corollary 3.2.8 that for every category D the induced map

$$\mathrm{Map}(D, \langle C^{\simeq} \subseteq C\rangle) \to \mathrm{Map}(D, C)$$

is an equivalence. The claim thus follows from Proposition 1.4.11. □

Lemma 3.2.10. *Let C and D be categories, and let $j\colon \Gamma \hookrightarrow C^{\simeq}$ and $k\colon \Lambda \hookrightarrow D^{\simeq}$ be embeddings. Then there is a preferred equivalence*

$$\langle \Gamma \times \Lambda \subseteq C \times D\rangle \xrightarrow{\sim} \langle \Gamma \subseteq C\rangle \times \langle \Lambda \subseteq D\rangle.$$

Proof. This is an exercise in unwinding definitions and is left to the reader: see Exercise 3.8.2. □

Lemma 3.2.11. *Consider a functor $f : B \to C$. Given an embedding $\Gamma \hookrightarrow C^{\simeq}$, we define $\Lambda \hookrightarrow B^{\simeq}$ via the pullback*

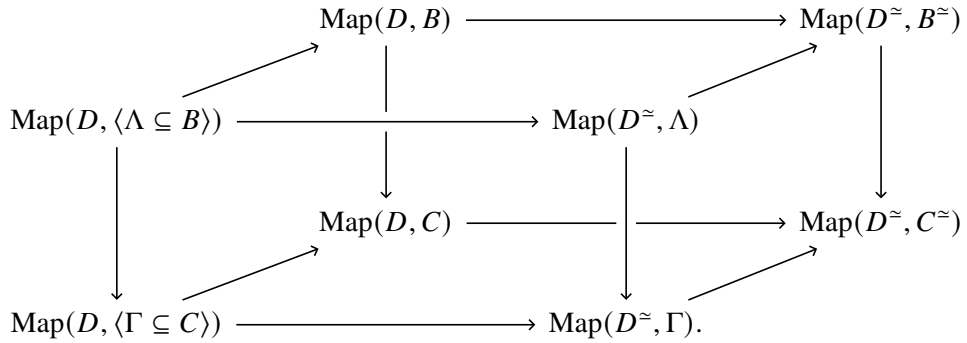
$$\begin{array}{ccc} \Lambda & \hookrightarrow & B^{\simeq} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f^{\simeq}} \\ \Gamma & \hookrightarrow & C^{\simeq}. \end{array}$$

Then the square

$$\begin{array}{ccc} \langle \Lambda \subseteq B \rangle & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ \langle \Gamma \subseteq C \rangle & \longrightarrow & C \end{array}$$

is a pullback square.

Proof. By Proposition 1.6.3, it will suffice to check that this square becomes a pullback square after applying the functor $\mathrm{Map}(D, -)$ for every category D. To this end, consider the following commutative diagram:



Here the top and bottom face of the cube are pullback squares by Corollary 3.2.8, while the right face is a pullback square since $\mathrm{Map}(D^{\simeq}, -)$ preserves pullback squares by Proposition 1.6.3. It follows from the pasting law of pullback squares that also the left face of the cube is a pullback square, finishing the proof. □

Lemma 3.2.12. *Let $f : B \to C$ be a functor exhibiting B as a full subcategory of C. Then f is an equivalence if and only if the induced map $f^{\simeq} : B^{\simeq} \to C^{\simeq}$ is an equivalence.*

Proof. The forward implication follows by functoriality of groupoid cores. Conversely, suppose that $f^{\simeq}$ is an equivalence. The defining pullback square for a full subcategory then shows that $\mathrm{Map}(D, B) \to \mathrm{Map}(D, C)$ is an equivalence for every category D. Hence f is an equivalence by Proposition 1.4.11. □

3.3 Localizations

In ordinary category theory, if one is given a category C equipped with a collection of morphisms W in C, one may form the *localization* of C at W: the initial functor $C \to C[W^{-1}]$ out of C which sends all morphisms in W to isomorphisms. Our goal in this section is to define localizations in the context of synthetic category theory.

Construction 3.3.1. Let C be a category equipped with a collection of morphisms $w\colon W \hookrightarrow \mathrm{Map}([1], C)$, in the sense of Definition 3.1.6. We will construct for every category D a full subcategory

$$\mathrm{Fun}^W(C, D) \subseteq \mathrm{Fun}(C, D)$$

In order to do this, we first define the anima $\mathrm{Map}^W(C, D)$ via the following pullback square:

$$\begin{array}{ccc} \mathrm{Map}^W(C, D) & \xhookrightarrow{\lambda^W_{C,D}} & \mathrm{Map}(C, D) \\ \downarrow & & \downarrow \\ & & \mathrm{Map}(\mathrm{Map}([1], C), \mathrm{Map}([1], D)) \\ & & \downarrow{\scriptstyle w^*} \\ \mathrm{Map}(W, \mathrm{Iso}(D)^{\simeq}) & \xhookrightarrow{(\pi_{\mathrm{Iso}})_*} & \mathrm{Map}(W, \mathrm{Map}([1], D)), \end{array}$$

where the top right vertical map is the map from Construction 3.1.1. The bottom map is induced by the embedding $\pi^{\simeq}_{\mathrm{Iso}}$ from Lemma 2.3.3 and hence is an embedding. It follows that also $\lambda^W_{C,D}$ is an embedding. We can now define

$$\mathrm{Fun}^W(C, D) := \langle \mathrm{Map}^W(C, D) \subseteq \mathrm{Fun}(C, D) \rangle.$$

In particular, the map $\lambda^W_{C,D}$ extends to a functor $\tilde{\lambda}^W_{C,D}\colon \mathrm{Fun}^W(C, D) \to \mathrm{Fun}(C, D)$. We may think of the category $\mathrm{Fun}^W(C, D)$ as the full subcategory of $\mathrm{Fun}(C, D)$ spanned by those functors sending the morphisms in W to isomorphisms in D.

Remark 3.3.2. The construction of $\mathrm{Fun}^W(C, D)$ is functorial in the pair (C, W): if C' is another category equipped with a collection of morphisms $w'\colon W' \hookrightarrow \mathrm{Map}([1], C')$ and $f\colon C \to C'$ is a functor which sends morphisms in W to morphisms in W' (in the sense of Definition 3.1.15), then for every category D there is an induced square

$$\begin{array}{ccc} \mathrm{Map}^{W'}(C', D) & \xrightarrow{f^*} & \mathrm{Map}^W(C, D) \\ {\scriptstyle \lambda^{W'}_{C',D}}\downarrow & & \downarrow{\scriptstyle \lambda^W_{C,D}} \\ \mathrm{Map}(C', D) & \xrightarrow{f^*} & \mathrm{Map}(C, D), \end{array}$$

which in turn induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}^{W'}(C', D) & \xrightarrow{f^*} & \mathrm{Fun}^W(C, D) \\ {\scriptstyle \tilde{\lambda}^{W'}_{C',D}}\downarrow & & \downarrow{\scriptstyle \tilde{\lambda}^W_{C,D}} \\ \mathrm{Fun}(C', D) & \xrightarrow{f^*} & \mathrm{Fun}(C, D) \end{array}$$

via Construction 3.1.16.

Before moving on to the localization axiom, we will establish some basic results that express that the categories $\mathrm{Fun}^W(C, D)$ behave as we expect. As a first result, we show that every functor $C \to D$ lies in $\mathrm{Fun}^W(C, D)$ whenever D is a groupoid:

Lemma 3.3.3. *Let C be a category equipped with a collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$, and let Γ be a groupoid. Then the functor*

$$\mathrm{Fun}^W(C, \Gamma) \to \mathrm{Fun}(C, \Gamma)$$

is an equivalence.

Proof. The functor $\mathrm{Iso}(\Gamma)^{\simeq} \to \mathrm{Map}([1], \Gamma)$ is an equivalence, since Γ is a groupoid. It follows from its defining property that also the map $\mathrm{Map}^W(C, \Gamma) \to \mathrm{Map}(C, \Gamma)$ is an equivalence, and hence both sides define the same full subcategory of $\mathrm{Fun}(C, \Gamma)$. Since the full subcategory determined by $\mathrm{Map}(C, \Gamma)$ is just $\mathrm{Fun}(C, \Gamma)$ by Lemma 3.2.9, this finishes the proof. □

The following lemma expresses the fact that any functor $C \to D$ takes isomorphisms in C to isomorphisms in D:

Proposition 3.3.4. *In the case of $W := \mathrm{Iso}(C)^{\simeq} \hookrightarrow \mathrm{Map}([1], C)$, the forgetful functor*

$$\mathrm{Fun}^W(C, D) \to \mathrm{Fun}(C, D)$$

is an equivalence.

Proof. Since the comparison map is the embedding of a full subcategory by construction, it suffices to prove that it induces an equivalence on objects. In other words, it suffices to check that the induced map

$$\mathrm{Map}^W(C, D) \to \mathrm{Map}(C, D)$$

is an equivalence. Since this is an embedding of animae, it suffices to check that any functor of the form $f : C \to D$ has the property that the induced map

$$f_* : \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$$

sends isomorphisms of C to isomorphisms of D. For this, since there is a canonical equivalence $C^{\simeq} \simeq \mathrm{Iso}(C)^{\simeq}$ given by the rule $x \mapsto \mathrm{id}_x$, it suffices to check that $f(\mathrm{id}_x) = \mathrm{id}_{f(x)}$ holds for any object x of C. This is the identity comparison for the functor f. □

Construction 3.3.5. Let $l : C \to L$ be a functor which sends all morphisms in W to isomorphisms in L, in the sense of Definition 3.1.15. Then we see that for every category D the induced map $l^* : \mathrm{Fun}(L, D) \to \mathrm{Fun}(C, D)$ factors through $\mathrm{Fun}^W(C, D)$, using the following composite:

$$\mathrm{Fun}(L, D) \simeq \mathrm{Fun}^{\mathrm{Iso}(L)^{\simeq}}(L, D) \xrightarrow{l^*} \mathrm{Fun}^W(C, D).$$

Here the map l^* is obtained via Remark 3.3.2.

Definition 3.3.6 (Localization)**.** Given a collection of morphisms W in C, we say that a functor $l : C \to L$ is a *localization of C at W* if it sends all morphisms in W to isomorphisms in L and is universal with respect to this property: for every category D, the functor

$$l^* : \mathrm{Fun}(L, D) \to \mathrm{Fun}^W(C, D)$$

is an equivalence. A functor $l : C \to L$ is called a *localization functor* if it is a localization of C at some collection of morphisms W.

If a localization exists, it is unique; see Exercise 3.8.4. We will frequently denote it by $C[W^{-1}]$, and refer to the functor $l\colon C \to C[W^{-1}]$ as the *localization functor*.

Axiom I (Localization axiom)**.** For every collection of morphisms $w\colon W \hookrightarrow \mathrm{Map}([1], C)$ in C there exists a localization $l\colon C \to C[W^{-1}]$ of C at W.

Proposition 3.3.7. *Given a collection of morphisms W in a category C as well as a functor $\lambda\colon C \to L$ sending all morphisms in W to isomorphisms in L, assume that, for every category D, the induced map*

$$\lambda^*\colon \mathrm{Map}(L, D) \to \mathrm{Map}^W(C, D)$$

is an equivalence. Then λ exhibits L as a localization of C at W.

Proof. Let $l\colon C \to C[W^{-1}]$ be a localization of C at W. Then the induced map

$$l^*\colon \mathrm{Map}(C[W^{-1}], D) \to \mathrm{Map}^W(C, D)$$

is an equivalence because it is the map on groupoid cores induced by the equivalence $l^*\colon \mathrm{Fun}(C[W^{-1}], D) \xrightarrow{\sim} \mathrm{Fun}^W(C, D)$. Since l inverts W, the assumption on L implies that there exists a functor $u\colon L \to C[W^{-1}]$ equipped with an identification $u \circ \lambda = l$. To see that u is an equivalence, consider for every category D the commutative triangle

$$\begin{array}{ccc} \mathrm{Map}(C[W^{-1}], D) & \xrightarrow{\quad u^* \quad} & \mathrm{Map}(L, D) \\ & {\scriptstyle l^*}\searrow \quad \swarrow{\scriptstyle \lambda^*} & \\ & \mathrm{Map}^W(C, D). & \end{array}$$

By Proposition 1.4.11, it suffices to show that u^* is an equivalence for every category D. This is immediate by the 2-out-of-3 property as both diagonal maps are equivalences.

Thus u is an equivalence compatible with the maps out of C, and the universal property of l transfers to λ. □

Remark 3.3.8. The previous proposition and the localization axiom give a pushout square

$$\begin{array}{ccc} [1] \times W & \xrightarrow{\tilde{w}} & C \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \ulcorner & \downarrow \\ W & \longrightarrow & C[W^{-1}], \end{array}$$

where the map $\tilde{w}$ is obtained by uncurrying the composite $W \xrightarrow{w} \mathrm{Map}([1], C) \to \mathrm{Fun}([1], C)$; see Exercise 3.8.5.

Construction 3.3.9. The construction $C \mapsto C[W^{-1}]$ is functorial in the pair (C, W): if D is another category equipped with a collection of morphisms W' and $f\colon C \to D$ is a functor which sends W

to W', the composite $C \to D \to D[W'^{-1}]$ sends all morphisms in W to isomorphisms, and hence induces a canonical dashed functor making the square commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l} \\ C[W^{-1}] & \dashrightarrow & D[W'^{-1}]. \end{array}$$

3.4 Geometric realizations

The localization of a category C at *all* its morphisms deserves special notation and terminology:

Definition 3.4.1 (Geometric realization)**.** In the case where $W := \mathrm{Map}([1], C)$ consists of all morphisms in C, we write

$$\|C\| \quad := \quad C[W^{-1}],$$

and we denote the localization functor by $\lambda_C \colon C \to \|C\|$. We refer to $\|C\|$ as the *geometric realization* of C.

In other words, the geometric realization is characterized by the property that for every category D, the functor

$$\lambda_C^* \colon \mathrm{Fun}(\|C\|, D) \to \mathrm{Fun}^{\mathrm{Map}([1],C)}(C, D)$$

is an equivalence.

By Construction 3.3.9, the assignment $C \mapsto \|C\|$ is functorial in C: every functor $f \colon C \to D$ induces a functor

$$\|f\| \colon \|C\| \to \|D\|.$$

Lemma 3.4.2. *Assume that X is a groupoid. Then the localization functor $\lambda_X \colon X \to \|X\|$ is an equivalence.*

Proof. Using that $\mathrm{Iso}(X) \simeq \mathrm{Fun}([1], X)$, we obtain by Proposition 3.3.4 equivalences

$$\mathrm{Fun}^{\mathrm{Map}([1],X)}(X, D) \simeq \mathrm{Fun}^{\mathrm{Iso}(X)^{\simeq}}(X, D) \simeq \mathrm{Fun}(X, D),$$

which shows that the identity map $X \to X$ exhibits X as a localization of X at all its morphisms. This gives the claim. □

Lemma 3.4.3. *If C is a category and Γ is a groupoid, then precomposition with the localization functor $\lambda_C \colon C \to \|C\|$ defines an equivalence*

$$\lambda_C^* \colon \mathrm{Fun}(\|C\|, \Gamma) \xrightarrow{\sim} \mathrm{Fun}(C, \Gamma)$$

Proof. We may simply compose the equivalence $\mathrm{Fun}(\|C\|, \Gamma) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Map}([1],C)}(C, \Gamma)$ with the equivalence $\mathrm{Fun}^{\mathrm{Map}([1],C)}(C, \Gamma) \xrightarrow{\sim} \mathrm{Fun}(C, \Gamma)$ from Lemma 3.3.3. □

Since we have obtained $\|C\|$ from C by inverting all its morphisms, one may expect it to be a groupoid. While this is indeed the case, this question is somewhat more subtle than it may seem at first, since a priori the localization may create new morphisms that do not lift to C itself. This problem is closely related to a variety of other statements we may expect to hold. Let us first formulate these and prove that they are equivalent; we will see afterwards that they are in fact satisfied.

Proposition 3.4.4. *The following conditions are equivalent:*

(1) *For every category C, the category $\|C\|$ is a groupoid;*

(2) *The geometric realization $\|[1]\|$ of the walking morphism $[1]$ is contractible;*

(3) *For every category C, the functor $p^*_{[1]} \colon C \to \mathrm{Fun}([1], C)$ is a full subcategory of $\mathrm{Fun}([1], C)$ (in the sense of Definition 3.2.2);*

(4) *For every category C, the functor $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1], C)$ is a full subcategory;*

(5) *Every category C for which the map $C^{\simeq} \simeq \mathrm{Map}(*, C) \xrightarrow{p^*_{[1]}} \mathrm{Map}([1], C)$ is an equivalence is a groupoid;*

(6) *For every category C, the subcategory $\langle \mathrm{Iso}(C)^{\simeq} \rangle_C \subseteq C$ spanned by all the isomorphisms in C is a groupoid;*

(7) *For every category C, the inclusion $C^{\simeq} \hookrightarrow C$ induces an equivalence $C^{\simeq} \xrightarrow{\sim} \langle \mathrm{Iso}(C)^{\simeq} \rangle_C$ between the groupoid core of C and the subcategory of C spanned by all its isomorphisms.*

Proof. We start by showing that (1) implies (2). By (1), the category $\|[1]\|$ is a groupoid, so it suffices by Proposition 1.4.11 to show that for every groupoid Γ the induced map

$$\mathrm{Map}(*, \Gamma) \to \mathrm{Map}(\|[1]\|, \Gamma)$$

is an equivalence. By Lemma 3.4.3, this map corresponds under the canonical equivalences to the map

$$p^*_{[1]} \colon \Gamma \to \mathrm{Map}([1], \Gamma).$$

But this map is an equivalence since Γ is a groupoid, finishing the proof.

For (2) $\implies$ (3), the equivalence $\|[1]\| \xrightarrow{\sim} *$ and the universal property of geometric realization show that precomposition with $p_{[1]} \colon [1] \to *$ induces an equivalence

$$C \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Map}([1],[1])}([1], C),$$

where the right-hand side is a full subcategory of $\mathrm{Fun}([1], C)$ by construction. Thus $p^*_{[1]}$ exhibits C as that full subcategory.

The Rezk equivalence $C \xrightarrow{\sim} \mathrm{Iso}(C)$ identifies $p^*_{[1]}$ with the composite $C \to \mathrm{Iso}(C) \xrightarrow{\pi_{\mathrm{Iso}}} \mathrm{Fun}([1], C)$. Hence (3) and (4) are equivalent.

For (3) $\implies$ (5), recall that C is a groupoid if and only if the map $C \to \mathrm{Fun}([1], C)$ is an equivalence. Since this functor is a full subcategory by (3), it follows from Lemma 3.2.12 that we may check this on objects, giving the criterion from (5).

We will now deduce (6) from (5). To show $\langle \mathrm{Iso}(C)^{\simeq} \rangle_C$ is a groupoid, it suffices by (5) to show that the map $(\langle \mathrm{Iso}(C)^{\simeq} \rangle_C)^{\simeq} \to \mathrm{Map}([1], \langle \mathrm{Iso}(C)^{\simeq} \rangle_C)$ is an equivalence. To this end, consider the following commutative diagram, where the two horizontal maps are induced by the inclusion $i \colon \langle \mathrm{Iso}(C)^{\simeq} \rangle_C \hookrightarrow C$:

$$\begin{array}{ccccc} (\langle \mathrm{Iso}(C)^{\simeq} \rangle_C)^{\simeq} & & \xrightarrow{\quad i^{\simeq} \quad} & & C^{\simeq} \\ \downarrow{\scriptstyle p^*_{[1]}} & & & \overset{\simeq}{\swarrow} & \downarrow{\scriptstyle p^*_{[1]}} \\ \mathrm{Map}([1], \langle \mathrm{Iso}(C)^{\simeq} \rangle_C) & \xrightarrow{\simeq} & \mathrm{Iso}(C)^{\simeq} & \xrightarrow[\pi^{\simeq}_{\mathrm{Iso}}]{} & \mathrm{Map}([1], C). \end{array}$$

By the 2-out-of-3 property it remains to show that the top map is an equivalence. To construct an inverse, we consider the inclusion $\iota_C \colon C^{\simeq} \to C$. It sends all morphisms to isomorphisms, so induces a functor $C^{\simeq} \to \langle \mathrm{Iso}(C)^{\simeq} \rangle_C$. Since the source is a groupoid, it factors through a map $\varphi \colon C^{\simeq} \to (\langle \mathrm{Iso}(C)^{\simeq} \rangle_C)^{\simeq}$. To check the relations $\varphi \circ i^{\simeq} = \mathrm{id}$ and $i^{\simeq} \circ \varphi = \mathrm{id}$, we may do so after composing with the embedding $\iota_C \colon C^{\simeq} \hookrightarrow C$ from Corollary 2.5.3 and the embeddings $(\langle \mathrm{Iso}(C)^{\simeq} \rangle_C)^{\simeq} \hookrightarrow \langle \mathrm{Iso}(C)^{\simeq} \rangle_C$ and $\langle \mathrm{Iso}(C)^{\simeq} \rangle_C \hookrightarrow C$ (Lemma 3.1.5) after which the claim becomes a tautology. This establishes (6).

Assume (6). The functor $C^{\simeq} \to \langle \mathrm{Iso}(C)^{\simeq} \rangle_C$ constructed above is then a functor between groupoids. Its map on groupoid cores is inverse to the equivalence $i^{\simeq}$ constructed in the preceding paragraph. Since the groupoid-core inclusions of its source and target are equivalences by Axiom G and Proposition 1.4.15, the functor itself is an equivalence by 2-out-of-3. This proves (7).

Finally, we show that (7) implies (1). The localization functor $\lambda_C \colon C \to \|C\|$ sends all morphisms of C to isomorphisms, hence it factors through the subcategory of $\|C\|$ spanned by its isomorphisms. By (7) this subcategory is equivalent to the groupoid core of $\|C\|$. The resulting functor $C \to \|C\|^{\simeq}$ still inverts all morphisms in C, hence induces a functor $\|C\| \to \|C\|^{\simeq}$. By uniqueness of the localization factorization, the composite $\|C\| \to \|C\|^{\simeq} \hookrightarrow \|C\|$ is identified with the identity. Thus the embedding $\|C\|^{\simeq} \hookrightarrow \|C\|$ admits a section and is an equivalence by Lemma 1.6.24. It follows that $\|C\|$ is a groupoid. □

Proposition 3.4.5. *All of the equivalent conditions from Proposition 3.4.4 are satisfied.*

Proof. It will suffice to show that the map $\|[1]\| \to *$ is an equivalence. By Proposition 1.4.11, it will suffice to show that for any category D, the induced functor $D^{\simeq} \simeq \mathrm{Map}(*, D) \to \mathrm{Map}(\|[1]\|, D) \simeq \mathrm{Map}^{\mathrm{Map}([1],[1])}([1], D)$ is an equivalence. Recall from Remark 2.1.29 that the groupoid $\mathrm{Map}([1], [1])$ is a disjoint union of three points, corresponding to the identity on 0, the identity on 1, and the canonical map from 0 to 1 in $[1]$. Spelling out the definition of $\mathrm{Map}^{\mathrm{Map}([1],[1])}([1], D)$, we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Map}^{\mathrm{Map}([1],[1])}([1], D) & \longrightarrow & \mathrm{Map}([1], D) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \substack{(f\colon x \to y) \\ \downarrow \\ (\mathrm{id}_x, \mathrm{id}_y, f)}} \\ D^{\simeq 3} & \longrightarrow & \mathrm{Map}([1], D)^3. \end{array}$$

The bottom map is the third power of the embedding $p^*_{[1]} \colon D^{\simeq} \to \mathrm{Map}([1], D)$. Hence the pullback consists of those morphisms $f \colon x \to y$ for which id_x, id_y, and f lie in the image of $p^*_{[1]}$. The first two conditions are automatic, while the third says precisely that f is an isomorphism. The Rezk equivalence $D^{\simeq} \xrightarrow{\sim} \mathrm{Iso}(D)^{\simeq}$ therefore identifies the pullback with $D^{\simeq}$, compatibly with the displayed map from $D^{\simeq}$. This proves the claim. □

Proposition 3.4.6 (Objectwise criterion for natural isomorphisms)**.** *Let $f, g\colon C \to D$ be functors and let $\alpha\colon f \to g$ be a natural transformation. Then α is a natural isomorphism if and only if $\alpha_x\colon f(x) \to g(x)$ is an isomorphism for every object x of C.*

Proof. By Propositions 3.4.4 and 3.4.5, the functor $\mathrm{Iso}(D) \to \mathrm{Fun}([1], D)$ is a full subcategory. Unwinding, this means we have a pullback square

$$\begin{array}{ccc} \mathrm{Map}(C, \mathrm{Iso}(D)) & \longrightarrow & \mathrm{Map}(C, \mathrm{Fun}([1], D)) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}(C^{\simeq}, \mathrm{Iso}(D)^{\simeq}) & \longrightarrow & \mathrm{Map}(C^{\simeq}, \mathrm{Map}([1], D)). \end{array}$$

The transformation α determines a term of the upper right corner, and the right-hand map restricts it to its components. If every component is invertible, applying this hypothesis to the universal object in the anima $C^{\simeq}$ gives a lift to the lower left corner. The pullback property then supplies a lift $C \to \mathrm{Iso}(D)$ of the functor encoding α. It follows from Lemma 1.7.11 that $\mathrm{Fun}(C, \mathrm{Iso}(D)) \simeq \mathrm{Iso}(\mathrm{Fun}(C, D))$, so that this lift provides the inverse data for α in $\mathrm{Fun}(C, D)$, showing that α is a natural isomorphism. Conversely, evaluation at each object sends an inverse of α to an inverse of its component. □

Corollary 3.4.7. *For categories C and D, the following square is a pullback square:*

$$\begin{array}{ccc} \mathrm{Map}(C, D) & \longrightarrow & \mathrm{Fun}(C, D) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}(C^{\simeq}, D) & \longrightarrow & \mathrm{Fun}(C^{\simeq}, D). \end{array}$$

Proof. By part (7) of Proposition 3.4.4, the two horizontal functors are the subcategories on the isomorphisms. The statement is thus equivalent to the statement that a natural transformation $\alpha\colon f \to g$ is a natural isomorphism if and only if its restriction to $C^{\simeq}$ is, which is the content of Proposition 3.4.6. □

Lemma 3.4.8. *For every category D, the following square is a pullback:*

$$\begin{array}{ccc} \mathrm{Map}([1], D) & \longrightarrow & \mathrm{Fun}([1], D) \\ {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)}\downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ D^{\simeq} \times D^{\simeq} & \longrightarrow & D \times D. \end{array}$$

Proof. This is Corollary 3.4.7 applied to $C = [1]$. □

Corollary 3.4.9. *For objects x, y of a category C, there is a pullback square*

$$\begin{array}{ccc} C(x, y) & \longrightarrow & \mathrm{Map}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ * & \xrightarrow{(x,y)} & C^{\simeq} \times C^{\simeq}. \end{array}$$

Proof. This follows from Lemma 3.4.8 and the pasting law for pullback squares. □

Lemma 3.4.10. *Consider a pushout square of categories*

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ D' & \longrightarrow & D. \end{array}$$

Then the induced square

$$\begin{array}{ccc} \|C'\| & \longrightarrow & \|C\| \\ \downarrow & & \downarrow \\ \|D'\| & \longrightarrow & \|D\| \end{array}$$

is again a pushout square.

Proof. By Definition 1.8.1, it will suffice to show that the square becomes a pullback square after applying $\mathrm{Map}(-, E)$ for every category E. By Corollary 1.4.13 and Lemma 3.4.3, there are equivalences

$$\mathrm{Map}(\|C\|, E) \simeq \mathrm{Map}(\|C\|, E^{\simeq}) \simeq \mathrm{Map}(C, E^{\simeq}),$$

and similarly for C replaced with C', D and D'. The claim now follows from the assumption on the original square. □

3.5 Dependent products along exponentiable functors

Recall from Section 1.9 the notion of a category over a base category S. Given a functor $f\colon S \to T$, base change along f provides an assignment

$$\begin{array}{ccc} \{\text{categoriesover } T\ \} & \to & \{\text{categories over } S\} \\ p\colon C \to T & \mapsto & f^*(p)\colon C \times_T S \to S. \end{array}$$

Semantically, we may think of this as the base change functor $f^*\colon \mathrm{Cat}_{/T} \to \mathrm{Cat}_{/S}$, where Cat denotes the universe of small categories (to be introduced in Chapter 8). This base change functor always admits a left adjoint $f_!\colon \mathrm{Cat}_{/S} \to \mathrm{Cat}_{/T}$, given by postcomposition with f:

$$\begin{array}{ccc} \{\text{categories over } S\ \} & \to & \{\text{categories over } T\ \} \\ q\colon D \to S & \mapsto & f \circ q\colon D \to T. \end{array}$$

However, the functor $f^*\colon \mathrm{Cat}_{/T} \to \mathrm{Cat}_{/S}$ will not in general admit a *right* adjoint; indeed, base change along f does not necessarily preserve all colimits. Let us illustrate this with the following simple counterexample:

Example 3.5.1. Consider the functor $f = d_1\colon [1] \to [2]$ given by $d_1(0) = 0$ and $d_1(1) = 2$. We will argue that base change along f does not preserve pushouts of categories, and hence cannot admit a right adjoint. To see this, consider the following commutative square:

$$\begin{array}{ccc} [0] & \xrightarrow{d_2} & [1] \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ [1] & \xrightarrow[d_2]{} & [2]. \end{array} \tag{3.1}$$

We see as a consequence of the Segal axiom Axiom E that this is a pushout square of categories. If we pull back this square along the functor $d_1 \colon [1] \to [2]$, we obtain the commutative square

$$\begin{array}{ccc} \emptyset & \longrightarrow & [0] \\ \downarrow & & \downarrow{\scriptstyle d_0} \\ [0] & \xrightarrow[d_1]{} & [1]. \end{array} \tag{3.2}$$

But this is no longer a pushout square of categories, unless all categories are contractible. Indeed, if it were a pushout square, the two inclusions $* \to * \sqcup *$ would extend to a functor

$$h \colon [1] \longrightarrow * \sqcup *$$

whose values at 0 and 1 are identified with the two respective inclusions. Since $* \sqcup *$ is an anima, h factors through $\|[1]\| \simeq *$, hence is identified with a constant functor. This means it takes the same values on 0 and 1, and we deduce that the two inclusions $* \to * \sqcup *$ agree.

By the universal property of the coproduct, it follows that the identity functor of $* \sqcup *$ is identified with the constant functor at either point. Thus $* \sqcup *$ is contractible, which, by Lemma 1.6.18, implies that all categories are contractible. Consequently, the functor d_1 fails to be exponentiable in every non-trivial model of our theory.

While this example means that f^* cannot admit a right adjoint for arbitrary f, there is a large class of functors, called the *exponentiable functors*, for which this right adjoint does exist. This class includes cartesian and cocartesian fibrations, to be introduced in Chapter 4. In this case, we refer to the right adjoint as the *dependent product along* f. We may already give its universal property right now in terms of animae of relative functors.

Definition 3.5.2 (Dependent product)**.** Consider a functor $f \colon S \to T$ and let $p \colon C \to S$ be a category over S. A *dependent product of* C *along* f is a category $q \colon D \to T$ over T equipped with a functor $\varepsilon \colon D \times_T S \to C$ over S which satisfies the property that for any other category $E \to T$ over T the composite functor

$$\mathrm{Map}_T(E, D) \xrightarrow{f^*} \mathrm{Map}_S(E \times_T S, D \times_T S) \xrightarrow{\varepsilon \circ -} \mathrm{Map}_S(E \times_T S, C)$$

is an equivalence.

If a dependent product of C along f exists, it is unique; see Exercise 3.8.6. We will denote it by $f_*(p) \colon f_*(C) \to T$.

Terminology 3.5.3. Assume that C admits a dependent product $f_*(C)$ along f. Given a functor $\varphi \colon D \to f_*(C)$ over T, we obtain a functor

$$\varphi^u := \varepsilon \circ f^*(\varphi) \colon D \times_T S \to C$$

over S that we will refer to as the *relative uncurrying of* φ. Conversely, there is for every functor $\psi \colon D \times_T S \to C$ over S a *relatively curried* functor $\psi_c \colon D \to f_*(C)$, satisfying

$$(\varphi^u)_c = \varphi \qquad \text{and} \qquad (\psi_c)^u = \psi.$$

In this way, we may think of the equivalence in Definition 3.5.2 as a *relative* version of currying/uncurrying.

Construction 3.5.4. Consider a pullback square

$$\begin{array}{ccc} S' & \xrightarrow{h} & S \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

of categories, and let $p\colon C \to S$ be a category over S. Assume that:

- There exists a dependent product $f_*(C) \to T$ of C along f;
- There exists a dependent product $f'_*(C \times_S S') \to T'$ of $C \times_S S'$ along f'.

We will construct a functor

$$\mathrm{BC}_*\colon f_*(C) \times_T T' \to f'_*(C \times_S S')$$

over T' called the *Beck-Chevalley map*. By relative uncurrying, we may equivalently define BC_* by specifying a map $(f_*(C) \times_T T') \times_{T'} S' \to C \times_S S'$, which we take to be the map

$$(f_*(C) \times_T T') \times_{T'} S' \simeq (f_*(C) \times_T S) \times_S S' \xrightarrow{\varepsilon \times_S S'} C \times_S S'.$$

We are now ready to introduce the class of exponentiable functors.

Definition 3.5.5. Let $f\colon S \to T$ be a functor. We say that f is *exponentiable* if the following two conditions are satisfied:

(1) For every pullback square

$$\begin{array}{ccc} S' & \xrightarrow{h} & S \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{g} & T \end{array}$$

and every category C' over S', there exists a dependent product $f'_*(C') \to T'$ of C' along f'.

(2) For every pullback diagram

$$\begin{array}{ccccc} S'' & \xrightarrow{h'} & S' & \xrightarrow{h} & S \\ {\scriptstyle f''}\downarrow & \lrcorner & {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ T'' & \xrightarrow{g'} & T' & \xrightarrow{g} & T \end{array}$$

and every $p'\colon C' \to S'$, the Beck-Chevalley map $\mathrm{BC}_*\colon f'_*(C') \times_{T'} T'' \to f''_*(C' \times_{S'} S'')$ is an equivalence.

Remark 3.5.6. Note that exponentiable functors are by definition closed under base change: if $f\colon S \to T$ is exponentiable, then so is any base change $f'\colon S' \to T'$ of f.

Construction 3.5.7. Let $f\colon S \to T$ be an exponentiable functor, let $p\colon C \to S$ and $q\colon D \to S$ be categories over S, and write $\varepsilon_C\colon f_*(C) \times_T S \to C$ and $\varepsilon_D\colon f_*(D) \times_T S \to D$ for the evaluation functors. A functor $\varphi\colon C \to D$ over S determines a composite

$$f_*(C) \times_T S \xrightarrow{\varepsilon_C} C \xrightarrow{\varphi} D$$

over S. Its relative currying is a functor $f_*(\varphi)\colon f_*(C) \to f_*(D)$ over T. The uniqueness part of relative currying supplies identifications

$$f_*(\mathrm{id}_C) = \mathrm{id}_{f_*(C)} \qquad \text{and} \qquad f_*(\psi \circ \varphi) = f_*(\psi) \circ f_*(\varphi).$$

Thus dependent products are functorial in categories over S.

Lemma 3.5.8. *Let $f\colon S \to T$ be an exponentiable functor. Then the assignment $p \mapsto f_*(p)$ preserves pullbacks: given functors $C_1 \to C_3 \leftarrow C_2$ over S, there exists a preferred equivalence*

$$f_*(C_1 \times_{C_3} C_2) \xrightarrow{\sim} f_*(C_1) \times_{f_*(C_3)} f_*(C_2)$$

over T.

Proof. Functoriality of dependent products applied to the two projections constructs the displayed comparison functor. For every category E over T, the universal properties of dependent products and pullbacks give equivalences

$$\begin{aligned}
\mathrm{Map}_T\big(E, f_*(C_1 \times_{C_3} C_2)\big) &\simeq \mathrm{Map}_S\big(E \times_T S, C_1 \times_{C_3} C_2\big) \\
&\simeq \mathrm{Map}_S(E \times_T S, C_1) \times_{\mathrm{Map}_S(E \times_T S, C_3)} \mathrm{Map}_S(E \times_T S, C_2) \\
&\simeq \mathrm{Map}_T(E, f_*(C_1)) \times_{\mathrm{Map}_T(E, f_*(C_3))} \mathrm{Map}_T(E, f_*(C_2)) \\
&\simeq \mathrm{Map}_T\big(E, f_*(C_1) \times_{f_*(C_3)} f_*(C_2)\big).
\end{aligned}$$

These equivalences are induced by the comparison functor and are natural in E. Taking E to be either side of the comparison constructs an inverse over T and identifications between the two composites and the respective identity functors, exactly as in the proof of Proposition 1.4.11. Hence the comparison is an equivalence. □

Corollary 3.5.9. *Let $f\colon S \to T$ be an exponentiable functor. If $p\colon C \to S$ is an embedding, then also $f_*(p)\colon f_*(C) \to T$ is an embedding.*

Proof. The identity functor of T exhibits $T \to T$ as the dependent product of $S \to S$ along f, so there is a preferred equivalence $f_*(S) \xrightarrow{\sim} T$. Since p is an embedding, its diagonal square is a pullback. Applying Lemma 3.5.8 shows that the diagonal square of $f_*(p)\colon f_*(C) \to f_*(S) \simeq T$ is a pullback. Hence $f_*(p)$ is an embedding. □

The terminology 'exponentiable' comes from the fact that it allows us to construct certain 'exponential objects' relative to a base category S:

Definition 3.5.10. Let $p\colon C \to S$ and $q\colon D \to S$ be two functors and assume that p is an exponentiable functor. We define the *relative functor category* $\underline{\mathrm{Fun}}_S(C, D) \to S$ as

$$\begin{array}{c} \underline{\mathrm{Fun}}_S(C, D) \\ \big\downarrow \\ S \end{array} \quad := \quad p_* \left(\begin{array}{c} C \times_S D \\ \big\downarrow{\scriptstyle p^*(q)} \\ C \end{array} \right).$$

It comes equipped with an evaluation map

$$\mathrm{ev}\colon \underline{\mathrm{Fun}}_S(C, D) \times_S C \to D$$

over S satisfying the property that for every other functor $E \to S$ the composite functor

$$\mathrm{Map}_S(E, \underline{\mathrm{Fun}}_S(C,D)) \xrightarrow{-\times_S C} \mathrm{Map}_S(E \times_S C, \underline{\mathrm{Fun}}_S(C,D) \times_S C) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Map}_S(E \times_S C, D).$$

is an equivalence. In other words, $\underline{\mathrm{Fun}}_S(C,D)$ is an 'internal hom object' or 'exponential object' for categories over S.

Remark 3.5.11. Note that, for a category T, the data of a functor $T \to \underline{\mathrm{Fun}}_S(C,D)$ is equivalent to the data of a pair (f, φ), where $f\colon T \to S$ is a functor and $\varphi\colon T \times_S C \to T \times_S D$ is a functor over T.

Observation 3.5.12. For any functor $f\colon T \to S$ and categories C and D over S such that $C \to S$ is exponentiable, there is a pullback square

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_T(T \times_S C, T \times_S D) & \longrightarrow & \underline{\mathrm{Fun}}_S(C,D) \\ \downarrow & \lrcorner & \downarrow \\ T & \xrightarrow{\quad f \quad} & S. \end{array}$$

In particular, the fiber of $\underline{\mathrm{Fun}}_S(C,D) \to S$ over an object s of S is the functor category $\mathrm{Fun}(C_s, D_s)$, where C_s and D_s are the fibers of C and D over s, respectively.

Proposition 3.5.13. *For every category S, the terminal functor $f_S\colon S \to *$ is exponentiable. Moreover, if $p\colon C \to S$ is exponentiable and $q\colon D \to S$ is any functor, there is a canonical equivalence*

$$\mathrm{Fun}_S(C,D) \xrightarrow{\sim} (f_S)_* \underline{\mathrm{Fun}}_S(C,D).$$

Proof. Let T be a category. The base change of f_S along $T \to *$ is the projection $\mathrm{pr}_T\colon T \times S \to T$. Given a category $r\colon E \to T \times S$, let $\delta_T\colon T \to \mathrm{Fun}(S, T \times S)$ be the currying of $\mathrm{id}_{T\times S}$. Define $\Pi_S(E) \to T$ by the pullback square

$$\begin{array}{ccc} \Pi_S(E) & \longrightarrow & \mathrm{Fun}(S,E) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle r_*} \\ T & \xrightarrow[\delta_T]{} & \mathrm{Fun}(S, T \times S). \end{array}$$

Uncurrying the upper horizontal functor gives an evaluation functor $\Pi_S(E) \times S \to E$ over $T \times S$. For every category $K \to T$, the pullback property and currying identify

$$\mathrm{Map}_T(K, \Pi_S(E)) \xrightarrow{\sim} \mathrm{Map}_{T\times S}(K \times S, E).$$

Thus $\Pi_S(E)$ is a dependent product of E along pr_T. If $T' \to T$ is a functor, the pullback defining $\Pi_S(E)$ base changes to the analogous pullback over T', since functor categories preserve pullbacks. The resulting Beck–Chevalley map is therefore an equivalence. This proves that f_S is exponentiable.

Now let $p\colon C \to S$ and $q\colon D \to S$ be as in the statement. For every category K, the universal properties of the dependent product and the relative functor category give equivalences

$$\begin{aligned} \mathrm{Map}\big(K, (f_S)_* \underline{\mathrm{Fun}}_S(C,D)\big) &\simeq \mathrm{Map}_S(K \times S, \underline{\mathrm{Fun}}_S(C,D)) \\ &\simeq \mathrm{Map}_S(K \times C, D) \\ &\simeq \mathrm{Map}(K, \mathrm{Fun}_S(C,D)). \end{aligned}$$

These equivalences are induced by evaluation and are natural in K. Hence the canonical functor $\mathrm{Fun}_S(C,D) \to (f_S)_* \underline{\mathrm{Fun}}_S(C,D)$ is an equivalence by Proposition 1.4.11. □

Corollary 3.5.14. *Let $p\colon C \to S$ be exponentiable and let $q\colon D \to S$ be any functor. Then there is a canonical equivalence*

$$\mathrm{Map}_S(C,D) \xrightarrow{\sim} \big((f_S)_* \underline{\mathrm{Fun}}_S(C,D)\big)^{\simeq}.$$

Proof. The definition of $\mathrm{Map}_S(C,D)$ and preservation of pullbacks by groupoid cores give $\mathrm{Map}_S(C,D) \xrightarrow{\sim} \mathrm{Fun}_S(C,D)^{\simeq}$. Now apply the preceding proposition. □

3.6 Joins

A final category constructor we introduce is the *join* $C \star D$ of two categories. This construction admits two universal properties:

- There is a mapping-in universal property: the join comes equipped with a functor to $[1]$, and a map living over $[1]$ from a category T into the join is the same as a pair of maps $T_0 \to C$ and $T_1 \to D$, where the sources are the fibes of $T \to [1]$.
- There is a mapping-out universal property: the join is obtained from the product $[1] \times C \times D$ by collapsing $\{0\} \times C \times D$ onto C and $\{1\} \times C \times D$ onto D.

In order to obtain joins in arbitrary categorical contexts, we will in fact introduce versions of the join over an arbitrary base category Γ.

Using the join construction we define $C^{\triangleright} := C \star *$ and $C^{\triangleleft} := * \star C$. We may then inductively define the categories $[n]$ by $[0] = *$ and $[n+1] := [n]^{\triangleright}$.

Two universal properties

We start with the universal property of mapping out of $C \star D$:

Axiom J.1 (Join axiom)**.** Let Γ be an anima and let $p\colon C \to \Gamma$ and $q\colon D \to \Gamma$ be functors. There is a category $C \star_\Gamma D$ over Γ and a pushout square

$$\begin{array}{ccc} (\{0\} \sqcup \{1\}) \times (C \times_\Gamma D) & \longrightarrow & [1] \times (C \times_\Gamma D) \\ \downarrow & & \downarrow \\ C \sqcup D & \xrightarrow{i_{C,D}} & C \star_\Gamma D, \end{array}$$

over Γ. Here the top map is induced by the inclusion $\{0\} \sqcup \{1\} \to [1]$, and the left vertical map is given by

$$(\{0\} \sqcup \{1\}) \times (C \times_\Gamma D) \xrightarrow{\sim} (C \times_\Gamma D) \sqcup (C \times_\Gamma D) \xrightarrow{\mathrm{pr}_C \sqcup \mathrm{pr}_D} C \sqcup D.$$

The displayed maps to $\Gamma \times [1]$ induce a height functor

$$h_{C,D}\colon C \star_\Gamma D \to \Gamma \times [1]$$

whose composite with the projection to Γ is the structural functor of the relative join.

For $\Gamma = *$, we write $C \star D$ for $C \star_* D$.

Remark 3.6.1. In other words, for every third category E, the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(C \star D, E) & \longrightarrow & \mathrm{Fun}(C \times [1] \times D, E) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(C \sqcup D, E) & \longrightarrow & \mathrm{Fun}(C \times \partial[1] \times D, E) \end{array}$$

is a pullback square.

Example 3.6.2. For $C = D = [0]$, we see that $[0] \star [0] \xrightarrow{\sim} [1]$.

Example 3.6.3. By Corollary 2.2.8, the commutative squares

$$\begin{array}{ccc} [1] \times \{0\} & \longrightarrow & [1] \times [1] \\ \downarrow & & \downarrow{\scriptstyle p_0} \\ * & \xrightarrow{0} & [2] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} [1] \times \{1\} & \longrightarrow & [1] \times [1] \\ \downarrow & & \downarrow{\scriptstyle p_2} \\ * & \xrightarrow{2} & [2] \end{array}$$

are pushout squares. Exchanging the two factors puts the interval coordinate of the join construction first. It follows that there are equivalences

$$[0] \star [1] \xrightarrow{\sim} [2] \xleftarrow{\sim} [1] \star [0].$$

Notation 3.6.4. For any functor $g : C \star D \to E$, we denote by

$$g_C : C \to E \qquad \text{and} \qquad g_D : D \to E$$

the composites of g with the inclusions $C \to C \star D$ and $D \to C \star D$, respectively.

Construction 3.6.5 (Functoriality of joins)**.** Consider functors $f : C \to C'$ and $g : D \to D'$. Then the functors $f \times \mathrm{id}_{[1]} \times g : C \times [1] \times D \to C' \times [1] \times D'$ and $f \sqcup g : C \sqcup D \to C' \sqcup D'$ induce a functor $f \star g : C \star D \to C' \star D'$. The universal property supplies identifications

$$\mathrm{id}_C \star \mathrm{id}_D = \mathrm{id}_{C \star D} \qquad \text{and} \qquad (f' \star g') \circ (f \star g) = (f' \circ f) \star (g' \circ g).$$

Example 3.6.6. Taking $f = p_C : C \to [0]$ and $g = p_D : D \to [0]$, we obtain a map $p_C \star p_D : C \star D \to [0] \star [0] = [1]$ fitting in a commutative diagram as follows:

$$\begin{array}{ccc} C \sqcup D & \xrightarrow{i_{C,D}} & C \star D \\ \downarrow{\scriptstyle f \sqcup g} & & \downarrow{\scriptstyle p_C \star p_D} \\ \partial[1] = [0] \sqcup [0] & \hookrightarrow & [1]. \end{array}$$

We now come to the second universal property of $C \star D$:

Axiom J.2 (Join as dependent product). Let Γ be a category and let $C \to \Gamma$ and $D \to \Gamma$ be functors. There is a category $C \star_\Gamma D$ over Γ and a height functor $h_{C,D} \colon C \star_\Gamma D \to \Gamma \times [1]$. When Γ is an anima, we require these to agree with the category and height functor supplied by Axiom J.1. Write

$$j_\Gamma := \mathrm{id}_\Gamma \times j \colon \Gamma \times \partial[1] \longrightarrow \Gamma \times [1]$$

for the weakening of the boundary inclusion. Then the canonical functor

$$C \sqcup D \longrightarrow (C \star_\Gamma D) \times_{\Gamma \times [1]} (\Gamma \times \partial[1])$$

is an equivalence, and its inverse exhibits the height functor $h_{C,D} \colon C \star_\Gamma D \to \Gamma \times [1]$ as a dependent product of the category $C \sqcup D$ over $\Gamma \times \partial[1]$ along j_Γ, in the sense of Definition 3.5.2.

Definition 3.6.7. Let $p \colon C \to S$ and $q \colon D \to S$ be functors. The category $C \star_S D \to S$ supplied by Axiom J.2 is called the *relative join* of C and D over S.

Remark 3.6.8. Let us spell out the resulting universal property more explicitly. Consider a category E over $\Gamma \times [1]$ and denote by E_0 and E_1 its restrictions along the two sections $\Gamma \to \Gamma \times [1]$. Then Axiom J.2 gives an equivalence

$$\mathrm{Map}_{\Gamma \times [1]}(E, C \star_\Gamma D) \simeq \mathrm{Map}_{\Gamma \times \partial[1]}(E \times_{\Gamma \times [1]} (\Gamma \times \partial[1]), C \sqcup D) \simeq \mathrm{Map}_\Gamma(E_0, C) \times \mathrm{Map}_\Gamma(E_1, D).$$

In other words, giving a functor $\varphi \colon E \to C \star_\Gamma D$ over $\Gamma \times [1]$ amounts to giving functors $\varphi_0 \colon E_0 \to C$ and $\varphi_1 \colon E_1 \to D$ over Γ.

Proposition 3.6.9. *Let A and B be two categories. The canonical embedding $A \sqcup B \hookrightarrow A \star B$ induces an equivalence*

$$A^\simeq \sqcup B^\simeq \xrightarrow{\sim} (A \star B)^\simeq .$$

Proof. Let $h \colon A \star B \to [1]$ be the height functor. By Axiom J.2, there is a pullback square

$$\begin{array}{ccc} A \sqcup B & \hookrightarrow & A \star B \\ \downarrow & & \downarrow{\scriptstyle h} \\ \partial[1] & \hookrightarrow & [1] \end{array}$$

The canonical equivalence $\partial[1] \xrightarrow{\sim} [1]^\simeq$ from Axiom C.1 identifies this with a pullback square

$$\begin{array}{ccc} A \sqcup B & \hookrightarrow & A \star B \\ \downarrow & & \downarrow{\scriptstyle h} \\ [1]^\simeq & \hookrightarrow & [1] \end{array}$$

up to the specified comparison. The square formed by the two groupoid-core inclusions and h therefore induces a functor

$$u \colon (A \star B)^\simeq \longrightarrow (A \sqcup B)^\simeq \xrightarrow{\sim} A^\simeq \sqcup B^\simeq.$$

The functor u is an embedding: after composing with the groupoid-core inclusion of $A \sqcup B$ and then with $A \sqcup B \hookrightarrow A \star B$, it agrees with the groupoid-core inclusion of $A \star B$. Since these inclusions are embeddings, so is u by Lemma 1.6.26.

Applying groupoid cores to $A \sqcup B \hookrightarrow A \star B$ and using Lemma 2.1.28 gives a section

$$A^{\simeq} \sqcup B^{\simeq} \longrightarrow (A \star B)^{\simeq}$$

of u. Hence u is an equivalence by Lemma 1.6.24. □

Unitality and associativity

We will now show that the join operation $(C, D) \mapsto C \star D$ is unital and associative.

Lemma 3.6.10 (Unitality of joins)**.** *For every category C, the maps $C \to \emptyset \star C$ and $C \to C \star \emptyset$ are equivalences.*

Proof. By definition the category $\emptyset \star C$ sits in a pushout square

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ C & \longrightarrow & \emptyset \star C \end{array}$$

and hence the bottom map is an equivalence. A similar argument applies to $C \star \emptyset$. □

Construction 3.6.11 (Associativity of joins)**.** Given categories C, D and E, we construct an equivalence

$$a_{C,D,E} \colon C \star (D \star E) \xrightarrow{\sim} (C \star D) \star E.$$

Applying the mapping-out property of joins twice shows that a functor from either parenthesization to a category F consists of the same data: functors from C, D, and E to F; families of connecting morphisms from C to D, from D to E, and from C to E; and a commutative triangle relating these three families over $C \times D \times E$. The two pullback descriptions are identified using the equivalences

$$[0] \star [1] \xrightarrow{\sim} [2] \xleftarrow{\sim} [1] \star [0]$$

from Example 3.6.3.

The universal property therefore gives a comparison $a_{C,D,E} \colon C \star (D \star E) \to (C \star D) \star E$ which is the identity on C, D, and E. For every category F, precomposition with $a_{C,D,E}$ identifies the two pullback descriptions above and is consequently an equivalence of functor categories. Passing to groupoid cores and applying Proposition 1.4.11 shows that $a_{C,D,E}$ is an equivalence.

No triangle or pentagon compatibility among the chosen equivalences is asserted here. Such a statement would require additional specified comparison data, which is not part of the present naive language.

Definition 3.6.12. Let C be a category. We define its *left cone* $C^{\triangleleft}$ and its *right cone* $C^{\triangleright}$ as follows:

$$C^{\triangleleft} := [0] \star C, \qquad \text{and} \qquad C^{\triangleright} := C \star [0].$$

3.7 Slice categories

The join operation can be used to describe functors into slice categories.

Construction 3.7.1 (Slice category). Let $p_C\colon C \to \Gamma$ and $p_Y\colon Y \to \Gamma$ be categories over Γ, let $\psi\colon Y \to C$ be a functor over Γ, and assume that p_Y is exponentiable. Relative currying of ψ gives a section $\bar{\psi}\colon \Gamma \to \underline{\mathrm{Fun}}_\Gamma(Y, C)$. Relative currying of the constant-family functor $(\mathrm{pr}_C, \mathrm{pr}_C)\colon C \times_\Gamma Y \to C \times_\Gamma C$ gives a functor $p_Y^*\colon C \to \underline{\mathrm{Fun}}_\Gamma(Y, C)$ over Γ. We define the *slice* $C_{/\psi} \to \Gamma$ by the pullback square of categories over Γ

$$\begin{array}{ccc} C_{/\psi} & \longrightarrow & \underline{\mathrm{Fun}}_\Gamma([1]_\Gamma \times_\Gamma Y, C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ C & \xrightarrow[(p_Y^*, \bar{\psi} \circ p_C)]{} & \underline{\mathrm{Fun}}_\Gamma(Y, C) \times_\Gamma \underline{\mathrm{Fun}}_\Gamma(Y, C). \end{array}$$

Here $[1]_\Gamma := \Gamma \times [1] \to \Gamma$. The relative functor category in the upper right is constructed by taking the dependent product along p_Y of the arrow category of $C \times_\Gamma Y$ over Y; thus its existence uses only the assumed exponentiability of p_Y.

For $\Gamma = *$, the terminal functor $Y \to *$ is exponentiable by Proposition 3.5.13. Thus the absolute slice exists without an additional exponentiability hypothesis.

Remark 3.7.2. We will mostly be interested in the case $Y = \Gamma$ and $p_Y = \mathrm{id}_\Gamma$. For a section $x\colon \Gamma \to C$, we write $C_{/x}$ for the resulting slice over Γ.

Lemma 3.7.3. *For every category X and every functor $\psi\colon Y \to C$, there is a pullback square of the form*

$$\begin{array}{ccc} \mathrm{Fun}(X, C_{/\psi}) & \longrightarrow & \mathrm{Fun}(X \star Y, C) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Fun}(X, C) & \xrightarrow[1 \times \psi]{} & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C). \end{array}$$

Proof. Since the assignment $C \mapsto \mathrm{Fun}(X, C)$ preserves pullbacks, applying it to the defining pullback square of $C_{/\psi}$ gives

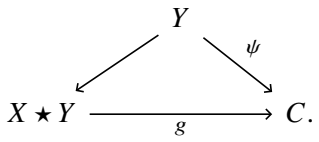
$$\begin{array}{ccccc} \mathrm{Fun}(X, C_{/\psi}) & & \longrightarrow & & \mathrm{Fun}(X \times [1] \times Y, C) \\ \downarrow & & \lrcorner & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathrm{Fun}(X, C) & \xrightarrow{1 \times \psi} & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C) & \xrightarrow{\mathrm{pr}_X^* \times \mathrm{pr}_Y^*} & \mathrm{Fun}(X \times Y, C) \times \mathrm{Fun}(X \times Y, C). \end{array}$$

Because of the factorization of the bottom map, we may compute this pullback by first pulling back along the map $\mathrm{pr}_X^* \times \mathrm{pr}_Y^*$, giving a pullback square

$$\begin{array}{ccccc} \mathrm{Fun}(X, C_{/\psi}) & & \longrightarrow & & \mathrm{Fun}(X \sqcup Y, C) \times_{\mathrm{Fun}(X \times \partial[1] \times Y, C)} \mathrm{Fun}(X \times [1] \times Y, C) \\ \downarrow & & \lrcorner & & \downarrow \\ & & & & \mathrm{Fun}(X \sqcup Y, C) \\ & & & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Fun}(X, C) = \mathrm{Fun}(X, C) \times * & & \xrightarrow{1 \times \psi} & & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C). \end{array}$$

Since the upper right corner is equivalent to $\mathrm{Fun}(X \star Y, C)$ by Remark 3.6.1, this finishes the proof. □

Remark 3.7.4. Informally speaking, the previous lemma says that a functor $X \to C_{/\psi}$ into the slice consists of a map $g\colon X \star Y \to C$ fitting in a commutative diagram as follows:



Completely analogously we obtain a dual slice construction.

Construction 3.7.5. Let $p_C\colon C \to \Gamma$ and $p_X\colon X \to \Gamma$ be categories over Γ, let $\varphi\colon X \to C$ be a functor over Γ, and assume that p_X is exponentiable. Let $\bar{\varphi}\colon \Gamma \to \underline{\mathrm{Fun}}_\Gamma(X, C)$ be its relative currying. Define the coslice $C_{\varphi/} \to \Gamma$ by the pullback square of categories over Γ

$$\begin{array}{ccc} C_{\varphi/} & \longrightarrow & \underline{\mathrm{Fun}}_\Gamma([1]_\Gamma \times_\Gamma X, C) \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow{(\bar{\varphi}\circ p_C, p_X^*)} & \underline{\mathrm{Fun}}_\Gamma(X, C) \times_\Gamma \underline{\mathrm{Fun}}_\Gamma(X, C). \end{array}$$

3.8 Exercises Chapter 3

Exercise 3.8.1. Consider a category Γ, a category D and a category $q\colon E \to \Gamma$ over Γ. Put $D_\Gamma := D \times \Gamma \to \Gamma$. Define a category $\underline{\mathrm{Fun}}_\Gamma(D_\Gamma, E)$ over Γ by the pullback square

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_\Gamma(D_\Gamma, E) & \longrightarrow & \mathrm{Fun}(D, E) \\ \downarrow & \lrcorner & \downarrow q_* \\ \Gamma & \xrightarrow{\delta_\Gamma} & \mathrm{Fun}(D, \Gamma), \end{array}$$

where δ_Γ is the currying of the projection $\Gamma \times D \to \Gamma$. Show that this category has the universal property of the relative functor category from D_Γ to E over Γ.

Exercise 3.8.2. Prove Lemma 3.2.10: show that for embeddings $\Gamma \hookrightarrow C^{\simeq}$ and $\Lambda \hookrightarrow D^{\simeq}$ the projections $C \times D \to C$ and $C \times D \to D$ induce an equivalence

$$\langle \Gamma \times \Lambda \subseteq C \times D \rangle \xrightarrow{\sim} \langle \Gamma \subseteq C \rangle \times \langle \Lambda \subseteq D \rangle.$$

Exercise 3.8.3. Prove the assertions from Remark 3.1.17: given a functor $f\colon C \to D$ sending a collection of morphisms M in C to a collection of morphisms N in D, we have:

(1) The map $f|\colon \langle M \rangle_C \to \langle N \rangle_D$ is uniquely determined by f.

(2) Given an identification $f = f'$, we obtain an induced identification $f| = f'|$.

(3) The construction is compatible with identities and composition: we have identifications $(\mathrm{id}_C)| = \mathrm{id}_{\langle M \rangle_C}$ and $(g \circ f)| = g| \circ f|$ if $g\colon D \to E$ is another functor preserving a collection of morphisms.

(4) If the functors $f\colon C \to D$ and $\tilde{f}\colon M \to N$ are equivalences, then also $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ is an equivalence.

Exercise 3.8.4. Let W be a collection of morphisms in a category C, and let $l\colon C \to L$ and $l'\colon C \to L'$ be two localizations of C at W. Construct an equivalence $L \xrightarrow{\sim} L'$.

Exercise 3.8.5. Let W be a collection of morphisms in a category C, and let $C \to C[W^{-1}]$ be a localization of C at W. Construct a pushout square

$$\begin{array}{ccc} [1]\times W & \xrightarrow{\tilde{w}} & C \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \ulcorner & \downarrow \\ W & \longrightarrow & C[W^{-1}], \end{array}$$

where the map $\tilde{w}$ is obtained by uncurrying the composite $W \xrightarrow{w} \mathrm{Map}([1], C) \to \mathrm{Fun}([1], C)$.

Exercise 3.8.6. Let $p\colon S \to T$ be a functor, and let $f\colon C \to S$ be a category over S. Let $g\colon D \to T$ and $g'\colon D' \to T$ be categories over T that are both a dependent product of C along p. Show that there is an equivalence $D \xrightarrow{\sim} D'$ of categories over T (in the sense of Definition 1.9.3).

4 Cartesian and cocartesian fibrations

In this chapter, we introduce synthetic analogues of the notions of *cartesian* and *cocartesian fibrations*, which play an important foundational role in this book.

In order to appreciate the relevance of these notions within our theory, let us start by recalling their role in classical category theory. Cartesian and cocartesian fibrations come up whenever one is interested in *functorial families of categories*. To make precise what this means, assume we are given some category Cat whose objects are small categories and whose morphisms are functors between them; we will introduce such a category in Chapter 8. For a category C, we may then think of functors $F\colon C \to \mathrm{Cat}$ as 'C-indexed families of (small) categories'. Informally, they consist of a category $F(c)$ for each object c of C, a 'transport functor' $F(u)\colon F(c) \to F(d)$ for each morphism $u\colon c \to d$, and various compatibilities of these transport functors with composition in C. In practice, these compatibilities are often not strict and need to be tracked; in classical language such functors are referred to as *pseudofunctors*. Constructing such functors by hand is often hard, and it becomes practically impossible in the setting of higher category theory.

Fortunately, there is an alternative way of packaging the data in such functors F via their *Grothendieck construction* $\int F$. The objects of this category are given by pairs (c, x), where c is an object of C and x is an object of $F(c)$. A morphism from (c, x) to (d, y) consists of a morphism $u\colon c \to d$ together with a morphism

$$F(u)(x) \to y$$

in $F(d)$. It comes equipped with a projection functor $p_F\colon \int F \to C$ which remembers the object or morphism in C. The individual categories $F(c)$ for $c \in C$ may be recovered from p_F by forming the fiber over c. The transport functors $F(u)$ can also be recovered from p_F as follows. Given a morphism $u\colon c \to d$ and an object $x \in F(c)$, there is a distinguished morphism over u of the form

$$\widetilde{u}\colon (c, x) \to (d, F(u)(x)),$$

determined by the identity of $F(u)(x)$. Moreover, every other morphism $(c, x) \to (d, y)$ over u factors uniquely through $\widetilde{u}$, followed by a morphism living in the fiber over d. More generally, a morphism from (c, x) over a composite morphism vu in C factors uniquely through $\widetilde{u}$, followed by a morphism over v. This universal property characterizes all these distinguished morphisms $\widetilde{u}$ purely in terms of p_F; they are called the *cocartesian morphisms*. We may thus recover $F(u)(x)$ from p_F by considering the unique cocartesian morphism over u in $\int F$ starting at (c, x) and passing to its second component. The compatibilities of the functor F with composition are recorded by p_F in an analogous manner.

The Grothendieck construction thus allows functors $F\colon C \to \mathrm{Cat}$ to be packaged in terms of functors $p\colon E \to C$. The functors arising from this construction are precisely the *cocartesian fibrations*,

characterized by the property that every morphism $u\colon c \to d$ in C admits cocartesian lifts in E starting at an arbitrary object x in the fiber over c. In practice, functors into Cat are often obtained by constructing a category E over C and checking it is a cocartesian fibration; since the latter is merely a *property*, this does not require any higher coherences and is therefore usually an easier task. One then obtains a functor $C \to \mathrm{Cat}$ by applying the Grothendieck correspondence. The construction can be dualized, leading to a notion of *cartesian fibrations* encoding contravariantly functorial families of categories.

Cocartesian fibrations already play an important role in the classical treatments of higher category theory, but they take on an even more foundational role within our synthetic setup:

- We need them to introduce the category Cat of small categories. Since our theory does not allow us to define categories by giving their objects and morphisms, we need to find a way to *characterize* Cat, and we will do so via the correspondence between functors $C \to \mathrm{Cat}$ and cocartesian fibrations over C.
- Moreover, they are essential for formulating the *functoriality of universals* property in Chapter 6. This crucial property is what makes precise the idea that various universal properties may be checked objectwise, and then automatically assemble into coherent constructions.

Since we do not yet have these tools at our disposal at this point, we need to be careful in how to precisely formulate the definition of a cocartesian fibration in our synthetic setup to guarantee that it really contains all the required higher coherences. We do so by considering a certain directed evaluation functor

$$\mathrm{Ar}(E) \to E \mathbin{\vec{\times}}_p C,$$

which sends a morphism in E to its source object and its image morphism in C. The target category records precisely the data for which we seek a cocartesian lift. We may then define p to be a cocartesian fibration by requiring this functor to admit a so-called left adjoint section: the section provides the lifts functorially, while the adjunction expresses their universal property.

We begin with directed pullbacks and left and right fibrations, followed by slice categories. We then develop adjunctions, including adjoint sections and relative adjunctions, before introducing cartesian and cocartesian fibrations. The remaining sections study the source and target fibrations, cocartesian functors, and locally cocartesian fibrations.

4.1 Directed pullbacks

We begin by introducing the notion of the *directed pullback* of two functors.

Construction 4.1.1 (Directed pullback). Consider two functors $f\colon A \to C$ and $g\colon B \to C$. We define the *directed pullback* $A \mathbin{\vec{\times}}_{f,g} B$ *of* A *and* B *over* C (sometimes called 'lax pullback') via the pullback square

$$\begin{array}{ccc} A \mathbin{\vec{\times}}_{f,g} B & \longrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ A \times B & \xrightarrow{f \times g} & C \times C. \end{array}$$

Note that a term of $A \vec{\times}_{f,g} B$ consists of a triple (a, b, φ) where a is a term of A, b is a term of B, and $\varphi\colon f(a) \to g(b)$ is a morphism in C.

We often abuse notation and write $A \vec{\times}_g B$ in the case where $C = A$ and $f = \mathrm{id}_A\colon A \to A$ and dually write $A \vec{\times}_f B$ when $C = B$ and $g = \mathrm{id}_B\colon B \to B$.

Given a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xleftarrow{g} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow[f']{} & C' & \xleftarrow[g']{} & B', \end{array}$$

we obtain an induced functor

$$\begin{aligned} \alpha \vec{\times}_\gamma \beta\colon A \vec{\times}_{f,g} B &\to A' \vec{\times}_{f',g'} B', \\ (a, b, \varphi) &\mapsto (\alpha(a), \beta(b), \gamma(\varphi)), \end{aligned}$$

where $\gamma(\varphi)$ is the map

$$f'(\alpha(a)) = \gamma(f(a)) \to \gamma(g(b)) = g'(\beta(b))$$

Remark 4.1.2. Combining Lemma 1.6.13 with the Rezk axiom (Axiom F), we obtain a pullback square

$$\begin{array}{ccc} A \times_C B & \longrightarrow & \mathrm{Iso}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle(\mathrm{ev}_0,\mathrm{ev}_1)} \\ A \times B & \xrightarrow{f\times g} & C \times C. \end{array}$$

We will see in Corollary 6.4.12 below that the inclusion $\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(C) \hookrightarrow \mathrm{Fun}([1], C)$ is fully faithful, and Proposition 6.4.6 then identifies it as a full subcategory. Hence it induces a full subcategory $A \times_C B \hookrightarrow A \vec{\times}_{f,g} B$. Therefore, we may think of the pullback $A \times_C B$ as the full subcategory of $A \vec{\times}_{f,g} B$ spanned by those triples (a, b, γ) such that the map $\gamma\colon f(a) \xrightarrow{\sim} g(b)$ is an isomorphism in C.

Construction 4.1.3 (Directed evaluation maps). Consider a functor $f\colon A \to B$. We define the *directed evaluation maps*

$$\begin{aligned} \vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) &\to A \vec{\times}_f B \\ (\varphi\colon a \to a') &\mapsto (a, f(a'), f(\varphi)) \end{aligned} \quad \text{and} \quad \begin{aligned} \vec{\mathrm{ev}}_1^f\colon \mathrm{Fun}([1], A) &\to B \vec{\times}_f A \\ (\varphi\colon a \to a') &\mapsto (f(a), a', f(\varphi)) \end{aligned}$$

via the following two commutative diagrams:

$$\begin{array}{ccccc} \mathrm{Fun}([1], A) & \overset{\vec{\mathrm{ev}}_0^f}{\dashrightarrow} & A \vec{\times}_f B & \longrightarrow & \mathrm{Fun}([1], B) \\ {\scriptstyle(\mathrm{ev}_0,\mathrm{ev}_1)}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle(\mathrm{ev}_0,\mathrm{ev}_1)} \\ A \times A & \xrightarrow{\mathrm{id}_A\times f} & A \times B & \xrightarrow{f\times \mathrm{id}_B} & B \times B \end{array} \qquad (f_*\ \text{over the top})$$

and

$$\begin{array}{ccccc} \mathrm{Fun}([1], A) & \overset{\vec{\mathrm{ev}}_1^f}{\dashrightarrow} & B \vec{\times}_f A & \longrightarrow & \mathrm{Fun}([1], B) \\ {\scriptstyle(\mathrm{ev}_0,\mathrm{ev}_1)}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle(\mathrm{ev}_0,\mathrm{ev}_1)} \\ A \times A & \xrightarrow{f\times \mathrm{id}_A} & B \times A & \xrightarrow{\mathrm{id}_B\times f} & B \times B. \end{array} \qquad (f_*\ \text{over the top})$$

Remark 4.1.4. In case $B = *$ there are equivalences $A \vec{\times}_f B \simeq A \simeq B \vec{\times}_f A$ and the directed evaluation maps $\vec{\mathrm{ev}}_0^f$ and $\vec{\mathrm{ev}}_1^f$ simplify to the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}([1], A) \to A$.

For later use, we record various properties of the directed evaluation maps.

Lemma 4.1.5. *Assume that $f \colon A \to B$ is an equivalence. Then also $\vec{\mathrm{ev}}_0^f$ and $\vec{\mathrm{ev}}_1^f$ are equivalences.*

Proof. For $\vec{\mathrm{ev}}_0^f$, this follows from the 2-out-of-3 property applied to the following two maps:

$$\mathrm{Fun}([1], A) \xrightarrow{\vec{\mathrm{ev}}_0^f} A \vec{\times}_f B \to \mathrm{Fun}([1], B).$$

Indeed, the composite is induced by f and is hence an equivalence, and the second map is a base change of f and thus also an equivalence. The proof for $\vec{\mathrm{ev}}_1^f$ is similar. □

Lemma 4.1.6 (Base change). *Consider a pullback square of the form*

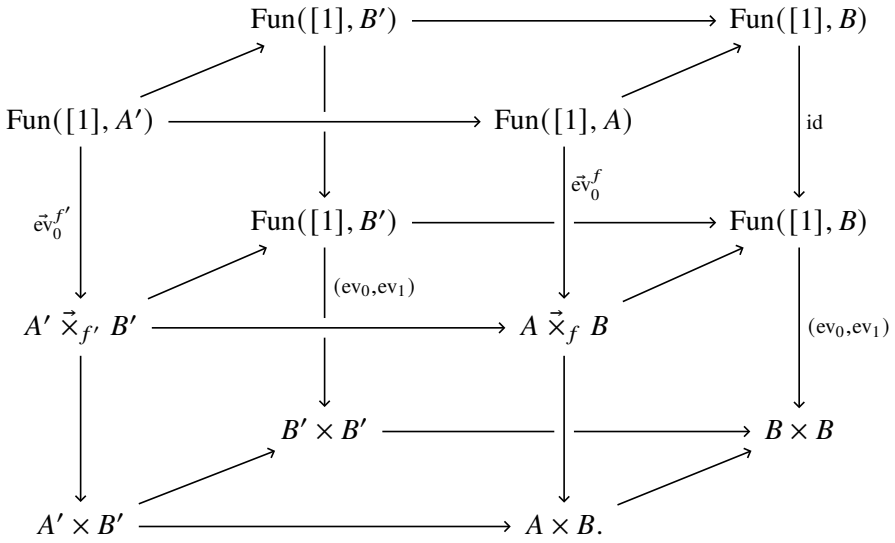
$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{v} & B. \end{array}$$

Then the commutative square

$$\begin{array}{ccc} \mathrm{Fun}([1], A') & \xrightarrow{u_*} & \mathrm{Fun}([1], A) \\ {\scriptstyle \vec{\mathrm{ev}}_0^{f'}}\downarrow & & \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^{f}} \\ A' \vec{\times}_{f'} B' & \xrightarrow{v \vec{\times}_f u} & A \vec{\times}_f B \end{array}$$

is a pullback square. The dual result for $\vec{\mathrm{ev}}_1^f$ and $\vec{\mathrm{ev}}_1^{f'}$ also holds.

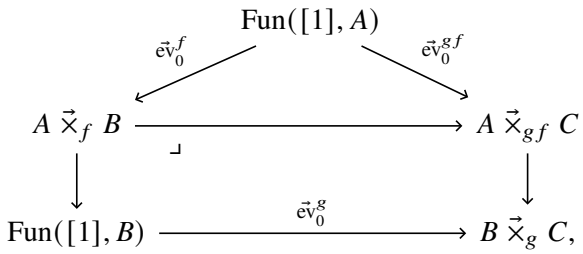
Proof. Consider the following commutative diagram:



Observe that the bottom square is a pullback square, while the lower vertical squares on the left and right sides are pullback squares by definition of $A' \vec{\times}_{f'} B'$ and $A \vec{\times}_f B$. It follows by the pasting

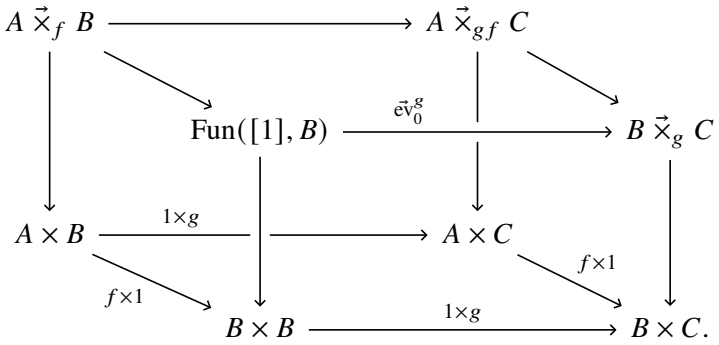
law of pullback squares that also the middle horizontal square is a pullback square. Since also the upper square at the back and the top square are pullback squares, it thus follows that the upper front square is a pullback square, as desired. □

Lemma 4.1.7 (Composition). *For functors $f\colon A \to B$ and $g\colon B \to C$, there is a commutative diagram*

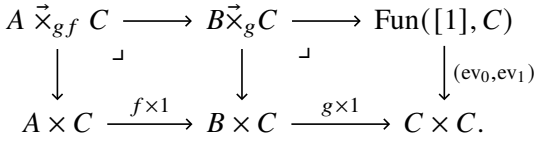


where the bottom square is a pullback square.

Proof. It is easy to verify that the diagram commutes, hence it remains to show that the bottom square is a pullback square. To this end, consider the following commutative diagram:

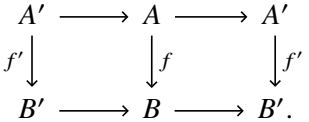


We want to show that the top square is a pullback square. The left square is a pullback square by definition and the bottom square is easily seen to be a pullback square, hence by the pasting law of pullback squares it will suffice to show that the right square is a pullback square. This follows from another instance of the pasting law, applied to the following commutative diagram:



Indeed, by the definitions of $A \vec{\times}_{gf} C$ and $B \vec{\times}_g C$ the right and outer squares are pullback squares and hence so is the left square. □

Lemma 4.1.8 (Retracts). *Consider a retract diagram of functors:*



Then the map $\vec{\mathrm{ev}}_0^{f'}$ *is a retract of* $\vec{\mathrm{ev}}_0^{f}$*:*

$$\begin{array}{ccccc} \mathrm{Fun}([1],A') & \longrightarrow & \mathrm{Fun}([1],A) & \longrightarrow & \mathrm{Fun}([1],A') \\ \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^{f'}} & & \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^{f}} & & \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^{f'}} \\ A' \vec{\times}_{f'} B' & \longrightarrow & A \vec{\times}_{f} B & \longrightarrow & A' \vec{\times}_{f'} B'. \end{array}$$

Proof. This is an immediate consequence of the definitions. □

4.2 Left and right fibrations

We define the notions of *left* and *right fibrations*, and show that they admit a notion of *covariant/contravariant transport*.

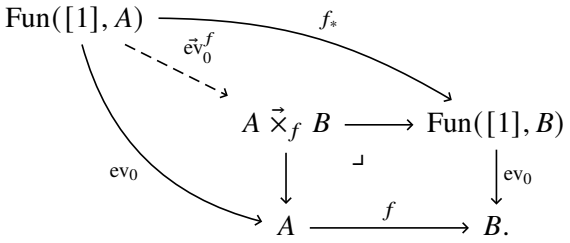
Definition 4.2.1 (Left/right fibration). A functor $f\colon A \to B$ is called a *left fibration* if the directed evaluation map $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$ is an equivalence. Dually, we say that f is a *right fibration* if $\vec{\mathrm{ev}}_1^f\colon \mathrm{Fun}([1],A) \to B \vec{\times}_f A$ is an equivalence.

Lemma 4.2.2. *A functor* $f\colon A \to B$ *is a left fibration if and only if the commutative square*

$$\begin{array}{ccc} \mathrm{Fun}([1],A) & \xrightarrow{f_*} & \mathrm{Fun}([1],B) \\ \downarrow{\scriptstyle \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback. A dual characterization for right fibrations is obtained by replacing $\vec{\mathrm{ev}}_0^f$ *by* $\vec{\mathrm{ev}}_1^f$ *and* ev_0 *by* ev_1.

Proof. This is immediate from the fact that the directed evaluation map $\vec{\mathrm{ev}}_0^f$ is defined as the unique dashed map filling the following diagram:



The proof for right fibrations is dual. □

Corollary 4.2.3. *Let* $f\colon A \to B$ *be a left (resp. right) fibration. Then the commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle (p_{[1]})^*} & & \downarrow{\scriptstyle (p_{[1]})^*} \\ \mathrm{Fun}([1],A) & \xrightarrow{f_*} & \mathrm{Fun}([1],B) \end{array}$$

is a pullback square.

Proof. We prove the claim for left fibrations; the claim for right fibrations is dual. Consider the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle (p_{[1]})^*}\downarrow & & \downarrow{\scriptstyle (p_{[1]})^*} \\ \mathrm{Fun}([1], A) & \xrightarrow{f_*} & \mathrm{Fun}([1], B) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ A & \xrightarrow{f} & B. \end{array}$$

The outer square is clearly cartesian, and the bottom square is cartesian by Lemma 4.2.2. It thus follows from the pasting law of pullback squares that also the top square is cartesian. □

Proposition 4.2.4. *Let $p\colon X \to Y$ be a functor and assume that Y is a groupoid. Then the following conditions are equivalent:*

(1) The category X is a groupoid;

(2) The map p is a left fibration;

(3) The map p is a right fibration.

Proof. We prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Fun}([1], X) & \xrightarrow{p_*} & \mathrm{Fun}([1], Y) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \sim\downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow{p} & Y. \end{array}$$

Since Y is a groupoid, the right vertical map is an equivalence. It follows that this square is a pullback square if and only if the left vertical map is an equivalence. But the former is by Lemma 4.2.2 equivalent to (2), while the latter is equivalent to (1). □

Corollary 4.2.5. *If $f\colon X \to Y$ is a left (resp. right) fibration and Y is a groupoid, then also X is a groupoid.* □

Corollary 4.2.6. *Let $f\colon A \to B$ be a left (resp. right) fibration. Then every fiber of f is a groupoid.*

Proof. For every term b of B, the map $A_b \to \{b\}$ is again a left (resp. right) fibration, hence the claim follows from the previous corollary. □

4.2.1 Covariant transport

For a left fibration $f\colon A \to B$, the fibers A_b are suitably *functorial* in the object $b \in B$: every morphism $\beta\colon b \to b'$ in B induces a functor $\beta_!\colon A_b \to A_{b'}$ between fibers, called the *covariant transport functor*. Dually, right fibrations admit *contravariant transport functors*. We will now construct these functors and prove various of their basic properties.

Definition 4.2.7 (Category of lifts). Let $f\colon A \to B$ be a functor, let Γ be a category, and consider a term $\Gamma \vdash (a,\beta) : A \vec{\times}_f B$, given by a term a of A and a morphism $\beta\colon f(a) \to b'$ in B. We define the category $\mathrm{Lift}_0^f(a,\beta)$ of *covariant lifts* as the pullback

$$\begin{array}{ccc} \mathrm{Lift}_0^f(a,\beta) & \longrightarrow & \mathrm{Fun}([1],A) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^f} \\ \Gamma & \xrightarrow{(a,\beta)} & A \vec{\times}_f B. \end{array}$$

Its structural functor to Γ makes it a category in context Γ. A term of $\mathrm{Lift}_0^f(a,\beta)$ is called a *covariant lift of β starting in a* and consists of a morphism $\alpha\colon a \to a'$ in A whose image under f is identified with β, i.e. there is an isomorphism between β and $f(\alpha)$ as objects of the relative slice $B_{f(a)/}$.

Dually, for a term $\Gamma \vdash (a',\beta) : B \vec{\times}_f A$, we define the category $\mathrm{Lift}_1^f(a',\beta)$ of *contravariant lifts* by the pullback

$$\begin{array}{ccc} \mathrm{Lift}_1^f(a',\beta) & \longrightarrow & \mathrm{Fun}([1],A) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \vec{\mathrm{ev}}_1^f} \\ \Gamma & \xrightarrow{(a',\beta)} & B \vec{\times}_f A. \end{array}$$

A term of $\mathrm{Lift}_1^f(a',\beta)$ is called a *contravariant lift of β ending in a'* and consists of a morphism $\alpha\colon a \to a'$ in A whose image under f is identified with β, i.e. there is an isomorphism between β and $f(\alpha)$ as objects of the relative slice $B_{/f(a')}$.

Corollary 4.2.8. *A functor $f\colon A \to B$ is a left fibration if and only if, for every category Γ and every term $\Gamma \vdash (a,\beta) : A \vec{\times}_f B$, the structural functor $\mathrm{Lift}_0^f(a,\beta) \to \Gamma$ is an equivalence. Dually, f is a right fibration if and only if $\mathrm{Lift}_1^f(a',\beta) \to \Gamma$ is an equivalence for every term $\Gamma \vdash (a',\beta) : B \vec{\times}_f A$.*

Proof. If $\vec{\mathrm{ev}}_0^f$ is an equivalence, then every pullback of it is an equivalence, so every structural functor $\mathrm{Lift}_0^f(a,\beta) \to \Gamma$ is an equivalence. Conversely, take (a,β) to be the universal term of $A \vec{\times}_f B$. The corresponding pullback is $\vec{\mathrm{ev}}_0^f$ itself, so the hypothesis says precisely that $\vec{\mathrm{ev}}_0^f$ is an equivalence. The right-fibration statement is dual. □

Remark 4.2.9. Objectwise recognition of left fibrations will be treated later, using functoriality of universals.

In what follows, we will spell out various constructions in the case of left fibrations, and be more terse about the dual constructions for right fibrations.

Construction 4.2.10 (Covariant transport). Let $f\colon A \to B$ be a left fibration. We denote by

$$\mathrm{lift}_0^f\colon A \vec{\times}_f B \to \mathrm{Fun}([1],A)$$

the inverse of the functor $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$. For a term (a,β) of $A \vec{\times}_f B$, we denote the morphism $\mathrm{lift}_0^f(a,\beta)$ in A by

$$\mathrm{lift}_0^f(a,\beta)\colon a \to \beta_!(a),$$

and refer to its target $\beta_!(a)$ as the *covariant transport of a along β*. Observe that $\mathrm{lift}_0^f(a,\beta)$ is a covariant lift of β starting in a, and in particular we see that $\beta_!(a)$ is a term of $A_{b'}$. Fixing a

morphism $\beta\colon b \to b'$ of B, the assignment $a \mapsto \beta_!(a)$ thus provides a functor

$$\beta_!\colon A_b \to A_{b'}.$$

Dually, if f is instead a *right* fibration, we denote the inverse of $\vec{\mathrm{ev}}_1^f$ by lift_1^f. Its value at a term (a', β) of $B \mathbin{\vec{\times}}_f A$ is denoted by $\mathrm{lift}_1^f(a', \beta)\colon \beta^*(a') \to a'$ and we refer to $\beta^*(a')$ as the *contravariant transport of a' along β*. If we again fix a morphism $\beta\colon b \to b'$ of B, this construction assembles into a functor $\beta^*\colon A_{b'} \to A_b$.

Remark 4.2.11. We remind the reader that we may construct functors like $\beta_!\colon A_b \to A_{b'}$ simply by providing for every term of A_b a term of $A_{b'}$: a 'term' is defined as a functor $a\colon \Gamma \to A_b$ from an arbitrary category Γ, and Construction 4.2.10 provides for us another functor $\beta_!(a)\colon \Gamma \to A_{b'}$. Applying this to the identity functor $a = \mathrm{id}_{A_b}\colon A_b \to A_b$ thus provides the desired functor $\beta_!\colon A_b \to A_{b'}$.

Since this is the first time we use this method for constructing a functor, let us give an alternative definition of $\beta_!\colon A_b \to A_{b'}$ that does not use the language of terms. As a first step, we observe that the assignment $a \mapsto (a, \beta)$ extends to a functor $A_b \to A \mathbin{\vec{\times}}_f B$, defined as the unique dashed arrow making the following diagram commute:

$$\begin{array}{ccccc} A_b & \overset{a\mapsto(a,\beta)}{\dashrightarrow} & A \mathbin{\vec{\times}}_f B & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow f \\ \{\beta\} & \xrightarrow{\beta} & \mathrm{Fun}([1], B) & \xrightarrow{\mathrm{ev}_0} & B. \end{array}$$

Here the outer rectangle is the commutative square defining the fiber A_b. We may now consider the composite

$$A_b \xrightarrow{a\mapsto(a,\beta)} A \mathbin{\vec{\times}}_f B \xrightarrow{\mathrm{lift}_0^f} \mathrm{Fun}([1], A) \xrightarrow{\mathrm{ev}_1} A.$$

Observe that the composite of ev_1 with f is isomorphic to the composite $\mathrm{Fun}([1], A) \xrightarrow{\vec{\mathrm{ev}}_0^f} A \mathbin{\vec{\times}}_f B \to \mathrm{Fun}([1], B) \xrightarrow{\mathrm{ev}_1} B$, and since lift_0^f is a section of $\vec{\mathrm{ev}}_0$ we see that this composite factors uniquely through the fiber $A_{b'}$. All in all we obtain a functor $\beta_!\colon A_b \to A_{b'}$.

The next proposition shows that the covariant transport functors $\beta_!\colon A_b \to A_{b'}$ are suitably 'functorial':

Proposition 4.2.12. *Let $f\colon A \to B$ be a left fibration.*

(1) For $\beta = \mathrm{id}_b\colon b \to b$, the functor $(\mathrm{id}_b)_!\colon A_b \to A_b$ is identified with the identity functor;

(2) For morphisms $\beta\colon b \to b'$ and $\beta'\colon b' \to b''$ in B, the composite functor

$$\beta'_! \circ \beta_!\colon A_b \to A_{b''}$$

is identified with $(\beta' \circ \beta)_!\colon A_b \to A_{b''}$.

The dual result for the contravariant transport functors of a right fibration is also valid.

Proof. For (1), observe that for any term $a : A_b$ the map $\mathrm{id}_a \colon a \to a$ is a covariant lift of $\mathrm{id}_b \colon b \to b$ starting in a. By uniqueness of covariant lifts, it follows that $\mathrm{id}_a = \mathrm{lift}_0^f(a, \mathrm{id}_b)$, and in particular we have $a = (\mathrm{id}_b)_! a$. Applying this to the universal term of A_b provides the desired identification $(\mathrm{id}_b)_! = \mathrm{id}_{A_b}$.

For (2), we have for a term $a : A_b$ a covariant lift

$$\mathrm{lift}_0^f(a, \beta) \colon a \to \beta_!(a)$$

of β starting in a, which in turn leads to a covariant lift

$$\mathrm{lift}_0^f(\beta_!(a), \beta') \colon \beta_!(a) \to \beta'_!(\beta_!(a)).$$

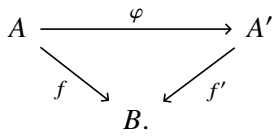
Since the composite of these two morphisms is a covariant lift of $\beta' \circ \beta$ starting in a, it follows that there is an isomorphism

$$\mathrm{lift}_0^f(\beta_!(a), \beta') \circ \mathrm{lift}_0^f(a, \beta) = \mathrm{lift}_0^f(a, \beta' \circ \beta),$$

and in particular we get $(\beta' \circ \beta)_!(a) = \beta'_!(\beta_!(a))$. Taking a to be the universal term of A_b then provides the desired identification. □

The covariant transport functors are compatible with maps of left fibrations:

Lemma 4.2.13. *Consider a morphism of left fibrations over B:*



Then there is for every morphism $\beta \colon b \to b'$ in B and every term a of A_b an identification

$$\beta_!(\varphi_b(a)) = \varphi_{b'}(\beta_!(a))$$

of functors $A_b \to A'_{b'}$.

Proof. This is clear from the observation that the morphism

$$\varphi(\mathrm{lift}_0^f(a, \beta)) \colon \varphi_b(a) \to \varphi_{b'}(\beta_!(a))$$

in A' is a covariant lift of β starting in $\varphi_b(a)$, so that it must be isomorphic to the morphism $\mathrm{lift}_0^f(\varphi_b(a), \beta) \colon \varphi_b(a) \to \beta_!(\varphi_b(a))$. □

4.2.2 Closure properties left/right fibrations

We now prove various preservation properties of left and right fibrations. While the results below are only stated for left fibrations, all of them admit dual versions for right fibrations whose formulations we leave to the reader.

Lemma 4.2.14. *Every equivalence $f \colon A \xrightarrow{\sim} B$ is a left fibration.*

Proof. This is immediate from Lemma 4.1.5. □

Proposition 4.2.15 (Base change). *Consider a pullback square of categories*

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{v} & B. \end{array}$$

(1) If f is a left fibration, then also f' is a left fibration;

(2) For a morphism $\beta\colon b \to b'$ in B', the following square commutes:

$$\begin{array}{ccc} A'_b & \xrightarrow{\beta_!} & A'_{b'} \\ {\scriptstyle u_b}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle u_{b'}} \\ A_{v(b)} & \xrightarrow{v(\beta)_!} & A_{v(b')}. \end{array}$$

Proof. Part (1) is immediate from Lemma 4.1.6, which says that the map $\vec{\mathrm{ev}}_0^{f'}$ is a base change of $\vec{\mathrm{ev}}_0^{f}$. This implies that the inverse map $\mathrm{lift}_0^{f'}$ is a base change of lift_0^{f}: given a morphism $\beta\colon b \to b'$ in B' and a term a of A'_b, the covariant lift $\mathrm{lift}_0^{f'}(a,\beta)\colon a \to \beta_!(a)$ of β starting in a is sent under u to the covariant lift $\mathrm{lift}_0^{f}(u(a), v(\beta))\colon u(a) \to v(\beta)_!(u(a))$ of $v(\beta)\colon v(b) \to v(b')$ starting in $u(a)$. This proves part (2). □

Proposition 4.2.16 (Composition). *Consider a commutative diagram*

$$\begin{array}{ccccc} & & B & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ A & & \xrightarrow[gf]{} & & C \end{array}$$

of functors, and assume that g is a left fibration. Then f is a left fibration if and only if gf is a left fibration.

Proof. By Lemma 4.1.7, there is a commutative triangle

$$\begin{array}{ccc} & \mathrm{Fun}([1], A) & \\ {\scriptstyle \vec{\mathrm{ev}}_0^{f}}\swarrow & & \searrow{\scriptstyle \vec{\mathrm{ev}}_0^{gf}} \\ A \mathbin{\vec{\times}}_f B & \longrightarrow & A \mathbin{\vec{\times}}_{gf} C, \end{array}$$

where the bottom map is a base change of the map $\vec{\mathrm{ev}}_0^{g}$ and hence is an equivalence as g is a left fibration. It follows by 2-out-of-3 that the map $\vec{\mathrm{ev}}_0^{f}$ is an equivalence if and only if $\vec{\mathrm{ev}}_0^{gf}$ is an equivalence. This finishes the proof. □

Lemma 4.2.17. *The class of left fibrations is stable under retracts.*

Proof. This is immediate from Lemma 4.1.8. □

Lemma 4.2.18. *Let $f\colon C \to D$ be a left fibration. Then for every category E also the induced functor $f_*\colon \mathrm{Fun}(E, C) \to \mathrm{Fun}(E, D)$ is a left fibration.*

Proof. Observe that the directed evaluation map for f_* is obtained by applying $\mathrm{Fun}(E, -)$ to the directed evaluation map of f, and hence is again an equivalence. □

4.3 Slice categories

In this section we provide two fundamental examples of left and right fibrations: the forgetful functors $C_{/x} \to C$ and $C_{x/} \to C$ for a category C and a term x of C. We further show that resulting functoriality on the fibers $C(x, y)$ of these two fibrations is given by precomposition and postcomposition in C, respectively.

Recall first from Definition 2.1.8 the definitions of the slice categories $C_{/x}$ and $C_{x/}$:

$$\begin{array}{ccc} C_{/x} & \longrightarrow & \mathrm{Fun}([1], C) \\ {\scriptstyle p_x}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ C & \xrightarrow{\mathrm{id}_C \times x} & C \times C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \{x\} & \xrightarrow{x} & C. \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C_{x/} & \longrightarrow & \mathrm{Fun}([1], C) \\ {\scriptstyle q_x}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ C & \xrightarrow{x \times \mathrm{id}_C} & C \times C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \{x\} & \xrightarrow{x} & C. \end{array}$$

Remark 4.3.1. The slice category $C_{/x}$ is a special case of the more general slice category construction $C_{/\psi}$ from Construction 3.7.1 by taking $Y = *$. It is also a special case of the directed pullback construction Construction 4.1.1 by taking $f = \mathrm{id}_C : C \to C$ and $g = x : * \to C$.

Lemma 4.3.2. *Consider categories C and D, let x be a term of C, and let $\tilde{x} \in \mathrm{Fun}(D, C)$ denote the constant functor on x. Then there are equivalences*

$$\mathrm{Fun}(D, C_{/x}) \xrightarrow{\sim} \mathrm{Fun}(D, C)_{/\tilde{x}} \qquad \textit{and} \qquad \mathrm{Fun}(D, C_{x/}) \xrightarrow{\sim} \mathrm{Fun}(D, C)_{\tilde{x}/}.$$

Proof. Since $\mathrm{Fun}(D, -)$ preserves pullbacks, we have a pullback square

$$\begin{array}{ccc} \mathrm{Fun}(D, C_{/x}) & \longrightarrow & \mathrm{Fun}([1], \mathrm{Fun}(D, C)) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathrm{Fun}(D, C) \times \{\tilde{x}\} & \xrightarrow{\mathrm{id}_{\mathrm{Fun}(D,C)} \times \tilde{x}} & \mathrm{Fun}(D, C) \times \mathrm{Fun}(D, C), \end{array}$$

showing that $\mathrm{Fun}(D, C_{/x}) \xrightarrow{\sim} \mathrm{Fun}(D, C)_{/\tilde{x}}$. The equivalence $\mathrm{Fun}(D, C_{x/}) \xrightarrow{\sim} \mathrm{Fun}(D, C)_{\tilde{x}/}$ is analogous. □

The following is the main result of this section:

Theorem 4.3.3. *Let C be a category and let x be a term of C. Then the forgetful functor $p_x : C_{/x} \to C$ is a right fibration. Dually, the map $q_x : C_{x/} \to C$ is a left fibration.*

Proof. By symmetry it will suffice to prove the claim for $C_{/x} \to C$. We have to show that the functor

$$\vec{\mathrm{ev}}_1^{p_x} : \mathrm{Fun}([1], C_{/x}) \to C \vec{\times}_{p_x} C_{/x}$$

is an equivalence. By Lemma 4.3.2, the source of this functor sits in a pullback square of the form

$$\begin{array}{ccc} \mathrm{Fun}([1], C_{/x}) & \longrightarrow & \mathrm{Fun}([1], \mathrm{Fun}([1], C)) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathrm{Fun}([1], C) \times \{x\} & \xrightarrow{1 \times x} \mathrm{Fun}([1], C) \times C & \xrightarrow{1 \times p^*_{[1]}} \mathrm{Fun}([1], C) \times \mathrm{Fun}([1], C), \end{array}$$

and as a consequence of Corollary 2.2.8 we thus obtain a pullback square

$$\begin{array}{ccc} \mathrm{Fun}([1], C_{/x}) & \longrightarrow & \mathrm{Fun}(\Delta^2, C) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_2} \\ \{x\} & \xrightarrow{x} & C. \end{array}$$

On the other hand, the target $C \vec{\times}_{p_x} C_{/x}$ is defined via the pullback square

$$\begin{array}{ccc} C \vec{\times}_{p_x} C_{/x} & \longrightarrow & \mathrm{Fun}([1], C) \times \{x\} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_1 \times \mathrm{id}} \\ C_{/x} & \longrightarrow & C \times \{x\} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle 1 \times x} \\ \mathrm{Fun}([1], C) & \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} & C \times C, \end{array}$$

and consequently we obtain a pullback square

$$\begin{array}{ccc} C \vec{\times}_{p_x} C_{/x} & \longrightarrow & \mathrm{Fun}(\Lambda^2_1, C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_2} \\ \{x\} & \xrightarrow{x} & C. \end{array}$$

Under these two identifications, the map $\vec{\mathrm{ev}}_1^{p_x} \colon \mathrm{Fun}([1], C_{/x}) \to C \vec{\times}_{p_x} C_{/x}$ is obtained by pulling back the map $\mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$ along $x \colon \{x\} \to C$, and thus it is an equivalence by the Segal axiom Axiom E. □

Remark 4.3.4. In more down-to-earth terms, the above proof is capturing the equivalence between three ways of encoding maps in the slice $C_{/x}$:

$$\begin{array}{ccc} y & \longrightarrow & z \\ \downarrow & & \downarrow \\ x & \xrightarrow{\mathrm{id}_x} & x \end{array} \quad \overset{Corollary\ 2.2.8}{\Longleftrightarrow} \quad \begin{array}{ccc} & z & \\ \nearrow & & \searrow \\ y & \longrightarrow & x \end{array} \quad \overset{Axiom\ E}{\Longleftrightarrow} \quad \begin{array}{ccc} & z & \\ \nearrow & & \searrow \\ y & & x. \end{array}$$

Corollary 4.3.5. *For two terms x and y in a category C, the hom anima $C(x, y)$ satisfies the groupoid condition, and hence is an anima.*

Proof. Since $C(x, y)$ is the fiber over x of the right fibration $C_{/y} \to C$ from Theorem 4.3.3, this follows from Corollary 4.2.6. □

Since the fiber of the left fibration $C_{x/} \to C$ over a term y of C is the hom anima $C(x, y)$ from Definition 2.1.3, Construction 4.2.10 produces for every morphism $g \colon y \to z$ a covariant transport functor

$$g_! \colon C(x, y) \to C(x, z).$$

This functor may be identified with a functor we have already encountered:

Proposition 4.3.6. *For every morphism $g\colon y \to z$, the functor $g_!\colon C(x,y) \to C(x,z)$ agrees with the composition functor*

$$g \circ -\colon C(x,y) \to C(x,z)$$

from Construction 2.2.3. Dually, the contravariant transport functor $g^*\colon C(z,x) \to C(y,x)$ associated to the right fibration $C_{/x} \to C$ agrees with the composition functor $- \circ g\colon C(z,x) \to C(y,x)$.

Proof. As we have seen in the proof of Theorem 4.3.3, the directed evaluation map $\vec{\mathrm{ev}}_0^{q_x}\colon \mathrm{Fun}([1], C_{x/}) \to C \mathbin{\vec{\times}}_{q_x} C_{x/}$ is a base change of the map $\mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$, and hence the map $\mathrm{lift}_0^{q_x}$ is given by the inverse of this map. But this inverse precisely defines composition in C. In more detail, fixing the morphism $g\colon y \to z$, the covariant transport functor $g_!$ is given as the composite

$$C(x,y) \xrightarrow{f \mapsto (f,g)} C \mathbin{\vec{\times}}_{q_x} C_{x/} \xrightarrow{\mathrm{lift}_0^{q_x}} \mathrm{Fun}([1], C_{x/}) \xrightarrow{\mathrm{ev}_1} C_{x/},$$

where one checks that this lands in the fiber $C(x,z)$ over z. Due to the above identification of the middle map, we may rewrite this as

$$\begin{aligned} C(x,y) \to \mathrm{Fun}([1], C) &\xrightarrow{f \mapsto (f,g)} \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C) \\ &\simeq \mathrm{Diag}(\Lambda^2_1, C) \xrightarrow{\sim} \mathrm{Diag}(\Delta^2, C) \xrightarrow{\mathrm{ev}_{\{1,2\}}} \mathrm{Fun}([1], C), \end{aligned}$$

where again one checks that this lands in $C(x,z)$. But this is precisely the map $g \circ -\colon C(x,y) \to C(x,z)$. □

The previous results directly imply versions for the *relative slice categories* $C_{/d}$ for a functor $f\colon C \to D$ and a term d of D:

Definition 4.3.7 (Relative slice). Given a functor $f\colon C \to D$ and a term d of D we define the *relative slices* $C_{/d}$ and $C_{d/}$ via the following pullback squares:

$$\begin{array}{ccc} C_{/d} & \longrightarrow & D_{/d} \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow{f} & D \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C_{d/} & \longrightarrow & D_{d/} \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow{f} & D. \end{array}$$

Corollary 4.3.8. *For any functor $f\colon C \to D$ and any term d of D, the functor $C_{/d} \to C$ is a right fibration and the functor $C_{d/} \to C$ is a left fibration.*

Proof. This is immediate from Theorem 4.3.3 and Proposition 4.2.15. □

Corollary 4.3.9. *For a functor $\psi\colon X \to C$, the forgetful functor $C_{/\psi} \to C$ from Construction 3.7.1 is a right fibration. Dually, $C_{\psi/} \to C$ is a left fibration.*

Proof. We prove the first statement; the second statement is dual. Unwinding definitions, we see that the category $C_{/\psi}$ sits in a pullback square of the form

$$\begin{array}{ccc} C_{/\psi} & \longrightarrow & \mathrm{Fun}(X, C)_{/\psi} \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow{p_X^*} & \mathrm{Fun}(X, C), \end{array}$$

hence the result is a special case of the previous corollary. □

4.3.1 Assembling slice fibrations

To assemble the slice projections while allowing their distinguished object to vary, we use fibrations relative to a base category. All constructions below are categories and functors over that base.

For $p\colon E \to S$, put $[1]_S = S \times [1] \to S$ and write

$$\underline{\mathrm{Fun}}_S([1]_S, E) := S \times_{\mathrm{Fun}([1],S)} \mathrm{Fun}([1], E),$$

where $S \to \mathrm{Fun}([1], S)$ assigns identity morphisms and the other functor is induced by p. This category over S describes arrows of E equipped with an identification of their image in S with an identity. Its category of sections is the absolute relative functor category $\mathrm{Fun}_S([1]_S, E)$ from Definition 1.9.2.

For a functor $f\colon E \to D$ over S, with structural functors $p\colon E \to S$, $q\colon D \to S$, and identification $qf = p$, define

$$E \mathbin{\vec{\times}}_{f,S} D := S \times_{\mathrm{Fun}([1],S)} (E \mathbin{\vec{\times}}_f D).$$

The map $E \mathbin{\vec{\times}}_f D \to \mathrm{Fun}([1], S)$ sends $f(e) \to d$ to $p(e) \to q(d)$, using the specified identification $qf = p$. Directed evaluation induces a functor over S

$$\vec{\mathrm{ev}}^f_{0,S}\colon \underline{\mathrm{Fun}}_S([1]_S, E) \longrightarrow E \mathbin{\vec{\times}}_{f,S} D.$$

Dually, the map $D \mathbin{\vec{\times}}_f E \to \mathrm{Fun}([1], S)$ sends $d \to f(e)$ to $q(d) \to p(e)$, giving $D \mathbin{\vec{\times}}_{f,S} E$ and $\vec{\mathrm{ev}}^f_{1,S}$.

Definition 4.3.10 (Relative left and right fibrations)**.** A functor $f\colon E \to D$ over S is a *left fibration relative to* S if $\vec{\mathrm{ev}}^f_{0,S}$ is an equivalence over S. It is a *right fibration relative to* S if $\vec{\mathrm{ev}}^f_{1,S}$ is an equivalence over S.

Proposition 4.3.11. *Let C be a category. Then the commutative diagram*

$$\begin{array}{ccc} \mathrm{Fun}([1], C) & \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} & C \times C \\ & {\scriptstyle \mathrm{ev}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2} & \\ & C & \end{array}$$

determines a right fibration relative to C. Dually, the diagram

$$\begin{array}{ccc} \mathrm{Fun}([1], C) & \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} & C \times C \\ & {\scriptstyle \mathrm{ev}_0} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & C & \end{array}$$

determines a left fibration relative to C.

Proof. We prove the first statement. Put $E = \mathrm{Fun}([1], C)$, $D = C \times C$, and $f = (\mathrm{ev}_0, \mathrm{ev}_1)$, with structural maps ev_1 and pr_2. The source of relative directed evaluation is

$$\underline{\mathrm{Fun}}_C([1]_C, E) = C \times_{\mathrm{Fun}([1],C)} \mathrm{Fun}([1], \mathrm{Fun}([1], C)),$$

where the map from the square category is restriction to the target side. By Corollary 2.2.8, the defining pullback identifies this category with $\mathrm{Triang}(C)$ over C, with structural functor ev_2. The

comparison replaces a triangle by the square whose target side is the identity of its last vertex. It includes the identification of that side with the constant arrow, as part of the pullback cone.

We next identify the target $D \vec{\times}_{f,C} E$. Using preservation of products by $\mathrm{Fun}([1], -)$ in its defining pullbacks gives

$$D \vec{\times}_{f,C} E \simeq C \times_{\mathrm{Fun}([1],C)} \big((\mathrm{Fun}([1], C) \times \mathrm{Fun}([1], C)) \times_{C\times C} \mathrm{Fun}([1], C)\big).$$

In the inner pullback the two functors are $(\mathrm{ev}_1, \mathrm{ev}_1)$ and $(\mathrm{ev}_0, \mathrm{ev}_1)$. The outer pullback uses the second arrow in the product and the identity-arrow functor from C. A diagram in this category therefore consists of an arrow $v\colon b \to c$ of C, an arrow

$$(u, w)\colon (a, c') \longrightarrow (b, c)$$

in $C \times C$, and the identification of w with an identity supplied by the pullback over $\mathrm{Fun}([1], C)$. Preservation of products by $\mathrm{Fun}([1], -)$ separates the two coordinates. Pasting the pullback that identifies the second coordinate with an identity leaves the pullback

$$\mathrm{Fun}([1], C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_0} \mathrm{Fun}([1], C).$$

Its two projections record $u\colon a \to b$ and $v\colon b \to c$, with the identification of their common vertex. The structural functor to C is the target of v. In particular, this calculation retains both the matching of the two arrows and the identification of the second-coordinate arrow with an identity.

Under these two pullback comparisons, relative directed evaluation becomes the short-edge restriction

$$\mathrm{Triang}(C) \longrightarrow \mathrm{Fun}([1], C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_0} \mathrm{Fun}([1], C).$$

Indeed, its projection to E records the target arrow v, and the first coordinate of its other projection records u. The remaining coordinate and its identification are the identity side of the square defining the triangle comparison. Thus the identification of functors is an identification of the complete pullback cones, and is over C via the last vertex. The Segal axiom makes this restriction an equivalence. Its inverse acquires its structural identification over C from the inverse identification of this equivalence, so it is an equivalence over C. This proves the right-fibration assertion.

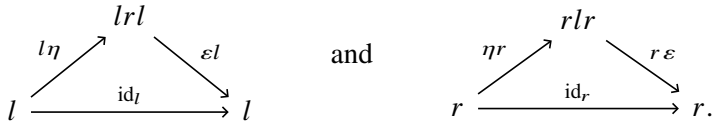
For the second assertion use the source side of the square and the first pullback in Corollary 2.2.8. Relative directed evaluation at the source is identified with the same short-edge restriction, now over C via the first vertex. The Segal axiom again gives an equivalence over C. □

4.4 Adjunctions

Adjunctions play a crucial role in category theory by encoding pairs of functors $l\colon C \to D$ and $r\colon D \to C$ equipped with a natural equivalence of hom animae $D(lx, y) \simeq C(x, ry)$. Since we do not yet have the hom functor available, we introduce adjunctions instead through their equivalent formulation in terms of units and counits. We then study Bousfield localizations, initial and terminal objects, and adjunctions between categories over a base.

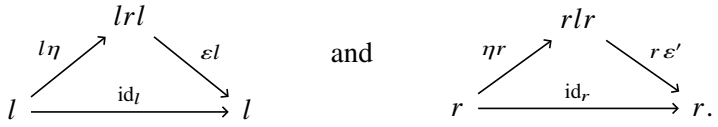
Definition 4.4.1 (Adjunction). Consider two functors $l\colon C \to D$ and $r\colon D \to C$. An *adjunction* $l \dashv r$ consists of natural transformations $\eta\colon \mathrm{id}_C \to rl$ and $\varepsilon\colon lr \to \mathrm{id}_D$ satisfying the *triangle*

identities:



We call η the *unit* of the adjunction and ε the *counit*. We often write $(l \dashv r)\colon C \rightleftarrows D$ to indicate the data $(l, r, \eta, \varepsilon)$ of an adjunction.

Warning 4.4.2. To obtain the correct *category of adjunctions*, Definition 4.4.1 needs to be modified as follows: an adjunction $l \dashv r$ consists of a natural transformation $\eta\colon \mathrm{id}_C \to rl$ and two natural transformations $\varepsilon, \varepsilon'\colon lr \to \mathrm{id}_D$ satisfying the triangle identities:



One may then deduce a posteriori that the counits ε and ε' are equivalent to each other, much like the left and right inverses of an equivalence are canonically equivalent.

Since we will not have much interest in the category of adjunctions in this book, we will keep working with the somewhat simpler and more familiar definition of an adjunction given in Definition 4.4.1.

Proposition 4.4.3. *Let $(l \dashv r)\colon C \rightleftarrows D$ and $(l' \dashv r')\colon D \rightleftarrows E$ be adjunctions. Then the composite functors $l'l\colon C \to E$ and $rr'\colon E \to C$ constitute an adjunction $(l'l \dashv rr')\colon C \rightleftarrows E$.*

Proof. The corresponding unit and counit are given as the composites

$$\mathrm{id}_C \xrightarrow{\eta} rl \xrightarrow{r\eta' l} rr'l'l \qquad \text{and} \qquad l'lrr' \xrightarrow{l'\varepsilon r'} l'r' \xrightarrow{\varepsilon'} \mathrm{id}_E\,.$$

We leave the verification of the triangle identity to the reader, see Exercise 4.9.1 □

Proposition 4.4.4. *Let $(l \dashv r)\colon C \leftrightarrows D$ be an adjunction. Then for every category E we obtain adjunctions*

$$l_*\colon \mathrm{Fun}(E, C) \rightleftarrows \mathrm{Fun}(E, D) : r_* \qquad \textit{and} \qquad r^*\colon \mathrm{Fun}(D, E) \rightleftarrows \mathrm{Fun}(C, E) : l^*.$$

Proof. We construct the first adjunction and leave the second adjunction to the reader. To construct the unit $\eta_*\colon \mathrm{id}_{\mathrm{Fun}(E,C)} \to r_*l_*$, we may by currying equivalently produce a transformation between the two curried functors $E \times \mathrm{Fun}(E, C) \to C$, which we do by whiskering the transformation $\eta\colon \mathrm{id}_C \to rl$ with the evaluation functor $\mathrm{ev}\colon E \times \mathrm{Fun}(E, C) \to C$. The counit $\varepsilon_*\colon l_*r_* \to \mathrm{id}_{\mathrm{Fun}(E,D)}$ is constructed similarly. The triangle identities for η_* and ε_* follow directly from the triangle identities of η and ε after whiskering. □

Proposition 4.4.5. *Adjoints are unique: if $(l \dashv r)\colon C \rightleftarrows D$ and $(l \dashv r')\colon C \rightleftarrows D$ are adjunctions, there is a preferred identification $r = r'$.*

Proof. Consider the following two composites:

$$r \xrightarrow{\eta' r} r'lr \xrightarrow{r'\varepsilon} r' \qquad \text{and} \qquad r' \xrightarrow{\eta r'} rlr' \xrightarrow{r\varepsilon'} r.$$

These two maps can be shown to be inverse to each other, finishing the proof. We leave the details to the reader, see Exercise 4.9.2 □

Proposition 4.4.6. *Let $f\colon C \to D$ be an equivalence with inverse $g\colon D \to C$. Then there exists an adjunction $f \dashv g$ between f and g.*

Proof. Choose an identification $\eta : \mathrm{id}_C = gf$. By Lemma 1.3.10, prewhiskering with f induces an equivalence of animae of identifications

$$(fg = \mathrm{id}_D) \longrightarrow (fgf = f), \qquad \varepsilon \longmapsto \varepsilon f.$$

We may therefore choose an identification $\varepsilon : fg = \mathrm{id}_D$ together with an identification $\varepsilon f = (f\eta)^{-1}$. Thus

$$(\varepsilon f) \circ (f\eta) = \mathrm{id}_f,$$

which is the first triangle identity.

By naturality (Lemma 2.2.10) we obtain an identification

$$(gf\eta) \circ \eta = (\eta gf) \circ \eta.$$

Cancelling the invertible transformation η gives $gf\eta = \eta gf$, and we compute

$$\begin{aligned}\big((g\varepsilon) \circ (\eta g)\big)f &= g(\varepsilon f) \circ (\eta gf)\\ &= g(\varepsilon f) \circ (gf\eta)\\ &= g\big((\varepsilon f) \circ (f\eta)\big) = \mathrm{id}_{gf} = (\mathrm{id}_g)f.\end{aligned}$$

Since prewhiskering with f is an equivalence on animae of identifications, this identification lifts to $(g\varepsilon) \circ (\eta g) = \mathrm{id}_g$. This is the second triangle identity, proving $f \dashv g$. □

Given an adjunction $l \vdash r$, we obtain for each absolute object $x \in C$ and $y \in D$ an equivalence

$$D(lx, y) \xrightarrow{\sim} C(x, ry)$$

by sending a morphism $f\colon lx \to y$ to the composite $x \xrightarrow{\eta_x} rlx \xrightarrow{r(f)} ry$. If one remembers that this equivalence is 'natural in x and y', it in fact provides an alternative encoding of adjunctions. Since we do not yet have hom functors available at this point, we cannot use this formulation yet. The following proposition is a variant of this idea, phrased in terms of the directed pullbacks $C \mathbin{\vec{\times}_l} D$ and $C \mathbin{\vec{\times}_r} D$ over $C \times D$, whose fibers over (x, y) are $D(lx, y)$ and $C(x, ry)$, respectively.

Proposition 4.4.7. *Let $l\colon C \to D$ and $r\colon D \to C$ be functors. Then the following two conditions are equivalent:*

(1) There exist a unit $\eta\colon \mathrm{id}_C \to rl$ and counit $\varepsilon\colon lr \to \mathrm{id}_D$ which exhibit r as a right adjoint of l;

(2) There exists an equivalence $\alpha\colon C \mathbin{\vec{\times}_l} D \xrightarrow{\sim} C \mathbin{\vec{\times}_r} D$ over $C \times D$.

Proof. To prove that (1) implies (2), let $\eta\colon \mathrm{id}_C \to rl$ and $\varepsilon\colon lr \to \mathrm{id}_D$ exhibit an adjunction. On the universal term (x, y, u) of $C \mathbin{\vec{\times}_l} D$, we have a morphism $u\colon lx \to y$. Applying r and composing with η_x gives a morphism $r(u) \circ \eta_x\colon x \to ry$. By Construction 2.2.3, this defines a functor to $\mathrm{Ar}(C)$ with the indicated source and target identifications. The defining pullback therefore supplies a functor

$$\alpha\colon C \mathbin{\vec{\times}_l} D \longrightarrow C \mathbin{\vec{\times}_r} D, \qquad \alpha_{x,y}(u) = r(u) \circ \eta_x,$$

over $C \times D$. Similarly, applying l to the universal morphism $v\colon x \to ry$ and composing with ε_y gives a functor

$$\beta\colon C \mathbin{\vec{\times}_r} D \longrightarrow C \mathbin{\vec{\times}_l} D, \qquad \beta_{x,y}(v) = \varepsilon_y \circ l(v),$$

over $C \times D$.

On the respective directed pullbacks, we compute

$$\beta_{x,y}(\alpha_{x,y}(u)) = \varepsilon_y \circ lr(u) \circ l\eta_x = u \circ \varepsilon_{lx} \circ l\eta_x = u,$$
$$\alpha_{x,y}(\beta_{x,y}(v)) = r\varepsilon_y \circ rl(v) \circ \eta_x = r\varepsilon_y \circ \eta_{ry} \circ v = v.$$

Here we use preservation of composition by functors, naturality of ε and η, and the triangle identities. These identifications fix the displayed sources and targets: preservation of composition, associativity, and unitality do so by Lemmas 2.2.5 and 2.2.6 and Proposition 2.2.7, while the naturality identifications lie in the hom animae specified by Lemma 2.2.10. Thus the two calculations give identifications in $D(lx, y)$ and $C(x, ry)$, respectively, on the respective total categories. Together with the unchanged projections x and y, the pullback property gives identifications $\beta\alpha = \mathrm{id}$ and $\alpha\beta = \mathrm{id}$ over $C \times D$. This proves (2).

Conversely, let α be an equivalence as in (2), and choose an inverse β with inverse identifications over $C \times D$. The identity morphisms define functors

$$C \longrightarrow C \mathbin{\vec{\times}_l} D, \quad x \longmapsto (x, lx, \mathrm{id}_{lx}), \qquad D \longrightarrow C \mathbin{\vec{\times}_r} D, \quad y \longmapsto (ry, y, \mathrm{id}_{ry}).$$

Composing with α and β, respectively, and then projecting to $\mathrm{Ar}(C)$ and $\mathrm{Ar}(D)$ gives natural transformations

$$\eta\colon \mathrm{id}_C \to rl, \qquad \eta_x = \alpha_{x,lx}(\mathrm{id}_{lx}),$$
$$\varepsilon\colon lr \to \mathrm{id}_D, \qquad \varepsilon_y = \beta_{ry,y}(\mathrm{id}_{ry}).$$

Their source and target identifications are supplied by the fact that α and β are functors over $C \times D$.

For the converse we use the theory of cocartesian transport developed later in this chapter. To verify the triangle identities, regard α as an equivalence over D between the two cocartesian directed-pullback projections. It preserves cocartesian transport by Corollary 4.7.5. Similarly, β, regarded over C, preserves the cartesian transport of the two source projections. The transport formulas for slices and their pullbacks give

$$\alpha_{x,y'}(b \circ u) = r(b) \circ \alpha_{x,y}(u),$$
$$\beta_{x',y}(v \circ a) = \beta_{x,y}(v) \circ l(a),$$

for $a\colon x' \to x$, $b\colon y \to y'$, $u\colon lx \to y$, and $v\colon x \to ry$. These are identifications of the actual transport functors on the directed pullbacks. They are compatible with the specified source and target identifications.

Applying these compatibilities to identity morphisms and using unitality, over D and C, respectively, gives

$$r\varepsilon_y \circ \eta_{ry} = \alpha_{ry,y}(\varepsilon_y) = \alpha_{ry,y}\big(\beta_{ry,y}(\mathrm{id}_{ry})\big) = \mathrm{id}_{ry},$$
$$\varepsilon_{lx} \circ l\eta_x = \beta_{x,lx}(\eta_x) = \beta_{x,lx}\big(\alpha_{x,lx}(\mathrm{id}_{lx})\big) = \mathrm{id}_{lx}\,.$$

The last identifications use the inverse identifications of α and β over $C \times D$. Consequently, these calculations take place in $C(ry, ry)$ and $D(lx, lx)$ with their prescribed sources and targets. Currying gives the two triangle identities as identifications of natural transformations, proving (1). □

4.4.1 Bousfield localizations

Bousfield localizations arise from adjunctions whose unit or counit is invertible. We establish their basic closure properties, relate them to localizations, and give a criterion in terms of an idempotent functor.

Definition 4.4.8 (Left/right adjoint section). Let $f\colon C \to D$ be a functor. A *left adjoint section* of f is a left adjoint $s\colon D \to C$ such that the unit transformation $\eta\colon \mathrm{id}_D \to fs$ is a natural isomorphism. Dually, a *right adjoint section* of f is a right adjoint $s\colon D \to C$ such that the counit transformation $\varepsilon\colon fs \to \mathrm{id}_D$ is a natural isomorphism.

In other words, f admits a left adjoint section if and only if it admits a section $s\colon D \to C$ (i.e. a functor satisfying $fs = \mathrm{id}_D$) which comes equipped with a natural transformation $\varepsilon\colon sf \to \mathrm{id}_C$ of functors $C \to C$ satisfying the following two conditions:

- The map ε is a transformation living over D: the induced map $f\varepsilon\colon f = fsf \to f$ is identified with the identity transformation.
- The map ε restricts to the identity on D: the map $\varepsilon s\colon s = sfs \to s$ is identified with the identity transformation.

Definition 4.4.9 (Left/right Bousfield localization). A functor $f\colon C \to D$ is called a *left Bousfield localization* if it admits a right adjoint section. We call f a *right Bousfield localization* if it admits a left adjoint section.

We will see in Proposition 6.4.10 that a left/right adjoint section $s\colon D \to C$ of a functor $f\colon C \to D$ is automatically a full subcategory. We may then think of f as reflecting the category C into this subcategory D.

Lemma 4.4.10. *Every equivalence $C \xrightarrow{\sim} D$ is both a left and a right Bousfield localization.*

Proof. This is immediate from Proposition 4.4.6. □

Lemma 4.4.11. *For a category C, the target functor $\mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$ is a left Bousfield localization with right adjoint section given by*

$$s = p^*_{[1]}\colon C \to \mathrm{Fun}([1], C).$$

Dually, the source functor $\mathrm{ev}_0\colon \mathrm{Fun}([1], C) \to C$ is a right Bousfield localization.

Proof. We prove the claim for ev_1; the claim for ev_0 is dual. Consider the following functor:

$$\mathrm{Fun}([1], C) \times [1] \to \mathrm{Fun}([1], C); \qquad (f, t) \mapsto (s \mapsto f(\max(t, s))).$$

Using Lemma 2.1.12, we see that $f(\max(0, y)) = f(y)$ and $f(\max(1, y)) = f(1)$, so that this assignment defines a natural transformation

$$\eta\colon \mathrm{id}_{\mathrm{Fun}([1],C)} \to s\mathrm{ev}_1$$

of functors $\mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C)$. It also follows from Lemma 2.1.12 that:

- The evaluation of η at 1 gives the identity of ev_1;

- Applying η to $s(c)$ gives the identity of $s(c)$.

This shows that η exhibits s as a right adjoint section of ev_1, finishing the proof. □

We can relax the condition on the transformation ε as follows:

Proposition 4.4.12. *Let $f\colon C \to D$ be a functor. Assume that there exists a section $s\colon D \to C$ of f and a natural transformation $\varepsilon\colon sf \to \mathrm{id}_C$ over D such that the transformation $\varepsilon s\colon s = sfs \to s$ is a natural isomorphism. Then s is a left adjoint section of f.*

The analogous statement for right adjoint sections also holds.

Proof. Let $(\varepsilon s)^{-1}\colon s \to s$ be an inverse of $\varepsilon s\colon s \to s$. We define a new transformation $\varepsilon'\colon sf \to \mathrm{id}_C$ as the following composite:

$$sf \xrightarrow{(\varepsilon s)^{-1} f} sf \xrightarrow{\varepsilon} \mathrm{id}_C\,.$$

This is again a transformation over D, and it satisfies

$$\varepsilon' s = (\varepsilon s) \circ ((\varepsilon s)^{-1} fs) = (\varepsilon s) \circ (\varepsilon s)^{-1} = \mathrm{id}_s\,.$$

This shows that the map $\varepsilon'\colon sf \to \mathrm{id}_C$ exhibits s as a left adjoint section of f. □

Left and right Bousfield localizations are closed under a variety of operations.

Proposition 4.4.13 (Composition). *Left (resp. right) Bousfield localizations are closed under composition.*

Proof. By Proposition 4.4.3 the composite functors form again an adjunction, and it is immediate from the descriptions of the unit and counit in the proof that the composite is again a left/right Bousfield localization. □

Proposition 4.4.14 (Functor categories). *Let $f\colon C \to D$ be a left (resp. right) Bousfield localization. Then also the functor $f_*\colon \mathrm{Fun}(A, C) \to \mathrm{Fun}(A, D)$ is a left (resp. right) Bousfield localization for any category A.*

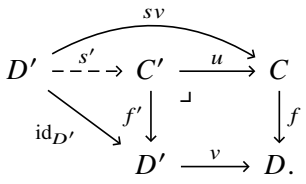
Proof. This is immediate from Proposition 4.4.4. □

Proposition 4.4.15 (Base change). *Left (resp. right) Bousfield localizations are stable under base change.*

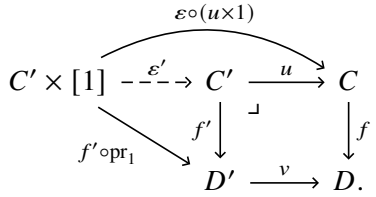
Proof. We prove the claim for right Bousfield localizations; the claim for left Bousfield localizations is dual. Consider a pullback square

$$\begin{array}{ccc} C' & \xrightarrow{u} & C \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ D' & \xrightarrow{v} & D \end{array}$$

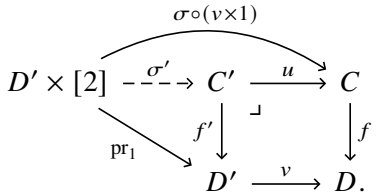
where f is a right Bousfield localization. Let (s, ε) be a left adjoint section of f. By the universal property of the pullback, we obtain a section $s' : D' \to C'$ of f' satisfying $us' = sv$:



Similarly, we define the map $\varepsilon' : C' \times [1] \to C'$ over D' via the following commutative diagram:



Note that ε' is indeed a transformation from $s'f'$ to $\mathrm{id}_{C'}$ over D'. Finally, we construct the desired commutative diagram $\sigma' : D' \times [2] \to C'$ via the following commutative diagram:



□

Proposition 4.4.16. *Let $f : C \to D$ be a left (resp. right) Bousfield localization. Then f is a localization, in the sense of Definition 3.3.6.*

Proof. Let $s : D \to C$ be a left adjoint section of f, with identification $\eta : \mathrm{id}_D = fs$ and counit $\varepsilon : sf \to \mathrm{id}_C$. For any category E, the induced functor

$$f^* : \mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$$

is a left adjoint section of $s^* : \mathrm{Fun}(C, E) \to \mathrm{Fun}(D, E)$, see Proposition 4.4.4. Let $W \hookrightarrow \mathrm{Map}([1], C)$ denote the collection of morphisms in C that are sent to isomorphisms in D under f, i.e.

$$\begin{array}{ccc} W & \hookrightarrow & \mathrm{Map}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f_*} \\ \mathrm{Iso}(D) & \hookrightarrow & \mathrm{Map}([1], D). \end{array}$$

Since the map $f(\varepsilon_x) : f(s(f(x))) \to f(x)$ is an isomorphism by the triangle identity, it follows that the map $\varepsilon_x : s(f(x)) \to x$ lies in W for any object x of C. In particular, for any functor $F : C \to E$ in $\mathrm{Fun}^W(C, E)$, the counit of the adjunction $f^* \dashv s^*$ is a natural isomorphism: $f^*s^*F \xrightarrow{\sim} F$. It follows that this adjunction restricts to an adjunction

$$f^* : \mathrm{Fun}(D, E) \rightleftarrows \mathrm{Fun}^W(C, E) : s^*$$

in which both the unit and counit are equivalences. We conclude that the induced functor

$$f^*\colon \mathrm{Fun}(D,E) \to \mathrm{Fun}^W(C,E)$$

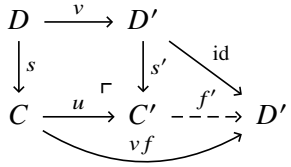
is an equivalence, showing that f exhibits D as a localization of C at W as desired. □

Proposition 4.4.17. *Left (respectively right) adjoint sections are stable under pushout.*

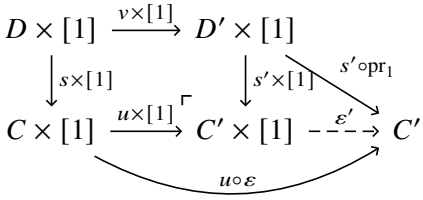
Proof. We prove the claim for right adjoint sections; the other claim is dual. Consider a pushout square

$$\begin{array}{ccc} D & \xrightarrow{v} & D' \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ C & \xrightarrow{u} & C' \end{array} \tag{4.1}$$

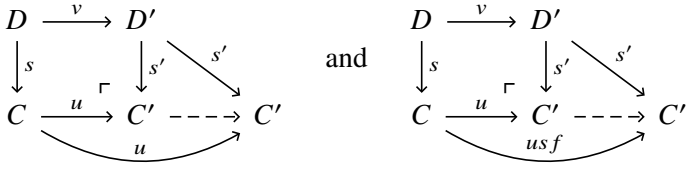
where $(s, \varepsilon\colon \mathrm{id}_C \to sf)$ is a right adjoint section of $f\colon C \to D$. By the universal property of the pushout



we obtain a map f' with $f's' = \mathrm{id}_{D'}$. Similarly, we define the map $\varepsilon'\colon C' \times [1] \to C'$ over D' via the following commutative diagram (over D')



using that $- \times [1]$ preserves pushouts. At $0, 1 \in [1]$ this diagram restricts to



respectively, which have (unique) solutions $\mathrm{id}_{C'}$ and $s'f'$ (using the given identifications $s'f'u = s'vf = usf$); this shows that ε' is indeed a transformation $\mathrm{id}_{C'} \to s'f'$ over D'. Finally, $\varepsilon' \circ (s' \times [1]) = s' \circ \mathrm{pr}_1$ by construction. Thus (s', ε') is a right adjoint section of f'. □

Proposition 4.4.18 (Idempotent localization criterion)**.** *Let $L\colon C \to C$ be a functor equipped with a natural transformation $\eta\colon \mathrm{id}_C \Rightarrow L$. Assume that both*

$$\eta_{Lx}\colon Lx \to LLx \qquad \textit{and} \qquad L\eta_x\colon Lx \to LLx$$

are isomorphisms for every object x of C. Let $D \subseteq C$ be the full subcategory spanned by those objects z for which η_z is an isomorphism. Then L factors through a functor $l\colon C \to D$ which is left adjoint to the inclusion $i\colon D \hookrightarrow C$, with unit η. In particular, l is a left Bousfield localization.

Proof. We first construct D by the pullback

$$\begin{array}{ccc} D & \longrightarrow & \mathrm{Iso}(C) \\ {\scriptstyle i}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{Iso}}} \\ C & \xrightarrow[\eta]{} & \mathrm{Fun}([1], C). \end{array}$$

Here the bottom functor is the one encoding the transformation η. By Propositions 3.4.4 and 3.4.5, the right-hand functor is a full subcategory. Its pullback therefore exhibits D as the claimed full subcategory of C. Since every η_{Lx} is invertible, we may use the characterization of Corollary 3.2.8 to get a factorization $l\colon C \to D$ of L through D. Note that every $z : D$ is isomorphic to Lz via η_z, so D contains precisely the objects in the essential image of L.

By the pointwise criterian for natural isomorphisms (Proposition 3.4.6), the transformations ηL and $L\eta$ are invertible. We claim that $\eta L = L\eta$. Applying Lemma 2.2.10 to η twice gives

$$(L\eta) \circ \eta = (\eta L) \circ \eta.$$

Whiskering with L on the left gives $(LL\eta) \circ (L\eta) = (L\eta L) \circ (L\eta)$, and since $L\eta$ is invertible we obtain $L^2\eta = L\eta L$. Similarly, naturality of ηL gives

$$(L\eta L) \circ (\eta L) = (\eta L^2) \circ (\eta L),$$

and since ηL is invertible we deduce $L\eta L = \eta L^2$. Finally, naturality applied to $L\eta$ gives

$$(L^2\eta) \circ (\eta L) = (\eta L^2) \circ (L\eta).$$

Plugging in the preceding two identifications and cancelling the invertible transformation ηL^2 then proves the claim.

We now establish the adjunction $l \vdash i$. Consider the restricted transformation $\eta i\colon i \to ili$, which is invertible by construction. By the universal property of the subcategory inclusion $i\colon D \hookrightarrow C$, it lifts to an invertible transformation $\mathrm{id}_D \to li$, and we let $\varepsilon\colon li \to \mathrm{id}_D$ be its inverse. Then we have

$$(i\varepsilon) \circ (\eta i) = \mathrm{id}_i$$

by construction. For the other triangle identity $(\varepsilon l) \circ (l\eta) = \mathrm{id}_l$, we may again check this after applying i. But there we have

$$(i\varepsilon l) \circ (il\eta) = (\eta L)^{-1} \circ (L\eta) = \mathrm{id}_L\,.$$

We conclude that we have an adjunction $l \dashv i$, with unit η and invertible counit ε, as required. □

4.4.2 Initial and terminal objects

Initial and terminal objects are the special cases of left and right adjoint sections for the functor $C \to *$. We use this description to characterize these objects and give basic examples.

Lemma 4.4.19. *Let C be a category and let x be an absolute object of C. Then the following conditions are equivalent:*

(1) The object x is initial in the sense of Definition 2.1.9: the forgetful functor $C_{x/} \to C$ is an equivalence;

(2) The object x defines a left adjoint section $x : * \to C$ of the functor $p_C : C \to *$;

(3) There exists a transformation $\varepsilon : \mathrm{const}_x \to \mathrm{id}_C$ and an identification $\varepsilon_x = \mathrm{id}_x$ of functors $* \to C$. Here const_x is defined as the composite $C \xrightarrow{p_C} * \xrightarrow{x} C$.

(4) There exists a transformation $\varepsilon : \mathrm{const}_x \to \mathrm{id}_C$ such that the map $\varepsilon_x : x \to x$ is an isomorphism in C.

The dual statement for terminal objects is also valid.

Proof. Let $q : C_{x/} \to C$ be the slice projection. If (1) holds, an inverse of q gives a transformation $\varepsilon : \mathrm{const}_x \to \mathrm{id}_C$. The fiber of q over x is contractible and contains id_x, so $\varepsilon_x = \mathrm{id}_x$. This proves (3). Conversely, a transformation as in (3) defines a section $j : C \to C_{x/}$ of q. Naturality at the universal arrow $u : x \to y$ of $C_{x/}$ gives $\varepsilon_y = u\varepsilon_x = u$, with the specified endpoints. These identifications define $jq = \mathrm{id}_{C_{x/}}$ over C, so q is an equivalence. Thus (1) and (3) are equivalent. To see that (2) is equivalent to (3), note that there is only a single functor $C \times [1] \to *$, and hence the unit transformation $\eta : \mathrm{id}_* \to p_C \circ x$ is unique. One of the two triangle identities is automatic, and the remaining triangle identity amounts to condition (3). The equivalence between (3) and (4) is a special case of Proposition 4.4.12. □

Remark 4.4.20. Let 0 be an initial object of C. Point (2) says that the functor $C_{0/} \to C$ is an equivalence, and in particular for every $y \in C$ the fiber $C(0, y)$ of this map over y is contractible. Informally, this says that for every y there exists a unique morphism $0 \to y$ in C.

The converse will be treated later, using functoriality of universals.

Lemma 4.4.21. *Let $f : C \to D$ be a right Bousfield localization with left adjoint section $s : D \to C$. For every object d of D, the object $s(d) \in C_d$ is an initial object of the fiber $C_d := \{d\} \times_D C$.*

The dual statement for left Bousfield localizations and terminal objects also holds.

Proof. It follows from Proposition 4.4.15 that the base change $f|_{C_d} : C_d \to \{d\}$ of f along $\{d\} \to D$ is again a right Bousfield localization with left adjoint section given by the restriction $s|_{\{d\}} : \{d\} \to C_d$. By definition this means that $s(d)$ is an initial object of C_d. □

We will now provide some basic examples of initial and terminal objects. The most important example will be the walking morphism:

Lemma 4.4.22. *The category $[1]$ has both an initial object $0 : [1]$ and a terminal object $1 : [1]$.*

Proof. This is true as Axiom C.2. □

This lets us compute all the hom animae of $[1]$:

Corollary 4.4.23. *There are equivalences*

$$\mathrm{Hom}_{[1]}(0,0) \simeq *, \qquad \mathrm{Hom}_{[1]}(0,1) \simeq * \qquad \text{and} \qquad \mathrm{Hom}_{[1]}(1,1) \simeq *.$$

Proof. This is immediate from Lemma 4.4.22. □

Corollary 4.4.24. *There is an equivalence* $\mathrm{Hom}_{[1]}(1,0) \simeq \emptyset$.

Proof. We first show that every morphism $f\colon 1 \to 0$ in $[1]$ is an isomorphism with inverse the canonical morphism $\mathrm{can}\colon 0 \to 1$. Indeed, by Corollary 4.4.23 there are unique identifications $f \circ \mathrm{can} = \mathrm{id}_0$ and $\mathrm{can} \circ f = \mathrm{id}_1$, so that can provides both a left and a right inverse to f. Since $[1] \simeq \mathrm{Iso}([1]) \hookrightarrow \mathrm{Fun}([1],[1])$ is an embedding by Lemma 2.3.3, we deduce that $\mathrm{Hom}_{[1]}(1,0)$ sits in a pullback square as follows:

$$\begin{array}{ccc} \mathrm{Hom}_{[1]}(1,0) & \longrightarrow & [1] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ * & \xrightarrow{(1,0)} & [1] \times [1]. \end{array}$$

Furthermore, since $[1]^{\simeq} \hookrightarrow [1]$ is an embedding Corollary 2.5.3, the right-hand square in the following commutative diagram is a pullback square, and hence by the pasting law so must be the left-hand square:

$$\begin{array}{ccccc} \mathrm{Hom}_{[1]}(1,0) & \longrightarrow & [1]^{\simeq} & \longrightarrow & [1] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ * & \xrightarrow{(1,0)} & [1]^{\simeq} \times [1]^{\simeq} & \longrightarrow & [1] \times [1]. \end{array}$$

But since $[1]^{\simeq} \simeq \{0,1\}$ is a disjoint union of two points, we see that

$$[1]^{\simeq} \times [1]^{\simeq} \simeq \{(0,0),(0,1),(1,0),(1,1)\}$$

is a disjoint union of four points, and the diagonal map corresponds to the inclusion

$$\{(0,0),(1,1)\} \hookrightarrow \{(0,0),(0,1),(1,0),(1,1)\}\,.$$

It thus follows from the universality of disjoint unions, Axiom B.5', that the pullback over the point $(1,0)$ is empty. □

Lemma 4.4.25. *For a term x of a category C, the slice category $C_{/x}$ admits a terminal object $(x, \mathrm{id}_x\colon x \to x)$. Similarly, $C_{x/}$ admits an initial object $(x, \mathrm{id}_x\colon x \to x)$.*

Proof. We showed in Lemma 4.4.11 that the evaluation map $\mathrm{ev}_1\colon \mathrm{Fun}([1],C) \to C$ admits a right adjoint section given by $p^*_{[1]}\colon C \to \mathrm{Fun}([1],C)$. Since $C_{/x}$ is the fiber over x of this map, the claim follows from the dual statement in Lemma 4.4.21. □

If one already knows the existence of an initial object x, then other initial objects are particularly easy to recognize:

Exercise 4.4.26. Let x and y be objects in a category C and assume that x is an initial object. Then y is an initial object if and only if the canonical map $x \to y$ from (the dual of) Remark 4.4.20 is an isomorphism.

The dual statement for terminal objects also holds.

Corollary 4.4.27. *Let $C' \subseteq C$ be the full subcategory spanned by the terminal objects. Then C' is an anima and $C' \to *$ is an embedding. In particular, if C admits a terminal object, C' is contractible. The dual statement holds for initial objects.*

Proof. For objects x, y of C', fullness and the dual of Remark 4.4.20 give

$$C'(x, y) \simeq C(x, y) \simeq *.$$

Every morphism $x \to y$ is therefore invertible: the unique morphism $y \to x$ is an inverse, since both endomorphism animae are contractible. Thus C' is a groupoid, hence an anima by the groupoid recognition axiom. By Corollary 2.3.5, the anima $(x = y)$ is contractible as well. Applying this in the anima context $(x, y) : C' \times C'$ identifies the two projections to C', so the first projection is inverse to the diagonal $C' \to C' \times C'$. This proves that $C' \to *$ is an embedding. If C admits a terminal object, this embedding has a section and is therefore an equivalence by Lemma 1.6.24. □

4.4.3 Relative adjunctions

A relative adjunction is an ordinary adjunction between categories over a fixed base whose unit and counit are natural transformations over that base.

Recall from Definition 1.9.2 that $\mathrm{Fun}_S(C, D)$ denotes the category of functors from C to D over S.

Definition 4.4.28 (Relative adjunction). Let $p\colon C \to S$ and $q\colon D \to S$ be functors, and let $L\colon C \to D$ be a functor over S. A *right adjoint to L relative to S* consists of a functor $R\colon D \to C$ over S and a counit

$$\varepsilon\colon LR \longrightarrow \mathrm{id}_D$$

in $\mathrm{Fun}_S(D, D)$ which exhibits the underlying functor $R\colon D \to C$ as a right adjoint to the underlying functor $L\colon C \to D$. We then call $L \dashv R$ an *adjunction relative to S*.

Thus a relative counit is a counit whose components lie over identity morphisms in S. Equivalently, one may impose this condition on the unit.

Proposition 4.4.29 (Unit formulation of relative adjunctions). *Let C and D be categories over S, and let $L\colon C \to D$ and $R\colon D \to C$ be functors over S. The following data are equivalent:*

(1) A counit $\varepsilon\colon LR \to \mathrm{id}_D$ in $\mathrm{Fun}_S(D, D)$ which exhibits R as a right adjoint to L;

(2) A unit $\eta\colon \mathrm{id}_C \to RL$ in $\mathrm{Fun}_S(C, C)$ which exhibits L as a left adjoint to R.

Proof. Suppose first that the counit ε lies over S, and let η be the unit of the underlying adjunction. Apply q to the triangle identity

$$(\varepsilon L)(L\eta) = \mathrm{id}_L\,.$$

After using the structure identifications $qL = p$ and $pR = q$, the first factor lies over an identity morphism. It follows that $p\eta$ is the identity transformation of p, so η lies over S. Conversely, if η lies over S, applying p to the other triangle identity

$$(R\varepsilon)(\eta R) = \mathrm{id}_R$$

shows that ε lies over S. □

Proposition 4.4.30 (Relative adjunctions on functor categories). *Let $L\colon C \rightleftarrows D : R$ be an adjunction relative to S. For every functor $E \to S$, postcomposition induces an adjunction*

$$L_*\colon \mathrm{Fun}_S(E, C) \rightleftarrows \mathrm{Fun}_S(E, D) : R_*.$$

Proof. Whiskering the relative unit and counit with functors out of E gives a unit and counit over S. Their triangle identities are inherited from those of $L \dashv R$. □

Proposition 4.4.31 (Composition and base change). *Relative adjunctions compose and are preserved by base change. More precisely:*

(1) If $L\colon C \rightleftarrows D : R$ and $L'\colon D \rightleftarrows E : R'$ are adjunctions relative to S, then $L'L \dashv RR'$ is an adjunction relative to S;

(2) If $L \dashv R$ is an adjunction relative to S and $T \to S$ is any functor, then the induced functors between the base changes to T form an adjunction relative to T.

Proof. The usual formulas for the unit and counit of a composite adjunction use only whiskering and composition. They therefore remain transformations over S. For the second assertion, pull back the relative unit, counit, and their triangle identifications along $T \to S$. □

4.5 Cartesian and cocartesian fibrations

In Section 4.2 we introduced the notion of a *left fibration*: a functor $f\colon A \to B$ such that every morphism $\beta\colon b \to b'$ admits a unique covariant lift starting in a given lift a of b. The goal of this section is to introduce a variant of this notion, called a *cocartesian fibration*, in which the covariant lift is not necessarily unique but where there exists always an *initial* covariant lift with given source. Dually, we obtain a notion of *cartesian fibration*, in which there exists a *terminal* contravariant lift with given target.

Definition 4.5.1 ((Co)cartesian fibration). A functor $f\colon A \to B$ is called a *cocartesian fibration* if the directed evaluation map $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) \to A \mathbin{\vec{\times}_f} B$ admits a left adjoint section $\mathrm{lift}_0^f\colon A \mathbin{\vec{\times}_f} B \to \mathrm{Fun}([1], A)$.

Dually, f is called a *cartesian fibration* if the map $\vec{\mathrm{ev}}_1^f\colon \mathrm{Fun}([1], A) \to B \mathbin{\vec{\times}_f} A$ admits a right adjoint section $\mathrm{lift}_1^f\colon B \mathbin{\vec{\times}_f} A \to \mathrm{Fun}([1], A)$.

Throughout this section, we will frequently state results only for cocartesian fibrations and leave their dual formulation for cartesian fibrations to the reader.

Example 4.5.2. The following are the first few basic examples of (co)cartesian fibrations:

- Every left fibration is a cocartesian fibration: the map $\vec{\mathrm{ev}}_0^f$ is an equivalence and thus in particular admits a left adjoint section by Lemma 4.4.10. Dually, every right fibration is a cartesian fibration.
- In particular, every equivalence is both a cartesian fibration as well as a cocartesian fibration.
- For a category C, the map $p_C\colon C \to *$ is both a cartesian fibration and a cocartesian fibration. Indeed, since the map $\vec{\mathrm{ev}}_0^{p_C}\colon \mathrm{Fun}([1], C) \to C \mathbin{\vec{\times}_{p_C}} *$ is isomorphic to $\mathrm{ev}_0\colon \mathrm{Fun}([1], C) \to C$ by Remark 4.1.4, it admits a left adjoint section by Lemma 4.4.11, showing that p_C is a cocartesian fibration. The cartesian case is dual.
- For categories C and D, the projection maps $\mathrm{pr}_1\colon C \times D \to C$ and $\mathrm{pr}_2\colon C \times D \to D$ are both a cartesian fibration and a cocartesian fibration. This can again be argued directly, but also follows from the previous example and Proposition 4.5.15 below.

4.5.1 Covariant transport

Cocartesian fibrations allow for a notion of *covariant transport* generalizing the situation for left fibration:

Notation 4.5.3 (Covariant transport). For a term (a, β) of $A \vec{\times}_f B$, we denote the morphism $\mathrm{lift}_0^f(a, \beta)$ in A as

$$\mathrm{lift}_0^f(a, \beta) \colon a \to \beta_!(a)$$

and refer to its target $\beta_!(a)$ as the *covariant transport of a along β*. Observe that this map is a term of the category $\mathrm{Lift}_0^f(a, \beta)$ of covariant lifts of β starting in a, defined in Definition 4.2.7. In fact, $\mathrm{lift}_0^f(a, \beta)$ is even an *initial object* of this category, see Lemma 4.4.21.

Just like in Construction 4.2.10, the assignment $a \mapsto \beta_!(a)$ determines a functor

$$\beta_! \colon A_b \to A_{b'}.$$

called *covariant transport along β*.

Remark 4.5.4. When we write $\mathrm{lift}_0^f(a, \beta)$ we allow a and β to be a term of A and a morphism of B, respectively, defined not just in a global or anima context, but in an arbitrary context Γ. Concretely, this means that for every category Γ and any solid commutative square of the form

$$\begin{array}{ccc} \Gamma \times \{0\} & \xrightarrow{a} & A \\ \downarrow & \nearrow & \downarrow f \\ \Gamma \times [1] & \xrightarrow{\beta} & B \end{array}$$

we have the dashed lift defined as the uncurrying of the composite

$$\Gamma \xrightarrow{(a,\beta)} A \vec{\times}_f B \xrightarrow{\mathrm{lift}_0^f} \mathrm{Fun}([1], A)$$

(where we are implicitly currying β). We call the dashed arrow a *cocartesian lift in context* Γ and simply denote it by $\mathrm{lift}_0^f(a, \beta)$. We can view it as a transformation $a \to \beta_!(a)$, whose codomain we call the *cocartesian transport in context* Γ of a along β.

Note that by construction, cocartesian lifts are compatible with change of context. Explicitly, for every $g \colon \Gamma' \to \Gamma$, reindexing identifies $\mathrm{lift}_0^f(a, \beta)|_g$ with $\mathrm{lift}_0^f(a|_g, \beta|_g)$, and analogously for cocartesian transport.

Finally, note that the definition of left adjoint section means precisely that for each term of $A \vec{\times}_f B$, i.e. each pair (a, β) in arbitrary context Γ as above, the object $\mathrm{lift}_0^f(a, \beta)$ is characterized as an initial object of the category $\mathrm{Lift}_0^f(a, \beta)$ in context Γ.

We would like to show that the covariant transport functors satisfy the relations $(\mathrm{id}_b)_! = \mathrm{id}$ and $(\beta' \circ \beta)_! = \beta'_! \circ \beta_!$ as in Proposition 4.2.12. While the statement for left fibrations was immediate due to uniqueness of covariant lifts, the statement for cocartesian fibrations is slightly more involved. We will need the following auxiliary notion of *cocartesian morphisms*:

Definition 4.5.5 (Cocartesian and cartesian morphisms)**.** Let $f\colon A \to B$ be a functor. A morphism $\alpha\colon a \to a'$ in A is called *f-cocartesian* if for every object z of A the square

$$\begin{array}{ccc} A(a',z) & \xrightarrow{-\circ\alpha} & A(a,z) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B(f(a'),f(z)) & \xrightarrow{-\circ f(\alpha)} & B(f(a),f(z)) \end{array}$$

is a pullback square. Dually, α is called *f-cartesian* if for every object z of A the square

$$\begin{array}{ccc} A(z,a) & \xrightarrow{\alpha\circ-} & A(z,a') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B(f(z),f(a)) & \xrightarrow{f(\alpha)\circ-} & B(f(z),f(a')) \end{array}$$

is a pullback square. We omit f from the terminology when it is clear from context.

Construction 4.5.6. Let $f\colon A \to B$ be a cocartesian fibration. Since the functor lift_0^f is a left adjoint section of $\vec{\mathrm{ev}}_0^f$, it comes equipped with a natural transformation $\varepsilon\colon \mathrm{lift}_0^f \circ \vec{\mathrm{ev}}_0^f \to \mathrm{id}_{\mathrm{Fun}([1],A)}$. In particular, for a morphism $\alpha\colon a \to a'$ in A with underlying morphism $\beta := f(\alpha)\colon f(a) \to f(a')$ in B, we obtain a morphism $\varepsilon_\alpha\colon \mathrm{lift}_0^f(a,\beta) \to \alpha$ in $\mathrm{Fun}([1],A)$, corresponding to a commutative square in A of the form

$$\begin{array}{ccc} a & \xrightarrow{\mathrm{lift}_0^f(a,\beta)} & \beta_!(a) \\ {\scriptstyle \mathrm{id}_a}\downarrow & & \vdots{\scriptstyle \varepsilon_\alpha} \\ a & \xrightarrow{\alpha} & a'. \end{array}$$

The image of this square under $\vec{\mathrm{ev}}_0^f$ is identified with the identity of (a,β) by the triangle identity of the adjunction. In particular, ε_α lies over $\mathrm{id}_{f(a')}$. By Corollary 2.2.8, we may think of this square in A as a commutative triangle in A of the form

$$\begin{array}{ccccc} & & \beta_!(a) & & \\ & {\scriptstyle \mathrm{lift}_0^f(a,\beta)}\nearrow & & \searrow{\scriptstyle \varepsilon_\alpha} & \\ a & & \xrightarrow[\alpha]{} & & a'. \end{array}$$

Lemma 4.5.7. *Let $f\colon A \to B$ be a cocartesian fibration. Fix a term a of A and composable morphisms $\beta\colon b := f(a) \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, both in some context Γ. Then for every $\gamma\colon a \to a''$ over $\beta'\beta$ there exists a dashed filling*

$$\begin{array}{ccc} a & \xrightarrow{\mathrm{lift}_0^f(a,\beta)} & \beta_!(a) \\ {\scriptstyle \gamma}\downarrow & & \vdots{\scriptstyle \bar{\gamma}} \\ a'' & \xrightarrow[\mathrm{id}_{a''}]{} & a'' \end{array} \qquad \text{over} \qquad \begin{array}{ccc} b & \xrightarrow{\beta} & b' \\ {\scriptstyle \beta'\beta}\downarrow & & \downarrow{\scriptstyle \beta'} \\ b'' & \xrightarrow[\mathrm{id}_{b''}]{} & b'' \end{array}$$

which is unique. In particular, the chosen lift $\mathrm{lift}_0^f(a,\beta)$ is f-cocartesian.

Proof. The adjunction $\mathrm{lift}_0^f \dashv \vec{\mathrm{ev}}_0^f$ gives an equivalence

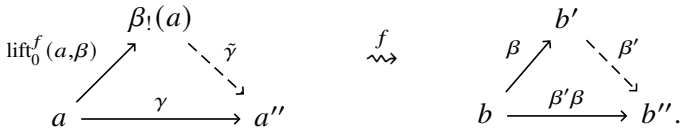
$$\mathrm{Hom}_{\mathrm{Fun}([1],A)}(\mathrm{lift}_0^f(a,\beta), \mathrm{id}_{a''}) \simeq \mathrm{Hom}_{A \,\vec{\times}_f\, B}((a,\beta), \vec{\mathrm{ev}}_0^f(\mathrm{id}_{a''}))$$

in the common context of the terms involved. On the left, a commutative square with target $\mathrm{id}_{a''}$ is determined by its right-hand morphism, by Corollary 2.2.8. On the right, a morphism consists of $\gamma\colon a \to a''$, a morphism $b' \to f(a'')$, and the commutative triangle between their images in B. Thus the displayed equivalence identifies with

$$A(\beta_! a, a'') \simeq A(a, a'') \times_{B(b, f(a''))} B(b', f(a'')),$$

whose components are precomposition with $\mathrm{lift}_0^f(a,\beta)$ and application of f. This proves that the chosen lift is f-cocartesian. Its fiber over the prescribed pair (γ, β') is the anima of dashed fillings, which is therefore contractible. □

We will often use Corollary 2.2.8 to rewrite such lifting problems in the following form:



Lemma 4.5.8. *Let $f\colon A \to B$ be a cocartesian fibration. A morphism $\alpha\colon a \to a'$ over β is f-cocartesian if and only if the morphism $\varepsilon_\alpha\colon \beta_! a \to a'$ from Construction 4.5.6 is an isomorphism.*

Proof. By Lemma 4.5.7, the chosen lift $a \to \beta_! a$ is f-cocartesian. Since α factors through it by ε_α, and $f(\varepsilon_\alpha) = \mathrm{id}_{f(a')}$, pasting pullback squares shows that α is f-cocartesian if and only if

$$- \circ \varepsilon_\alpha\colon A(a', z) \to A(\beta_! a, z)$$

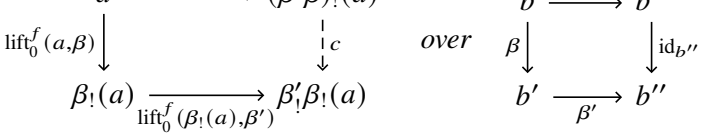
is an equivalence for every object z of A. By Exercise 2.6.2, this is equivalent to ε_α being an isomorphism. □

Remark 4.5.9. By abuse of notation, we used ε_α both for the morphism $\mathrm{lift}_0^f(a,\beta) \to \alpha$ in $\mathrm{Fun}([1], A)$ and for its image under ev_1. The preceding criterion concerns the latter morphism, from $\beta_! a$ to a'.

Proposition 4.5.10. *Let $f\colon A \to B$ be a cocartesian fibration.*

(1) For each term a of A, with $b \coloneqq f(a)$, the cocartesian lift $a \to (\mathrm{id}_b)_! a$ of the identity can be chosen to be the identity map $\mathrm{id}_a\colon a \to a$.

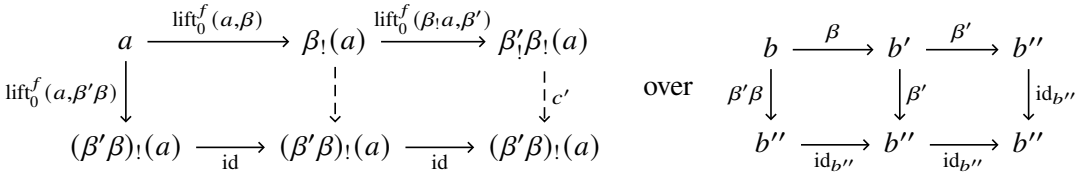
(2) Given a term a of A and composable morphisms $b \coloneqq f(a) \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, the canonical map $c\colon (\beta'\beta)_!(a) \to \beta'_!\beta_!(a)$ over b'' obtained by filling



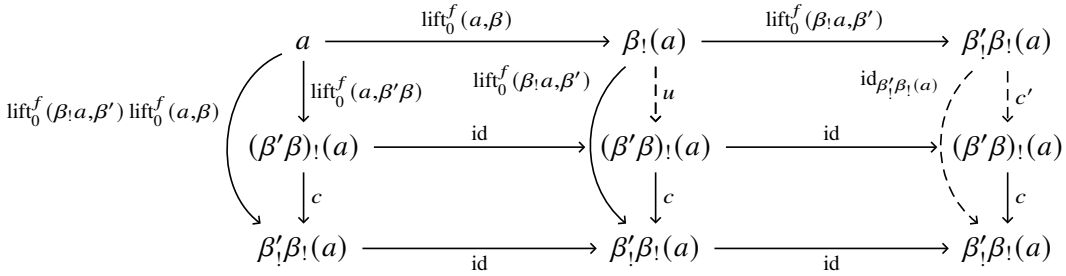
is an isomorphism.

Proof. (1) Recall that $(\beta_! a, \mathrm{lift}_0^f)$ is an initial object of the category $\mathrm{Lift}_0^f(a,\beta)$. In the case where $\beta = \mathrm{id}_b$ is the identity, this category is exactly the slice $(A_b)_{a/}$ which already has an initial object given by (a, id_a). By Exercise 4.4.26, the canonical morphism between these initial objects is an isomorphism, proving the claim.

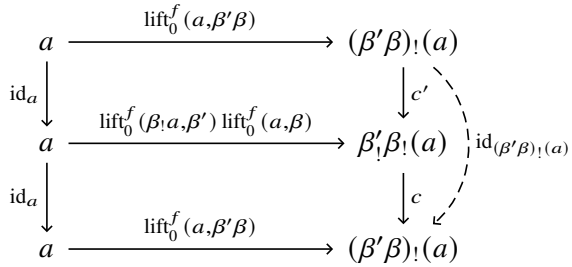
(2) We construct an explicit inverse $c' : \beta'_!\beta_!(a) \to (\beta'\beta)_!(a)$ as follows. Consider the solid diagram



By applying Lemma 4.5.7 twice, we obtain the indicated dashed extensions, yielding the desired map c'. The fact that this is an inverse follows by the uniqueness part of Lemma 4.5.7 applied to the diagrams



(twice, yielding $\mathrm{lift}_0^f(\beta_! a, \beta') = cu$ and then $\mathrm{id} = cc'$) and



(yielding $\mathrm{id} = c'c$). □

Corollary 4.5.11. *Let $f : A \to B$ be a cocartesian fibration. Covariant transport is functorial in the following sense:*

(1) For every object b of B, covariant transport $(\mathrm{id}_b)_! : A_b \to A_b$ along the identity is identified with the identity functor.

(2) Given composable morphisms $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$, there is a canonical identification $(\beta'\beta)_! = \beta'_!\beta_! : A_b \to A_{b''}$.

Proof. This follows directly from Proposition 4.5.10 applied to the tautological term $a : A_b \to A$ of A in context A_b (which is just the fiber inclusion). □

Lemma 4.5.12. *Let $f\colon A \to B$ be a cocartesian fibration.*

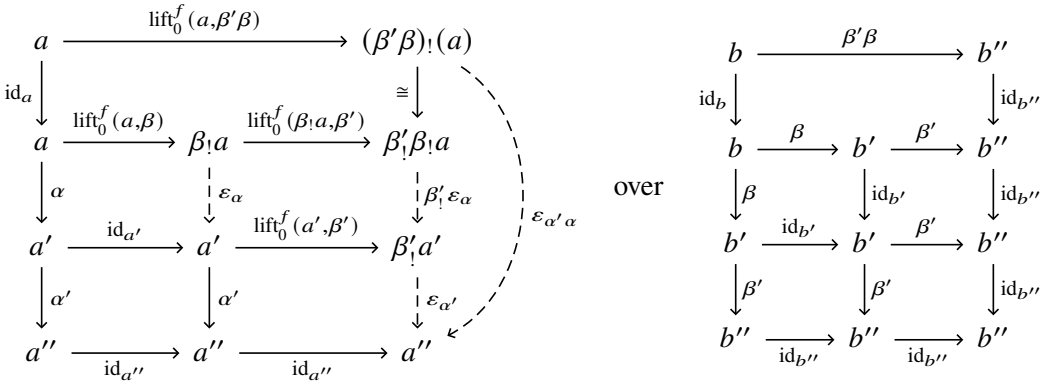
(1) Every isomorphism of A is f-cocartesian.

(2) Let $a \xrightarrow{\alpha} a' \xrightarrow{\alpha'} a''$ be composable arrows of A over $b \xrightarrow{\beta} b' \xrightarrow{\beta'} b''$. Assume that α is f-cocartesian. Then α' is f-cocartesian if and only if $\alpha'\alpha$ is f-cocartesian.

Proof. We use Lemma 4.5.8 to test cocartesian morphisms.

(1) Without loss of generality we may treat the case of an identity $\mathrm{id}_a\colon a \to a$. In this case, we can choose $\mathrm{lift}_0^f(a, \mathrm{id}_{f(a)})$ to be the identity, so that ε_α must be the identity too.

(2) Consider the following diagram



where the rightmost semicircle commutes by the uniqueness statement of Lemma 4.5.7. Since α is cocartesian by assumption, ε_α is an isomorphism, hence also its cocartesian transport along β'. We conclude that $\varepsilon_{\alpha'\alpha}$ is an isomorphism if and only if $\varepsilon_{\alpha'}$ is an isomorphism, which is what we wanted to show. □

4.5.2 Closure properties of cartesian and cocartesian fibrations

We will now show that the classes of cartesian and cocartesian fibrations satisfy various closure properties.

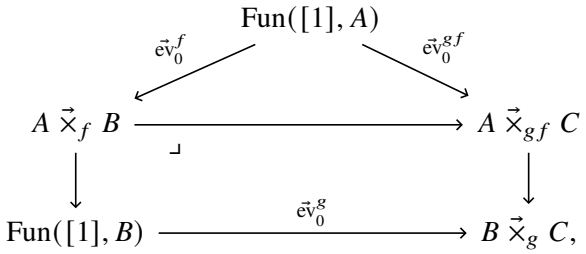
Proposition 4.5.13. *Let $f\colon A \to B$ be a cocartesian fibration. Then for every category E, the functor $f_*\colon \mathrm{Fun}(E, A) \to \mathrm{Fun}(E, B)$ is a cocartesian fibration.*

Proof. Since $\mathrm{Fun}(E, -)$ preserves pullbacks, one observes that the directed evaluation map of f_* is given by applying $\mathrm{Fun}(E, -)$ to $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) \to A \,\vec{\times}_f\, B$. The claim is now an immediate consequence from the fact that $\mathrm{Fun}(E, -)$ preserves adjunctions, see Proposition 4.4.4. □

Proposition 4.5.14. *The classes of cartesian and cocartesian fibrations are stable under composition.*

Proof. We will only prove the claim about cocartesian fibrations, leaving the dual claim for cartesian fibrations to the reader. If $f\colon A \to B$ and $g\colon B \to C$ are two cocartesian fibrations, we have by

Lemma 4.1.7 a commutative diagram



where the bottom square is a pullback square. As f and g are cocartesian fibrations, the maps $\vec{\mathrm{ev}}_0^g$ and $\vec{\mathrm{ev}}_0^f$ admit left adjoint sections, and it then follows from Proposition 4.4.13 and Proposition 4.4.15 that also $\vec{\mathrm{ev}}_0^{gf}$ admits a left adjoint section. □

Proposition 4.5.15. *The classes of cartesian and cocartesian fibrations are stable under base change.*

Proof. Again we only treat the case of cocartesian fibrations, leaving the dual statement for cartesian fibrations to the reader. Consider a pullback square of the form

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{v} & B, \end{array}$$

and assume that f is a cocartesian fibration. By Lemma 4.1.6, there is a pullback square of the form

$$\begin{array}{ccc} \mathrm{Fun}([1], A') & \longrightarrow & \mathrm{Fun}([1], A) \\ {\scriptstyle \vec{\mathrm{ev}}_0^{f'}}\downarrow & & \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^{f}} \\ A' \vec{\times}_{f'} B' & \longrightarrow & A \vec{\times}_f B, \end{array}$$

and thus it follows from Proposition 4.4.15 that also $\vec{\mathrm{ev}}_0^{f'}$ admits a left adjoint section. □

4.5.3 Relative cocartesian fibrations

Let S be a category. Using relative directed evaluation from Section 4.3.1 and the relative adjunctions of Section 4.4.3, we now define cartesian and cocartesian fibrations relative to S.

Definition 4.5.16. A functor $f : C \to D$ over S is a *cocartesian fibration relative to S* if $\vec{\mathrm{ev}}_{0,S}^f$ admits a left adjoint section relative to S. It is a *cartesian fibration relative to S* if $\vec{\mathrm{ev}}_{1,S}^f$ admits a right adjoint section relative to S. The corresponding notions of relative left and right fibrations were defined in Definition 4.3.10.

Lemma 4.5.17. *Let $f : C \to D$ be a functor over S. Then f admits a left (resp. right) adjoint section relative to S if and only if its underlying functor admits a left (resp. right) adjoint section.*

Proof. Forgetting the structure over S gives one implication. Conversely, let $s\colon D \to C$ be a left adjoint section of f. Write its section identification as $fs = \mathrm{id}_D$, with counit $\varepsilon\colon sf \to \mathrm{id}_C$. The normalized triangle identifications are

$$f\varepsilon = \mathrm{id}_f, \qquad \varepsilon s = \mathrm{id}_s\,.$$

The identification $fs = \mathrm{id}_D$ makes s a functor over D, and hence over S. The first triangle identification makes ε a transformation over D, and postcomposition with $q\colon D \to S$ makes it a transformation over S. Thus the same counit exhibits s as a left adjoint relative to S by Definition 4.4.28. Its section identification is over S with the structure just specified. The right adjoint case is dual. □

Lemma 4.5.18. *For a functor $f\colon C \to D$ over S, there is a pullback square*

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_S([1]_S, C) & \longrightarrow & \mathrm{Fun}([1], C) \\ {\scriptstyle \vec{\mathrm{ev}}^f_{0,S}}\downarrow & \lrcorner & \downarrow{\scriptstyle \vec{\mathrm{ev}}^f_0} \\ C \mathbin{\vec{\times}}_{f,S} D & \longrightarrow & C \mathbin{\vec{\times}}_f D. \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_S([1]_S, C) & \longrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & & \downarrow{\scriptstyle \vec{\mathrm{ev}}^f_0} \\ C \mathbin{\vec{\times}}_{f,S} D & \longrightarrow & C \mathbin{\vec{\times}}_f D \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Fun}([1], S). \end{array}$$

The bottom square and the outer rectangle are the defining pullback squares. The top square is therefore a pullback by the pasting law. □

Proposition 4.5.19. *If the underlying functor of a functor $f\colon C \to D$ over S is a left (resp. right) fibration, then f is a left (resp. right) fibration relative to S.*

Proof. By Lemma 4.5.18, relative directed evaluation is a pullback of absolute directed evaluation. A pullback of an equivalence is an equivalence, with its inverse and identifications over the base. The right fibration case is dual. □

Proposition 4.5.20. *If the underlying functor of a functor $f\colon C \to D$ over S is a cocartesian (resp. cartesian) fibration, then f is a cocartesian (resp. cartesian) fibration relative to S.*

Proof. The functor $\vec{\mathrm{ev}}^f_{0,S}$ is a pullback of $\vec{\mathrm{ev}}^f_0$ by Lemma 4.5.18. It therefore admits a left adjoint section by Proposition 4.4.15. By Lemma 4.5.17, this is a left adjoint section relative to S. The cartesian case is dual. □

Proposition 4.5.21. *Let A be an anima and let $f\colon C \to D$ be a functor over A. Then f is a cocartesian (resp. cartesian) fibration relative to A if and only if its underlying functor is a cocartesian (resp. cartesian) fibration.*

Proof. Since A is an anima, the identity-arrow functor $A \to \mathrm{Fun}([1], A)$ is an equivalence. Its pullbacks

$$\underline{\mathrm{Fun}}_A([1]_A, C) \longrightarrow \mathrm{Fun}([1], C), \qquad C \vec{\times}_{f,A} D \longrightarrow C \vec{\times}_f D$$

are therefore equivalences. The square in Lemma 4.5.18 identifies the two directed evaluation functors through these equivalences. One admits a left adjoint section if and only if the other does; apply Lemma 4.5.17 to obtain the relative assertion. The cartesian case is dual. □

Corollary 4.5.22. *For an anima A, every functor $p\colon C \to A$ is both a cartesian and a cocartesian fibration.*

Proof. Regarding p as a functor over A, its relative directed evaluation at the source is

$$\mathrm{ev}_0\colon \underline{\mathrm{Fun}}_A([1]_A, C) \longrightarrow C.$$

The identity-arrow functor is a left adjoint section: the contraction of an arrow to its source used in Lemma 4.4.11 restricts to arrows over identities in A. The contraction and its triangle identifications remain over A. Dually, evaluation at the target has a right adjoint section. Thus p is both cartesian and cocartesian relative to A, and the result follows from Proposition 4.5.21. □

4.6 The source and target fibrations

In this section, we provide two fundamental examples of (co)cartesian fibrations: the source and target maps $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$ for a category C. The fibers of these two functors are the slice categories $C_{/x}$ and $C_{x/}$ constructed in Section 4.3, and the fact that ev_0 and ev_1 are a cartesian and a cocartesian fibration, respectively, encodes the functoriality of these slices in x.

We start with an auxiliary result, for which we need to recall the maps $j_0, j_1\colon \Delta^2 \to [1] \times [1]$ and $p_0, p_2\colon [1] \times [1] \to \Delta^2$ from Definition 2.1.21 and Construction 2.1.23, pictorially represented by the following diagrams:

$$j_0 = \begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ & \searrow & \downarrow \\ & & (1,1), \end{array} \qquad j_1 = \begin{array}{ccc} (0,0) & & \\ \downarrow & \searrow & \\ (1,0) & \longrightarrow & (1,1), \end{array}$$

$$p_0 = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ {\scriptstyle \mathrm{id}_0}\downarrow & & \downarrow \\ 0 & \longrightarrow & 2, \end{array} \qquad p_2 = \begin{array}{ccc} 0 & \longrightarrow & 2 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}_2} \\ 1 & \longrightarrow & 2. \end{array}$$

Proposition 4.6.1. *The map j_0 is a left adjoint section of the map p_0. The map j_1 is a right adjoint section of the map p_2.*

Proof. We prove the claim for j_1 and p_2 and leave the analogous case for j_0 and p_0 to the reader. For simplicity we write $j := j_1$ and $p := p_2$ for the remainder of this proof. By construction j is a section of p. We will construct a natural transformation $\eta\colon \mathrm{id}_{[1]\times[1]} \to jp$, or equivalently a map

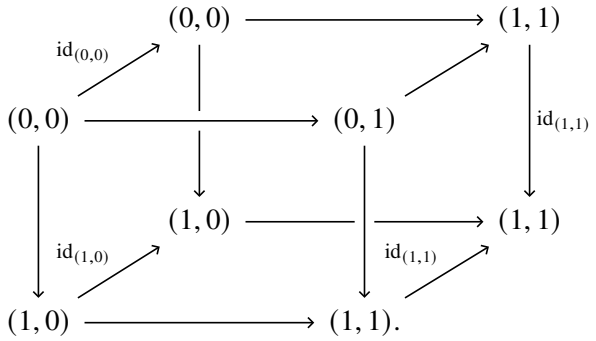
$$h\colon ([1] \times [1]) \times [1] \to ([1] \times [1])$$

such that $h(x, y, 0) = (x, y)$ and $h(x, y, 1) = jp(x, y) = (\max(x, y), y)$, where the last equality holds by Lemma 2.1.26. Formally, we may define h by

$$h(x, y, t) = (\max(\min(t, y), x), y),$$

using the maps max and min from Construction 2.1.11. Indeed, when $t = 0$, we have $\min(t, y) = \min(0, y) = 0$, so that the first component is $\max(0, x) = x$. When $t = 1$, we have $\min(t, y) = \min(1, y) = y$, so that the first component is $\max(y, x) = \max(x, y)$.

Pictorially, we may display h as the following commutative cube in $[1] \times [1]$ (read from front to back):



An explicit computation using the relations from Lemma 2.1.12 shows that the composite ph is the identity transformation $\mathrm{id}_p \colon p \to p$. Similarly, one can show that the composite $h(j \times 1)$ is the identity transformation $\mathrm{id}_j \colon j \to j$. This finishes the proof. □

Construction 4.6.2. Given a natural transformation $H \colon [1] \times C \to D$ between two functors $f, g \colon C \to D$, and a category A, we construct a natural transformation

$$H^* \colon [1] \times \mathrm{Fun}(D, A) \to \mathrm{Fun}(C, A)$$

between the precomposition functors $f^*, g^* \colon \mathrm{Fun}(D, A) \to \mathrm{Fun}(C, A)$ as follows:

$$\mathrm{Fun}(D, A) \xrightarrow{H^*} \mathrm{Fun}([1] \times C, A) \simeq \mathrm{Fun}([1], \mathrm{Fun}(C, A)).$$

It is clear that this construction sends identity transformations to identity transformations and compositions of transformations to compositions of transformations.

Theorem 4.6.3. *For any category C, the functor* $\mathrm{ev}_0 \colon \mathrm{Fun}([1], C) \to C$ *is a cartesian fibration. Dually, the functor* $\mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$ *is a cocartesian fibration.*

Proof. We show the statement for ev_0 and leave the dual case of ev_1 to the reader. We need to show that the map

$$\vec{\mathrm{ev}}_1^{\mathrm{ev}_0} \colon \mathrm{Fun}([1], \mathrm{Fun}([1], C)) \to C \vec{\times}_{\mathrm{ev}_0} \mathrm{Fun}([1], C)$$

admits a right adjoint section. We may identify the source of this map with $\mathrm{Fun}([1] \times [1], C)$. The target sits by definition in a pullback square

$$\begin{array}{ccc} C \vec{\times}_{\mathrm{ev}_0} \mathrm{Fun}([1], C) & \longrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathrm{Fun}([1], C) & \xrightarrow{\mathrm{ev}_0} & C, \end{array}$$

and thus we may identify it with $\mathrm{Fun}(\Lambda^2_1, C)$. Unwinding definitions, we see that the map $\vec{\mathrm{ev}}_1^{\mathrm{ev}_0}$ factors as

$$\mathrm{Fun}([1]\times[1], C) \xrightarrow{j_1^*} \mathrm{Fun}(\Delta^2, C) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Sp}_2, C),$$

where the second map is a trivial fibration by the Segal axiom. It will thus suffice to show that j_1^* admits a right adjoint section.

We claim that p_2^* is a right adjoint section of j_1^*. Indeed, we have $j_1^* p_2^* = (p_2 j_1)^* = (\mathrm{id})^* = \mathrm{id}$. The transformation $h\colon 1 \to j_1 p_2$ gives rise to a transformation $h^*\colon 1 \to (j_1 p_2)^* = p_2^* j_1^*$ via Construction 4.6.2. Furthermore, we have

$$j_1^* h^* = (h j_1)^* = \mathrm{id}_{j_1^*} \qquad \text{and} \qquad h^* p_2^* = (p_2 h)^* = \mathrm{id}_{p_2^*}.$$

This finishes the proof. □

Remark 4.6.4. Pictorially, the functor $\vec{\mathrm{ev}}_1^{\mathrm{ev}_0}$ may be thought of as the following assignment:

$$\begin{array}{ccc} x & \xrightarrow{u} & y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ x' & \xrightarrow[u']{} & y' \end{array} \quad\mapsto\quad \begin{array}{ccc} x & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{u'\alpha} & \\ x' & \xrightarrow[u']{} & y' \end{array} \quad\mapsto\quad \begin{array}{ccc} x & & \\ {\scriptstyle\alpha}\downarrow & & \\ x' & \xrightarrow[u']{} & y'. \end{array}$$

Its right adjoint section is given informally by the following assignment:

$$\begin{array}{ccc} x & & \\ {\scriptstyle\alpha}\downarrow & & \\ x' & \xrightarrow[u']{} & y' \end{array} \quad\mapsto\quad \begin{array}{ccc} x & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{u'\alpha} & \\ x' & \xrightarrow[u']{} & y' \end{array} \quad\mapsto\quad \begin{array}{ccc} x & \xrightarrow{u'\alpha} & y' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mathrm{id}_{y'}} \\ x' & \xrightarrow[u']{} & y'. \end{array}$$

We will in fact need a slight generalization of Theorem 4.6.3. Given a functor $f\colon C \to D$, we may consider the directed pullbacks $C \vec{\times}_f D$ and $D \vec{\times}_f C$. They both come equipped with canonical maps to D, which we claim to be (co)cartesian fibrations:

Proposition 4.6.5. *For a functor $f\colon C \to D$, the projection $\mathrm{ev}_1\colon C \vec{\times}_f D \to D$ is a cocartesian fibration, while the projection $\mathrm{ev}_0\colon D \vec{\times}_f C \to D$ is a cartesian fibration.*

Proof. By symmetry, it suffices to prove the claim for $\mathrm{ev}_1\colon C \vec{\times}_f D \to D$. We thus need to show that the map

$$\vec{\mathrm{ev}}_0^{\mathrm{ev}_1}\colon \mathrm{Fun}([1], C \vec{\times}_f D) \to (C \vec{\times}_f D) \vec{\times}_{\mathrm{ev}_1} D$$

admits a left adjoint section. The proof is analogous to the proof of Theorem 4.6.3; to make the proof more readable we will explain what happens on objects and leave it to the reader to write a formal proof.

First observe that terms of the category $\mathrm{Fun}([1], C \vec{\times}_f D)$ are pairs (α, σ) where $\alpha\colon x \to x'$ is a morphism in C, and σ is a commutative square in D of the form

$$\begin{array}{ccc} f(x) & \xrightarrow{u} & y \\ {\scriptstyle f(\alpha)}\downarrow & & \downarrow{\scriptstyle\beta} \\ f(x') & \xrightarrow{u'} & y'. \end{array}$$

On the other hand, terms of $(C \vec{\times}_f D) \vec{\times}_\pi D$ are of the form $(x, f(x) \xrightarrow{u} y \xrightarrow{v} z)$, which we might alternatively display as

$$\begin{pmatrix} & f(x) & \xrightarrow{u} & y \\ x, & & & \downarrow{\scriptstyle v} \\ & & & z \end{pmatrix}.$$

The map $\vec{\mathrm{ev}}_0^{\mathrm{ev}_1}$ is then given by the assignment

$$\begin{pmatrix} x & & f(x) & \xrightarrow{u} & y \\ {\scriptstyle\alpha}\downarrow & , & {\scriptstyle f(\alpha)}\downarrow & & \downarrow{\scriptstyle\beta} \\ x' & & f(x') & \xrightarrow{u'} & y' \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} & f(x) & \xrightarrow{u} & y \\ x, & & & \downarrow{\scriptstyle\beta} \\ & & & y' \end{pmatrix}$$

A left adjoint section is then given by the assignment

$$\begin{pmatrix} & f(x) & \xrightarrow{u} & y \\ x, & & & \downarrow{\scriptstyle v} \\ & & & z \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} x & & f(x) & \xrightarrow{u} & y \\ {\scriptstyle \mathrm{id}_x}\downarrow & , & {\scriptstyle \mathrm{id}_{f(x)}}\downarrow & & \downarrow{\scriptstyle v} \\ x & & f(x) & \xrightarrow{vu} & z \end{pmatrix}.$$

The precise definition of this functor and the fact that it is a left adjoint section proceed just like the (dual version of the) proof of Theorem 4.6.3, using the functors $j_0 \colon \Delta^2 \to [1] \times [1]$ and $p_0 \colon [1] \times [1] \to \Delta^2$. □

4.7 Cocartesian functors

In Lemma 4.2.13, we saw that any functor between left fibrations is compatible with the covariant transport functors. The situation is different for cocartesian fibrations: a functor between cocartesian fibrations is compatible with the covariant transport functors if and only if it preserves cocartesian morphisms. Functors satisfying this property are called *cocartesian functors* and will be made precise in this section.

Construction 4.7.1 (Beck-Chevalley transformation). Let $p \colon C \to S$ and $q \colon D \to S$ be two cocartesian fibrations, and consider a commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S. & \end{array}$$

By definition of cocartesian fibrations, the map $\vec{\mathrm{ev}}_0^p \colon \mathrm{Fun}([1], C) \to C \vec{\times}_p S$ admits a left adjoint section lift_0^p, and similarly $\vec{\mathrm{ev}}_0^q$ admits a left adjoint section lift_0^q. In particular, we have identifications $\vec{\mathrm{ev}}_0^p \, \mathrm{lift}_0^p = \mathrm{id}$ and $\vec{\mathrm{ev}}_0^q \, \mathrm{lift}_0^q = \mathrm{id}$, and transformations $\varepsilon_p \colon \mathrm{lift}_0^p \, \vec{\mathrm{ev}}_0^p \to \mathrm{id}$ and $\varepsilon_q \colon \mathrm{lift}_0^q \, \vec{\mathrm{ev}}_0^q \to \mathrm{id}$. Observe that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}([1], C) & \xrightarrow{f_*} & \mathrm{Fun}([1], D) \\ {\scriptstyle \vec{\mathrm{ev}}_0^p}\downarrow & & \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^q} \\ C \vec{\times}_p S & \xrightarrow{\vec{f}} & D \vec{\times}_q S. \end{array}$$

We now define the *Beck-Chevalley transformation* $\mathrm{BC}_!\colon \mathrm{lift}_0^q\, \vec{f} \to f_* \,\mathrm{lift}_0^p$ as the following composite:

$$\mathrm{lift}_0^q\, \vec{f} = \mathrm{lift}_0^q\, \vec{f}\vec{\mathrm{ev}}_0^p\, \mathrm{lift}_0^p = \mathrm{lift}_0^q\, \vec{\mathrm{ev}}_0^q f_* \,\mathrm{lift}_0^p \xrightarrow{\varepsilon_q} f_* \,\mathrm{lift}_0^p\,.$$

Definition 4.7.2 (Cocartesian functor). The functor f over S as above is said to be a *cocartesian functor over S* if the natural transformation $\mathrm{BC}_!\colon \mathrm{lift}_0^q\, \vec{f} \to f_* \,\mathrm{lift}_0^p$ is invertible.

There is a dual notion of a *cartesian functor* between cartesian fibrations over S.

We can make the Beck-Chevalley transformation a bit more concrete as follows. For every morphism $v\colon x \to y$ in S and every term a of C_x, the map $\mathrm{BC}_!(v)\colon \mathrm{lift}_0^q\, \vec{f}(a, v) \to f_* \,\mathrm{lift}_0^p(a, v)$ takes the form of a commutative square

$$\begin{array}{ccc} f(a) & \xrightarrow{\mathrm{id}_{f(a)}} & f(a) \\ {\scriptstyle \mathrm{lift}_0^q(f(a),v)}\downarrow & & \downarrow{\scriptstyle f(\mathrm{lift}_0^p(a,v))} \\ v_!(f(a)) & \xrightarrow{\overline{v}} & f(v_!(a)). \end{array}$$

In this sense, we see that the Beck-Chevalley transformation induces a canonical morphism of functors

$$\overline{v}\colon v_!\, f_x \to f_y v_!$$

where $v_!\colon C_x \to C_y$ and $v_!\colon D_x \to D_y$ are the covariant transport functors and $f_x\colon C_x \to D_x$ is the functor induced by f on each fiber.

Lemma 4.7.3. *Given two cocartesian fibrations $p\colon C \to S$ and $q\colon D \to S$, and a functor $f\colon C \to D$ over S, the following assertions are equivalent:*

(i) The functor f is cocartesian over S;

(ii) For any morphism $v\colon x \to y$ in S, the Beck-Chevalley comparison $\overline{v}$ in the commutative square

$$\begin{array}{ccc} C_x & \xrightarrow{f_x} & D_x \\ {\scriptstyle v_!}\downarrow & & \downarrow{\scriptstyle v_!} \\ C_y & \xrightarrow{f_y} & D_y \end{array}$$

is a natural isomorphism;

(iii) The functor f sends locally p-cocartesian morphisms to locally q-cocartesian morphisms;

(iv) The functor f sends p-cocartesian morphisms to q-cocartesian morphisms.

Proof. Conditions (i) and (ii) are equivalent by the definition of the Beck-Chevalley transformation. The equivalence of (ii) and (iii) follows from the description of locally cocartesian lifts in Remark 4.8.11. Finally, (iii) and (iv) are equivalent by Lemma 4.8.10. □

Example 4.7.4. For $S = *$, every functor $f\colon C \to D$ is a cocartesian functor over $*$. Indeed, we have $C \,\vec{\times}_p * \simeq C$, with the directed evaluation map corresponding to $\mathrm{ev}_0\colon \mathrm{Fun}([1], C) \to C$, which admits a left adjoint section $p^*_{[1]}\colon C \to \mathrm{Fun}([1], C)$. The same holds for D, and the claim follows from naturality of $p^*_{[1]}$.

Corollary 4.7.5. *Every equivalence $C \xrightarrow{\sim} D$ over S is a cocartesian functor.* □

Example 4.7.6. If p and q are left fibrations, then any functor f over S from C to D is a cocartesian functor over S: this is a direct consequence of the fact that the two adjunctions $\mathrm{lift}_0^p \dashv \vec{\mathrm{ev}}_0^p$ and $\mathrm{lift}_0^q \dashv \vec{\mathrm{ev}}_0^q$ are adjoint equivalences.

Lemma 4.7.7. *Consider cocartesian fibrations $f\colon C \to D$ and $q\colon D \to S$, and let $p := qf\colon C \to S$ denote the composite cocartesian fibration from Proposition 4.5.14. Then f is a cocartesian functor over S.*

Proof. By construction of the Beck-Chevalley map, it will suffice to show that the map

$$\mathrm{lift}_0^q \vec{\mathrm{ev}}_0^q f_* \mathrm{lift}_0^p \xrightarrow{\varepsilon_q} f_* \mathrm{lift}_0^p$$

is an equivalence. Spelling out the left adjoint section lift_0^p obtained from Proposition 4.5.14, we see that it sits in the following commutative diagram:

$$\begin{array}{ccccc}
 & & \overset{\mathrm{lift}_0^{qf}}{\frown} & & \\
\mathrm{Fun}([1],C) & \underset{\vec{\mathrm{ev}}_0^f}{\overset{\mathrm{lift}_0^f}{\rightleftarrows}} & C \vec{\times}_f D & \underset{}{\overset{}{\rightleftarrows}} & C \vec{\times}_{qf} S \\
 & {\scriptstyle f_*}\searrow & \downarrow & \lrcorner & \downarrow \\
 & & \mathrm{Fun}([1],D) & \underset{\vec{\mathrm{ev}}_0^q}{\overset{\mathrm{lift}_0^q}{\rightleftarrows}} & D \vec{\times}_q S.
\end{array}$$

In particular, we see that $f_* \mathrm{lift}_0^p$ factors through lift_0^q, and thus it suffices to show that the map

$$\mathrm{lift}_0^q \vec{\mathrm{ev}}_0^q \mathrm{lift}_0^q \xrightarrow{\varepsilon_q \mathrm{lift}_0^q} \mathrm{lift}_0^q$$

is an equivalence. But this is immediate from the triangle identities and the fact that the unit $\mathrm{id} \to \vec{\mathrm{ev}}_0^q \mathrm{lift}_0^q$ of the adjunction $\mathrm{lift}_0^q \dashv \vec{\mathrm{ev}}_0^q$ is a natural isomorphism. □

Proposition 4.7.8. *For any category C the diagram*

$$\begin{array}{ccc}
\mathrm{Fun}([1],C) & \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} & C \times C \\
 & {\scriptstyle \mathrm{ev}_1}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\
 & C &
\end{array}$$

is a cocartesian functor over C.

Since locally ev_1-cocartesian morphisms in $\mathrm{Fun}([1],C)$ precisely are those commutative squares in C of the form

$$\begin{array}{ccc}
x' & \xrightarrow{f'} & y' \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
x & \xrightarrow{f} & y
\end{array}$$

in which u is an isomorphism, this is a direct consequence of:

Proposition 4.7.9. *Let $p\colon X \to B$ be any cocartesian fibration and $u\colon X \to C$ any functor. Then the diagram*

$$\begin{array}{ccc} X & \xrightarrow{(p,u)} & B \times C \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_B} & \\ & B & \end{array}$$

is a cocartesian functor over B if and only if the functor u sends locally p-cocartesian morphisms to isomorphisms.

Proof. We know from Example 4.8.5 that the (locally) pr_B-cocartesian morphisms in $B \times C$ are nothing else than pairs (u, v) with $u\colon b_0 \to b_1$ any morphism in B and $v\colon c_0 \to c_1$ any isomorphism in C. □

Definition 4.7.10. Let us consider a commutative square of categories of the form below.

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xrightarrow{u} & T \end{array}$$

We assume that both p and q are cocartesian (cartesian) fibrations. We say that f is a *cocartesian functor over u* (a *cartesian functor over u*) if the induced functor $C \to S \times_T D$ is a cocartesian functor over S (a cartesian functor over S, respectively).

Remark 4.7.11. For a commutative square as in the definition above, the property that f is a (co)cartesian functor over u is equivalent to the property that it sends p-(co)cartesian morphisms in C to q-(co)cartesian morphisms in D.

Proposition 4.7.12. *For any functor $f\colon C \to D$, the diagram*

$$\begin{array}{ccc} \mathrm{Fun}([1], C) & \xrightarrow{\vec{\mathrm{ev}}_1^f} & D \mathbin{\vec{\times}_f} C \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ C & \xrightarrow{\mathrm{id}_C} & C \end{array}$$

determines $\vec{\mathrm{ev}}_1^f$ as a cocartesian functor over C.

Proof. Recall that terms of $D \mathbin{\vec{\times}_f} C$ are pairs (u, x) where x is a term of C and $u\colon y \to f(x)$ is a morphism in D. In particular, morphisms in $D \mathbin{\vec{\times}_f} C$ are essentially characterized as morphisms $\alpha\colon x \to x'$ in C together with commutative squares of the form

$$\begin{array}{ccc} y & \xrightarrow{u} & f(x) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle f(\alpha)} \\ y' & \xrightarrow{u'} & f(x') \end{array}$$

in D. Such data yield a pr_2-cocartesian morphism if and only if β is an isomorphism in D (more precisely, the proof of Proposition 4.6.5 tells us that these are the morphisms isomorphic to morphisms obtained from such data with β an identity). On the other hand ev_1-cocartesian

morphisms in $\mathrm{Fun}([1], C)$ correspond to commutative squares in which the left vertical map is an isomorphism. Its image under $\vec{\mathrm{ev}}_1^f$ has left vertical map obtained by applying f, hence again an isomorphism. It follows that the functor $\vec{\mathrm{ev}}_1^f$ preserves ev_1-cocartesian morphisms, so that Lemma 4.7.3 applies to this situation. □

Lemma 4.7.13. *For a functor $f : C \times [1] \to D$, the induced functor $(f, \mathrm{pr}_{[1]}) : C \times [1] \to D \times [1]$ over $[1]$ is a cocartesian functor over $[1]$ if and only if f is a natural isomorphism of functors $C \to D$.*

Proof. Recall from Example 4.8.5 that the C-component of a cocartesian morphism in $C \times [1]$ is necessarily invertible, and similarly for cocartesian morphisms in $D \times [1]$. It follows that $(f, \mathrm{pr}_{[1]})$ is a cocartesian functor over $[1]$ if and only if for every term x of C the morphism $f_x : \{x\} \times [1] \to D$ is invertible in D. But this is equivalent to the condition that f is a natural isomorphism. □

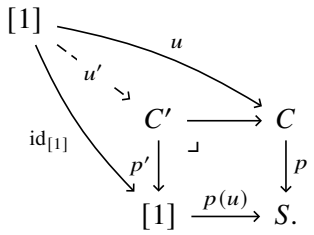
Proposition 4.7.14. *Let $p : C \to S$ and $q : D \to S$ be cocartesian fibrations and let $f : C \to D$ be a functor over S. If f is a cocartesian functor over S, then for every functor $T \to S$ the induced functor $f \times_S T : C \times_S T \to D \times_S T$ is a cocartesian functor over T.*

Proof. Let us abbreviate $C_T := C \times_S T$, $D_T := D \times_S T$, and $f_T := f \times_S T$. By Proposition 4.5.15, the functors $p_T : C_T \to T$ and $q_T : D_T \to T$ are cocartesian fibrations. A morphism in C_T is p_T-cocartesian if and only if its image in C is p-cocartesian, by Observation 4.8.6 and Lemma 4.8.10. Since f preserves cocartesian morphisms, f_T sends every p_T-cocartesian morphism to a q_T-cocartesian morphism. The claim now follows from Lemma 4.7.3. □

4.8 Locally cocartesian fibrations

In this section, we introduce the notion of a locally (co)cartesian fibration. Our exposition closely follows that of [Lur, Section 5.1.3, 01TW].

Definition 4.8.1. Let $p : C \to S$ be a functor, and consider a morphism $u : x \to y$ in C, say in context Γ. This morphism uniquely lifts to a morphism u' in the category C' defined by the following pullback square in context Γ:



We say that u is *locally p-cocartesian* if the morphism u' is p'-cocartesian. We say that p is a *locally cocartesian fibration* if for every morphism $w : s \to t$ in S and any $x : C_s$ there exists a locally p-cocartesian lift $u : x \to y$ of w starting in x, equipped with a specified identification $p(u) = w$.

One may dually define the notion of a *locally p-cartesian* morphism and the associated notion of locally cartesian fibrations.

Remark 4.8.2. Notice that the morphism $u\colon x \to y$ is locally p-cocartesian if for every object $z \in C_{p(y)}$ the functor $v \mapsto vu$ given by composing with u induces an equivalence

$$C_{p(y)}(y,z) \to C(x,z)_{p(u)},$$

where $C_{p(y)}$ is the fiber of p over $p(y)$ and $C(x,z)_{p(u)}$ is defined as the fiber

$$\begin{array}{ccc} C(x,z)_{p(u)} & \longrightarrow & C(x,z) \\ \downarrow & \lrcorner & \downarrow p \\ * & \xrightarrow{p(u)} & S(p(x),p(y)). \end{array}$$

More informally, this says that for any morphism $u'\colon x \to z$ equipped with an identification $p(u') = p(u)$, there is a unique morphism $v\colon y \to z$ in $C_{p(y)}$ fitting in a commutative triangle

$$\begin{array}{ccc} & y & \\ {\scriptstyle u}\nearrow & & \searrow{\scriptstyle \exists! v} \\ x & \xrightarrow[u']{} & z \end{array}$$

in C which is sent under p to the degenerate commutative triangle on $p(u)$ in S.

Example 4.8.3. Any p-cartesian morphism is locally p-cartesian.

Example 4.8.4. Any isomorphism is p-cartesian, hence in particular locally p-cartesian.

Example 4.8.5. Let A and B be two categories. A morphism (u,v), consisting of a pair of morphisms $u\colon x_0 \to x_1$ and $v\colon y_0 \to y_1$ in A and B, respectively, is locally pr_A-cocartesian if and only if v is an isomorphism. To see this, we observe that $(u, \mathrm{id}_b)\colon (x_0,b) \to (x_1,b)$ is pr_A-cocartesian for any object b by direct inspection of the universal properties. On the other hand, for any morphism $u\colon x_0 \to x_1$ in A any object of the form (x_0,b) in $A \times B$, the morphism (u,id_b) is a locally pr_A-cocartesian lift of u. This implies that a morphism of $A \times B$ is pr_A-cocartesian if and only if it is isomorphic to a morphism of the form (u,id_b) with v an isomorphism in B. It is straightforward to see that the latter conditions characterizes the morphisms of $A \times B$ whose second component is an isomorphism.

Observation 4.8.6. Consider a pullback square

$$\begin{array}{ccc} C' & \xrightarrow{f} & C \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \longrightarrow & S \end{array}$$

and let $u'\colon x' \to y'$ be a morphism in C' with image $u := f(u')\colon x \to y$ in C. Then u' is locally p'-cocartesian if and only if u is locally p-cocartesian.

Proposition 4.8.7. *Let $p\colon C \to S$ be a functor and let $u\colon x \to y$ be a morphism in C. Assume that $p(u)\colon p(x) \to p(y)$ is an isomorphism in S. Then the following conditions are equivalent:*

(1) u is locally p-cocartesian;

(2) u is locally p-cartesian;

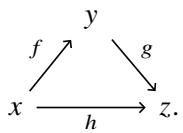
(3) u is an isomorphism in C.

Proof. By the Rezk Axiom, the functor $p(u)\colon [1] \to S$ is isomorphic to the constant functor at $p(x)$. Pulling back p along these isomorphic functors identifies the lift of u with a morphism in the product projection $C_{p(x)} \times [1] \to [1]$. By Example 4.8.5, this lift is locally cocartesian if and only if its component in $C_{p(x)}$ is an isomorphism. The latter is equivalent to u being an isomorphism in C. The dual argument gives the equivalence with local cartesianness. □

Lemma 4.8.8. *Assume that $S = T^{\triangleright} = T \star [0]$ for some category T. Let $p\colon C \to T^{\triangleright}$ be a functor and let $u\colon x \to y$ be a morphism in C such that $p(y)$ is the cocone point of $T^{\triangleright}$. Then u is locally p-cocartesian if and only if it is p-cocartesian.*

Proof. Every p-cocartesian morphism is locally p-cocartesian. Conversely, suppose that u is locally p-cocartesian. There are no morphisms from the cocone point of $T^{\triangleright}$ to an object coming from T, since there is no morphism from 1 to 0 in $[1]$. Thus any morphism $p(y) \to p(z)$ lies in the full subcategory $\emptyset \star [0] \simeq [0]$ and is identified with the identity of the cocone point. The lifting problems in the definition of a p-cocartesian morphism therefore reduce to those over this identity. These are precisely the lifting problems in Remark 4.8.2, so u is p-cocartesian. □

Corollary 4.8.9. *Let $p\colon C \to S$ be a functor and consider a commutative triangle $\sigma\colon [2] \to C$ of the form*



Assume that f is p-cocartesian. Then g is locally p-cocartesian if and only if h is locally p-cocartesian.

Proof. By pulling back along the map $p(\sigma)\colon [2] \to S$ it will suffice to prove the claim when $S = [2]$ and $p(\sigma) = \mathrm{id}_{[2]}$. In this case Lemma 4.8.8 tells us that the morphisms g and h are locally p-cocartesian if and only if they are p-cocartesian. The claim thus follows from Lemma 4.5.12. □

Lemma 4.8.10. *Let $p\colon C \to S$ be a cocartesian fibration. A morphism in C is locally p-cocartesian if and only if it is p-cocartesian.*

Proof. Every p-cocartesian morphism is locally p-cocartesian. Conversely, let $f\colon x \to y$ be a locally p-cocartesian morphism, and let $g\colon x \to y'$ be the chosen p-cocartesian lift of $p(f)$. Since both f and g are locally p-cocartesian lifts of $p(f)$ with the same source, there is a unique isomorphism $u\colon y' \to y$ in the fiber over $p(y)$ such that $f = ug$. Thus f is p-cocartesian. □

Remark 4.8.11. Let $p\colon C \to S$ be a cocartesian fibration. Given any morphism $v\colon y_0 \to y_1$ in S and any object x_0 in C such that $p(x_0) = y_0$, there is a (locally) p-cocartesian morphism $u\colon x_0 \to x_1$ in C such that $p(u) = v$ in $S(y_0, y_1)$. On the other hand, there is a functor

$$v_!\colon C_{y_0} \to C_{y_1}$$

defined by covariant transport. Each object x_0 of C_{y_0} comes with a canonical map

$$\varepsilon_{x_0} := \mathrm{lift}_0^p(x_0, v)\colon x_0 \to v_!(x_0)$$

and there is a canonical map $c_u : v_!(x_0) \to x_1$ in C_{y_1} such that $c_u \varepsilon_{x_0} = u$ holds in C. By uniqueness of locally p-cocartesian lifts, the property that u is a locally p-cocartesian lift of v is equivalent to the property that it is a p-cocartesian morphism in C, which in turn means that c_u is an isomorphism in C. In other words, every (locally) p-cocartesian morphism is isomorphic in $\mathrm{Fun}([1], C)$ to a morphism of the form ε_{x_0} for some object x_0 of C and some morphism v in S as above.

4.9 Exercises Chapter 4

Exercise 4.9.1. Spell out the missing details in the proof of Proposition 4.4.3.

Exercise 4.9.2. Spell out the missing details in the proof of Proposition 4.4.5.

5 Synthetic categories and contexts

Throughout the preceding chapters, an object of a category C has meant a functor $x\colon A \to C$ from an arbitrary anima A, called the *context* of the object. General anima contexts are essential for allowing interpretations of synthetic categories such as categories internal to a topos $\mathcal{B}$. But in this generality, it is not immediately clear how to formulate and work with constructions involving objects.

For example, given a functor $p\colon E \to C$ and an absolute object $x\colon * \to C$, the fiber E_x is the pullback $E \times_C *$. For an arbitrary object $x\colon A \to C$, the corresponding construction is a category over A:

$$\begin{array}{ccc} A \times_C E & \longrightarrow & E \\ {\scriptstyle p|_A}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ A & \xrightarrow{x} & C. \end{array}$$

Questions about the fiber E_x translate into properties of the functor $p|_A$. For example, asking the fiber E_x to be contractible translates to asking $p|_A$ to be an equivalence. Asking it to admit a terminal object translates to asking $p|_A$ to admit a right adjoint section.

In this chapter, we introduce the language of *categories in context* A, and use them to express such objectwise constructions and properties in the usual categorical language. Semantically, categories in context A correspond to categories over A. But by making them into primitive notions, we can require them to satisfy the same constructions and axioms as the absolute theory, allowing all results established so far to carry over. Every category C give rise to a category $W_A C$ in context A, called its *weakening*, which semantically corresponds to the constant family $\mathrm{pr}_2\colon C \times A \to A$. Functors $x\colon A \to C$ can then be reinterpreted as absolute objects of $W_A C$ in context A, allowing us to make statements about arbitrary objects using the language and notations of absolute objects.

We introduce categories in anima contexts and weakening in Section 5.1. In Section 5.2, we introduce dependent products and sums, which describe the category of sections and the total category of a family. We use these constructions in Section 5.3 to relate categories in context A to categories over A. Finally, Section 5.4 discusses substitution along maps of animae and the conventions for leaving anima contexts implicit.

5.1 Categories in anima contexts

Axiom K.1 (Categories in context)**.** For every anima A, there are categories, animae, functors, and identifications in context A. These have the same primitives, operations, and specified coherences as the absolute theory.

We use the phrase *absolute context* for the theory considered so far. A category in context A may be denoted by B, or by $B(a)$ if we wish to display its dependence on $a : A$. Similarly, we write $f(a)\colon B(a) \to C(a)$ for a functor in context A. When it is useful to display the context separately, we write

$$a : A \vdash B(a) \text{ category}, \qquad a : A \vdash f(a)\colon B(a) \to C(a).$$

We denote the terminal category in this context by 1_A. The notations $\mathrm{Map}_A(B, C)$ and $\mathrm{Fun}_A(B, C)$ refer to the mapping anima and functor category formed in context A.

Axiom K.2 (Axioms in context)**.** Every axiom of the theory is available in an arbitrary anima context. This includes the rules of the present chapter and all future chapters.

Convention 5.1.1 (Iterated contexts)**.** Since Axiom K.2 in particular applies to Axiom K.1, we may form iterated contexts: given an anima A in context Γ, we may work in the further context $\Gamma, a : A$. Thus a general context Γ may take the form of a finite string $(a_1 : A_1, a_2 : A_2(a_1), \ldots, a_n : A_n(a_1, \ldots, a_{n-1}))$. When such context Γ is fixed, we often leave it implicit in the notation; in this case the phrase "absolute context" refers to Γ.

In light of Axiom K.2, an argument using the axioms introduced so far can be repeated in any anima context. For example, categories $B(a)$ and $C(a)$ have a product $B(a) \times C(a)$ in context A, with the projections and universal property of Axiom B.2. Similarly, the commutative-square, Segal, and Rezk axioms are available in each such context.

Axiom K.3 (Weakening)**.** For every anima A, there is a *weakening operation* W_A governed by the following rules. For an absolute category C, we obtain a category $W_A C$ in context A, which is an anima whenever C is. For an absolute functor $f\colon C \to D$, we obtain a functor

$$W_A f\colon W_A C \longrightarrow W_A D.$$

For parallel absolute functors f, g, there is an equivalence in context A of the form

$$\iota_{f,g}\colon W_A(f = g) \xrightarrow{\sim} (W_A f = W_A g)_A.$$

For every primitive coherence in the theory, we require identifications relating the weakening of the absolute form of this coherence to its contextual form applied to the weakened inputs; see the explanation below. For each category constructor, the 'canonical' comparison functor in context A is an equivalence.

Convention 5.1.2 (Objects in context)**.** When working in a fixed context, an object of a category C means an absolute object in that context, that is, a functor from its terminal category to C. This also allows us to work with objects $x\colon A \to C$ in the earlier sense: after passing to context $a : A$, such a functor corresponds to an absolute object of $W_A C$, as we will prove in Corollary 5.3.6.

Convention 5.1.3 (Omitting weakening)**.** In context A, we usually write C and f for the weakenings $W_A C$ and $W_A f$. Constructions involving these inputs are formed in context A; their compatibility with weakening is understood as in Section 5.1.1. We write W_A explicitly when the distinction matters.

5.1.1 Preservation of specified structure

The weakening axiom, Axiom K.3, requires that the weakening operation is compatible with all postulated constructions of categories. For example, the weakening of a product is equivalent to the product of the weakenings, compatibly with the projections. It also requires that weakening preserves the specified identifications of the theory in the appropriate sense. The rest of this subsection spells out these requirements in more detail. On a first reading, one may continue with Section 5.2 and return here when a proof uses a particular compatibility.

The compatibilities we need will show up again in the discussion of substitution in Section 5.4. We will therefore explain them in the following abstract setting: We let S and T be two copies of the theory of synthetic categories, and describe the structure and rules determining a 'change of theory' W from S to T. Weakening and substitution will then be instances of such change of theory.

Given a category C of S, W determines a specified category WC of T, which is an anima whenever C is. Similarly, a functor $f\colon C \to D$ in S determines a functor $Wf\colon WC \to WD$ of T. We require specified identifications

$$W(\mathrm{id}_C) = \mathrm{id}_{WC}, \qquad W(gf) = (Wg)(Wf).$$

For parallel functors $f, g\colon C \to D$ in S, we require an equivalence $W(f =_S g) \xrightarrow{\sim} Wf =_T Wg$; in particular, every identification $\alpha\colon f = g$ in S produces an identification $W\alpha\colon Wf = Wg$ in T.

Consider now an expression $X(F)$ in the abstract theory which depends on a tuple of input variables $F = (f_1, \dots, f_n)$. We write $X_S(F)$ for its interpretation in theory S and $X_T(WF)$ for its interpretation in theory T, applied to the tuple $WF = (Wf_1, \dots, Wf_n)$. If X is a functor expression formed from variables using identities and composition, we recursively obtain an identification

$$X_W(F)\colon W(X_S(F)) = X_T(WF),$$

as follows. If $X(F) = f_i$ is a variable, we take this to be the identity identification. If $X(f)$ is an identity or a composite, we take it to be the given specified identification. For an expression of the form $Y \circ X$, it is defined as the composite identification

$$W(Y_S X_S) = (WY_S)(WX_S) \xrightarrow{Y_W * X_W} Y_T X_T.$$

Next, consider identifications $\alpha\colon X = Y$ between two such expressions. Again this admits an interpretation α_S in theory S, and an interpretation α_T in theory T. In order to compare the two

instantiations of this coherence, we must first 'normalize' α_S to give it the correct source and target: we define

$$N^1_W(\alpha_S) \quad := \quad Y_W \cdot \big(W(\alpha_S) \cdot X_W^{-1}\big),$$

which is an identification of the form $N^1_W(\alpha_S)\colon X_T = Y_T$. Here we write $\beta \cdot \alpha$ for the vertical composite of α followed by β. Now, for each primitive coherence α of the abstract theory, we require W to come equipped with an identification

$$\alpha_W\colon N^1_W(\alpha_S) = \alpha_T.$$

If α is a more complicated identification expression formed by composition, inversion, or whiskering, we obtain an identification $\alpha_W\colon N^1_W(\alpha_S) = \alpha_T$ by recursion on its structure in terms of those primitive expressions.

Finally, we consider identifications of the form $p\colon \alpha = \beta$, with α and β both expressions in the abstract theory for identifications $X = Y$. Generalizing the normalization procedure for α and β results in an identification

$$N^1_W(p_S)\colon N^1_W(\alpha_S) = N^1_W(\beta_S).$$

By combining this with the already constructed identifications for α and β, this in turn yields a 'second-order normalization' of the form

$$N^2_W(p_S) \quad := \quad \beta_W \cdot \big(N^1_W(p_S) \cdot \alpha_W^{-1}\big),$$

which is now an identification of the form $N^2_W(p_S)\colon \alpha_T = \beta_T$. We then require W to come equipped with identifications

$$p_W\colon N^2_W(p_S) = p_T$$

for each primitive coherence $p\colon \alpha = \beta$. The same rule applies at each further identification level present in the vocabulary: first compare the boundaries, and then compare the transported witness with the specified target witness.

Example 5.1.4 (The pentagon)**.** Consider four composable functors f, g, h and k, as in Axiom A.8. The two paths around the pentagon determine identification expressions

$$\alpha \quad := \quad \alpha_{g\circ f,h,k} \cdot \alpha_{f,g,k\circ h}, \qquad \text{and} \qquad \beta \quad := \quad (\mathrm{id}_k \star \alpha_{f,g,h}) \cdot \alpha_{f,h\circ g,k} \cdot (\alpha_{g,h,k} \star \mathrm{id}_f),$$

providing identifications between $((k \circ h) \circ g) \circ f$ and $k \circ (h \circ (g \circ f))$. If we instantiate these four functors in theory S, there are now four ways to obtain an identification $((Wk \circ Wh) \circ Wg) \circ Wf = Wk \circ (Wh \circ (Wg \circ Wf))$ in theory T:

α_T Instantiate α directly in theory T, applied to the tuple $WF = (Wf, Wg, Wh, Wk)$;

β_T Instantiate β directly in theory T, applied to WF;

$N^1_W(\alpha_S)$ Instantiate α in theory S, apply W, and compose with the previously constructed identifications of its source and target

$$W(((k \circ h) \circ g) \circ f) = ((Wk \circ Wh) \circ Wg) \circ Wf$$
$$W(k \circ (h \circ (g \circ f))) = Wk \circ (Wh \circ (Wg \circ Wf)).$$

$N^1_W(\beta_S)$ Instantiate β in theory S, apply W, and compose with the same two identifications.

Theories S and T provide pentagon identifications $p_S\colon \alpha_S = \beta_S$ and $p_T\colon \alpha_T = \beta_T$. The coherence data $p_W\colon N^2_W(p_S) = p_T$ that W needs to provide can then be interpreted as requiring the following square to commute:

$$\begin{array}{ccc} \alpha_T & \xrightarrow{\alpha_W^{-1}} & N^1_W(\alpha_S) \\ {\scriptstyle p_T}\downarrow & & \downarrow{\scriptstyle N^1_W(p_S)} \\ \beta_T & \xleftarrow[\beta_W]{} & N^1_W(\beta_S). \end{array}$$

Let us now make precise the compatibility requirements of W with category constructors. In each case, we require a certain 'canonical' comparison functor to be an equivalence. For instance, the universal property of the terminal category 1_T in theory T provides a functor $W1_S \to 1_T$, which we require to be an equivalence. Similarly, the projections induce functors $W\,\mathrm{pr}_1\colon W(C \times D) \to WC$ and $W\,\mathrm{pr}_2\colon W(C \times D) \to WD$, and we require the induced functor

$$W(C \times D) \xrightarrow{\sim} WC \times WD$$

to be an equivalence.

The general procedure is as follows. Each category constructor X comes equipped with specified structure. We may instantiate this structure in theories S and T, giving categories X_S and X_T. The structure in S may be transported to T through W. The universal property of X_T then provides a functor of the form $W(X_S) \to X_T$ or $X_T \to W(X_S)$, depending on X, compatibly with the structure morphisms on both sides. We require this resulting functor to be an equivalence.

Remark 5.1.5. Some of the primitive coherences are functors on a product of identification animae. In order to express the compatibility of W with these coherences, we must combine the above normalization procedure with the equivalences $W(C \times D) \xrightarrow{\sim} WC \times WD$ and $W(f = g) \xrightarrow{\sim} Wf = Wg$.

If we have three instantiations S, T and U of the abstract theory, and two changes of theory $W\colon S \to T$ and $V\colon T \to U$, we may combine them into another change of theory $VW\colon S \to U$ by first using the compatibility for W and then the compatibility for V:

$$VW(X_S) \xrightarrow{V_*(X_W)} V(X_T(WF)) \xrightarrow{X_V} X_U(VWF).$$

5.2 Dependent products and sums

Fix an anima A. We introduce two constructions that assign an absolute category to every category in context A: the dependent product and the dependent sum. We start with the former.

Axiom K.4 (Dependent product)**.** For a category B in context A, there is an absolute category $\Pi_A B$, equipped with an evaluation functor

$$\varepsilon_B\colon W_A(\Pi_A B) \longrightarrow B$$

in context A. For an absolute functor $g\colon T \to \Pi_A B$, define $g^u := \varepsilon_B \circ W_A g$. We require:

(1) For every absolute category T and functor $f\colon W_A T \to B$ in context A, there is an absolute functor $f_c\colon T \to \Pi_A B$, equipped with an identification $f = (f_c)^u$ in context A.

(2) If B is an anima in context A, then $\Pi_A B$ is an anima.

(3) For any $h, k\colon T \to \Pi_A B$, the functor

$$(h = k) \longrightarrow \Pi_A(h^u = k^u)_A$$

is an equivalence. This functor is obtained by currying the functor $W_A(h = k) \to (h^u = k^u)_A$ induced by weakening and postwhiskering with evaluation.

We call $\Pi_A B$ the *dependent product*, or *category of sections*, of B over A. We also write $\prod_{a:A} B(a)$. Its absolute objects correspond to functors $1_A \to B$ in context A, using the equivalence $W_A(*) \simeq 1_A$. The second clause guarantees that both sides in the last clause are animae in the absolute context.

Together, the first and third clause specify a currying/uncurrying correspondence between functors $W_A T \to B$ in context A and absolute functors $T \to \Pi_A B$. Semantically, we think of Π_A as a right adjoint to the weakening operation W_A.

Construction 5.2.1 (Functoriality of dependent products)**.** A functor $u\colon B \to C$ in context A gives a functor

$$\Pi_A u\colon \Pi_A B \to \Pi_A C$$

by currying the composite $W_A(\Pi_A B) \xrightarrow{\varepsilon_B} B \xrightarrow{u} C$. By construction, there is a commutative square

$$\begin{array}{ccc} W_A(\Pi_A B) & \xrightarrow{W_A(\Pi_A u)} & W_A(\Pi_A C) \\ {\scriptstyle \varepsilon_B}\downarrow & & \downarrow{\scriptstyle \varepsilon_C} \\ B & \xrightarrow{\quad u \quad} & C. \end{array}$$

The uniqueness clause of currying shows that this construction is compatible with identities and composites. Moreover, identifications $\alpha\colon f = g$ between functors in context A induce identifications $\Pi_A \alpha\colon \Pi_A f = \Pi_A g$ between their dependent products.

Lemma 5.2.2. *Dependent products preserve terminal categories, products, and pullbacks.*

Proof. Consider $B \to D \leftarrow C$ in context A. Using the projections, we obtain a functor

$$\Pi_A(B \times_D C) \longrightarrow \Pi_A B \times_{\Pi_A D} \Pi_A C.$$

To construct an inverse, write $P := \Pi_A B \times_{\Pi_A D} \Pi_A C$. We first construct a functor $W_A P \to B \times_D C$. The projections of P induce functors $W_A P \to W_A \Pi_A B$ and $W_A P \to W_A \Pi_A C$, and composing them with the evaluations give functors $W_A P \to B$ and $W_A P \to C$. Their composites to D are identified by the defining identification of P and the compatibility identifications of evaluation. This results in a functor $W_A P \to B \times_D C$, which in turn curries to a functor $P \to \Pi_A(B \times_D C)$. The fact that this

is an inverse follows similarly by uncurrying and using the respective pullback properties; we leave the details to the interested reader. The proofs for products and the terminal category are the same, with the corresponding universal properties. □

We now introduce the dependent sum, or 'total category', of a contextual category. Its universal property is dual to that of the dependent product.

Axiom K.5 (Dependent sum)**.** For a category B in context A, there is an absolute category $\Sigma_A B$, equipped with a *pair functor*

$$\eta_B \colon B \longrightarrow W_A(\Sigma_A B)$$

in context A. For an absolute functor $k \colon \Sigma_A B \to D$, define $k^\flat := W_A k \circ \eta_B$.

(1) For every absolute category D and functor $f \colon B \to W_A D$ in context A, there is an absolute functor $\bar{f} \colon \Sigma_A B \to D$, equipped with an identification $f = (\bar{f})^\flat$ in context A.

(2) For functors $k, h \colon \Sigma_A B \to D$, the functor

$$(k = h) \longrightarrow \Pi_A(k^\flat = h^\flat)_A$$

is an equivalence. It is obtained by currying the functor on identification animae induced by weakening and prewhiskering with η_B.

We also write $\sum_{a:A} B(a)$ for $\Sigma_A B$ and $(a, -)$ for η_B. The two universal properties express that Σ_A is left adjoint to weakening.

Construction 5.2.3 (Functoriality of dependent sums)**.** For a functor $u \colon B \to C$ in context A, applying the sum axiom to $\eta_C \circ u$ gives a functor $\Sigma_A u \colon \Sigma_A B \to \Sigma_A C$ and a commutative square

$$\begin{array}{ccc} B & \xrightarrow{\eta_B} & W_A(\Sigma_A B) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle W_A(\Sigma_A u)} \\ C & \xrightarrow{\eta_C} & W_A(\Sigma_A C). \end{array} \tag{5.1}$$

The functors $\Sigma_A u$ are compatible with identities, composites, and identifications. For an absolute category D, extending $\mathrm{id}_{W_A D}$ gives a functor

$$d_D \colon \Sigma_A W_A D \longrightarrow D$$

equipped with an identification $W_A d_D \circ \eta_{W_A D} = \mathrm{id}_{W_A D}$.

Proposition 5.2.4 (Mapping animae for dependent products and sums)**.** *For absolute categories T, D and a category B in context A, evaluation and restriction along the pair functor induce equivalences of absolute animae*

$$\mathrm{Map}(T, \Pi_A B) \simeq \Pi_A \,\mathrm{Map}_A(W_A T, B), \qquad \mathrm{Map}(\Sigma_A B, D) \simeq \Pi_A \,\mathrm{Map}_A(B, W_A D).$$

Proof. For the first map, weaken the evaluation functor $\mathrm{Map}(T, \Pi_A B) \times T \to \Pi_A B$ and compose with ε_B to obtain a functor $W_A \,\mathrm{Map}(T, \Pi_A B) \times W_A T \to B$. Currying first in the variable $W_A T$ and then in the A-variable produces the functor. Conversely, if we denote $Q := \Pi_A \,\mathrm{Map}_A(W_A T, B)$, then evaluation provides a functor $W_A(Q \times T) \simeq W_A Q \times W_A T \to B$. Currying along A gives $Q \times T \to \Pi_A B$, which in turn curries to $Q \to \mathrm{Map}(T, \Pi_A B)$.

For the second map, we weaken the evaluation functor $\mathrm{Map}(\Sigma_A B, D) \times \Sigma_A B \to D$ and restrict its second input along η_B to obtain $W_A \,\mathrm{Map}(\Sigma_A B, D) \times B \to W_A D$. Currying in B and then along A produces the functor. For its inverse, write $Q := \Pi_A \,\mathrm{Map}_A(B, W_A D)$. Evaluation provides a functor $W_A Q \times B \longrightarrow W_A D$. Currying in $W_A Q$ and using preservation of functor categories gives

$$B \longrightarrow \mathrm{Fun}_A(W_A Q, W_A D) \simeq W_A \,\mathrm{Fun}(Q, D),$$

and the axiom for dependent sums extends this to $\Sigma_A B \to \mathrm{Fun}(Q, D)$. Uncurrying gives $\Sigma_A B \times Q \to D$, and then currying in $\Sigma_A B$, with anima parameter Q, gives $Q \to \mathrm{Map}(\Sigma_A B, D)$.

In each case, the computation identifications for these successive currying operations identify the two composites with the identity after evaluation, which suffices to obtain the identifications at the level of the displayed mapping animae. □

Proposition 5.2.5. *For a category B in context A and an absolute category C, the functor induced by taking the sum of the first projection and extending the second projection is an equivalence*

$$\Sigma_A(B \times W_A C) \xrightarrow{\sim} \Sigma_A B \times C.$$

Proof. For every absolute category D, the preceding proposition, currying, and preservation of functor categories give a chain of equivalences

$$\begin{aligned}
\mathrm{Map}(\Sigma_A(B \times W_A C), D) &\simeq \Pi_A \,\mathrm{Map}_A(B \times W_A C, W_A D) \\
&\simeq \Pi_A \,\mathrm{Map}_A(B, \mathrm{Fun}_A(W_A C, W_A D)) \\
&\simeq \Pi_A \,\mathrm{Map}_A(B, W_A \,\mathrm{Fun}(C, D)) \\
&\simeq \mathrm{Map}(\Sigma_A B, \mathrm{Fun}(C, D)) \\
&\simeq \mathrm{Map}(\Sigma_A B \times C, D).
\end{aligned}$$

The composite equivalence can be identified with precomposition with the functor in the proposition statement. We may now obtain an inverse to the given functor by taking D to be its source and its target, respectively. □

Remark 5.2.6 (Compatibility with change of theory)**.** Let S and T be two instances of the abstract theory, and let $W\colon S \to T$ be a change of theory, as detailed in Section 5.1.1. Considering an anima A in theory S, we then obtain a theory S^A of categories of S in context A, and a theory T^{WA} of categories of T in context WA.

We assume that W comes equipped with a change of theory $W^{[A]}$ from S^A to T^{WA}. This change of theory is compatible with weakening, in that we have $W^{[A]}(W_A C) \xrightarrow{\sim} W_{WA}(WC)$ for every category C in S, compatibly with functors and specified identifications. Moreover, we require compatibility with dependent products and sums: the canonical functors

$$W(\Pi_A B) \to \Pi_{WA}(W^{[A]} B), \qquad \Sigma_{WA}(W^{[A]} B) \to W(\Sigma_A B)$$

are required to be equivalences, respecting the evaluation and pair functors and their specified identifications. Since $W^{[A]}$ is itself a change of theory, the same requirements apply in further anima contexts.

5.3 Categories over an anima

The dependent sum of a category in context A should come with a projection to A, and the original category should be its fiber at the generic point of A. We now impose axioms that express these properties. They will allow us to pass between categories in context A and categories over A, together with their functors and identifications.

Axiom K.6 (The sum of the terminal category)**.** For every anima A, there is a specified equivalence

$$e_A \colon \Sigma_A 1_A \xrightarrow{\sim} A.$$

Construction 5.3.1 (Projection and generic point)**.** Let B be a category in context A, and write $p_B \colon B \to 1_A$ for its functor to the terminal category. We call $\Sigma_A B$ its *total category*. Its *first projection* is the composite

$$\mathrm{pr}_1^B \colon \Sigma_A B \xrightarrow{\Sigma_A p_B} \Sigma_A 1_A \xrightarrow{e_A} A.$$

We usually omit the superscript B when its source is clear.

To recover a family from its total category, we need to take the fiber at the variable $a : A$. The following object of $W_A A$ represents this variable and is called the *generic point* of A:

$$a_A \colon 1_A \xrightarrow{\eta_{1_A}} W_A(\Sigma_A 1_A) \xrightarrow{W_A e_A} W_A A.$$

Applying (5.1) to p_B gives an identification

$$W_A(\mathrm{pr}_1^B) \circ \eta_B = a_A \circ p_B.$$

In what follows, we use categories and functors over A in the sense of Definitions 1.9.1 and 1.9.3. In particular, a functor over A includes an identification between the two composites to A. We write $\mathrm{Map}_{/A}(X, Y)$ for their absolute mapping anima, as in Definition 1.9.2.

Construction 5.3.2 (Generic fiber)**.** Let $p \colon X \to A$ be an absolute functor. Its *generic fiber*, denoted $F_A(p)$, is the fiber of $W_A p$ at a_A:

$$\begin{array}{ccc} F_A(p) & \xrightarrow{j_p} & W_A X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle W_A p} \\ 1_A & \xrightarrow{a_A} & W_A A. \end{array}$$

A functor over A induces a functor between generic fibers by the universal property of this pullback. The identification to A supplies the commutative square needed for this construction. Identifications between functors over A likewise induce identifications between the functors on generic fibers.

We can now relate a category to the generic fiber of its total category. For a category B in context A, the identity $W_A(\mathrm{pr}_1^B)\eta_B = a_A p_B$ induces a functor

$$r_B \colon B \longrightarrow F_A(\mathrm{pr}_1^B).$$

In the other direction, the dependent-sum axiom extends $j_p\colon F_A(p) \to W_A X$ to an absolute functor

$$s_p\colon \Sigma_A F_A(p) \longrightarrow X.$$

This functor lies over A. Indeed, restricting $p \circ s_p$ along the pair functor gives $W_A p \circ j_p = a_A \circ p_{F_A(p)}$, which is also the restriction of $\mathrm{pr}_1^{F_A(p)}$. The uniqueness clause for dependent sums therefore gives an identification $p \circ s_p = \mathrm{pr}_1^{F_A(p)}$.

Axiom K.7 (Recovery from the total category and the generic fiber)**.** For every category B in context A, the functor r_B is an equivalence in context A. For every absolute functor $p\colon X \to A$, the functor s_p is an equivalence over A.

In particular, the square

$$\begin{array}{ccc} B & \xrightarrow{\eta_B} & W_A(\Sigma_A B) \\ {\scriptstyle p_B}\downarrow & \lrcorner & \downarrow{\scriptstyle W_A(\mathrm{pr}_1^B)} \\ 1_A & \xrightarrow{a_A} & W_A A \end{array}$$

is a pullback. In other words: $B(a)$ is the fiber of its total category over the generic point $a : A$. We next show that a functor between families is the same as a functor between their total categories over A. To prove this, we first establish a relative form of the mapping-anima description of dependent sums.

Lemma 5.3.3 (Relative mapping animae for dependent sums)**.** *Let B be a category in context A, and let $p\colon \Sigma_A B \to Z$ and $q\colon Y \to Z$ be absolute functors. Regard B as a category over $W_A Z$ by the functor $W_A p \circ \eta_B$. Then restriction along η_B induces an equivalence*

$$\mathrm{Map}_{/Z}((\Sigma_A B, p), (Y, q)) \xrightarrow{\sim} \Pi_A \mathrm{Map}_{/W_A Z}((B, W_A p \circ \eta_B), (W_A Y, W_A q)),$$

where the relative mapping anima on the right is formed in context A.

Proof. The restriction equivalences of Proposition 5.2.4 fit into the following square, whose vertical functors are induced by postcomposition with q:

$$\begin{array}{ccc} \mathrm{Map}(\Sigma_A B, Y) & \longrightarrow & \Pi_A \mathrm{Map}_A(B, W_A Y) \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle \Pi_A(W_A q)_*} \\ \mathrm{Map}(\Sigma_A B, Z) & \longrightarrow & \Pi_A \mathrm{Map}_A(B, W_A Z). \end{array}$$

The square commutes by currying the identification

$$W_A(qh) \circ \eta_B = (W_A q \circ W_A h) \circ \eta_B = W_A q \circ (W_A h \circ \eta_B)$$

on the whole mapping anima of functors $h\colon \Sigma_A B \to Y$. The lower horizontal functor sends p to the section represented by $W_A p \circ \eta_B$. Taking the fibers of the vertical functors at these points therefore

gives an equivalence. The source is the displayed relative mapping anima. Since Π_A preserves pullbacks and terminal categories, the target is

$$\Pi_A \operatorname{Map}_{/W_A Z}((B, W_A p \circ \eta_B), (W_A Y, W_A q)).$$

More explicitly, a point $(h, \alpha\colon qh = p)$ in the source is sent to the functor $W_A h \circ \eta_B$ equipped with the identification

$$W_A q \circ (W_A h \circ \eta_B) = W_A(qh) \circ \eta_B \xrightarrow{(W_A \alpha)\star \eta_B} W_A p \circ \eta_B.$$

Since we obtained this map by taking fibers, it also sends identifications between functors to identifications compatible with these triangles. Its inverse recovers both the functor and its commutative triangle. □

Proposition 5.3.4. *For categories B, C in context A, dependent sum induces an equivalence of absolute animae*

$$\Pi_A \operatorname{Map}_A(B, C) \xrightarrow{\sim} \operatorname{Map}_{/A}(\Sigma_A B, \Sigma_A C).$$

Its inverse is induced by taking generic fibers and using r_B and r_C.

Proof. We apply Lemma 5.3.3 with $Z = A$, $Y = \Sigma_A C$, $p = \mathrm{pr}_1^B$, and $q = \mathrm{pr}_1^C$. Since $W_A(\mathrm{pr}_1^B) \circ \eta_B = a_A \circ p_B$, the relative mapping anima in context A is

$$\operatorname{Map}_A(B, F_A(\mathrm{pr}_1^C)).$$

Indeed, a functor to this generic fiber consists of a functor to $W_A \Sigma_A C$ and an identification of its composite to $W_A A$ with $a_A p_B$. The equivalence $r_C\colon C \xrightarrow{\sim} F_A(\mathrm{pr}_1^C)$ identifies this anima with $\operatorname{Map}_A(B, C)$.

For a functor $u\colon B \to C$ in context A, the defining square for $\Sigma_A u$ identifies its restriction with $\eta_C u$. Composing this square with $W_A(\mathrm{pr}_1^C)$ and using the identities for the first projections gives the triangle over $W_A A$. By the uniqueness clause for restriction, it determines the required identification

$$\mathrm{pr}_1^C \circ \Sigma_A u = \mathrm{pr}_1^B.$$

Thus the functor over A obtained above is $\Sigma_A u$ together with its natural identification to the base. In the other direction, we take generic fibers and use r_B and r_C. □

In light of the previous result, we may freely pass back and forth between categories in context A and categories over A, including their functors and identifications. We first spell out what this means for constant families and objects.

Corollary 5.3.5. *For every absolute category C, there is an equivalence over A*

$$(\mathrm{pr}_1^{W_A C}, d_C)\colon \Sigma_A W_A C \xrightarrow{\sim} A \times C.$$

Proof. Apply Proposition 5.2.5 with $B = 1_A$ and use e_A. The resulting components are the first projection and d_C. □

Corollary 5.3.6. *Let B be a category in context A. Objects of B in that context, or equivalently absolute objects of $\Pi_A B$, correspond to sections of $\mathrm{pr}_1^B\colon \Sigma_A B \to A$, compatibly with identifications.*

Proof. Apply Proposition 5.3.4 to 1_A and B, and use e_A. □

Corollary 5.3.7. *For an absolute category C, objects of $W_A C$ in context A correspond to absolute functors $A \to C$.*

Proof. Apply Corollary 5.3.5. □

Remark 5.3.8. For $A = *$, we obtain equivalences $\Sigma_* W_* C \simeq C$ for an absolute category C, and $B \simeq W_* \Sigma_* B$ for a category B in context $*$. The first follows from Corollary 5.3.5, while the second follows from the generic-fiber axiom, since $W_*(*)$ is terminal. Together with their action on functors and identifications, these equivalences allow us to pass between the absolute theory and the theory in terminal anima context.

The correspondence also describes diagrams in a family: these are diagrams in its total category equipped with an identification of their projection to A with a constant diagram. For a fixed absolute shape, this is expressed by the following pullback.

Lemma 5.3.9. *Let C be a category in context A, and let K be an absolute category. There is an equivalence over A*

$$\Sigma_A \operatorname{Fun}_A(W_A K, C) \simeq A \times_{\operatorname{Fun}(K,A)} \operatorname{Fun}(K, \Sigma_A C),$$

where $A \to \operatorname{Fun}(K, A)$ assigns constant diagrams. It is compatible with restriction along functors between the shapes K.

Proof. Take the generic fiber of the category on the right. Preservation of functor categories and pullbacks by weakening identifies it with the pullback

$$1_A \times_{\operatorname{Fun}_A(W_A K, W_A A)} \operatorname{Fun}_A(W_A K, W_A \Sigma_A C).$$

Here the map from 1_A selects the constant diagram at a_A. Since $\operatorname{Fun}_A(W_A K, -)$ preserves pullbacks, this is equivalent to

$$\operatorname{Fun}_A(W_A K, 1_A \times_{W_A A} W_A \Sigma_A C) \simeq \operatorname{Fun}_A(W_A K, C).$$

By Axiom K.7, taking total categories now gives the required equivalence over A. For a functor $L \to K$, restriction preserves constant diagrams and commutes with the projections in the displayed pullbacks, and so the resulting equivalence respects restriction. □

To apply this correspondence to adjunctions, we also need compatibility with precomposition and postcomposition. We check this using the description of functors between total categories proved above.

Lemma 5.3.10. *The correspondence between functors in context A and functors over A respects precomposition and postcomposition, including the identifications to the base. Moreover, applying these two operations in either order results in identified functors, compatibly with the correspondence.*

Proof. We work with the relative mapping animae in the proof of Proposition 5.3.4. For functors $u\colon B' \to B$ and $h\colon \Sigma_A B \to Y$, compatibility with precomposition is expressed by the identification

$$W_A(h \circ \Sigma_A u) \circ \eta_{B'} = W_A h \circ W_A(\Sigma_A u) \circ \eta_{B'} = W_A h \circ \eta_B \circ u.$$

The second step uses the defining square for $\Sigma_A u$. Compatibility with postcomposition by a functor $q\colon Y \to Z$ is expressed by an identification between two pastings of this square. Using the abbreviations $v = \Sigma_A u$, $\eta' = \eta_{B'}$, and $\eta = \eta_B$, we need to compare the composite identification

$$W_A(q(hv))\eta' = W_A q\big(W_A(hv)\eta'\big) = W_A q\big(W_A h(W_A v \eta')\big) = W_A q\big(W_A h(\eta u)\big)$$

with the composite identification

$$W_A(q(hv))\eta' = W_A((qh)v)\eta' = W_A(qh)(W_A v\eta') = W_A(qh)(\eta u) = W_A q\big(W_A h(\eta u)\big).$$

This is done by combining the various identifications specified in Section 5.1.1. For example, the fact that W_A preserves the associator identifies the two ways of expanding $W_A(q(hv))$. Naturality of the identification $W_A(hv) = (W_A h)(W_A v)$, together with whiskering applied to $W_A v\eta' = \eta u$, identifies the subsequent steps. Finally, pasting these identifications with the image of $qh = p$ under weakening shows that the two calculations agree over the base.

All these constructions are made on mapping animae. Evaluating on a parameterized family therefore gives the same identities for families of functors and for identifications between them. □

We can now describe which sections of a total category correspond to terminal objects in the family.

Proposition 5.3.11 (Terminal objects and adjoint sections)**.** *Let C be a category in context A, and let $t\colon 1_A \to C$ be an object in that context. Let $s\colon A \to \Sigma_A C$ be the corresponding section of pr_1^C, with identification $\varepsilon\colon \mathrm{pr}_1^C\, s = \mathrm{id}_A$. The following conditions are equivalent:*

(1) The object t is terminal in C in context A.

(2) In context A, the functor t is a right adjoint section of $p_C\colon C \to 1_A$.

(3) The functor s is a right adjoint section of pr_1^C, with counit ε.

The dual statements hold for initial objects and left adjoint sections.

Proof. The equivalence of (1) and (2) is the dual of Lemma 4.4.19, applied in context A.

To compare (2) and (3), we also need the correspondence for natural transformations. For categories B, D in context A, apply Proposition 5.3.4 to $B \times W_A[1]$ and D. The Frobenius equivalence

$$\Sigma_A(B \times W_A[1]) \simeq \Sigma_A B \times [1]$$

identifies a natural transformation in context A with a natural transformation of the corresponding total functors over A. By Lemma 5.3.9, this correspondence respects restriction of diagrams, and thus in particular sources, targets, identities, and composition. The required compatibility with precomposition and postcomposition is supplied by Lemma 5.3.10. It therefore also transports units, counits, and their triangle identities.

Write $\mathrm{pr}_1 = \mathrm{pr}_1^C$ for the rest of the proof. A unit $\mathrm{id}_C \to t p_C$ in context A corresponds to a transformation

$$\eta\colon \mathrm{id}_{\Sigma_A C} \longrightarrow s\,\mathrm{pr}_1$$

over A. This being a transformation over A means that $(\varepsilon\,\mathrm{pr}_1)(\mathrm{pr}_1\,\eta) = \mathrm{id}_{\mathrm{pr}_1}$, which is the triangle identity at pr_1. The triangle identity at t corresponds to an identification $(s\varepsilon)(\eta s) = \mathrm{id}_s$. We conclude

that the adjunction in context A is precisely an adjunction $\mathrm{pr}_1 \dashv s$ with counit ε. Conversely, that counit and the first triangle identity turn its unit into a transformation over A, so the same correspondence recovers the adjunction in context A. □

By the proposition, a terminal object $t(a)$ of a family of categories $C(a)$ over A is encoded by a right adjoint section of $\mathrm{pr}_1^C : \Sigma_A C \to A$. Similarly, C is contractible in context A if and only if pr_1^C is an equivalence. To see this, apply the correspondence to the terminal functor $p_C : C \to 1_A$: its total functor is $\Sigma_A p_C$, and this becomes pr_1^C under the equivalence $e_A : \Sigma_A 1_A \xrightarrow{\sim} A$.

Example 5.3.12 (Fibers)**.** The preceding results allow us to speak of the fiber at an arbitrary object just as we would at an absolute object. Let us spell out how statements about this fiber translate into statements about categories over its anima of parameters.

Let $f : C \to D$ be a functor and $x : A \to D$ an object. In context $a : A$, we may regard x as an absolute object of D, omitting weakening from the notation, and form the fiber C_x. Its total category is equivalent over A to the pullback

$$A \times_D C \longrightarrow A.$$

Indeed, preservation of pullbacks by weakening identifies the generic fiber of this projection with C_x, so this follows from Axiom K.7.

By Corollary 5.3.6, an object of C_x in context A corresponds to a section of this projection. The terminal-object correspondence of Proposition 5.3.11 and the contractibility criterion following it give:

(1) The fiber C_x is contractible in context A if and only if $A \times_D C \to A$ is an equivalence in the original context.

(2) An object of C_x is terminal if and only if its corresponding section is a right adjoint section, with the section identification as counit.

Thus, when we write "let x be an object of D and form the fiber C_x", we work in the context of x. The translations above express these fiberwise assertions in the original context, where x is a functor $A \to D$.

We next check how taking total categories interacts with pullbacks and with the condition of being an anima.

Proposition 5.3.13 (Pullbacks of total categories)**.** *Dependent sums preserve pullbacks. Moreover, every square* (5.1) *is a pullback.*

Proof. Pullbacks of categories over A are computed on their total categories. Taking generic fibers preserves these pullbacks: weakening preserves pullbacks, and taking the fiber at a_A does so as well. Since generic fibers and total categories are inverse constructions by Axiom K.7 and Proposition 5.3.4, taking total categories also preserves pullbacks.

For the second assertion, express both B and C as the pullbacks of $W_A \Sigma_A B$ and $W_A \Sigma_A C$ along a_A. The square in question is the resulting pullback square between these two fibers. □

Lemma 5.3.14. *A category B in an anima context A is an anima if and only if its total category $\Sigma_A B$ is an anima.*

Proof. By Lemma 5.3.9, the total category of the arrow category of B is given by a pullback square

$$\begin{array}{ccc} \Sigma_A \operatorname{Fun}_A(W_A[1], B) & \longrightarrow & \operatorname{Fun}([1], \Sigma_A B) \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & \operatorname{Fun}([1], A). \end{array}$$

The bottom map sends an object to its identity arrow and is an equivalence since A is an anima. Thus the top map is an equivalence. It identifies the total functor of the identity-arrow functor of B with the identity-arrow functor of $\Sigma_A B$. Taking total categories preserves and reflects equivalences, so the assertion follows from the groupoid recognition criterion. □

Corollary 5.3.15 (Isomorphisms in a family)**.** *For a category C in context A, the total category of its category of isomorphisms is*

$$\Sigma_A \operatorname{Iso}_A(C) \simeq A \times_{\operatorname{Iso}(A)} \operatorname{Iso}(\Sigma_A C),$$

where $A \to \operatorname{Iso}(A)$ assigns the identity isomorphism. We also denote the category on the right, over A, by $\operatorname{Iso}_{/A}(\Sigma_A C)$.

Proof. Apply Lemma 5.3.9 to the diagram categories in the definition of Iso. The equivalences respect restriction maps, and generic fibers and total categories preserve the pullbacks used in that definition. Thus the category on the right has generic fiber $\operatorname{Iso}_A(C)$, and Axiom K.7 gives the asserted equivalence. □

Remark 5.3.16 (Diagrams over a base)**.** The pullback description of diagrams in a family also makes sense over an arbitrary base category. Let $p\colon X \to \Gamma$ be a functor and let K be an absolute category. The category

$$\Gamma \times_{\operatorname{Fun}(K,\Gamma)} \operatorname{Fun}(K, X)$$

describes K-shaped diagrams in X equipped with an identification of their image in Γ with a constant diagram. Here $\Gamma \to \operatorname{Fun}(K, \Gamma)$ assigns constant diagrams. Thus these are diagrams lying over a single object of Γ.

For example, the category of isomorphisms in X equipped with an identification of their image with an identity in Γ is

$$\Gamma \times_{\operatorname{Iso}(\Gamma)} \operatorname{Iso}(X).$$

The identity-isomorphism functor induces an equivalence from X to this category over Γ. Indeed, the naturality square

$$\begin{array}{ccc} X & \longrightarrow & \operatorname{Iso}(X) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \operatorname{Iso}(p)} \\ \Gamma & \longrightarrow & \operatorname{Iso}(\Gamma) \end{array}$$

has equivalences as its horizontal functors by Axiom F, and is therefore a pullback. This illustrates how an equivalence of diagram categories that respects constant diagrams gives an equivalence for diagrams over a fixed object of the base.

5.4 Changing the context

Let $g\colon B \to A$ be a map of animae. Given a category $C(a)$ in context $a : A$, we would like to regard $C(g(b))$ as a category in context $b : B$. The operation that allows us to do this is called *substitution*. We will show that it corresponds to pulling back total categories along g.

Substitution should leave constant families unchanged and send the variable $a : A$ to $g(b)$. The following axiom expresses these requirements.

Axiom K.8 (Substitution)**.** There is a change of theory g^* from the theory in context A to the theory in context B, in the sense of Section 5.1.1. For every absolute category C, it is equipped with a specified equivalence

$$g^* W_A C \xrightarrow{\sim} W_B C.$$

These equivalences are compatible with absolute functors, operations, and specified coherences: the required identifications are formed after identifying their sources and targets as in Section 5.1.1. After identifying $g^* 1_A$ with 1_B and $g^* W_A A$ with $W_B A$, we also require a specified identification

$$g^* a_A = W_B g \circ a_B$$

of functors $1_B \to W_B A$.

We write $C(g(b))$ for $g^* C$, and similarly for functors and identifications. The last identification says that substitution sends the generic point of A to the point of A represented by g.

Proposition 5.4.1 (Substitution and pullback)**.** *For a category C in context A, there is a pullback square*

$$\begin{array}{ccc} \Sigma_B(g^* C) & \longrightarrow & \Sigma_A C \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_1} \\ B & \xrightarrow{\ g\ } & A. \end{array}$$

This square is natural in functors $C \to D$ in context A. For an object t of C in that context, the section corresponding to $g^* t$ is the pullback of the section corresponding to t, together with its identification after projection to the base.

Proof. Apply g^* to the generic-fiber square for C. Since substitution preserves pullbacks and is compatible with weakening and the generic point, this identifies $g^* C$ with

$$1_B \times_{W_B A} W_B(\Sigma_A C),$$

where the map from 1_B is $W_B g \circ a_B$. Taking the generic fiber of $B \times_A \Sigma_A C \to B$ gives the same pullback: weakening preserves its defining square, and we then take the fiber at a_B. The required equivalence follows by taking total categories and applying Axiom K.7.

For a functor $u\colon C \to D$ in context A, apply g^* to the naturality square for η_C and η_D, including its identifications over $W_A A$. We obtain a map between the pullback squares above. It is also the map

obtained by taking the generic fiber of

$$B \times_A \Sigma_A C \longrightarrow B \times_A \Sigma_A D.$$

Thus Proposition 5.3.4 identifies this functor with $\Sigma_B(g^* u)$ over B, proving naturality.

For $t\colon 1_A \to C$, the total category of $g^* 1_A$ is equivalent to B over B. The preceding naturality statement therefore identifies the section associated with $g^* t$ with the pullback of the section associated with t. Since this is an identification of functors over B, it includes their identifications after projection to B. □

For example, consider the fiber C_x of $f\colon C \to D$ at an object $x\colon A \to D$, as in Example 5.3.12. Substituting along $g\colon B \to A$ gives C_{xg}, since substitution preserves the defining pullback. Its total category is

$$\Sigma_B C_{xg} \simeq B \times_A (A \times_D C) \simeq B \times_D C.$$

A terminal object of C_x gives a terminal object of C_{xg}, and substitution transports the corresponding adjunction in context. Under Proposition 5.3.11, this gives the pullback of the original right adjoint section, including its unit and counit.

Construction 5.4.2 (Successive substitutions)**.** For maps of animae $D \xrightarrow{h} B \xrightarrow{g} A$ and a category C in context A, the preceding proposition and the pullback universal property give equivalences

$$\mathrm{id}_A^* C \simeq C, \qquad h^* g^* C \simeq (g \circ h)^* C.$$

For the identity map, pulling back the total category leaves it unchanged. For the composite, the equivalence of total categories is

$$D \times_B (B \times_A \Sigma_A C) \simeq D \times_A \Sigma_A C.$$

The universal property of pullbacks makes these equivalences natural in functors and identifications in C.

When we apply a construction after successive substitutions, we use the equivalences and identifications for preservation obtained by first applying g^* and then h^*. Relating these to the specified preservation data for $(gh)^*$ requires further coherence data, as explained in Section 5.1.1.

Construction 5.4.3 (Iterated anima contexts)**.** Let Γ be an anima and let A be an anima in context Γ. We will show that the iterated context $\Gamma, a : A$ can also be regarded as the single anima context $\Sigma_\Gamma A$.

For a category C in context $\Gamma, a : A$, its dependent sum over A is a category in context Γ, with a projection to A. Taking the total category once more gives an absolute category equipped with a chain

$$\Sigma_\Gamma \Sigma_A C \longrightarrow \Sigma_\Gamma A \longrightarrow \Gamma.$$

The middle category is an anima by Lemma 5.3.14. Thus the first projection determines a category in context $\Sigma_\Gamma A$.

Conversely, start with a category in context $\Sigma_\Gamma A$ and take its total category over $\Sigma_\Gamma A$. Composing its projection with the projection to Γ and taking generic fibers gives a category over A in context Γ. Its generic fiber, now taken within that theory, is a category in context $\Gamma, a : A$. Applying

Axiom K.7 and Proposition 5.3.4 at each step shows that these constructions are inverse, including on functors and identifications. We may therefore pass between the iterated context $\Gamma, a : A$ and the single anima context $\Sigma_\Gamma A$.

We now explain how changes of context will be left implicit in statements and proofs.

Convention 5.4.4 (Statements in an anima context)**.** Unless a context is specified, all data in a statement belong to a common implicit anima context, and objects are absolute objects in that theory, as in Convention 5.1.2. The phrase "for all objects $x \in C$" means that we pass to the context of the universal object

$$C^{\simeq} \longrightarrow C$$

and make the assertion there. The same interpretation applies to a construction "for each object $x \in C$". In both cases, we weaken the other data to context $C^{\simeq}$ and interpret x as the corresponding absolute object of $W_{C^{\simeq}} C$.

Note that for any object $x \colon A \to C$, factoring x through $A \to C^{\simeq}$ and applying substitution gives the statement or construction for this particular object in context A. These evaluations are compatible with substitution along maps of animae $B \to A$, by the equivalences for successive substitutions. This justifies the terminology "for all objects".

For two independently varying objects $x, y \in C$, we use the universal pair in context $C^{\simeq} \times C^{\simeq}$. For a family of categories $D(x)$ in context $x : C^{\simeq}$, the phrase "for all $x \in C$ and $y \in D(x)$" is interpreted in the iterated context $x : C^{\simeq}, y : D(x)^{\simeq}$. Similarly, "for every morphism in C" refers to the universal morphism in context $\mathrm{Map}([1], C)$. For a fixed category I, "for every diagram $I \to C$" refers to the universal diagram in context $\mathrm{Map}(I, C)$. All these parameter animae are formed in the current ambient theory.

Remark 5.4.5 (Reasoning with objects)**.** By Axiom K.2, the usual axioms are available in these contexts, so we can reason with the universal objects using the usual categorical constructions. An objectwise construction made in this way already respects isomorphisms and their higher identifications. To extend it along noninvertible morphisms of C, we need a functoriality result; this is the subject of the next chapter.

5.5 Exercises Chapter 5

Exercise 5.5.1. Show that $\Pi_A W_A C \simeq \mathrm{Fun}(A, C)$ for an absolute category C, and identify evaluation under this equivalence.

Exercise 5.5.2. Show that every category in the empty anima context is equivalent to its terminal category. Deduce that $\Sigma_\emptyset B \simeq \emptyset$ and $\Pi_\emptyset B \simeq *$.

Exercise 5.5.3. For an absolute functor $f \colon C \to D$, show that the naturality square for the functors d_C and d_D is a pullback, by comparing it with the square of second projections from $A \times C$ and $A \times D$.

Exercise 5.5.4. Let $g \colon B \to A$ and $h \colon D \to B$ be maps of animae, and let $x \colon 1_A \to C$ be an object in context A. Describe the sections corresponding to $h^* g^* x$ and $(gh)^* x$. Using the equivalence between pullbacks in Construction 5.4.2, identify these sections with their section identifications.

6 Functoriality of universals

In ordinary category theory, categories and functors are defined analytically: a category has sets of objects and morphisms, and a functor consists of assignments on objects and morphisms that are compatible with composition; in particular, functors can be constructed 'by hand' by writing down specifications on objects and on morphisms and verifying that the compatibility conditions are satisfied. This approach no longer makes sense in synthetic category theory, where functors are taken as a primitive concept; the only way to produce functors is to apply some rule, axiom or theorem that constructs it wholesale. Indeed, all of the functors we defined in the previous chapters are ultimately built out of a couple of basic ones by combining them in various ways.

The main goal of this chapter is to introduce a fundamentally new[1] way of constructing functors, through a functoriality principle. Roughly speaking, a *functoriality principle* is any special condition that provides a sufficient criterion for the following question.

Question (Functoriality)**.**

- (Absolute form) Let C, D be categories and assume that for each object $x \in C$ we have an object $f_x \in D$. When can we extend this objectwise assignment to a functor $f : C \to D$?
- (Dependent form) Let $p : D \to C$ be a functor and assume that for each $x \in C$ we have an object $s_x \in D_x$ in the fiber. When can we extend this objectwise assignment to a section of p, i.e., a functor $s : C \to D$ with $ps = \mathrm{id}_C$?

Here "for each object $x \in C$" refers to the universal object in context $C^{\simeq}$, as explained in Convention 5.4.4 and Remark 5.4.5. Thus the absolute assignment is a functor $C^{\simeq} \to D^{\simeq}$, and the dependent assignment is a section of $D^{\simeq} \to C^{\simeq}$. These assignments already respect isomorphisms and their higher identifications. The question is whether they extend to noninvertible morphisms, giving a functor on C, or a section over C, respectively. When C is an anima, there is nothing further to construct.

In this chapter, we will be interested in a particular instance of a functoriality principle, which we refer to as *functoriality of universals*. This principle says that terminal objects in the fibers of a cocartesian fibration assemble into a right adjoint section, and dually for initial objects in the fibers

[1] The closest we have seen to a functoriality principle is the defining property of the (full) subcategory C of D spanned by a subanima of objects and/or arrows. It says that *given a preexisting functor* $F : T \to D$ we can construct a functor $T \to C$ by the object- and/or arrow-wise verification that F takes values in C.

The reason we do not consider this a real functoriality principle is that it only allows to construct functors into a very special kind of category, namely one that is constructed by the (full) subcategory construction. So far, we are lacking a way to recognize a (differently constructed or a priori given) category as a (full) subcategory of some other category. Using the functoriality principle introduced in this chapter we will remedy the issue in Proposition 6.4.5 and Proposition 6.4.6.

of a cartesian fibration. Informally, it expresses that the individual solutions to a functorial family of 'universality problems' automatically assemble into a functor.

One of the main results of this chapter is that this principle is equivalent to the *Fundamental Theorem of category theory*: the fact that a functor $f\colon C \to D$ is an equivalence if and only if it is fully faithful (i.e. induces equivalences on hom animae $C(x,y) \xrightarrow{\sim} D(fx, fy)$) and strongly surjective (i.e. each object $d \in D$ is isomorphic to $f(c)$ for some $c \in C$.) It is also equivalent to the condition that the walking morphism detects equivalences: a functor f is an equivalence if and only if it induces an equivalence after applying $\mathrm{Map}([1], -)$.

After establishing these equivalent properties, we discuss various applications: characterizations of subcategories and full subcategories as embeddings and fully faithful functors, a characterization of conservative functors, pointwise and fiberwise criteria for fibrations, and a criterion for relative adjunctions in terms of fiberwise data.

6.1 Functoriality of universals

In this section, we introduce the functoriality of universals principle and discuss some elementary consequences.

6.1.1 Unique arrow lifts

Let us start by discussing a slightly simpler form of a functoriality principle.

Informally, categories are supposed to be governed by objects and arrows. Thus, one special situation in which we expect a functoriality principle to hold is when the values of the section s on arrows is uniquely constrained, in the sense that for each arrow $\gamma\colon x \to y$ in C there is a *unique* arrow $s_\gamma\colon s_x \to s_y$ in D with $p(s_\gamma) = \gamma$ (compatibly with the given identifications $p(s_x) = x$ and $p(s_y) = y$). The heuristic is that any associativity or higher coherence constraint will take place in a contractible anima. For example, given an identification $\gamma = \gamma' \circ \gamma''$ there will be a unique identification $s_\gamma = s_{\gamma'} \circ s_{\gamma''}$ living over it because both sides are (identified with) objects of the anima of lifts of γ, which is contractible by assumption.

We will relate this principle to detection of equivalences by $\mathrm{Map}([1], -)$, using the description of hom animae as fibers of the anima of arrows from Corollary 3.4.9.

Lemma 6.1.1. *The following are logically equivalent:*

(1) *($[1]$ detects equivalences)*

Let $p\colon D \to C$ be a functor. If $p_*\colon \mathrm{Map}([1], D) \to \mathrm{Map}([1], C)$ is an equivalence, then so is p.

(2) *(Functoriality principle for sections with unique arrow lifts)*

Let $p\colon D \to C$ be a functor and $(x \in C) \mapsto (s_x \in D_x)$ an objectwise section. Assume that s has unique arrow lifts*, i.e., for each arrow $\gamma\colon x \to y$ in C there is a unique arrow $\delta\colon s_x \to s_y$ with $p\delta = \gamma$. Then there is a section s of p that extends the given objectwise assignment $x \mapsto s_x$.*

Remark 6.1.2. Note that each $s_x \in D_x$ is really a pair consisting of $s_x \in D$ and an identification $\varepsilon_x \colon p(s_x) = x$.

Making this unpacking explicit, the assumption in (2) translates to saying that the map

$$D(s_x, s_y) \xrightarrow{p} C(p(s_x), p(s_y)) \xrightarrow[\simeq]{\varepsilon_y \circ - \circ \varepsilon_x^{-1}} C(x, y) \tag{6.1}$$

of animae is an equivalence for all $x, y \in C$.

Similarly unpacking the conclusion, it consists of a functor $s \colon C \to D$ an identification $\varepsilon : ps = \mathrm{id}_C$ (jointly comprising a section of p), which extend the objectwise section in the sense that we have identifications $s(x) = s_x$ and $(p(s(x)) = p(s_x) \xrightarrow[=]{\varepsilon_x} x) = \varepsilon(x)$ for all $x : C^{\simeq}$.

Proof.

"(1) → (2)" Assume that we have given an objectwise section $x \mapsto (s_x, \varepsilon_x)$ of p such that the map (6.1) is an equivalence. For each $x, y : C^{\simeq}$, consider the composite map

$$(x = y) \xrightarrow{s} (s_x = s_y) \to (x = y) \tag{6.2}$$

where the first map is induced by $s \colon C^{\simeq} \to D^{\simeq}$ and the second one is obtained by restricting (6.1) to the subanimae of isomorphisms, i.e., identifications. This restriction is an equivalence: an inverse to the image of an arrow lifts through the given hom equivalence, and the two inverse identities lift through the corresponding hom equivalences as well. Thus an arrow between selected objects is invertible precisely when its image is invertible. For each $x, y : C^{\simeq}$, the map (6.2) is (identified with) the identity; indeed it suffices to observe that $\mathrm{id}_x \mapsto \varepsilon_x \circ p(s(\mathrm{id}_x)) \circ \varepsilon_x^{-1} = \mathrm{id}_x$ for all x. By 2-out-of-3 we conclude that each $s \colon (x = y) \to (s_x = s_y)$ is an equivalence, i.e., that $s \colon C^{\simeq} \to D^{\simeq}$ is an embedding.

Following Definition 3.2.5 we may thus define the full subcategory $C' \coloneqq \langle s \colon C^{\simeq} \subseteq D \rangle$ and consider the composite

$$p' \colon C' \xhookrightarrow{i} D \xrightarrow{p} C \tag{6.3}$$

We claim that $\mathrm{Map}([1], p')$ is an equivalence. Indeed, by the universal property of the full subcategory (Definition 3.2.2), applying $\mathrm{Map}([1], -)$ to the sequence (6.3) and taking the fiber in the anima context of an arrow $\gamma \colon x \to y$ yields

$$\begin{aligned}
&(\gamma' : \mathrm{Map}([1], C')) \times (p'(\gamma') = \gamma) \\
&\simeq ((x', y') : \mathrm{Map}([1]^{\simeq}, C^{\simeq})) \times (\delta : \mathrm{Map}([1], D)) \times ((s_{x'}, s_{y'}) = \partial\delta) \times (p\delta = \gamma) \\
&\simeq (\delta : D(s_x, s_y)) \times (p(\delta) =_{C(x,y)} \gamma)
\end{aligned}$$

(keeping the identifications $p(s_x) = x$ and $p(s_y) = y$ implicit), which is contractible by assumption.

Thus we may apply (1) to p' to conclude that the composite $p' \colon C' \to C$ is an equivalence. The composite $s \colon C \simeq C' \hookrightarrow iD$ is then the desired section of p; and it is straightforward to verify that it extends the given objectwise assignment.

"(2) → (1)" Let p be such that $\mathrm{Map}([1], p)$ is an equivalence. Since $p^{\simeq} = \mathrm{Map}([0], p)$ is a retract of $\mathrm{Map}([1], p)$ (because $[0]$ is a retract of $[1]$), it follows that $p^{\simeq}$ is an equivalence. Thus there

is a unique section $x \mapsto (s_x, ps = x)$ of $p^{\simeq}$. We claim that this assignment satisfies the unique-arrow-lifting assumption. Indeed, for each $x, y \in C$, the map (6.1) is identified with the map on vertical fibers at (s_x, s_y) of the square

$$\begin{array}{ccc} \mathrm{Map}([1], D) & \xrightarrow{p_*} & \mathrm{Map}([1], C) \\ \downarrow & & \downarrow \\ D^{\simeq} \times D^{\simeq} & \xrightarrow{p^{\simeq} \times p^{\simeq}} & C^{\simeq} \times C^{\simeq} \end{array} \tag{6.4}$$

hence it is an equivalence by assumption. Applying (2) we thus obtain a section s of p.

Finally, note that $\mathrm{Map}([1], s)$ is a section of $\mathrm{Map}([1], p)$, hence itself an equivalence. Applying the just proven statement to s instead of p, we get that s itself also has a section. It follows that s, hence p, is an equivalence. □

6.1.2 Universal objects and the functoriality axiom

The functoriality principle for sections with unique arrow lifts admits an important special case:

Lemma 6.1.3. *Let $p\colon D \to C$ be a cocartesian fibration, and suppose that for every object x of C we are given a terminal object s_x of the fiber D_x. Then for each object $y \in C$ and $d \in D$ and each arrow $\gamma\colon p(d) \to y$, there is a unique lift of γ to an arrow $d \to s_y$. In particular, s has unique arrow lifts.*

Proof. We have the equivalence

$$(\delta : D(d, s_y)) \times (p\delta = \gamma) \xleftarrow{\simeq} (\bar{\delta} : D(\gamma_! d, s_y)) \times (p\bar{\delta} = \mathrm{id}_y) \simeq D_y(\gamma_! d, s_y)$$

of animae (given by precomposing with $\mathrm{lift}_0^p(d, \gamma)$, see Lemma 4.5.7). The right side is contractible because s_y is a terminal object in the fiber D_y.

To see that s has unique arrow lifts of an arrow $\gamma\colon x \to y$, simply take $d \coloneqq s_x$. □

A universal object is one which (together with some auxiliary data) is initial/terminal (among all objects equipped with similar data). For example, a limit is a terminal cone, a colimit is an initial cocone, and a tensor product of V, W is equipped with an initial bilinear map from $V \times W$, etc.

We can now say what it means for such universal objects to be functorial: Let's say that for each object $x \in C$ we have a category $D(x)$ in the anima context of x, encoding some data that depends on x, and assume that in each $D(x)$ we have a universal (initial or terminal) object s_x. If the category $D(x)$ itself varies sufficiently functorially in x, then we expect the s_x to "be functorial", i.e., assemble to a functor s out of C in a canonical way. The way we express this is through the language of cartesian/cocartesian fibrations.

Definition 6.1.4 (Functoriality of universals)**.** The *functoriality of universals* principle is the following assertion. Let $p\colon E \to C$ be a cocartesian (resp. cartesian) fibration. Suppose that for every object x of C we are given a terminal (resp. initial) object s_x of the fiber E_x. Then this family extends to a right (resp. left) adjoint section s of p.

More explicitly, the family s_x is a terminal (resp. initial) object of E_x in context $x : C^{\simeq}$. The total category of this contextual category is the pullback

$$\begin{array}{ccc} E_0 = C^{\simeq} \times_C E & \longrightarrow & E \\ {\scriptstyle p_0}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C^{\simeq} & \xrightarrow{\iota_C} & C. \end{array}$$

By Proposition 5.3.11, the given family corresponds to a right (resp. left) adjoint section s_0 of p_0. The extension includes an identification of the restriction of s with s_0 as sections over $C^{\simeq}$, including their section identifications.

Example 6.1.5. If C is the category of pairs of vector spaces, and D is the category of triples V, W, Z together with a bilinear map $V \times W \to Z$, then we have the cartesian projection $p\colon D \to C$. The fact that each fiber $D(V, W)$ has an initial object, namely a tensor product $V \times W \to V \otimes W$, means that by functoriality of universals one obtains a left adjoint section $s\colon C \to D$ that maps $(V, W) \mapsto V \otimes W$.

Proposition 6.1.6. *Assume that* $[1]$ *detects equivalences. Then universals are functorial.*

Proof. By Lemma 6.1.3 the objectwise section $x \mapsto s_x$ has unique arrow lifts; hence by Lemma 6.1.1 there is a section $s\colon C \to D$ extending it (including the implicit identification $\varepsilon\colon ps = \mathrm{id}_C$). It remains to show that the section s is indeed right adjoint (or left adjoint). Let us treat the cocartesian/right adjoint case.

Consider the following pullback

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Fun}([1], D) \times_{\mathrm{Fun}([1],C)} C \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow \\ D & \xrightarrow{(\mathrm{id}_D, sp, \varepsilon p)} & D \times_C D \end{array} \tag{6.5}$$

where the right vertical map is obtained by taking horizontal pullbacks in the diagram

$$\begin{array}{ccccc} \mathrm{Fun}([1], D) & \xrightarrow{p_*} & \mathrm{Fun}([1], C) & \longleftarrow & C \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_C} \\ D \times D & \xrightarrow{p \times p} & C \times C & \longleftarrow & C \end{array}$$

Explicitly, the terms of E are tuples $(d : D, \eta\colon d \to spd, p\eta = \varepsilon^{-1}p)$ and $q\colon E \to D$ is the projection onto the first component.

We claim that $\mathrm{Map}([1], q)$ is an equivalence of animae. For this, we must show that given an arrow $\delta\colon d \to d'$ in D, there is a unique commutative square

$$\begin{array}{ccc} d & \overset{\eta}{\dashrightarrow} & spd \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle sp\delta} \\ d' & \overset{\eta'}{\dashrightarrow} & spd' \end{array} \quad \text{over} \quad \begin{array}{ccc} pd & \overset{\varepsilon^{-1}pd}{=\!=} & pspd \\ {\scriptstyle p\delta}\downarrow & & \downarrow{\scriptstyle psp\delta} \\ pd' & \overset{\varepsilon^{-1}pd'}{=\!=} & pspd' \end{array} \tag{6.6}$$

This follows by applying Lemma 6.1.3 three times: twice to find the unique lifts η and η', and once more to find the unique identification $sp\delta \circ \eta = \eta' \circ \delta$ filling the commutative square (both sides live in the contractible anima of lifts $d \to spd'$ over the given arrow $pd \to pspd'$).

Since [1] detects equivalences, we conclude that q is an equivalence. We claim that the resulting functor

$$\eta\colon D \simeq E \to \mathrm{Fun}([1], D) \times_{\mathrm{Fun}([1],C)} C \tag{6.7}$$

exhibits s as a right adjoint section of p. By construction, η is a transformation over C. By the dual of Proposition 4.4.12, it remains to show that the restricted transformation

$$C \times [1] \xrightarrow{s\times[1]} D \times [1] \xrightarrow{\eta} D \tag{6.8}$$

is a natural isomorphism. By Proposition 3.4.6, we may check this objectwise. For each object $x \in C$, naturality of ε gives $ps(\varepsilon_x) = \varepsilon_{psx}$. Hence the isomorphism $s(\varepsilon_x^{-1})\colon s_x \to s_{psx}$ and η_{s_x} lie over the same base arrow. By uniqueness of lifts into the selected terminal object, they are identified. Thus η_{s_x} is invertible, as required. □

In the remainder of the book, we assume functoriality of universals.

Axiom L (Functoriality of universals)**.** Universals are functorial in the sense of Definition 6.1.4: a family of terminal (resp. initial) objects in the fibers of a cocartesian (resp. cartesian) fibration extends to a right (resp. left) adjoint section, with the specified identification with the given family.

Any section extending the given family is automatically an adjoint section. To see this in the cocartesian case, let t be the right adjoint section supplied by the axiom, and let s be another section extending the family, with identification $\varepsilon\colon ps = \mathrm{id}_C$. The unit η of $p \dashv t$ gives a comparison map

$$s \xrightarrow{\eta s} tps \xrightarrow{t\varepsilon} t$$

over C. Its components are morphisms between terminal objects in the same fiber, hence are isomorphisms. By Proposition 3.4.6, this comparison is a natural isomorphism, so s is also a right adjoint section of p. The cartesian case is dual.

In the next two sections, we will prove a converse to the preceding proposition: we show that functoriality of universals implies the Fundamental Theorem, which in turn implies that [1] detects equivalences.

6.1.3 Universal objects in families

We first apply functoriality of universals to a left or right fibration with contractible fibers. This will give a criterion for initial and terminal objects in terms of their hom animae, which we can apply in any anima context.

Proposition 6.1.7. *Let $f\colon E \to B$ be a left (resp. right) fibration. Assume that the fiber E_b of f at b is contractible for every object b of B. Then f is an equivalence.*

Proof. We treat left fibrations; the right-fibration case is dual. In context $b : B^{\simeq}$, the contractible category E_b has a terminal object. Since a left fibration is cocartesian, functoriality of universals

gives a right adjoint section $s\colon B \to E$ of f. Its unit $\eta\colon \mathrm{id}_E \to sf$ lies over identities in B, using the section identification $fs = \mathrm{id}_B$ and the triangle identity.

Left fibrations reflect isomorphisms, by Corollary 4.2.3. Thus every component of η is invertible, and Proposition 3.4.6 makes η a natural isomorphism. Together with $fs = \mathrm{id}_B$, this exhibits s as an inverse to f. □

Lemma 6.1.8. *Let $f\colon E \to B$ be a cocartesian fibration. Then f is a left fibration if and only if every fiber E_b is an anima in the context of b. Equivalently, the pullback $B^{\simeq} \times_B E$ is an anima. Dually, a cartesian fibration with anima fibers is a right fibration.*

Proof. By Lemma 5.3.14, the contextual fiber condition is equivalent to the stated condition on the total category over $B^{\simeq}$. If f is a left fibration, its pullback to $B^{\simeq}$ is a left fibration over an anima and hence is an anima by Corollary 4.2.5. Conversely, factor a morphism $u\colon x \to y$ as its chosen cocartesian lift followed by the vertical morphism c_u. This map belongs to the pullback over $B^{\simeq}$ and is therefore invertible. These morphisms are the components of the counit of the left adjoint section of $\vec{\mathrm{ev}}_0^f$. By the objectwise criterion for natural isomorphisms from Proposition 3.4.6, that counit is thus a natural isomorphism, showing that $\vec{\mathrm{ev}}_0^f$ is an equivalence. It follows that f is a left fibration. □

Lemma 6.1.9 (Anima-indexed slices)**.** *Let A be an anima and $t\colon A \to X$ a functor. The projection*

$$X \vec{\times}_t A \longrightarrow X$$

from the category of arrows $x \to t(a)$ is a right fibration.

Proof. It is a cartesian fibration by the dual of Proposition 4.6.5. In context $x : X^{\simeq}$, its fiber is the dependent sum $\Sigma_{a:A} X(x, t(a))$. This is an anima by Lemma 5.3.14. The claim follows from the dual of Lemma 6.1.8. □

Lemma 6.1.10. *Let x be an object of a category C. Then x is initial if and only if for every other object y of C the hom anima $C(x, y)$ is contractible. Dually, x is terminal if and only if $C(y, x)$ is contractible for every object y of C.*

Proof. By Lemma 4.4.19, the object x is initial if and only if the forgetful functor $C_{x/} \to C$ is an equivalence. Since this functor is a left fibration by Theorem 4.3.3, it follows from Proposition 6.1.7 that it is an equivalence if and only if all of its fibers $C(x, y)$ are contractible. □

Lemma 6.1.11 (Recognizing terminal families)**.** *Let $q\colon X \to A$ be a functor to an anima, and let $t\colon A \to X$ be a section with identification $\varepsilon\colon qt = \mathrm{id}_A$. In context $a : A$, suppose that*

$$X_a(x, t(a))$$

is contractible for every object x of X_a. Then t is a right adjoint section of q, with counit ε. There is a dual criterion for initial families.

Proof. In context $a : A$, the preceding lemma shows that $t(a)$ is terminal in X_a. By Proposition 5.3.11, this is equivalent to t being a right adjoint section of q with the given counit. The initial case is dual. □

6.1.4 Fiberwise equivalences of left and right fibrations

The contractible-fiber criterion from Proposition 6.1.7 also lets us recognize equivalences and pullback squares between left or right fibrations on their fibers.

Lemma 6.1.12. *Consider a commutative triangle*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & C, & \end{array}$$

where p and q are left (resp. right) fibrations. Assume that the induced map $A_c \to B_c$ on fibers is an equivalence for every object c of C. Then f is an equivalence.

Proof. The functor f is also a left (resp. right) fibration by Proposition 4.2.16. It suffices by Proposition 6.1.7 to show that its fibers are contractible. Work in the anima context of an object b of B, and put $c = q(b)$. There is a pullback square in this context

$$\begin{array}{ccc} A_b & \longrightarrow & A_c \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{b} & B_c. \end{array}$$

The right vertical functor is an equivalence by hypothesis, so $A_b \to *$ is an equivalence. Thus A_b is contractible, as required. □

Corollary 6.1.13. *Consider a commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{g} & D \end{array}$$

and assume that p and q are left (resp. right) fibrations. Then the square is a pullback square if and only if for every object x of C the induced map $f_x \colon A_x \to B_{g(x)}$ on fibers is an equivalence.

Proof. Since left (resp. right) fibrations are closed under base change by Proposition 4.2.15, the functor $B \times_D C \to C$ is a left (resp. right) fibration. The square is a pullback square if and only if the map $(f, p) \colon A \to B \times_D C$ is an equivalence over C. By Lemma 6.1.12, we may check this condition fiberwise, where it precisely becomes the claim that the induced map $A(x) \to B(g(x))$ is an equivalence. □

Corollary 6.1.14. *Let $f \colon E \to C$ be a left (resp. right) fibration. Then the commutative square*

$$\begin{array}{ccc} E^{\simeq} & \xrightarrow{\iota_E} & E \\ {\scriptstyle f^{\simeq}}\downarrow & & \downarrow{\scriptstyle f} \\ C^{\simeq} & \xrightarrow{\iota_C} & C \end{array}$$

is a pullback square.

Proof. Note that the functor $f^{\simeq}$ is again a left (resp. right) fibration by Proposition 4.2.4, since both its source and target are animae. By Corollary 6.1.13 it will thus suffice to show that ι_E induces equivalences on all fibers over $C^{\simeq}$. But given an object x of C, the induced map on fibers is $\iota_{E_x} : (E_x)^{\simeq} \to E_x$, which is an equivalence since E_x is an anima by Corollary 4.2.6. □

6.2 Universal arrows and adjoints

An adjoint assigns a universal arrow to each object of its target. Functoriality of universals lets us assemble these arrows into an adjunction. We first express this through terminal objects in relative slices, then give the corresponding criterion on hom animae. For an object d of D, the relative slice $C_{/d}$ and its terminal object are taken in the anima context of d.

Theorem 6.2.1. *Let $f : C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor f admits a right adjoint $g : D \to C$;

(2) The projection functor $p : C \vec{\times}_f D \to D$ admits a right adjoint section $s : D \to C \vec{\times}_f D$;

(3) For every object d in D, the relative slice $C_{/d} = C \times_D D_{/d}$ admits a terminal object s_d.

Dually, f admits a left adjoint if and only if the projection functor $D \vec{\times}_f C \to D$ admits a left adjoint section, or equivalently if and only if the relative coslice $C_{d/}$ admits an initial object for every object d of D.

Remark 6.2.2. We will in fact prove the following somewhat more refined statement: assume that for every object d of D, the relative slice $C_{/d}$ admits a terminal object $s_d = (g_d, \varepsilon_d)$, where g_d is an object of C and $\varepsilon_d : f(g_d) \to d$ is a morphism in D. Then the right adjoint $g : D \to C$ of f may be chosen in such a way that for every object d of D the following two conditions are satisfied:

- There is an identification $g_d = g(d)$;
- Under this isomorphism, the counit map $\varepsilon : f(g(d)) \to d$ of the adjunction corresponds to the given map $\varepsilon_d : f(g_d) \to d$.

Proof of Theorem 6.2.1. We will prove the equivalence between (1), (2) and (3), and leave the dual claim to the reader. Recall from Construction 4.1.1 the directed pullback $C \vec{\times}_f D$, defined via the following pullback diagram:

$$\begin{array}{ccc} C \vec{\times}_f D & \longrightarrow & \mathrm{Fun}([1], D) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ C \times D & \xrightarrow{f \times \mathrm{id}_D} & D \times D \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_1} \\ C & \xrightarrow{f} & D. \end{array}$$

Objects of $C \vec{\times}_f D$ are pairs (x, u) where x is an object in C and u is an object in $\mathrm{Fun}([1], D)$ of the form $u : f(x) \to y$. There are two canonical projection maps

$$p : C \vec{\times}_f D \to D \qquad \text{and} \qquad q : C \vec{\times}_f D \to C$$

which are given on terms by $p(x, u\colon f(x) \to y) = y$ and $q(x, u) = x$. We have:

- The functor p is a cocartesian fibration by Proposition 4.6.5;
- The functor q is a base change of the evaluation map $\mathrm{ev}_0\colon \mathrm{Fun}([1], D) \to D$. Since ev_0 admits a left adjoint section given by $y \mapsto \mathrm{id}_y$ by Lemma 4.4.11, we get that also q admits a left adjoint section $i\colon C \to C \vec{\times}_f D$ given by $i(x) = (x, f(x) \xrightarrow{=} f(x))$.
- The composite
$$C \xrightarrow{i} C \vec{\times}_f D \xrightarrow{p} D$$
is identified with f.

For (1) $\implies$ (2), recall from Proposition 4.4.7 that the data of an adjunction $f \dashv g$ is equivalent to the data of an equivalence $C \vec{\times}_f D \simeq C \vec{\times}_g D$ over $C \times D$. We may therefore equivalently show that the projection $C \vec{\times}_g D \to D$ admits a right adjoint section. But this follows from Lemma 4.4.11, as this map is a base change of the evaluation functor $\mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$.

For (2) $\implies$ (1), assume that p admits a right adjoint section $s\colon D \to C \vec{\times}_f D$. Since adjunctions compose by Proposition 4.4.3, we conclude that the composite $g := qs\colon D \to C$ is the desired right adjoint of $f = pi$:

$$C \underset{q}{\overset{i}{\rightleftarrows}} C \vec{\times}_f D \underset{s}{\overset{p}{\rightleftarrows}} D. \qquad \text{(with } f \text{ and } g \text{ as the outer arrows)}$$

A right adjoint section restricts to terminal objects in the fibers, by Lemma 4.4.21 and its dual. Conversely, apply functoriality of universals to the terminal objects in (3), considered in context $d : D^{\simeq}$. Since p is cocartesian, they extend to a right adjoint section of p, proving (2).

For the refined statement from Remark 6.2.2, we use that the terminal objects $s_d = (g_d, \varepsilon_d)$ of the relative slices $C_{/d}$ assemble into the right adjoint section $s\colon D \to C \vec{\times}_f D$, with counit $ps = \mathrm{id}_D$. The right adjoint g of f is then given by qs, with counit $fg \to \mathrm{id}_D$ given by the composite

$$fg(d) = piqs(d) \xrightarrow{p\varepsilon'_{s(d)}} ps(d) = d,$$

where the transformation $\varepsilon'\colon iq \to \mathrm{id}_{C \vec{\times}_f D}$ is the counit of the adjunction $i \dashv q$. After unwinding the definitions, one sees that this map is precisely the given map $\varepsilon_d\colon f(g_d) \to d$, as claimed. □

Proposition 6.2.3 (Pointwise criterion for adjunctions)**.** *Let $f\colon C \to D$ be a functor. Assume that, for every object $d \in D$, we are given an object $g(d)$ in C together with a morphism $\varepsilon_d\colon f(g(d)) \to d$ (resp. $\eta_d\colon d \to f(g(d))$) such that for any object c in C the map*

$$C(c, g(d)) \xrightarrow{f} D(f(c), f(g(d))) \xrightarrow{\varepsilon_d \circ -} D(f(c), d)$$

is an equivalence (resp. for every c in C the map

$$C(g(d), c) \xrightarrow{f} D(f(g(d)), f(c)) \xrightarrow{-\circ \eta_d} D(d, f(c))$$

is an equivalence.) Then there is a right adjoint (resp. left adjoint) $g\colon D \to C$ of f such that $g(d)$ coincides on objects with the data above, and the counit (resp. unit) is on objects given by the maps ε_d (resp. η_d).

Proof. We will only prove the first case; the other case is dual. By Theorem 6.2.1 it suffices to check that, for each object d of D, the pair $(g(d), \varepsilon_d)$ is a terminal object of the relative slice $C_{/d} = C \times_D D_{/d}$. By Lemma 6.1.10, it suffices to show that the hom anima $\mathrm{Hom}_{C_{/d}}((c,\alpha),(g(d),\varepsilon_d))$ is contractible for any object $(c, \alpha\colon f(c) \to d)$ of $C_{/d}$. Work in the iterated context $d : D^{\simeq}, (c,\alpha) : (C_{/d})^{\simeq}$ of the universal object of D and the universal object of its relative slice. By definition of $C_{/d}$ as a pullback, we may compute this hom anima via the following pullback diagram, where $*$ is the terminal category in this context:

$$\begin{array}{ccccc}
\mathrm{Hom}_{C_{/d}}((c,\alpha),(g(d),\varepsilon_d)) & \longrightarrow & \mathrm{Hom}_{D_{/d}}((f(c),\alpha),(f(g(d)),\varepsilon_d)) & \longrightarrow & * \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{Hom}_C(c, g(d)) & \xrightarrow{f} & \mathrm{Hom}_D(f(c), f(g(d))) & \xrightarrow{(\varepsilon_d)_*} & \mathrm{Hom}_D(f(c), d).
\end{array}$$

But since the bottom composite is assumed to be an equivalence, we see that the top composite is also an equivalence, and so the left-upper corner is contractible as desired. This finishes the proof. □

Proposition 6.2.4 (Base change of a right adjoint along a cocartesian fibration)**.** *Consider a pullback square*

$$\begin{array}{ccc}
C' & \xrightarrow{u} & C \\
{\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
D' & \xrightarrow{v} & D
\end{array}$$

such that p is a cocartesian fibration. If v admits a left adjoint, then also u admits a left adjoint.

Proof. By the dual statement of Theorem 6.2.1, the projection $D \mathbin{\vec{\times}}_v D' \to D$ admits a left adjoint section. Consider the pullback square

$$\begin{array}{ccc}
C \mathbin{\vec{\times}}_{p,v} D' & \longrightarrow & D \mathbin{\vec{\times}}_v D' \\
\downarrow & \lrcorner & \downarrow \\
C & \xrightarrow{p} & D.
\end{array}$$

It follows that the projection $C \mathbin{\vec{\times}}_{p,v} D' \to C$ admits a left adjoint section. A term of $C \mathbin{\vec{\times}}_{p,v} D'$ is a triple (c, d', α) with $\alpha\colon p(c) \to v(d')$. There is also a pullback square

$$\begin{array}{ccc}
C \mathbin{\vec{\times}}_u C' & \longrightarrow & \mathrm{Fun}([1], C) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \vec{\mathrm{ev}}_0^p} \\
C \mathbin{\vec{\times}}_{p,v} D' & \longrightarrow & C \mathbin{\vec{\times}}_p D.
\end{array}$$

The lower horizontal map sends (c, d', α) to $(c, v(d'), \alpha)$. Thus the upper left corner consists of a morphism $c \to c_1$ in C, an object d' of D', and a specified identification $p(c_1) = v(d')$, which is precisely a term of $C \mathbin{\vec{\times}}_u C'$. Since p is a cocartesian fibration, $\vec{\mathrm{ev}}_0^p$ admits a left adjoint section. Functors admitting left adjoint sections are stable under base change and composition, so the composite projection $C \mathbin{\vec{\times}}_u C' \to C \mathbin{\vec{\times}}_{p,v} D' \to C$ admits a left adjoint section. Applying the dual statement of Theorem 6.2.1 once more shows that u admits a left adjoint. □

6.3 The Fundamental Theorem of category theory

We apply the adjoint criterion to construct inverses to functors that are fully faithful and strongly surjective. This gives the Fundamental Theorem and, as a consequence, detection of equivalences by the anima of arrows.

Definition 6.3.1 (Fully faithful)**.** A functor $f\colon C \to D$ is said to be *fully faithful* if for all objects x and y of C the induced functor

$$f\colon C(x,y) \to D(f(x),f(y))$$

on hom animae is an equivalence.

Remark 6.3.2. The map $C(x,y) \to D(f(x),f(y))$ is obtained by passing to fibers over (x,y) in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Map}([1],C) & \xrightarrow{f_*} & \mathrm{Map}([1],D) \\ \downarrow & & \downarrow \\ C^{\simeq}\times C^{\simeq} & \xrightarrow{f^{\simeq}\times f^{\simeq}} & D^{\simeq}\times D^{\simeq}, \end{array} \tag{6.9}$$

where we use Corollary 3.4.9 to identify the fibers of the two vertical maps with $C(x,y)$ and $D(f(x),f(y))$, respectively. It follows that fully faithfulness of f is equivalent to the condition that Equation (6.9) is a pullback square.

Corollary 6.3.3. *Let $f\colon C \to D$ and $g\colon D \to E$ be two functors such that g is fully faithful. Then f is fully faithful if and only if $gf\colon C \to E$ is fully faithful.*

Proof. This follows from applying the pasting lemma for pullbacks to the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Map}([1],C) & \xrightarrow{f_*} & \mathrm{Map}([1],D) & \xrightarrow{g_*} & \mathrm{Map}([1],E) \\ \downarrow & & \downarrow & & \downarrow \\ C^{\simeq}\times C^{\simeq} & \xrightarrow{f^{\simeq}\times f^{\simeq}} & D^{\simeq}\times D^{\simeq} & \xrightarrow{g^{\simeq}\times g^{\simeq}} & E^{\simeq}\times E^{\simeq}. \end{array}$$

□

Example 6.3.4. Every equivalence $u\colon C \xrightarrow{\sim} D$ is fully faithful, see Exercise 6.7.1.

Proposition 6.3.5. *If a functor $f\colon C \to D$ is fully faithful, then it reflects isomorphisms: if $f(u)$ is invertible, then u is invertible.*

Proof. Consider a morphism $u\colon x \to y$ in C whose image $f(u)$ in D is invertible. We will show that u admits a left inverse; the argument for the existence of a right inverse will be analogous. Since $f(u)$ is invertible in D, it admits a left inverse $v\colon f(y) \to f(x)$, i.e. we have $vf(u) = \mathrm{id}_{f(x)}$. Since f is fully faithful, there exists a morphism $\tilde{v}\colon y \to x$ in C satisfying $f(\tilde{v}) = v$. We claim that the relation $\tilde{v}u = \mathrm{id}_x$ is satisfied. Since f is fully faithful, it suffices to show that $f(\tilde{v}u) = f(\mathrm{id}_x)$. But since f preserves identities and compositions, this is equivalent to the relation $f(\tilde{v})f(u) = vf(u) = \mathrm{id}_{f(x)}$, which holds by assumption. □

Definition 6.3.6 (Strongly surjective functor)**.** We say that a functor $f : C \to D$ is *strongly surjective* if it comes equipped with a section $s : D^{\simeq} \to C^{\simeq}$ of the underlying map $f^{\simeq} : C^{\simeq} \to D^{\simeq}$ on groupoid cores.

Remark 6.3.7. Using the contextual convention from Convention 5.4.4, note that f is strongly surjective if and only if for every object d of D there exists an object c in C together with an isomorphism $f(c) \xrightarrow{\sim} d$ in D. For this reason, this notion was previously called 'essential surjectivity' in this text.

Theorem 6.3.8 (Fundamental theorem of category theory)**.** *A functor $f : C \to D$ is an equivalence if and only if it is fully faithful and strongly surjective.*

Proof. It is clear that every equivalence is fully faithful and strongly surjective. Conversely, let $f : C \to D$ be a functor which is fully faithful and strongly surjective. Our goal is to show that f is an equivalence.

We apply Theorem 6.2.1 to construct a right adjoint of f. Strong surjectivity gives a functor $c : D^{\simeq} \to C$ and an isomorphism $e : fc = \iota_D$. For each object d of D, consider the object (c_d, e_d) of the relative slice $C_{/d}$.

For an object $(x, u : f(x) \to d)$ of this slice, its hom anima into (c_d, e_d) is the fiber over u of

$$C(x, c_d) \xrightarrow{f} D(f(x), f(c_d)) \xrightarrow{e_d \circ -} D(f(x), d).$$

This is an equivalence by full faithfulness and invertibility of e_d, so the fiber is contractible. This calculation takes place in the iterated context $d : D^{\simeq}, (x, u) : (C_{/d})^{\simeq}$ of the universal object of D and the universal object of its relative slice. By Lemma 6.1.10, (c_d, e_d) is terminal in $C_{/d}$.

The general adjoint criterion and its refinement, Remark 6.2.2, therefore give a right adjoint g of f with values c_d and counit components e_d. We use the construction in that proof: writing $E = C \vec{\times}_f D$ with projections $p : E \to D$ and $q : E \to C$, the chosen terminal objects extend to a right adjoint section $s : D \to E$ of p. The functor q has the left adjoint section $i(x) = (x, \mathrm{id}_{f(x)})$, and $f = pi$ has right adjoint $g = qs$. We retain the specified identification of s with the chosen section over $D^{\simeq}$.

Recall that the adjoint g we obtain is given by a composite of adjoints, as displayed as follows:

$$C \underset{q}{\overset{i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} C \vec{\times}_f D \underset{s}{\overset{p}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} D.$$

with f the composite $C \to D$ and g the composite $D \to C$.

Let $\alpha : \mathrm{id}_C \to qi$ be the unit isomorphism of the left adjoint section i of q, let $\eta : \mathrm{id}_{C \vec{\times}_f D} \to sp$ be the unit of the adjunction $p \dashv s$, let $\varepsilon : iq \to \mathrm{id}_{C \vec{\times}_f D}$ be the counit of the adjunction $i \dashv q$, and let $\delta : ps \to \mathrm{id}_D$ be the counit isomorphism of the right adjoint section s of p. The counit and unit of the composite adjunction $f = pi \dashv qs = g$ are

$$fg = piqs \xrightarrow{p\varepsilon s} ps \xrightarrow{\delta} \mathrm{id}_D$$

and

$$\mathrm{id}_C \xrightarrow{\alpha} qi \xrightarrow{q\eta i} qspi = gf,$$

respectively. Since α and δ are already natural isomorphisms, it remains to show that $p\varepsilon s$ and $q\eta i$ are natural isomorphisms.

For an object $(x, u\colon f(x) \to y)$ of $C \mathbin{\vec{\times}_f} D$, the component $p\varepsilon_{(x,u)}$ is the morphism $u\colon f(x) \to y$. In particular, at the chosen object $s(y) = (c_y, e_y)$ it is the isomorphism

$$p\varepsilon_{s(y)} = e_y\colon f(c_y) \xrightarrow{\sim} y.$$

Thus $p\varepsilon s$ is objectwise an isomorphism, and hence is a natural isomorphism by Proposition 3.4.6.

To show that $q\eta i$ is a natural isomorphism, it will by Proposition 3.4.6 suffice to do so on objects. We first describe the transformation $\eta\colon \mathrm{id}_{C \mathbin{\vec{\times}_f} D} \to sp$ at an object $(x, u\colon f(x) \to y)$. Write $s(y) = (c_y, e_y)$ as above. By full faithfulness, there is a morphism $v\colon x \to c_y$ and an identification $e_y f(v) = u$. The map $\eta_{(x,u)}\colon (x, u) \to (c_y, e_y)$ is given by the following morphism (to be read from top to bottom):

$$\begin{pmatrix} x & f(x) & \xrightarrow{u} & y \\ {\scriptstyle v}\downarrow & \downarrow{\scriptstyle f(v)} & & \| \\ c_y & f(c_y) & \xrightarrow{e_y} & y \end{pmatrix}.$$

Now evaluate at $i(x) = (x, \mathrm{id}_{f(x)})$. The resulting morphism $q\eta_{i(x)}$ is a map $v\colon x \to c_{f(x)}$ such that $e_{f(x)} f(v) = \mathrm{id}_{f(x)}$. Thus $f(v)$ is an isomorphism. Since a fully faithful functor is conservative by Proposition 6.3.5, v is an isomorphism. Hence $q\eta i$ is objectwise an isomorphism and therefore a natural isomorphism. □

Corollary 6.3.9. *A functor $f\colon C \to D$ is an equivalence if and only if the induced map*

$$f_*\colon \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$$

is an equivalence.

Proof. If f is an equivalence, it is clear that f_* is an equivalence. Conversely, assume that f_* is an equivalence. By Theorem 6.3.8, it will suffice to show that f is strongly surjective and fully faithful. Consider the following commutative diagram:

$$\begin{array}{ccccc} C^{\simeq} & \xrightarrow{p^*_{[1]}} & \mathrm{Map}([1], C) & \xrightarrow{\mathrm{ev}_0} & C^{\simeq} \\ {\scriptstyle f^{\simeq}}\downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f^{\simeq}} \\ D^{\simeq} & \xrightarrow{p^*_{[1]}} & \mathrm{Map}([1], D) & \xrightarrow{\mathrm{ev}_0} & D^{\simeq}. \end{array}$$

It follows that $f^{\simeq}$ is a retract of f_*, and thus in particular it is an equivalence. In particular, f is strongly surjective. Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Map}([1], C) & \xrightarrow{f_*} & \mathrm{Map}([1], D) \\ {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)}\downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ C^{\simeq} \times C^{\simeq} & \xrightarrow{f^{\simeq} \times f^{\simeq}} & D^{\simeq} \times D^{\simeq}. \end{array}$$

Since the top and bottom maps are equivalences, it follows that the square is a pullback square, and thus f is fully faithful by Remark 6.3.2. □

Remark 6.3.10. The preceding arguments show that the following principles are equivalent, relative to the axioms of the previous chapters:

(1) Universals are functorial, in the sense of Definition 6.1.4.

(2) The functor $\mathrm{Map}([1], -)$ detects equivalences.

(3) A functor is an equivalence if and only if it is fully faithful and strongly surjective.

The implication (2) $\implies$ (1) is Proposition 6.1.6. Under the hypothesis (1), Theorem 6.3.8 proves (3). Finally, the proof of Corollary 6.3.9 uses only the criterion (3), together with the preceding chapters and the elementary description of full faithfulness, to deduce (2). The proof of (2) $\implies$ (1) precedes Axiom L and does not use it.

6.4 Subcategories and conservative functors

The Fundamental Theorem lets us recognize subcategories and full subcategories among arbitrary functors. We then study full faithfulness under passage to functor categories and in adjunctions, and characterize conservative functors.

6.4.1 Subcategories and full subcategories

Lemma 6.4.1. *Let X and Y be animae. Then a map $u\colon X \to Y$ is fully faithful if and only if it is an embedding.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ \mathrm{Map}([1], X) & \xrightarrow{u_*} & \mathrm{Map}([1], Y) \\ \downarrow & & \downarrow \\ X \times X & \xrightarrow{u \times u} & Y \times Y. \end{array}$$

Since the top two vertical maps are equivalences, we see that the outer rectangle is a pullback square if and only if the bottom square is a pullback square. Since the former means that u is an embedding and the latter that u is fully faithful, this finishes the proof. □

Corollary 6.4.2. *A fully faithful functor $f\colon C \to D$ induces an embedding $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$.*

Proof. The equivalences $C(x, y) \simeq D(fx, fy)$ restrict to equivalences between their subanimae of isomorphisms: functors preserve isomorphisms, and f reflects them by Proposition 6.3.5. By the Rezk axiom these are precisely the animae of identifications $x = y$ and $f(x) = f(y)$. Thus $f^{\simeq}$ is an embedding. □

Corollary 6.4.3. *Let $f\colon C \to D$ be a fully faithful functor such that $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ admits a section. Then the functor $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an equivalence.*

Proof. We have seen in Corollary 6.4.2 that the functor $f^{\simeq}$ is an embedding. Since it also admits a section, it follows that $f^{\simeq}$ is an equivalence by Lemma 1.6.24. □

Lemma 6.4.4. *For a functor $f : C \to D$, the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) For every object x in C, the following commutative square is a pullback square:

$$\begin{array}{ccc} C_{/x} & \xrightarrow{f_*} & D_{/f(x)} \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D. \end{array}$$

(3) For every object x in C, the following commutative square is a pullback square:

$$\begin{array}{ccc} C_{x/} & \xrightarrow{f_*} & D_{f(x)/} \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D. \end{array}$$

Proof. We will prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. The vertical two maps in (2) are both right fibrations by Theorem 4.3.3, hence by Corollary 6.1.13 the square is a pullback square if and only if it induces an equivalence on all fibers. But for an object y of C, the induced map on fibers is the map $C(y,x) \to D(f(y), f(x))$, so this condition is satisfied if and only if f is fully faithful. □

Proposition 6.4.5. *Consider a functor $f : C \to D$. Then the following conditions are equivalent:*

(1) The functor f is an embedding, in the sense of Definition 1.6.19;

(2) The induced map $f_* : M_C := \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$ is an embedding and the induced map $C \to \langle M_C \rangle_D$ is an equivalence;

(3) The functor f exhibits C as a subcategory of D, in the sense of Definition 3.1.2.

Proof. It is clear that (2) implies (3), and we showed in Lemma 3.1.5 that (3) implies (1). To see that (1) implies (2), assume that f is an embedding. Then also the induced map $f_* : \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$ is an embedding. To show that the functor $C \to \langle M_C \rangle_D$ is an equivalence, it suffices by Corollary 6.3.9 to show that the induced functor $\mathrm{Map}([1], C) \to \mathrm{Map}([1], \langle M_C \rangle_D)$ is an equivalence. But this is clear since under the fixed identification $\mathrm{Map}([1], \langle M_C \rangle_D) \xrightarrow{\sim} M_C = \mathrm{Map}([1], C)$ this map corresponds to the identity map on $\mathrm{Map}([1], C)$. □

Proposition 6.4.6. *Consider a functor $f : C \to D$. Then the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) The functor $f^{\simeq} : C^{\simeq} \to D^{\simeq}$ is an embedding and the canonical functor $C \to \langle C^{\simeq} \subseteq D \rangle$ is an equivalence;

(3) The functor f exhibits C as a full subcategory of D, in the sense of Definition 3.2.2.

Proof. First note that in all three cases the induced map $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an embedding, where in case (1) we use Corollary 6.4.2.

To prove that (1) implies (2), assume that f is fully faithful. We already argued that $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an embedding. To show that $C \to \langle C^{\simeq} \subseteq D\rangle$ is an equivalence, it suffices by Corollary 6.3.9 to show that the induced functor $\mathrm{Map}([1], C) \to \mathrm{Map}([1], \langle C^{\simeq} \subseteq D\rangle)$ is an equivalence. By Corollary 3.2.8, this is equivalent to the condition that the top square in the following commutative diagram is a pullback square:

$$\begin{array}{ccc}
\mathrm{Map}([1], C) & \xrightarrow{f_*} & \mathrm{Map}([1], D) \\
\downarrow & & \downarrow \\
\mathrm{Map}(([1])^{\simeq}, C^{\simeq}) & \xrightarrow{(f^{\simeq})_*} & \mathrm{Map}(([1])^{\simeq}, D^{\simeq}) \\
{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\
C^{\simeq} \times C^{\simeq} & \xrightarrow{f^{\simeq} \times f^{\simeq}} & D^{\simeq} \times D^{\simeq}.
\end{array}$$

Since the bottom two vertical maps are equivalences by Axiom C.1, the bottom square is always a pullback square, hence by the pasting law for pullback squares it will suffice to show that the outer square is a pullback square. But this is precisely the full faithfulness of f.

The fact that (2) implies (3) follows from Proposition 3.2.7, which shows that the functor $\langle C^{\simeq} \subseteq D\rangle \to D$ is a full subcategory, together with the equivalence $C \xrightarrow{\sim} \langle C^{\simeq} \subseteq D\rangle$ from (2).

Finally, condition (3) demands that for every category B the commutative square

$$\begin{array}{ccc}
\mathrm{Map}(B, C) & \xrightarrow{f_*} & \mathrm{Map}(B, D) \\
\downarrow & & \downarrow \\
\mathrm{Map}(B^{\simeq}, C^{\simeq}) & \xrightarrow{(f^{\simeq})_*} & \mathrm{Map}(B^{\simeq}, D^{\simeq})
\end{array}$$

is a pullback square, and as explained above this square for $B = [1]$ precisely says that f is fully faithful, giving (1). □

6.4.2 Full faithfulness and functor categories

Theorem 6.4.7. *Let $f\colon C \to D$ be any functor. Then the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) The functor f induces a pullback square

$$\begin{array}{ccc}
\mathrm{Fun}([1], C) & \xrightarrow{f_*} & \mathrm{Fun}([1], D) \\
{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)}\downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\
C \times C & \xrightarrow{f \times f} & D \times D.
\end{array}$$

(3) For every category E, the induced functor $f_*\colon \mathrm{Fun}(E, C) \to \mathrm{Fun}(E, D)$ is fully faithful.

Proof. We first show that (2) and (3) are equivalent. By Proposition 1.6.3, we see that condition (2) is equivalent to the condition that for every category E the commutative square

$$\begin{array}{ccc} \mathrm{Map}(E, \mathrm{Fun}([1], C)) & \longrightarrow & \mathrm{Map}(E, \mathrm{Fun}([1], D)) \\ \downarrow & & \downarrow \\ \mathrm{Map}(E, C \times C) & \longrightarrow & \mathrm{Map}(E, D \times D) \end{array}$$

is a pullback square. Notice that this square is isomorphic to the following square:

$$\begin{array}{ccc} \mathrm{Map}([1], \mathrm{Fun}(E, C)) & \longrightarrow & \mathrm{Map}([1], \mathrm{Fun}(E, D)) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(E, C)^{\simeq} \times \mathrm{Fun}(E, C)^{\simeq} & \longrightarrow & \mathrm{Fun}(E, D)^{\simeq} \times \mathrm{Fun}(E, D)^{\simeq}. \end{array}$$

But this square is a pullback square if and only if the functor $f_* \colon \mathrm{Fun}(E, C) \to \mathrm{Fun}(E, D)$ is fully faithful, which is condition (3).

Condition (2) implies (1) by taking groupoid cores. Conversely, assume (1). By Proposition 6.4.6, f exhibits C as a full subcategory of D.

Let $J = [1] \sqcup [1]$ be the two edges $\{0, 1\} \times [1]$ of $[1] \times [1]$. The inclusion $J \to [1] \times [1]$ is an equivalence on groupoid cores. Apply the defining mapping-anima pullback property of a full subcategory to J and $[1] \times [1]$. Pasting these pullbacks gives

$$\begin{array}{ccc} \mathrm{Map}([1] \times [1], C) & \longrightarrow & \mathrm{Map}([1] \times [1], D) \\ \downarrow & & \downarrow \\ \mathrm{Map}(J, C) & \longrightarrow & \mathrm{Map}(J, D). \end{array} \tag{6.10}$$

Equivalently, a square in D with its two indicated edges in C lifts uniquely to a square in C, since those edges already specify all four vertices.

By currying and preservation of pullbacks by mapping animae, this says that applying $\mathrm{Map}([1], -)$ to the comparison functor

$$\mathrm{Fun}([1], C) \longrightarrow (C \times C) \times_{D \times D} \mathrm{Fun}([1], D)$$

gives an equivalence. Detection by $\mathrm{Map}([1], -)$ makes the comparison an equivalence, proving (2). □

Corollary 6.4.8. *Let $u \colon A \hookrightarrow C$ be a full subcategory, in the sense of Definition 3.2.2. Then for every category E, the induced functor $\mathrm{Fun}(E, u) \colon \mathrm{Fun}(E, A) \hookrightarrow \mathrm{Fun}(E, C)$ is again a full subcategory, with essential image those functors $F \colon E \to C$ satisfying the property that, for any object x of E, the object $F(x)$ of C is contained in A.*

Proof. The functor u is fully faithful by Proposition 6.4.6, hence so is $\mathrm{Fun}(E, u)$ by Theorem 6.4.7. Another application of Proposition 6.4.6 thus shows that $\mathrm{Fun}(E, A)$ is a full subcategory of $\mathrm{Fun}(E, C)$. The characterization of its essential image in $\mathrm{Fun}(E, C)$ is then a consequence of the following pullback square from Corollary 3.2.8:

$$\begin{array}{ccc} \mathrm{Map}(E, A) & \longrightarrow & \mathrm{Map}(E, C) \\ \downarrow & & \downarrow \\ \mathrm{Map}(E^{\simeq}, A^{\simeq}) & \longrightarrow & \mathrm{Map}(E^{\simeq}, C^{\simeq}). \end{array}$$

□

Remark 6.4.9. For every category E, we have a pullback square

$$\begin{array}{ccc} \mathrm{Fun}(E, \langle \Gamma \subseteq C \rangle) & \longrightarrow & \mathrm{Fun}(E, C) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(E^{\simeq}, \langle \Gamma \subseteq C \rangle) & \longrightarrow & \mathrm{Fun}(E^{\simeq}, C). \end{array}$$

Indeed, consider the comparison map from the upper-left corner to the pullback. The defining pullback square for the full subcategory $\langle \Gamma \subseteq C \rangle$ shows that this comparison induces an equivalence on groupoid cores, and hence it is strongly surjective. Both horizontal functors in the displayed square are fully faithful by Theorem 6.4.7. The cancellation property of fully faithful functors from Corollary 6.3.3 therefore shows that the comparison map is fully faithful. The fundamental theorem of category theory, Theorem 6.3.8, now implies that it is an equivalence.

6.4.3 Fully faithful adjoints

Proposition 6.4.10. *Let $f\colon C \to D$ be a functor and let $s\colon D \to C$ be a left (resp. right) adjoint of f. The following conditions are equivalent:*

(i) The functor s is a left (resp. right) adjoint section of f.

(ii) The functor s is fully faithful.

(iii) For every object y of D, the unit map $y \to f(s(y))$ (resp. the counit map $f(s(y)) \to y$) is an isomorphism.

Proof. We prove the case in which s is right adjoint to f; the other case is dual. Let $\eta\colon \mathrm{id}_C \to sf$ and $\varepsilon\colon fs \to \mathrm{id}_D$ denote the unit and counit. Conditions (i) and (iii) are equivalent by the objectwise criterion for natural isomorphisms, Proposition 3.4.6.

Given objects x and y of D, consider the counit map $\varepsilon_x\colon fsx \to x$. By the triangle identity, the composite

$$sx \xrightarrow{\eta_{sx}} sfsx \xrightarrow{s\varepsilon_x} sx$$

is the identity and we may now consider the following commutative diagram:

$$\begin{array}{ccc} D(x,y) & \xrightarrow{s_*} & C(sx,sy) \\ \downarrow{\scriptstyle (\varepsilon_x)^*} & & \downarrow{\scriptstyle (s\varepsilon_x)^*} \\ D(fsx,y) & \xrightarrow{s_*} & C(sfsx,sy) \\ & \searrow^{\simeq} & \downarrow{\scriptstyle (\eta_{sx})^*} \\ & & C(sx,sy). \end{array}$$

As discussed in the proof of Proposition 4.4.7, the diagonal composite

$$D(fsx,y) \xrightarrow{s_*} C(sfsx,sy) \xrightarrow{\eta_{sx}^*} C(sx,sy)$$

is precisely the adjunction equivalence. This shows that the map $(\varepsilon_x)^*$ is an equivalence for all x and y if and only if s_* is an equivalence for all x and y. The latter property expresses precisely the property for s to be fully faithful. The first property expresses the invertibility of ε_x for all x (Exercise 2.6.2). This shows that (ii) and (iii) are equivalent. □

By virtue of Lemma 4.4.11, Proposition 6.4.10 immediately implies:

Corollary 6.4.11. *The functor $p^*_{[1]} \colon C \to \mathrm{Fun}([1], C)$ is fully faithful.* □

Corollary 6.4.12. *The inclusion functor $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1], C)$ is fully faithful.*

Proof. Let $i \colon C \to \mathrm{Iso}(C)$ be the equivalence from the Rezk axiom. The composite $\pi_{\mathrm{Iso}} i$ is the constant-diagram functor $p^*_{[1]}$, which is fully faithful by the previous corollary. Given objects a and b of $\mathrm{Iso}(C)$, choose objects x and y of C and isomorphisms $i(x) \xrightarrow{\sim} a$ and $i(y) \xrightarrow{\sim} b$. Conjugation by these isomorphisms identifies the map

$$\mathrm{Iso}(C)(a, b) \longrightarrow \mathrm{Fun}([1], C)(\pi_{\mathrm{Iso}}(a), \pi_{\mathrm{Iso}}(b))$$

with the map on hom animae induced by $p^*_{[1]}$ from x to y, and the latter is an equivalence. Thus π_{Iso} is fully faithful. □

6.4.4 Conservative functors

We introduce the notion of a *conservative* functor and prove some basic properties about them.

Definition 6.4.13 (Conservative functor)**.** A functor $f \colon C \to D$ is said to be *conservative* if the following condition is satisfied: for every pair of objects x and y of C and every morphism $u \colon x \to y$ in C, if the image $f(u) \colon f(x) \to f(y)$ is invertible in D, then u is already invertible in C.

Conservative functors may be characterized in a variety of different ways:

Proposition 6.4.14. *Let $f \colon C \to D$ be any functor. The following conditions are equivalent:*

(i) The functor f is conservative.

(ii) The commutative square

$$\begin{array}{ccc} \mathrm{Iso}(C)^{\simeq} & \xrightarrow{(\pi_{\mathrm{Iso}})^{\simeq}} & \mathrm{Map}([1], C) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{Iso}(D)^{\simeq} & \xrightarrow{(\pi_{\mathrm{Iso}})^{\simeq}} & \mathrm{Map}([1], D) \end{array}$$

is a pullback square.

(iii) The commutative square

$$\begin{array}{ccc} C^{\simeq} = \mathrm{Map}(*, C) & \xrightarrow{(p_{[1]})^*} & \mathrm{Map}([1], C) \\ \downarrow & & \downarrow{\scriptstyle f_*} \\ D^{\simeq} = \mathrm{Map}(*, D) & \xrightarrow{(p_{[1]})^*} & \mathrm{Map}([1], D) \end{array}$$

is a pullback square.

(iv) For any functor $A \to D$ with A an anima, the pullback $A \times_D C$ is an anima;

(v) The pullback $D^{\simeq} \times_D C$ is an anima;

(vi) A given commutative square

$$\begin{array}{ccc} X & \longrightarrow & C \\ \downarrow & & \downarrow f \\ Y & \longrightarrow & D \end{array}$$

with X and Y animae is a pullback square whenever the associated square

$$\begin{array}{ccc} X & \longrightarrow & C^{\simeq} \\ \downarrow & & \downarrow f^{\simeq} \\ Y & \longrightarrow & D^{\simeq} \end{array}$$

is a pullback square.

(vii) The square

$$\begin{array}{ccc} C^{\simeq} & \longrightarrow & C \\ {\scriptstyle f^{\simeq}}\downarrow & & \downarrow{\scriptstyle f} \\ D^{\simeq} & \longrightarrow & D \end{array}$$

is a pullback square.

Proof. We first show the equivalence between (i) and (ii). Recall from Lemma 2.3.3 that the map

$$(\pi_{\mathrm{Iso}})^{\simeq} \colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Map}([1], C)$$

is an embedding for every category C. It thus follows from Lemma 1.6.26 that also the comparison map

$$\mathrm{Iso}(C)^{\simeq} \to \mathrm{Iso}(D)^{\simeq} \times_{\mathrm{Map}([1],D)} \mathrm{Map}([1], C)$$

is an embedding. It thus follows from Lemma 1.6.24 that it is an equivalence if and only if it admits a section. Since this is a map of animae with proposition-valued fibers, the latter is equivalent to the existence of a lift for every object of the target, by applying this condition to the generic object in the context of that target anima. But since the objects in the target of this map are precisely those morphisms $u \colon x \to y$ in C whose image in D is invertible, this lifting condition is precisely the condition on f that every such morphism u in fact comes from an isomorphism in C, which is the definition of conservativity.

The equivalence between (ii) and (iii) is immediate from the equivalence $C^{\simeq} \xrightarrow{\sim} \mathrm{Iso}(C)^{\simeq}$ from the Rezk axiom, Axiom F. Conditions (iv) and (v) are equivalent: every map from an anima to D factors through $D^{\simeq}$, and animae are closed under pullbacks. Under these conditions, $Y \times_D C$ is an anima whenever Y is an anima. Since groupoid cores preserve pullbacks, there are equivalences

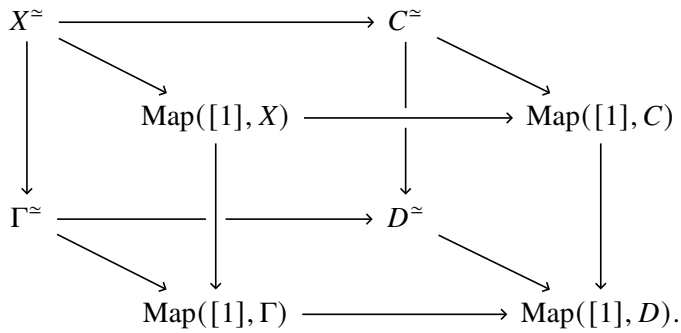
$$Y \times_D C \simeq (Y \times_D C)^{\simeq} \simeq Y \times_{D^{\simeq}} C^{\simeq}.$$

This proves (vi). The implication (vi) $\implies$ (vii) is clear. To see that (vii) implies (iii), we use that the functor $\mathrm{Map}([1], -)$ preserves pullback squares, and apply the equivalence $C^{\simeq} \xrightarrow{\sim} \mathrm{Map}([1], C^{\simeq})$ coming from the fact that $C^{\simeq}$ is an anima.

It thus remains to show that (iii) implies (v). Consider a pullback square

$$\begin{array}{ccc} X = \Gamma \times_D C & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \Gamma & \xrightarrow{\;b\;} & D \end{array}$$

with Γ an anima. We must show that X is an anima as well. To this end, consider the following commutative diagram:



The back and front faces are pullback squares, and the right face is a pullback square by the hypothesis from (iii). We conclude from the pasting lemma of pullback squares that the left face

$$\begin{array}{ccc} X^{\simeq} & \longrightarrow & \mathrm{Map}([1], X) \\ \downarrow & & \downarrow \\ \Gamma^{\simeq} & \xrightarrow{\simeq} & \mathrm{Map}([1], \Gamma) \end{array}$$

is a pullback square. Since the bottom map $\Gamma^{\simeq} \to \mathrm{Map}([1], \Gamma)$ is an equivalence, it follows that also the top map $X^{\simeq} \to \mathrm{Map}([1], X)$ is an equivalence, and thus we conclude from part (5) of Proposition 3.4.4 that X is an anima as desired. □

Lemma 6.4.15. *Compositions of conservative functors are conservative.*

Proof. This is clear from the definition. □

Lemma 6.4.16. *Let $f\colon C \to D$ be a functor admitting a retraction $r\colon D \to C$, i.e. we have $rf = \mathrm{id}_C$. Then f is conservative.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Iso}(C)^{\simeq} & \xrightarrow{(\pi_{\mathrm{Iso}})^{\simeq}} & \mathrm{Map}([1], C) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{Iso}(D)^{\simeq} & \xrightarrow{(\pi_{\mathrm{Iso}})^{\simeq}} & \mathrm{Map}([1], D) \\ {\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle r_*} \\ \mathrm{Iso}(C)^{\simeq} & \xrightarrow{(\pi_{\mathrm{Iso}})^{\simeq}} & \mathrm{Map}([1], C). \end{array}$$

Since the horizontal maps are embeddings and the outer square is clearly a pullback square, it follows from Lemma 1.6.28 that the top square is a pullback square. The claim thus follows from Proposition 6.4.14. □

Corollary 6.4.17. *For every category C, the functor $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}([1], C) \to C \times C$ is conservative.*

Proof. We will check condition (iii) from Proposition 6.4.14. Under the currying-uncurrying equivalence from Lemma 1.7.12, this translates to showing that the square

$$\begin{array}{ccc} \mathrm{Map}([1], C) & \longrightarrow & \mathrm{Map}([1] \times [1], C) \\ \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} & & \downarrow \\ C^{\simeq} \times C^{\simeq} & \longrightarrow & \mathrm{Map}([1] \sqcup [1], C) \end{array}$$

is a pullback square. For this, it will in turn suffice to express the category $[1]$ as a pushout of $[0] \sqcup [0]$ and $[1] \times [1]$ along $[1] \sqcup [1]$. For this, we consider the following commutative square:

$$\begin{array}{ccccc} [0] \sqcup [0] & \longrightarrow & [1] \sqcup [1] & \longrightarrow & [1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} \\ [1] \sqcup [1] & \longrightarrow & [2] \sqcup [2] & \longrightarrow & [1] \times [1] \\ \downarrow & & & & \downarrow \\ [0] \sqcup [0] & & \longrightarrow & & [1]. \end{array}$$

In this diagram, both of the top two squares are pushouts as a consequence of the Commutative Square Axiom, Axiom D, where for the top right square we use the dual of Lemma 1.6.13. Since the outer square is vertically degenerate and hence also a pushout square by the dual of Lemma 1.6.12, it follows from the pasting law for pushout squares that the bottom rectangle is a pushout square, as desired. □

Lemma 6.4.18. *Every left (resp. right) fibration is conservative.*

Proof. In light of Corollary 4.2.3, this follows immediately from characterization (iii) in Proposition 6.4.14. □

Lemma 6.4.19. *Let $f : C \to S$ and $g : D \to S$ be functors and assume that g is conservative. Then the canonical inclusion map*

$$\mathrm{Map}_S(C, D) \to \mathrm{Fun}_S(C, D)$$

from Definition 1.9.2 is an equivalence.

Proof. The induced functor $g_* : \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, S)$ is conservative. Indeed, if $g_*\alpha$ is a natural isomorphism, then conservativity of g implies that every component of α is invertible. By Proposition 3.4.6, α is a natural isomorphism. It thus follows from Proposition 6.4.14 that we have a pullback square of the form

$$\begin{array}{ccc} \mathrm{Map}(C, D) & \hookrightarrow & \mathrm{Fun}(C, D) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathrm{Map}(C, S) & \hookrightarrow & \mathrm{Fun}(C, S). \end{array}$$

Passing to vertical fibers over f, it follows that the canonical map $\mathrm{Map}_S(C, D) \to \mathrm{Fun}_S(C, D)$ is an equivalence, as claimed. □

Corollary 6.4.20. *For categories C and D, the canonical embedding $\iota_C : C^{\simeq} \to C$ induces a conservative functor $\iota_C^* : \mathrm{Fun}(C, D) \to \mathrm{Fun}(C^{\simeq}, D)$.*

Proof. If a natural transformation becomes invertible after restriction to $C^{\simeq}$, each of its components is invertible. It is therefore invertible by Proposition 3.4.6. □

6.5 Pointwise and fiberwise criteria for fibrations

We apply the pointwise adjunction criterion to recognize fibrations and then study their local and fiberwise behavior.

Proposition 6.5.1. *Let $f : C \to D$ be a functor. Then f is a cartesian fibration if and only if, for every object x of C and every morphism $v : y \to f(x)$ in D, there exists an f-cartesian lift $u : \tilde{y} \to x$ of v. Dually, f is a cocartesian fibration if and only if every morphism $v : f(x) \to y$ admits an f-cocartesian lift with source x.*

Proof. The forward implication follows from the dual of Lemma 4.5.7. For the converse, write $F : \mathrm{Fun}([1], C) \to D \mathbin{\vec{\times}}_f C$ for the directed evaluation map $\vec{\mathrm{ev}}_1^f$. By Theorem 6.2.1, it suffices to show that, for every object (v, x) of $D \mathbin{\vec{\times}}_f C$, the relative slice $\mathrm{Fun}([1], C)_{/(v,x)}$ admits a terminal object.

Work with the universal object (v, x) in context $(D \mathbin{\vec{\times}}_f C)^{\simeq}$. An object of this relative slice consists of a morphism $a : z_0 \to z_1$ in C, morphisms $\beta : f(z_0) \to y$ and $\gamma : z_1 \to x$, and an identification

$$v\beta = f(\gamma) f(a).$$

Choose an f-cartesian lift $u : \tilde{y} \to x$ of v. The specified identification $F(u) = (v, x)$ defines an object of the relative slice. A morphism from the object above to this chosen object amounts to a morphism $\delta : z_0 \to \tilde{y}$ satisfying

$$u\delta = \gamma a \qquad \text{and} \qquad f(\delta) = \beta,$$

compatibly with the displayed identification. Since u is f-cartesian, the defining pullback square applied to z_0 says precisely that the anima of such choices of δ is contractible. Hence the chosen object is terminal. We conclude that F admits a right adjoint. By the refined form of Theorem 6.2.1 recorded in Remark 6.2.2, these chosen terminal objects assemble into a right adjoint G of F whose counit at (v, x) is the specified identification $F(u) = (v, x)$. Hence the counit is a natural isomorphism, so G is a right adjoint section of F. Therefore f is a cartesian fibration. The cocartesian statement is dual. □

Lemma 6.5.2. *Let $f : A \to B$ be a functor and let $\alpha : a \to a'$ be a morphism in A. Then α is f-cocartesian if and only if the square*

$$\begin{array}{ccc} A_{a'/} & \xrightarrow{\alpha^*} & A_{a/} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B_{f(a')/} & \xrightarrow{f(\alpha)^*} & B_{f(a)/} \end{array}$$

is a pullback square.

Proof. The square is a pullback if and only if the canonical map $A_{a'/} \to A_{a/} \times_{B_{f(a)/}} B_{f(a')/}$ is an equivalence. This map fits into a commutative triangle

$$\begin{array}{ccc} A_{a'/} & \longrightarrow & A_{a/} \times_{B_{f(a)/}} B_{f(a')/} \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

of left fibrations over A. By Lemma 6.1.12, it is an equivalence if and only if its map on fibers

$$A(a', x) \to A(a, x) \times_{B(f(a), f(x))} B(f(a'), f(x))$$

is an equivalence for every object x of A. By Definition 4.5.5, this is precisely the condition that α is f-cocartesian. □

Remark 6.5.3. In more informal terms, the definition says that a morphism $\alpha\colon a \to a'$ is f-cocartesian if and only if for any morphism $v\colon a \to x$ in A and morphism $w\colon f(a') \to f(x)$ in B such that the triangle

$$\begin{array}{ccccc} & & f(a') & & \\ & {\scriptstyle f(\alpha)}\nearrow & & \searrow{\scriptstyle w} & \\ f(a) & & \xrightarrow{\quad f(v) \quad} & & f(x) \end{array}$$

commutes, there exists a unique morphism $\tilde{w}\colon a' \to x$ such that the triangle

$$\begin{array}{ccccc} & & a' & & \\ & {\scriptstyle \alpha}\nearrow & & \searrow{\scriptstyle \tilde{w}} & \\ a & & \xrightarrow{\quad v \quad} & & x \end{array}$$

commutes and is sent to the previous triangle under f.

Proposition 6.5.4. *Let $p\colon C \to S$ be a locally cocartesian fibration. The following conditions are equivalent:*

(1) The functor p is a cocartesian fibration;

(2) Every locally p-cocartesian morphism of C is p-cocartesian;

(3) Locally p-cocartesian morphisms are closed under composition: if $f\colon x \to y$ and $g\colon y \to z$ are locally p-cocartesian morphisms in C then so is $gf\colon x \to z$.

Proof. If p is a cocartesian fibration, then condition (2) holds: indeed, if $f\colon x \to y$ is a locally p-cocartesian morphism, then there is a unique p-cocartesian lift $g\colon x \to y'$ of $p(f)$. Since both f and g are locally p-cocartesian lifts of $p(f)$ with the same source, there is a unique isomorphism $u\colon y' \to y$ in the fiber of $p(y)$ such that $f = ug$. In particular, f is the composition of two p-cocartesian morphisms, hence is p-cocartesian as well. Since p-cocartesian morphisms are stable under composition, it is clear that condition (3) follows from condition (2). It remains to prove that condition (1) follows from condition (3). Let us assume that condition (3) holds and that

$f\colon x \to y$ is a locally p-cocartesian morphism. Definition 4.5.5 tells us that, in order to prove that f is p-cocartesian, it suffices to prove that, for any object z in C,

$$\begin{array}{ccc} C(y,z) & \xrightarrow{-\circ f} & C(x,z) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S(p(y),p(z)) & \xrightarrow{-\circ p(f)} & S(p(x),p(z)) \end{array}$$

is a pullback square of animae. This means that we have to check that the map induced by this pullback square

$$C(y,z) \to S(p(y),p(z)) \times_{S(p(x),p(z))} C(x,z)$$

is an equivalence. In other words, by virtue of Proposition 1.3.1, if we consider any morphism $g\colon x \to z$ in C as well as a morphism $v\colon p(y) \to p(z)$ in S such that $p(g)$ is the composition of v and $p(f)$, we have to prove that there is a unique morphism $u\colon y \to z$ in C such that $p(u) = v$ and such that g is the composition of u and f. We define $u'\colon y \to z'$ as the unique locally p-cocartesian lift of v with source y. Since $u'f$ is a locally p-cocartesian lift of $vp(f)$, there is a unique morphism $h\colon z' \to z$ in the fiber over $p(z)$ such that $hu'f$ is identified with g. Define $u := hu'$. If we have another morphism of the form $w\colon y \to z$ in C such that $wf = g$, and such that $p(w) = v$, then we have a unique morphism $k\colon z' \to z$ in the fiber over $p(z)$ such that $ku' = w$. In particular, we must have

$$h\,u'\,f = g \quad \text{and} \quad k\,u'\,f = g$$

and the universal property of $u'\,f$ as locally p-cocartesian lift of $v\,p(f)$ implies that k and h must agree as terms of $C(z', z)$. Therefore, $u = w$ must hold true. We have proved that the locally cocartesian lifts are cocartesian. The dual of Proposition 6.5.1 therefore shows that p is a cocartesian fibration. □

Proposition 6.5.5 (Interval test for local fibrations)**.** *A functor $p\colon E \to S$ is a locally cocartesian fibration if and only if, for every anima A and functor $A \times [1] \to S$, its pullback to $A \times [1]$ is a cocartesian fibration.*

Proof. Suppose that all the indicated pullbacks are cocartesian fibrations. Apply this to the evaluation functor $\mathrm{Map}([1], S) \times [1] \to S$. In context $\mathrm{Map}([1], S)$, cocartesian lifts over $0 \to 1$ give locally p-cocartesian lifts of the universal morphism in S. Hence p is locally cocartesian. Conversely, suppose that p has local cocartesian lifts and put $p'\colon E' \to A \times [1]$ for such a pullback. Local cocartesian lifts are stable under base change. Up to the identifications in A, a morphism in $A \times [1]$ is either invertible or goes from 0 to 1 in the second factor. A local lift of an invertible morphism is invertible by Proposition 4.8.7. A local lift ending over 1 is cocartesian: every subsequent base morphism is invertible, so the local universal property gives its full universal property after transport along that identification. This is the same argument as Lemma 4.8.8. Hence all these local lifts are cocartesian. The recognition principle gives that p' is a cocartesian fibration. □

Next, we show that various properties we have introduced can be checked at the level of objects. To start with, recall that a functor $f\colon C \to D$ is a left fibration if and only if the commutative square

$$\begin{array}{ccc} \mathrm{Fun}([1], C) & \xrightarrow{f_*} & \mathrm{Fun}([1], D) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ C & \xrightarrow{f} & D \end{array}$$

is a pullback square, and dually for right fibrations. We may interpret this as saying that every morphism in D admits a unique lift to C with prescribed source, functorially in the morphism in D. The next lemma says that it suffices to find these lifts objectwise:

Lemma 6.5.6. *Let $f\colon C \to D$ be a functor. Then f is a left (resp. right) fibration if and only if the square*

$$\begin{array}{ccc} \mathrm{Map}([1], C) & \xrightarrow{f_*} & \mathrm{Map}([1], D) \\ {\scriptstyle \mathrm{ev}_i}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_i} \\ C^{\simeq} & \xrightarrow{f^{\simeq}} & D^{\simeq} \end{array}$$

is a pullback for $i = 0$ (resp. $i = 1$).

Proof. By symmetry, it suffices to treat the case of left fibrations. One direction is clear by Lemma 4.2.2, so assume conversely that the square in the statement is a pullback. Write

$$\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], C) \to C \mathbin{\vec{\times}}_f D$$

for the directed evaluation map. By assumption, its underlying map on groupoid cores

$$(\vec{\mathrm{ev}}_0^f)^{\simeq}\colon \mathrm{Map}([1], C) = \mathrm{Fun}([1], C)^{\simeq} \to C^{\simeq} \times_{D^{\simeq}} \mathrm{Fun}([1], D)^{\simeq} = (C \mathbin{\vec{\times}}_f D)^{\simeq}$$

is an equivalence, and in particular $\vec{\mathrm{ev}}_0^f$ is strongly surjective. By Theorem 6.3.8, it remains to show that it is fully faithful.

The pullback hypothesis also says that for every object a of C and every morphism $\gamma\colon f(a) \to d$ in D, the anima of lifts of γ with source a is contractible. Consider objects $u\colon x_0 \to x_1$ and $v\colon y_0 \to y_1$ of $\mathrm{Fun}([1], C)$. We show that the induced map

$$\mathrm{Hom}_{\mathrm{Fun}([1],C)}(u, v) \to \mathrm{Hom}_{C \mathbin{\vec{\times}}_f D}((x_0, f(u)), (y_0, f(v)))$$

is an equivalence by showing that each of its fibers is contractible. Fix a morphism in the target. It consists of a morphism $\alpha\colon x_0 \to y_0$ in C, a morphism $\beta\colon f(x_1) \to f(y_1)$ in D, and a commutative square

$$\begin{array}{ccc} f(x_0) & \xrightarrow{f(u)} & f(x_1) \\ {\scriptstyle f(\alpha)}\downarrow & & \downarrow{\scriptstyle \beta} \\ f(y_0) & \xrightarrow{f(v)} & f(y_1). \end{array}$$

The anima of lifts of β with source x_1 is contractible, so choose a lift $\beta'\colon x_1 \to z$. The two composites $\beta' u$ and $v\alpha$ are lifts with source x_0 of the same morphism $\beta f(u) = f(v) f(\alpha)$. Contractibility of the corresponding lift anima supplies a unique isomorphism $i\colon z \xrightarrow{\sim} y_1$ over $\mathrm{id}_{f(y_1)}$ such that $i\beta' u = v\alpha$. The composite $\overline{\beta} = i\beta'$ therefore defines a lift of the chosen target morphism to a morphism $u \to v$ in $\mathrm{Fun}([1], C)$. Both choices were made in contractible animae, so the anima of such lifts is contractible. Thus $\vec{\mathrm{ev}}_0^f$ is fully faithful and hence an equivalence. □

Proposition 6.5.7. *Let $f\colon C \to D$ be a right fibration, let c be an object of C, and set $d := f(c)$. Then the induced functor*

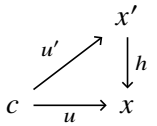
$$q\colon C_{c/} \to D_{d/}, \qquad (x, u\colon c \to x) \mapsto (f(x), f(u)\colon d \to f(x)),$$

is a right fibration as well.

Proof. We apply the right-fibration criterion of Lemma 6.5.6. Fix an object (x, u) of $C_{c/}$ and write $y = f(x)$ and $v = f(u)\colon d \to y$. A morphism in $D_{d/}$ with codomain (y, v) consists of a diagram

$$d \xrightarrow{v'} y' \xrightarrow{k} y$$

together with an identification $t : kv' = v$. Since f is a right fibration, the anima of lifts of k with codomain x is contractible; choose such a lift $h\colon x' \to x$. Similarly, choose a lift $u_0\colon c_0 \to x'$ of v' with codomain x'. Then hu_0 and u are lifts with codomain x of the morphisms kv' and v, identified by t. Contractibility of the corresponding lift anima supplies a unique isomorphism $i\colon c \xrightarrow{\sim} c_0$ over id_d such that $hu_0 i = u$. Setting $u' = u_0 i$ gives a commutative triangle



whose image under f is the prescribed triangle in D. Every choice made above belongs to a contractible anima, so the anima of such lifts in $C_{c/}$ is contractible. The objectwise criterion now shows that q is a right fibration. □

Corollary 6.5.8 (Pointwise cocartesian transformations)**.** *Let $p\colon E \to C$ be a cocartesian fibration and let I be a category. Then*

$$p_*\colon \mathrm{Fun}(I, E) \longrightarrow \mathrm{Fun}(I, C)$$

is a cocartesian fibration. A natural transformation $\alpha\colon X \to Y$ in $\mathrm{Fun}(I, E)$ is p_*-cocartesian if and only if its component $\alpha_i\colon X(i) \to Y(i)$ is p-cocartesian for every object i of I.

Proof. The first assertion is Proposition 4.5.13. Under the canonical identifications

$$\mathrm{Fun}([1], \mathrm{Fun}(I, E)) \simeq \mathrm{Fun}(I, \mathrm{Fun}([1], E))$$

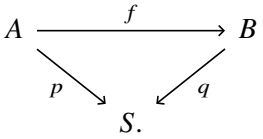
and

$$\mathrm{Fun}(I, E) \mathbin{\vec{\times}_{p_*}} \mathrm{Fun}(I, C) \simeq \mathrm{Fun}(I, E \mathbin{\vec{\times}_p} C),$$

the proof of that proposition identifies the left adjoint section of the directed evaluation map for p_* with the functor obtained by applying $\mathrm{Fun}(I, -)$ to the left adjoint section for p. Consequently, the comparison morphism ε_α associated with α is the natural transformation whose component at i is the comparison morphism ε_{α_i} associated with α_i. By the definition of cocartesian morphisms, α is p_*-cocartesian if and only if ε_α is a natural isomorphism. By Corollary 6.4.20, this holds if and only if every ε_{α_i} is an isomorphism, which is equivalent to every α_i being p-cocartesian. □

We may use this theorem to give a more concrete description of when a functor between cocartesian fibrations is a cocartesian functor.

Corollary 6.5.9. *Let $p\colon A \to S$ and $q\colon B \to S$ be two cocartesian fibrations, and consider a commutative triangle*



Then f is a cocartesian functor over S if and only if for every morphism $v\colon x \to y$ in S and every object a of $A(x)$, the map $\overline{v}\colon v_!(f(a)) \to f(v_!(a))$ is an isomorphism.

Proof. By definition, f is a cocartesian functor if and only if the natural transformation

$$\mathrm{BC}_!\colon \mathrm{lift}_0^q \vec{f} \to f_* \mathrm{lift}_0^p$$

constructed in Construction 4.7.1 is a natural isomorphism. By Corollary 6.4.20, this may be tested pointwise. Thus, it is equivalent to asking that for every morphism $v\colon x \to y$ in S and every object a of $A(x)$, the morphism $\mathrm{BC}_!(a, v)$ in $\mathrm{Fun}([1], B)$ is an isomorphism. This morphism takes the form of a commutative square

$$\begin{array}{ccc} f(a) & = & f(a) \\ {\scriptstyle \mathrm{lift}_0^q(f(a),v)}\downarrow & & \downarrow{\scriptstyle f(\mathrm{lift}_0^p(a,v))} \\ v_!(f(a)) & \xrightarrow{\overline{v}} & f(v_!(a)), \end{array}$$

and by Corollary 6.4.17 this is in turn equivalent to asking both horizontal maps to be isomorphisms. Since the top horizontal morphism is an identity map and thus always an isomorphism, this finishes the proof. □

Proposition 6.5.10. *Let $p\colon A \to S$ and $q\colon B \to S$ be cocartesian fibrations, and let $f\colon A \to B$ be a functor over S. Then the following conditions are equivalent:*

(1) The functor f is a cocartesian functor over S;

(2) For any functor $T \to S$, the induced functor $f_T\colon T \times_S A \to T \times_S B$ is a cocartesian functor over T;

(3) Condition (2) holds whenever T is of the form $T = [1] \times \Gamma$ for some anima Γ;

(4) Condition (2) holds for $T = [1] \times \mathrm{Map}([1], S)$ equipped with the evaluation map $[1] \times \mathrm{Map}([1], S) \to S$.

The dual conditions characterize cartesian functors between cartesian fibrations.

Proof. We showed that (1) implies (2) in Proposition 4.7.14. The implications (2) $\implies$ (3) $\implies$ (4) are clear. To see that (4) implies (3), consider a map $t\colon [1] \times \Gamma \to S$. We then obtain by currying a map $\tilde{t}\colon \Gamma \to \mathrm{Fun}([1], S)^{\simeq} = \mathrm{Map}([1], S)$, and the map t is given as the composite

$$[1] \times \Gamma \xrightarrow{1 \times \tilde{t}} [1] \times \mathrm{Map}([1], S) \xrightarrow{\mathrm{ev}} S.$$

In particular, if the pullback of f to $[1] \times \mathrm{Map}([1], S)$ is a cocartesian functor, then its pullback to $[1] \times \Gamma$ will also be a cocartesian functor by Proposition 4.7.14, as desired.

It remains to prove (3) $\implies$ (1). By Corollary 6.5.9, we test the Beck–Chevalley transformation at the universal morphism $v\colon x \to y$ in context $\mathrm{Map}([1], S)$ and the universal source object in the further context $a : A(x)^{\simeq}$. By (3), pulling back along the evaluation functor $[1] \times \mathrm{Map}([1], S) \to S$ gives a cocartesian functor. Its Beck–Chevalley transformation over $0 \to 1$ is therefore invertible. Base change identifies its component at a with $\overline{v}\colon v_!(f(a)) \to f(v_!(a))$, proving (1). □

Proposition 6.5.11. *Let $p\colon E \to B$ be a cocartesian fibration. Then p is also a cartesian fibration if and only if its covariant transport functors $u_!\colon E_a \to E_b$ admit right adjoints $u^*\colon E_b \to E_a$, as a family indexed by $u \in \mathrm{Map}([1], B)$. Dually, a cartesian fibration is cocartesian if and only if its contravariant transport functors admit left adjoints.*

Proof. Work in context $u : \mathrm{Map}([1], B)$, and write a and b for the source and target of u. The hypothesis gives an adjunction $u_! \dashv u^*$ between the contextual categories E_a and E_b. Suppose first that such right adjoints are given. For $y \in E_b$, compose the chosen cocartesian lift of u from u^*y with the counit:

$$u^*y \longrightarrow u_!u^*y \xrightarrow{\varepsilon_y} y.$$

For $v\colon c \to a$ and $z \in E_c$, the transport universal properties give

$$E(z, u^*y)_v \simeq E_a(v_!z, u^*y) \simeq E_b(u_!v_!z, y) \simeq E_b((uv)_!z, y) \simeq E(z, y)_{uv}.$$

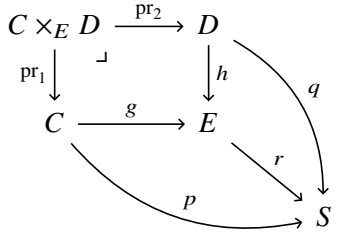
The composite is induced by composition with the displayed lift. These equivalences, tested on fibers over the anima of base morphisms, give its cartesian hom pullback square. Thus we have a family of cartesian lifts. By Proposition 6.5.1, p is cartesian.

Conversely, suppose that p is both cartesian and cocartesian. The chosen cartesian lifts define the functor $u^*\colon E_b \to E_a$ in context $u : \mathrm{Map}([1], B)$. For $x \in E_a$ and $y \in E_b$, their universal properties give

$$E_b(u_!x, y) \simeq E(x, y)_u \simeq E_a(x, u^*y).$$

The cartesian lift of u ending at y factors through the cocartesian lift starting at u^*y, giving the counit $u_!u^*y \to y$. The displayed equivalence identifies its universal-arrow map. The factorization is natural in y, since the chosen lifts are functorial, so it gives a natural transformation $\varepsilon\colon u_!u^* \to \mathrm{id}_{E_b}$. Apply Proposition 6.2.3 in context $u : \mathrm{Map}([1], B)$ to obtain a right adjoint R of $u_!$. The adjoint of ε is a natural transformation $u^* \to R$. For each y, it induces the displayed equivalence on hom animae, followed by the adjunction equivalence, and is therefore an isomorphism by Exercise 2.6.2. The objectwise isomorphism criterion makes $u^* \to R$ a natural isomorphism. Thus u^* itself is right adjoint to $u_!$ with the indicated counit. The dual argument proves the last assertion. □

Proposition 6.5.12. *Consider a commutative diagram*



of categories, and assume that the functors p, q, r are cocartesian fibrations and g and h are cocartesian functors over S.

(1) The induced functor $C \times_E D \to S$ is a cocartesian fibration;

(2) The projection functors pr_1 and pr_2 are cocartesian functors over S;

(3) For every other cocartesian fibration $T \to S$ and cocartesian functors $t_C : T \to C$ and $t_D : T \to D$ over S equipped with an identification $gt_C = ht_D$ over S, the induced functor $(t_C, t_D) : T \to C \times_E D$ is a cocartesian functor over S.

Proof. Write $\pi\colon C \times_E D \to S$ for the induced functor. An object of the pullback consists of objects x of C and y of D together with a specified identification $g(x) = h(y)$. Let $w\colon s_0 \to s_1$ be a morphism in S, and choose p- and q-cocartesian lifts

$$u\colon x_0 \to x_1 \qquad \text{and} \qquad v\colon y_0 \to y_1$$

of w. Since g and h are cocartesian functors, $g(u)$ and $h(v)$ are r-cocartesian. Their sources are identified by the pullback datum, so uniqueness of cocartesian lifts supplies a unique identification $g(x_1) = h(y_1)$ compatible with $g(u)$ and $h(v)$. Hence (u, v) defines a morphism in $C \times_E D$.

We verify the local universal property of (u, v). Consider a compatible pair of morphisms from (x_0, y_0) to an object (x', y') over s_1 whose image under π is w. The universal properties of u and v give unique vertical factorizations $x_1 \to x'$ and $y_1 \to y'$. Their images in E are compatible: after precomposition with the r-cocartesian morphisms $g(u)$ and $h(v)$, the required compatibility is exactly that of the original pair, and the universal property of these r-cocartesian morphisms detects the compatibility uniquely. Thus (u, v) is locally π-cocartesian.

Now let a be any locally π-cocartesian lift of w with source (x_0, y_0). Comparing a with the componentwise lift just constructed, uniqueness of locally π-cocartesian lifts gives an isomorphism between their targets in the fiber over s_1. Hence the projections of a are p- and q-cocartesian. We have proved that the locally π-cocartesian morphisms are precisely the compatible pairs of p- and q-cocartesian morphisms. Such pairs are stable under composition, so Proposition 6.5.4 implies that π is a cocartesian fibration. The same characterization shows that both projections preserve cocartesian morphisms, and hence are cocartesian functors.

Finally, let $T \to S$ and (t_C, t_D) be as in (3). A cocartesian morphism of T is sent by t_C and t_D to a compatible pair of p- and q-cocartesian morphisms. By the characterization just proved, its image in $C \times_E D$ is π-cocartesian. Hence (t_C, t_D) is a cocartesian functor over S. □

Proposition 6.5.13 (Fiberwise criterion for cocartesian fibrations)**.** *Let $p\colon E \to S$ and $q\colon D \to S$ be cocartesian fibrations, and let $u\colon E \to D$ be a cocartesian functor over S. Then u is a cocartesian fibration if and only if the following conditions hold:*

(1) For every object s of S, the induced functor $u_s\colon E_s \to D_s$ is a cocartesian fibration.

(2) For every morphism $v\colon s \to t$ in S, the transport functor $v^E_!\colon E_s \to E_t$ sends u_s-cocartesian morphisms to u_t-cocartesian morphisms.

Proof. Condition (1) is interpreted at the universal object in context $S^{\simeq}$, and condition (2) at the universal morphism in context $\mathrm{Map}([1], S)$, with the fiber categories obtained by substitution along its source and target. First observe that every p-cocartesian morphism is u-cocartesian: its image under u is q-cocartesian, so this follows by cancellation of pullback squares in Definition 4.5.5.

Suppose that u is a cocartesian fibration. Condition (1) follows by base change. Moreover, a morphism in E_s is u_s-cocartesian if and only if its image in E is u-cocartesian, by Observation 4.8.6 and Proposition 6.5.4. For such a morphism $\alpha\colon x \to y$ and a morphism $v\colon s \to t$, naturality of the

p-cocartesian lift gives a commutative square

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ \downarrow & & \downarrow \\ v_!^E(x) & \xrightarrow[v_!^E(\alpha)]{} & v_!^E(y). \end{array}$$

Both vertical morphisms and the upper horizontal morphism are u-cocartesian. Composition and cancellation of cocartesian morphisms show that the lower horizontal morphism is u-cocartesian as well. It is therefore u_t-cocartesian, proving (2).

Conversely, suppose that (1) and (2) hold. We first show that every u_s-cocartesian morphism $\alpha\colon x \to y$ in E_s is u-cocartesian in E. For an object z of E, write $t = p(z)$ and consider the square

$$\begin{array}{ccc} E(y,z) & \xrightarrow{-\circ\alpha} & E(x,z) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ D(u(y),u(z)) & \xrightarrow[-\circ u(\alpha)]{} & D(u(x),u(z)). \end{array}$$

This is a square of animae over $S(s,t)$. On its fiber over $v\colon s \to t$, the universal properties of cocartesian transport identify the upper row with precomposition by $v_!^E(\alpha)$ in E_t, and the lower row with precomposition by $v_!^D(u_s(\alpha))$ in D_t. Since u is cocartesian over S, the identification $u_t v_!^E = v_!^D u_s$ identifies this fiber square with the square expressing that $v_!^E(\alpha)$ is u_t-cocartesian, and hence it is a pullback by (2). The displayed square is therefore a pullback as well, since this can be checked on fibers over the anima $S(s,t)$ by Lemma 6.1.12. Now Definition 4.5.5 proves the claim.

Let $\beta\colon u(e) \to d$ be a morphism in D, and write $v = q(\beta)\colon s \to t$. Choose a p-cocartesian lift $\eta\colon e \to v_!^E(e)$. Its image under u is q-cocartesian, so β factors as $\overline{\beta}\,u(\eta)$ for a morphism $\overline{\beta}\colon u_t(v_!^E(e)) \to d$ in D_t. By (1), choose a u_t-cocartesian lift $\lambda\colon v_!^E(e) \to e'$ of $\overline{\beta}$. Both η and λ are u-cocartesian by the preceding arguments, and pasting their mapping-anima pullback squares shows that $\lambda\eta$ is u-cocartesian. Moreover, it is a lift of β with source e.

The dual of Proposition 6.5.1 now shows that u is a cocartesian fibration. □

Corollary 6.5.14 (Fiberwise criterion for left fibrations)**.** *Let $p\colon E \to S$ and $q\colon D \to S$ be cocartesian fibrations, and let $u\colon E \to D$ be a cocartesian functor over S. Then u is a left fibration if and only if $u_s\colon E_s \to D_s$ is a left fibration for every object s of S.*

Proof. The forward implication follows from stability of left fibrations under base change. Conversely, suppose that every u_s is a left fibration. Every morphism in E_s is then u_s-cocartesian, so both conditions of Proposition 6.5.13 hold. Hence u is a cocartesian fibration. For an object d of D, put $s = q(d)$. In the context of d, its fiber is

$$E_d \simeq E_s \times_{D_s} *,$$

where $* \to D_s$ selects d. This is an anima since u_s is a left fibration. Thus u is a left fibration by Lemma 6.1.8. □

Proposition 6.5.15. *Let $p\colon C \to A \times B$ be a functor. We denote by $\mathrm{pr}_A\colon A \times B \to A$ and $\mathrm{pr}_B\colon A \times B \to B$ the obvious projections, and we assume that $\mathrm{pr}_B\, p\colon C \to B$ is a cocartesian fibration. The property that p is a cocartesian fibration is equivalent to the following list of properties to be fulfilled altogether:*

(i) *The functor* $p\colon C \to A \times B$ *is cocartesian over* B*, meaning that it sends* pr_B *p-cocartesian morphisms to* pr_B*-cocartesian morphisms;*

(ii) *For any object* b *of* B*, the induced functor* $p_b\colon C_b \to A$ *is a cocartesian fibration;*

(iii) *For any morphism* $v\colon b_0 \to b_1$ *in* B*, the covariant transport functor*

$$v_!\colon C_{b_0} \to C_{b_1}$$

sends p_{b_0}*-cocartesian morphisms to* p_{b_1}*-cocartesian morphisms.*

Proof. Write $q = \mathrm{pr}_B\, p$. By Example 4.8.5 and Proposition 6.5.4, the pr_B-cocartesian morphisms are the pairs (u, v) for which u is an isomorphism in A and v is an arbitrary morphism in B.

Suppose first that p is a cocartesian fibration. We repeatedly use the pointwise characterization from Definition 4.5.5: pasting the relevant pullback squares shows that a p-cocartesian morphism whose image is pr_B-cocartesian is q-cocartesian, and that a q-cocartesian morphism whose image is pr_B-cocartesian is p-cocartesian. Given a q-cocartesian morphism with source x and image v in B, compare it with a p-cocartesian lift at x of $(\mathrm{id}_{\mathrm{pr}_A\, p(x)}, v)$. The latter is q-cocartesian by the first implication, so uniqueness of q-cocartesian lifts identifies the two morphisms. Its image under p is therefore pr_B-cocartesian, which proves (i). Condition (ii) follows from stability of cocartesian fibrations under base change.

For (iii), let α be a p_{b_0}-cocartesian morphism and form its naturality square under transport along $v\colon b_0 \to b_1$. Its vertical morphisms are q-cocartesian and have pr_B-cocartesian images by (i), hence they are p-cocartesian by the second implication above. The upper horizontal morphism is also p-cocartesian, by the base-change characterization of cocartesian morphisms. Consequently the two composites around the square are p-cocartesian, and cancellation in Lemma 4.5.12 shows that the lower horizontal morphism is p-cocartesian. Since it lies over the identity of b_1, it is p_{b_1}-cocartesian. This proves (iii).

Conversely, assume all three conditions.

In the context of an object b of B, the fiber of pr_B is A. For a morphism $v\colon b_0 \to b_1$ in B, transport between these fibers is the identity of A, considered in the context of v. Condition (i) thus ensures that we have canonical commutative triangles of the form

$$\begin{array}{ccc} C_{b_0} & \xrightarrow{v_!} & C_{b_1} \\ & {\scriptstyle p_{b_0}}\searrow \quad \swarrow{\scriptstyle p_{b_1}} & \\ & A & \end{array}$$

for each morphism $v\colon b_0 \to b_1$ in B. Condition (ii) ensures that each projection p_b is a cocartesian fibration and condition (iii) says that each of these functors $v_!$ in the triangles above preserves cocartesian morphisms over A.

Let us consider a morphism $f\colon x_0 \to x_1$ in C. Its image by p is of the form $p(f) = (u, v)$ with $u\colon a_0 \to a_1$ and $v\colon b_0 \to b_1$ morphisms in A and in B, respectively. This yields a commutative diagram of the following form in $A \times B$:

$$\begin{array}{ccc} (a_0, b_0) & \xrightarrow{(u, \mathrm{id}_{b_0})} & (a_1, b_0) \\ {\scriptstyle (\mathrm{id}_{a_0}, v)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_{a_1}, v)} \\ (a_0, b_1) & \xrightarrow{(u, \mathrm{id}_{b_1})} & (a_1, b_1) \end{array}$$

We consider the morphism $\varepsilon_{x_0} : x_0 \to u_!(x_0)$ obtained as the image in C of the p_{b_0}-cocartesian lift of u. The lift comes with an identification of $\mathrm{pr}_B(p(\varepsilon_{x_0}))$ with the identity of b_0. Seeing $u_!(x_0)$ as an object of the fiber C_{b_0} we may now apply the functor $v_!$ and get an object $v_!u_!(x_0)$ in the fiber C_{b_1}. Alternatively, we may as well apply covariant transport along v to x_0, yielding an object $v_!(x_0)$ in the fiber C_{b_1}, using the identification $\mathrm{pr}_A\, p(v_!(x_0)) = a_0$ supplied by condition (i) and then apply covariant transport along u. The fact that $v_!(\varepsilon_{x_0})$ is a p_{b_1}-cocartesian morphism ensures that we have a canonical identification of the form

$$v_!u_!(x_0) \xrightarrow{\sim} u_!v_!(x_0)$$

inducing a commutative square of the form

$$\begin{array}{ccc} x_0 & \xrightarrow{\varepsilon_{x_0}} & u_!(x_0) \\ {\scriptstyle \eta_{x_0}}\downarrow & & \downarrow{\scriptstyle \eta_{u_!(x_0)}} \\ v_!(x_0) & \xrightarrow{\varepsilon_{v_!(x_0)}} & u_!v_!(x_0) \end{array}$$

in which the vertical maps are $\mathrm{pr}_B\, p$-cocartesian morphisms (the commutativity of the square comes from the fact that the formation of $v_!(x_0)$ is functorial in x_0 and from the identification $\varepsilon_{v_!(x_0)} = v_!(\varepsilon_{x_0})$ due to condition (iii)). There is a unique morphism $g : v_!(x_0) \to x_1$ such that $g\eta_{x_0} = f$. This factorization includes an identification of $\mathrm{pr}_B\, p(g)$ with the identity of b_1. In other words, we can see g as a map in C_{b_1}. There is thus a unique morphism of the form $h : u_!v_!(x_0) \to x_1$ in C such that $h\varepsilon_{v_!(x_0)} = g$ and such that $\mathrm{pr}_B\, p(h) = \mathrm{id}_{b_1}$. In other words, the morphism $\varepsilon_{v_!(x_0)}\eta_{x_0}$ is a locally p-cocartesian lift of (u, v). This proves that p is a locally cocartesian fibration.

By Proposition 6.5.4, it remains to prove that locally p-cocartesian morphisms are stable under composition in C. Every locally p-cocartesian morphism is isomorphic to one of the form $\varepsilon_{v_!(x_0)}\eta_{x_0}$ constructed above. Consider two composable morphisms in this normal form, with composable B-components v and v'. Naturality of the q-cocartesian lift gives a commutative square as above which moves the fiberwise cocartesian part of the first morphism past the B-transport part of the second. By condition (iii), the transported horizontal morphism is again cocartesian in the target fiber. We may therefore compose the two horizontal parts using closure of p_b-cocartesian morphisms under composition, and compose the two vertical parts using closure of q-cocartesian morphisms under composition. The canonical identifications

$$(v'v)_! = v'_!v_!$$

from Corollary 4.5.11 identify the resulting composite with the normal-form lift for the composite morphism in $A \times B$. Thus locally p-cocartesian morphisms are closed under composition, and p is a cocartesian fibration. □

Our next goal is to provide a fiberwise criterion for a map between (co)cartesian fibrations to be an equivalence. In contrast to the statement for left (resp. right) fibrations from Lemma 6.1.12, the statement for (co)cartesian fibrations requires that we restrict attention to *(co)cartesian functors*, a notion that was defined in Section 4.7.

Lemma 6.5.16. *Let*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative triangle such that p and q are cartesian (resp. cocartesian) fibrations and f is a cartesian (resp. cocartesian) functor over S. If the induced functor $f_s\colon C_s \to D_s$ is fully faithful for every object s of S, then f is fully faithful.

Proof. We treat the cartesian case; the cocartesian case is dual. Let x_0 and x_1 be objects of C, and write $s_i = p(x_i)$. The induced map

$$C(x_0, x_1) \longrightarrow D(f(x_0), f(x_1))$$

is a map of animae over $S(s_0, s_1)$. We show that it is an equivalence on every fiber. Fix a morphism $w\colon s_0 \to s_1$ and choose a p-cartesian lift

$$\widetilde{w}\colon w^* x_1 \to x_1.$$

Since f is a cartesian functor, $f(\widetilde{w})$ is a q-cartesian lift of w. The universal properties of these cartesian lifts give equivalences

$$C(x_0, x_1)_w \simeq C_{s_0}(x_0, w^* x_1)$$

and

$$D(f(x_0), f(x_1))_w \simeq D_{s_0}(f(x_0), f(w^* x_1)).$$

Under these equivalences, the map on fibers is induced by f_{s_0} and is therefore an equivalence by hypothesis. Thus the map on hom animae is an equivalence, and f is fully faithful. □

Theorem 6.5.17. *A (co)cartesian functor between (co)cartesian fibrations is an equivalence if and only if it is an equivalence on fibers. More precisely, consider a commutative triangle*

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

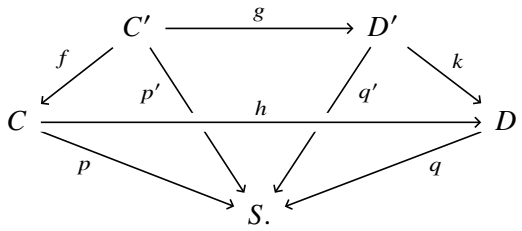
such that p and q are cartesian (resp. cocartesian) fibrations and that f is a cartesian (resp. cocartesian) functor over S. If for every object s of S the induced functor $f_s\colon C_s \to D_s$ is an equivalence, then f is an equivalence.

Proof. If f is an equivalence, then every functor f_s on fibers is an equivalence by base change. Conversely, assume that $f_s\colon C_s \to D_s$ is an equivalence for every object s of S. Work in context $y : D^{\simeq}$ and set $s = q(y)$. There is a pullback square

$$\begin{array}{ccc} C_y & \longrightarrow & C_s \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f_s} \\ * & \xrightarrow{\quad y \quad} & D_s. \end{array}$$

Since f_s is an equivalence, C_y is contractible in this context. Its object gives a lift of y to C, hence, by Corollary 5.3.6, a section of $D^{\simeq} \times_D C \to D^{\simeq}$. Projecting to C and factoring through $C^{\simeq}$ gives the required strong-surjectivity section. Moreover, f is fully faithful by Lemma 6.5.16. The fundamental theorem of category theory, Theorem 6.3.8, now implies that f is an equivalence. □

Corollary 6.5.18. *Consider a commutative square of categories over S, i.e. a commutative diagram*



Assume that all four functors p, p', q, q' are cocartesian fibrations and that the functors f, g, h, k are cocartesian functors over S. Further assume that for every object s of S the induced square

$$\begin{array}{ccc} C'_s & \xrightarrow{g_s} & D'_s \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle k_s} \\ C_s & \xrightarrow{h_s} & D_s \end{array}$$

is a pullback square. Then also the top square of the original diagram is a pullback square.

Proof. Define the category E as the pullback $C \times_D D'$. By Proposition 6.5.12, the resulting functor $C \times_D D' \to S$ is a cocartesian fibration and the induced functor $C' \to E$ is a cocartesian functor. Our task is to show that it is an equivalence. By Theorem 6.5.17 it will suffice to show that the induced map $C'_s \to E_s$ on fibers is an equivalence for every object s of S. But since taking the fiber commutes with pullbacks, this map is the canonical comparison map $C'_s \to C_s \times_{D_s} D'_s$ coming from the induced square on fibers. Since this square was assumed to be a pullback square, this map is indeed an equivalence, finishing the proof. □

6.6 Relative adjunctions

We finish by asking when adjunctions on the fibers of categories over a base assemble into a relative adjunction. The universal-arrow criterion reduces this question to compatibility with transport.

Let $p\colon C \to S$ be a functor, let $x : C_s$ and $y : C_t$, and let $u\colon s \to t$ be a morphism in S. As before, we write $C(x,y)_u$ for the fiber over u of $C(x,y) \to S(s,t)$, the anima of morphisms from x to y lying over u. This fiber and the categories C_s and C_t are formed in the common anima context of x, y, and u.

Lemma 6.6.1 (Relative hom animae and cocartesian transport). *Let $p\colon C \to S$ be a cocartesian fibration and let $u\colon s \to t$. For $x \in C_s$ and $y \in C_t$, composition with the canonical cocartesian lift $x \to u_!x$ induces an equivalence*

$$C_t(u_!x, y) \xrightarrow{\sim} C(x,y)_u.$$

Proof. Apply Lemma 4.5.7 to the chosen lift $x \to u_!x$ and the factorization $u = \mathrm{id}_t \circ u$. □

Now let $p\colon C \to S$ and $q\colon D \to S$, and let $L\colon C \to D$ and $R\colon D \to C$ be functors over S. Their structure identifications induce functors

$$C \mathbin{\vec{\times}}_L D \longrightarrow C \mathbin{\vec{\times}}_{p,q} D \qquad \text{and} \qquad C \mathbin{\vec{\times}}_R D \longrightarrow C \mathbin{\vec{\times}}_{p,q} D,$$

where

$$C \vec{\times}_{p,q} D \coloneqq (C \times D) \times_{S \times S} \mathrm{Fun}([1], S).$$

These send a morphism $Lx \to y$ or $x \to Ry$, respectively, to the pair (x, y) together with its image $p(x) \to q(y)$ in S.

Proposition 6.6.2 (Relative hom-anima criterion). *The data of a relative adjunction $L \dashv R$ are equivalent to an equivalence*

$$C \vec{\times}_L D \simeq C \vec{\times}_R D$$

over $C \vec{\times}_{p,q} D$. In particular, taking fibers gives equivalences

$$D(Lx, y) \simeq C(x, Ry)$$

over $S(p(x), q(y))$.

Proof. By Proposition 4.4.7, an adjunction between the underlying functors is equivalent to an equivalence between the two directed pullbacks over $C \times D$. For a relative adjunction, the usual equivalence sends $f : Lx \to y$ to $R(f)\eta_x : x \to Ry$. Since R lies over S and η_x lies over an identity morphism, these two arrows have the same image in S, compatibly with their sources and targets. The equivalence is therefore over $C \vec{\times}_{p,q} D$.

Conversely, the proof of Proposition 4.4.7 constructs the unit and counit by applying the equivalence and its inverse to identity morphisms. If the equivalence is over $C \vec{\times}_{p,q} D$, these components lie over identity morphisms in S. Hence the resulting adjunction is relative. □

In particular, a relative adjunction restricts for every $u : s \to t$ to equivalences

$$D(L_s x, y)_u \simeq C(x, R_t y)_u. \tag{6.11}$$

Let $p : C \to S$ and $q : D \to S$ be cocartesian fibrations. We first ask when a functor $L : C \to D$ over S admits a relative right adjoint. We will then consider the converse problem: given a functor $R : D \to C$ over S, when do left adjoints to its fiber functors extend to a relative left adjoint?

A family of adjunctions $L_s \dashv R_s$ on the fibers is an adjunction between C_s and D_s at the universal object in context $s : S^{\simeq}$, including its unit, counit, and triangle identifications. As in the proof of Proposition 5.3.11, passing to total categories identifies this with a relative adjunction over $S^{\simeq}$. For the universal morphism $u : s \to t$ in context $\mathrm{Map}([1], S)$, we substitute this adjunction along the source and target maps to $S^{\simeq}$. The transport transformations below are formed in that context.

Theorem 6.6.3 (Fiberwise recognition of relative right adjoints)**.** *Let $p : C \to S$ and $q : D \to S$ be cocartesian fibrations, and let $L : C \to D$ be a functor over S. The following conditions are equivalent:*

(1) The functor L admits a right adjoint relative to S;

(2) The functor L is cocartesian over S, and every fiber functor $L_s : C_s \to D_s$ admits a right adjoint $R_s : D_s \to C_s$.

Under these conditions, the restriction of the relative right adjoint to D_s is right adjoint to L_s.

Proof. Suppose first that $L \dashv R$ is an adjunction relative to S. In context $s : S^{\simeq}$, base change along $s\colon * \to S$ gives an adjunction $L_s \dashv R_s$ by Proposition 4.4.31.

To show that L preserves cocartesian morphisms, let $a\colon x \to x'$ be p-cocartesian over $u\colon s \to t$. For any object y of D, the relative adjunction gives a commutative diagram

$$\begin{array}{ccc} D(Lx', y) & \xrightarrow{-\circ L(a)} & D(Lx, y) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ C(x', Ry) & \xrightarrow{-\circ a} & C(x, Ry) \\ \downarrow & & \downarrow \\ S(t, q(y)) & \xrightarrow{-\circ u} & S(s, q(y)). \end{array}$$

The lower square is a pullback since a is p-cocartesian. The vertical arrows in the upper square are equivalences, so the outer square is a pullback as well. Thus $L(a)$ is q-cocartesian, and L is cocartesian by Lemma 4.7.3.

Conversely, suppose that L is cocartesian and fix fiberwise adjunctions $L_t \dashv R_t$. For $y : D_t$, let

$$\varepsilon_y\colon L_t R_t y \longrightarrow y$$

be the fiberwise counit. We claim that this is also a universal morphism from L to y: for every $x : C_s$, the map

$$C(x, R_t y) \longrightarrow D(Lx, y), \qquad h \longmapsto \varepsilon_y L(h)$$

is an equivalence. Since ε_y lies over id_t, this map commutes with the projections to $S(s,t)$. We may therefore check the claim on fibers over $u\colon s \to t$, where cocartesian transport and the fiber adjunction give

$$C(x, R_t y)_u \simeq C_t(u_! x, R_t y) \simeq D_t(L_t u_! x, y) \simeq D_t(u_! L_s x, y) \simeq D(Lx, y)_u.$$

Here the third equivalence uses that L preserves cocartesian transport. To identify the composite, write $h = \bar{h}\lambda$, where $\lambda\colon x \to u_! x$ is the chosen lift. The fiber adjunction sends $\bar{h}$ to $\varepsilon_y L(\bar{h})$; the remaining maps compose this with $L(\lambda)$. Their composite is therefore $\varepsilon_y L(h)$, as required. The map is an equivalence on every fiber, hence an equivalence by Lemma 6.1.12. The pointwise criterion Proposition 6.2.3 therefore produces a right adjoint $R\colon D \to C$ whose value on $y : D_t$ is $R_t y$ and whose counit has component ε_y.

Applying q to the counit and using $qL = p$ gives a transformation $pR \to q$ whose component at $y : D_t$ is id_t. By Proposition 3.4.6, it is a natural isomorphism and equips R with the structure of a functor over S. With this structure, the counit lies in $\mathrm{Fun}_S(D, D)$. Thus $L \dashv R$ is relative to S by Definition 4.4.28. □

We now consider a functor $R\colon D \to C$ over S and fiberwise adjunctions $L_s \dashv R_s$. We will show that these extend to a relative adjunction precisely when the L_s commute with cocartesian transport. To formulate this condition, recall that the Beck–Chevalley transformation of R gives

$$\overline{u}^R\colon u_! R_s \longrightarrow R_t u_!. \tag{6.12}$$

Write η^s for the unit of $L_s \dashv R_s$. Under the adjunction on functor categories induced by $L_t \dashv R_t$, the composite

$$u_! \xrightarrow{u_!\eta^s} u_! R_s L_s \xrightarrow{\overline{u}^R L_s} R_t u_! L_s$$

corresponds to a natural transformation

$$\theta_u \colon L_t u_! \longrightarrow u_! L_s. \tag{6.13}$$

By Proposition 4.4.4, it is characterized by the identity

$$R_t\theta_u \circ \eta^t u_! = \overline{u}^R L_s \circ u_!\eta^s. \tag{6.14}$$

Proposition 6.6.4 (Relative left-adjoint criterion)**.** *Let $p \colon C \to S$ and $q \colon D \to S$ be cocartesian fibrations, and let $R \colon D \to C$ be a functor over S. Fix fiberwise adjunctions $L_s \dashv R_s$. Then R admits a left adjoint relative to S inducing these fiberwise adjunctions if and only if the transformation θ_u of* (6.13) *is invertible for every morphism $u \colon s \to t$ in S. When these conditions hold, the relative left adjoint is cocartesian, and its Beck–Chevalley transformation over u is θ_u^{-1}.*

Proof. Suppose first that $L \dashv R$ is a relative adjunction inducing the chosen fiberwise adjunctions. Then Theorem 6.6.3 shows that L is cocartesian. Let $\overline{u}^L \colon u_! L_s \to L_t u_!$ be its Beck–Chevalley transformation. For the chosen lift $\lambda \colon x \to u_! x$ in C, naturality of the unit gives

$$\begin{array}{ccc} x & \xrightarrow{\eta^s_x} & R_s L_s x \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle RL(\lambda)} \\ u_! x & \xrightarrow{\eta^t_{u_! x}} & R_t L_t u_! x. \end{array}$$

The lift functor makes this a square of natural transformations of functors $C_s \to C$. Factoring through the chosen cocartesian lifts, the right-hand morphism gives $R_t\overline{u}^L \circ \overline{u}^R L_s$, while the upper morphism gives $u_!\eta^s$. The universal property of these lifts therefore gives

$$\eta^t u_! = R_t\overline{u}^L \circ \overline{u}^R L_s \circ u_!\eta^s$$

in $\mathrm{Fun}(C_s, C_t)$. Since $\overline{u}^L$ is invertible, we obtain

$$R_t(\overline{u}^L)^{-1} \circ \eta^t u_! = \overline{u}^R L_s \circ u_!\eta^s.$$

The characterization (6.14) therefore identifies θ_u with $(\overline{u}^L)^{-1}$. In particular, θ_u is invertible.

Conversely, suppose that every θ_u is invertible. For $x : C_s$, we claim that the fiberwise unit $\eta^s_x \colon x \to R_s L_s x$ is a universal morphism from x to R. For every $y : D_t$, we must therefore show that

$$D(L_s x, y) \longrightarrow C(x, Ry), \qquad g \longmapsto R(g)\eta^s_x$$

is an equivalence. This map again commutes with the projections to $S(s,t)$. On the fiber over $u \colon s \to t$, cocartesian transport and the fiber adjunction give

$$D(L_s x, y)_u \simeq D_t(u_! L_s x, y) \simeq D_t(L_t u_! x, y) \simeq C_t(u_! x, R_t y) \simeq C(x, Ry)_u,$$

where the second equivalence is precomposition with θ_u. To identify the composite, write $g = \bar{g}\lambda$ for the chosen lift $\lambda \colon L_s x \to u_! L_s x$ in D. The middle two equivalences send $\bar{g}$ to

$$R_t(\bar{g}) R_t(\theta_{u,x})\eta^t_{u_! x} = R_t(\bar{g})\overline{u}^R_{L_s x} u_!(\eta^s_x)$$

by (6.14). Composing with the lift $x \to u_! x$ gives $R(g)\eta^s_x$, by the definition of $\overline{u}^R$ and naturality of the chosen lifts. Thus the displayed map is an equivalence on every fiber, hence an equivalence by Lemma 6.1.12. The left-adjoint form of Proposition 6.2.3 produces a left adjoint $L\colon C \to D$ with the prescribed values and unit components.

Applying p to the unit and using $pR = q$ gives a natural transformation $p \to qL$. Its components are identities, so it is invertible by Proposition 3.4.6. Its inverse equips L with the structure of a functor over S, with respect to which the unit lies over S. By Proposition 4.4.29, the adjunction is therefore relative to S. Applying Theorem 6.6.3 shows that L is cocartesian. The necessity argument above, applied to this relative adjunction, then identifies θ_u with $(\overline{u}^L)^{-1}$ for every u. □

Corollary 6.6.5 (Cartesian dual)**.** *Let $p\colon C \to S$ and $q\colon D \to S$ be cartesian fibrations, and let $R\colon D \to C$ be a functor over S. The functor R admits a left adjoint relative to S if and only if R is cartesian and every fiber functor $R_s\colon D_s \to C_s$ admits a left adjoint. The relative left adjoint restricts to these fiberwise left adjoints. If $L\colon C \to D$ denotes this relative left adjoint, then L is cartesian if and only if the canonical Beck–Chevalley transformations*

$$L_s u^* \longrightarrow u^* L_t$$

are invertible for every $u\colon s \to t$ in S.

Proof. The first assertion follows by dualizing the proof of Theorem 6.6.3. The last assertion is the cartesian version of Lemma 4.7.3. □

6.7 Exercises Chapter 6

Exercise 6.7.1. Show that every equivalence $u\colon C \xrightarrow{\sim} D$ is fully faithful.

Exercise 6.7.2. Show that a composite of fully faithful functors is fully faithful.

Exercise 6.7.3. Given categories C and D, construct a canonical pullback square of the form

$$\begin{array}{ccc} \mathrm{Equiv}(C,D) & \longrightarrow & \mathrm{Equiv}(\mathrm{Map}([1],C),\mathrm{Map}([1],D)) \\ \downarrow & & \downarrow \\ \mathrm{Map}(C,D) & \longrightarrow & \mathrm{Map}(\mathrm{Map}([1],C),\mathrm{Map}([1],D)) \end{array}$$

7 Limits and colimits

This chapter has recently been split off from Chapter 4 into a standalone chapter. It is still somewhat of a stub right now and should be improved in the future.

Limits and colimits can be defined in synthetic category theory as terminal cones and initial cocones, respectively. Using the functoriality results of the preceding chapter, we may pass from these 'objectwise' universal properties to coherent limit and colimit functors. We will use their description as adjoints to study preservation questions and to compute limits and colimits in functor categories.

7.1 Cones, limits, and colimits

Definition 7.1.1. Let I and C be categories. We refer to objects of the functor category $\mathrm{Fun}(I, C)$ as *I-indexed diagrams in C*. Given such a diagram $F \in \mathrm{Fun}(I, C)$, we define the category $C_{/F}$ of *cones over F* as the following relative slice category:

$$\begin{array}{ccc} C_{/F} & \longrightarrow & \mathrm{Fun}(I, C)_{/F} \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow[x \mapsto \mathrm{const}_x]{} & \mathrm{Fun}(I, C). \end{array}$$

In other words, a cone over F consists of a pair (x, ε) where $x \in C$ and $\varepsilon \colon \mathrm{const}_x \to F$ is a natural transformation. A cone (x, ε) over F is called a *limit cone* if it is a terminal object of the cone category $C_{/F}$. If it exists, a limit cone is unique. The underlying object x will then be called the *limit of F* and denoted by $\lim(F)$, or $\lim_I F$, or $\lim_{i \in I} F(i)$.

Dually, we define a *cocone under F* to be an object of the dually defined category $C_{F/}$. A *colimit cocone* is an initial object of $C_{F/}$, and its underlying object, if it exists, will be denoted by $\mathrm{colim}(F)$, or $\mathrm{colim}_I F$ or $\mathrm{colim}_{i \in I} F(i)$.

7.1.1 Examples

Throughout, we let C denote a category.

Observation 7.1.2. Consider the empty diagram $I = \emptyset$. Then the category $\mathrm{Fun}(\emptyset, C)$ is contractible by Lemma 1.7.14, providing a unique functor $F \colon \emptyset \to C$. Observe that $C_{/F} \simeq C \simeq C_{F/}$. So cones and cocones on F are simply given by objects $x \in C$, and as a consequence we see:

- Terminal objects in C are the same as $\emptyset$-indexed limits.

- Initial objects in C are the same as $\emptyset$-indexed colimits.

Definition 7.1.3. Consider the two-point set $I = \{0,1\} := * \sqcup *$. Then the two evaluation maps define an equivalence $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}(\{0,1\}, C) \xrightarrow{\sim} C \times C$, so that a functor $\{0,1\} \to C$ is the same as a pair (x, y) of objects of C.

- A *product of x and y*, denoted $x \times y$, is a limit of the $\{0,1\}$-indexed diagram (x, y).
- A *coproduct of x and y*, denoted $x \sqcup y$, is a colimit of the $\{0,1\}$-indexed diagram (x, y).

Unwinding the definitions, we see that $x \times y$ is terminal among objects equipped with maps $\mathrm{pr}_x : x \times y \to x$ and $\mathrm{pr}_y : x \times y \to y$. Dually, $x \sqcup y$ is initial among objects equipped with maps $x \to x \sqcup y$ and $y \to x \sqcup y$.

Definition 7.1.4. We define categories $\lrcorner$ and $\ulcorner$ as follows:

$$\lrcorner := \{0,1\} \star * \qquad \text{and} \qquad \ulcorner := * \star \{0,1\},$$

using the join operation from Axiom J.1.

The notations for these categories reflect the shape of the diagrams they classify: a functor $\lrcorner \to C$ corresponds to a diagram in C of the form

$$\begin{array}{ccc} & & x \\ & & \downarrow{\scriptstyle f} \\ y & \xrightarrow{g} & z, \end{array}$$

while a functor $\ulcorner \to C$ corresponds to a diagram of the form

$$\begin{array}{ccc} z & \xrightarrow{f} & x \\ {\scriptstyle g}\downarrow & & \\ y. & & \end{array}$$

More precisely, there are pullback squares

$$\begin{array}{ccc} \mathrm{Fun}(\lrcorner, C) & \longrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathrm{Fun}([1], C) & \xrightarrow{\mathrm{ev}_1} & C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Fun}(\ulcorner, C) & \longrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathrm{Fun}([1], C) & \xrightarrow{\mathrm{ev}_0} & C. \end{array}$$

Definition 7.1.5. Consider a $\lrcorner$-indexed diagram $x \xrightarrow{f} z \xleftarrow{g} y$ in C. We refer to its limit as the *pullback of f and g*, and denote it by $x \times_z y$.

Dually, given a $\ulcorner$-indexed diagram $x \xleftarrow{f} z \xrightarrow{g} y$, we refer to its colimit in C as the *pushout of f and g*, and denote it by $x \sqcup_z y$.

7.2 Limit and colimit functors

The results of Chapter 6 allow us to assemble limits and colimits into functors. We express this using adjoints to the constant-diagram functor.

Definition 7.2.1. Let I and C be categories. We say that *C admits I-indexed limits (resp. colimits)* if the constant-diagram functor $p_I^* \colon C \to \mathrm{Fun}(I, C)$ admits a right (resp. left) adjoint

$$\lim_I \colon \mathrm{Fun}(I, C) \to C \qquad (\text{ resp. } \operatorname{colim}_I \colon \mathrm{Fun}(I, C) \to C).$$

For $I = \{0, 1\}$, we say that *C admits binary products* or *binary coproducts*, respectively. Likewise, *C admits pullbacks* if it admits $\lrcorner$-indexed limits, and *admits pushouts* if it admits $\ulcorner$-indexed colimits.

As in Convention 5.4.4, "for every diagram $F \colon I \to C$" refers to the universal diagram in context $\mathrm{Map}(I, C)$, with I and C weakened to that context.

Proposition 7.2.2. *A category C admits I-indexed colimits if and only if for every diagram $F \colon I \to C$, there exists a colimit $\operatorname{colim}_I F$. The dual statement holds for limits.*

Proof. Apply Theorem 6.2.1 and its dual to the constant-diagram functor $C \to \mathrm{Fun}(I, C)$. Its relative slices and coslices are precisely the categories of cones and cocones. □

Definition 7.2.3. Let I be a category and let $f \colon C \to D$ be a functor. If C and D admit I-indexed colimits, we say that *f preserves I-indexed colimits* if the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ {\scriptstyle \mathrm{const}_C}\downarrow & & \downarrow{\scriptstyle \mathrm{const}_D} \\ \mathrm{Fun}(I, C) & \xrightarrow{f_*} & \mathrm{Fun}(I, D) \end{array}$$

is left-adjointable, in the sense that the Beck–Chevalley map

$$\mathrm{BC} \colon \operatorname{colim}_I \circ f_* \to f \circ \operatorname{colim}_I$$

is a natural isomorphism of functors $\mathrm{Fun}(I, C) \to D$. Preservation of I-indexed limits is defined dually.

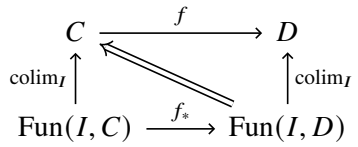
We can also express preservation in terms of individual colimit cocones.

Proposition 7.2.4. *Let C and D admit I-indexed colimits. A functor $f \colon C \to D$ preserves I-indexed colimits if and only if for every I-indexed diagram $F \colon I \to C$ and every colimit cocone $(\operatorname{colim}_I F, \varepsilon)$ for F, the induced cocone $(f(\operatorname{colim}_I F), f(\varepsilon))$ is a colimit cocone for $f \circ F$.*

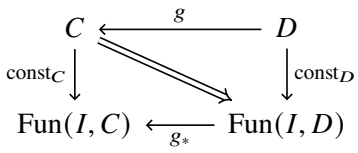
Proof. The component of the Beck–Chevalley transformation at F is the morphism $\operatorname{colim}_I(f \circ F) \to f(\operatorname{colim}_I F)$ induced by the displayed cocone. It is an isomorphism precisely when that cocone is a colimit cocone. The assertion follows from Proposition 3.4.6. □

Lemma 7.2.5 (Adjoints preserve (co)limits)**.** *Let $f \colon C \to D$ be a functor and let I be a category such that both C and D admit I-indexed colimits. If f admits a right adjoint $g \colon D \to C$, then f preserves I-indexed colimits. The dual statement that right adjoints preserve limits also holds.*

Proof. We have to show that the Beck–Chevalley transformation in the diagram



is a natural isomorphism. By passing to right adjoints, this is equivalent to showing that the Beck–Chevalley transformation in the diagram



is a natural isomorphism. But this is clear since the adjunction $f_* \colon \mathrm{Fun}(I,C) \rightleftarrows \mathrm{Fun}(I,D) : g_*$ is natural in I. □

7.3 Limits and colimits in functor categories

Limits and colimits in a functor category are computed at each object. We first prove this using adjunctions, and then apply it to iterated limits and colimits.

Lemma 7.3.1. *Let D and I be categories and assume that D admits I-indexed colimits. Then for every category C also* $\mathrm{Fun}(C,D)$ *admits I-indexed colimits and for every functor $f \colon C' \to C$ the restriction functor $f^* \colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C',D)$ preserves I-indexed colimits.*

The dual statement for I-indexed limits also holds true.

Proof. By assumption, there is an adjunction of the form

$$\mathrm{colim}_I \colon \mathrm{Fun}(I,D) \rightleftarrows D \colon \mathrm{const}_I \ .$$

For any category C, it induces an adjunction of the form

$$(\mathrm{colim}_I)_* \colon \mathrm{Fun}(C, \mathrm{Fun}(I,D)) \rightleftarrows \mathrm{Fun}(C,D) \colon (\mathrm{const}_I)_* \, .$$

By virtue of Proposition 4.4.4, this yields an adjunction of the form

$$\mathrm{colim}_I \colon \mathrm{Fun}(I, \mathrm{Fun}(C,D)) \rightleftarrows \mathrm{Fun}(C,D) \colon \mathrm{const}_I \ .$$

By construction, the unit and counit of the latter adjunction are those in D, termwise with respect to C. This implies right away that, for any functor $f \colon C' \to C$ the functor

$$f^* \colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C',D)$$

preserves I-indexed colimits by Proposition 7.2.4. □

Lemma 7.3.2 (Fubini for indexed limits)**.** *Let I, J, and C be categories. If C admits I-indexed and J-indexed limits, then it admits $(I \times J)$-indexed limits. For every functor $F\colon I \times J \to C$, there are canonical isomorphisms*

$$\lim_{(i,j):I\times J} F(i,j) \simeq \lim_{i:I} \lim_{j:J} F(i,j) \simeq \lim_{j:J} \lim_{i:I} F(i,j),$$

natural in F. The analogous statement holds for colimits.

Proof. Under the exponential-law equivalence

$$\mathrm{Fun}(I \times J, C) \simeq \mathrm{Fun}(I, \mathrm{Fun}(J, C)),$$

the constant-diagram functor $C \to \mathrm{Fun}(I \times J, C)$ is the composite

$$C \xrightarrow{\mathrm{const}_J} \mathrm{Fun}(J, C) \xrightarrow{\mathrm{const}_I} \mathrm{Fun}(I, \mathrm{Fun}(J, C)).$$

By the dual statement in Lemma 7.3.1, the second functor admits the pointwise I-indexed limit functor as a right adjoint, while the first admits $\lim_J$ as a right adjoint. Their composite is therefore right adjoint to the constant $(I \times J)$-diagram functor and computes $\lim_J \lim_I$. Reversing I and J gives the other iterated limit. The displayed isomorphisms, including their naturality in F, follow from uniqueness of right adjoints. The colimit statement is dual. □

8 Universes and directed univalence

This chapter develops the universe formalism needed in the rest of the book. We first construct categories of cocartesian functors and study straightening over the walking morphism. This first form of straightening extracts transport along a single arrow and supplies the input for directed univalence. We then formulate directed univalence, introduce the universe Cat of small categories, and develop category theory internal to Cat. After constructing the universe An of animae and the groupoid-core functor, we assemble the object-level and morphism-level classifications into the straightening/unstraightening equivalence. We conclude the discussion of straightening by recording some computations with unstraightening, and then turn to geometric realizations and localizations, including their fiberwise forms.

In more detail, the main purpose of this chapter is to introduce a synthetic category Cat called the *universe of small categories*. It comes equipped with a cocartesian fibration $\pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ called the *universal small cocartesian fibration*. The conditions on Cat and π_{univ} will imply that all the constructions of synthetic category theory can be performed internally in the universe Cat. We will also construct the universe of small animae An (historically often denoted $\mathcal{S}$ and called the *'∞-category of spaces'* in the literature), together with its natural functors to and from Cat. The historical phrase does not identify animae with topological spaces.

$$\mathrm{An} \underset{(-)^{\simeq}}{\overset{\|-\|}{\rightleftarrows}} \mathrm{Cat}.$$

A fundamental property that π_{univ} is required to satisfy is *directed univalence*. To state what that says, note that to every object c of Cat we may assign a synthetic category C by forming the following pullback:

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{c} & \mathrm{Cat}. \end{array}$$

We will call a synthetic category C *small* if it is obtained in this way for a given object c of Cat. Directed univalence then demands that for two small categories C and D, corresponding to objects c and d of Cat, the anima of morphisms from c to d in Cat is equivalent to the anima of functors $C \to D$; more precisely, we require the map

$$\mathrm{Hom}_{\mathrm{Cat}}(c, d) \to \mathrm{Map}(C, D)$$

to be an equivalence. Informally, this means that working with small categories and functors between them is the same as working with objects and morphisms of Cat, justifying the terminology 'universe of small categories'.

8.1 The subcategory of cocartesian functors

In order to phrase directed univalence, we need to have a category of cocartesian functors between two cocartesian fibrations $p\colon C \to S$ and $q\colon D \to S$. Recall from Definition 4.7.2 that a functor $f\colon C \to D$ over S is a cocartesian functor if it sends p-cocartesian morphisms in C to q-cocartesian morphisms in D, see Lemma 4.7.3. The goal of this section is to construct a category $\underline{\mathrm{CoCart}}_S(C, D)$ over S whose global sections $S \to \underline{\mathrm{CoCart}}_S(C, D)$ correspond to cocartesian functors from C to D over S.

We will construct this category as a subcategory of the relative functor category $\underline{\mathrm{Fun}}_S(C, D)$ of functors $C \to D$ over S, defined in Definition 3.5.10. For the latter to make sense, recall that we need the functor p to be an *exponentiable functor*, in the sense of Definition 3.5.5. In light of this, we will impose the following axiom:

Axiom M (Exponentiability Axiom)**.** Let $p\colon S \to T$ be a cartesian or cocartesian fibration. Then p is an exponentiable functor.

Example 8.1.1. Assume that $S = \Gamma$ is an anima. By Corollary 4.5.22 any functor $p\colon C \to \Gamma$ is a cocartesian fibration, hence in particular exponentiable. It follows that the relative functor category $\underline{\mathrm{Fun}}_\Gamma(C, D)$ exists for *any* categories C and D over Γ.

Construction 8.1.2. Let $p\colon C \to S$ and $q\colon D \to S$ be two cocartesian fibrations, and let $f\colon C \to D$ be a functor over S. We will construct a category $\texttt{is_Cocart}(f)$ whose objects are proofs that f is a cocartesian functor.

Recall from Corollary 6.5.9 that a functor $f\colon C \to D$ over S is a cocartesian functor over S if and only if for every morphism $v\colon x \to y$ in S and for every object $c \in C(x)$ a certain canonical map $\mathrm{BC}_!(v, c)\colon v_! f(c) \to f(v_! c)$ is an isomorphism in D. This map is natural in the pair (v, x), in the sense that it is part of a natural transformation of functors $C \mathbin{\vec{\times}}_p S \to D$. Precomposing with the directed evaluation functor $\vec{\mathrm{ev}}\colon \mathrm{Fun}([1], C) \to C \mathbin{\vec{\times}}_p S$, we thus obtain a functor

$$\mathrm{BC}_!\colon \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], D)$$

and f is a cocartesian functor over S if and only if $\mathrm{BC}_!$ factors through the subcategory $D \simeq \mathrm{Iso}(D) \hookrightarrow \mathrm{Fun}([1], D)$ of isomorphisms.

Consider now the following pullback square:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Iso}(D)^{\simeq} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{Iso}}} \\ \mathrm{Map}([1], C) & \xrightarrow{\mathrm{BC}_!} & \mathrm{Map}([1], D). \end{array}$$

Consider also the map $p = p_{\mathrm{Map}([1],C)} \colon \mathrm{Map}([1], C) \to *$. We then define

$$\texttt{is_Cocart}(f) := p_*(P),$$

where $p_*(P)$ denotes the dependent product of P along p as defined in Definition 3.5.2.

Lemma 8.1.3. *For every f as above, the functor* $\texttt{is_Cocart}(f) \to *$ *is an embedding. In particular, if* $\texttt{is_Cocart}(f)$ *admits an (absolute) object, then it is contractible.*

Proof. Note that $P \to \mathrm{Map}([1], C)$ is an embedding, since it is the base change of the embedding π_{Iso} from Lemma 2.3.3. It follows from Corollary 3.5.9 that also the map $\texttt{is_Cocart}(f) \to *$ is an embedding. An absolute object of $\texttt{is_Cocart}(f)$ determines a section of this embedding, which then by Lemma 1.6.24 implies that this embedding is an equivalence. □

Observation 8.1.4. Note that $\texttt{is_Cocart}(f)$ admits an absolute object if and only if f is a cocartesian functor over S. Indeed, an object $* \to \texttt{is_Cocart}(f)$ corresponds by adjunction to a section of the map $P \to \mathrm{Map}([1], C)$, which in turn is equivalent to a lift of $\mathrm{BC}_!$ along π_{Iso}:

$$\begin{array}{ccc} & & \mathrm{Iso}(D)^{\simeq} \\ & \nearrow & \downarrow{\scriptstyle \pi_{\mathrm{Iso}}} \\ \mathrm{Map}([1], C) & \xrightarrow[\mathrm{BC}_!]{} & \mathrm{Map}([1], D). \end{array}$$

Such a section precisely says that the map $\mathrm{BC}_!(\varphi)$ is an isomorphism for every morphism φ in C, which is the condition for f to be a cocartesian functor.

Remark 8.1.5. The above constructions go through in the context of an arbitrary anima Γ: for a category S in context Γ, cocartesian fibration $C \to S$ and $D \to S$ in context Γ, and a functor $f \colon C \to D$ over S in context Γ, we obtain a category $\texttt{is_Cocart}_\Gamma(f)$ in context Γ.

We continue to fix two cocartesian fibrations $p \colon C \to S$ and $q \colon D \to S$ over S. Our next goal is to define a subcategory $\underline{\mathrm{CoCart}}_S(C, D) \subseteq \underline{\mathrm{Fun}}_S(C, D)$ spanned by the cocartesian functors $C \to D$ over S. More precisely, we would like that for any other functor $T \to S$, a functor $T \to \underline{\mathrm{Fun}}_S(C, D)$ over S factors through $\underline{\mathrm{CoCart}}_S(C, D)$ if and only if the associated functor $T \times_S C \to T \times_S D$ over T from Remark 3.5.11 is a cocartesian functor over T.

Definition 8.1.6. Consider a morphism in $\underline{\mathrm{Fun}}_S(C, D)$, i.e. an object of $\mathrm{Fun}([1], \underline{\mathrm{Fun}}_S(C, D))$. Such an object is given by an anima Γ together with a functor $[1] \times \Gamma \to S$ and a functor over $[1] \times \Gamma$ of the form

$$f \colon ([1] \times \Gamma) \times_S C \to ([1] \times \Gamma) \times_S D.$$

We say that the morphism is *cocartesian* if the associated map f is a cocartesian functor over $[1] \times \Gamma$. Equivalently, we may regard f as a functor over Γ and ask that it is a cocartesian functor over $[1]_\Gamma$ in context Γ. By Remark 8.1.5, this is classified by a category $\texttt{is_Cocart}_\Gamma(f)$ in context Γ.

We will apply this to the *universal* morphism of $\underline{\mathrm{Fun}}_S(C, D)$:

Construction 8.1.7. Define $T := [1] \times \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D))$. Then evaluation determines a map

$$\mathrm{ev} \colon T \to \underline{\mathrm{Fun}}_S(C, D),$$

which corresponds to a functor $T \to S$ together with a functor

$$f_{\mathrm{univ}} \colon C_T \to D_T.$$

Applying Definition 8.1.6 to the case $\Gamma = \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D))$, we thus obtain a category $\mathtt{is_Cocart}_\Gamma(f_{\mathrm{univ}})$ in context Γ, i.e. a functor

$$\mathtt{is_Cocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D)).$$

Lemma 8.1.8. *The above functor is an embedding.*

Proof. This is a special case of Lemma 8.1.3, applied in context Γ. □

Note that a morphism f in $\underline{\mathrm{Fun}}_S(C, D)$ lies in the image of this embedding if and only if the corresponding functor $([1] \times \Gamma) \times_S C \to ([1] \times \Gamma) \times_S D)$ over $[1] \times \Gamma$ is a cocartesian functor.

Lemma 8.1.9. *The embedding $M := \mathtt{is_Cocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(C, D))$ forms a collection of morphisms in $\underline{\mathrm{Fun}}_S(C, D)$ which is closed under composition, in the sense of Definition 3.1.6.*

Proof. We first show that M is closed under identity maps. But this is clear: given a morphism f in $\underline{\mathrm{Fun}}_S(C, D)$, the corresponding functors over $[1] \times \Gamma$ corresponding to $\mathrm{id}_{f(0)}$ and $\mathrm{id}_{f(1)}$ are given by pulling back the corresponding functor to f along the two maps

$$[1] \times \Gamma \to \{0\} \times \Gamma \hookrightarrow [1] \times \Gamma \qquad \text{and} \qquad [1] \times \Gamma \to \{1\} \times \Gamma \hookrightarrow [1] \times \Gamma,$$

and cocartesian functors are closed under base change, see Proposition 6.5.10.

We now show that M is closed under composition. Consider two morphisms f and g in $\underline{\mathrm{Fun}}_S(C, D)$. By the Segal Axiom, we may just as well assume they arise as the restrictions of some map $\sigma \colon [2] \times \Gamma \to \underline{\mathrm{Fun}}_S(C, D)$ along the two inclusions $[1]_{01} \hookrightarrow [2]$ and $[1]_{12} \hookrightarrow [2]$, where for $0 \leq i < j \leq 2$ we write $[1]_{ij} \subseteq [2]$ for the subcategory spanned by i and j. This map σ corresponds to a functor

$$\begin{array}{ccc} C_{[2]\times\Gamma} & \xrightarrow{\quad\sigma\quad} & D_{[2]\times\Gamma} \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & [2] \times \Gamma & \end{array}$$

over $[2] \times \Gamma$. By assumption on f and g, if we form the pullbacks of this map along the inclusions $[1]_{01} \times \Gamma \hookrightarrow [2] \times \Gamma$ and $[1]_{12} \times \Gamma \hookrightarrow [2] \times \Gamma$ we obtain cocartesian functors over $[1] \times \Gamma$. Our task is to show that also the pullback of this map along the inclusion $[1]_{02} \times \Gamma \hookrightarrow [2] \times \Gamma$ is a cocartesian functor over $[1] \times \Gamma$. To this end, consider a cocartesian morphism $x_0 \to x_2$ in $C|_{[1]_{02}\times\Gamma}$ living over the edge $0 \to 2$. By cocartesian lifting, we may factor this morphism as a composite of two morphisms $x_0 \to x_1$ and $x_1 \to x_2$ living over $0 \to 1$ and $1 \to 2$, respectively. By the assumption on f and g, these morphisms are then sent to cocartesian morphisms in $D_{[1]_{01}\times\Gamma}$ and $D_{[1]_{12}\times\Gamma}$, respectively. But then their composite must be a cocartesian morphism in $D_{[1]_{02}\times\Gamma}$. This finishes the proof. □

Definition 8.1.10. We define the (non-full) subcategory $\underline{\mathrm{CoCart}}_S(C, D) \subseteq \underline{\mathrm{Fun}}_S(C, D)$ as the subcategory spanned by M:

$$\underline{\mathrm{CoCart}}_S(C, D) := \langle M \rangle_{\underline{\mathrm{Fun}}_S(C,D)} \subseteq \underline{\mathrm{Fun}}_S(C, D).$$

Lemma 8.1.11. *Let C and D be categories, regarded as cocartesian fibrations over the terminal category $S = *$. Then we have an equivalence*

$$\underline{\mathrm{CoCart}}_*(C, D) \simeq \mathrm{Map}(C, D) \subseteq \mathrm{Fun}(C, D).$$

Proof. We need to show that a morphism $f\colon [1] \to \mathrm{Fun}(C, D)$ lies in the image of $M \hookrightarrow \mathrm{Map}([1], \mathrm{Fun}(C, D))$ if and only if it is a natural isomorphism of functors $C \to D$. By definition, the former is the case if and only if the corresponding functor $[1] \times C \to [1] \times D$ is a cocartesian functor of cocartesian fibrations over $[1]$. The claim then follows from Lemma 4.7.13. □

Lemma 8.1.12. *Let $p\colon X \to S$ and $q\colon Y \to S$ be two left fibrations. Then the inclusion*

$$\underline{\mathrm{CoCart}}_S(X, Y) \hookrightarrow \underline{\mathrm{Fun}}_S(X, Y)$$

is an equivalence.

Proof. Since every functor between two left fibrations is a cocartesian functor, it follows that the embedding $\mathtt{is_Cocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(X, Y))$ is an equivalence. The claim thus follows from Corollary 3.1.14. □

Proposition 8.1.13. *Consider a functor $T \to S$ and let $\varphi\colon T \to \underline{\mathrm{Fun}}_S(C, D)$ be a functor over S, corresponding to a functor $f\colon T \times_S C \to T \times_S D$ over T. Then f is cocartesian over T if and only if φ factors through the subcategory $\underline{\mathrm{CoCart}}_S(C, D)$.*

Proof. By definition of $\underline{\mathrm{CoCart}}_S(C, D)$ as a subcategory, φ factors through $\underline{\mathrm{CoCart}}_S(C, D)$ if and only if the induced map $\mathrm{Map}([1], T) \to \mathrm{Map}([1], \mathrm{Fun}_S(C, D)) = \mathrm{Map}_S([1] \times C, D)$ factors through M. This is in turn equivalent to the condition that for every Γ and every map $\alpha\colon \Gamma \to \mathrm{Map}_S([1], T)$, the object

$$\varphi(\alpha)\colon \Gamma \to \mathrm{Map}_S([1], \mathrm{Fun}_S(C, D))$$

factors through M. The map α corresponds to a map $\tilde{\alpha}\colon [1] \times \Gamma \to T$, and by definition of M this happens if and only if the induced functor

$$f_{\Gamma \times [1]}\colon (\Gamma \times [1]) \times_S C \to (\Gamma \times [1]) \times_S D$$

is a cocartesian functor over $\Gamma \times [1]$. Now, observe that this map is obtained from $f_T\colon T \times_S C \to T \times_S D$ by pulling back along $\tilde{\alpha}\colon \Gamma \times [1] \to T$, and thus it follows from Proposition 6.5.10 that the condition is equivalent to the condition that f_T is a cocartesian functor over T. This finishes the proof. □

Corollary 8.1.14. *For every functor $T \to S$, there is a pullback square of the form*

$$\begin{array}{ccc} \underline{\mathrm{CoCart}}_T(T \times_S C, T \times_S D) & \longrightarrow & \underline{\mathrm{CoCart}}_S(C, D) \\ \downarrow & \lrcorner & \downarrow \\ T & \xrightarrow{\quad t \quad} & S. \end{array}$$

Proof. The pullback square from Observation 3.5.12 restricts to a pullback square on the subcategories of cocartesian functors by Proposition 8.1.13. □

Thus $\underline{\mathrm{CoCart}}_S(C, D)$ represents cocartesian functors from C to D over S, and this representation is stable under change of base. It will provide the target of the straightening map used to formulate directed univalence.

8.2 Straightening over the walking morphism

Recall that for a cocartesian fibration $W \to [1]$, the canonical morphism $v\colon 0 \to 1$ in $[1]$ determines a covariant transport functor $v_!\colon W_0 \to W_1$. We will now construct a version of covariant transport that works over an arbitrary base category.

Construction 8.2.1. Let S be a category, let $p\colon W \to [1] \times S$ be a cocartesian fibration, and denote by $p_i\colon W_i \to S$ for $i = 0, 1$ the base change of p along the inclusion $(i, \mathrm{id})\colon S \to [1] \times S$. We will construct a cocartesian functor over $[1] \times S$ of the form

$$\begin{array}{ccc} [1] \times W_0 & \xrightarrow{\quad l \quad} & W \\ & {\scriptstyle \mathrm{id}_{[1]} \times p_0} \searrow \quad \swarrow {\scriptstyle p} & \\ & [1] \times S. & \end{array}$$

The functor l will be constructed as a dashed filler fitting in the following commutative diagram:

$$\begin{array}{ccc} W_0 & \xrightarrow{\iota_0} & W \\ {\scriptstyle (0,\mathrm{id}_{W_0})}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ [1] \times W_0 & \xrightarrow[\mathrm{id}_{[1]} \times p_0]{} & [1] \times S. \end{array}$$

Observe that such a choice of lift is equivalent to finding a filler for the following commutative diagram:

$$\begin{array}{ccc} \{0\} & \xrightarrow{\iota_0} & \mathrm{Fun}(W_0, W) \\ \downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle p_*} \\ [1] & \longrightarrow & \mathrm{Fun}(W_0, [1] \times S), \end{array}$$

where the right vertical functor is given by postcomposition with p and the bottom map is adjoint to $[1] \times p_0\colon [1] \times W_0 \to [1] \times S$. Equivalently, we need to find a morphism in $\mathrm{Fun}(W_0, W)$ whose source is the inclusion $\iota_0\colon W_0 \to W$ and which lifts the morphism in $\mathrm{Fun}(W_0, [1] \times S)$ adjoint to $\mathrm{id}_{[1]} \times p_0\colon [1] \times W_0 \to [1] \times S$. Since p_* is a cocartesian fibration by Proposition 4.5.13, we may simply choose k to be a cocartesian lift which starts in ι_0.

Unwinding the proof of Proposition 4.5.13, we obtain an explicit description of l at the level of objects: given an object $(\varepsilon, w) \in [1] \times W_0$, write $s := p_0(w) \in S$ and let $u\colon 0 \to \varepsilon$ denote the morphism in $[1]$ obtained from the fact that $0 \in [1]$ is an initial object, see Lemma 4.4.22. Then we have

$$l(\varepsilon, w) = (u, \mathrm{id}_s)_!(w),$$

where the functor $(u, \mathrm{id}_s)_!\colon W_{(0,s)} \to W_{(\varepsilon,s)}$ denotes cocartesian transport along the morphism $(u, \mathrm{id}_s)\colon (0, s) \to (\varepsilon, s)$ in $[1] \times S$.

Lemma 8.2.2. *The functor $l\colon [1] \times W_0 \to W$ constructed in Construction 8.2.1 is a cocartesian functor over $[1] \times S$.*

Proof. Consider a morphism $\psi\colon (\varepsilon, s) \to (\varepsilon', s')$ in $[1] \times S$ and an object $(\varepsilon, w) \in \{\varepsilon\} \times (W_0)_s = ([1] \times W_0)_{(\varepsilon,s)}$. By Corollary 6.5.9, it will suffice to show that the canonical map

$$\overline{\psi}\colon \psi_!(l(\varepsilon, w)) \to l(\psi_!(\varepsilon, w))$$

is an isomorphism in W. Observe that we may factor ψ as a composite

$$(\varepsilon, s) \to (\varepsilon', s) \to (\varepsilon', s'),$$

and hence since the covariant transport functors compose, Corollary 4.5.11, it will suffice to show the claim in the two cases where either $\varepsilon = \varepsilon'$ or $s = s'$:

- Assume that $s = s'$ and $\psi = (v, \mathrm{id}_s) \colon (\varepsilon, s) \to (\varepsilon', s)$ for some morphism $v \colon \varepsilon \to \varepsilon'$ in $[1]$. Letting $u_\varepsilon \colon 0 \to \varepsilon$ and $u_{\varepsilon'} \colon 0 \to \varepsilon'$ denote the canonical maps in $[1]$, the map $\overline{\psi}$ then simplifies to the map
$$(v, \mathrm{id}_s)_!(u_\varepsilon, \mathrm{id}_s)_!(w) \to (u_{\varepsilon'}, \mathrm{id}_s)_!(w).$$
But since $u_{\varepsilon'} \colon 0 \to \varepsilon'$ is the composite of $u_\varepsilon \colon 0 \to \varepsilon$ and $v \colon \varepsilon \to \varepsilon'$, this map is an isomorphism by Corollary 4.5.11.

- Assume that $\varepsilon = \varepsilon'$ and that $\psi = (\mathrm{id}_\varepsilon, v)$ for some morphism $v \colon s \to s'$ in S. The canonical comparison $(\varepsilon, w) \xrightarrow{\sim} (u_\varepsilon, \mathrm{id}_s)_!(0, w)$, together with the previous case, reduces the claim to $\varepsilon = 0$. The map $\overline{\psi}$ then simplifies to the map
$$(\mathrm{id}_0, v)_!(u_0, \mathrm{id}_s)_!(0, w) \to (u_0, \mathrm{id}_{s'})_!(\mathrm{id}_0, v)_!(0, w),$$
which is an isomorphism by the unit coherence for cocartesian transport, since $u_0 \colon 0 \to 0$ is the identity.

This finishes the proof. □

Definition 8.2.3 (Straightening)**.** Let $p \colon W \to [1] \times S$ be a cocartesian fibration, and denote by $p_i \colon W_i \to S$ for $i = 0, 1$ the base change of p along the inclusion $(i, \mathrm{id}) \colon S \to [1] \times S$. We define the *straightening* of p over $[1]$ to be the cocartesian functor over S of the form

$$\begin{array}{ccc} W_0 & \xrightarrow{\mathrm{St}(p)} & W_1 \\ & {\scriptstyle p_0}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & S, & \end{array}$$

obtained from the cocartesian functor $l \colon [1] \times W_0 \to W$ over $[1] \times S$ by forming the base change along $(1, \mathrm{id}) \colon S \to [1] \times S$. Note that $\mathrm{St}(p)$ corresponds to a section

$$\sigma_p \colon S \to \underline{\mathrm{CoCart}}_S(W_0, W_1)$$

of the structural functor $\underline{\mathrm{CoCart}}_S(W_0, W_1) \to S$.

Remark 8.2.4. If $S' \to S$ is any functor and if $p \colon W \to [1] \times S$ is a cocartesian fibration, we may form the pullback

$$\begin{array}{ccc} W' & \longrightarrow & W \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ [1] \times S' & \longrightarrow & [1] \times S. \end{array}$$

We obtain a pullback square

$$\begin{array}{ccc} \underline{\mathrm{CoCart}}_{S'}(W_0', W_1') & \longrightarrow & \underline{\mathrm{CoCart}}_{S}(W_0, W_1) \\ \downarrow & \lrcorner & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

There are now a priori two ways of constructing a cocartesian functor $W_0' \to W_1'$ over S': we can either apply Definition 8.2.3 directly to p', or we first apply Definition 8.2.3 to p to get a cocartesian functor $W_0 \to W_1$ over S and then pull back along $S' \to S$. One can show that these two functors agree. In other words, the diagram

$$\begin{array}{ccc} \underline{\mathrm{CoCart}}_{S'}(W_0', W_1') & \longrightarrow & \underline{\mathrm{CoCart}}_{S}(W_0, W_1) \\ {\scriptstyle \sigma_{p'}}\uparrow & & \uparrow{\scriptstyle \sigma_p} \\ S' & \longrightarrow & S \end{array}$$

is commutative.

8.2.1 Weak functoriality of straightening over $[1]$

Given a cocartesian fibration $W \to [2]$, we obtain functors $W_0 \to W_1$, $W_1 \to W_2$ and $W_0 \to W_2$ by performing covariant transport along the three maps $0 \to 1$, $1 \to 2$ and $0 \to 2$ in $[2]$. We have seen in Corollary 4.5.11 that these three functors fit into a commutative triangle

$$\begin{array}{ccc} & W_1 & \\ \nearrow & & \searrow \\ W_0 & \longrightarrow & W_2. \end{array}$$

Similarly, if the cocartesian fibration $W \to [1]$ were pulled back along the map $[1] \to [0]$ from some functor $W' \to [0]$, then the corresponding covariant transport functor $W' \to W'$ is identified with the identity functor. This expresses the functoriality of covariant transport.

As we shall now see, the analogous statements still hold true over an arbitrary base S.

Lemma 8.2.5. *Consider a cocartesian fibration $p' \colon W' \to S$, and let $p \colon [1] \times W' \to [1] \times S$ be its pullback along the projection $\mathrm{pr}_S \colon [1] \times S \to S$:*

$$\begin{array}{ccc} [1] \times W' & \longrightarrow & W' \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle p'} \\ [1] \times S & \xrightarrow{\mathrm{pr}_S} & S. \end{array}$$

Then the straightening functor $\mathrm{St}(p) \colon W' \to W'$ of p is identified with the identity of W' over S.

Proof. This is clear from the fact that the dashed diagonal map in the diagram in Construction 8.2.1 may be taken to be the identity map on $[1] \times W'$. □

Proposition 8.2.6. *Consider a cocartesian fibration $p\colon W \to [2] \times S$. For $0 \leq i < j \leq 2$, define the cocartesian fibration $p_{ij}\colon W_{ij} \to [1]_{ij} \times S$ via the following pullback square:*

$$\begin{array}{ccc} W_{ij} & \longrightarrow & W \\ {\scriptstyle p_{ij}}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ [1]_{ij} \times S & \hookrightarrow & [2] \times S, \end{array}$$

where $[1]_{ij}$ is the subcategory of $[2]$ spanned by i and j. Then there is an identification

$$\mathrm{St}(p_{02}) = \mathrm{St}(p_{12}) \circ \mathrm{St}(p_{01})$$

of cocartesian functors $W_0 \to W_2$ over S.

Proof. The proof has two steps. We first extend the three transport constructions from the edges of $[2]$ to cocartesian functors over $[2] \times S$. We then compare the extension associated to $0 \to 2$ with the composite of the extensions associated to $0 \to 1$ and $1 \to 2$.

Via Construction 8.2.1, we obtain cocartesian functors

$$\begin{array}{ccc} [1]_{01} \times W_0 & \xrightarrow{l_{01}} & W_{01} \\ & {\scriptstyle \mathrm{id} \times p_0}\searrow \quad \swarrow{\scriptstyle p_{01}} & \\ & [1]_{01} \times S, & \end{array} \qquad \begin{array}{ccc} [1]_{12} \times W_1 & \xrightarrow{l_{12}} & W_{12} \\ & {\scriptstyle \mathrm{id} \times p_1}\searrow \quad \swarrow{\scriptstyle p_{12}} & \\ & [1]_{12} \times S & \end{array}$$

and

$$\begin{array}{ccc} [1]_{02} \times W_0 & \xrightarrow{l_{02}} & W_{02} \\ & {\scriptstyle \mathrm{id} \times p_0}\searrow \quad \swarrow{\scriptstyle p_{02}} & \\ & [1]_{02} \times S. & \end{array}$$

Restricting these three functors to $\{j\} \times S \hookrightarrow [1]_{ij} \times S$ produces the three functors $\mathrm{St}(p_{01})$, $\mathrm{St}(p_{12})$ and $\mathrm{St}(p_{02})$. We claim that these three diagrams can all be extended to cocartesian functors over $[2] \times S$ of the form

$$\begin{array}{ccc} [2] \times W_0 & \xrightarrow{\bar{l}_{01}} & W_{01} \times_{[1]_{01}} [2] \\ & {\scriptstyle \mathrm{id} \times p_0}\searrow \quad \swarrow & \\ & [2] \times S, & \end{array} \qquad \begin{array}{ccc} W_{01} \times_{[1]_{01}} [2] & \xrightarrow{\bar{l}_{12}} & W \\ & {\scriptstyle \mathrm{id} \times p_1}\searrow \quad \swarrow{\scriptstyle p} & \\ & [2] \times S & \end{array}$$

and

$$\begin{array}{ccc} [2] \times W_0 & \xrightarrow{\bar{l}_{02}} & W \\ & {\scriptstyle \mathrm{id} \times p_0}\searrow \quad \swarrow{\scriptstyle p} & \\ & [2] \times S. & \end{array}$$

The map $\bar{l}_{01}$ is obtained from l_{01} by pulling back along the map $s_1\colon [2] \to [1]$ (given by $0 \mapsto 0$, $1 \mapsto 1$ and $2 \mapsto 1$). To obtain the map $\bar{l}_{12}$, we write $[2]$ as a pushout of the span $[1]_{01} \leftarrow \{1\} \to$

$[1]_{12}$, giving a pushout square

$$\begin{array}{ccc} W_1 & \hookrightarrow & W_{01} \\ {\scriptstyle (1,\mathrm{id})}\downarrow & \ulcorner & \downarrow \\ [1]_{12} \times W_1 & \longrightarrow & W_{01} \times_{[1]_{01}} [2]. \end{array}$$

The map $\bar{l}_{12}$ is then given on W_{01} by the inclusion $W_{01} \hookrightarrow W$ and on $[1]_{12} \times W_1$ by the composite

$$[1]_{12} \times W_1 \xrightarrow{l_{12}} W_{12} \hookrightarrow W;$$

this is well-defined since the restriction of both functors to W_1 is the inclusion $W_1 \hookrightarrow W$. Since both l_{12} and the inclusion preserve cocartesian morphisms, it follows that $\bar{l}_{12}$ preserves cocartesian morphisms lying over either $[1]_{01}$ or $[1]_{12}$, from which it follows that $\bar{l}_{12}$ is indeed a cocartesian functor over $[2] \times S$.

It remains to extend the transport associated to $0 \to 2$. The map $\bar{l}_{02}$ is defined by finding a lift in the following commutative diagram:

$$\begin{array}{ccccc} \{1\} & \longrightarrow & [1]_{01} & \xrightarrow{k_{01}} & \mathrm{Fun}(W_0, W) \\ \downarrow & & \downarrow & \overset{\overline{k_{02}}}{\nearrow} & \downarrow{\scriptstyle p_*} \\ [1]_{12} & \longrightarrow & [2] & \longrightarrow & \mathrm{Fun}(W_0, [2] \times S). \end{array}$$

Since the left square is a pushout square, it suffices to construct a diagonal map $[1] \to \mathrm{Fun}(W_0, W)$ making the top and bottom triangle commute, but such a map exists uniquely due to the fact that p_* is a cocartesian fibration. Observe that the pullback of the map $\bar{l}_{02}$ along the inclusion $[1]_{02} \times S \hookrightarrow [2] \times S$ is indeed the map l_{02}: the restriction of its curried functor $\overline{k_{02}}$ to $[1]_{02} \hookrightarrow [2]$ is a solution of the lifting problem

$$\begin{array}{ccc} \{0\} & \longrightarrow & \mathrm{Fun}(W_0, W_{02}) \\ \downarrow & \overset{\overline{k_{02}}}{\nearrow} & \downarrow{\scriptstyle p_*} \\ [1]_{02} & \longrightarrow & \mathrm{Fun}(W_0, [1]_{02} \times S) \end{array}$$

and must hence agree with the (unique) solution k_{02}.

To finish the proof, observe that the composite $\bar{l}_{12} \circ \bar{l}_{01} \colon [2] \times W_0 \to W$ is another cocartesian functor over $[2] \times S$ that satisfies the same lifting criterion as $\bar{l}_{02}$. By uniqueness, we thus obtain an identification

$$\bar{l}_{02} = \bar{l}_{12} \circ \bar{l}_{01}.$$

Pulling back along the inclusion $\{2\} \times S \hookrightarrow [2] \times S$ then provides the desired identification

$$\mathrm{St}(p_{02}) = \mathrm{St}(p_{12}) \circ \mathrm{St}(p_{01}). \qquad \square$$

Consequently, straightening over $[1]$ respects identities and composition. This is the coherence needed to interpret the arrows and composable arrows of a universe as functors and composable functors between the categories they classify.

8.3 Directed univalence

In this section, we introduce a class of cocartesian fibrations called *directed univalent* cocartesian fibrations. The guiding idea is that directed univalence identifies morphisms in U with functors between the U-small categories classified by their source and target, compatibly in families.

To formulate this precisely, we fix a cocartesian fibration $\pi\colon U_\bullet \to U$. We first define U-small cocartesian fibrations and construct a category classifying cocartesian functors between them. We then use straightening over $[1]$ to define directed univalence.

8.3.1 Small cocartesian fibrations

We start by defining what it means for a cocartesian fibration to be 'small', and by introducing various categories that classify small cocartesian fibrations. Throughout this section, we fix a cocartesian fibration $\pi\colon U_\bullet \to U$.

Definition 8.3.1. We say that a cocartesian fibration $p\colon E \to S$ is *U-small* if it comes equipped with a pullback square of the form

$$\begin{array}{ccc} E & \longrightarrow & U_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\ulcorner p \urcorner} & U. \end{array}$$

We say that the map $\ulcorner p \urcorner$ *classifies the fibration* p, or that p *is classified by the map* $\ulcorner p \urcorner$. We say that a category C is *U-small* if the functor $p_C\colon C \to *$ is U-small.

Terminology 8.3.2. We say that the category U *classifies* U-small cocartesian fibrations, and that the fibration $\pi\colon U_\bullet \to U$ is the *universal U-small cocartesian fibration*.

Observe that the category $U \times U$ classifies *pairs* of U-small cocartesian fibrations:

Observation 8.3.3. Let S be a category equipped with two U-small cocartesian fibrations $p_1\colon E_1 \to S$ and $p_2\colon E_2 \to S$, classified by functors $\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner\colon S \to U$. Then p_1 and p_2 may be obtained via pullback from the functors $\pi \times \mathrm{id}\colon U_\bullet \times U \to U \times U$ and $\mathrm{id} \times \pi\colon U \times U_\bullet \to U \times U$:

$$\begin{array}{ccccc} E_1 & \longrightarrow & U_\bullet \times U & \xrightarrow{\mathrm{pr}_1} & U_\bullet \\ {\scriptstyle p_1}\downarrow & \lrcorner & {\scriptstyle \pi\times\mathrm{id}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow[(\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner)]{} & U \times U & \xrightarrow{\mathrm{pr}_1} & U \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} E_2 & \longrightarrow & U \times U_\bullet & \xrightarrow{\mathrm{pr}_2} & U_\bullet \\ {\scriptstyle p_2}\downarrow & \lrcorner & {\scriptstyle \mathrm{id}\times\pi}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow[(\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner)]{} & U \times U & \xrightarrow{\mathrm{pr}_2} & U. \end{array}$$

We may think of the pair of cocartesian fibrations $\pi \times \mathrm{id}\colon U_\bullet \times U \to U \times U$ and $\mathrm{id} \times \pi\colon U \times U_\bullet \to U \times U$ as the *universal pair of U-small cocartesian fibrations*.

Observation 8.3.4. Given the pair of cocartesian fibrations $\pi \times \mathrm{id}\colon U_\bullet \times U \to U \times U$ and $\mathrm{id} \times \pi\colon U \times U_\bullet \to U \times U$, Definition 3.5.10 provides a category $\underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ equipped with a functor

$$\underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \to U \times U.$$

Given a category S, it follows from Remark 3.5.11 that the data of a functor $S \to \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ is the same as the data of a functor $(\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner)\colon S \to U \times U$ together with a functor

$E_1 \to E_2$ over S between the cocartesian fibrations $p_1 \colon E_1 \to S$ and $p_2 \colon E_2 \to S$ classified by $\lceil p_1 \rceil$ and $\lceil p_2 \rceil$. In other words: the category $\underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ classifies *functors between two U-small cocartesian fibrations*.

Observation 8.3.5. Consider the subcategory

$$\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \subseteq \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$$

of cocartesian functors over $U \times U$, defined in Definition 8.1.10. By Proposition 8.1.13, a functor $S \to \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ factors through this subcategory if and only if the functor $E_1 \to E_2$ over S from Observation 8.3.4 is a cocartesian functor over S. In other words: the category $\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ classifies *cocartesian* functors between two U-small cocartesian fibrations.

Remark 8.3.6. Recall from Observation 3.5.12 and Corollary 8.1.14 that the constructions $\underline{\mathrm{Fun}}$ and $\underline{\mathrm{CoCart}}$ are compatible with pullbacks. It thus follows that for any pair (p_1, p_2) of U-small cocartesian fibrations over S, classified by functors $\lceil p_1 \rceil, \lceil p_2 \rceil \colon S \to U$, there are pullback squares of the form

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_S(E_1, E_2) & \longrightarrow & \underline{\mathrm{Fun}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \\ \downarrow & \lrcorner & \downarrow \\ S & \xrightarrow{(\lceil p_1 \rceil, \lceil p_2 \rceil)} & U \times U \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathrm{CoCart}}_S(E_1, E_2) & \longrightarrow & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \\ \downarrow & \lrcorner & \downarrow \\ S & \xrightarrow{(\lceil p_1 \rceil, \lceil p_2 \rceil)} & U \times U. \end{array}$$

Using this remark, we may be more precise about the way that the category $\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ 'classifies' cocartesian functors between two cocartesian fibrations.

Definition 8.3.7. Write $T := \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ for short. Define cocartesian fibrations $\overline{\pi}_1 \colon \overline{U}_1 \to T$ and $\overline{\pi}_2 \colon \overline{U}_2 \to T$ via the following pullback squares:

$$\begin{array}{ccc} \overline{U}_1 & \longrightarrow & U_\bullet \times U \\ {\scriptstyle \overline{\pi}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi \times \mathrm{id}} \\ T & \longrightarrow & U \times U \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \overline{U}_2 & \longrightarrow & U \times U_\bullet \\ {\scriptstyle \overline{\pi}_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{id} \times \pi} \\ T & \longrightarrow & U \times U. \end{array}$$

Since the map $T \to U \times U$ lifts to a map $T \to \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ (namely the identity map on T), it follows from Observation 8.3.5 that the cocartesian fibrations $\overline{\pi}_1$ and $\overline{\pi}_2$ come equipped with a cocartesian functor

$$\begin{array}{ccc} \overline{U}_1 & \xrightarrow{f_{\mathrm{univ}}} & \overline{U}_2 \\ & {\scriptstyle \overline{\pi}_1} \searrow \quad \swarrow {\scriptstyle \overline{\pi}_2} & \\ & T. & \end{array}$$

Lemma 8.3.8. *Let $p_1 \colon E_1 \to S$ and $p_2 \colon E_2 \to S$ be two U-small cocartesian fibrations, and let $f \colon E_1 \to E_2$ be a cocartesian functor over S. Then there exists a functor $\lceil f \rceil \colon S \to T :=$*

$\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ *such that* f *is the base change of* f_{univ} *along* $\ulcorner f \urcorner$*:*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & S & \end{array} \qquad \simeq \qquad \begin{array}{ccc} \ulcorner f \urcorner^*(\overline{U}_1) & \xrightarrow{\ulcorner f \urcorner^*(f_{\mathrm{univ}})} & \ulcorner f \urcorner^*(\overline{U}_2) \\ & {\scriptstyle \ulcorner f \urcorner^*(\overline{\pi}_1)}\searrow \quad \swarrow{\scriptstyle \ulcorner f \urcorner^*(\overline{\pi}_2)} & \\ & S. & \end{array}$$

Proof. We saw in Observation 8.3.5 that the cocartesian functors between U-small cocartesian fibrations over a category S are classified by functors $S \to T = \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$. This process is functorial in S: given functors $\varphi\colon S \to S'$ and $\psi\colon S' \to T$, the cocartesian functor over S classified by the composite $\psi \circ \varphi\colon S \to T$ is obtained from the one over S' classified by ψ by pulling back along φ. Taking $S' = T$ now gives the claim. □

8.3.2 Directed univalence

We now make the guiding idea precise, still fixing a cocartesian fibration $\pi\colon U_\bullet \to U$. A morphism $c \to d$ in U may be regarded as a functor $[1] \to U$. Pulling back π along this functor gives a U-small cocartesian fibration over $[1]$, whose straightening is a functor $C \to D$ between the U-small categories classified by c and d. Applying this construction to the family of all morphisms in U produces the comparison functor used in the definition.

Construction 8.3.9 (Universal straightening)**.** Let $\pi\colon U_\bullet \to U$ be a cocartesian fibration. We will construct a *straightening functor*

$$\begin{array}{ccc} \mathrm{Fun}([1], U) & \xrightarrow{\mathrm{St}_U} & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \\ & {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)}\searrow \quad \swarrow & \\ & U \times U. & \end{array}$$

To this end, consider the cocartesian fibration $p\colon W \to [1] \times \mathrm{Fun}([1], U)$ defined as the base change of π along the evaluation functor:

$$\begin{array}{ccc} W & \longrightarrow & U_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ [1] \times \mathrm{Fun}([1], U) & \xrightarrow{\mathrm{ev}} & U. \end{array}$$

(Observe that p is the universal U-small cocartesian fibration of the form $W \to [1] \times S$ for some S.) Applying the straightening procedure from Section 8.2, we thus obtain a cocartesian functor

$$\begin{array}{ccc} W_0 & \xrightarrow{\mathrm{St}(p)} & W_1 \\ & {\scriptstyle p_0}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & \mathrm{Fun}([1], U) & \end{array}$$

of cocartesian fibrations over $\mathrm{Fun}([1], U)$, where W_0 and W_1 are defined as the following base changes of π:

$$\begin{array}{ccc} W_i & \longrightarrow & U_\bullet \\ {\scriptstyle p_i}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ \mathrm{Fun}([1], U) & \xrightarrow{\mathrm{ev}_i} & U. \end{array}$$

Applying Observation 8.3.5 to the case $S = \mathrm{Fun}([1], U)$, we see that the cocartesian functor $\mathrm{St}(p)$ over S is precisely classifying a lift St_U in the following commutative diagram:

$$\begin{array}{ccc} & & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \\ & \overset{\mathrm{St}_U}{\nearrow} & \downarrow \\ \mathrm{Fun}([1], U) & \xrightarrow[(\mathrm{ev}_0, \mathrm{ev}_1)]{} & U \times U. \end{array}$$

This finishes the construction of St_U.

Definition 8.3.10 (Directed univalence)**.** A cocartesian fibration $\pi\colon U_\bullet \to U$ is called *directed univalent* if the functor $\mathrm{St}_U\colon \mathrm{Fun}([1], U) \to \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$ is an equivalence.

Perhaps the most important consequence of directed univalence is that it provides a translation between morphisms in U and functors of U-small categories. To see this, consider two objects c and d of U, and consider their corresponding U-small categories C and D:

$$\begin{array}{ccc} C & \longrightarrow & U_\bullet \\ \downarrow & \lrcorner & \downarrow \pi \\ * & \xrightarrow{c} & U \end{array} \qquad \text{and} \qquad \begin{array}{ccc} D & \longrightarrow & U_\bullet \\ \downarrow & \lrcorner & \downarrow \pi \\ * & \xrightarrow{d} & U. \end{array}$$

Since the map St_U is a functor over $U \times U$, it induces a functor on fibers over (c, d). This functor takes the form

$$\mathrm{Hom}_U(c, d) \to \underline{\mathrm{CoCart}}_{\{(c,d)\}}(C, D) \simeq \mathrm{Map}(C, D),$$

where the equivalence comes from Lemma 8.1.11.

Corollary 8.3.11. *Let $\pi\colon U_\bullet \to U$ be a directed univalent cocartesian fibration. For every pair of objects c and d of U with corresponding U-small categories C and D, the induced map*

$$\mathrm{Hom}_U(c, d) \to \mathrm{Map}(C, D)$$

is an equivalence of animae. □

In other words, morphisms in U from c to d are nothing but functors from C to D. It then follows from Proposition 8.2.6 and Lemma 8.2.5 that under this correspondence identities and compositions in U correspond to identity functors and composition of functors. All in all we get the following dictionary:

Objects c of U	$\Longleftrightarrow$	U-small categories C
Morphisms $c \to d$ in U	$\Longleftrightarrow$	Functors $C \to D$
Identity id_c in U	$\Longleftrightarrow$	The identity functor $\mathrm{id}_C\colon C \to C$
Composition in U	$\Longleftrightarrow$	Composition of functors
Commutative triangles $[2] \to U$	$\Longleftrightarrow$	Commutative triangles $\begin{array}{ccc} & D & \\ \nearrow & & \searrow \\ C & \longrightarrow & E \end{array}$
Commutative squares $[1] \times [1] \to U$	$\Longleftrightarrow$	Commutative squares $\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ C' & \longrightarrow & D' \end{array}$

8.3.3 Consequences of directed univalence

We continue to fix a directed univalent cocartesian fibration $\pi : U_\bullet \to U$. We draw three consequences from directed univalence. First, it identifies natural transformations between classifying functors with cocartesian functors between the corresponding families. Second, it shows that a small structure on a cocartesian fibration is a property. Finally, it implies univalence in Voevodsky's sense.

Definition 8.3.12. Consider two cocartesian fibrations $p_0 : E_0 \to S$ and $p_1 : E_1 \to S$. We define the *cocartesian mapping anima over S* as

$$\mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1) := (p_S)_* \begin{pmatrix} \underline{\mathrm{CoCart}}_S(E_0, E_1) \\ \downarrow \\ S \end{pmatrix}^{\simeq} .$$

We observe that by virtue of Corollary 3.5.14, there is a canonical equivalence of the form

$$\mathrm{Map}_S(E_0, E_1) \xrightarrow{\sim} (p_S)_* \begin{pmatrix} \underline{\mathrm{Fun}}_S(E_0, E_1) \\ \downarrow \\ S \end{pmatrix}^{\simeq} .$$

There is thus an embedding $\mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1) \hookrightarrow \mathrm{Map}_S(E_0, E_1)$.

Lemma 8.3.13. *Given two left fibrations $p_0 : X_0 \to S$ and $p_1 : X_1 \to S$, the inclusion*

$$\mathrm{Map}_S^{\mathrm{CoCart}}(X_0, X_1) \hookrightarrow \mathrm{Map}_S(X_0, X_1)$$

is an equivalence.

Proof. This is an immediate consequence of Lemma 8.1.12. □

Now assume that the cocartesian fibrations $p_i : E_i \to S$ are U-small: there are functors $F_i : S \to U$ for $i = 0, 1$ and pullback squares

$$\begin{array}{ccc} E_i & \xrightarrow{\widetilde{F_i}} & U_\bullet \\ {\scriptstyle p_i}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{F_i} & U. \end{array}$$

Lemma 8.3.14. *Given functors $F_0, F_1 : S \to U$, the fiber of the functor*

$$\mathrm{Fun}(S, \mathrm{CoCart}_{U\times U}(U_\bullet \times U, U \times U_\bullet)) \to \mathrm{Fun}(S, U) \times \mathrm{Fun}(S, U)$$

at the pair (F_0, F_1) is $\mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1)$.

Proof. The fibers of this functor are animae: indeed, we are considering the fibers of a functor equivalent to

$$\mathrm{Fun}(S, \mathrm{Fun}([1], U)) \simeq \mathrm{Fun}([1], \mathrm{Fun}(S, U)) \to \mathrm{Fun}(S, U) \times \mathrm{Fun}(S, U)\,,$$

and since the latter is known to be conservative, this proves our claim. Consider an anima Γ. Then a map $\Gamma \to \mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1)$ corresponds by definition to a cocartesian functor $f\colon \Gamma \times E_0 \to \Gamma \times E_1$ over $\Gamma \times S$.

On the other hand, a map from Γ to the fiber is given as a map $Y_0 \to Y_1$ over $\Gamma \times S$ together with a proof that it is a cocartesian functor over $\Gamma \times S$, where Y_0 and Y_1 are defined via the pullback squares

$$\begin{array}{ccccc} Y_i & \longrightarrow & & & U_\bullet \\ \downarrow & \lrcorner & & & \downarrow{\scriptstyle \pi} \\ \Gamma \times S & \xrightarrow{\mathrm{pr}} & S & \xrightarrow{F_i} & U \end{array}$$

But it is easy to see that Y_0 and Y_1 are equivalent to $\Gamma \times E_0$ and $\Gamma \times E_1$, respectively. □

Theorem 8.3.15. *For U-small cocartesian fibrations $p_i\colon E_i \to S$ classified by functors $F_i\colon S \to U$, there is an equivalence of animae*

$$\mathrm{Hom}_{\mathrm{Fun}(S,U)}(F_0, F_1) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1).$$

Proof. By definition, the hom anima $\mathrm{Hom}_{\mathrm{Fun}(S,U)}(F_0, F_1)$ is defined via the following pullback square:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Fun}(S,U)}(F_0, F_1) & \longrightarrow & \mathrm{Fun}([1], \mathrm{Fun}(S,U)) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{(F_0,F_1)} & \mathrm{Fun}(S,U) \times \mathrm{Fun}(S,U). \end{array}$$

By the previous lemma, it will thus suffice to show that the right vertical map is equivalent to the map in the lemma. But this follows directly from the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Fun}([1], \mathrm{Fun}(S,U)) & \xrightarrow{\simeq} & \mathrm{Fun}(S, \mathrm{Fun}([1], U)) & \xrightarrow{\mathrm{St}_*} & \mathrm{Fun}(S, \mathrm{CoCart}_{U\times U}(U_\bullet \times U, U \times U_\bullet)) \\ \downarrow & & & & \downarrow \\ \mathrm{Fun}(S,U) \times \mathrm{Fun}(S,U) & = & = & = & \mathrm{Fun}(S,U) \times \mathrm{Fun}(S,U), \end{array}$$

since the top two functors are equivalences. □

Corollary 8.3.16. *For U-small cocartesian fibrations $p_i\colon E_i \to S$ classified by functors $F_i\colon S \to U$, the equivalence of Theorem 8.3.15 restricts to an equivalence of animae*

$$\mathrm{Fun}(S,U)^{\simeq}(F_0, F_1) \xrightarrow{\sim} \mathrm{Equiv}_S(E_0, E_1).$$

Proof. Let $u\colon E_0 \to E_1$ be an equivalence over S, with inverse $v\colon E_1 \to E_0$. Both are cocartesian functors by Corollary 4.7.5, and hence correspond to natural transformations $\varphi\colon F_0 \to F_1$ and $\psi\colon F_1 \to F_0$. Compatibility of straightening with composition, Proposition 8.2.6, shows that $\varphi \circ \psi$ and $\psi \circ \varphi$ are identified with the respective identity transformations. Thus φ is a natural isomorphism.

Conversely, let $\varphi\colon F_0 \to F_1$ be a natural isomorphism, with inverse ψ. Under Theorem 8.3.15, these transformations correspond to cocartesian functors $u\colon E_0 \to E_1$ and $v\colon E_1 \to E_0$. Compatibility with composition and identities shows that $u \circ v$ and $v \circ u$ are identified over S with the respective identity functors. Therefore u is an equivalence over S. □

Definition 8.3.17 (U-small structures)**.** Let $p\colon C \to S$ be a cocartesian fibration. We define the *anima of U-small structures on p* by

$$\mathrm{Small}_U(p) := \sum_{F:\mathrm{Fun}(S,U)^{\simeq}} \mathrm{Equiv}_S\big(C, F^*U_\bullet\big).$$

Thus a term consists of a functor $F\colon S \to U$ and an equivalence over S from C to the pullback of $U_\bullet$ along F. Equivalently, it is a U-small structure in the sense of Definition 8.3.1.

Theorem 8.3.18 (Smallness is a property)**.** *Let $\pi\colon U_\bullet \to U$ be directed univalent, and let $p\colon C \to S$ be a cocartesian fibration. The canonical functor*

$$\mathrm{Small}_U(p) \to *$$

is an embedding. If p is U-small, then $\mathrm{Small}_U(p)$ *is contractible.*

Proof. Consider two terms (F, e) and (F', e') of $\mathrm{Small}_U(p)$. Their mapping anima in $\mathrm{Small}_U(p)$ is the fiber over the equivalence $e' \circ e^{-1}$ of the equivalence

$$\mathrm{Fun}(S,U)^{\simeq}(F,F') \xrightarrow{\sim} \mathrm{Equiv}_S(F^*U_\bullet, F'^*U_\bullet)$$

from Corollary 8.3.16. This fiber is contractible. The same argument applies in every anima context, so $\mathrm{Small}_U(p) \to *$ is an embedding by Proposition 1.6.20. If p is U-small, then this embedding has a section, so it is an equivalence by Lemma 1.6.24. □

Remark 8.3.19 (Directed univalence implies ordinary univalence)**.** The preceding theorem has the following universal form. Consider the diagram

$$\begin{array}{ccc} \mathrm{Iso}(U) & \dashrightarrow & \mathrm{Equiv}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Fun}([1],U) & \xrightarrow{\mathrm{St}} & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet). \end{array}$$

We claim that this is a pullback square and thus that the dashed map is an equivalence. Indeed, given any category S, we obtain a commutative square of the form

$$\begin{array}{ccc} \mathrm{Map}(S, \mathrm{Iso}(U)) & \dashrightarrow & \mathrm{Map}(S, \mathrm{Equiv}_{U\times U}(U_\bullet \times U, U \times U_\bullet)) \\ \downarrow & & \downarrow \\ \mathrm{Map}(S, \mathrm{Fun}([1],U)) & \xrightarrow{\mathrm{St}_*} & \mathrm{Map}(S, \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)). \end{array}$$

This is a pullback square: since this is a diagram of animae over $\mathrm{Map}(S, U \times U)$, it suffices to check this property fiber-wise over each object of $\mathrm{Map}(S, U \times U)$, which we already know.

The equivalence

$$\mathrm{Iso}(U) \xrightarrow{\sim} \mathrm{Equiv}_{U\times U}(U_\bullet \times U, U \times U_\bullet)$$

precisely means that the cocartesian fibration $\pi\colon U_\bullet \to U$ is univalent in Voevodsky's sense. In practical terms, this is the precise way to say that there is a correspondence between U-small

cocartesian fibrations of the form $p\colon X \to S$ and functors of the form $F\colon S \to U$: given a pullback square of the form

$$\begin{array}{ccc} X & \longrightarrow & U_\bullet \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{F} & U \end{array}$$

p and F determine each other uniquely.

8.4 The universe of categories

Our next goal is to define the *universe of all (small) categories*, i.e. a category Cat whose objects classify (small) categories. We require that Cat comes equipped with a directed univalent cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$, so that we may identify the objects and morphisms of Cat with (small) categories and functors between them. To ensure that Cat does indeed behave like a universe of categories, we will have to enforce that all the categorical constructions of categories we have introduced so far admit analogues internal to Cat.

8.4.1 Universes

Let us start by introducing some general terminology regarding universes.

Definition 8.4.1 ((Regular) Universe)**.** A *universe* is a category U equipped with a directed univalent cocartesian fibration $\pi\colon U_\bullet \to U$.

A universe U is called *regular* if U-small cocartesian fibrations are closed under composition.

Remark 8.4.2. By Theorem 8.3.18, we may suppress the contractibly unique U-small structure and treat U-smallness as a property.

While there can a priori be many different universes, the collection of universes behaves like a partially ordered set:

Theorem 8.4.3. *Let U be a universe with directed univalent cocartesian fibration $\pi\colon U_\bullet \to U$, and let $i\colon V \to U$ be a functor. Then the following two conditions are equivalent:*

(1) The base change $V_\bullet := V \times_U U_\bullet \to V$ exhibits V as a universe;

(2) The functor i is fully faithful.

Proof. Consider the map $\pi'\colon V_\bullet \to V$ defined as the base change of π along i:

$$\begin{array}{ccc} V_\bullet & \xrightarrow{j} & U_\bullet \\ {\scriptstyle \pi'}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{i} & U. \end{array}$$

By taking the cartesian product of this square with a degenerate pullback square, we obtain two pullback squares

$$\begin{array}{ccc} V_\bullet \times V & \xrightarrow{i\times j} & U_\bullet \times U \\ {\scriptstyle \pi'\times \mathrm{id}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi\times \mathrm{id}} \\ V \times V & \xrightarrow{i\times i} & U \times U \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V \times V_\bullet & \xrightarrow{j\times i} & U \times U_\bullet \\ {\scriptstyle \mathrm{id}\times \pi'}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{id}\times \pi} \\ V \times V & \xrightarrow{i\times i} & U \times U. \end{array}$$

As a consequence, we obtain from Corollary 8.1.14 a pullback square

$$\begin{array}{ccc} \underline{\mathrm{CoCart}}_{V\times V}(V_\bullet \times V, V \times V_\bullet) & \longrightarrow & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \\ \downarrow & \lrcorner & \downarrow \\ V \times V & \xrightarrow{\qquad i\times i \qquad} & U \times U. \end{array}$$

Since the square

$$\begin{array}{ccc} [1] \times \mathrm{Fun}([1], V) & \xrightarrow{\mathrm{ev}} & V \\ {\scriptstyle 1\times i_*}\downarrow & & \downarrow{\scriptstyle i} \\ [1] \times \mathrm{Fun}([1], U) & \xrightarrow{\mathrm{ev}} & U \end{array}$$

commutes, we obtain pullback squares

$$\begin{array}{ccccc} W_V & \longrightarrow & W_U & \longrightarrow & U_\bullet \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ [1] \times \mathrm{Fun}([1], V) & \xrightarrow{1\times i_*} & [1] \times \mathrm{Fun}([1], U) & \xrightarrow{\mathrm{ev}} & U. \end{array}$$

Consider now the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Fun}([1], V) & \xrightarrow{\qquad i_* \qquad} & \mathrm{Fun}([1], U) \\ \downarrow{\scriptstyle \mathrm{St}_V} & & \downarrow{\scriptstyle \mathrm{St}_U} \\ \underline{\mathrm{CoCart}}_{V\times V}(V_\bullet \times V, V \times V_\bullet) & \longrightarrow & \underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet) \\ \downarrow & \lrcorner & \downarrow \\ V \times V & \xrightarrow{\qquad i\times i \qquad} & U \times U. \end{array}$$

Since the bottom square is a pullback square, it follows from the pasting rule for pullback squares that i is fully faithful if and only if the top square is cartesian. Since the upper right vertical map St_U is an equivalence, this happens if and only if the left vertical map St_V is also an equivalence, i.e. if the cocartesian fibration $V_\bullet \to V$ is directed univalent. This finishes the proof. □

Lemma 8.4.4. *Let Γ be an anima and let U be a universe. Then the category U_Γ in context Γ is a universe for categories in context Γ.*

Proof. It is clear that U_Γ is a cocartesian fibration. Since Γ is an anima, a family of functors $f(x)\colon A(x) \to B(x)$ parametrized by terms $x : \Gamma$ is a cocartesian fibration in context Γ if and only if its associated functor

$$\Sigma_{x:\Gamma} f(x)\colon \Sigma_{x:\Gamma} A(x) \to \Sigma_{x:\Gamma} B(x)$$

is cocartesian in the absolute context. Similarly, if $S(x)$ is a family of categories indexed by terms $x : \Gamma$ with associated category in the absolute context $S = \Sigma_{x:\Gamma} S(x)$, a functor between cocartesian fibrations over $S(x)$ in context Γ is cocartesian over $S(x)$ in context Γ if and only if it is cocartesian in context S in the absolute context. This implies that

$$\underline{\mathrm{CoCart}}_{U\times U}(U_\bullet \times U, U \times U_\bullet)_\Gamma \to (U \times U)_\Gamma = U_\Gamma \times_\Gamma U_\Gamma$$

classifies cocartesian functors between cocartesian fibrations that are pullbacks of the cocartesian fibration $U_\bullet \to U$ in context Γ. Since we also have

$$\mathrm{Fun}([1], U)_\Gamma = \mathrm{Fun}_\Gamma([1], U_\Gamma)$$

this implies right away that U_Γ is a directed univalent universe in context Γ simply because the operation $(-)_\Gamma$ preserves equivalences. □

8.4.2 The universe of categories

We will now introduce the universe Cat. The preceding lemma records that a universe remains a universe under anima parameters; this will justify the parametrized uses of the constructions below.

Definition 8.4.5 (Universe of categories)**.** We say that a universe U is a *universe of categories* if the following conditions are satisfied:

(Cat1) (Regularity) The composite of two U-small cocartesian fibrations is U-small;

(Cat2) (Terminal category) The category $*$ is U-small;

(Cat3) (Pullbacks) Given U-small cocartesian fibrations $C \to S$, $D \to S$ and $E \to S$, and cocartesian functors $C \to E$ and $D \to E$ over S, the cocartesian fibration $C \times_E D \to S$ from Proposition 6.5.12 is U-small;

(Cat4) (Subcategories) Let $p : C \to S$ be a U-small cocartesian fibration and let $M \hookrightarrow \mathrm{Map}([1], C)$ be a collection of morphisms closed under composition such that the composite $p \circ i_M : \langle M \rangle_C \to S$ is a cocartesian fibration. Then $p \circ i_M$ is a U-small cocartesian fibration;

(Cat5) (Joins) If $C \to S$ and $D \to S$ are U-small cocartesian fibrations, then their relative join $C \star_S D \to S$ (in the sense of Definition 3.6.7) exists and is again a U-small cocartesian fibration;

(Cat6) (Dependent products) For any U-small category S, any U-small cocartesian fibration $f : C \to D$ and any functor $D \to S$, the functor $(p_S)_*(f) : (p_S)_*(C) \to (p_S)_*(D)$ is a U-small cocartesian fibration;

(Cat7) (Localizations) If C is U-small and $W \to \mathrm{Map}([1], C)$ is an embedding, then there exists a localization of C at W which is U-small;

(Cat8) (Regular subuniverses) For any U-small subcategory V of U there exists a U-small subcategory $V \subseteq V' \subseteq U$ such that V' is a regular universe;

(Cat9) (Small fibers) A cocartesian fibration with U-small fibers is U-small.

(Cat10) A synthetic category C is U-small if and only if the groupoid $\mathrm{Map}([1], C)$ is U-small.

Conceptually, conditions (Cat1)–(Cat7) provide closure under the categorical constructions used in this chapter, condition (Cat8) provides enlargement inside a regular universe, and condition (Cat9) recognizes small families from their fibers.

Proposition 8.4.6 (Equivalent forms of subcategory closure)**.** *Item (Cat4) is equivalent to the following condition: if $p\colon C \to S$ is U-small and $i\colon C' \hookrightarrow C$ is a subcategory such that $p \circ i\colon C' \to S$ is a cocartesian fibration, then $p \circ i$ is U-small.*

Proof. The displayed condition implies (Cat4) by Corollary 3.1.12. Conversely, given $i\colon C' \hookrightarrow C$, apply (Cat4) to

$$M_{C'} := \mathrm{Map}([1], C') \hookrightarrow \mathrm{Map}([1], C).$$

By Lemma 3.1.13, the resulting subcategory $\langle M_{C'} \rangle_C$ is equivalent to C' over C, hence over S. The claim follows from equivalence-invariance of smallness. □

In (Cat9) we used the following terminology.

Definition 8.4.7 (U-small fibers)**.** Let U be a universe in the sense of Definition 8.4.1, and let $f\colon C \to D$ be a functor. We say that f *has U-small fibers* if for any U-small category T and any functor $T \to D$ the pullback $C \times_D T$ is U-small.

Directed univalence and the closure conditions describe an individual universe. The following axiom asserts that sufficiently large universes carrying this structure exist.

Axiom N (Directed Univalence Axiom)**.** For any cocartesian fibration $q\colon E \to B$, there exists a universe of categories U such that B is U-small and q has U-small fibers.

In practice, we will mostly just need a single universe of categories; we will frequently fix a choice and denote it by Cat. We denote the associated universal cocartesian fibration by

$$\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}.$$

We will say that a category C is *small* if it is Cat-small in the sense of Definition 8.3.1, i.e. if there exists an object $c\colon * \to \mathrm{Cat}$ and a pullback square

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{c} & \mathrm{Cat}. \end{array}$$

We will similarly speak of *small cocartesian fibrations*, etcetera. The reader should keep in mind that all results stated for the fixed universe Cat will hold for any universe of categories U.

We now derive the closure properties needed to construct initial and terminal objects, products, coproducts, pullbacks, and functor categories internally in Cat.

Lemma 8.4.8 (Identity functors)**.** *For every category S, the identity functor $\mathrm{id}_S\colon S \to S$ is a small cocartesian fibration.*

Proof. The identity functor on S is a base change along $p_S\colon S \to *$ of the functor $\mathrm{id}_*\colon * \to *$, and hence this follows from (Cat2). □

Lemma 8.4.9 (Initial category)**.** *For every category S, the functor $\emptyset \to S$ is a small cocartesian fibration.*

Proof. This follows from Proposition 8.4.6, applied to the embedding $\emptyset \hookrightarrow S$, together with the fact that $\mathrm{id}_S\colon S \to S$ is small by the previous lemma. □

Lemma 8.4.10 (Products)**.** *Let $p\colon C \to S$ and $q\colon D \to S$ be small cocartesian fibrations. Then also $C \times_S D \to S$ is a small cocartesian fibration.*

Proof. This functor may be written as a composite $C \times_S D \xrightarrow{\mathrm{pr}_C} C \xrightarrow{p} S$ of two small cocartesian fibrations, hence the claim follows from (Cat1). □

Lemma 8.4.11 (Groupoid cores)**.** *Let C be a small category. Then also its groupoid core $C^{\simeq}$ is small.*

Proof. This follows from Proposition 8.4.6, applied to the embedding $C^{\simeq} \hookrightarrow C$ from Corollary 2.5.3. □

Lemma 8.4.12 (Universality of coproducts)**.** *Let S and T be categories. A cocartesian fibration $p\colon C \to S \sqcup T$ is small if and only if its pullbacks $p_S\colon C_S \to S$ and $p_T\colon C_T \to T$ are small.*

Proof. The 'only if'-direction is clear since U-small cocartesian fibrations are closed under base change. Conversely, assume that the cocartesian fibrations $p_S\colon C_S \to S$ and $p_T\colon C_T \to T$ are represented by functors $F_S\colon S \to \mathrm{Cat}$ and $F_T\colon T \to \mathrm{Cat}$, respectively. We claim that p is represented by the functor $F := \langle F_S, F_T\rangle\colon S \sqcup T \to \mathrm{Cat}$. Indeed, if $D \to S \sqcup T$ denotes the small cocartesian fibration classified by F, then D pulls back to $C_S \to S$ and $C_T \to T$. By Axiom B.5', we therefore have equivalences $D \simeq C_S \sqcup C_T \simeq C$ over $S \sqcup T$. □

Lemma 8.4.13 (Coproducts)**.** *Given small cocartesian fibrations $p\colon C \to S$ and $q\colon D \to S$, their coproduct $\langle p, q\rangle\colon C \sqcup D \to S$ is a small cocartesian fibration.*

Proof. We may factor this functor as $C \sqcup D \xrightarrow{p\sqcup q} S \sqcup S \xrightarrow{\langle \mathrm{id}_S, \mathrm{id}_S\rangle} S$. The first functor is a small cocartesian fibration by Lemma 8.4.12. The second functor is a base change of the functor $* \sqcup * \to *$, hence it remains to show that $* \sqcup *$ is small. But since $* \sqcup * = [1]^{\simeq}$, this follows from applying Lemma 8.4.11 to the small category $[1]$. □

Lemma 8.4.14. *The product $p \times q\colon E \times F \to C \times D$ of two small cocartesian fibrations $p\colon E \to C$ and $q\colon F \to D$ is again small.*

Proof. We may factor $p \times q$ as a composite

$$E \times F \xrightarrow{p \times \mathrm{id}_F} C \times F \xrightarrow{\mathrm{id}_C \times q} C \times D.$$

The maps $p \times \mathrm{id}_F$ and $\mathrm{id}_C \times q$ are small, and hence so is their composite by (Cat1). □

Lemma 8.4.15. *Let $p\colon C \to S$ be a small cocartesian fibration. Then p has small fibers.*

Proof. Let T be a small category equipped with a functor $T \to S$. The pullback $C_T \to T$ is small by base-change stability. Since $T \to *$ is small, regularity implies that the composite $C_T \to T \to *$ is small. Thus p has small fibers in the sense of Definition 8.4.7. □

Lemma 8.4.16. *Let C be a small category and $p\colon D \to S$ a small cocartesian fibration. Then also $p_*\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, S)$ is a small cocartesian fibration.*

Proof. The functor $p \times \mathrm{id}_C\colon D \times C \to S \times C$ is a small cocartesian fibration, being a base change of p. The projection maps to C exhibit it as a cocartesian fibration over C. By (Cat6), the functor $(p_C)_*(D \times C) \to (p_C)_*(S \times C)$ is again a small cocartesian fibration. But this functor is precisely $p_*\colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, S)$, hence we are done. □

Lemma 8.4.17 (Functor categories)**.** *If C and D are small categories, then also* $\mathrm{Fun}(C, D)$ *is small.*

Proof. This is a special case of Lemma 8.4.16 by taking $S = *$. □

8.5 Category theory internal to the universe

In the previous chapters of this book, we have introduced various axioms for constructing and manipulating categories, and we used these axioms to build up some basic category theory. We will sometimes refer to this theory as the 'external' category theory: the notion of 'category' is taken for granted, and it is up to the user of the theory to interpret it in different contexts.

In this section, we will study 'internal' category theory. Recall that we introduced the *universe of categories* Cat in the previous section, whose objects we may think of as small categories. We construct the internal versions of the basic categorical operations needed below: initial and terminal objects, products and coproducts, pullbacks, and functor categories. These operations will be used to define the universe An of animae and to express the cocartesian-fibration condition internally in Cat. We refer to this collection of constructions as 'internal category theory in Cat'.

8.5.1 Constructions of categories

We start by discussing internal versions of initial/terminal categories, products/coproducts, pullbacks and functor categories.

Products and coproducts

Notation 8.5.1 (Initial/terminal categories)**.** By condition (Cat2) and Lemma 8.4.9, the terminal category $*$ and the initial category $\emptyset$ are both small, hence are classified by objects $\lfloor * \rfloor$ and $\lfloor \emptyset \rfloor$ of Cat:

$$\begin{array}{ccc} * & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{\lfloor * \rfloor} & \mathrm{Cat} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \emptyset & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{\lfloor \emptyset \rfloor} & \mathrm{Cat}. \end{array}$$

Lemma 8.5.2. *The object* $\lfloor * \rfloor \in \mathrm{Cat}$ *is terminal. The object* $\lfloor \emptyset \rfloor \in \mathrm{Cat}$ *is initial.*

Proof. We start with the statement about $\lfloor * \rfloor$. By Lemma 6.1.10, it suffices to show that for any other object $c \in \mathrm{Cat}$ the hom anima $\mathrm{Hom}_{\mathrm{Cat}}(c, \lfloor * \rfloor)$ is contractible. By directed univalence, this anima is equivalent to $\mathrm{Map}(C, *)$, where C is the small category classified by c. But we have $\mathrm{Fun}(C, *) \xrightarrow{\sim} *$ by Lemma 1.7.9 and $*^{\simeq} \xrightarrow{\sim} *$ by Example 2.4.4.

The argument for $\lfloor \emptyset \rfloor$ is analogous, except that we use the equivalence $\mathrm{Fun}(\emptyset, C) \xrightarrow{\sim} *$ from Lemma 1.7.14. □

Notation 8.5.3 (Products)**.** By Lemma 8.4.14, the product functor $\pi_{\mathrm{univ}} \times \pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \times \mathrm{Cat}_\bullet \to \mathrm{Cat} \times \mathrm{Cat}$ is a small cocartesian fibration, and hence is classified by some functor $- \times - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{array}{ccc} \mathrm{Cat}_\bullet \times \mathrm{Cat}_\bullet & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle \pi_{\mathrm{univ}} \times \pi_{\mathrm{univ}}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} \times \mathrm{Cat} & \overset{\times}{\dashrightarrow} & \mathrm{Cat}. \end{array}$$

Given two objects c and d of Cat corresponding to small categories C and D, note that the fiber of $\pi_{\mathrm{univ}} \times \pi_{\mathrm{univ}}$ over (c, d) is the product category $C \times D$, so that the functor $- \times - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$ does indeed encode the product of categories.

Notation 8.5.4 (Coproducts)**.** By Lemma 8.4.13, the coproduct functor

$$\langle \pi_{\mathrm{univ}} \times \mathrm{id}, \mathrm{id} \times \pi_{\mathrm{univ}} \rangle \colon (\mathrm{Cat}_\bullet \times \mathrm{Cat}) \sqcup (\mathrm{Cat} \times \mathrm{Cat}_\bullet) \to \mathrm{Cat} \times \mathrm{Cat}$$

is a small cocartesian fibration, and hence is classified by some functor $- \sqcup - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{array}{ccc} (\mathrm{Cat}_\bullet \times \mathrm{Cat}) \sqcup (\mathrm{Cat} \times \mathrm{Cat}_\bullet) & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle \langle \pi_{\mathrm{univ}} \times \mathrm{id}, \mathrm{id} \times \pi_{\mathrm{univ}} \rangle}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} \times \mathrm{Cat} & \overset{\sqcup}{\dashrightarrow} & \mathrm{Cat}. \end{array}$$

Given two objects c and d of Cat corresponding to small categories C and D, note that the fiber of the left vertical map over (c, d) is the coproduct category $C \sqcup D$, so that the functor $- \sqcup - \colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$ does indeed encode the coproduct of categories.

Lemma 8.5.5. *For objects* $c, d \in \mathrm{Cat}$*, the object* $c \times d$ *is a product in* Cat*, and the object* $c \sqcup d$ *is a coproduct in* Cat*.*

Proof. The argument is similar to Lemma 8.5.2. For the product, we need to show that for any other $e \in \mathrm{Cat}$ the two projection maps induce an equivalence

$$\mathrm{Hom}_{\mathrm{Cat}}(e, c \times d) \simeq \mathrm{Hom}_{\mathrm{Cat}}(e, c) \times \mathrm{Hom}_{\mathrm{Cat}}(e, d) \simeq \mathrm{Hom}_{\mathrm{Cat} \times \mathrm{Cat}}((e, e), (c, d)).$$

Under directed univalence this follows from the equivalence $\mathrm{Map}(E, C \times D) \xrightarrow{\sim} \mathrm{Map}(E, C) \times \mathrm{Map}(E, D)$ from Lemma 1.7.10. The analogous argument for the coproduct uses Lemma 1.7.15. □

Pullbacks

Recall from Definition 7.1.4 the category $\lrcorner$, which for the purposes of this section we will display as follows:

$$\lrcorner \quad = \quad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ 1 & \longrightarrow & 2. \end{array}$$

In particular we denote its three objects by 0, 1 and 2. We would like to construct a pullback functor $\lim\colon \mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$.

Construction 8.5.6. Let us write $\mathrm{Cat}^{\lrcorner} := \mathrm{Fun}(\lrcorner, \mathrm{Cat})$. For $i = 0, 1, 2$, we denote by $\mathrm{Cat}^{\lrcorner}_{\bullet,i} \to \mathrm{Cat}^{\lrcorner}$ the small cocartesian fibration defined via the following pullback diagram:

$$\begin{array}{ccc} \mathrm{Cat}^{\lrcorner}_{\bullet,i} & \longrightarrow & \mathrm{Cat}_{\bullet} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat}^{\lrcorner} & \xrightarrow{\mathrm{ev}_i} & \mathrm{Cat}. \end{array}$$

Using directed univalence to translate the morphisms $0 \to 2$ and $1 \to 2$ in $\lrcorner$ into cocartesian functors, we obtain

$$\begin{array}{ccc} \mathrm{Cat}^{\lrcorner}_{\bullet,0} & \longrightarrow & \mathrm{Cat}^{\lrcorner}_{\bullet,2} \\ & \searrow \quad \swarrow & \\ & \mathrm{Cat}^{\lrcorner} & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Cat}^{\lrcorner}_{\bullet,1} & \longrightarrow & \mathrm{Cat}^{\lrcorner}_{\bullet,2} \\ & \searrow \quad \swarrow & \\ & \mathrm{Cat}^{\lrcorner}. & \end{array}$$

It then follows from condition (Cat3) that the resulting cocartesian fibration

$$\mathrm{Cat}^{\lrcorner}_{\bullet,0} \times_{\mathrm{Cat}^{\lrcorner}_{\bullet,2}} \mathrm{Cat}^{\lrcorner}_{\bullet,1} \to \mathrm{Cat}^{\lrcorner}$$

is again a small cocartesian fibration, and hence is classified by some functor $\lim\colon \mathrm{Cat}^{\lrcorner} \to \mathrm{Cat}$:

$$\begin{array}{ccc} \mathrm{Cat}^{\lrcorner}_{\bullet,0} \times_{\mathrm{Cat}^{\lrcorner}_{\bullet,2}} \mathrm{Cat}^{\lrcorner}_{\bullet,1} & \longrightarrow & \mathrm{Cat}_{\bullet} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat}^{\lrcorner} & \xrightarrow{\lim} & \mathrm{Cat}. \end{array}$$

Observation 8.5.7. Note that an object of $\mathrm{Cat}^{\lrcorner}$ corresponds to a pair of morphisms $c \to e$ and $d \to e$ in Cat, which in turn corresponds under directed univalence to a pair of functors $C \to E$ and $D \to E$. A computation shows that the fiber of the left vertical map over this object is the pullback $C \times_E D$, showing that the functor $\lim\colon \mathrm{Cat}^{\lrcorner} \to \mathrm{Cat}$ does indeed classify pullbacks of categories.

Lemma 8.5.8. *The functor* $\lim\colon \mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$ *is right adjoint to the diagonal map* $p^*_{\lrcorner}\colon \mathrm{Cat} \to \mathrm{Fun}(\lrcorner, \mathrm{Cat})$.

Proof. Consider a cospan $c \to e \leftarrow d$ in Cat and an object x of Cat. We need to show that the map

$$\mathrm{Hom}_{\mathrm{Cat}}(x, c \times_e d) \to \mathrm{Hom}_{\mathrm{Cat}^{\lrcorner}}(x \to x \leftarrow x, c \to e \leftarrow d)$$

induced by the cone diagram of $c \times_e d$ is an equivalence. A computation shows that the right-hand side is equivalent to

$$\mathrm{Hom}_{\mathrm{Cat}}(x, c) \times_{\mathrm{Hom}_{\mathrm{Cat}}(x,e)} \mathrm{Hom}_{\mathrm{Cat}}(x, d).$$

By directed univalence, the map is then equivalent to the map

$$\mathrm{Map}(X, C \times_E D) \to \mathrm{Map}(X, C) \times_{\mathrm{Map}(X,E)} \mathrm{Map}(X, D),$$

which is an equivalence by Lemma 1.7.11 and Lemma 1.6.4. □

Corollary 8.5.9. *The category* Cat *admits pullbacks.* □

Functor categories

We will now discuss functor categories internal to Cat.

Construction 8.5.10. Let c be an object of Cat and let C be the associated small category. By Lemma 8.4.16, the functor

$$(\pi_{\mathrm{univ}})_* \colon \mathrm{Fun}(C, \mathrm{Cat}_\bullet) \to \mathrm{Fun}(C, \mathrm{Cat})$$

is a small cocartesian fibration, hence it is classified by some functor $\Gamma_C \colon \mathrm{Fun}(C, \mathrm{Cat}) \to \mathrm{Cat}$:

$$\begin{array}{ccc} \mathrm{Fun}(C, \mathrm{Cat}_\bullet) & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle (\pi_{\mathrm{univ}})_*}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Fun}(C, \mathrm{Cat}) & \overset{\Gamma_C}{\dashrightarrow} & \mathrm{Cat}. \end{array}$$

We define the functor $(-)^c \colon \mathrm{Cat} \to \mathrm{Cat}$ as the following composite:

$$\mathrm{Cat} \xrightarrow{p_C^*} \mathrm{Fun}(C, \mathrm{Cat}) \xrightarrow{\Gamma_C} \mathrm{Cat}.$$

Note that for an object $d \in \mathrm{Cat}$ corresponding to a small category D, the image $d^c \in \mathrm{Cat}$ classifies the small category $\mathrm{Fun}(C, D)$.

Lemma 8.5.11. *The functor* $(-)^c \colon \mathrm{Cat} \to \mathrm{Cat}$ *is right adjoint to* $- \times c \colon \mathrm{Cat} \to \mathrm{Cat}$.

Proof. Using directed univalence, this follows from Lemma 1.7.12, similarly to the proofs of Lemmas 8.5.2, 8.5.5 and 8.5.8. □

Construction 8.5.12. Let $f \colon c \to d$ be a morphism in Cat corresponding under directed univalence to a functor $f \colon C \to D$ between small categories. We will construct a natural transformation

$$f^* \colon (-)^d \to (-)^c.$$

We denote by $\mathrm{Cat}_\bullet^c \to \mathrm{Cat}$ the small cocartesian fibration classified by $(-)^c \colon \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{array}{ccccc} \mathrm{Cat}_\bullet^c & \longrightarrow & \mathrm{Fun}(C, \mathrm{Cat}_\bullet) & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & {\scriptstyle (\pi_{\mathrm{univ}})_*}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{p_C^*} & \mathrm{Fun}(C, \mathrm{Cat}) & \xrightarrow{\Gamma_C} & \mathrm{Cat}. \\ & & & {\scriptstyle (-)^c}\searrow & \end{array}$$

Precomposition with f gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}(D, \mathrm{Cat}_\bullet) & \longrightarrow & \mathrm{Fun}(C, \mathrm{Cat}_\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(D, \mathrm{Cat}) & \longrightarrow & \mathrm{Fun}(C, \mathrm{Cat}), \end{array}$$

which induces a cocartesian functor $\mathrm{Cat}_\bullet^d \to \mathrm{Cat}_\bullet^c$ over Cat. Under directed univalence this corresponds to a natural transformation $f^* \colon (-)^d \to (-)^c$ as desired.

Lemma 8.5.13. *There is an identification* $(-)^* = \mathrm{id}_{\mathrm{Cat}}$ *of functors* $\mathrm{Cat} \to \mathrm{Cat}$.

Proof. This is immediate from the equivalence $\mathrm{Cat}_\bullet^* \simeq \mathrm{Cat}_\bullet$ over Cat, which follows from the equivalences $\mathrm{Fun}(*, \mathrm{Cat}) \simeq \mathrm{Cat}$ and $\mathrm{Fun}(*, \mathrm{Cat}_\bullet) \simeq \mathrm{Cat}_\bullet$ from Lemma 1.7.13. □

Lemma 8.5.14. *For small categories C and D there is an identification* $((-)^C)^D = (-)^{C \times D}$ *of functors* $\mathrm{Cat} \to \mathrm{Cat}$.

Proof. The small cocartesian fibration classified by $((-)^C)^D$ sits in a pullback square as follows:

$$\begin{array}{ccccc} (\mathrm{Cat}_\bullet^c)^d & & \longrightarrow & & \mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat}_\bullet)) \\ \downarrow & & \lrcorner & & \downarrow{\scriptstyle (\pi_{\mathrm{univ}})_*} \\ \mathrm{Cat} & \xrightarrow{p_D^*} & \mathrm{Fun}(D, \mathrm{Cat}) & \xrightarrow{(p_C^*)_*} & \mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat})). \end{array}$$

Under the equivalences $\mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat})) \simeq \mathrm{Fun}(C \times D, \mathrm{Cat})$ and $\mathrm{Fun}(D, \mathrm{Fun}(C, \mathrm{Cat}_\bullet)) \simeq \mathrm{Fun}(C \times D, \mathrm{Cat}_\bullet)$ from Lemma 1.7.12, the bottom functor is $p_{C \times D}^*$ and we see that the left vertical map agrees with the small cocartesian fibration $\mathrm{Cat}_\bullet^{c \times d} \to \mathrm{Cat}$ classifying $(-)^{c \times d}$. The claim thus follows from directed univalence. □

Remark 8.5.15. Proceeding in this way, one may prove each of the following identifications:

- For $c \in \mathrm{Cat}$ there is an identification $(- \times -)^c = (-)^c \times (-)^c$ of functors $\mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$, cf. Lemma 1.7.10;
- For $t \in \mathrm{Cat}$ there is an identification $(\lim(-))^t = \lim((-)^t)$ of functors $\mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$, cf. Lemma 1.7.11;
- There is an identification $(-)^\emptyset = \mathrm{const}_*$, cf. Lemma 1.7.14;
- There is an identification $(-)^{c \sqcup d} = (-)^c \times (-)^d$, cf. Lemma 1.7.15.

8.6 The universe of animae

In this section we construct the *universe of animae* An and equip it with the *universal small left fibration* $\pi_{\mathrm{univ}} \colon \mathrm{An}_\bullet \to \mathrm{An}$. We further show that the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}$ admits the groupoid-core functor $(-)^\simeq \colon \mathrm{Cat} \to \mathrm{An}$ as a right adjoint. Its left adjoint will be constructed in Section 8.8.

Recall that a category X is an anima if all its morphisms are invertible, or equivalently if the functor $p_{[1]}^* \colon X \to \mathrm{Fun}([1], X)$ is an equivalence. By the groupoid recognition axiom, the primitive animae

are precisely the categories satisfying this condition. We therefore use it to cut out their universe internally:

Definition 8.6.1. Applying Construction 8.5.12 to the projection map $[1] \to [0]$, we obtain a natural transformation

$$p^*_{[1]} \colon \mathrm{id}_{\mathrm{Cat}} = (-)^{[0]} \to (-)^{[1]}$$

of functors $\mathrm{Cat} \to \mathrm{Cat}$, which corresponds to a functor $\mathrm{Cat} \to \mathrm{Fun}([1], \mathrm{Cat})$ sending x to $p^*_{[1]} \colon x \to x^{[1]}$. We say that $x \in \mathrm{Cat}$ satisfies the *anima condition* if this morphism is an isomorphism in Cat. We define the category An of animae via the pullback square

$$\begin{array}{ccc} \mathrm{An} & \longrightarrow & \mathrm{Iso}(\mathrm{Cat}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Cat} & \xrightarrow[x \mapsto (x \to x^{[1]})]{} & \mathrm{Fun}([1], \mathrm{Cat}). \end{array}$$

Since the right vertical map is fully faithful by Corollary 6.4.12, we see that the resulting functor $\mathrm{An} \to \mathrm{Cat}$ is fully faithful. We define the cocartesian fibration $\pi_{\mathrm{univ}} \colon \mathrm{An}_\bullet \to \mathrm{An}$ via the pullback square

$$\begin{array}{ccc} \mathrm{An}_\bullet & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle \pi_{\mathrm{univ}}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{An} & \longrightarrow & \mathrm{Cat}. \end{array}$$

Note that the cocartesian fibration $\pi_{\mathrm{univ}} \colon \mathrm{An}_\bullet \to \mathrm{An}$ is directed univalent by Theorem 8.4.3, because it is classified by a fully faithful functor $\mathrm{An} \hookrightarrow \mathrm{Cat}$.

Definition 8.6.2. A small cocartesian fibration $p \colon X \to S$ which is a left fibration is called a *small left fibration*.

Theorem 8.6.3. *The functor $\pi_{\mathrm{univ}} \colon \mathrm{An}_\bullet \to \mathrm{An}$ is the universal small left fibration: given a small cocartesian fibration $p \colon X \to S$ classified by a functor $F \colon S \to \mathrm{Cat}$, we have that p is a left fibration if and only if F factors through the full subcategory* An *of* Cat.

Proof. Let $p \colon X \to S$ be a small cocartesian fibration classified by a functor F:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ S & \xrightarrow{F} & \mathrm{Cat}. \end{array}$$

We know from Lemma 6.1.8 that p is a left fibration if and only if its fibers are animae. The fiber of p at an object x of S is the category classified by $F(x)$, so p is a left fibration if and only if $F(x) \in \mathrm{An}$ for every object x of S. Since the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}$ is fully faithful, the latter condition is equivalent to F factoring through An. □

Proposition 8.6.4. *The category* An *admits pullbacks, and the inclusion* $\mathrm{An} \hookrightarrow \mathrm{Cat}$ *preserves pullbacks.*

Proof. Recall from Corollary 8.5.9 that Cat admits pullbacks: given morphisms $c \to e$ and $d \to e$ in Cat corresponding to functors $C \to E$ and $D \to E$, the pullback $c \times_e d$ in Cat corresponds to the pullback $C \times_E D$. We have seen in Lemma 2.4.9 that if C, D and E satisfy the anima condition then so does $C \times_E D$. By recognition, this shows that the functor $\lim\colon \mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$ restricts to a functor $\lim\colon \mathrm{Fun}(\lrcorner, \mathrm{An}) \to \mathrm{An}$ right adjoint to the diagonal functor. Thus An admits pullbacks and its inclusion into Cat preserves them. □

8.6.1 Groupoid cores

We would like to show that the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}$ admits a right adjoint

$$(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{An}$$

which assigns to every small category C its groupoid core $C^{\simeq}$. For this, we start by constructing the *fiberwise groupoid core* $C^{\simeq/S} \to S$ of a cocartesian fibration $p\colon C \to S$. It has the same objects as C and precisely the p-cocartesian morphisms. The following construction expresses this description as a subcategory that can be applied uniformly in families.

Construction 8.6.5. Let $p\colon C \to S$ be a cocartesian fibration. Consider a morphism $u\colon x_0 \to x_1$ in C, and write $v = p(u)$ for the resulting morphism in S. We then get a unique map $\psi(u)\colon v_!(x_0) \to x_1$ in C_{s_1} fitting in the following commutative diagram:

$$\begin{array}{ccc} & v_!(x_0) & \\ \nearrow & & \searrow^{\psi(u)} \\ x_0 & \longrightarrow & x_1. \end{array}$$

This process defines a functor

$$\psi\colon \mathrm{Map}([1], C) \to \mathrm{Map}([1], C)\colon u \mapsto \psi(u).$$

Observe that we have $p(\psi(u)) = \mathrm{id}_{p(x_1)}$. We now define a collection of maps in C via the following pullback square:

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{Iso}(C)^{\simeq} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}([1], C) & \xrightarrow{\psi} & \mathrm{Map}([1], C). \end{array}$$

Observe that a morphism is in M if and only if it is p-cocartesian. Since the p-cocartesian morphisms in C are closed under composition by Proposition 6.5.4, it follows that the collection of maps M is closed under composition, and thus we may make the following definition:

Definition 8.6.6. Given a cocartesian fibration $C \to S$, we define the *fiberwise groupoid core* of C over S as the subcategory

$$C^{\simeq/S} := \langle M \rangle_C \subseteq C.$$

Proposition 8.6.7. *The resulting map $p'\colon C^{\simeq/S} \to S$ is a left fibration and the functor $C^{\simeq/S} \hookrightarrow C$ is a cocartesian functor over S.*

Proof. A direct inspection of universal properties shows that any morphism in $C^{\simeq/S}$ whose image in C is p-cocartesian is in fact p'-cocartesian. Since all p-cocartesian morphisms of C belong to $C^{\simeq/S}$, a p-cocartesian lift in C provides the unique p'-cocartesian lift with prescribed source. Thus p' is a cocartesian fibration and the embedding $C^{\simeq/S} \hookrightarrow C$ is a cocartesian functor over S. By construction, every morphism of $C^{\simeq/S}$ is p'-cocartesian, so p' is a left fibration. □

Lemma 8.6.8. *Let $p\colon C \to S$ be a cocartesian fibration and let $f\colon T \to S$ be any functor. For any left fibration $q\colon Y \to T$, there is an equivalence*

$$\mathrm{Map}_T(Y, T \times_S C^{\simeq/S}) \xrightarrow{\sim} \mathrm{Map}_T^{\mathrm{CoCart}}(Y, T \times_S C).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} (T \times_S C)^{\simeq/T} & \xrightarrow{\simeq} & T \times_S C^{\simeq/S} & \longrightarrow & C^{\simeq/S} \\ & & \downarrow & \lrcorner & \downarrow \\ & & T & \longrightarrow & S. \end{array}$$

The horizontal equivalence identifies the fiberwise groupoid core after base change with the base change of the fiberwise groupoid core. Since both $Y \to T$ and $T \times_S C^{\simeq/S} \to T$ are left fibrations, every functor between them over T is cocartesian. The universal property of the subcategory $\langle M \rangle_{T \times_S C}$ therefore identifies maps from Y into $(T \times_S C)^{\simeq/T}$ with those functors $Y \to T \times_S C$ that send every morphism of Y to a cocartesian morphism over T. Because every morphism of the left fibration $Y \to T$ is cocartesian, this is precisely the anima of cocartesian functors $Y \to T \times_S C$ over T. □

We are now ready to construct the groupoid core functor $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Cat}$.

Construction 8.6.9 (Groupoid core)**.** Applying Proposition 8.6.7 to the universal small cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$, we obtain a left fibration

$$\begin{array}{ccc} \mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} & \hookrightarrow & \mathrm{Cat}_\bullet \\ & \searrow \quad \swarrow_{\pi_{\mathrm{univ}}} & \\ & \mathrm{Cat}. & \end{array}$$

Since the top functor is an embedding, it follows from Proposition 8.4.6 that the left fibration $\mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \to \mathrm{Cat}$ is a small cocartesian fibration, hence is classified by a functor $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Cat}$:

$$\begin{array}{ccc} \mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{(-)^{\simeq}} & \mathrm{Cat}. \end{array}$$

The inclusion map $\mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \to \mathrm{Cat}_\bullet$ is a cocartesian functor over Cat, and hence corresponds under directed univalence to a natural transformation

$$\iota\colon (-)^{\simeq} \to \mathrm{id}_{\mathrm{Cat}}$$

of functors $\mathrm{Cat} \to \mathrm{Cat}$.

Lemma 8.6.10. *The functor* $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{Cat}$ *factors through the full subcategory* $\mathrm{An} \subseteq \mathrm{Cat}$.

Proof. By definition of full subcategories, it suffices to show that for every object c of Cat the object $c^{\simeq}$ belongs to An. If C is the small category classified by c, then a direct computation shows that its groupoid core $C^{\simeq}$ is classified by $c^{\simeq}$ (here one uses the equivalence $C^{\simeq} \simeq \langle \mathrm{Iso}(C) \rangle_C$). Since $C^{\simeq}$ is an anima, the claim follows. □

Proposition 8.6.11. *The functor* $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{An}$ *is right adjoint to the inclusion* $\mathrm{An} \hookrightarrow \mathrm{Cat}$*, with counit given by the natural transformation* $\iota\colon (-)^{\simeq} \to \mathrm{id}_{\mathrm{Cat}}$.

Proof. Consider objects $c, x \in \mathrm{Cat}$ and assume that $x \in \mathrm{An}$. We have to show that postcomposition with the map $c^{\simeq} \to c$ induces an equivalence of animae

$$\mathrm{Hom}_{\mathrm{An}}(x, c^{\simeq}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(x, c).$$

Under directed univalence, this map agrees with the functor

$$\mathrm{Map}(X, C^{\simeq}) \to \mathrm{Map}(X, C)$$

induced by the inclusion $\iota_C\colon C^{\simeq} \to C$. The fact that this is an equivalence is Corollary 1.4.13. □

8.7 Straightening/unstraightening

Given a category S, functors $S \to \mathrm{Cat}$ correspond by their very definition to small cocartesian fibrations over S. Furthermore, directed univalence of π_{univ} implies that natural transformations of functors $S \to \mathrm{Cat}$ correspond to cocartesian functors between small cocartesian fibrations over S, and we have seen in Lemma 8.2.5 and Proposition 8.2.6 that this assignment is compatible with identity functors and with composition of functors.

The goal of this section is to enhance the correspondence between functors $S \to \mathrm{Cat}$ and small cocartesian fibrations over S to a fully fledged equivalence of categories: if S is a small category corresponding to an object s of Cat, we will define a (non-full) subcategory $\mathrm{CoCart}(S) \subseteq \mathrm{Cat}_{/s}$ of *(small) cocartesian fibrations over* S and construct an equivalence

$$\mathrm{Fun}(S, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{CoCart}(S)$$

called the *straightening/unstraightening equivalence*.

8.7.1 Directed univalence over a base

Let S be a small category with associated object $s : \mathrm{Cat}^{\simeq}$.

$$\begin{array}{ccc} S & \longrightarrow & \mathrm{Cat}_{\bullet} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{s} & \mathrm{Cat} \end{array}$$

We will write $\mathrm{Cat}_{/S} := \mathrm{Cat}_{/s}$. We consider similarly two small categories X and Y corresponding to

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{x} & \mathrm{Cat} \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{y} & \mathrm{Cat} \end{array}$$

as well as functors $p\colon X \to S$ and $q\colon Y \to S$. We will also denote by $p\colon x \to s$ and $q\colon y \to s$ the associated morphisms in Cat. Since straightening is compatible with composition of functors (by Proposition 8.2.6) the canonical embedding

$$\mathrm{Cat}_{/S}((x,p),(y,q)) \hookrightarrow \mathrm{Map}([1],\mathrm{Cat}) \xrightarrow{\sim} \underline{\mathrm{CoCart}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat} \times \mathrm{Cat}_\bullet)^{\simeq}$$

corresponds to a commutative triangle of the form below.

$$\begin{array}{ccc} X \times \mathrm{Cat}_{/S}((x,p),(y,q)) & \xrightarrow{\quad f_{\mathrm{univ}} \quad} & Y \times \mathrm{Cat}_{/S}((x,p),(y,q)) \\ & {\scriptstyle p\times\mathrm{id}} \searrow \quad \swarrow {\scriptstyle q\times\mathrm{id}} & \\ & S \times \mathrm{Cat}_{/S}((x,p),(y,q)) & \end{array}$$

Therefore, in context $\mathrm{Cat}_{/S}((x,p),(y,q))$, there is a unique term of $\mathrm{Map}_S(X,Y)$ corresponding to this diagram. In other words, this defines a canonical map

$$\mathrm{Cat}_{/S}((x,p),(y,q)) \to \mathrm{Map}_S(X,Y)\,.$$

Lemma 8.7.1. *There is a canonical equivalence* $\mathrm{Cat}_{/S}((x,p),(y,q)) \xrightarrow{\sim} \mathrm{Map}_S(X,Y)$.

Proof. By definition, the two sides are the fibers over p of the vertical maps in the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Cat}}(x,y) & \xrightarrow{\mathrm{St}} & \mathrm{Map}(X,Y) \\ {\scriptstyle q\circ -}\downarrow & & \downarrow{\scriptstyle q\circ -} \\ \mathrm{Hom}_{\mathrm{Cat}}(x,s) & \xrightarrow{\mathrm{St}} & \mathrm{Map}(X,S). \end{array}$$

The horizontal maps are equivalences by directed univalence, and the square commutes by Proposition 8.2.6. The induced map on the fibers over p is therefore an equivalence. □

8.7.2 The subcategory of cocartesian fibrations

We now construct the subcategory $\mathrm{CoCart}(S)$ of $\mathrm{Cat}_{/s}$. There are three stages: we first express the cocartesian-fibration condition internally in Cat, then cut out the objects and morphisms of $\mathrm{CoCart}(S)$, and finally construct the unstraightening functor.

Recognizing cocartesian fibrations internally

Remark 8.7.2. In what follows, we will use the following previously established facts about the category Cat:

- The walking morphism $[1]$ is small and thus defines an object $[1] \in \mathrm{Cat}$;

- There is a pullback functor $\mathrm{Fun}(\lrcorner, \mathrm{Cat}) \to \mathrm{Cat}$ right adjoint to the constant diagram functor, see Corollary 8.5.9;
- There is a product functor $-\times-\colon \mathrm{Cat}\times\mathrm{Cat} \to \mathrm{Cat}$ right adjoint to the diagonal $\mathrm{Cat} \to \mathrm{Cat}\times\mathrm{Cat}$;
- For every $d \in \mathrm{Cat}$, the functor $-\times d\colon \mathrm{Cat} \to \mathrm{Cat}$ admits a right adjoint $(-)^d\colon \mathrm{Cat} \to \mathrm{Cat}$.
- For any object c in Cat and any object x of c (i.e. any morphism $x\colon [0] \to c$), one can define the slice $c_{/x}$ by forming the following pullback square in Cat.

$$\begin{array}{ccc} c_{/x} & \longrightarrow & c^{[1]} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ c\times[0] & \xrightarrow{\mathrm{id}_c\times x} & c\times c. \end{array}$$

- An object x of an object c of Cat is terminal if the projection $c_{/x} \to c$ is an equivalence.

Construction 8.7.3. For two morphisms $f\colon a \to c$ and $g\colon b \to c$ in Cat, we define their *directed pullback* $a \vec{\times}_{f,g} b$ via the following pullback square in Cat:

$$\begin{array}{ccc} a \vec{\times}_{f,g} b & \longrightarrow & c^{[1]} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ a\times b & \xrightarrow{f\times g} & c\times c. \end{array}$$

When $b = c$ and $g = \mathrm{id}_b$, we write $a \vec{\times}_f b := a \vec{\times}_{f,g} b$. For a morphism $f\colon a \to b$ in Cat, we define the *directed evaluation map* $\vec{\mathrm{ev}}_0^f\colon a^{[1]} \to a \vec{\times}_f b$ as the map whose two components are $\mathrm{ev}_0\colon a^{[1]} \to a$ and $f \circ \mathrm{ev}_1\colon a^{[1]} \to a \to b$.

These constructions allow us to express the (co)cartesian-fibration condition internally in Cat. The external language will remain useful for defining the collection of all (co)cartesian fibrations over a given base, as well as the collection of cocartesian functors between them.

The category of cocartesian fibrations

Construction 8.7.4. Consider an object s of Cat, corresponding to a small category S:

$$\begin{array}{ccc} S & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{s} & \mathrm{Cat}. \end{array}$$

Recall that we write $\mathrm{Cat}_{/S} = \mathrm{Cat}_{/s}$. In order to define $\mathrm{CoCart}(S)$ as a subcategory of $\mathrm{Cat}_{/S}$, we first define the collection $\mathrm{CoCart}(S)^{\simeq}$ of cocartesian fibrations between small categories. Given a functor $p\colon X \to A$ in an anima context Γ, the proposition asserting that a morphism f of X is p-cocartesian can be expressed as follows. For each morphism $f\colon x \to y$ of X in context Γ, one gets a commutative square of the form

$$\begin{array}{ccc} X_{y/} & \xrightarrow{f^*} & X_{x/} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ A_{p(y)/} & \xrightarrow{p(f)^*} & A_{p(x)/} \end{array}$$

hence a canonical functor

$$c_{f/p} \colon X_{y/} \to A_{p(y)/} \times_{A_{p(x)/}} X_{x/}$$

and we have the proposition asserting that the latter is an equivalence in context Γ:

$$\texttt{is_Equiv}_\Gamma(c_{f/p}) \hookrightarrow \Gamma\,.$$

The proposition $\texttt{is_Equiv}_\Gamma(c_{f/p})$ is also the assertion, in context Γ, that f is a p-cocartesian morphism in X. Let us consider the universal morphism in X, in context $\mathrm{Map}([1], X)$, defined through the evaluation functor:

$$\mathit{univ}_X \colon [1] \times \mathrm{Map}_\Gamma([1], X) \hookrightarrow [1] \times \mathrm{Fun}_\Gamma([1], X) \to X\,.$$

If we still denote by p the pullback of p along the projection of $\mathrm{Map}_\Gamma([1], X)$ to Γ, the proposition asserting that univ_X is p-cocartesian in context $\mathrm{Map}_\Gamma([1], X)$ defines the collection of p-cocartesian morphisms in X:

$$\mathrm{Map}_\Gamma([1], X)_{p\text{-coCart}} := \texttt{is_Equiv}_{\mathrm{Map}_\Gamma([1],X)}(c_{\mathit{univ}_X/p}) \hookrightarrow \mathrm{Map}_\Gamma([1], X)\,.$$

The property that p is a cocartesian fibration can be reformulated by asserting, for any object x_0 in X and any morphism $g \colon p(x_0) \to y_1$ in A, the existence of a unique p-cocartesian morphism in X of the form $f \colon x_0 \to x_1$ equipped with an isomorphism from $p(f)$ to g. In other words, evaluation at 0 induces a commutative square of the form

$$\begin{array}{ccc} \mathrm{Map}_\Gamma([1], X)_{p\text{-coCart}} & \xrightarrow{\mathrm{ev}_0} & X^{\simeq} \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Map}_\Gamma([1], A) & \xrightarrow{\mathrm{ev}_0} & A^{\simeq} \end{array}$$

hence a canonical functor

$$\varphi_p \colon \mathrm{Map}_\Gamma([1], X)_{p\text{-coCart}} \to \mathrm{Map}_\Gamma([1], A) \times_{A^{\simeq}} X^{\simeq}\,,$$

and the property for the latter to be an equivalence is equivalent to the property that the functor p is a cocartesian fibration in context Γ. We thus define

$$\texttt{is_coCartFib}(p) := \texttt{is_Equiv}_\Gamma(\varphi_p) \hookrightarrow \Gamma\,.$$

We consider now the universal functor from a small category to S. This is the functor of the form

$$p_{\mathrm{univ}/S} \colon X_{\mathrm{univ}/S} \to (\mathrm{Cat}_{/S})^{\simeq} \times S$$

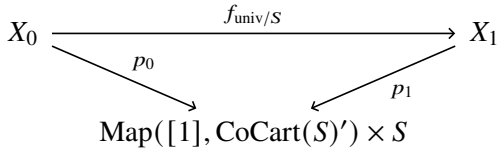
corresponding through directed univalence to the morphism

$$\mathrm{univ}_{/S} \colon [1] \times (\mathrm{Cat}_{/S})^{\simeq} \hookrightarrow [1] \times \mathrm{Map}([1], \mathrm{Cat}) \hookrightarrow [1] \times \mathrm{Fun}([1], \mathrm{Cat}) \to \mathrm{Cat}$$

induced by the obvious evaluation functor. The collection of cocartesian fibrations of the form $p \colon X \to S$ with X small is defined to be the proposition asserting that $p_{\mathrm{univ}/S}$ is a cocartesian fibration in context $(\mathrm{Cat}_{/S})^{\simeq}$:

$$\mathrm{CoCart}(S)^{\simeq} := \texttt{is_coCartFib}(p_{\mathrm{univ}/S}) \hookrightarrow (\mathrm{Cat}_{/S})^{\simeq}\,.$$

We define as an intermediate step the full subcategory $\mathrm{CoCart}(S)'$ of $\mathrm{Cat}_{/S}$ spanned by cocartesian fibrations with codomain S. By directed univalence, in context $\mathrm{Map}([1], \mathrm{CoCart}(S)')$, there is a universal functor between cocartesian fibrations over S:



Asserting that f_{univ} is a cocartesian functor over S in context $\mathrm{Map}([1], \mathrm{CoCart}(S)')$ thus defines a collection of morphisms in $\mathrm{CoCart}(S)'$:

$$\mathrm{Map}([1], \mathrm{CoCart}(S)')_{\mathrm{CoCart}} := \mathtt{is_Cocart}(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \mathrm{CoCart}(S)') \hookrightarrow \mathrm{Map}([1], \mathrm{Cat}_{/S}) .$$

We finally define $\mathrm{CoCart}(S)$ as the subcategory of $\mathrm{Cat}_{/S}$ generated by $\mathrm{Map}([1], \mathrm{CoCart}(S)')_{\mathrm{CoCart}}$:

$$\mathrm{CoCart}(S) = \langle \mathrm{Map}([1], \mathrm{CoCart}(S)')_{\mathrm{CoCart}} \rangle_{\mathrm{Cat}_{/S}} .$$

By definition, a functor $f\colon B \to \mathrm{Cat}_{/S}$ factors through $\mathrm{CoCart}(S)$ if and only if it sends each object of B to cocartesian fibrations over S and each morphism of B to cocartesian functors between cocartesian fibrations over S.

The unstraightening functor

Construction 8.7.5. Given S as above, we now construct a functor

$$\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{Cat}_{/S}.$$

The intended assignment is simple: a functor $S \to \mathrm{Cat}$ is sent to the small cocartesian fibration that it classifies, and a natural transformation is sent to the corresponding cocartesian functor. To assemble these assignments into a functor, we work uniformly in an arbitrary parameter category B.

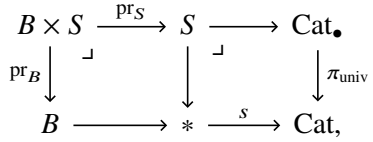
Given a category B and a functor $\Phi\colon B \to \mathrm{Fun}(S, \mathrm{Cat})$, we may uncurry Φ into a functor of the form

$$F\colon B \times S \to \mathrm{Cat}$$

and then consider the small cocartesian fibration classified by F:

$$p\colon E \to B \times S .$$

Observe that the projection map $\mathrm{pr}_B\colon B \times S \to B$ sits in a pullback square of the form



and hence pr_B is a small cocartesian fibration classified by the constant functor $\mathrm{const}_s\colon B \to \mathrm{Cat}$. Since Cat is a regular universe, the composite $q := \mathrm{pr}_B \circ p\colon E \to B$ is a small cocartesian fibration as well, thus classified by some functor $e\colon B \to \mathrm{Cat}$.

We may regard the functor $p\colon E \to B \times S$ as a functor over B in this way, and it follows from Lemma 4.7.7 that p is a cocartesian functor over B. By directed univalence of Cat, more specifically by Theorem 8.3.15, it follows that p corresponds to a natural transformation $e \to \mathrm{const}_s$ of functors $B \to \mathrm{Cat}$. But by Lemma 4.3.2 such a natural transformation precisely corresponds to a functor

$$\mathrm{Un}(\Phi)\colon B \to \mathrm{Cat}_{/S}$$

of the desired form. Consider now $B = \mathrm{Fun}(S, \mathrm{Cat})$ and the identity of B. The previous construction applied to the identity of $\mathrm{Fun}(S, \mathrm{Cat})$ defines a functor of the desired type:

$$\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{Cat}_{/S}\,.$$

Lemma 8.7.6. *The functor* $\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{Cat}_{/S}$ *lands in the subcategory* $\mathrm{CoCart}(S)$.

Proof. By definition of a subcategory it suffices to check this on individual morphisms in $\mathrm{Fun}(S, \mathrm{Cat})$.

Let us check what this functor does on objects. Note that the above construction is functorial in B, in the sense that if we precompose $F\colon B \times S \to \mathrm{Cat}$ by the functor $g \times \mathrm{id}_S\colon B' \times S \to B \times S$ induced by some functor $g\colon B' \to B$, we will end up with the composite $B' \xrightarrow{g} B \xrightarrow{\mathrm{Un}} \mathrm{Cat}_{/S}$. So to compute $\mathrm{Un}(f)$ for some functor $f\colon S \to \mathrm{Cat}$, we may directly do the above construction where we replace B by $*$. But then the natural transformation $e \to \mathrm{const}_s$ is just the morphism in Cat corresponding under directed univalence to the cocartesian fibration $p\colon E \to S$ classified by the given functor $f\colon S \to \mathrm{Cat}$. So on objects we see that it does indeed factor through $\mathrm{CoCart}(S)$.

On morphisms, the proof is similar. It suffices to interpret how the functor Un interprets cocartesian fibrations with small fibers of the form

$$p\colon E \to [1] \times S$$

as a morphism in $\mathrm{Cat}_{/S}$. Let l be a p-cocartesian lift of the term $\mathrm{id}_{[1]} \times p_0\colon [1] \times E_0 \to [1] \times S$ extending the term given by the canonical embedding ι_0 of the fiber E_0 into E obtained from Lemma 8.2.2.

$$\begin{array}{ccccc} E_0 & \xrightarrow{\iota_0} & E & \xrightarrow{\mathrm{id}_E} & E \\ {\scriptstyle (0,\mathrm{id}_{E_0})}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \mathrm{pr}_{[1]}\,p} \\ [1] \times E_0 & \xrightarrow[\mathrm{id}_{[1]} \times p_0]{} & [1] \times S & \xrightarrow{\mathrm{pr}_{[1]}} & [1] \end{array}$$

Taking fibers over 1 turns l into a functor $l_1\colon E_0 \to E_1$ over S that is a cocartesian functor over S. The right-hand square shows that, after forgetting the base S, directed univalence identifies l_1 with the morphism in Cat associated to p. Thus the induced morphism in $\mathrm{Cat}_{/S}$ is the morphism classified by l_1, together with its compatibility with the projections to S. In particular, it is a cocartesian functor over S. □

Lemma 8.7.7. *The functor* $\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat}) \to \mathrm{CoCart}(S)$ *is fully faithful.*

Proof. Let us consider for $i = 0$ and $i = 1$ a pullback square of the following form.

$$\begin{array}{ccc} X_i & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle p_i}\downarrow & & \downarrow \\ S & \xrightarrow{F_i} & \mathrm{Cat} \end{array}$$

If we denote by x_i the object of $\mathrm{CoCart}(S)$ corresponding to the cocartesian fibration p_i, we see that, by definition of $\mathrm{CoCart}(S)$, the equivalence

$$\mathrm{Cat}_{/S}(x_0, x_1) \xrightarrow{\sim} \mathrm{Map}_S(X_0, X_1)$$

from Lemma 8.7.1 induces a pullback square of the form

$$\begin{array}{ccc} \mathrm{CoCart}(S)(x_0, x_1) & \xrightarrow{\sim} & \mathrm{Map}_S^{\mathrm{CoCart}}(X_0, X_1) \\ \downarrow & & \downarrow \\ \mathrm{Cat}_{/S}(x_0, x_1) & \xrightarrow{\sim} & \mathrm{Map}_S(X_0, X_1) \end{array}$$

We observe that the map induced by Un

$$\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat})(F_0, F_1) \to \mathrm{CoCart}(S)(x_0, x_1)$$

composed with the equivalence

$$\mathrm{CoCart}(S)(x_0, x_1) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(X_0, X_1)$$

coincides termwise (in any anima context) with the canonical equivalence

$$\mathrm{Fun}(S, \mathrm{Cat})(F_0, F_1) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(X_0, X_1)\,.$$

provided by Theorem 8.3.15. This shows that this map is an equivalence by applying Proposition 1.3.1. □

Theorem 8.7.8 (Straightening/unstraightening)**.** *For every small category S, the functor*

$$\mathrm{Un}\colon \mathrm{Fun}(S, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{CoCart}(S)$$

is an equivalence.

Proof. By virtue of the preceding lemma and of the fundamental theorem of category theory (Theorem 6.3.8), it suffices to prove that the functor Un has a section. We observe that a functor $\Phi\colon B \to \mathrm{Cat}_{/S}$ admits a lift $\widetilde{\Phi}\colon B \to \mathrm{Fun}(S, \mathrm{Cat})$ along Un if and only if it factors through $\mathrm{CoCart}(S)$: this is precisely what Proposition 6.5.15 means. Considering $B = \mathrm{CoCart}(S)$, and Φ the canonical embedding of $\mathrm{CoCart}(S)$ into $\mathrm{Cat}_{/S}$, this proves that the functor Un has a section. □

Corollary 8.7.9. *For every small anima Γ, there is a canonical equivalence of the form*

$$\mathrm{Un}\colon \mathrm{Fun}(\Gamma, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{Cat}_{/\Gamma}\,.$$

Proof. For any functor $p\colon X \to \Gamma$, identity morphisms in X provide cocartesian lifts of morphisms in Γ: their images need only be isomorphic to the specified base morphisms. Moreover, every functor over Γ preserves these lifts. Thus

$$\mathrm{CoCart}(\Gamma) = \mathrm{Cat}_{/\Gamma}$$

for any small anima Γ. The corollary is thus a special case of the previous theorem. □

8.7.3 Straightening/unstraightening for left fibrations

Under straightening, anima-valued functors correspond to left fibrations. We now express this correspondence as an equivalence of categories.

Construction 8.7.10. Let A be a small category, with groupoid-core inclusion $\iota_A \colon A^{\simeq} \hookrightarrow A$. We define the category $\mathrm{LFib}(A)$ of *left fibrations over A* as the full subcategory

$$\mathrm{LFib}(A) \subseteq \mathrm{CoCart}(A) \subseteq \mathrm{Cat}_{/A}$$

spanned by those cocartesian fibrations $B \to A$ which are in fact left fibrations (i.e. have anima-valued fibers). In other words, it is defined by forming the pullback square below.

$$\begin{array}{ccccc} \mathrm{LFib}(A) & & \longrightarrow & & \mathrm{An}_{/A^{\simeq}} \\ \downarrow & \lrcorner & & & \downarrow \\ \mathrm{CoCart}(A) & \hookrightarrow & \mathrm{Cat}_{/A} & \xrightarrow{\iota_A^*} & \mathrm{Cat}_{/A^{\simeq}} \end{array}$$

Since the vertical functor at the right hand side is fully faithful, so is its counterpart at the left hand side. We observe that, since An is a full subcategory of Cat, there is a canonical pullback square of the form below.

$$\begin{array}{ccc} \mathrm{Fun}(A, \mathrm{An}) & \xrightarrow{\iota_A^*} & \mathrm{Fun}(A^{\simeq}, \mathrm{An}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Fun}(A, \mathrm{Cat}) & \xrightarrow{\iota_A^*} & \mathrm{Fun}(A^{\simeq}, \mathrm{Cat}) \end{array}$$

Corollary 8.7.11. *For every small anima Γ, there is a canonical equivalence of the form*

$$\mathrm{Un} \colon \mathrm{Fun}(\Gamma, \mathrm{An}) \xrightarrow{\sim} \mathrm{An}_{/\Gamma}\,.$$

Proof. Since left fibrations are precisely conservative cocartesian fibrations, and any functor with codomain an anima is a cocartesian fibration, left fibrations with anima codomain are precisely functors between animae. Thus Corollary 8.7.9 induces the claimed equivalence. □

Proposition 8.7.12 ((Un)Straightening for left fibrations)**.** *For every small category A, the straightening/unstraightening equivalence from Theorem 8.7.8 restricts to an equivalence*

$$\mathrm{Un} \colon \mathrm{Fun}(A, \mathrm{An}) \xrightarrow{\sim} \mathrm{LFib}(A).$$

Proof. For a cocartesian fibration $p \colon X \to A$, the property of being a left fibration is equivalent to being conservative or, equivalently, to all fibers of p being animae. In other words, p is a left fibration if and only if the pullback $A^{\simeq} \times_A X$ is an anima. In view of the two pullback squares above, in order to prove this proposition, by virtue of Theorem 8.7.8 and of Corollary 8.7.9, it suffices to prove the special case where A is an anima. This is Corollary 8.7.11. □

8.7.4 Computations with unstraightening

Straightening allows us to pass between constructions on Cat-valued functors and constructions on cocartesian fibrations. We now make this relation explicit for functor categories and coslices. We also record results about products in an unstraightening.

Proposition 8.7.13 (Functor categories in cocartesian unstraightenings)**.** *Let $p\colon E \to C$ be a small cocartesian fibration classified by a functor $F\colon C \to \mathrm{Cat}$, let I be a small category, and let i be its associated object of* Cat*. Define $q\colon Q \to C$ by the pullback square*

$$\begin{array}{ccc} Q & \longrightarrow & \mathrm{Fun}(I, E) \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p_*} \\ C & \xrightarrow[p_I^*]{} & \mathrm{Fun}(I, C). \end{array}$$

Then q is a small cocartesian fibration classified by the composite

$$C \xrightarrow{F} \mathrm{Cat} \xrightarrow{(-)^i} \mathrm{Cat},$$

where $(-)^i$ is the internal functor-category functor of Construction 8.5.10.

Proof. Apply $\mathrm{Fun}(I, -)$ to the pullback square classifying p. Since functor categories preserve pullbacks, both squares in the following diagram are pullbacks:

$$\begin{array}{ccccc} Q & \longrightarrow & \mathrm{Fun}(I, E) & \longrightarrow & \mathrm{Fun}(I, \mathrm{Cat}_\bullet) \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p_*} & \lrcorner & \downarrow{\scriptstyle (\pi_{\mathrm{univ}})_*} \\ C & \xrightarrow[p_I^*]{} & \mathrm{Fun}(I, C) & \xrightarrow[F_*]{} & \mathrm{Fun}(I, \mathrm{Cat}). \end{array}$$

The bottom composite is $F_* p_I^* = p_I^* F$. Thus the outer rectangle identifies q with the pullback along F of the small cocartesian fibration used to define $(-)^i$ in Construction 8.5.10. Its classifying functor is therefore $(-)^i \circ F$. □

Proposition 8.7.14 (Coslices in cocartesian unstraightenings)**.** *Let $p\colon E \to B$ be a small cocartesian fibration classified by a functor $F\colon B \to \mathrm{Cat}$, and let X be an object of the fiber $F(b)$. The induced functor*

$$p_{X/}\colon E_{X/} \longrightarrow B_{b/}$$

is a cocartesian fibration classified by the functor

$$B_{b/} \longrightarrow \mathrm{Cat}, \qquad (f\colon b \to c) \longmapsto F(c)_{f_! X/}.$$

Proof. There is a pullback square

$$\begin{array}{ccc} E_{X/} & \longrightarrow & B_{b/} \times_{\mathrm{Fun}([1], B)} \mathrm{Fun}([1], E) \\ {\scriptstyle p_{X/}}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{pr}_1, \mathrm{ev}_0)} \\ B_{b/} & \xrightarrow[(\mathrm{id}, X)]{} & B_{b/} \times E_b. \end{array}$$

The upper right corner is a small cocartesian fibration over $B_{b/}$ by Lemma 8.4.16 and base change. Both lower corners are also small cocartesian fibrations over $B_{b/}$.

By Corollary 6.5.8, the source evaluation functor sends a cocartesian morphism in the upper right corner to a p-cocartesian morphism over the identity of b, hence to an isomorphism in E_b. The right vertical map is therefore a cocartesian functor over $B_{b/}$, as is the bottom map. It follows from (Cat3)

that $p_{X/}$ is a small cocartesian fibration. We define the functor in the statement to be its classifying functor.

To compute its values, let $f\colon b \to c$. Composition with a p-cocartesian lift $X \to f_!X$ gives an equivalence

$$F(c)_{f_!X/} \simeq (E_{X/})_f$$

where the right-hand side denotes the fiber of $p_{X/}$ over f. Indeed, the square of coslices expressing that $X \to f_!X$ is p-cocartesian is a pullback; taking its fiber over id_c gives the displayed equivalence.

Finally, let $u\colon f \to f'$ be a morphism in $B_{b/}$, represented by a morphism $u\colon c \to c'$ and an identification $f' = uf$. The pullback construction above shows that a cocartesian lift starting in $X \to Y$ is given by postcomposition with the p-cocartesian lift $Y \to u_!Y$. Under the displayed equivalences, transport therefore sends a map $f_!X \to Y$ to the composite

$$f'_!X = u_!f_!X \longrightarrow u_!Y. \qquad \square$$

Proposition 8.7.15 (Products in cocartesian unstraightenings)**.** *Let $p\colon E \to B$ be a small cocartesian fibration classified by a functor $F\colon B \to \mathrm{Cat}$. If B admits finite products and F preserves them, then E admits finite products and p preserves them. More precisely, let $(X_a \in F(b_a))_{a\in A}$ be a finite family, and write $b = \prod_{a\in A} b_a$. Under the equivalence*

$$F(b) \simeq \prod_{a\in A} F(b_a),$$

let $X \in F(b)$ correspond to the tuple $(X_a)_{a\in A}$. Then the p-cocartesian lifts $X \to X_a$ of the product projections $b \to b_a$ exhibit X as the product of the X_a in E.

Proof. For an object $Z \in F(c)$, the proposed product cone gives a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_E(Z, X) & \longrightarrow & \displaystyle\prod_{a\in A} \mathrm{Hom}_E(Z, X_a) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_B(c, b) & \xrightarrow{\sim} & \displaystyle\prod_{a\in A} \mathrm{Hom}_B(c, b_a). \end{array}$$

For every $f\colon c \to b$, the induced map on fibers is

$$\mathrm{Hom}_{F(b)}(f_!Z, X) \longrightarrow \prod_{a\in A} \mathrm{Hom}_{F(b_a)}((\mathrm{pr}_a f)_!Z, X_a).$$

Under the product equivalence for $F(b)$, the object $f_!Z$ corresponds to the tuple $((\mathrm{pr}_a f)_!Z)_{a\in A}$, by compatibility of cocartesian transport with composition. Thus the map on fibers is an equivalence. The bottom horizontal map is an equivalence because b is the product of the b_a, so the top horizontal map is an equivalence as well. This proves that the chosen lifts exhibit X as the product of the X_a. Their images under p form the product cone with vertex b. $\square$

Remark 8.7.16. The dual statements describe slices and finite coproducts in cartesian unstraightenings. We can obtain them once opposite categories and cartesian straightening have been constructed in Section 12.1 and Theorem 12.4.10.

8.8 Localizations

We now construct geometric realization as the left adjoint to An $\hookrightarrow$ Cat and then pass to localization at a chosen collection of morphisms. We subsequently perform these constructions fiberwise and identify the resulting relative construction with an absolute localization.

8.8.1 Geometric realizations and localizations

The groupoid-core functor constructed in Proposition 8.6.11 is right adjoint to the inclusion An $\hookrightarrow$ Cat. We now construct its left adjoint

$$\|-\|\colon \mathrm{Cat} \to \mathrm{An}$$

which sends a small category to its geometric realization. We then extend the same construction to localization at a chosen subcategory.

Proposition 8.8.1. *The inclusion functor* $i\colon \mathrm{An} \hookrightarrow \mathrm{Cat}$ *admits a left adjoint* $\|-\|\colon \mathrm{Cat} \to \mathrm{An}$.

Proof. We will verify the conditions of Proposition 6.2.3. Let c be an object of Cat and let C denote its associated small category. By condition (Cat7) there exists a small category $\|C\|$ together with a functor $\eta_C\colon C \to \|C\|$ that exhibits it as a localization of C at all its morphisms. Thus $\|C\|$ is the geometric realization of C and is an anima by Proposition 3.4.5. By recognition, it is classified by some object $\|C\| \in \mathrm{An}$, and under directed univalence the localization map $C \to \|C\|$ is classified by a morphism $c \to \|C\|$ in Cat.

We claim the assignment $c \mapsto \|C\|$ enhances to a functor $\|-\|\colon \mathrm{Cat} \to \mathrm{An}$ which is left adjoint to the inclusion functor. By Proposition 6.2.3, it will suffice to show that for any object $x \in \mathrm{An}$ the map

$$\mathrm{Hom}_{\mathrm{An}}(\|C\|, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(i\|C\|, ix) \xrightarrow{-\circ\eta_C} \mathrm{Hom}_{\mathrm{Cat}}(c, ix)$$

is an equivalence. Under directed univalence, this map agrees with the map

$$-\circ\eta_C\colon\ \mathrm{Map}(\|C\|, X) \to \mathrm{Map}(C, X).$$

The fact that this map is an equivalence then follows from Lemma 3.4.3. □

The geometric realization functor is a special case of the process of localizing by a family of maps.

Construction 8.8.2. Let C be a category with pullbacks. There is a functor

$$\delta\colon\ \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], C)$$

sending each morphism $f\colon x \to y$ in C to the diagonal map $\delta_f\colon x \to x \times_y x$, induced by the canonical commutative square

$$\begin{array}{ccc} x & \xrightarrow{\mathrm{id}_x} & x \\ {\scriptstyle \mathrm{id}_x}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{f} & y \end{array}$$

in C. We define the category of embeddings of C as the full subcategory of $\mathrm{Fun}([1], C)$ spanned by embeddings in C. In other words, this is the category obtained by forming the following pullback square.

$$\begin{array}{ccc} \mathrm{Emb}(C) & \hookrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & & \downarrow{\scriptstyle \delta} \\ \mathrm{Iso}(C) & \hookrightarrow & \mathrm{Fun}([1], C) \end{array}$$

For $C = \mathrm{Cat}$, we define the categories of pairs of small categories as

$$\mathrm{Pair} = \mathrm{Emb}(\mathrm{Cat})$$

Thus Pair is the category of small categories equipped with a chosen subcategory. The localization functor constructed below will universally invert the morphisms in that chosen subcategory. By a slight abuse of notation, we will often think of objects in Pair as a pair (C, W) where C is a small category and W a subcategory of C (which makes sense, using Proposition 6.4.5). There is a functor

$$\mathrm{ev}_1 : \mathrm{Pair} \to \mathrm{Cat}$$

sending (C, W) to C. The latter has a unique section

$$(-)^\flat : \mathrm{Cat} \to \mathrm{Pair}$$

with the property of sending each small category C to the pair $C^\flat = (C, C^\simeq)$. The functor $(-)^\flat$ is easily seen to be fully faithful (this boils down to the fact that any functor sends isomorphisms to isomorphisms).

Proposition 8.8.3. *The functor $C \mapsto (C, C^\simeq)$ has a left adjoint*

$$L : \mathrm{Pair} \to \mathrm{Cat}\,.$$

More precisely, for a pair (C, W), the category $L(C, W)$ corresponds to the localization of C by the family of maps $\mathrm{Map}([1], W)$.

Proof. A morphism of pairs of the form $(C, W) \to (D, D^\simeq)$ corresponds to a commutative square in Cat of the form

$$\begin{array}{ccc} W & \longrightarrow & D^\simeq \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D \end{array}$$

in which both vertical maps are embeddings. But by virtue of Proposition 6.4.5, such a commutative square is totally determined by the functor $f : C \to D$ and the fact that the induced map

$$f_* : \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$$

sends $\mathrm{Map}([1], W)$ to $\mathrm{Iso}(D)^\simeq$. In other words, this is the same thing (in any anima context) as a functor from $L(C, W)$ to D, where $L(C, W)$ denotes the localization of C by $\mathrm{Map}([1], W)$. □

Remark 8.8.4. For any category, $i(\|C\|) = L(C, C)$, where $i : \mathrm{An} \to \mathrm{Cat}$ denotes the canonical embedding.

8.8.2 Fiberwise localizations

As a consequence of the adjunction between $(-)^\flat$ and L from the preceding proposition, we get for every category S an adjunction between the induced functors

$$L_S\colon \mathrm{Fun}(S,\mathrm{Pair}) \to \mathrm{Fun}(S,\mathrm{Cat}) \qquad \text{and} \qquad (-)^\flat_S\colon \mathrm{Fun}(S,\mathrm{Cat}) \to \mathrm{Fun}(S,\mathrm{Pair}).$$

The existence of the functor L_S allows us to form *fiberwise localizations* of (small) cocartesian fibrations:

Proposition 8.8.5. *Let $p\colon C \to S$ be a small cocartesian fibration. We consider, for each object $x : S^\simeq$ a subcategory $W_x \hookrightarrow C_x$ and assume that, for each morphism $f\colon x \to y$ in S, the induced functor $f_!\colon C_x \to C_y$ restricts to a functor from W_x to W_y. Then there exists a commutative triangle*

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & L_S(C,W) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle L_S(p)} & \\ & S & \end{array}$$

satisfying the following properties:

(1) The functor $L_S(p)$ is a cocartesian fibration;

(2) For any other small cocartesian fibration $D \to S$, precomposition with η_C induces an equivalence

$$\eta_C^*\colon \mathrm{Map}_S^{\mathrm{CoCart}}(L_S(C,W),D) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart},W}(C,D)$$

where $\mathrm{Map}_S^{\mathrm{CoCart},W}(C,D)$ is the collection of cocartesian functors $\varphi\colon C \to D$ over S with the property that, for any object $x : S^\simeq$, the induced functor on fibers $\varphi_x\colon C_x \to D_x$ sends all morphisms in W_x to isomorphisms in D_x.

(3) The formation of η_C and $L_S(C,W)$ is stable under base change along any functor $S' \to S$. In particular, for any object $s : S^\simeq$, the functor $(\eta_C)_s\colon C_s \to L_S(C,W)_s$ induces an equivalence

$$L(C_s,W_s) \xrightarrow{\sim} L_S(C,W)_s\,.$$

Proof. Using cocartesian transport, each morphism u of C factors uniquely into a p-cocartesian morphism $v\colon x \to y$ followed by a morphism $w\colon y \to y'$ in the fiber over $p(y)$. We define W to have the objects x that belong to $W_{p(x)}$ and the morphisms u between such objects for which the vertical factor w belongs to $W_{p(y)}$. The assumption that cocartesian transport restricts to the W_x implies that W is closed under composition, that $W \to S$ is a cocartesian fibration, and that $W \hookrightarrow C$ is a cocartesian functor over S. The fibers W_x are small by subcategory closure, so $W \to S$ is small by (Cat9). Thus directed univalence gives a functor $F\colon S \to \mathrm{Pair}$. Then the composite

$$L \circ F\colon S \to \mathrm{Cat}$$

classifies a cocartesian fibration $L_S(p)\colon L_S(C,W) \to S$:

$$\begin{array}{ccccc} L_S(C,W) & \longrightarrow & & & \mathrm{Cat}_\bullet \\ {\scriptstyle L_S(p)}\downarrow & \lrcorner & & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ S & \xrightarrow{F} & \mathrm{Pair} & \xrightarrow{L} & \mathrm{Cat}. \end{array}$$

The natural transformation $\mathrm{id}_{\mathrm{Pair}} \to (L(-))^\flat$ provides a natural transformation $F \to (L(F))^\flat$, which, by directed univalence, yields a cocartesian functor from C to $L_S(C, W)$ over S:

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & L_S(C,W) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle L_S(p)} & \\ & S. & \end{array}$$

This produces the desired commutative triangle and proves property (1). For (2), we note that postcomposition with the functors L and $(-)^\flat$ extends to an adjunction

$$L_S = (L)_* \colon \mathrm{Fun}(S, \mathrm{Pair}) \rightleftarrows \mathrm{Fun}(S, \mathrm{Cat}) : ((-)^\flat)_*$$

whose unit is induced by the unit map of the adjoint pair $(L, (-)^\flat)$, by Proposition 4.4.4. In particular, for functors $F \colon S \to \mathrm{Pair}$ and $G \colon S \to \mathrm{Cat}$ the map

$$\mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Cat})}(L \circ F, G) \xrightarrow{(-)^\flat \circ -} \mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Pair})}((L \circ F)^\flat, G^\flat) \xrightarrow{- \circ \eta} \mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Pair})}(F, G^\flat)$$

is an equivalence. The functor $F \colon S \to \mathrm{Pair}$ corresponds to a small cocartesian fibration $C \to S$, equipped with a subcategory $W \hookrightarrow C$ such that the restriction of p to W remains a cocartesian fibration and the embedding of W into C a cocartesian functor over S, while the functor $G \colon S \to \mathrm{Cat}$ corresponds to a small cocartesian fibration $D \to S$. Under directed univalence, the previous functor then translates into the functor

$$\eta_C^* \colon \mathrm{Map}_S^{\mathrm{CoCart}}(L_S(C,W), D) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart},W}(C, D),$$

from part (2), showing that it is an equivalence.

For part (3), we observe that the map η_C can in fact be recovered as a pullback from the universal case $S = \mathrm{Pair}$ and $F = \mathrm{id}_{\mathrm{Pair}}$. More precisely, the embedding

$$\mathrm{Pair} \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat}) \xrightarrow{\sim} \underline{\mathrm{CoCart}}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat} \times \mathrm{Cat}_\bullet)$$

yields a commutative triangle of the form

$$\begin{array}{ccc} W_{\mathrm{univ}} & \xhookrightarrow{i_{\mathrm{univ}}} & C_{\mathrm{univ}} \\ & \searrow \quad \swarrow{\scriptstyle p_{\mathrm{univ}}} & \\ & \mathrm{Pair} & \end{array}$$

in which p_{univ} as well as its restriction to W_{univ} are cocartesian fibrations and i_{univ} is a cocartesian functor over Pair and an embedding (fiberwise, hence globally as well). Pulling back the diagram above along $F \colon S \to \mathrm{Pair}$ recovers $W \hookrightarrow C$ over S and we have canonical pullback squares of the form:

$$\begin{array}{ccccc} L_S(C,W) & \to & L_{\mathrm{Pair}}(C_{\mathrm{univ}}, W_{\mathrm{univ}}) & \to & \mathrm{Cat}_\bullet \\ {\scriptstyle L_S(p)}\downarrow & & {\scriptstyle L_{\mathrm{Pair}}(p_{\mathrm{univ}})}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ S & \xrightarrow{F} & \mathrm{Pair} & \xrightarrow{L} & \mathrm{Cat}. \end{array}$$

This readily implies the final statement in part (3). □

Corollary 8.8.6. *Let $p\colon C \to S$ be a small cocartesian fibration. Then there exists a commutative triangle*

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & \|C\|_S \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \|p\|_S} & \\ & S & \end{array}$$

satisfying the following properties:

(1) The functor $\|p\|_S$ is a small left fibration;

(2) For any other small left fibration $X \to S$, precomposition with η_C induces an equivalence

$$\eta_C^*\colon \operatorname{Map}_S^{\mathrm{CoCart}}(\|C\|_S, X) \xrightarrow{\sim} \operatorname{Map}_S^{\mathrm{CoCart}}(C, X).$$

(3) The formation of η_C and $\|C\|_S$ is stable under base change along any functor $S' \to S$. In particular, for any object s of S the functor $(\eta_C)_s\colon C_s \to (\|C\|_S)_s$ induces an equivalence

$$\|C_s\| \xrightarrow{\sim} (\|C\|_S)_s.$$

Proof. Under the assumptions of the preceding proposition, if we suppose furthermore that $W_x = C_x$ for all $x : S^{\simeq}$, we define

$$\|C\|_S = L_S(C, C)\,.$$

We see that each fiber of $L_S(p)$ at $x : S^{\simeq}$ is the geometric realization of the fiber of p:

$$\|C_x\| = L_S(C, C)_x\,.$$

Thus the fibers of the cocartesian fibration $L_S(p)$ are animae, so $L_S(p)$ is a left fibration by Lemma 6.1.8. This corollary is an immediate consequence of Proposition 8.8.5. □

8.8.3 Fiberwise localization as a localization

In the previous subsection we constructed the relative localization $\eta_C\colon C \to L_S(C, W)$ and established its fiberwise universal property. We now prove that it is the absolute localization at those morphisms of W whose images in S are isomorphisms. The special case $W = C$ recovers fiberwise geometric realization.

The main ingredient for the proof will be a certain result about relative functor categories. Let $p\colon C \to S$ be a cocartesian fibration. For any other category D, we may consider the projection map $S \times D \to S$, which defines another category over S. Since p is exponentiable by Axiom M, we may form the relative functor category over S:

$$\pi\colon \underline{\mathrm{Fun}}_S(C, S \times D) \to S.$$

Given another functor $q\colon E \to S$, a map $f\colon E \to \underline{\mathrm{Fun}}_S(C, S \times D)$ over S corresponds to a map $\varphi\colon E \times_S C \to S \times D$ over S:

$$\begin{array}{ccc} E & \xrightarrow{f} & \underline{\mathrm{Fun}}_S(C, S \times D) \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array} \qquad \Longleftrightarrow \qquad \begin{array}{ccc} E \times_S C & \xrightarrow{\varphi} & S \times D \\ & \searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & S. & \end{array}$$

Proposition 8.8.7. *Let $p\colon C \to S$ be a cocartesian fibration and let D be a category. Then the functor*

$$\pi\colon T := \underline{\mathrm{Fun}}_S(C, D \times S) \to S$$

is a cartesian fibration whose fiber over an object s of S is given by $\mathrm{Fun}(C_s, D)$.

For any cocartesian functor $C \to C'$ over S, the induced functor $\underline{\mathrm{Fun}}_S(C', D\times S) \to \underline{\mathrm{Fun}}_S(C, D\times S)$ is cartesian over S. Dually, a cartesian fibration $C \to S$ gives a cocartesian relative functor category, and precomposition along a cartesian functor is cocartesian over S.

Proof. We prove the cocartesian case. The cartesian case is dual, by exchanging the roles of 0 and 1 in the interval $[1]$.

Recall that for a map $\alpha\colon S' \to S$, we have a pullback square

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{S'}(C', S' \times D) & \longrightarrow & \underline{\mathrm{Fun}}_S(C, S \times D) \\ {\scriptstyle \pi'}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\quad\alpha\quad} & S, \end{array}$$

where we write $C' := S' \times_S C \to S'$. In particular, it follows directly that the fibers of π are given by $\mathrm{Fun}(C_s, D)$, and it remains to show that π is a cartesian fibration.

We will first show that π is a locally cartesian fibration and then show that locally cartesian morphisms compose. By the dual of Proposition 6.5.4, this will imply the claim.

Consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Fun}([1], T) & \xrightarrow{\vec{\mathrm{ev}}_1^{\pi}} & S \vec{\times}_{\pi} T & \longrightarrow & T \\ {\scriptstyle \pi_*}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ \mathrm{Fun}([1], S) & = & \mathrm{Fun}([1], S) & \xrightarrow{\mathrm{ev}_1} & S. \end{array}$$

Given a morphism $u\colon a \to b$ in S, we may consider the fiber of the map $\vec{\mathrm{ev}}_1^{\pi}$ over $u \in \mathrm{Fun}([1], S)$ to obtain a functor

$$l := (\vec{\mathrm{ev}}_1^{\pi})_u\colon \mathrm{Fun}([1], T)_u \to (S \vec{\times}_{\pi} T)_u \simeq T_b.$$

We claim that π is a locally cartesian fibration if and only if this functor l admits a right adjoint section $r\colon T_b \hookrightarrow \mathrm{Fun}([1], T)_u$. Indeed, the existence of r means that for every object $t \in T_b$ there exists a morphism $r(t)\colon t' \to t$ over u such that for every diagram

$$\begin{array}{ccc} s' & \longrightarrow & s \\ \vdots & & \downarrow \\ t' & \xrightarrow{r(t)} & t \end{array}$$

in T living over $u\colon a \to b$, there exists a unique dashed arrow filling the square. This is equivalent to $r(t)\colon t' \to t$ being a locally cartesian lift of u.

This characterization of locally cartesian fibrations works for arbitrary functors $T \to S$. For our specific T we must now compute the map $\mathrm{Fun}([1], T)_u \to T_b$. Its source is $\mathrm{Fun}(C_u, D)$, where C_u

is defined via the following pullback diagram:

$$\begin{array}{ccc} C_u & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow p \\ [1] & \xrightarrow{u} & S. \end{array}$$

We also have $T_b \xrightarrow{\sim} \mathrm{Fun}(C_b, D)$. The map $\mathrm{Fun}(C_u, D) \to \mathrm{Fun}(C_b, D)$ is given by restriction along the inclusion $C_b \hookrightarrow C_u$. Our goal now is to find a right adjoint section of this functor.

Recall from Proposition 4.4.4 that if $\ell \dashv s \colon A \rightleftarrows B$ and s is a right adjoint section, then precomposition gives a right adjoint section

$$s^* \colon \mathrm{Fun}(A, D) \rightleftarrows \mathrm{Fun}(B, D) \colon \ell^*.$$

Here ℓ^* is the right adjoint section. Thus it suffices to show that the inclusion $C_b \hookrightarrow C_u$ is a right adjoint section. This is a general fact about cocartesian fibrations: we have a pullback square

$$\begin{array}{ccc} C_b & \hookrightarrow & C_u \\ {\scriptstyle p_b}\downarrow & \lrcorner & \downarrow{\scriptstyle p_u} \\ \{1\} & \hookrightarrow & [1], \end{array}$$

Since p_u is a cocartesian fibration and $\{1\} \hookrightarrow [1]$ admits a left adjoint $[1] \to \{1\}$, Proposition 6.2.4 shows that $C_b \hookrightarrow C_u$ also admits a left adjoint. Hence the inclusion is a right adjoint section, as desired. This finishes the proof that $\pi \colon T \to S$ is a locally cartesian fibration.

To see that locally π-cartesian morphisms are closed under composition, we may equivalently show that the locally cartesian transport functors compose. In other words, given two morphisms $u \colon a \to b$ and $v \colon b \to c$ in S, we need to show that the composite

$$T_c = \mathrm{Fun}(C_c, D) \xrightarrow{v^*} \mathrm{Fun}(C_b, D) \xrightarrow{u^*} \mathrm{Fun}(C_a, D) = T_a$$

agrees with $(vu)^* \colon T_c \to T_a$. To see this, we give an explicit description of v^* and u^*.

We start by giving an explicit description of the left adjoint $t \colon C_u \to C_b$ to the inclusion functor $C_b \to C_u$: on objects, it sends a pair (t, x) consisting of an object t of $[1]$ and an object x of $C_{u(t)}$ to the object $u(t \to 1)_!(x)$, given by the cocartesian transport of x along the morphism $u(t \to 1)$, where $t \to 1$ is the unique map in $[1]$ from t to 1. The locally cartesian transport $u^* \colon T_b \to T_a$ along $u \colon a \to b$ is then given as the following composite:

$$u^* \colon \mathrm{Fun}(C_b, D) \xrightarrow{t^*} \mathrm{Fun}(C_u, D) \xrightarrow{\mathrm{res}} \mathrm{Fun}(C_a, D).$$

The first functor sends $F \colon C_b \to D$ to the assignment

$$(t, x) \mapsto F(u(t \to 1)_!(x)).$$

Restricting to $t = 0$ thus shows that the composite sends F to the functor $x \mapsto F(u_!(x))$, which is the precomposition of F with $u_! \colon C_a \to C_b$.

Now, given two morphisms $u \colon a \to b$ and $v \colon b \to c$ in S, the composite

$$\mathrm{Fun}(C_c, D) \xrightarrow{v^*} \mathrm{Fun}(C_b, D) \xrightarrow{u^*} \mathrm{Fun}(C_a, D)$$

is given by precomposition with $v_! \circ u_!$. Since p is a cocartesian fibration, this functor is identified with $(v \circ u)_! \colon C_a \to C_c$, so precomposition with it agrees with $(vu)^*$.

It remains to prove the assertion concerning functoriality. Let $f \colon C \to C'$ be a cocartesian functor over S. For a morphism $u \colon a \to b$ in S, the preceding computation identifies cartesian transport in the two relative functor categories with precomposition by $u_!^C \colon C_a \to C_b$ and $u_!^{C'} \colon C'_a \to C'_b$, respectively. Since f is cocartesian, there is an identification

$$f_b \circ u_!^C = u_!^{C'} \circ f_a.$$

Consequently, the square

$$\begin{array}{ccc} \mathrm{Fun}(C'_b, D) & \xrightarrow{f_b^*} & \mathrm{Fun}(C_b, D) \\ {\scriptstyle (u_!^{C'})^*}\downarrow & & \downarrow{\scriptstyle (u_!^C)^*} \\ \mathrm{Fun}(C'_a, D) & \xrightarrow[f_a^*]{} & \mathrm{Fun}(C_a, D) \end{array}$$

commutes by the induced identification. Thus precomposition with f preserves cartesian transport, and hence defines a cartesian functor over S. This finishes the proof. □

Remark 8.8.8. More generally, if $p \colon C \to S$ is merely *locally* cocartesian, $\pi \colon T \to S$ will be locally cartesian. Indeed, for a functor $S' \to S$ we have a pullback square

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{S'}(C \times_S S', D \times S') & \longrightarrow & \underline{\mathrm{Fun}}_S(C, D \times S) \\ \downarrow & \lrcorner & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

Taking $S' = [1]$ then shows that π being locally cartesian only depends on the pullbacks of $C \to S$ to $[1]$.

Theorem 8.8.9. *Let $p \colon C \to S$ be a small cocartesian fibration. We assume given a subcategory $i \colon W \hookrightarrow C$ such that $p \circ i$ is a cocartesian fibration and i is a cocartesian functor over S. Then the functor $\eta_C \colon C \to L_S(C, W)$ exhibits $L_S(C, W)$ as the localization of C by the morphisms in W which are sent to isomorphisms in S.*

Proof. By Proposition 8.4.6, the cocartesian fibration $W \to S$ is small, so the fiberwise localization of Proposition 8.8.5 is defined. We must show that precomposition with η_C induces for every category D a fully faithful functor

$$\eta_C^* \colon \mathrm{Fun}(L_S(C, W), D) \hookrightarrow \mathrm{Fun}(C, D)$$

whose image consists of the functors which invert the morphisms in W sent to isomorphisms in S. To this end, consider the following functor over S:

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_S(L_S(C, W), S \times D) & \xrightarrow{\eta_C^*} & \underline{\mathrm{Fun}}_S(C, S \times D) \\ & \searrow \quad \swarrow & \\ & S. & \end{array}$$

By Proposition 8.8.7 this is a cartesian functor between cartesian fibrations over S, which is fiberwise given by the fully faithful functors

$$\mathrm{Fun}(L(C_x, W_x), D) \hookrightarrow \mathrm{Fun}(C_x, D)\,.$$

Recall from Lemma 6.5.16 that a cartesian functor $T' \to T$ over S is fully faithful if the induced functors $T'_x \to T_x$ on fibers are fully faithful for all $x : S^{\simeq}$. Since this image is a full subcategory stable under cartesian transport, it is detected fiberwise. We conclude that $\eta_C^* : \underline{\mathrm{Fun}}_S(L_S(C, W), S \times D) \to \underline{\mathrm{Fun}}_S(C, S \times D)$ is fully faithful, and that a functor $f \in \mathrm{Fun}(C_x, D) = \underline{\mathrm{Fun}}_S(C, S \times D)_x$ is in the essential image if and only if it inverts all morphisms in W_x. Now, applying $(p_S)_*(-)$ with $p_S : S \to [0]$, we obtain the functor

$$\eta_C^* :\ \mathrm{Fun}(L_S(C, W), D) \hookrightarrow \mathrm{Fun}(C, D),$$

which is again fully faithful. Its essential image consists of those functors $f : C \to D$ whose restriction to each fiber C_x inverts all morphisms in W_x. A morphism of W sent to an isomorphism in S factors as a cocartesian lift of that isomorphism followed by a morphism in a fiber. The first factor is an isomorphism, so the fiberwise condition is equivalent to requiring that f invert every morphism in W sent to an isomorphism in S. This is the asserted universal property. □

The constructions of this chapter have now produced the two universal classifiers needed later in the book. The universe Cat classifies small cocartesian fibrations and the universe An classifies small left fibrations. Directed univalence upgrades these classifications to the straightening/unstraightening equivalences, while geometric realization and fiberwise localization illustrate how categorical constructions can be performed uniformly on the classified families.

9 Cofinality

Initial and terminal functors are characterized by orthogonality to left and right fibrations. The corresponding factorization of an arbitrary functor turns cofinality questions into equivalence questions between fibrations. Applied to the universal left fibration, it constructs anima-valued left Kan extensions. Its behavior under base change leads to proper and smooth functors, the relative form of Quillen's theorem A, and the Beck–Chevalley theorems.

We call the two classes *initial* and *terminal*.[1] Every functor factors as an initial functor followed by a left fibration, and dually as a terminal functor followed by a right fibration. This factorization is the main tool throughout the chapter.

9.1 Initial and terminal functors

Let S be a category and consider two categories over S given by functors $u\colon C \to S$ and $p\colon X \to S$. Recall from Definition 1.9.2 the anima $\mathrm{Map}_S(C,X)$ of functors $C \to X$ over S, defined via the pullback square

$$\begin{array}{ccc} \mathrm{Map}_S(C,X) & \longrightarrow & \mathrm{Map}(C,X) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_*} \\ * & \xrightarrow{u} & \mathrm{Map}(C,S). \end{array}$$

Definition 9.1.1. A *covariant equivalence* (resp. a *contravariant equivalence*) over S is a commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & S & \end{array}$$

such that for any left (resp. right) fibration $p\colon X \to S$ the induced functor

$$-\circ f\colon \mathrm{Map}_S(D,X) \to \mathrm{Map}_S(C,X)$$

is an equivalence.

Definition 9.1.2 (Initial/terminal functors)**.** A functor $f\colon C \to D$ is called *initial* (or *limit-cofinal*) if for any category S and any functor $v\colon D \to S$, f is a covariant equivalence over S (with respect to

[1] The words *final*, *cofinal*, *initial*, and *coinitial* occur in several mutually incompatible conventions in the literature. In particular, different authors use *final* and *cofinal* for opposite classes. Our terminology is symmetric, and its meaning is easy to remember from Proposition 9.1.13.

the composite $C \xrightarrow{f} D \xrightarrow{v} S$). Dually, f is called *terminal* (or *colimit-cofinal*) if for any $v\colon D \to S$, f is a contravariant equivalence over S.

Lemma 9.1.3. *Consider a covariant (resp. contravariant) equivalence*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & S & \end{array}$$

over S. Then for every functor $\varphi\colon S \to T$, the functor f is also a covariant (resp. contravariant) equivalence over T.

Proof. Let $X \to T$ be a left (resp. right) fibration. We obtain a commutative square as follows:

$$\begin{array}{ccc} \mathrm{Map}_T(D, X) & \xrightarrow{-\circ f} & \mathrm{Map}_T(C, X) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{Map}_S(D, S \times_T X) & \xrightarrow{-\circ f} & \mathrm{Map}_S(C, S \times_T X). \end{array}$$

We need to show that the top map is an equivalence. But this follows from the fact that the bottom map is an equivalence, since the functor $S \times_T X \to S$ is again a left (resp. right) fibration. □

Proposition 9.1.4. *Let $f\colon C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor f is initial (resp. terminal).

(2) For every left (resp. right) fibration $X \to D$ the induced map

$$-\circ f\colon \mathrm{Map}_D(D, X) \to \mathrm{Map}_D(C, X)$$

is an equivalence.

(3) For every left (resp. right) fibration $p\colon X \to S$ the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(D, X) & \xrightarrow{f^*} & \mathrm{Fun}(C, X) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathrm{Fun}(D, S) & \xrightarrow[f^*]{} & \mathrm{Fun}(C, S) \end{array}$$

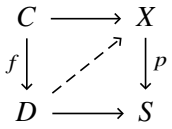
is a pullback square.

Proof. The equivalence between (1) and (2) is immediate from Lemma 9.1.3. For the equivalence between (1) and (3), note that the two vertical maps labeled p_* are still left (resp. right) fibrations by Lemma 4.2.18. Thus the square is a pullback square if and only if it induces equivalences on fibers, by Corollary 6.1.13. By Lemma 6.4.19, the map on fibers is precisely the map

$$-\circ f\colon \mathrm{Map}_S(D, X) \to \mathrm{Map}_S(C, X)$$

appearing in the definition of initial (resp. terminal) functors. □

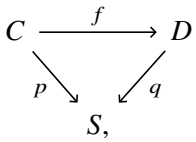
Remark 9.1.5. We may understand part (3) as saying that for every solid commutative square



there exists a unique diagonal filler making both of the triangles commute. One sometimes also says that the initial functors are *left orthogonal* to the left fibrations, and similarly that the terminal functors are left orthogonal to the right fibrations.

We first record the closure and recognition properties that will be used to construct the factorization.

Lemma 9.1.6. *Consider a commutative triangle*



where p and q are left (resp. right) fibrations. Then the following three conditions are equivalent:

(1) The functor f is an equivalence.

(2) The functor f is initial (resp. terminal).

(3) The functor f is a covariant (resp. contravariant) equivalence over S.

Proof. The implications (1) $\implies$ (2) and (2) $\implies$ (3) are immediate. Suppose (3) holds. Under straightening, p, q, and f correspond to functors $P, Q\colon S \to \mathrm{An}$ and a natural transformation $P \to Q$. For every left fibration $X \to S$ classified by $H\colon S \to \mathrm{An}$, the defining equivalence for f identifies with

$$\mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{An})}(Q, H) \to \mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{An})}(P, H).$$

This is an equivalence for every H, so Exercise 2.6.2 shows that $P \to Q$ is an isomorphism in $\mathrm{Fun}(S, \mathrm{An})$. Under straightening, it follows from Corollary 8.3.16 that f is an equivalence. The right-fibration case is dual. □

Lemma 9.1.7. *Let $f\colon C \to D$ be a left (resp. right) fibration. Then f is initial (resp. terminal) if and only if it is an equivalence.*

Proof. This follows from Lemma 9.1.6 by taking $q = \mathrm{id}_D$ and $p = f$. □

Proposition 9.1.8. *(1) Let $u\colon C \to D$ be an initial (resp. terminal) functor. Then for every category E, also the functor $u \times \mathrm{id}_E\colon C \times E \to D \times E$ is initial (resp. terminal).*

(2) Consider two functors $u\colon C \to D$ and $v\colon D \to E$ and assume that u is initial (resp. terminal). Then v is initial (resp. terminal) if and only if vu is initial (resp. terminal).

Proof. For (1), consider a functor $f\colon D \times E \to S$ and a left (resp. right) fibration $X \to S$. We want to show that the induced functor

$$-\circ(u \times \mathrm{id}_E)\colon \mathrm{Map}_S(D \times E, X) \to \mathrm{Map}_S(C \times E, X)$$

is an equivalence. By uncurrying, the functor f corresponds to a functor $\tilde{f}\colon D \to \mathrm{Fun}(E, S)$, and the above functor is equivalent to the functor

$$\mathrm{Map}_{\mathrm{Fun}(E,S)}(D, \mathrm{Fun}(E, X)) \xrightarrow{-\circ u} \mathrm{Map}_{\mathrm{Fun}(E,S)}(C, \mathrm{Fun}(E, X)).$$

Since the functor $\mathrm{Fun}(E, X) \to \mathrm{Fun}(E, S)$ is still a left (resp. right) fibration by Lemma 4.2.18, it follows from the assumption on u that this functor is an equivalence.

For (2), let $X \to E$ be a left (resp. right) fibration, and consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Map}_E(E, X) & \xrightarrow{-\circ v} & \mathrm{Map}_E(D, X) \\ & {\scriptstyle -\circ vu} \searrow \quad \swarrow {\scriptstyle -\circ u} & \\ & \mathrm{Map}_E(C, X). & \end{array}$$

By assumption, the right diagonal map is an equivalence. It follows by 2-out-of-3 that the left diagonal map is an equivalence if and only if the top horizontal map is an equivalence. □

Proposition 9.1.9. *Let $f\colon C \to D$ be a localization, so it factors as $C \to C[W^{-1}] \xrightarrow{\sim} D$ for some collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$. Then f is both initial and terminal.*

Proof. Recall that $\mathrm{Map}^W(C, T)$ denotes the full subanima of $\mathrm{Map}(C, T)$ on the functors that send all morphisms in W to isomorphisms in T. Consider a left (resp. right) fibration $p\colon X \to D$. By Lemma 6.4.18, p is conservative. Hence a functor $F\colon C \to X$ sends the morphisms in W to isomorphisms in X if and only if $pF\colon C \to D$ sends them to isomorphisms in D. It follows that the right square in the following diagram is a pullback:

$$\begin{array}{ccccc} \mathrm{Map}(D, X) & \xrightarrow{\simeq} & \mathrm{Map}^W(C, X) & \hookrightarrow & \mathrm{Map}(C, X) \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathrm{Map}(D, D) & \xrightarrow{\simeq} & \mathrm{Map}^W(C, D) & \hookrightarrow & \mathrm{Map}(C, D). \end{array}$$

The left square is also a pullback square since both horizontal functors are equivalences. By passing to fibers over the identity $\mathrm{id}_D\colon D \to D$, this shows that the induced functor

$$\mathrm{Map}_D(D, X) \to \mathrm{Map}_D(C, X)$$

is an equivalence, which is what we needed to show. □

The endpoint inclusions provide the basic examples from which the adjoint examples will follow.

Example 9.1.10. The inclusion $\{1\} \hookrightarrow [1]$ is terminal. The inclusion $\{0\} \hookrightarrow [1]$ is initial.

Proof. Let us treat the case of $\{0\} \hookrightarrow [1]$, the other case being dual. Consider a left fibration $p\colon X \to S$. By Proposition 9.1.4, it will suffice to show that the commutative square

$$\begin{array}{ccc} \mathrm{Fun}([1], X) & \xrightarrow{p_*} & \mathrm{Fun}([1], S) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ \mathrm{Fun}(\{0\}, X) & \xrightarrow{p_*} & \mathrm{Fun}(\{0\}, S) \end{array}$$

is a pullback square. But since the bottom functor agrees with $p\colon X \to S$, this holds true by Lemma 4.2.2. □

We next show that right adjoints are terminal and left adjoints are initial. We begin with the fully faithful case.

Proposition 9.1.11. *Let $v\colon D \to C$ be a left Bousfield localization with right adjoint section $u\colon C \hookrightarrow D$. Then u is a terminal functor. Dually, a left adjoint section of v is an initial functor.*

Proof. We prove the first case, and leave the second dual case to the reader. Consider a right fibration $p\colon X \to D$. The unit transformation $\eta\colon \mathrm{id} \to uv$ corresponds to a functor $\eta\colon [1]\times D \to D$, allowing us to regard $[1] \times D$ as a category over D. Consider now the following square:

$$\begin{array}{ccc} \mathrm{Map}_D([1] \times D, X) & \longrightarrow & \mathrm{Map}_D([1] \times C, X) \\ \downarrow & & \downarrow \\ \mathrm{Map}_D(\{1\} \times D, X) & \longrightarrow & \mathrm{Map}_D(\{1\} \times C, X). \end{array}$$

Since the inclusion $\{1\} \hookrightarrow [1]$ is terminal by Example 9.1.10, so is $\{1\} \times C \to [1] \times C$ by Proposition 9.1.8, and hence both vertical functors are equivalences. It follows that the square is a pullback square, i.e. we have an equivalence

$$\mathrm{Map}_D([1] \times D, X) \xrightarrow{\sim} \mathrm{Map}_D([1] \times C, X) \times_{\mathrm{Map}_D(\{1\}\times C, X)} \mathrm{Map}_D(\{1\} \times D, X).$$

We will finish the proof by exhibiting the map $- \circ u\colon \mathrm{Map}_D(D, X) \to \mathrm{Map}_D(C, X)$ as a retract of this equivalence, implying that it is itself an equivalence as desired. To this end, we consider the following diagram:

$$\begin{array}{ccccc} \mathrm{Map}_D(D, X) & \xrightarrow{\mathrm{pr}_2^*} & \mathrm{Map}_D([1] \times D, X) & \xrightarrow{i_0^*} & \mathrm{Map}_D(D, X) \\ {\scriptstyle -\circ u}\downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle -\circ u} \\ \mathrm{Map}_D(C, X) & \xrightarrow{(\mathrm{pr}_2^*, v^*)} & \mathrm{Map}_D([1] \times C, X) \times_{\mathrm{Map}_D(\{1\}\times C, X)} \mathrm{Map}_D(\{1\} \times D, X) & \xrightarrow{i_0^* \circ \mathrm{pr}_1} & \mathrm{Map}_D(C, X). \end{array}$$

The fact that all these maps are well-defined and that the two squares commute is a consequence of the following commutative diagram over D:

$$\begin{array}{ccccc} & & C & \xrightarrow{u} & D \\ & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_0} \\ \{1\} \times C & \xrightarrow{i_1} & [1] \times C & \xrightarrow{[1]\times u} & [1] \times D \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \eta} \\ \{1\} \times D & \xrightarrow{v} & C & \xrightarrow{u} & D. \end{array}$$

This finishes the proof. □

Proposition 9.1.12. *Any right adjoint is terminal. Any left adjoint is initial.*

Proof. By symmetry it suffices to do the right adjoint case, so let $f\colon C \to D$ be a right adjoint. As in the proof of Theorem 6.2.1, we may factorize f as a composite

$$C \xhookrightarrow{i} C \vec{\times}_f D \xrightarrow{\pi} D,$$

where i is a fully faithful right adjoint and where π admits a left adjoint section, i.e. is a right Bousfield localization. The functor i is terminal by Proposition 9.1.11. The functor π is a localization by Proposition 4.4.16, and hence also terminal by Proposition 9.1.9. It follows from Proposition 9.1.8 that also their composite f is terminal. □

Proposition 9.1.13. *Let x be an object of a category C. Then x is an initial (resp. terminal) object of C if and only if the functor $x\colon * \to C$ is initial (resp. terminal).*

Proof. If x is initial, then $x\colon * \to C$ is a left adjoint and is therefore initial by Proposition 9.1.12. Conversely, assume that $x\colon * \to C$ is initial. The functor $(x, \mathrm{id}_x)\colon * \to C_{x/}$ is a left adjoint and hence initial. Since its composite with the projection $C_{x/} \to C$ is x, cancellation shows that $C_{x/} \to C$ is initial. This projection is a functor between left fibrations over C, so Lemma 9.1.6 implies that it is an equivalence. Thus x is an initial object. The terminal case is dual. □

Corollary 9.1.14. *Let x be an object of a category C. For any left (resp. right) fibration $p\colon A \to C$ with fiber A_x over x, restriction along $(x, \mathrm{id}_x)\colon * \to C_{x/}$ (resp. $(x, \mathrm{id}_x)\colon * \to C_{/x}$) induces a canonical isomorphism*

$$\mathrm{Map}_C(C_{x/}, A) \xrightarrow{\sim} A_x \qquad \big(\textit{resp. } \mathrm{Map}_C(C_{/x}, A) \xrightarrow{\sim} A_x\big).$$

Proof. The functor $(x, \mathrm{id}_x)\colon * \to C_{x/}$ is initial by Proposition 9.1.13. Since $A \to C$ is a left fibration, precomposition therefore induces an equivalence

$$\mathrm{Map}_C(C_{x/}, A) \xrightarrow{\sim} \mathrm{Map}_C(*, A) = A_x.$$

The right-fibration statement is dual. □

9.2 The cofinality factorization

We now refine the graph factorization of a functor in three stages. The first map is a fully faithful adjoint, the second localizes the fibers, and the third is a left or right fibration. The first two stages combine to give the initial or terminal part of the factorization.

Theorem 9.2.1 (Refined cofinality factorization)**.** *Any functor $f\colon C \to D$ admits a factorization of the form*

$$f\colon C \xrightarrow{i} C' \xrightarrow{f'} E \xrightarrow{p} D,$$

where

- *The functor i is a fully faithful left (resp. right) adjoint.*

- *The functor f' is a localization and remains a localization after base change along an arbitrary functor $D' \to D$.*
- *The functor p is a left (resp. right) fibration.*

Proof. We start by factoring f as

$$C \xrightarrow{i} C \vec{\times}_f D \xrightarrow{\pi} D,$$

where i is a fully faithful left (resp. right) adjoint and π is a cocartesian (resp. cartesian) fibration. We may now factor π as

$$C \vec{\times}_f D \xrightarrow{f'} \Pi_\infty(C \vec{\times}_f D/D) \xrightarrow{\Pi_\infty(\pi/D)} D.$$

We showed in Proposition 8.8.5 that $\Pi_\infty(\pi/D)$ is a left fibration, and we showed in Theorem 8.8.9 that f' is a localization. Since this construction is compatible with base change along $D' \to D$ (part (3) of Proposition 8.8.5) we see that in fact any base change of f' along $D' \to D$ is still a localization. □

Remark 9.2.2 (Smallness)**.** The construction preserves the smallness needed below. If f has small fibers, then the factorization using $\Pi_\infty(-/D)$ has small fibers, without any smallness assumption on the source or target. If C and D are small, all intermediate categories in the displayed factorizations are small.

Corollary 9.2.3. *Any functor $f : C \to D$ admits a factorization of the form*

$$C \xrightarrow{j} X \xrightarrow{p} D,$$

where $j : C \to X$ is an initial (resp. terminal) functor, and $p : X \to D$ is a left (resp. right) fibration.

Proof. In the factorization of Theorem 9.2.1, the first functor is a left (resp. right) adjoint and the second is a localization. Hence both are initial (resp. terminal), so their composite has the same property. □

Proposition 9.2.4. *Consider a commutative triangle*

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S, & \end{array}$$

where q is a left (resp. right) fibration. Then f is initial (resp. terminal) if and only if it is a covariant (resp. contravariant) equivalence over S.

Proof. If f is initial, then it is a covariant equivalence over S. Conversely, suppose that f is a covariant equivalence over S, and factor it as

$$C \xrightarrow{i} X \xrightarrow{r} D,$$

where i is initial and r is a left fibration. The functor i is a covariant equivalence over S, so 2-out-of-3 implies that r is also a covariant equivalence over S. Both $X \to S$ and $D \to S$ are left fibrations. Hence Lemma 9.1.6 shows that r is an equivalence, and therefore $f = ri$ is initial. The right-fibration case is dual. □

9.3 Colimits of animae

The cofinality factorization immediately implies that An admits all small colimits. More generally, it constructs left Kan extensions of anima-valued functors. Given a functor $F\colon C \to \mathrm{An}$ and a functor $u\colon C \to D$, we unstraighten F to a left fibration over C, factor its total category over D into an initial functor followed by a left fibration, and straighten the latter. Initiality gives the required universal property. Taking $D = *$ then identifies the colimit of F with the geometric realization of the total category of its left fibration.

Theorem 9.3.1 (Anima-valued left Kan extensions)**.** *Let $u\colon C \to D$ be a functor which has small fibers.*

(1) Then the pullback functor

$$u^*\colon \mathrm{Fun}(D, \mathrm{An}) \to \mathrm{Fun}(C, \mathrm{An})$$

admits a left adjoint $u_!\colon \mathrm{Fun}(C, \mathrm{An}) \to \mathrm{Fun}(D, \mathrm{An})$.

(2) This left adjoint is given on objects as follows: for a functor $F\colon C \to \mathrm{An}$, consider the pullback

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{F} & \mathrm{An}, \end{array}$$

and apply Corollary 9.2.3 to factor the composite $u \circ p\colon E \to D$ as a composite

$$E \xrightarrow{j} Y \xrightarrow{q} D,$$

where j is initial and q is a left fibration. Since u has small fibers, also q has small fibers, and thus there is a pullback square of the form

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{u_!(F)} & \mathrm{An}. \end{array}$$

Under directed univalence, the functor $(p, j)\colon E \to C \times_D Y$ over C corresponds to a natural transformation $\eta_F\colon F \to u^* u_!(F)$. This is the unit component exhibiting $u_!(F)$ as the value of the left adjoint at F.

Proof. Consider a functor $G\colon D \to \mathrm{An}$. We have to show that the canonical map

$$\mathrm{Hom}_{\mathrm{Fun}(D,\mathrm{An})}(u_!(F), G) \to \mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{An})}(F, u^*(G))$$

is an equivalence. Define the left fibration $r\colon Z \to D$ via the following pullback square:

$$\begin{array}{ccc} Z & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle r}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{G} & \mathrm{An}. \end{array}$$

Directed univalence identifies the displayed map with

$$\mathrm{Map}_D(Y, Z) \to \mathrm{Map}_C(E, C \times_D Z) \simeq \mathrm{Map}_D(E, Z)$$

induced by precomposition with $j\colon E \to Y$. Here we use Theorem 8.3.15 and Lemma 8.3.13. Since j is initial and $r\colon Z \to D$ is a left fibration, this map is an equivalence. □

Corollary 9.3.2. *The category* An *admits all small colimits.*

Proof. For a small category C, apply Theorem 9.3.1 to the functor $p_C\colon C \to *$. □

Suppose that C admits a terminal object ω. Then $p_C\colon C \to *$ is left adjoint to $\omega\colon * \to C$, so Proposition 4.4.4 gives an adjunction

$$\mathrm{ev}_\omega\colon \mathrm{Fun}(C, \mathrm{An}) \rightleftarrows \mathrm{An} : p_C^*.$$

Thus evaluation at a terminal object is left adjoint to the constant-diagram functor.

Notation 9.3.3. For a small category C with $p_C\colon C \to *$, we write

$$\mathrm{colim}_C := (p_C)_!\colon \mathrm{Fun}(C, \mathrm{An}) \to \mathrm{An}.$$

Sometimes this is also denoted by $\varinjlim$.

Remark 9.3.4. Consider a pullback square

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C & \xrightarrow[F]{} & \mathrm{An}. \end{array}$$

The colimit $\mathrm{colim}_C F$ is the geometric realization $\Pi_\infty(X)$ of the total category of the left fibration classified by F. Indeed, it is obtained by factoring $X \to *$ into the initial functor $X \to \Pi_\infty(X)$ followed by the left fibration $\Pi_\infty(X) \to *$. Equivalently, there is a pullback square

$$\begin{array}{ccc} \Pi_\infty(X) & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow[\mathrm{colim}_C F]{} & \mathrm{An}. \end{array}$$

9.4 Proper and smooth functors

The factorization from the preceding section becomes useful under base change only when its initial or terminal part is preserved. Proper and smooth functors record precisely these two stability properties: proper functors preserve terminal functors under pullback, while smooth functors preserve initial functors under pullback. This will be the input for the two Beck–Chevalley theorems in the final section.

Definition 9.4.1 (Proper/smooth functors, Grothendieck)**.** A functor $p\colon E \to S$ is called *proper* (resp. *smooth*) if for any pullback diagram

$$\begin{array}{ccccc} E'' & \xrightarrow{u'} & E' & \xrightarrow{u} & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ S'' & \xrightarrow{v'} & S' & \xrightarrow{v} & S \end{array}$$

in which v' is terminal (resp. initial), also u' is terminal (resp. initial).

Remark 9.4.2 (Slice predicates)**.** The definition belongs to a more general pattern. A predicate on categories determines a corresponding notion of initiality by requiring the relevant relative slices to satisfy that predicate, and hence a corresponding notion of smoothness by requiring stability under base change. For example, a functor $u\colon C \to D$ is a left adjoint precisely when every relative slice $C_{/x}$ admits a terminal object. The resulting notion of smoothness is that of a locally cocartesian fibration. In the next section, weak contractibility of the relative slices will instead characterize initial functors.

Remark 9.4.3 (Relation with right Kan extensions)**.** Later we will prove that for a category E with small limits and any functor $u\colon C \to D$ with C and D small, the restriction functor

$$u^*\colon \mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$$

admits a right adjoint $u_*\colon \mathrm{Fun}(C, E) \to \mathrm{Fun}(D, E)$. Then u is proper if and only if, for every E with small limits and every pullback square

$$\begin{array}{ccc} C' & \xrightarrow{v} & C \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ D' & \xrightarrow{w} & D \end{array}$$

the canonical map $w^*u_* \to u'_*v^*$ is a natural isomorphism of functors $\mathrm{Fun}(C, E) \to \mathrm{Fun}(D', E)$, and in fact the same property holds for any base change of u.

Lemma 9.4.4. *The collection of proper (resp. smooth) functors is closed under composition and base change.*

Proof. Closure under base change is immediate from the definition. For composition, let $p\colon X \to B$ and $q\colon B \to C$ be proper functors. For functors $C'' \to C' \to C$ with $C'' \to C'$ terminal, consider

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ B'' & \longrightarrow & B' & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow q \\ C'' & \longrightarrow & C' & \longrightarrow & C. \end{array}$$

Since q is proper, also $B'' \to B'$ is terminal, but then by properness of p also $X'' \to X'$ is terminal. This finishes the proof. □

Lemma 9.4.5 (Base change of fully faithful right adjoints)**.** *Let $p' \colon X' \to S'$ be a functor such that, for every functor $h \colon [1] \times T \to S'$, the base change of the endpoint inclusion $\{1\} \times T \hookrightarrow [1] \times T$ along the induced functor $X' \times_{S'} ([1] \times T) \to [1] \times T$ is terminal. Consider a pullback square*

$$\begin{array}{ccc} X'' & \xrightarrow{u'} & X' \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p'} \\ S'' & \xrightarrow{v'} & S' \end{array}$$

in which v' is a fully faithful right adjoint. Then u' is terminal.

Proof. Let $w \colon S' \to S''$ be a left adjoint of v', with counit isomorphism $\varepsilon \colon wv' \xrightarrow{\sim} \mathrm{id}_{S''}$ and unit $\eta \colon \mathrm{id}_{S'} \to v'w$. Form the pullback

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\eta}} & X' \\ {\scriptstyle \tilde{p}}\downarrow & \lrcorner & \downarrow{\scriptstyle p'} \\ [1] \times S' & \xrightarrow{\eta} & S'. \end{array}$$

Consider the diagram

$$\begin{array}{ccccc} \{1\} \times S'' & \xrightarrow{v'} & \{1\} \times S' \simeq S' & & \\ \downarrow & & \downarrow & \searrow^{w} & \\ [1] \times S'' & \xrightarrow{\mathrm{id}_{[1]} \times v'} & [1] \times S' & & S'' \\ & \searrow^{\mathrm{pr}_2} & & \searrow^{\eta} & \downarrow{\scriptstyle v'} \\ & & S'' & \xrightarrow{v'} & S'. \end{array}$$

Pulling back this diagram along p' gives

$$\begin{array}{ccccc} \{1\} \times X'' & \longrightarrow & S' \times_{S''} X'' & & \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\ [1] \times X'' & \longrightarrow & \tilde{X} & \searrow^{\tilde{\eta}} & \\ & & & & X'. \end{array}$$

The map i is terminal, and j is terminal by the endpoint hypothesis since it is a pullback of $\{1\} \times S' \hookrightarrow [1] \times S'$:

$$\begin{array}{ccccc} S' \times_{S''} X'' & \longrightarrow & \tilde{X} & \longrightarrow & X' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p'} \\ \{1\} \times S' & \hookrightarrow & [1] \times S' & \xrightarrow{\eta} & S'. \end{array}$$

To show that $u' \colon X'' \to X'$ is terminal, let $q \colon Z \to X'$ be a right fibration. The preceding commutative square over X' induces a commutative square

$$\begin{array}{ccc} \mathrm{Map}_{X'}(\tilde{X}, Z) & \longrightarrow & \mathrm{Map}_{X'}([1] \times X'', Z) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathrm{Map}_{X'}(S' \times_{S''} X'', Z) & \longrightarrow & \mathrm{Map}_{X'}(\{1\} \times X'', Z), \end{array}$$

where the two vertical maps are equivalences since i and j are terminal. Hence there is an equivalence

$$\mathrm{Map}_{X'}(\tilde{X}, Z) \xrightarrow{\sim} \mathrm{Map}_{X'}(S' \times_{S''} X'', Z) \times_{\mathrm{Map}_{X'}(X'', Z)} \mathrm{Map}_{X'}([1] \times X'', Z).$$

We now describe the retract explicitly. Denote the fiber product on the right by P_Z, and denote the displayed equivalence by

$$e_Z \colon \mathrm{Map}_{X'}(\tilde{X}, Z) \longrightarrow P_Z.$$

Let $r \colon \tilde{X} \to X'$ be the structural functor. Restricting the defining pullback of $\tilde{X}$ to $\{0\} \times S'$, where η is the identity, gives a section $s \colon X' \to \tilde{X}$ with $rs = \mathrm{id}_{X'}$.

Define

$$\pi_Z \colon P_Z \longrightarrow \mathrm{Map}_{X'}(X'', Z)$$

by projecting to $\mathrm{Map}_{X'}([1] \times X'', Z)$ and restricting along $\{0\} \times X'' \hookrightarrow [1] \times X''$. Conversely, the counit isomorphism identifies the composites of the projections below with the displayed structural maps to X'. Precomposition with the projections

$$S' \times_{S''} X'' \longrightarrow X'' \qquad \text{and} \qquad [1] \times X'' \longrightarrow X''$$

defines a map

$$\sigma_Z \colon \mathrm{Map}_{X'}(X'', Z) \longrightarrow P_Z.$$

The two resulting maps agree after restriction to $\{1\} \times X''$, so σ_Z lands in the fiber product and $\pi_Z \sigma_Z = \mathrm{id}$. The agreement at the 1-endpoint uses the counit isomorphism $v'\varepsilon \colon v'wv' \xrightarrow{\sim} v'$ together with the triangle identity $v'\varepsilon \circ \eta v' = \mathrm{id}_{v'}$. Thus, after pulling back to X'', the 1-endpoint of the path classified by η is identified with the structural map of X'' over X'.

By restriction to the two factors defining P_Z, these maps form a commutative diagram in the arrow category of animae:

$$\begin{array}{ccccc}
\mathrm{Map}_{X'}(X', Z) & \xrightarrow{r^*} & \mathrm{Map}_{X'}(\tilde{X}, Z) & \xrightarrow{s^*} & \mathrm{Map}_{X'}(X', Z) \\
{\scriptstyle (u')^*}\downarrow & & \downarrow{\scriptstyle e_Z} & & \downarrow{\scriptstyle (u')^*} \\
\mathrm{Map}_{X'}(X'', Z) & \xrightarrow{\sigma_Z} & P_Z & \xrightarrow{\pi_Z} & \mathrm{Map}_{X'}(X'', Z).
\end{array}$$

The horizontal composites are identities since $rs = \mathrm{id}_{X'}$ and $\pi_Z \sigma_Z = \mathrm{id}$. Thus $(u')^*$ is a retract of the equivalence e_Z, and is therefore an equivalence. This proves that u' is terminal. □

Theorem 9.4.6. *Let $p \colon E \to S$ be a functor, and assume that p satisfies the property that for any pullback diagram of the form*

$$\begin{array}{ccccc}
E'' & \xrightarrow{u'} & E' & \xrightarrow{u} & E \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
S'' = \{\varepsilon\} \times S'' & \xhookrightarrow{v'} & [1] \times S'' & \xrightarrow{v} & S
\end{array}$$

for $\varepsilon = 1$ (resp. $\varepsilon = 0$) the base change u' is terminal (resp. initial). Then p is proper (resp. smooth).

Proof. We prove the statement for proper functors; the smooth case is dual. The strategy is to factor an arbitrary terminal functor into a fully faithful right adjoint, a localization, and a right fibration.

The localization remains terminal after base change, the right-fibration factor is forced to be an equivalence, and the endpoint hypothesis handles the fully faithful factor.

Consider an arbitrary pullback diagram

$$\begin{array}{ccccc} E'' & \xrightarrow{u'} & E' & \xrightarrow{u} & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ S'' & \xrightarrow{v'} & S' & \xrightarrow{v} & S \end{array}$$

in which v' is terminal. We need to show that also u' is terminal. To this end, we factor v' as

$$S'' = S''_0 \xrightarrow{j} S''_1 \xrightarrow{\lambda} S''_2 = \Pi_\infty(S''_1/S') \xrightarrow{q} S',$$

where j is a fully faithful right adjoint, λ is a localization and remains so after pullback, and q is a right fibration. Consider now the following commutative triangle:

$$\begin{array}{ccc} S'' & \longrightarrow & S''_2 \\ & {\scriptstyle v'} \searrow \quad \swarrow {\scriptstyle q} & \\ & S'. & \end{array}$$

The functor j is terminal because it is a right adjoint, and λ is terminal because it is a localization. Thus their composite $S'' \to S''_2$ is terminal. Since $v' = q\lambda j$ is terminal, cancellation shows that q is terminal. As q is also a right fibration, it is an equivalence by Lemma 9.1.7. Now consider the following pullback diagram:

$$\begin{array}{ccccccccc} E'' & \longrightarrow & E''_1 & \xrightarrow{\lambda'} & E''_2 & \xrightarrow{\simeq} & E' & \xrightarrow{u} & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ S'' & \xrightarrow{j} & S''_1 & \xrightarrow{\lambda} & S''_2 & \xrightarrow[\simeq]{q} & S' & \xrightarrow{v} & S. \end{array}$$

Since λ' is a localization, it is terminal by Proposition 9.1.9. The functor $E'' \to E''_1$ is the pullback of the fully faithful right adjoint j. The relevant base change of p satisfies the endpoint hypothesis because that hypothesis is stable under base change. Hence Lemma 9.4.5 shows that $E'' \to E''_1$ is terminal. The remaining map is an equivalence, so the composite $u' \colon E'' \to E'$ is terminal. □

Corollary 9.4.7. *Any cocartesian fibration is proper. Any cartesian fibration is smooth.*

Proof. Let $p \colon E \to S$ be a cocartesian fibration and consider a pullback square of the following form:

$$\begin{array}{ccccc} E'' & \xrightarrow{\quad u' \quad} & E' & \xrightarrow{u} & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ S'' = \{\varepsilon\} \times S'' & \xhookrightarrow{v'} & [1] \times S'' & \xrightarrow{v} & S \end{array}$$

Let $p' \colon E' \to [1] \times S''$ denote the indicated base change of p. Since p' is a cocartesian fibration and v' is a right adjoint, Proposition 6.2.4 shows that u' is a right adjoint, and hence terminal by Proposition 9.1.12. The claim now follows from Theorem 9.4.6. □

Remark 9.4.8. The preceding corollary remains valid for locally cocartesian and locally cartesian fibrations.

Corollary 9.4.9. *Any left fibration is proper. Any right fibration is smooth.* □

Corollary 9.4.10. *Any projection of the form $C \times D \to C$ is both smooth and proper.* □

9.5 Weak homotopy equivalences and Quillen's theorem A

Weak homotopy equivalences are the absolute shadow of covariant and contravariant equivalences. Over the terminal category, left and right fibrations are both precisely animae, so the two notions agree and are detected by geometric realization. The main result of this section is a relative form of Quillen's theorem A: a functor over S is a covariant equivalence precisely when it is a weak homotopy equivalence on every relative slice.

Definition 9.5.1. A functor $f : C \to B$ is called a *weak homotopy equivalence* if it is a covariant equivalence over $*$. A category C is called *weakly contractible* if the functor $p_C : C \to *$ is a weak homotopy equivalence.

Lemma 9.5.2. *For a functor $f : C \to B$, the following are equivalent:*

(1) The functor f is a weak homotopy equivalence.

(2) For every anima X, the induced map

$$\mathrm{Map}(B, X) \to \mathrm{Map}(C, X)$$

is an equivalence.

(3) The induced map $\Pi_\infty(f) : \Pi_\infty(C) \to \Pi_\infty(B)$ is an equivalence.

Proof. The equivalence between (1) and (2) is immediate, since by Proposition 4.2.4 a functor $X \to *$ is a left fibration if and only if X is an anima. By Lemma 3.4.3, condition (2) is equivalent to requiring that f induce an equivalence

$$\mathrm{Map}(\Pi_\infty(B), X) \xrightarrow{\sim} \mathrm{Map}(\Pi_\infty(C), X)$$

for every anima X. This is equivalent to (3) by Proposition 1.4.11. □

Corollary 9.5.3. *A category C is weakly contractible if and only if its geometric realization $\Pi_\infty(C)$ is contractible.* □

Example 9.5.4. Any initial (or terminal) functor is a weak homotopy equivalence.

The comparison between fibers and relative slices is where the proper and smooth functors from the preceding section enter. Smoothness preserves the initial map into a left-fibration replacement, while properness identifies the fiber with the corresponding slice up to weak homotopy equivalence.

Lemma 9.5.5. *Consider a commutative triangle of the form*

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle p} & \\ & S & \end{array}$$

where i is initial and where p is a left fibration. Then, for any object s of S, the following two functors are weak homotopy equivalences:

$$A_{/s} \to X_{/s} \leftarrow X_s.$$

In fact, $A_{/s} \to X_{/s}$ is initial and $X_s \to X_{/s}$ is terminal.

Proof. Consider the following pullback diagram:

$$\begin{array}{ccccc} X_s & \longrightarrow & X_{/s} & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ * & \xrightarrow{(s,\mathrm{id}_s)} & S_{/s} & \longrightarrow & S. \end{array}$$

Since p is a left fibration, it is proper, so the terminal functor $* \to S_{/s}$ pulls back to a terminal functor $X_s \to X_{/s}$. Consider now the following commutative diagram:

$$\begin{array}{ccccc} A_{/s} & \longrightarrow & X_{/s} & \longrightarrow & S_{/s} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{i} & X & \xrightarrow{p} & S. \end{array}$$

Since the functor $S_{/s} \to S$ is a right fibration, it is in particular smooth, so the initial functor $i\colon A \to X$ pulls back to an initial functor $A_{/s} \to X_{/s}$. □

Proposition 9.5.6 (Relative Quillen theorem A)**.** *Consider a commutative triangle*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & S. & \end{array}$$

Then f is a covariant equivalence over S if and only if for every object x of S the induced map

$$C_{/x} \to D_{/x}$$

is a weak homotopy equivalence.

Proof. The first step is to construct a commutative diagram of the form

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow & & \downarrow \\ C' & \xrightarrow{f'} & D' \\ & {\scriptstyle u'}\searrow \quad \swarrow{\scriptstyle v'} & \\ & S, & \end{array}$$

where the two vertical maps are initial and the two diagonal maps u' and v' are left fibrations. The vertical maps are covariant equivalences over S, so 2-out-of-3 shows that f is a covariant equivalence if and only if f' is. Since u' and v' are left fibrations, Lemma 9.1.6 identifies the latter condition

with f' being an equivalence. By Lemma 6.1.12, this is equivalent to requiring that $C'_x \to D'_x$ be an equivalence for every x.

Finally, Lemma 9.5.5 gives weak homotopy equivalences

$$C_{/x} \to C'_{/x} \leftarrow C'_x \qquad \text{and} \qquad D_{/x} \to D'_{/x} \leftarrow D'_x.$$

Thus $C_{/x} \to D_{/x}$ is a weak homotopy equivalence if and only if $C'_x \to D'_x$ is a weak homotopy equivalence. Since C'_x and D'_x are animae, the latter is equivalent to $C'_x \to D'_x$ being an equivalence. This proves the claim. □

Corollary 9.5.7. *A functor $u\colon C \to D$ is initial if and only if, for every object d of D, the relative slice $C_{/d}$ is weakly contractible. Dually, u is terminal if and only if every relative coslice $C_{d/}$ is weakly contractible.*

Proof. The functor u is initial if and only if the diagram

$$\begin{array}{ccc} C & \xrightarrow{u} & D \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_D} & \\ & D & \end{array}$$

is a covariant equivalence. The statement now follows as the relative slice of the identity functor id_D at any object of D has a terminal object and is thus weakly contractible. □

Corollary 9.5.8 (Quillen's theorem A)**.** *Let $f\colon C \to D$ be a functor such that the relative slice $C_{/d}$ is weakly contractible for any object d of D. Then f is a weak homotopy equivalence.*

Proof. The preceding corollary shows that f is initial, and every initial functor is a weak homotopy equivalence by Example 9.5.4. □

9.6 Proper and smooth base change

This section concerns base change for anima-valued left Kan extensions. Given a pullback square, there are two complementary hypotheses. Smoothness of the horizontal functor makes the initial-left-fibration factorization stable under pullback. Properness of the vertical functor instead gives a pointwise formula on strict fibers. We first identify the Beck–Chevalley map under straightening and prove smooth base change. We then characterize terminal functors by colimit cofinality, derive the pointwise formula, and prove proper base change.

9.6.1 The Beck–Chevalley map and smooth base change

Let $f\colon A \to B$ be a functor between small categories. Recall from Theorem 9.3.1 that the restriction functor

$$f^*\colon \mathrm{Fun}(B, \mathrm{An}) \to \mathrm{Fun}(A, \mathrm{An})$$

admits a left adjoint $f_!$.

Construction 9.6.1 (The Beck–Chevalley map under straightening)**.** Consider a commutative square of small categories

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{v} & B \end{array}$$

and a functor $F\colon A \to \mathrm{An}$. This data determines the Beck–Chevalley map

$$\mathrm{BC}_!\colon f'_! u^*(F) \to v^* f_!(F)$$

in $\mathrm{Fun}(B', \mathrm{An})$. The Beck–Chevalley map is defined as the composite

$$f'_! u^*(F) \xrightarrow{\eta} f'_! u^* f^* f_!(F) = f'_! f'^* v^* f_!(F) \xrightarrow{\varepsilon} v^* f_!(F),$$

where $\eta\colon \mathrm{id}_{\mathrm{Fun}(A,\mathrm{An})} \to f^* f_!$ is the unit of $f_! \dashv f^*$ and $\varepsilon\colon f'_! f'^* \to \mathrm{id}_{\mathrm{Fun}(B',\mathrm{An})}$ is the counit of $f'_! \dashv f'^*$. Equivalently, $\mathrm{BC}_!$ is adjoint to

$$u^*(F) \xrightarrow{\eta} u^* f^* f_! F = f'^* v^* f_!(F).$$

We next describe this map under straightening. Consider the pullback square

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{F} & \mathrm{An}, \end{array}$$

where the right vertical map is the universal left fibration with small fibers. Choose a factorization of $f \circ p\colon X \to B$ as

$$X \xrightarrow{j} Y \xrightarrow{q} B,$$

where j is initial and q is a left fibration. In particular, q is given via a pullback square

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & \lrcorner & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{f_!(F)} & \mathrm{An} \end{array}$$

for some functor $f_!(F)$. The commutative square on the left exhibits the unit $F \to f^* f_! F$. Pulling back along v and u, we obtain the commutative diagram

$$\begin{array}{ccc} A' \times_A X & \xrightarrow{f' \times_f j} & B' \times_B Y \\ {\scriptstyle u^* p}\downarrow & & \downarrow{\scriptstyle v^* q} \\ A' & \xrightarrow{f'} & B'. \end{array}$$

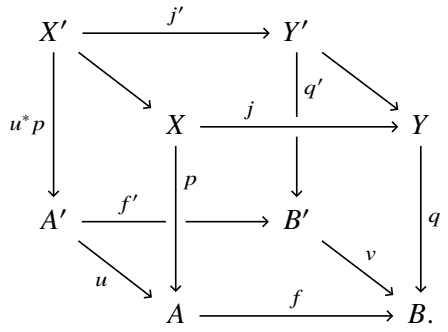
Write $X' := A' \times_A X$. We may factor $f' \times_f j$ as

$$X' \xrightarrow{j'} Y' \xrightarrow{r} B' \times_B Y,$$

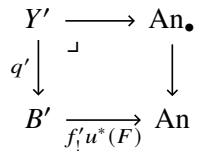
where j' is initial and r is a left fibration. Then the composite

$$q' := v^* q \circ r \colon Y' \to B'$$

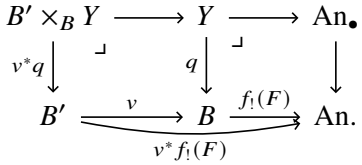
is again a left fibration. These factorizations fit into the commutative diagram



We have pullback squares



and



The map $r \colon Y' \to B' \times_B Y$ over B' corresponds under directed univalence to a natural transformation

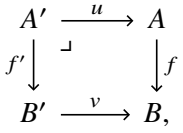
$$f'_! u^*(F) \to v^* f_!(F).$$

Its adjunct along $f'_! \dashv f'^*$ is represented by the composite $r j' \colon X' \to B' \times_B Y$. This composite is induced by $j \colon X \to Y$, which represents the unit $F \to f^* f_!(F)$. Its adjunct is therefore the map

$$u^*(F) \to u^* f^* f_!(F) = f'^* v^* f_!(F)$$

that defines the Beck–Chevalley map. Hence r corresponds to $\mathrm{BC}_! \colon f'_! u^*(F) \to v^* f_!(F)$.

Theorem 9.6.2 (Smooth base change)**.** *Consider a pullback square of small categories*



where v is smooth. Then for any functor $F \colon A \to \mathrm{An}$, the Beck–Chevalley map

$$\mathrm{BC}_! \colon f'_! u^*(F) \to v^* f_!(F)$$

is an isomorphism.

Proof. Use the notation of the preceding construction. The composite

$$X' = A' \times_A X \xrightarrow{j'} Y' \xrightarrow{r} B' \times_B Y$$

is the pullback of $j\colon X \to Y$ along the projection $B' \times_B Y \to Y$. This projection is a base change of the smooth functor v, and is therefore smooth by Lemma 9.4.4. Since j is initial, its pullback rj' is initial. The functor j' is initial by construction, so Proposition 9.1.8 implies that r is initial. It is also a left fibration. Hence Lemma 9.1.7 shows that r is an equivalence. Under the preceding directed-univalence identification, r classifies $\mathrm{BC}_!$, so Corollary 8.3.16 shows that the Beck–Chevalley map is an isomorphism. □

9.6.2 Cofinality and pointwise Kan extensions

The Beck–Chevalley map for the terminal functor to $*$ is the canonical comparison between two colimits. The following result therefore identifies our terminal functors with the usual colimit-cofinal functors.

Proposition 9.6.3. *Let A and B be small categories. A functor $i\colon A \to B$ is terminal if and only if, for any functor $F\colon B \to \mathrm{An}$, the Beck–Chevalley map*

$$\mathrm{BC}_!\colon \operatorname{colim}_A i^*F \to \operatorname{colim}_B F$$

from Construction 9.6.1 is an isomorphism.

Proof. We first rewrite the Beck–Chevalley map as a morphism of left fibrations. Consider the pullback diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle q'}\downarrow & \lrcorner & \downarrow{\scriptstyle q} & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ A & \xrightarrow{i} & B & \xrightarrow{F} & \mathrm{An}. \end{array}$$

Since q is a left fibration, it is proper. Hence, if i is terminal, its pullback j is terminal. Writing $p_B\colon B \to *$ for the terminal functor, the Beck–Chevalley map is

$$\operatorname{colim}_A i^*F \xrightarrow{\sim} (p_B)_! i_! i^* F \xrightarrow{(p_B)_!(\mathrm{counit})} (p_B)_! F = \operatorname{colim}_B F.$$

Under the description of Remark 9.3.4, this is the map

$$\Pi_\infty(j)\colon \Pi_\infty(X) \to \Pi_\infty(Y).$$

It is an equivalence because every terminal functor is a weak homotopy equivalence.

Conversely, suppose that the Beck–Chevalley map is an equivalence for every F. For an object b of B, take F to classify the left fibration $B_{b/} \to B$. Its pullback along i is $A_{b/} \to A$, and the corresponding Beck–Chevalley map is

$$\Pi_\infty(A_{b/}) \to \Pi_\infty(B_{b/}) \xrightarrow{\sim} *.$$

Hence every relative coslice $A_{b/}$ is weakly contractible. The dual statement in Corollary 9.5.7 now shows that i is terminal. □

Example 9.6.4 (The colimit of a morphism)**.** Let $F\colon [1] \to \mathrm{An}$ be a morphism of animae. Since the inclusion $\{1\} \hookrightarrow [1]$ is terminal, Proposition 9.6.3 gives a canonical isomorphism

$$F(1) \xrightarrow{\sim} \mathrm{colim}_{[1]} F.$$

For a proper functor, the general slice formula for a left Kan extension simplifies to a colimit over the strict fiber. This is the pointwise input for proper base change.

Lemma 9.6.5. *Let $f\colon A \to B$ be a proper functor between small categories. Consider an absolute object b of B and let A_b denote the fiber of f over b. Then for any functor $F\colon A \to \mathrm{An}$, there is a canonical isomorphism*

$$\mathrm{colim}_{A_b} i^* F \xrightarrow{\sim} b^* f_!(F) = f_!(F)(b)$$

of animae, where $i\colon A_b \to A$ denotes the inclusion of the fiber.

Proof. We will again translate this to a statement about left fibrations. Consider the left fibration $p\colon X \to A$ associated to F:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ A & \xrightarrow{F} & \mathrm{An}. \end{array}$$

Consider now the following pullback diagram:

$$\begin{array}{ccccc} X_b & \xrightarrow{k_b} & X_{/b} & \xrightarrow{u_b} & X \\ \downarrow & \lrcorner & \downarrow & & \downarrow{\scriptstyle p} \\ A_b & \xrightarrow{i_b} & A_{/b} & \xrightarrow{v_b} & A \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f_{/b}} & & \downarrow{\scriptstyle f} \\ * & \xrightarrow{(b,\mathrm{id}_b)} & B_{/b} & \xrightarrow{w_b} & B. \end{array}$$

(with the composite $b\colon * \to B$ drawn as an arc from $*$ to B)

Since w_b is a right fibration, it is in particular smooth, hence by smooth base change (Theorem 9.6.2) we see that the canonical map

$$\mathrm{BC}_!\colon (f_{/b})_! v_b^*(F) \xrightarrow{\sim} w_b^* f_!(F)$$

is an isomorphism. Passing to colimits over $B_{/b}$, we obtain an equivalence

$$\mathrm{colim}_{A_{/b}} v_b^*(F) \simeq \mathrm{colim}_{B_{/b}} (f_{/b})_! v_b^*(F) \simeq \mathrm{colim}_{B_{/b}} w_b^* f_!(F).$$

Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{colim}_{A_b} i^* F & \longrightarrow & b^* f_!(F) \\ \downarrow & & \downarrow \\ \mathrm{colim}_{A_{/b}} v_b^* F & \longrightarrow & \mathrm{colim}_{B_{/b}} w_b^* f_!(F). \end{array}$$

The square commutes by functoriality of Beck–Chevalley maps under pasting: the bottom horizontal map is the mate obtained from the smooth base-change isomorphism above, and its restrictions along

the two terminal inclusions are precisely the two vertical maps and the canonical top comparison map. Since the inclusion $(b, \mathrm{id}_b) \colon * \to B_{/b}$ is terminal, the right vertical map is an equivalence by Proposition 9.6.3. But also i_b is terminal, since it is a base change of $(b, \mathrm{id}_b) \colon * \to B_{/b}$ along the proper functor $f_{/b}$, hence also the left vertical map is an equivalence. By 2-out-of-3 it follows that also the top map is an equivalence, finishing the proof. □

9.6.3 Proper base change

The proper base-change theorem says that left Kan extension along a proper functor commutes with arbitrary pullback. It follows from the pointwise formula of Lemma 9.6.5.

Theorem 9.6.6 (Proper base change)**.** *Consider a pullback square of small categories*

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{v} & B, \end{array}$$

where f is proper. Then for every functor $F \colon A \to \mathrm{An}$, the Beck–Chevalley map

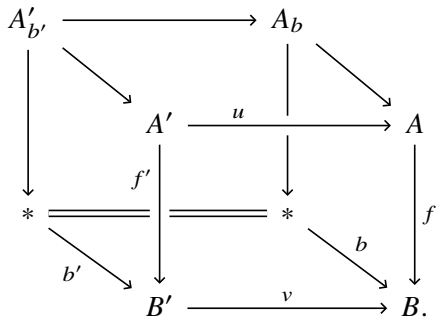
$$f'_! u^*(F) \to v^* f_!(F)$$

is a natural isomorphism.

Proof. By the pointwise criterion for isomorphisms of functors, it suffices to evaluate the Beck–Chevalley map at an object b' of B'. Write $b := v(b')$. We must show that

$$f'_! u^*(F)(b') \to v^* f_!(F)(b') = f_!(F)(v(b'))$$

is an isomorphism in An. The pullback square induces a commutative diagram



in which all non-horizontal faces are pullbacks. In particular, the induced functor $A'_{b'} \to A_b$ is an equivalence. Since proper functors are stable under base change, f' is proper. The pointwise formula of Lemma 9.6.5 and functoriality of Beck–Chevalley maps under pasting therefore give a commutative square

$$\begin{array}{ccc} \mathrm{colim}_{A'_{b'}}\, i'^* u^* F & \longrightarrow & \mathrm{colim}_{A_b}\, i^* F \\ \downarrow & & \downarrow \\ f'_! u^*(F)(b') & \longrightarrow & f_!(F)(b) \end{array}$$

whose bottom horizontal map is the Beck–Chevalley map evaluated at b'. The two vertical maps are equivalences, and the top map is an equivalence because $A'_{b'} \to A_b$ is. Hence the bottom map is an equivalence. □

9.6.4 Fiberwise criteria for later use

We finish with two consequences that recast the preceding base-change theorem as criteria for initiality. They will be used in Chapter 12 to analyze relative twisted-arrow constructions and the Yoneda embedding.

Corollary 9.6.7 (Proper base change in disguise)**.** *Let $p\colon A \to B$ be a proper functor and consider a commutative square of the form*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi} & B \end{array}$$

where f is initial and both φ and ψ are left fibrations. Then, for any functor $v\colon B' \to B$, the induced functor

$$g\colon B' \times_B X \to B' \times_B Y$$

is again initial.

Proof. Choose a universe in which all the displayed categories and functors are small. The left fibration φ corresponds to a functor $F\colon A \to \mathrm{An}$:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle \varphi}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ A & \xrightarrow{F} & \mathrm{An}. \end{array}$$

Since $p\varphi = \psi f$ and f is initial, the pair $X \xrightarrow{f} Y \xrightarrow{\psi} B$ is an initial-left-fibration factorization of $p\varphi$. Consequently, ψ is classified by $p_!(F)\colon B \to \mathrm{An}$. Let $p'\colon A' \to B'$ be defined by the pullback square

$$\begin{array}{ccc} A' & \xrightarrow{u} & A \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{v} & B, \end{array}$$

Factor g as

$$B' \times_B X \xrightarrow{j'} Z \xrightarrow{r} B' \times_B Y,$$

where j' is initial and r is a left fibration over B'. Under directed univalence, r classifies the Beck–Chevalley map

$$p'_! u^*(F) \longrightarrow v^* p_!(F).$$

This map is an isomorphism by Theorem 9.6.6, so r is an equivalence. Hence $g = r j'$ is initial. □

The next lemma promotes initiality on the fibers over one factor of a product base to global initiality. This is the form needed for the relative twisted-arrow construction.

Lemma 9.6.8. *Consider a commutative square of the form*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A \times C & \xrightarrow{\mathrm{id}_A \times g} & A \times B, \end{array}$$

where p and q are left fibrations. Assume that f is fiberwise initial over A, meaning that for every object a in A, f induces an initial functor

$$f_a \colon X_a \to Y_a.$$

Then f is initial.

Proof. Since q is a left fibration, it suffices to check that f is a covariant equivalence over $A \times B$, see Proposition 9.2.4. For any object b of B, we may consider the following commutative square

$$\begin{array}{ccc} X_{/b} & \xrightarrow{f_{/b}} & Y_{/b} \\ {\scriptstyle p_{/b}}\downarrow & & \downarrow{\scriptstyle q_{/b}} \\ A \times C_{/b} & \xrightarrow{\mathrm{id}_A \times g_{/b}} & A \times B_{/b} \end{array}$$

obtained from the above commutative square by pulling back along $B_{/b} \to B$. For any object a of A, the commutative square

$$\begin{array}{ccc} X_a & \xrightarrow{f_a} & Y_a \\ {\scriptstyle p_a}\downarrow & & \downarrow{\scriptstyle q_a} \\ C & \xrightarrow{g} & B \end{array}$$

induces a commutative square

$$\begin{array}{ccc} (X_a)_{/b} & \xrightarrow{(f_a)_{/b}} & (Y_a)_{/b} \\ {\scriptstyle (p_a)_{/b}}\downarrow & & \downarrow{\scriptstyle (q_a)_{/b}} \\ C_{/b} & \xrightarrow{g_{/b}} & B_{/b} \end{array}$$

by pulling back along $B_{/b} \to B$. The top map agrees with the map obtained from

$$\begin{array}{ccc} X_{/b} & \xrightarrow{f_{/b}} & Y_{/b} \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

by passing to fibers over a. In particular, $(f_{/b})_a = (f_a)_{/b}$ is a pullback of the initial functor f_a along a base change of the smooth functor $B_{/b} \to B$, and hence is initial.

The functor $X_{/b} \to A$ induced by p is proper: it is the composite of a left fibration with the proper projection $A \times C_{/b} \to A$. Pulling back the terminal inclusion of the terminal object in $A_{/a}$ therefore gives a terminal functor

$$(X_{/b})_a = (X_a)_{/b} \to (X_{/b})_{/a} = X_{/(a,b)}\,,$$

hence a weak homotopy equivalence. Similarly, the functor $Y_{/b} \to A$ induced by q yields a weak homotopy equivalence

$$(Y_{/b})_a = (Y_a)_{/b} \to (Y_{/b})_{/a} = Y_{/(a,b)}\,.$$

This means that we have a commutative square of the form

$$\begin{array}{ccc} (X_a)_{/b} & \xrightarrow{(f_a)_{/b}} & (Y_a)_{/b} \\ \downarrow & & \downarrow \\ X_{/(a,b)} & \xrightarrow{f_{/(a,b)}} & Y_{/(a,b)} \end{array}$$

The top horizontal map and both vertical maps are weak homotopy equivalences. Hence 2-out-of-3 shows that $f_{/(a,b)}$ is a weak homotopy equivalence. Since this holds for every (a, b), Proposition 9.5.6 implies that f is a covariant equivalence over $A \times B$, as required. □

The chapter has established the cofinality factorization and used it in three ways. It constructs anima-valued left Kan extensions, recognizes initial functors by weakly contractible relative slices, and turns stability of initial and terminal functors under base change into the smooth and proper Beck–Chevalley theorems.

10 Natural numbers and simplices

We construct the natural numbers from the free category on one endomorphism, and then the simplices $[n]$ by repeatedly adjoining a terminal cone point. The resulting simplex category has sifted opposite. Consequently, geometric realization preserves finite products whenever products in the target preserve geometric realizations separately in each variable.

10.1 Sets and propositions

We begin with the elementary properties of propositions and sets needed for the construction of the natural numbers.

Definition 10.1.1. An anima P is called a *proposition* if the map $p_P \colon P \to *$ is an embedding. Equivalently, the diagonal $\Delta \colon P \to P \times P$ is an equivalence.

If a proposition P admits an absolute object $x \colon * \to P$, then x is a section of the embedding $p_P \colon P \to *$. Hence p_P is an equivalence by Lemma 1.6.24. In particular, if an absolute object of P exists, it is unique.

Definition 10.1.2. A *set* is an anima X such that the diagonal $\Delta \colon X \to X \times X$ is an embedding, in the sense of Definition 1.6.19. We denote by $\mathrm{Set} \subseteq \mathrm{An}$ the full subcategory spanned by the sets.

Thus a set is an anima in which equality of two objects is a property: if an isomorphism between them exists, it is unique.

Lemma 10.1.3 (Identity types of sets)**.** *An anima X is a set if and only if for every two objects x and y of X the mapping anima $X(x, y)$ is a proposition.*

Proof. Since X satisfies the groupoid condition, we have $X \xrightarrow{\sim} \mathrm{Fun}([1], X)$, and consequently the mapping anima $X(x, y)$ fits in a pullback square as follows:

$$\begin{array}{ccc} X(x,y) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ \{(x,y)\} & \xrightarrow{(x,y)} & X \times X. \end{array}$$

Since embeddings are closed under pullback, this proves the forward implication. Conversely, apply the hypothesis in context $X \times X$ to the objects $x = \mathrm{pr}_1$ and $y = \mathrm{pr}_2$. Their mapping anima is the diagonal of X, so the diagonal is an embedding. □

Notation 10.1.4. Let X be a set and let x and y be objects of X. We write $x = y$ for the mapping anima $X(x, y)$. Thus $x = y$ is a proposition, and to give $p : x = y$ is to identify x and y.

We use the following standard terminology for sets.

Terminology 10.1.5. Let X and Y be sets.

(i) An *element*[1] of X, written $x \in X$, is an object x of X.

(ii) A *function* $f\colon X \to Y$ is a functor from X to Y. Given an element x of X we write $f(x)$ for the resulting element in Y (given by composing x and f).

(iii) We say two functions $f, g\colon X \to Y$ are *equal*, written $f = g$, if there exists a (necessarily unique) identification between them.

(iv) We say that f is a *bijection* if it is an equivalence. This is the case if and only if there exists a function $g\colon Y \to X$ satisfying $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Observation 10.1.6. When X and Y are sets, the category $\mathrm{Fun}(X, Y)$ is again a set. Its elements are functions from X to Y.

Example 10.1.7. Note that the terminal category $*$ is a set: the diagonal $\Delta\colon * \to * \times *$ is even an equivalence. We refer to $*$ as the *one-point set*. Similarly the initial category $\emptyset$ is a set, which we will refer to as the *empty set*.

Example 10.1.8. We will prove in Corollary 10.2.18 that the anima of natural numbers $\mathbb{N}$ is a set. We will also see in Lemma 10.2.14 that the successor function identifies $\mathbb{N}$ with the positive natural numbers. Together with the element 0, this gives a bijection $\{0\} \sqcup \mathbb{N} \xrightarrow{\sim} \mathbb{N}$.

Lemma 10.1.9 (Characterizations of propositions)**.** *Let X be an anima. The following are logically equivalent:*

(1) X is a proposition.

(2) For any two $x, y : X$, the type $X(x, y)$ is contractible; in formulas $\prod_{x,y:X} \mathtt{is_Contr}\, X(x, y)$.

(3) For any two $x, y : X$ we have an identification $p : X(x, y)$; in formulas $\prod_{x,y:X} X(x, y)$.

(4) If we have an inhabitant $x : X$, then X is contractible; in formulas $X \to \mathtt{is_Contr}\, X$.

Proof. The equivalence between (1) and (2) follows from the characterization of embeddings between animae as fully faithful functors. The implication from (2) to (3) is obtained by forgetting the contraction.

Assume (3), and write $f : \prod_{x,y:X} X(x, y)$ for the given family. If $c : X$, then $f(c, -)$ contracts X onto c. This proves (4).

Finally, assume (4) and let $x, y : X$. Since x inhabits X, the anima X is contractible. It follows that $X(x, y)$ is contractible, which proves (2). □

[1] In [LR03] this is called a *generalized element*.

Lemma 10.1.10 (Propositions are closed under exponentiation)**.** *Let Z be an anima and let P be a proposition. Then the mapping anima* $\mathrm{Map}(Z, P)$ *is a proposition.*

Proof. Under the canonical equivalence

$$\mathrm{Map}(Z, P \times P) \simeq \mathrm{Map}(Z, P) \times \mathrm{Map}(Z, P),$$

the map induced by the diagonal $\Delta_P \colon P \to P \times P$ identifies with the diagonal of $\mathrm{Map}(Z, P)$. Since P is a proposition, Δ_P is an equivalence, and hence so is the diagonal of $\mathrm{Map}(Z, P)$. □

Example 10.1.11 (Negations are propositions)**.** For any anima Z, its *negation* is the anima

$$\neg Z \coloneqq \mathrm{Map}(Z, \emptyset).$$

Since $\emptyset$ is a proposition, Lemma 10.1.10 shows that $\neg Z$ is a proposition. For $x, y : X$, we abbreviate $x \neq y \coloneqq \neg(x = y)$. This does not require X to be a set.

To recognize a set, it suffices to consider automorphisms.

Lemma 10.1.12 (Characterizations of sets)**.** *Let X be an anima. The following are logically equivalent:*

(1) X is a set.

(2) For any $x, y : X$ and any two identities $p, q : x = y$, we have $p = q$.

(3) For every $x : X$ and every $p : X(x, x)$, we have $p = \mathrm{id}_x$.

Proof. The equivalence between (1) and (2) follows from Lemmas 10.1.3 and 10.1.9. The implication from (2) to (3) is the special case $x = y$ and $q = \mathrm{id}_x$.

Conversely, assume (3) and let $p, q : x = y$. The composite $q^{-1} \circ p$ is an automorphism of x, hence is equal to id_x. Composing this equality with q gives $p = q$, which proves (2). □

10.1.1 Decidable equality

Definition 10.1.13. An anima X is said to have *decidable equality* if for each $x, y : X$ we either have $x = y$ or $x \neq y$. In formulas

$$\texttt{hasDecidableEquality}(X) \coloneqq \prod_{x,y:X} (x = y) \sqcup (x \neq y).$$

Hedberg's theorem [Hed98] says that an anima with decidable equality is a set. Its proof uses the following notion.

Definition 10.1.14. We say that a map $f \colon X \to Y$ of animae is *wildly constant* if we have a section of $X \times_Y X \to X \times X$, i.e., an inhabitant of

$$\texttt{wconstant}(f) \coloneqq \prod_{x,y:X} f(x) = f(y).$$

Lemma 10.1.15 (Hedberg)**.** *Let X be an anima and assume that for each $x, y : X$ we have a wildly constant map $x = y \to x = y$. Then X is a set.*

Proof. Assume that for each x, y we have a wildly constant map $f_{x,y} : x = y \to x = y$. First we note that for each $p : x = y$ we have $f_{x,y}(p) = p \circ f_{x,x}(\mathrm{id}_x)$; indeed, path induction on p reduces this assertion to the reflexivity identification at x. Then for two arbitrary $p, q : x = y$ we obtain an identification

$$p = f_{x,y}(p) \circ f_{x,x}(\mathrm{id}_x)^{-1} = f_{x,y}(q) \circ f_{x,x}(\mathrm{id}_x)^{-1} = q$$

where we use the equality $f_{x,y}(p) = f_{x,y}(q)$ coming from the fact that $f_{x,y}$ is wildly constant.

Since p, q were arbitrary, it follows from Lemma 10.1.12 that X is a set. □

Corollary 10.1.16 (Decidable equality implies being a set)**.** *Let X be an anima with decidable equality. Then X is a set.*

Proof. By Hedberg's lemma it suffices, for each $x, y : X$, to construct a wildly constant map $f : x = y \to x = y$. Decidable equality gives two cases.

- If $p : x = y$, let f be the constant map with value p.
- If $e : x \neq y$, let f be the composite $x = y \xrightarrow{e} \emptyset \to x = y$.

In the first case the map is constant by construction. In the second its domain is empty, so it is again wildly constant. □

Exercise 10.1.17. Assuming the law of excluded middle (i.e. that there is a term $\mathrm{LEM}_P : P \sqcup \neg P$ for each proposition P), prove the converse of Corollary 10.1.16, i.e., that every set has decidable equality.

10.2 The natural numbers

We construct the anima $\mathbb{N}$ of natural numbers from the free category on one endomorphism, and show that it satisfies dependent recursion and is a set.

Definition 10.2.1 (Natural numbers)**.** We define the category $\mathrm{B}\,\mathbb{N}$ as the following pushout in Cat:

$$\begin{array}{ccc} [1]^{\simeq} & \longrightarrow & [1] \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \mathrm{one}} \\ * & \xrightarrow[\mathrm{base}]{} & \mathrm{B}\,\mathbb{N}\,. \end{array}$$

We define the anima $\mathbb{N}$ as

$$\mathbb{N} := \mathrm{Hom}_{\mathrm{B}\,\mathbb{N}}(\mathrm{base}, \mathrm{base}),$$

the anima of endomorphisms of the canonical basepoint of $\mathrm{B}\,\mathbb{N}$. The identity endomorphism defines an object $0 : \mathbb{N}$, while the generating endomorphism one defines an object $1 : \mathbb{N}$.

Construction 10.2.2 (Arithmetic operations)**.** Composition in $\mathrm{B}\,\mathbb{N}$ defines the *addition function*

$$+\colon\ \mathbb{N}\times\mathbb{N}\to\mathbb{N}, \qquad n+m := m\circ n.$$

Thus addition is unital and associative, with canonical identifications

$$(n+m)+k = n+(m+k), \qquad 0+n = n = n+0.$$

The *successor function* is right addition by the generator:

$$\mathrm{succ}\colon\ \mathbb{N}\to\mathbb{N}, \qquad \mathrm{succ}(n) = n+1.$$

Equivalently, it is the composite

$$\mathbb{N}\xrightarrow{\sim}\mathbb{N}\times\{1\}\to\mathbb{N}\times\mathbb{N}\xrightarrow{+}\mathbb{N}\,.$$

Lemma 10.2.3 (The free endomorphism)**.** *For any category C, there is an equivalence*

$$\mathrm{Map}(\mathrm{B}\,\mathbb{N}, C)\simeq\{f : \mathrm{Map}([1], C)\mid s(f) = t(f)\}\simeq\sum_{x:C^{\simeq}} C(x,x).$$

Proof. This is an immediate consequence of the universal property of the pushout. □

Construction 10.2.4. Let C be a category and let $\varphi\colon \mathrm{B}\,\mathbb{N}\to C$ classify an endomorphism $f\colon x\to x$ in C, so that $x = \varphi(\mathrm{base})$ and $f = \varphi(\mathrm{one})$. By passing to hom animae, φ induces a morphism

$$\varphi\colon\ \mathbb{N} = \mathrm{B}\,\mathbb{N}(\mathrm{base},\mathrm{base})\to C(x,x)$$

which we will denote by $n\mapsto f^n$. Since φ preserves identities and composition, we have:

- The endomorphism $f^0\colon x\to x$ is the identity of x in C;
- For $n, m\in\mathbb{N}$, the endomorphism f^{n+m} agrees with $f^m\circ f^n$.

We may therefore think of f^n as the n-fold iterate of f.

Example 10.2.5. As a special case of Construction 10.2.4, take $C = \mathrm{Cat}$ to be the category of small categories, and let $x = P$ be a small category equipped with an endomorphism $f\colon P\to P$. Then the previous construction provides a map

$$\mathbb{N}\to\mathrm{Cat}(P,P)\simeq\mathrm{Map}(P,P), \qquad n\mapsto f^n.$$

Uncurrying this functor, we obtain a functor $\mathbb{N}\times P\to P$, $(n,x)\mapsto f^n(x)$.

Remark 10.2.6. Construction 10.2.4 is functorial in C: if $F\colon C\to D$ is a functor, then the endomorphism $F(f^n)\colon F(x)\to F(x)$ agrees with $F(f)^n\colon F(x)\to F(x)$ for every natural number $n\in\mathbb{N}$.

10.2.1 Dependent recursion for natural numbers

Given categories P_n and functors $f_n\colon P_n \to P_{n+1}$ for $n : \mathbb{N}$, we will show that a recursive section $x_{n+1} = f_n(x_n)$ is uniquely determined by its value at zero. This holds at the level of categories: evaluation at zero is an equivalence between the category of recursive sections and P_0.

Definition 10.2.7 (Recursive sections)**.** Let $p\colon P \to \mathbb{N}$ be a functor and let $f\colon P \to P$ be an endofunctor over the successor function. For a functor $u\colon \mathbb{N} \to \mathbb{N}$, define the category

$$\mathrm{Sect}_u(p) \coloneqq \mathrm{Fun}(\mathbb{N}, P) \times_{\mathrm{Fun}(\mathbb{N},\mathbb{N})} \{u\},$$

where the functor from $\mathrm{Fun}(\mathbb{N}, P)$ is postcomposition with p. Thus $\mathrm{Sect}_{\mathrm{id}_{\mathbb{N}}}(p)$ is the category of sections of p, while $\mathrm{Sect}_{\mathrm{succ}}(p)$ is the category of functors $g\colon \mathbb{N} \to P$ over succ.

The category $\mathrm{Rec}(p, f)$ of *recursive sections* is defined by the pullback square

$$\begin{array}{ccc} \mathrm{Rec}(p,f) & \longrightarrow & \mathrm{Iso}\big(\mathrm{Sect}_{\mathrm{succ}}(p)\big) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathrm{Sect}_{\mathrm{id}_{\mathbb{N}}}(p) & \xrightarrow[s\mapsto(s\circ\mathrm{succ},\, f\circ s)]{} & \mathrm{Sect}_{\mathrm{succ}}(p) \times \mathrm{Sect}_{\mathrm{succ}}(p). \end{array}$$

An object is a section $s\colon \mathbb{N} \to P$ of p together with an identification

$$\vartheta\colon s \circ \mathrm{succ} \xrightarrow{\sim} f \circ s$$

over $\mathbb{N}$. A morphism $(s, \vartheta) \to (t, \psi)$ is a natural transformation $\alpha\colon s \to t$ over $\mathbb{N}$ equipped with a commutative square

$$\begin{array}{ccc} s \circ \mathrm{succ} & \xrightarrow{\vartheta} & f \circ s \\ {\scriptstyle \alpha\circ\mathrm{succ}}\downarrow & & \downarrow{\scriptstyle f\circ\alpha} \\ t \circ \mathrm{succ} & \xrightarrow[\psi]{} & f \circ t \end{array}$$

in $\mathrm{Sect}_{\mathrm{succ}}(p)$. Evaluation at zero gives a functor $\mathrm{ev}_0\colon \mathrm{Rec}(p, f) \to P_0$.

Remark 10.2.8. Let $\tilde{P}\colon \mathrm{B}\,\mathbb{N} \to \mathrm{Cat}$ be the functor classifying the endomorphism $f\colon P^{\simeq} \to P^{\simeq}$. Similarly, the endomorphism $\mathrm{succ}\colon \mathbb{N} \to \mathbb{N}$ is classified by the representable functor $\mathrm{B}\,\mathbb{N}(\mathrm{base}, -)\colon \mathrm{B}\,\mathbb{N} \to \mathrm{An}$. The functor p intertwining f and succ exactly amounts to a map $\tilde{p}\colon \tilde{P} \to \mathrm{B}\,\mathbb{N}(\mathrm{base}, -)$ in $\mathrm{Fun}(\mathrm{B}\,\mathbb{N}, \mathrm{An})$. With this translation, the anima $\mathrm{Rec}(p, f)^{\simeq}$ of recursive sections can be identified with the anima of (usual) sections of $p\colon \tilde{P} \to \mathrm{B}\,\mathbb{N}(\mathrm{base}, -)$ in $\mathrm{Fun}(\mathrm{B}\,\mathbb{N}, \mathrm{An})$. By Yoneda, this anima is equivalent to the anima $P_0^{\simeq}$ by evaluation.

We now upgrade the equivalence of Remark 10.2.8 to an equivalence of categories:

Theorem 10.2.9 (Dependent recursion for natural numbers)**.** *Let $p\colon P \to \mathbb{N}$ and $f\colon P \to P$ be functors equipped with a commutative square*

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{N} & \xrightarrow{\mathrm{succ}} & \mathbb{N}. \end{array}$$

Evaluation at zero defines an equivalence

$$\mathrm{ev}_0 \colon \mathrm{Rec}(p, f) \xrightarrow{\sim} P_0.$$

Consequently, for every object $x_0 : P_0$, the category of recursive sections equipped with an identification $s(0) = x_0$ is contractible.

Proof. Iterate the given square as an endomorphism of p in $\mathrm{Fun}([1], \mathrm{Cat})$, using Construction 10.2.4. Uncurrying gives a commutative square

$$\begin{array}{ccc} \mathbb{N} \times P & \xrightarrow{(n,x) \mapsto f^n(x)} & P \\ {\scriptstyle \mathrm{id}_{\mathbb{N}} \times p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{N} \times \mathbb{N} & \xrightarrow[(n,m) \mapsto \mathrm{succ}^n(m)]{} & \mathbb{N}, \end{array}$$

The covariant representable $\mathrm{B}\,\mathbb{N}(\mathrm{base}, -) \colon \mathrm{B}\,\mathbb{N} \to \mathrm{An}$ sends the generating endomorphism to succ. Its action on $\mathbb{N}$ is $(n, m) \mapsto n \circ m = m + n$, so $\mathrm{succ}^n(m) = m + n$. Restriction to $\mathbb{N} \times P_0$ therefore gives a functor to P over $\mathbb{N}$. Currying, together with the comparison $f^{n+1} = f \circ f^n$, defines

$$R \colon P_0 \longrightarrow \mathrm{Rec}(p, f), \qquad x \longmapsto (n \mapsto f^n(x)).$$

The comparison lies over $\mathbb{N}$ because we iterated the square, including its specified commutativity. Thus it gives an isomorphism in $\mathrm{Sect}_{\mathrm{succ}}(p)$, as required by the defining pullback. Since $f^0 = \mathrm{id}_P$, we have $\mathrm{ev}_0 R = \mathrm{id}_{P_0}$.

To construct $R\mathrm{ev}_0 = \mathrm{id}$, put $C = \mathrm{Rec}(p, f)$ and let $U \colon \mathbb{N} \times C \to P$ be the universal section. Write $K(n, c) = f^n(U(0, c))$ for the section underlying $R\mathrm{ev}_0$. Their recursive isomorphisms will be denoted by ϑ_U and ϑ_K. Form the pullback

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Iso}_{\mathbb{N}}(P) \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbb{N} \times C & \xrightarrow[(K,U)]{} & P \times_{\mathbb{N}} P, \end{array}$$

where $\mathrm{Iso}_{\mathbb{N}}(P) = \mathbb{N} \times_{\mathrm{Iso}(\mathbb{N})} \mathrm{Iso}(P)$ is the relative isomorphism category from Corollary 5.3.15. Thus E parametrizes isomorphisms $u \colon K(n, c) \xrightarrow{\sim} U(n, c)$ in P_n. Composition of relative isomorphisms defines an endofunctor h of E over $\mathrm{succ} \times \mathrm{id}_C$ by

$$h(u) = \vartheta_{U,n}^{-1} \circ f(u) \circ \vartheta_{K,n}.$$

The identity $f^0 = \mathrm{id}_P$ gives a functor $j \colon C \to E$ over $c \mapsto (0, c)$.

Now iterate the square $qh = (\mathrm{succ} \times \mathrm{id}_C)q$ in $\mathrm{Fun}([1], \mathrm{Cat})$ and evaluate its iterates on j. Since $\mathrm{succ}^n(0) = n$, this gives a section

$$L \colon \mathbb{N} \times C \to E, \qquad L(n, c) = h^n(j(c)),$$

together with an isomorphism $L \circ (\mathrm{succ} \times \mathrm{id}_C) = h \circ L$ over $\mathrm{succ} \times \mathrm{id}_C$. Projection to $\mathrm{Iso}_{\mathbb{N}}(P)$ gives an isomorphism $\alpha \colon K \xrightarrow{\sim} U$ over $\mathbb{N}$, and the iteration comparison says

$$\alpha_{n+1} = \vartheta_{U,n}^{-1} \circ f(\alpha_n) \circ \vartheta_{K,n}.$$

Equivalently, $\vartheta_{U,n} \circ \alpha_{n+1} = f(\alpha_n) \circ \vartheta_{K,n}$, which is the square required of a morphism of recursive sections. All these constructions are functorial over C, so α defines $R\mathrm{ev}_0 = \mathrm{id}_C$. Hence R is inverse to ev_0. The last assertion follows by taking the fiber at x_0. □

Remark 10.2.10. Applied to the family $\mathbb{N} \times C \to \mathbb{N}$ associated with an endomorphism $g\colon C \to C$, the theorem says that $(\mathbb{N}, 0, \mathrm{succ})$ is the free pointed endomorphism: for every $x : C$, the category of functors $u\colon \mathbb{N} \to C$ equipped with identifications $u(0) = x$ and $u \circ \mathrm{succ} = g \circ u$ is contractible.

Observation 10.2.11 (Dependent induction)**.** Let $Q(n)$ be an anima for every $n : \mathbb{N}$, and let $q\colon Q \to \mathbb{N}$ be the corresponding family. A family of maps $i_n\colon Q(n) \to Q(n+1)$ determines a commutative diagram

$$\begin{array}{ccccc} Q & \xrightarrow{i} & \mathrm{succ}^* Q & \xrightarrow{\mathrm{can}} & Q \\ & \searrow^{q} & {\scriptstyle \mathrm{succ}^* q}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ & & \mathbb{N} & \xrightarrow{\mathrm{succ}} & \mathbb{N} \end{array}$$

where i is given termwise by i_n. Equivalently, this corresponds to a commutative square

$$\begin{array}{ccc} Q & \xrightarrow{j} & Q \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{N} & \xrightarrow{\mathrm{succ}} & \mathbb{N}, \end{array}$$

where $j = \mathrm{can} \circ i$. Thus the lower square is an endomorphism of $q\colon Q \to \mathbb{N}$, regarded as an object of $\mathrm{Fun}([1], \mathrm{Cat})$.

Given $x_0 : Q(0)$, the universal property of $\mathrm{B}\,\mathbb{N}$ produces, for every $n \in \mathbb{N}$, a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{j^n} & Q \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{N} & \xrightarrow{\mathrm{succ}^n} & \mathbb{N}, \end{array}$$

hence a map $i^{(n)}\colon Q \to (\mathrm{succ}^n)^* Q$, and therefore maps $i^{(n)}_m\colon Q(m) \to Q(m+n)$. We may then define

$$x_n := (i^{(n)})_0(x_0) : Q(n).$$

Since statements are represented by animae and proofs by their objects, this construction explains how Theorem 10.2.9 recovers the usual induction principle.

10.2.2 Decidability of the natural numbers

We prove that $\mathbb{N}$ has decidable equality, so that Hedberg's theorem applies. First we show that zero is not a successor and that successor is an embedding.

Lemma 10.2.12. *For all $n : \mathbb{N}$ we have $0 \neq \mathrm{succ}\, n$.*

Proof. By recursion, we can construct a function $f\colon \mathbb{N} \to \mathrm{An}$ with $f(0) := *$ and $f(\mathrm{succ}\, n) := \emptyset$. Thus for each $n : \mathbb{N}$ we get the desired map $e\colon (0 = \mathrm{succ}\, n) \to \emptyset$ defined as $e(p) := f(p)(*)$, where $f(p)\colon * \to \emptyset$ is the map induced by the equality $p : 0 = \mathrm{succ}\, n$. □

Construction 10.2.13 (Positive natural numbers)**.** By Example 10.1.11, the negation $n \neq 0$ is a proposition for every $n : \mathbb{N}$. We define the anima of *positive natural numbers* by

$$\mathbb{N}_{\geq 1} := \{n : \mathbb{N} \mid n \neq 0\} := \sum_{n:\mathbb{N}} (n \neq 0).$$

Thus an object of $\mathbb{N}_{\geq 1}$ is a pair (n, e) consisting of a natural number n and a term $e : n \neq 0$. We write

$$j :\equiv \mathrm{pr}_1 \colon \mathbb{N}_{\geq 1} \hookrightarrow \mathbb{N}$$

for the first projection. Its fiber over n is the proposition $n \neq 0$, so j is an embedding.

Lemma 10.2.14. *The successor function induces an equivalence*

$$s \colon \mathbb{N} \xrightarrow{\sim} \mathbb{N}_{\geq 1}, \qquad n \longmapsto n + 1.$$

Proof. By Lemma 10.2.12, every successor is nonzero. Hence succ lifts through j to a function

$$s \colon \mathbb{N} \longrightarrow \mathbb{N}_{\geq 1}\,.$$

Dependent induction produces, for every $n : \mathbb{N}$, a function

$$(n \neq 0) \longrightarrow \sum_{m:\mathbb{N}} (\mathrm{succ}(m) = n).$$

For $n = 0$, a term of the source contradicts $\mathrm{id}_0 : 0 = 0$. For $n = \mathrm{succ}(m)$, send every term of the source to the pair $(m, \mathrm{id}_{\mathrm{succ}(m)})$. Passing to total animae and projecting to the first component defines a function

$$p \colon \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N}$$

together with identifications

$$\mathrm{succ}(p(x)) = j(x) \qquad (x : \mathbb{N}_{\geq 1}).$$

The successor clause of the construction gives $ps = \mathrm{id}_{\mathbb{N}}$. The displayed identifications give $jsp = j$, and since j is an embedding, they lift uniquely to an identification $sp = \mathrm{id}_{\mathbb{N}_{\geq 1}}$. Thus p is inverse to s. □

Construction 10.2.15 (Predecessor function)**.** We define the *predecessor function*

$$\mathrm{pred} \colon \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N}$$

to be the inverse of the successor equivalence from Lemma 10.2.14. Thus

$$\mathrm{pred}(s(n)) = n \qquad \text{and} \qquad \mathrm{succ}(\mathrm{pred}(x)) = j(x)$$

for $n : \mathbb{N}$ and $x : \mathbb{N}_{\geq 1}$.

Lemma 10.2.16 (Successor is an embedding)**.** *The successor function* $\mathrm{succ} \colon \mathbb{N} \to \mathbb{N}$ *is an embedding. Equivalently, for all* $n, m : \mathbb{N}$*, the induced function*

$$(n = m) \longrightarrow (\mathrm{succ}\, n = \mathrm{succ}\, m)$$

is an equivalence.

Proof. The successor function factors as

$$\mathbb{N} \xrightarrow{s} \mathbb{N}_{\geq 1} \xhookrightarrow{j} \mathbb{N}\,.$$

The first map is an equivalence by Lemma 10.2.14, and the second is an embedding by Construction 10.2.13. Hence their composite is an embedding. The description in terms of identity animae is the fully faithful characterization of embeddings between animae. □

Proposition 10.2.17. *The anima $\mathbb{N}$ of natural numbers has decidable equality.*

Proof. We need to show that for all $n, m : \mathbb{N}$ we have $n = m$ or $n \neq m$. By double induction on n and m it suffices to observe the following:

- We have $\mathrm{id}_0 : 0 = 0$.
- We have $0 \neq \mathrm{succ}\, m$ and $\mathrm{succ}\, n \neq 0$ by Lemma 10.2.12.
- If we assume that $n = m$ then we also have $\mathrm{succ}\, n = \mathrm{succ}\, m$, and similarly $n \neq m$ implies $\mathrm{succ}\, n \neq \mathrm{succ}\, m$; both by Lemma 10.2.16. □

Corollary 10.2.18. *The anima $\mathbb{N}$ is a set.*

Proof. This follows from Proposition 10.2.17 and Corollary 10.1.16. □

Proposition 10.2.19. *Denote by $E : \mathbb{N} \times \mathbb{N} \to \mathrm{Prop}$ the* observational equality *of natural numbers, defined by induction via*

$$E(0,0) \coloneqq *, \quad E(0, n+1) \coloneqq \emptyset, \quad E(n+1, 0) \coloneqq \emptyset \quad E(n+1, m+1) \coloneqq E(n, m). \tag{10.1}$$

Then for each $n, m \in N$ we have

$$(n = m) = E(n, m). \tag{10.2}$$

Proof. Each $E(n, m)$ is a proposition by definition, and each $(n = m)$ is a proposition by Corollary 10.2.18. Moreover, the two propositions can be shown to be logically equivalent by induction. The base case is

$$!: (0 = 0) \leftrightarrow * : \mathrm{id}_0 \tag{10.3}$$

and the successor steps follow by using Lemma 10.2.12 and Lemma 10.2.16. □

Proposition 10.2.20 (Zero-positive decomposition)**.** *The function induced by* 0 *and j is a bijection*

$$\{0\} \sqcup \mathbb{N}_{\geq 1} \xrightarrow{\sim} \mathbb{N}\,.$$

Proof. Let

$$q : \{0\} \sqcup \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N}$$

be induced by the element 0 and the embedding j. By universality of coproducts, its fiber over $n : \mathbb{N}$ is equivalent to

$$(0 = n) \sqcup (n \neq 0).$$

Both summands are propositions, and they are disjoint: an identification $0 = n$ and a term $n \neq 0$ together give a term of the empty anima. Consequently their coproduct is a proposition. It is inhabited by decidable equality for $\mathbb{N}$, using inversion to identify $n = 0$ with $0 = n$, and is therefore contractible. Since q is a functor between animae, it is a left fibration. It is thus an equivalence by Proposition 6.1.7. □

10.3 Posets

Before constructing the simplices, we study *posets*: categories whose mapping animae are propositions. A functor into a poset is determined by its values on objects together with the order relations forced by the arrows in its source. This functorial characterization will be used both here and in the following chapters.

10.3.1 Posets and joins

Definition 10.3.1. A category C is called a *partially ordered set*, or a *poset*, if $C(x, y)$ is a proposition for any objects $x, y : C^{\simeq}$. For a fixed universe Cat, we denote by Poset the full subcategory spanned by the posets in Cat.

Notation 10.3.2. When dealing with sets or posets, we use the following notation.

- We may write $x \in S$ instead of $x : S$ for an object of a set S.
- If P is a poset, we write $p \leq q$ for the proposition $P(p, q)$.

Lemma 10.3.3. *For any category C, the canonical functor $C \to C \star [0]$ is a full embedding.*

Proof. We have a pullback square of the form

$$\begin{array}{ccc} C & \longrightarrow & C \star [0] \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & [1] \end{array}$$

in which the lower horizontal map is a full embedding (since 0 is an initial object in $[1]$). Since full embeddings are stable under pullbacks, this proves the lemma. □

Lemma 10.3.4. *For any category C, the cone point $0 \colon [0] \hookrightarrow C \star [0]$ is a terminal object of $C \star [0]$.*

Proof. Let x be an object of C, regarded as an object of $C \star [0]$. The mapping anima $(C \star [0])(x, 0)$ is the anima of sections, with endpoints x and the cone point, of the canonical projection

$$C \star [0] \to [0] \star [0] = [1] .$$

By the universal property of the join as a dependent product, restriction to $[1]^{\simeq} = \{0\} \sqcup \{1\}$ is an equivalence on section categories. Its fiber at the prescribed endpoints is therefore contractible. Hence $(C \star [0])(x, 0)$ is contractible. The inclusion $[0] \hookrightarrow C \star [0]$ is fully faithful by the same pullback argument as in Lemma 10.3.3, so the endomorphism anima of the cone point is also contractible. Thus the cone point is terminal. □

Example 10.3.5. For each $n : \mathbb{N}$, the category $[n]$ is a poset. This is proven by induction using the fact that, for any poset P, the join $P \star [0]$ is a poset. Indeed we have $P^{\simeq} \sqcup [0] = (P \star [0])^{\simeq}$ by Proposition 3.6.9 and the canonical functor $P \hookrightarrow P \star [0]$ is a full embedding. Hence the mapping anima between two old objects is a proposition. For an old object x, the mapping anima from x to the cone point is contractible, while the mapping anima from the cone point to x is empty. Finally, the endomorphism anima of the cone point is contractible. Thus every mapping anima in $P \star [0]$ is a proposition.

10.3.2 Functorial characterizations of posets

We characterize posets using arrow, slice, and functor categories. The following lemma will be used to describe functors into a poset.

Lemma 10.3.6 (Upward-closed families)**.** *Let Q and A be categories and $\Gamma \subset Q^{\simeq} \times A^{\simeq}$ be a subanima such that, for each $q : Q^{\simeq}$ and every arrow $a \to a'$ in A, the proposition $\Gamma(q, a)$ implies $\Gamma(q, a')$. Then the projection*

$$R \coloneqq \langle \Gamma \subseteq Q^{\simeq} \times A^{\simeq} \rangle \to Q \times A \to A$$

is a cocartesian fibration.

Proof. Since $Q \times A \to A$ is tautologically a cocartesian fibration and $R \subseteq Q \times A$ is a full subcategory, it suffices to check that the cocartesian lift $(q, a) \to (q, a')$ of an arrow $a \to a'$ remains in R. This is exactly the closure property imposed on Γ. □

Lemma 10.3.7 (Characterizations of posets)**.** *Let P be a category. The following assertions are logically equivalent.*

(1) *P is a poset, i.e., for each $p, q : P^{\simeq}$, the mapping anima $P(p, q)$ is a proposition.*

(2) *The map*
$$\mathrm{Map}([1], P) \to P^{\simeq} \times P^{\simeq} \tag{10.4}$$
is an embedding of animae.

(3) *The functor*
$$\mathrm{Fun}([1], P) \to P \times P \tag{10.5}$$
is fully faithful.

(4) *For every category A, the induced map*
$$\mathrm{Fun}(A, P) \to \mathrm{Fun}(A^{\simeq}, P) \tag{10.6}$$
is fully faithful.

(5) *For every category A, the induced map* (10.6) *is an embedding.*

(6) *For any object $x : P^{\simeq}$, the canonical projection*
$$P_{/x} \to P \tag{10.7}$$
is fully faithful.

(7) *For any object $x : P^{\simeq}$, the canonical projection* (10.7) *is an embedding.*

(8) *P^{op} is a poset.*

Moreover, if P is a poset, then the essential image of the map (10.6) *consists of those collections $(p_a)_{a:A^{\simeq}}$ for which $p_a \leq p_b$ whenever we have an arrow $a \to b$ in A.*

Proof. Reversing all morphisms proves the equivalence between (1) and (8). The equivalence between (1) and (2) is precisely the fiberwise characterization of embeddings.

We next prove that (1) implies (3). Assume that P is a poset and fix objects $f = (f_0 \to f_1)$ and $g = (g_0 \to g_1)$ of $\mathrm{Fun}([1], P)$. We have to show that the map

$$\mathrm{Fun}([1], P)(f, g) \to P(f_0, g_0) \times P(f_1, g_1)$$

of animae is an equivalence. But the fiber over some pair (α_0, α_1) in the target is identified with the anima $P(f_0, g_1)(g \circ \alpha_0, \alpha_1 \circ f)$ which is contractible by the assumption that $P(f_0, g_1)$ is a proposition.

We now assume (3) and prove (4). If the functor (10.5) is fully faithful, then we may identify $\mathrm{Fun}([1], P)$ as a full subcategory of $P \times P$, which means that the following square is cartesian

$$\begin{array}{ccc} \mathrm{Map}(A, \mathrm{Fun}([1], P)) & \longrightarrow & \mathrm{Map}(A^{\simeq}, \mathrm{Fun}([1], P)^{\simeq}) \\ \downarrow & & \downarrow \\ \mathrm{Map}(A, P \times P) & \longrightarrow & \mathrm{Map}(A^{\simeq}, (P \times P)^{\simeq}) \end{array}$$

Note that we may replace $\mathrm{Fun}([1], P)^{\simeq}$ by $\mathrm{Fun}([1], P)$ in the upper right because $A^{\simeq}$ is an anima, so that this square is identified with the square

$$\begin{array}{ccc} \mathrm{Map}([1], \mathrm{Fun}(A, P)) & \longrightarrow & \mathrm{Map}([1], \mathrm{Fun}(A^{\simeq}, P)) \\ \downarrow & & \downarrow \\ \mathrm{Map}(A, P) \times \mathrm{Map}(A, P) & \longrightarrow & \mathrm{Map}(A^{\simeq}, P) \times \mathrm{Map}(A^{\simeq}, P) \end{array}$$

after curry-uncurry. This cartesian square expresses precisely that the functor (10.6) is fully faithful.

It remains to compute the essential image of the functor $\mathrm{Fun}(A, P) \to \mathrm{Fun}(A^{\simeq}, P)$. If $f^{\simeq} \colon A^{\simeq} \to P$ is obtained from a functor $f \colon A \to P$, then it satisfies $f(a) \le f(a')$ for each arrow $a \to a'$. To prove the converse, assume that we have a functor $f^{\simeq} \colon A^{\simeq} \to P^{\simeq}$, written $a \mapsto f_a$, satisfying $f_a \le f_{a'}$ whenever there is an arrow $a \to a'$ in A. Our task is to extend this objectwise functor to a functor $f \colon A \to P$.

Note that since P is a poset, we have the subanima

$$\Gamma \coloneqq \{(p, a) \mid p \le f_a\} \subseteq P^{\simeq} \times A^{\simeq},$$

If $p \le f_a$ and $a \to a'$, then $p \le f_a \le f_{a'}$. Hence Γ satisfies the condition of Lemma 10.3.6, so that the projection

$$\langle (p, a) \mid p \le f_a \rangle \subseteq P \times A \to A$$

is a cocartesian fibration. The fiber over a is the slice poset $P_{/f_a}$, whose terminal object is id_{f_a}. Thus the object (f_a, a) is terminal in its fiber, so that by functoriality of universals they assemble to a functor $f \colon A \to P$ extending the original family $(f_a)_{a : A^{\simeq}}$.

A fully faithful functor induces an embedding on groupoid cores by Corollary 6.4.2; equivalently, its object map has proposition-valued fibers, so the functor is an embedding. Thus (4) implies (5). Specializing the latter condition to $A = [1]$, and then passing to groupoid cores and using $[1]^{\simeq} = \{0\} \sqcup \{1\}$ proves (2).

The implication from (3) to (6) follows by pulling back the fully faithful source-target functor along $P \times \{x\} \to P \times P$. Since every fully faithful functor is an embedding, (6) implies (7). Finally, assume

(7). If $P_{/x} \to P$ is an embedding, then, for each object y of P, we get by pullback an embedding of the form

$$P(y,x) \hookrightarrow *\,.$$

Doing this for all $x : P^{\simeq}$ proves (1). □

Corollary 10.3.8 (Functor categories into posets)**.** *Let P be a poset and A any category. Then* $\mathrm{Fun}(A,P)$ *is a poset. For two functors $f, g\colon A \to P$ we have $f \leq g$ if and only if for all $a : A^{\simeq}$ we have $f(a) \leq g(a)$.*

Proof. The source-target characterization (3) of Lemma 10.3.7 shows that $\mathrm{Fun}(A,P)$ is a poset. Indeed, applying $\mathrm{Fun}(A,-)$ to the fully faithful source-target functor $\mathrm{Fun}([1],P) \to P \times P$ gives the fully faithful source-target functor for $\mathrm{Fun}(A,P)$. The second statement is just a reformulation of the characterization (4). □

10.3.3 Partially ordered sets

The name "poset" is of course short for "partially ordered set". We now turn this fact into a theorem.

Definition 10.3.9. Let S be a set. A *partially order* on S is a relation

$$\leq\colon S \times S \to \mathrm{Prop} \tag{10.8}$$

which is

- reflexive, i.e., $x \leq x$ for each $x \in S$,
- transitive, i.e., $(x \leq y) \times (y \leq z) \to (x \leq z)$ for all $x, y, z \in S$, and
- antisymmetric, i.e., $(x \leq y) \times (y \leq x) \to (x = y)$ for all $x, y \in S$.

The category of sets with partial order is the full subcategory of

$$\int_{S:\mathrm{Set}} \mathrm{Set}_{/S\times S} \tag{10.9}$$

spanned by those $(S, R \to S \times S)$ whose straightening is a partial order on the set S.

Theorem 10.3.10. *The functor*

$$F\colon \mathrm{Poset} \to \int_{S:\mathrm{Set}} \mathrm{Set}_{/S\times S} \tag{10.10}$$

that maps

$$P \mapsto (P^{\simeq}, \mathrm{Map}([1],P) \to P^{\simeq} \times P^{\simeq}) \tag{10.11}$$

is fully faithful; its essential image is the category of sets with partial order.

Proof. Abbreviate the codomain of F by $\mathcal{R}$. To prove fully faithfulness, fix two posets P, Q. We have a commutative triangle

$$\begin{array}{ccc} \mathrm{Poset}(P,Q) & \xrightarrow{\quad F_{P,Q} \quad} & \mathcal{R}(FP,FQ) \\ & \searrow \quad \swarrow & \\ & \mathrm{Set}(P^{\simeq}, Q^{\simeq}) & \end{array} \tag{10.12}$$

of animae coming from the fact that the first component of F is just $(-)^{\simeq}$. We claim that for each $g : \mathrm{Set}(P^{\simeq}, Q^{\simeq})$ the respective fibers of the diagonal maps are logically equivalent propositions, and hence $F_{P,Q}$ is an equivalence.

Using the characterization of posets (Lemma 10.3.7) the fiber on the left is the proposition

$$\prod_{p,p':P^{\simeq}} P(p,p') \to Q(gp,gp') \tag{10.13}$$

and the fiber on the right is the anima of dashed lifs

$$\left\{ \begin{array}{ccc} \mathrm{Map}([1],P) & \dashrightarrow & \mathrm{Map}([1],Q) \\ \downarrow & & \downarrow \\ P^{\simeq} \times P^{\simeq} & \xrightarrow{g \times g} & Q^{\simeq} \times Q^{\simeq} \end{array} \right\} \tag{10.14}$$

which is a proposition because the right vertical map is an embedding. The two propositions (10.13) and (10.14) are equivalent by currying and uncurrying because $\mathrm{Map}([1],P) \simeq \Sigma_{p,p':P^{\simeq}} P(p,p')$.

Clearly if P is a poset, then $P(-,-)$ is a partial order on $P^{\simeq}$. It remains to show that every partial order can be obtained this way. Let $(S,\leq)$ be a set with a partial order. We need to find a poset P with $(P^{\simeq},\leq) \cong (S,\leq)$. Concretely, let $P \subseteq \mathrm{Fun}(S,\mathrm{Prop})$ be the full subcategory of those predicates $U\colon S \to \mathrm{Prop}$ that are represented, i.e., satisfy the following property:

$$\sum_{u \in S} \prod_{s \in S} U(s) = (s \leq u) \tag{10.15}$$

(A priori, this anima does not look like a property; but note that any two $u,u' \in S$ representing U must satisfy $u \leq u'$ (because $U(u)$) and $u' \leq u$ (because $U(u')$), hence $u = u'$ by antisymmetry.)

Since Prop is a poset, so is P. Moreover, we have a map of sets

$$S \to \mathrm{Fun}(S,\mathrm{Prop}), \quad s \mapsto (- \leq s) \tag{10.16}$$

which tautologically factors through P, hence yields a map $S \to P^{\simeq}$. We claim that U is a poset isomorphism.

- The underlying map of sets is a bijection; the inverse maps a predicate U to its unique representing element.
- To show that this map is an isomorphism, it suffices to observe that for all $s,t : S$ we have a logical equivalence

$$(s \leq t) \leftrightarrow \left(\prod_{x:S} \mathrm{Prop}((x \leq s),(x \leq t)) \right) \simeq \mathrm{Fun}(S,\mathrm{Prop})((- \leq s),(- \leq t)) \tag{10.17}$$

  of propositions; this is just a reformulation of reflexivity and transitivity of $\leq$. □

10.4 The simplex category

We now internalize the construction of the simplices from Section 3.6, using the natural numbers.

10.4.1 The simplex functor

Construction 10.4.1. Consider the functor

$$- \star [0] \colon \mathrm{Cat} \to \mathrm{Cat}, \qquad C \mapsto C \star [0]$$

together with the empty object $\emptyset : \mathrm{Cat}$. By iterating, we obtain a functor $[-] \colon \mathbb{N} \to \mathrm{Cat}$, sending $n : \mathbb{N}$ to a category $[n]$ defined by the formula

$$[0] = \emptyset \star [0] \quad \text{and} \quad [n+1] = [n] \star [0] \quad \text{for } n : \mathbb{N}.$$

Proposition 10.4.2 (Cancellation after adjoining a cone point)**.** *For any posets C and D, the rule $\varphi \mapsto \varphi \star \mathrm{id}_{[0]}$ defines an equivalence*

$$\mathrm{Equiv}(C, D) \xrightarrow{\sim} \mathrm{Equiv}(C \star [0], D \star [0]) .$$

Proof. Let $\psi \colon C \star [0] \xrightarrow{\sim} D \star [0]$ be an equivalence. The cone point is terminal by Lemma 10.3.4, and terminal objects form a contractible anima. Hence ψ carries the cone point to the cone point.

Recall from Proposition 3.6.9 that we have

$$C^{\simeq} \sqcup [0] = (C \star [0])^{\simeq} .$$

Since coproducts of categories are universal, ψ carries the complementary summand $C^{\simeq}$ onto $D^{\simeq}$. Equivalently, an old object cannot be sent to the cone point: the cone point admits no morphism to an old object, whereas an equivalence preserves the terminal cone point. Since C is a full subcategory of $C \star [0]$, the functor ψ restricts to a fully faithful functor from C to D. Applying the same argument to an inverse of ψ shows that this restriction is an equivalence:

$$\psi_{|C} \colon C \xrightarrow{\sim} D$$

Since functors between posets are determined by their values on objects by Lemma 10.3.7, restriction and adjoining a cone point are inverse as maps of animae. This proves the asserted equivalence. □

Theorem 10.4.3 (Rigidity of the simplices)**.** *The functor $[-] \colon \mathbb{N} \to \mathrm{Cat}^{\simeq}$ is an embedding.*

Proof. Since every simplex and the empty category are posets, Proposition 10.4.2 gives equivalences, for all $m, n : \mathbb{N}$,

$$\mathrm{Equiv}([m+1], [n+1]) \simeq \mathrm{Equiv}([m], [n]).$$

Taking one of the two posets to be empty also gives

$$\mathrm{Equiv}([m+1], [0]) \simeq \mathrm{Equiv}([m], \emptyset) \simeq \emptyset,$$
$$\mathrm{Equiv}([0], [n+1]) \simeq \mathrm{Equiv}(\emptyset, [n]) \simeq \emptyset,$$

since every simplex has a cone point and is therefore nonempty. Finally, $\mathrm{Equiv}([0], [0])$ is contractible. Double induction shows that $\mathrm{Equiv}([m], [n])$ is contractible when $m = n$ and empty otherwise.

Since $\mathbb{N}$ is a set with decidable equality, the identity animae $m = n$ has the same description. Consequently, the canonical map

$$(m = n) \longrightarrow \mathrm{Equiv}([m], [n])$$

is an equivalence for every $m, n : \mathbb{N}$. The fiberwise characterization of embeddings now proves the claim. □

Definition 10.4.4 (Simplex category)**.** We define the *simplex category*

$$\mathbf{\Delta} \subseteq \mathrm{Cat}$$

as the full subcategory of Cat whose object anima is the image of the embedding $[-]\colon \mathbb{N} \hookrightarrow \mathrm{Cat}^{\simeq}$. Given a category C, a functor $F\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is called a *simplicial object* of C. We write $\mathrm{s}C := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ for the category of simplicial objects in C.

10.4.2 Geometric realization

Definition 10.4.5 (Geometric realization)**.** For a simplicial object $X_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to C$, we define its *geometric realization* $|X_\bullet|$ as

$$|X_\bullet| := \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n,$$

whenever this colimit exists in C.

Proposition 10.4.6 (Cosiftedness of simplex category)**.** *The diagonal functor $\delta\colon \mathbf{\Delta} \to \mathbf{\Delta}\times\mathbf{\Delta}$, $[n] \mapsto ([n],[n])$ is an initial functor.*

Proof. Let $m, n \in \mathbb{N}$ be two natural numbers. By Corollary 9.5.7, we have to show that the slice category

$$\mathbf{\Delta}_{/([m],[n])} = \mathbf{\Delta} \times_{\mathbf{\Delta}\times\mathbf{\Delta}} (\mathbf{\Delta}\times\mathbf{\Delta})_{/([m],[n])}$$

is weakly contractible. The forgetful functor $\mathbf{\Delta}_{/([m],[n])} \to \mathbf{\Delta}$ is a right fibration, hence there exists a pullback square

$$\begin{array}{ccc} \left(\mathbf{\Delta}_{/([m],[n])}\right)^{\mathrm{op}} & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & & \downarrow \\ \mathbf{\Delta}^{\mathrm{op}} & \dashrightarrow & \mathrm{An}. \end{array}$$

The classifying presheaf of this right fibration has value at $[p]$ equal to

$$\mathrm{Hom}_{\mathbf{\Delta}}([p],[m]) \times \mathrm{Hom}_{\mathbf{\Delta}}([p],[n]).$$

It is therefore the product $\mathbf{\Delta}^m \times \mathbf{\Delta}^n$ of the presheaves represented by $[m]$ and $[n]$.

Since $- \star [0]\colon \mathrm{Cat} \to \mathrm{Cat}$ preserves simplices by definition, it restricts to an endofunctor on $\mathbf{\Delta}$, hence we may consider the maps

$$\mathrm{id}_{\mathbf{\Delta}} \to (-) \star [0] \leftarrow \mathrm{const}_{[0]}$$

of endofunctors of $\mathbf{\Delta}$.

Now consider the product $[m]\times[n]$ in Cat. The simplicial presheaf $[p] \mapsto \mathrm{Map}([p],[m]\times[n])$ is canonically the product $\mathbf{\Delta}^m \times \mathbf{\Delta}^n$. The inclusion

$$[m]\times[n] \hookrightarrow ([m]\times[n]) \star [0]$$

admits a right adjoint

$$r\colon ([m]\times[n]) \star [0] \to [m]\times[n]$$

satisfying $r(a,b) = (a,b)$ for $(a,b) \in [m] \times [n]$ and $r(0) = (m,n)$. Thus, given a functor $(a,b)\colon [p] \to [m] \times [n]$, we may consider

$$\begin{array}{ccccc} [p] & \hookrightarrow & [p] \star [0] & & \\ {\scriptstyle (a,b)}\downarrow & & \downarrow{\scriptstyle (a,b)\star[0]} & & \\ [m]\times[n] & \hookrightarrow & ([m]\times[n]) \star [0] & \hookleftarrow & [0] \\ & & \downarrow{\scriptstyle r} & \swarrow{\scriptstyle (m,n)} & \\ & & [m]\times[n]. & & \end{array}$$

This provides homotopies between $\mathrm{id}_{\mathbf{\Delta}_{/([m],[n])}}$ and φ, and between φ and $\mathrm{const}_{(m,n)}$, where

$$\varphi([p],(a,b)) = ([p+1], r \circ ((a,b) \star \mathrm{id}_{[0]})).$$

It follows that $\mathbf{\Delta}_{/([m],[n])}$ is weakly contractible. □

Corollary 10.4.7. *The functor $\delta^{\mathrm{op}}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$ is a terminal functor.* □

Thus $\mathbf{\Delta}^{\mathrm{op}}$ is sifted. We obtain the following consequences for geometric realization.

Corollary 10.4.8 (Diagonal realization)**.** *Let C admit geometric realizations, and let $\Phi\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to C$ be a functor. Then the following colimits exist and the diagonal induces an equivalence*

$$\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Phi_{n,n} \xrightarrow{\sim} \mathrm{colim}_{([m],[n])\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} \Phi_{m,n}.$$

Proof. The iterated geometric realization exists and computes the colimit over $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$, by the universal property of colimits. The assertion follows from terminality of the diagonal, Corollary 10.4.7. □

Proposition 10.4.9 (Geometric realization and finite products)**.** *Let C be a category admitting geometric realizations and finite products, and assume that the functor $- \times -\colon C \times C \to C$ preserves geometric realizations in both variables. Then the colimit functor $|-|\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \to C$ preserves finite products.*

Proof. Since $[0]$ is terminal in $\mathbf{\Delta}$, the anima $\Pi_\infty(\mathbf{\Delta}^{\mathrm{op}})$ is contractible. Thus the constant terminal simplicial object has terminal realization. It remains to check binary products. For simplicial objects $X, Y\colon \mathbf{\Delta}^{\mathrm{op}} \to C$, the *box product* $X \boxtimes Y\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to C$ is the composite

$$\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X\times Y} C \times C \xrightarrow{-\times-} C.$$

By Corollary 10.4.8, we have

$$|X \times Y| \simeq \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} (X \boxtimes Y)_{n,n} \simeq \mathrm{colim}_{([m],[n])\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} X_m \times Y_n.$$

Since for a fixed object T of C, the functor $T \times -\colon C \to C$ preserves geometric realizations by assumption, we get

$$\mathrm{colim}_{([m],[n])\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}} X_m \times Y_n \simeq \mathrm{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} X_m \times \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} Y_n = |X| \times |Y|,$$

as desired. □

10.4.3 Split simplicial objects

A split simplicial object has an extra degeneracy, including a section of its augmentation. Its realization is preserved by every functor. To keep all the compatibilities in this condition, we describe it using an indexing category.

Let $\mathbf{\Delta}_{-\infty}$ be the category whose objects are the symbols $[n]$ for $n \geq -1$ and whose morphisms $[m] \to [n]$ are the order-preserving maps

$$\{-\infty\} \sqcup [m] \longrightarrow \{-\infty\} \sqcup [n]$$

that preserve $-\infty$. There are inclusions $\mathbf{\Delta} \subseteq \mathbf{\Delta}_{+} \subseteq \mathbf{\Delta}_{-\infty}$, where a morphism belongs to $\mathbf{\Delta}_{+}$ precisely when the inverse image of $-\infty$ is $\{-\infty\}$.

Definition 10.4.10 ([Lur17, Definition 4.7.2.2])**.** An augmented simplicial object $X\colon {\mathbf{\Delta}_{+}}^{\mathrm{op}} \to C$ is *split* if it extends to a functor ${\mathbf{\Delta}_{-\infty}}^{\mathrm{op}} \to C$. A simplicial object is *split* if it extends to a split augmented simplicial object.

Lemma 10.4.11 (Absolute realization of split simplicial objects)**.** *Every split augmented simplicial object is a colimit diagram. In particular, every split simplicial object admits a geometric realization, and every functor preserves this realization.*

Proof. The inclusion $j\colon \mathbf{\Delta} \to \mathbf{\Delta}_{-\infty}$ has a right adjoint: forget the distinguished point of $\{-\infty\} \sqcup [n]$ and regard the underlying ordered set as $[n+1]$. Indeed, a morphism from $j([m])$ to $[n]$ in $\mathbf{\Delta}_{-\infty}$ is precisely a monotone map from $[m]$ to this underlying ordered set. This identification is natural.

Consequently, j^{op} is a right adjoint and hence terminal by Proposition 9.1.12. The object $[-1]$ is terminal in ${\mathbf{\Delta}_{-\infty}}^{\mathrm{op}}$, so the colimit of an extension to that category is its value at $[-1]$. Restriction along j^{op} has the same colimit, which proves the first assertion. Every functor preserves the extension and its augmentation, so it also preserves this realization. □

11 Covers and truncations

11.1 Images and Čech complexes

Recall from Definition 1.6.19 the notion of an *embedding*, which we may think of as a monomorphism of categories. This notion is the first in a hierarchy of notions of *n-truncation* for every integer $n \geq -1$. In particular, we obtain notions of *n-truncated categories* for every n. For example, the (-1)-truncated animae are categories P such that the map $P \to *$ is an embedding; we will refer to such categories as *propositions* and denote their full subcategory by Prop.

The *Čech nerve* gives an explicit construction of (-1)-truncation: it factors a map of animae as a cover followed by an embedding. We then develop the general truncation hierarchy, set and poset truncation, and constructive notions of finite and π-finite animae.

Before defining the general hierarchy of truncation levels, we will discuss one special case: the (-1)-truncation operation. We will see that there is a very explicit description of this operation in terms of the *Čech nerve* of a morphism.

Definition 11.1.1. Since An admits small limits, the functor

$$\mathrm{ev}_{[0]} \colon \mathrm{PSh}(\mathbf{\Delta}) \to \mathrm{An}\,, \quad X \mapsto X_0 = X([0])$$

admits a right adjoint

$$\check{C} \colon \mathrm{An} \to \mathrm{PSh}(\mathbf{\Delta})$$

given by right Kan extension along the functor $i = [0] \colon * \to \mathbf{\Delta}^{\mathrm{op}}$. We refer to the simplicial anima $\check{C}(X)$ as the *Čech nerve* of X.

Remark 11.1.2. Using the pointwise formula for right Kan extensions, we may concretely describe the evaluation of $\check{C}(X)$ at $[n] : \mathbf{\Delta}^{\mathrm{op}}$ as follows.

$$\check{C}(X)_n \simeq X^{n+1}\,.$$

Indeed, the right Kan extension at $[n]$ is indexed by maps $[n] \to [0]$ in $\mathbf{\Delta}^{\mathrm{op}}$, or equivalently by maps $[0] \to [n]$ in $\mathbf{\Delta}$. This indexing anima is

$$(h_{[n]})_0 = \mathrm{Hom}_{\mathbf{\Delta}}([0],[n]) = [n]^{\simeq}\,;$$

using that $(C \star [0])^{\simeq} \simeq C^{\simeq} \sqcup e$, by Proposition 3.6.9, we observe furthermore by induction on n that

$$[n]^{\simeq} = \{0,\dots,n\}\,,$$

where $\{0, \dots, n\}$ is defined inductively by

$$\{0, \dots, 0\} = [0] \quad \text{and} \quad \{0, \dots, n+1\} = \{0, \dots, n\} \sqcup [0] \,.$$

It follows that, for $X \in \mathrm{An}^{\simeq}$, we have

$$\check{C}(X) \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}, \qquad \check{C}(X)_n = \mathrm{Hom}_{\mathrm{An}}(\{0, \dots, n\}, X) \,.$$

By a new induction, one can show that this simplifies to

$$\check{C}(X)_n \simeq X^{n+1} \,,$$

where the right-hand side is defined inductively via $X^0 = e$ and $X^{n+1} = X^n \times X$.

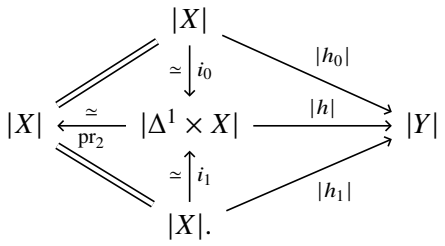
Remark 11.1.3. Since $i \colon * \to \mathbf{\Delta}^{\mathrm{op}}$ is fully faithful, also $i_!$ is fully faithful, so the unit $\mathrm{id} \xrightarrow{\sim} i^* i_!$ is an isomorphism. Passing to right adjoints, also the counit $i^* i_* \to \mathrm{id}$ is an isomorphism, so that i_* is fully faithful.

Lemma 11.1.4. *For any functor $Y \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$, the morphism $|\mathrm{pr}_2| \colon |\Delta^1 \times Y| \to |Y|$ is an isomorphism.*

Proof. Under the equivalence $|\Delta^1 \times Y| \simeq |\Delta^1| \times |Y|$ from Proposition 10.4.9, the map $|\mathrm{pr}_2| \colon |\Delta^1 \times Y| \to |Y|$ corresponds to the projection map $\mathrm{pr}_2 \colon |\Delta^1| \times |Y| \to |Y|$. So it remains to show that $|\Delta^1| \simeq *$. Since Δ^1 is the presheaf on $\mathbf{\Delta}$ represented by $[1] \in \mathbf{\Delta}$, this is an instance of Corollary 12.6.1. □

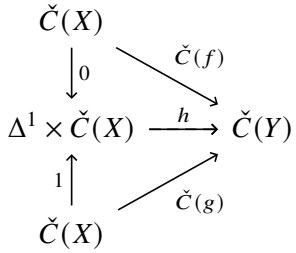
Corollary 11.1.5. *Given simplicial animae $X, Y \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ and a simplicial homotopy $h \colon \Delta^1 \times X \to Y$, we have $|h_0| = |h_1|$ as maps $|X| \to |Y|$.*

Proof. Consider the diagram



□

Consider morphisms $f \colon X \to Y$ and $g \colon X \to Y$ in An. Then there is a one-to-one correspondence between commutative diagrams



and extensions

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\langle f,g\rangle} & Y \\ \downarrow & \overset{\tilde{h}}{\nearrow} & \\ (\Delta^1 \times \check{C}(X))_0. & & \end{array}$$

Corollary 11.1.6. *If there is a map $x\colon e \to X$, then $|\check{C}(X)|$ is contractible.*

Proof. The extension preceding the statement gives a simplicial homotopy from the constant endomorphism $X \to e \xrightarrow{x} X$ of $\check{C}(X)$ to its identity. By Corollary 11.1.5, the identity of $|\check{C}(X)|$ is therefore equal to the constant map with value induced by x, which is a contraction. □

Construction 11.1.7. All of the above applies in any anima context. Let Y be a small anima and let $f\colon X \to Y$ be a morphism of small animae, i.e. an anima (X, f) in context Y. We have an adjunction

$$\begin{array}{ccc} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An}_{/Y}) & \underset{\check{C}_Y}{\overset{\mathrm{ev}_0}{\rightleftarrows}} & \mathrm{An}_{/Y} \\ {\scriptstyle \mathrm{fgt}}\downarrow & \swarrow_{(X,f)\mapsto \check{C}(f)} & \\ \mathrm{PSh}(\mathbf{\Delta}). & & \end{array}$$

Here $\check{C}(f)$ is the anima $\pi_f\colon \check{C}(X/Y) \to Y$ in context Y, obtained by applying the Čech nerve construction in context Y to the anima $f\colon X \to Y$ in context Y. We will refer to $\check{C}(f)$ as the *Čech nerve of f*.

Remark 11.1.8. Let C be a category that admits small colimits. Then also the slice $C_{/y}$ admits small colimits for any object y of C, and the forgetful functor $C_{/y} \to C$ preserves small colimits.

So for any morphism $f\colon X \to Y$ of small animae we obtain a map $|\pi_f|\colon |\check{C}(X/Y)| \to Y$.

Theorem 11.1.9. *Consider a morphism $f\colon X \to Y$ of small animae. If f admits a section, then the map $|\pi_f|\colon |\check{C}(X/Y)| \to Y$ is an equivalence.*

Proof. This is an instance of Corollary 11.1.6 applied in context Y. □

Let X and T be objects of An. We may consider the projection map $f = \mathrm{pr}_2\colon X \times T \to T$. Note that this is the image of the map $X \to *$ under the pullback functor $\mathrm{An} \to \mathrm{An}_{/T}$, and it follows that

$$\check{C}(X \times T/T) \simeq \check{C}(X) \times T$$

as simplicial objects in $\mathrm{An}_{/T}$. We then get

$$|\check{C}(X \times T/T)| \simeq |\check{C}(X) \times T| \simeq |\check{C}(X)| \times T.$$

Taking $T = X$, the projection map $\mathrm{pr}_2\colon X \times X \to X$ admits a section, given by the diagonal. By Theorem 11.1.9, it follows that

$$|\check{C}(X \times X/X)| \xrightarrow{\sim} X.$$

This proves:

Corollary 11.1.10. *For every small anima X the projection map $\mathrm{pr}_2 \colon |\check{C}(X)| \times X \to X$ is an equivalence.* □

Theorem 11.1.11. *Let X be a small anima. Then the anima $|\check{C}(X)|$ is a proposition, in the sense that $|\check{C}(X)| \to *$ is an embedding.*

Proof. Consider the anima $T = |\check{C}(X)|$. By the previous corollary, the projection map $f = \mathrm{pr}_1 \colon X \times T \to X$ is an equivalence. We thus get

$$\check{C}(X \times T/T) \xrightarrow{\sim} \check{C}(X) \times T \xrightarrow{\sim} \check{C}(X),$$

where the second equivalence is levelwise: in degree n it is the projection $X^{n+1} \times T \to X^{n+1}$, obtained by pulling back $X \times T \to X$ along the first coordinate $X^{n+1} \to X$. Taking geometric realizations gives

$$T \times T \xrightarrow{\sim} |\check{C}(X \times T/T)| \xrightarrow{\sim} T.$$

It follows that T is a proposition. □

Corollary 11.1.12. *Let $f \colon X \to Y$ be any morphism in* An. *Then the morphism $|\pi_f| \colon |\check{C}(X/Y)| \to Y$ is an embedding.*

Proof. Apply Theorem 11.1.11 in context Y. □

11.2 Covers

The notion of a Čech cover leads to the definition of a cover.

Definition 11.2.1. A morphism $f \colon X \to Y$ in An is said to be a *cover* if the induced map $|\pi_f| \colon |\check{C}(X/Y)| \to Y$ is an isomorphism.

In homotopy type theory, covers are usually called *surjections*.

Remark 11.2.2. For any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

there is a canonical isomorphism

$$\check{C}(X/Y) \times_Y Y' \simeq \check{C}(X'/Y')$$

and since $(-) \times_Y Y'$ commutes with small colimits, this induces a canonical isomorphism

$$|\check{C}(X/Y)| \times_Y Y' \simeq |\check{C}(X'/Y')|\,.$$

This implies in particular that covers are stable under pullbacks.

Definition 11.2.3 (Essential surjectivity)**.** A functor $F \colon C \to D$ of categories is called *essentially surjective* if the induced map $F^{\simeq} \colon C^{\simeq} \to D^{\simeq}$ on groupoid cores is a cover.

Proposition 11.2.4. *For any morphism $f\colon X \to Y$, the map $X \to |\check{C}(X/Y)|$ induced from the degree-zero term $\check{C}(X/Y)_0 = X$ is a cover. In particular, f factors as a composite*

$$X \xrightarrow{\text{cover}} |\check{C}(X/Y)| \xrightarrow{\text{embedding}} Y\,.$$

Proof. Write $I = |\check{C}(X/Y)|$. Since $I \hookrightarrow Y$ is an embedding, the relative Čech nerve of $X \to I$ agrees with that of $X \to Y$. Its realization is therefore I, so $X \to I$ is a cover. The second assertion now follows from Corollary 11.1.12. □

Definition 11.2.5. We refer to the anima $|\check{C}(X/Y)|$ as the *image* of f, and denote it by $\mathrm{Im}(f)$.

Proposition 11.2.6. *Let $f\colon X \to Y$ be a morphism in* An*. The following conditions are equivalent.*

(i) The map f is a cover.

(ii) The pullback functor

$$f^*\colon \mathrm{Cat}_{/Y} \to \mathrm{Cat}_{/X}$$

is conservative.

(iii) The pullback functor

$$f^*\colon \mathrm{An}_{/Y} \to \mathrm{An}_{/X}$$

is conservative.

Proof. Assume that f is a cover and consider a morphism $\varphi\colon Z \to W$ over Y:

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & W \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y, & \end{array}$$

and assume that the induced map $X \times_Y \varphi\colon X \times_Y Z \to X \times_Y W$ is an isomorphism. Then φ induces a map

$$\check{C}(X/Y) \times_Y Z \to \check{C}(X/Y) \times_Y W$$

of simplicial objects in $\mathrm{Cat}_{/Y}$ which is a levelwise isomorphism, hence an isomorphism in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_{/Y})$. Applying geometric realization in $\mathrm{Cat}_{/Y}$ thus produces an isomorphism

$$|\check{C}(X/Y) \times_Y Z| \xrightarrow{\sim} |\check{C}(X/Y) \times_Y W|\,.$$

On the other hand, for any category $T \to Y$, since the functor $(-) \times_Y T$ commutes with colimits, we have a canonical equivalence of the form

$$|\check{C}(X/Y) \times_Y T| \xrightarrow{\sim} |\check{C}(X/Y)| \times_Y T = T\,.$$

We obtain a commutative diagram of the form

$$\begin{array}{ccc} |\check{C}(X/Y) \times_Y Z| & \longrightarrow & |\check{C}(X/Y) \times_Y W| \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ Z & \xrightarrow{\varphi} & W \end{array}$$

proving that φ is an equivalence. This proves that (ii) follows from (i). For each small anima X, the full embedding $\mathrm{An}_{/X} \hookrightarrow \mathrm{Cat}_{/X}$ is conservative and is compatible with pullbacks. This implies right away that (iii) follows from (ii). Finally, observe that the pullback of $|\check{C}(X/Y)|$ along f is $|\check{C}(X \times_Y X/X)|$, through the first projection of $X \times_Y X$ to X, say. Since the first projection of $X \times_Y X$ to X has a section, this means that f^* sends the embedding of the image of f into Y to an equivalence in $\mathrm{An}_{/X}$. Therefore, assertion (i) follows from assertion (iii). □

Corollary 11.2.7. *Let $f : X \to Y$ and $g : Y \to Z$ be two functors between small animae.*

(a) If f and g are covers, so is $g \circ f$.

(b) If $g \circ f$ is a cover, so is g.

Proof. We will use characterization (iii) of Proposition 11.2.6. To prove assertion (a) we observe that if g^* and f^* are conservative functors, so is the composite $f^* \circ g^* \simeq (g \circ f)^*$. Similarly, to prove assertion (b), we check that if the composite $f^* \circ g^* \simeq (g \circ f)^*$ is conservative, then so is g^*. □

Corollary 11.2.8. *For any diagram $X \leftarrow A \to Y$ of animae, the map $X \amalg Y \to X \amalg_A Y$ is a cover.*

Proof. Corollary 13.2.3 says that the composite

$$X \amalg A \amalg Y \to X \amalg Y \to X \amalg_A Y \tag{11.1}$$

(where we use either $A \to X$ or $A \to Y$ in the first map) is a cover, hence also the second map by Corollary 11.2.7. □

Corollary 11.2.9. *Any retract of a cover is a cover.*

Proof. We will use again characterization (iii) of Proposition 11.2.6. Any retract of a conservative functor is conservative (this follows from Proposition 6.4.14, since any retract of a pullback square is a pullback square). Therefore, if f is a retract of g then f^* is a retract of g^*. □

The following is an important consequence of Proposition 11.2.6.

Proposition 11.2.10. *Let $f : X \to Y$ be a map between animae that is both a cover and an embedding. Then f is an equivalence.*

Proof. In general, in order to check that f is an equivalence, we may regard it as a morphism in $\mathrm{An}_{/Y}$. By Proposition 11.2.6, it will suffice to show that its pullback $\mathrm{pr}_2 : X \times_Y X \to X$ along itself is an equivalence. In other words, without loss of generality, we may assume furthermore that f has a section and apply Lemma 1.6.24. □

We now finally get the strengthening

Proposition 11.2.11. *Let $f : A \to B$ be a functor. If f is fully faithful and if the map $f^{\simeq} : A^{\simeq} \to B^{\simeq}$ is a cover, then the functor f is an equivalence.*

Proof. Since f is fully faithful, the functor $f^{\simeq}$ is an embedding, hence by Proposition 11.2.10 it is an equivalence. In particular, $f^{\simeq}$ admits a section, hence by Theorem 6.3.8 the functor f is an equivalence. □

Proposition 11.2.12. *Let $f\colon X \to Y$ be a morphism in* An*. Then the following are equivalent:*

(1) The morphism f is a cover;

(2) For every embedding $Z \to Y$, if we have $X \times_Y Z \xrightarrow{\sim} X \times_Y Y = X$ then $Z \xrightarrow{\sim} Y$.

Proof. The fact that (1) implies (2) is an instance of Proposition 11.2.6. The fact that (2) implies (1) follows by taking $Z = |\check{C}(X/Y)| \hookrightarrow Y$, so that $X \times_Y Z = |\check{C}(X \times_Y X/X)| \xrightarrow{\sim} X$. □

Theorem 11.2.13. *Let $u\colon A \to B$ be a functor. Assume there exists a cover $i\colon X \to B^{\simeq}$ such that for every object x of X, the relative slice category $A_{/i(x)} = A \times_B B_{/i(x)}$ admits a terminal object. Then u admits a right adjoint.*

Proof. For any anima Γ, we can consider the situation in context Γ: we may ask whether the resulting functor $u_\Gamma\colon A_\Gamma \to B_\Gamma$ admits a right adjoint in context Γ. If this holds for $\Gamma = B^{\simeq}$, it also holds in the absolute context.

If Y is an anima equipped with a map $b\colon Y \to B^{\simeq}$, then in context $B^{\simeq}$ this looks like

$$\begin{array}{ccccc} Y & \xrightarrow{(b,b)} & B^{\simeq} \times B^{\simeq} & \longrightarrow & B \times B^{\simeq} \\ & & & \searrow & \downarrow \mathrm{pr}_2 \\ & & & & B^{\simeq}. \end{array}$$

So we may express that $A_{/b}$ has a terminal object for every object b of B as a proposition P in context $B^{\simeq}$. The existence of a terminal object is then expressed as the assertion that this proposition P is equivalent to the terminal anima in context $B^{\simeq}$. To prove such an equivalence, by virtue of Proposition 11.2.6, it suffices to check that the structural map $P \to B^{\simeq}$ becomes an equivalence after pulling back along the cover $i\colon X \to B^{\simeq}$. The hypothesis says precisely that this pullback is terminal. We conclude that all relative slices $A_{/b}$ admit terminal objects in context $B^{\simeq}$, and hence that u admits a right adjoint by Theorem 6.2.1. □

Proposition 11.2.14. *Let $u\colon A \to B$ be an essentially surjective functor. Then, for every category C, the restriction functor*

$$u^*\colon \mathrm{Fun}(B, C) \longrightarrow \mathrm{Fun}(A, C)$$

is conservative.

Proof. We may assume that we work in a universe so that A, B and C are small. Since $u^{\simeq}$ is a cover, Proposition 11.2.6, applied objectwise, implies that, for any category X, the functor

$$u^*\colon \mathrm{Fun}(B, \mathrm{Fun}(X, \mathrm{An})) \to \mathrm{Fun}(A, \mathrm{Fun}(X, \mathrm{An}))\,.$$

is conservative (because one can check equivalences objectwise). We consider finally the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(B, C) & \hookrightarrow & \mathrm{Fun}(B, \mathrm{PSh}(C)) \\ {\scriptstyle u^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathrm{Fun}(A, C) & \hookrightarrow & \mathrm{Fun}(A, \mathrm{PSh}(C)) \end{array}$$

in which the horizontal arrows are the full embeddings induced by the Yoneda embedding of C. The right vertical functor is conservative by the preceding paragraph, and the horizontal functors reflect equivalences. It follows that the left vertical functor u^* is conservative. □

Warning 11.2.15. The converse of Proposition 11.2.14 is false, already for ordinary categories. Let B be the category generated by

$$b \xrightarrow{i} a \xrightarrow{r} b, \qquad ri = \mathrm{id}_b,$$

with no further relations, and let $A \subset B$ be the full subcategory on a. The inclusion is not essentially surjective, since b is not isomorphic to a. Nevertheless, restriction $\mathrm{Fun}(B, C) \to \mathrm{Fun}(A, C)$ is conservative for every C. Indeed, if $\alpha \colon F \to G$ is a natural transformation and α_a is invertible, then

$$F(r) \circ \alpha_a^{-1} \circ G(i)$$

is inverse to α_b, by naturality and the relation $ri = \mathrm{id}_b$.

Corollary 11.2.16. *For any category C and any family of morphisms W stable under composition in C, the localization functor $\gamma \colon C \to C[W^{-1}]$ induces a cover*

$$\gamma^{\simeq} \colon C^{\simeq} \to (C[W^{-1}])^{\simeq}.$$

Proof. Let $I \hookrightarrow (C[W^{-1}])^{\simeq}$ be the image of $\gamma^{\simeq}$, and let $j \colon L_I \hookrightarrow C[W^{-1}]$ be the corresponding full subcategory. The localization functor factors as $C \xrightarrow{\gamma'} L_I \xrightarrow{j} C[W^{-1}]$. Since L_I is full, γ' sends every morphism of W to an isomorphism. The universal property of the localization therefore gives a functor $s \colon C[W^{-1}] \to L_I$ satisfying $s\gamma = \gamma'$. Hence $js\gamma = \gamma$. Precomposition with γ is fully faithful, so $js = \mathrm{id}_{C[W^{-1}]}$. Thus the full embedding j is essentially surjective and hence an equivalence. It follows that $I \hookrightarrow (C[W^{-1}])^{\simeq}$ is an equivalence, which says exactly that $\gamma^{\simeq}$ is a cover. □

Corollary 11.2.17. *For every category C, the canonical localization functor $C \to \Pi_\infty(C)$ induces a cover*

$$C^{\simeq} \to \Pi_\infty(C).$$

Proof. Apply Corollary 11.2.16 to the localization of C at the subcategory of all its morphisms. □

Proposition 11.2.18 (Non-axiom of replacement)**.** *Let C be a locally small category and Γ a small anima. We consider a functor $f \colon \Gamma \to C^{\simeq}$. Then $\mathrm{Im}(f)$ is small. In particular, the full subcategory of C with anima of objects $\mathrm{Im}(f)$ is small.*

Proof. For any $x, y : C^{\simeq}$, we have an embedding

$$C^{\simeq}(x, y) \hookrightarrow C(x, y)$$

with small codomain. Therefore, the anima $C^{\simeq}$ is locally small. In other words, there is a pullback square of the form

$$\begin{array}{ccc} C^{\simeq} & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle\Delta}\downarrow & \lrcorner & \downarrow{\scriptstyle\pi_{\mathrm{univ}}} \\ C^{\simeq} \times C^{\simeq} & \longrightarrow & \mathrm{Cat}. \end{array}$$

Since cocartesian fibrations with small fibers are stable under composition, the diagonal $C^{\simeq} \to C^{\simeq} \times \cdots \times C^{\simeq}$ is a cocartesian fibration with small fibers for any finitely many cartesian product of copies of $C^{\simeq}$. This implies that, for any $n : \mathbb{N}$, the map

$$\underbrace{\Gamma \times_{C^{\simeq}} \cdots \times_{C^{\simeq}} \Gamma}_{n+1 \text{ times}} \longrightarrow \underbrace{\Gamma \times \cdots \times \Gamma}_{n+1 \text{ times}}$$

is a cocartesian fibration with small fibers and with small codomain. Therefore, $\check{C}(\Gamma/C^{\simeq})$ is a functor with values in small animae. Therefore, since the indexing category $\mathbf{\Delta}^{\mathrm{op}}$ is small as well, its colimit $|\check{C}(\Gamma/C^{\simeq})|$ is a small anima.

Denote by D the full subcategory of C spanned by $\mathrm{Im}(f)$. To see that D is small, we use 10, so that it suffices to show that the anima $\mathrm{Map}([1], D)$ is small. We have

$$\mathrm{Map}([1], D) \simeq \sum_{x,y:D^{\simeq}} D(x, y), \tag{11.2}$$

where we have just established that $D^{\simeq} \simeq \mathrm{Im}(f)$ is small. Moreover, for each $x, y : D^{\simeq}$, we have $D(x, y) \simeq C(ix, iy)$, where $i \colon D \to C$ is the defining embedding, each of which is a small anima by local smallness of C. Since small animae are closed under Sigma-types (by 9 and 1), it follows that $\mathrm{Map}([1], D)$ is small, as claimed. □

Remark 11.2.19. In homotopy type theory, the statement that the image of a map $f \colon X \to Y$ from a small type X to a locally small type Y is itself small is called the *principle of replacement*. It is either assumed as an axiom or can be derived in the presence of suitable higher inductive types.

When Y is the type of small types (which is indeed locally small by univalence), this is analogous to the replacement axiom schema of ZFC, that says that the image of any function $X \to V$, where X is a set and V is the set-theoretic universe (which is not itself a set, but only a class), is again a set.

11.3 Truncated categories and functors

We define $\mathbb{Z}_{\geq -2} = [0] \sqcup [0] \sqcup \mathbb{N}$. The two objects obtained by the contribution of $[0] \sqcup [0]$ are called -2 and -1. We define $-2 + 1 = -1$ and $-1 + 1 = 0$. Since $n + 1$ is already defined for $n : \mathbb{N}$, this defines a successor operation on $\mathbb{Z}_{\geq -2}$ starting at -2. Iterating Lemma 10.2.14 gives an equivalence $\mathbb{N} \xrightarrow{\sim} \mathbb{Z}_{\geq -2}$. In particular, we can do induction starting from -2. We will often write $n \geq -2$ instead of $n : \mathbb{Z}_{\geq -2}$.

We will now discuss the hierarchy of n-truncated functors for $n \geq -2$.

Definition 11.3.1 (n-truncated functors)**.** We fix a category U with pullbacks as well as with a terminal object $*$. For an integer $n \geq -2$, we inductively define what it means for a morphism $f \colon A \to B$ in U to be *n-truncated*:

- When $n = -2$, we say f is (-2)-truncated if it is an equivalence;
- For $n : \mathbb{Z}_{\geq -2}$, we say f is $(n + 1)$-truncated if the map $\Delta_f := (1_A, 1_A) \colon A \to A \times_B A$ to the pullback of f along itself is n-truncated.

An object A of U is called *n-truncated* if the map $A \to *$ is n-truncated.

Definition 11.3.2 (Locally n-truncated functors)**.** A functor $f \colon A \to B$ is *locally n-truncated* if for each pair of objects $x, y : A^{\simeq}$, the map of animae $A(x, y) \to B(fx, fy)$ is n-truncated. A category A is called locally n-truncated, if every hom anima $A(x, y)$ is n-truncated.

Remark 11.3.3.
- A category C is (-2)-truncated if and only if it is contractible (Definition 1.1.6)

- A functor $F\colon C \to D$ is (-1)-truncated if and only if it is an embedding (Definition 1.6.19)
- A functor $F\colon C \to D$ is locally (-2)-truncated if and only if it is fully faithful.

Exercise 11.3.4. Show that if a category C is 0-truncated, then $C \star [0]$ is also 0-truncated. Show moreover that if every automorphism in C is an identity, then the same holds in $C \star [0]$.

Proposition 11.3.5. *Any functor that commutes with pullbacks preserves n-truncated morphisms. Any functor that preserves pullbacks and terminal objects preserves n-truncated objects.*

Proof. Induct on n, using preservation of equivalences for $n = -2$ and the identification $F(\Delta_f) = \Delta_{F(f)}$ for the induction step. The object statement follows by applying this to $A \to *$. □

Proposition 11.3.6. *Assume that the ambient category U is locally small. A morphism $f\colon A \to B$ is n-truncated in a category U if and only if, for any object $X : U^{\simeq}$, the induced map*

$$f_*\colon \mathrm{Hom}_U(X, A) \to \mathrm{Hom}_U(X, B)$$

is n-truncated in An.

Proof. The Yoneda embedding of U is conservative and commutes with limits, whereas equivalences in a presheaf category are detected objectwise. Induction on n therefore gives the assertion. □

Lemma 11.3.7 (Truncation levels are cumulative)**.** *Let $-2 \leq n \leq m$. Then every n-truncated morphism is also m-truncated.*

Proof. It suffices to prove the case $m = n + 1$. Induct on n: the diagonal of an equivalence is an equivalence, and the induction step follows by applying the induction hypothesis to the relative diagonal. □

Proposition 11.3.8 (Closure properties of n-truncated maps)**.** *Let $n \geq -2$.*

(1) Consider morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and assume that g is n-truncated. Then f is n-truncated if and only if gf is n-truncated.

(2) n-truncated functors are closed under fiber products, retracts and base change.

(3) For every map of animae $u\colon \Gamma \to \Gamma'$, n-truncated functors are preserved by dependent sum and dependent product along u.

Proof. We prove the three assertions simultaneously by induction on n. For $n = -2$, they reduce to the corresponding closure and cancellation properties of equivalences. Dependent sum does not change the underlying morphism, while dependent product preserves equivalences.

Suppose the assertions have been proved for n, and consider $A \xrightarrow{f} B \xrightarrow{g} C$ with g $(n+1)$-truncated. There is a factorization of relative diagonals

$$A \xrightarrow{\Delta_f} A \times_B A \longrightarrow A \times_C A,$$

whose composite is Δ_{gf}. The second morphism is the base change of $\Delta_g \colon B \to B \times_C B$, hence is n-truncated by the induction hypothesis. The cancellation assertion in degree n now shows that Δ_f is n-truncated if and only if Δ_{gf} is n-truncated. This proves the first assertion in degree $n+1$.

The relative diagonal of a base change of a morphism is the corresponding base change of its relative diagonal. The induction hypothesis therefore proves stability under base change. Composites of $(n+1)$-truncated morphisms are $(n+1)$-truncated by the first assertion, and fiber products of such morphisms are obtained by composing base changes. Finally, the relative diagonal of a retract is a retract of the relative diagonal. Stability under retracts follows from the induction hypothesis.

It remains to consider a map of animae $u \colon \Gamma \to \Gamma'$. Truncation in a slice is computed on the underlying morphism, since its relative diagonals are formed by the same pullbacks as in Cat. Dependent sum along u is composition with u, so it leaves that underlying morphism unchanged. Dependent product along u exists because a map between animae is a cocartesian fibration by Corollary 4.5.22, and hence is exponentiable. It preserves pullbacks by Lemma 3.5.8, and it sends the terminal object id_Γ of $\mathrm{Cat}_{/\Gamma}$ to the terminal object $\mathrm{id}_{\Gamma'}$ of $\mathrm{Cat}_{/\Gamma'}$. Hence Proposition 11.3.5 shows that dependent product preserves $(n+1)$-truncated morphisms. □

Lemma 11.3.9 (Truncation is fiberwise)**.** *Consider functors $C \xrightarrow{f} D \to \Gamma$ with animae Γ and categories C and D. Then f is n-truncated if and only if f_γ is n-truncated for each object $\gamma : \Gamma$.*

Proof. If f is n-truncated, then every f_γ is n-truncated by stability under base change. Conversely, the assertion that f_γ is n-truncated is formulated in context $\gamma \colon \Gamma$; applying it to the universal element id_Γ proves that f is n-truncated. □

For animae, the truncation level can be fully determined by looking at the identity types.

Proposition 11.3.10. *Let $f \colon X \to Y$ be a map of animae and $n \geq -1$. Then f is n-truncated if and only if for all objects $x, y : X$, the induced map $X(x,y) \to Y(fx, fy)$ is $(n-1)$-truncated.*

Proof. Apply Lemma 11.3.9 to $X \xrightarrow{f} X \times_Y X \to X \times X$ and observe that base-changing the diagonal Δ_f to $(x,y) : X \times X$ precisely yields the map $X(x,y) \to Y(fx, fy)$. □

Remark 11.3.11. Let X and Y be animae. In type theoretic language we can therefore define

$$\texttt{isTruncated}(-2, X) := \texttt{is_Contr}(X)$$

and

$$\texttt{isTruncated}(n+1, X) := \prod_{x,y:X} \texttt{isTruncated}(n, X(x,y))$$

for all $n \geq -2$. Moreover, it follows from Lemma 11.3.9 (with $X = C$ and $Y = D = \Gamma$) that for any map $f \colon X \to Y$ we have the bi-implication

$$\texttt{isTruncated}(n, f) \leftrightarrow \prod_{y:Y} \texttt{isTruncated}(n, X_y).$$

Lemma 11.3.12 (Truncation of functor categories)**.** *For every n-truncated functor $C \to D$, the induced map*

$$\mathrm{Fun}(T, C) \to \mathrm{Fun}(T, D) \tag{11.3}$$

is again n-truncated for every category T. If C is n-truncated, then so is $\mathrm{Fun}(T, C)$.

Proof. This is a particular case of Proposition 11.3.5. □

For categories, the situation is slightly more complicated, since being truncated involves a condition both on identity types and on hom animae.

Proposition 11.3.13 (Characterization of n-truncated functors)**.** *Let $f\colon C \to D$ be a functor and $n \geq -2$. The following are equivalent:*

(1) The functor f is n-truncated.

(2) The map of animae $\mathrm{Map}([1], f)\colon \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$ is n-truncated.

(3) The functor f is locally n-truncated and induces an n-truncated map of animae $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$.

Proof. We first prove the equivalence between (1) and (2) by induction on n. The case $n = -2$ is Corollary 6.3.9. The induction step follows because $\mathrm{Map}([1], -)$ preserves pullbacks and therefore carries the relative diagonal of f to the relative diagonal of $\mathrm{Map}([1], f)$.

We next compare (2) and (3). If $\mathrm{Map}([1], f)$ is n-truncated, then so is $f^{\simeq}$, since the latter is a retract of the former. We may therefore assume that $f^{\simeq}$ is n-truncated. Applying Lemma 11.3.9 to

$$\mathrm{Map}([1], C) \xrightarrow{\alpha} \mathrm{Map}([1], D) \times_{D^{\simeq} \times D^{\simeq}} (C^{\simeq} \times C^{\simeq}) \longrightarrow C^{\simeq} \times C^{\simeq}$$

shows that f is locally n-truncated if and only if α is n-truncated. Moreover, $\mathrm{Map}([1], f)$ factors as

$$\mathrm{Map}([1], C) \xrightarrow{\alpha} \mathrm{Map}([1], D) \times_{D^{\simeq} \times D^{\simeq}} (C^{\simeq} \times C^{\simeq}) \longrightarrow \mathrm{Map}([1], D).$$

The second morphism is a base change of $f^{\simeq} \times f^{\simeq}$, hence is n-truncated. The cancellation property in Proposition 11.3.8 now shows that $\mathrm{Map}([1], f)$ is n-truncated if and only if α is n-truncated. This proves the remaining equivalence. □

Proposition 11.3.14. *Let $n \geq -2$. Every locally n-truncated functor is $(n+1)$-truncated. In particular, every locally n-truncated category is $(n+1)$-truncated.*

Proof. The case $n = -2$ is exactly the statement that every fully faithful functor is an embedding; this follows from Proposition 6.4.6 and Corollary 3.2.3. Now let $n \geq -1$ and let $f\colon C \to D$ be locally n-truncated. By Lemma 11.3.13 we only have to show that $f^{\simeq}$ is $n+1$-truncated. Let $x, y : C^{\simeq}$. Then we have a commutative diagram of animae

$$\begin{array}{ccc} C^{\simeq}(x,y) & \longrightarrow & C(x,y) \\ \downarrow{\scriptstyle f^{\simeq}} & & \downarrow{\scriptstyle f} \\ D^{\simeq}(fx,fy) & \longrightarrow & D(fx,fy) \end{array}$$

where the two horizontal maps are embeddings (i.e., (-1)-truncated) by Corollary 2.5.3 and the right vertical map is n-truncated by assumption. Since $n \geq -1$, it follows that the left vertical map is also n-truncated. Since $x, y : C^{\simeq}$ were arbitrary, we are done by Proposition 11.3.10. □

Corollary 11.3.15. *Every (-1)-truncated functor is conservative. In particular, every (-1)-truncated category is an anima.*

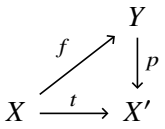
Proof. Let $f\colon C \to D$ be a (-1)-truncated functor. Let $x, y : C^{\simeq}$ and $u : C(x,y)$ such that $f(u)$ is invertible. Since the map $C^{\simeq}(x,y) \to D^{\simeq}(fx, fy)$ is an equivalence we find an invertible arrow $v : C^{\simeq}(x,y)$ with $f(v) = f(u)$. But that implies $v = u$ in $C(x,y)$ because f is an embedding. It follows that u is invertible. Since u was arbitrary, f is conservative.

The second statement is the special case where $D = *$. □

11.4 Truncation

Definition 11.4.1 (Induction principle of n-truncation)**.** Let X be an anima. We say that a map $t\colon X \to X'$ is an *n-approximation* if it satisfies the following *induction principle of n-truncation*:

- For every n-truncated map $p\colon Y \to X'$ of animae, every section of $X \times_{X'} Y \to X$ can be lifted to a section of $Y \to X'$. Explicitly this means that for every commutative triangle



  where p is n-truncated we have a section $f'\colon X' \to Y$ of p and an identification $f = f't$ over X'.

If additionally X' is n-truncated, we say that t exhibits X' as the *n-truncation* of X. If an n-truncation of X exists, we denote it by $t\colon X \to \|X\|_n$ and its action on objects by $t\colon x \mapsto [x]$.

Remark 11.4.2. A more type-theoretic way to think about the induction principle of n-truncation is as follows: Whenever we have a family of n-truncated animae $Y_{x'}$ indexed by $x' : X'$, to produce a section $x' : X' \mapsto f'(x') : Y_{x'}$ it suffices to do so on terms of the form $x' = tx$, by specifying the value $f'(tx) \coloneqq f(x)$.

Lemma 11.4.3. *In the setting of Definition 11.4.1, assume that we have two lifts $f_1', f_2'\colon X' \to Y$ of the section f. Then we have an identification $h' : f_1' = f_2'$ with $h't = \mathrm{id}_f$.*

Proof. Consider the family $Z\colon (x' : X') \mapsto Y_{x'}(f_1'x', f_2'x')$. If $n = -2$, the fibers of $Y \to X'$ are contractible, and so are the fibers of $Z \to X'$. If $n \geq -1$, each $Z_{x'}$ is $(n-1)$-truncated and hence n-truncated by Lemma 11.3.7. Thus in every case the projection $Z \to X'$ is n-truncated. Moreover, for each $x : X$ we have $h(x) \coloneqq \mathrm{id}_{f(x)} : Y_x(fx, fx) = Z(tx)$ (using that $f_i'tx = fx$). Therefore, we may apply the induction principle to $Z \to X'$ to obtain a lift $h'\colon X' \to Z$ of h, which is precisely the datum of an identification $f_1' = f_2'$ extending the identity of f. □

Exercise 11.4.4. Show that n-truncation of an anima is unique if it exists.

Lemma 11.4.5. *A map $X \to X'$ is an n-approximation of animae if and only if for every n-truncated map $Y \to X'$ of animae, the canonical map*

$$\mathrm{Map}_{X'}(X', Y) \to \mathrm{Map}_{X'}(X, Y) \tag{11.4}$$

is an equivalence.

Proof. For the forward direction, fix a section $f : X \to Y$ over X'. The induction principle gives a lift $f' : X' \to Y$, so the fiber of the displayed map over f is inhabited. Any two such lifts are identified over f by Lemma 11.4.3. Hence this fiber is a proposition by Lemma 10.1.9, and is therefore contractible. All fibers of the displayed map are contractible, so it is an equivalence.

The converse is trivial because being an n-approximation just requires the map (11.4) to have a section. □

Remark 11.4.6. Matching the terminology of Definition 10.3.1, the -1-truncation and 0-truncations are also called propositional truncation and set truncation, respectively.

We conclude by defining the local analog of truncation.

Definition 11.4.7. A functor $t : C \to C'$ is called a *local n-approximation* if for every locally n-truncated functor $p : D \to C'$, the induced map

$$\mathrm{Map}_{C'}(C', D) \to \mathrm{Map}_{C'}(C, D) \tag{11.5}$$

is an equivalence. If C' is additionally locally n-truncated, then we say that t exhibits C' as the *local n-truncation* of C, and write $C' = \|C\|_n^{\mathrm{loc}}$.

11.4.1 Propositional truncation

Let us refer to (-1)-truncation as *propositional truncation*. We will now show that the Čech nerve construction from Section 11.1 computes the propositional truncation.

Lemma 11.4.8. *A map of animae $t : X \to X'$ is an (-1)-approximation if and only if t is a cover.*

Proof. In the case $n = -1$ the induction principle of the n-truncation can be simplified by recalling that an (-1)-truncated map has a section if and only if it is an equivalence (and any two sections are equal in that case), see Lemma 1.6.24. Thus in the case $n = -1$, the condition of Definition 11.4.1 can be reformulated as saying that an embedding $Y \to X'$ is an equivalence whenever the pulled-back map $X \times_{X'} Y \to X$ is an equivalence. This is equivalent to $X \to X'$ being a cover by Proposition 11.2.12. □

Corollary 11.4.9 (Existence of propositional truncation)**.** *For every anima X, the map $X \to |\check{C}(X)|$ is a propositional truncation. (Here the Čech nerve is computed in any universe in which X is small.)*

Proof. The anima $|\check{C}(X)|$ is a proposition by Theorem 11.1.11 and the map $X \to |\check{C}(X)|$ is a cover by Proposition 11.2.4. □

As a consequence of the corollary, the (-1)-truncation $\|X\|_{-1}$ exists and is given by $|\check{C}(X)|$.

Definition 11.4.10. Let X be an anima. We say that there *merely exists* an object of X if $\|X\|_{-1}$ is inhabited.

Lemma 11.4.11. *A map $f : Y \to X$ of animae is a cover if and only if for each $x : X$ there merely exists a $y : Y$ with $fy = x$, i.e., if we have an inhabitant of*

$$\prod_{x:X} \left\| \sum_{y:Y} fy = x \right\|_{-1}.$$

Proof. If $x : X$ is a term defined in context Γ (with Γ an anima), we see that $\left\|\sum_{y:Y} fy = x\right\|_{-1}$ literally is $|\check{C}(x^*(f))|$ for $x^*(f)$ the pullback of f along $x\colon \Gamma \to X$. Therefore, if we have a term in

$$\prod_{x:X} \left\| \sum_{y:Y} fy = x \right\|_{-1},$$

we have a term in $|\check{C}(x^*(f))|$ for all x, in particular for x the universal term of X, also known as the identity. This means that $|\check{C}(Y/X)| \hookrightarrow X$ has a section and is thus an equivalence. The converse follows from the fact that covers are stable under base change. □

Remark 11.4.12. What the proof of Lemma 11.4.11 actually does is the construction of a fiberwise equivalence

$$|\check{C}(Y/X)|_x \xrightarrow{\sim} \left\| \sum_{y:Y} fy = x \right\|_{-1}$$

in context $x : X$. Both sides are propositions in that context, so this equivalence is unique.

Proposition 11.4.13. *Propositional truncation $X \mapsto \|X\|_{-1}$ yields a left adjoint to the inclusion* Prop $\hookrightarrow$ An.

Proof. By the pointwise construction of adjoints and directed univalence we need to show the following: for each anima X there exists a proposition X' and a map $t\colon X \to X'$ such that for all propositions Z the induced map

$$\mathrm{Map}(X', Z) \to \mathrm{Map}(X, Z) \tag{11.6}$$

is an equivalence. Choosing $X \to X' \coloneqq \|X\|_{-1}$ to be a propositional truncation, this is a special case of Lemma 11.4.5 for the constant family $Y \coloneqq X' \times Z \to X'$. □

11.4.2 Set and poset truncation

We saw that one can construct propositional truncations using the Čech nerve. We next give a criterion for the universal property of poset truncation in terms of covers on objects and mapping animae.

Lemma 11.4.14. *Let $p\colon A \to B$ be a functor that is a cover on objects. For every poset P, the induced functor*

$$\mathrm{Fun}(B, P) \to \mathrm{Fun}(A, P) \tag{11.7}$$

is fully faithful. Moreover a functor $f\colon A \to P$ lies in the essential image (i.e. factors through p) if and only if it satisfies $f(a) \leq f(a')$ whenever we have an arrow $pa \to pa'$ in B.

Proof.

- To establish fully faithfulness, by Corollary 10.3.8, it suffices to show for all $f, g\colon B \to P$ the logical equivalence between
  - for all $b : B^{\simeq}$ we have $f(b) \leq g(b)$ and
  - for all $a : A^{\simeq}$ we have $f(pa) \leq g(pa)$,

The forward direction is immediate by restriction. The converse follows from the induction principle for propositional truncation which holds because $p^{\simeq}\colon A^{\simeq} \to B^{\simeq}$ is a (-1)-approximation by assumption. Indeed it tells us that to show the proposition $f(b) \leq g(b)$ for all objects $b : B^{\simeq}$ it suffices to do so for those of the form $b = pa$.

- Let $f\colon A \to P$ satisfy $fa \leq fa'$ whenever we have an arrow $pa \to pa'$. We have to show that f factors through p. Conversely, any factorization through p has this property by functoriality.

  We start with the special case where $P = S$ is a set, and where $A = A^{\simeq}$ and $B = B^{\simeq}$ are animae. Let $R \to B \times S$ be the fiberwise -1-truncation of the map $(p, f)\colon A \to B \times S$; in other words, R consists of those pairs (b, s) for which there merely exists an $a : A$ with $pa = b$ and $fa = s$. We claim that the composite

  $$R \subseteq B \times S \to B \tag{11.8}$$

  is an embedding. To show this we have to construct, for any $b : B$ and any two $(b, s), (b, s') : R_b$, an equality $s = s'$ in S; but since S is a set, hence $S(s, s')$ is a proposition, mere existence is enough. By definition of R there merely exist $a, a' : A$ with $fa = s$ and $fa' = s'$ but also $pa = b = pa'$. By the assumption on the map f, we thus obtain $s = fa = fa' = s'$ as claimed. Now we may use that every cover is a (-1)-approximation (Lemma 11.4.8) to obtain the dashed filler

  $$\begin{array}{ccc} & R & \hookrightarrow B \times S \\ {\scriptstyle (p,f)}\nearrow & & \downarrow \\ A & \xrightarrow{\quad p \quad} & B \end{array} \tag{11.9}$$

  whose second component yields the desired factorization of f through p.

- To prove the general case consider the commutative square

  $$\begin{array}{ccc} \mathrm{Map}(B, P) & \longrightarrow & \mathrm{Map}(A, P) \\ \downarrow & & \downarrow \\ \mathrm{Map}(B^{\simeq}, P) & \longrightarrow & \mathrm{Map}(A^{\simeq}, P) \end{array} \tag{11.10}$$

  where all maps are embeddings; the horizontal ones by what we proved earlier, and the vertical ones by Lemma 10.3.7. Thus it suffices to show that $f^{\simeq}\colon A^{\simeq} \to P^{\simeq}$ factors through $p^{\simeq}\colon A^{\simeq} \to B^{\simeq}$ and that the resulting map $g^{\simeq}\colon B^{\simeq} \to P^{\simeq}$ extends to $g\colon B \to P$. We have the first fact by applying the step we already proved; indeed, $P^{\simeq}$ is a set and $A^{\simeq}, B^{\simeq}$ are animae. For the second step we have to show $g(b) \leq g(b')$ whenever we have an arrow $b \to b'$ in B (by Lemma 10.3.7). Since $p^{\simeq}$ is a cover, so is $p^{\simeq} \times p^{\simeq}\colon A^{\simeq} \times A^{\simeq} \to B^{\simeq} \times B^{\simeq}$. It is therefore a (-1)-approximation, and it suffices to show this in the case where $b = pa$ and $b' = pa'$ for $a, a' : A^{\simeq}$, in which case it is precisely the assumption on f. □

Corollary 11.4.15. *Let $p\colon A \to B$ be a functor which is a cover both on objects and on each mapping anima. Then the induced restriction functor*

$$\mathrm{Fun}(B, P) \to \mathrm{Fun}(A, P) \tag{11.11}$$

is an equivalence for each poset P.

Proof. By Lemma 11.4.14, restriction is fully faithful. Let $f\colon A \to P$. To apply the essential-image criterion, fix an arrow $u\colon pa \to pa'$ in B. Since $A(a,a') \to B(pa,pa')$ is a cover, there merely exists a lift of u to an arrow $a \to a'$ in A. Functoriality of f then gives $f(a) \leq f(a')$. This target is a proposition, so mere existence of the lift suffices. Consequently every f lies in the essential image, and restriction is an equivalence. □

Proposition 11.4.16. *A functor which is a cover on objects and on mapping animae is a local (-1)-approximation.*

Proof. Let $p\colon A \to B$ be such a functor, let $q\colon D \to B$ be locally (-1)-truncated, and fix $s\colon A \to D$ over B. Write D_s for the full essential image of s. For $a, a' : A^{\simeq}$, the composite

$$A(a,a') \longrightarrow D(sa,sa') \longrightarrow B(pa,pa')$$

is a cover and the second map is an embedding. By cancellation for covers, the second map is also a cover, hence an isomorphism. Since $A^{\simeq} \to D_s^{\simeq}$ is a cover, this proves that $D_s \to B$ is fully faithful. It is essentially surjective because p is, so it is an equivalence. Its inverse extends s to a section of q.

Every extension has full essential image D_s, since p is essentially surjective. It therefore factors uniquely through D_s. The equivalence $D_s \to B$ now makes the anima of extensions, with their identification on A, contractible. This proves the relative universal property of Definition 11.4.7. □

Remark 11.4.17. Every category C admits a *poset truncation*: a functor $C \to \|C\|_{\mathrm{pos}}$ to a poset such that restriction induces an equivalence

$$\mathrm{Fun}(\|C\|_{\mathrm{pos}}, P) \simeq \mathrm{Fun}(C,P)$$

for every poset P. We will prove this by giving an explicit construction in Construction 12.3.22, using the Yoneda embedding. For an anima C, this also gives its set truncation, as we will show in Corollary 12.3.24.

11.5 Connected and surjective functors

Definition 11.5.1. Let $f\colon C \to D$ be a functor between categories.

- For any $n \geq 0$, the functor f is called *surjective on n-cells*
  - for $n = 0$, if the induced map $f\colon C^{\simeq} \to D^{\simeq}$ on objects is a cover;
  - for $n \geq 1$, if for every pair $x, x' : C^{\simeq}$ the induced map $C(x,x') \to D(fx,fx')$ is surjective on $(n-1)$-cells.
- For any $-1 \leq n \leq \infty$, the functor f is called *n-surjective* if it is surjective on k-cells for all $0 \leq k \leq n$. (Note that this is vacuous for $n = -1$).
- For any $-2 \leq n < \infty$, the functor f is called *n-connected* if it is left orthogonal to all n-truncated functors; i.e., if for every solid commutative square of the form

$$\begin{array}{ccc} C & \longrightarrow & S \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow \\ D & \longrightarrow & T \end{array} \tag{11.12}$$

where $S \to T$ is n-truncated, there is a unique dashed lift as indicated.

- The functor is called ∞-*connected* if it is n-connected for all n.

Remark 11.5.2. If $f\colon X \to Y$ is a functor between animae, then surjectivity on 0-cells is just surjectivity; for $n \geq 1$, the map f is surjective on n-cells if and only if the diagonal $\Delta_f\colon X \to X \times_Y X$ is surjective on $(n-1)$-cells. It follows directly from the definition that being surjective on n-cells is a fiberwise condition.

Lemma 11.5.3. *Let $n \in \mathbb{Z}_{\geq -2}$ and let $f\colon X \to Y$ be a map of animae. The following are equivalent:*

(1) *f is $(n+1)$-surjective.*

(2) *For every n-truncated map $S \to Y$, the map*

$$\prod_{y:Y} S_y \to \prod_{y:Y} \mathrm{Map}(X_y, S_y), \quad s \mapsto (y \mapsto x \mapsto s_y) \tag{11.13}$$

is an equivalence.

(3) *For every n-truncated map $S \to Y$ and every $y : Y$, the map*

$$S_y \to \mathrm{Map}(X_y, S_y), \quad s \mapsto (x \mapsto s) \tag{11.14}$$

is an equivalence.

(4) *For every n-truncated anima T and every $y : Y$, the map*

$$T \to \mathrm{Map}(X_y, T), \quad t \mapsto (x \mapsto t) \tag{11.15}$$

is an equivalence.

Proof. First we prove that the three conditions 2, 3, and 4 are equivalent. The implications $4 \Rightarrow 3$ and $3 \Rightarrow 2$ are easy: for the first, simply choose $T :\equiv S_y$ and for the second take the dependent product over $y : Y$.

For the implication $2 \Rightarrow 4$, fix $y : Y$ and an n-truncated anima T, and define the family $S \to Y$ by $S_z :\equiv \mathrm{Map}((y = z), T)$. Each S_z is n-truncated because T is. Moreover, we have a commutative square

$$\begin{array}{ccccc} \prod_{z:Y} S_z & = & \prod_{z:Y} \mathrm{Map}((y=z),T) & \xrightarrow{\simeq} & T \\ \downarrow{\scriptstyle s \mapsto (z \mapsto x \mapsto s_z)} & & \downarrow & & \downarrow{\scriptstyle t \mapsto (x \mapsto t)} \\ \prod_{z:Y} \mathrm{Map}(X_z, S_z) & \xrightarrow{\simeq} & \prod_{x:X} \mathrm{Map}(y = fx, T) & \xrightarrow{\simeq} & \mathrm{Map}(X_y, T) \end{array} \tag{11.16}$$

identifying the desired vertical map on the right with the known equivalence on the left.

Next we prove the implication $1 \Rightarrow 4$. Since both conditions are fiberwise we may assume that $Y = *$, so that we must show that

$$c\colon T \to \mathrm{Map}(X, T), \quad t \mapsto (x \mapsto t) \tag{11.17}$$

is an equivalence for every $(n+1)$-surjective X and n-truncated T. For $n = -2$, both T and $\mathrm{Map}(X, T)$ are contractible. For $n = -1$, both sides are propositions; since X is merely inhabited, evaluation at a merely chosen point gives an inverse to c. Now let $n \geq 0$. By induction assume that we have proved the statement for $n-1$. We show that c is a surjective embedding:

- Embedding: For each $s, t : T$ we need to show that the map

$$s = t \to \mathrm{Map}(X, (s = t)), \quad (p \mapsto (x \mapsto p)) \tag{11.18}$$

  is an equivalence. Since $s = t$ is $(n-1)$-truncated and X is in particular n-surjective, this follows from the induction hypothesis (in context $s, t : T$).

- Surjectivity: Choose a point $x_0 : X$ (allowed, because X is 0-surjective and we are proving a proposition). Let $r\colon X \to T$. We claim that we have an equality $c(r(x_0)) = r$, i.e., that for each $x : X$ we have $rx_0 = rx$. For each $x : X$ consider the map

$$(rx_0 = rx) \to \mathrm{Map}(x_0 = x, (rx_0 = rx)), \quad p \mapsto (q \mapsto p) \tag{11.19}$$

  Since the anima $rx_0 = rx$ is $(n-1)$-truncated and $x_0 = x$ is n-surjective, by induction this map is an equivalence. Since the right side has an object (the action of r on equalities), so does the left.

It remains to show that the equivalent conditions 2, 3, and 4 jointly imply 1. Since conditions 4 and 1 are fiberwise, we may assume that $Y = *$. We again argue by induction on n. The case $n = -2$ is automatic.

For arbitrary $n \geq -1$, take $T \coloneqq \|X\|_{-1}$ to obtain the equivalence

$$\|X\|_{-1} \xrightarrow{\simeq} \mathrm{Map}(X, \|X\|_{-1}) \tag{11.20}$$

The canonical map $X \to \|X\|_{-1}$ on the right then gives a proof on the left that X is merely inhabited, i.e. 0-surjective.

Now let $n \geq 0$. We must show that all identity animae $x =_X z$, for $x, z : X$, are n-surjective. We will show that for each $x : X$, the map $x\colon * \to X$ satisfies condition 2 for $n-1$; by induction this implies that its fibers are n-surjective which is what we need.

Fix $x : X$, and let $p\colon S \to X$ be $(n-1)$-truncated. Choose a universe in which p has small fibers and consider it as a map $S\colon X \to \mathrm{An}_{\leq n-1}$. Here $\mathrm{An}_{\leq n-1}$ denotes the anima of $(n-1)$-truncated animae in the chosen universe. It is n-truncated by univalence, so we can apply condition 4 with $T \coloneqq \mathrm{An}_{\leq n-1}$ to obtain that the map

$$c\colon \mathrm{An}_{\leq n-1} \to \mathrm{Map}(X, \mathrm{An}_{\leq n-1}) \tag{11.21}$$

is an equivalence. In particular, S is equivalent to a constant family $z \mapsto S_0$ for some $(n-1)$-type S_0, i.e., $S \simeq S_0 \times X$ over X.

Thus the map in (11.13), applied to $x : * \to X$, is the restriction $\prod_{z:X} S_z \to S_x$. Under this identification of S with the constant family S_0, it becomes the evaluation map

$$\mathrm{ev}_x\colon \mathrm{Map}(X, S_0) \to S_0 \tag{11.22}$$

Since evaluation is a retraction of the constant-map equivalence (11.17), supplied by condition 4 with $T = S_0$, it is an equivalence, thus concluding the proof. □

Corollary 11.5.4. *Let $f\colon X \to Y$ be a map of animae and $n \in \mathbb{Z}_{\geq -2}$. The following are equivalent:*

(1) *The map f is $(n+1)$-surjective.*

(2) *The map f is an n-approximation.*

(3) *The map f is n-connected.*

Proof. "1 $\iff$ 2": Using condition 2 of Lemma 11.5.3, this is exactly Lemma 11.4.5.

"1 $\iff$ 3": When the map $Y \to T$ is an equivalence, the unique lifting statement (11.12) is exactly the equivalence (11.13) from Lemma 11.5.3; the general case can always be reduced to this case by pulling back along $Y \to T$, because n-truncated maps are closed under base change. □

Remark 11.5.5. For categories, a $(n+1)$-surjective functor need not be n-connected. For example, the functor $\{0,1\} \to [1]$ is 0-surjective and (-1)-truncated, but is not orthogonal to itself (i.e., not an isomorphism).

Lemma 11.5.6 (Closure properties of n-connected maps)**.** *Let $n \geq -2$. The collection of n-connected maps of animae is closed under:*

- *base change*
- *composition*
- *colimits*
- *precancellation: If gf and f are n-connected, then so is g.*

Proof. Base change follows from being fiberwise (Remark 11.5.2), composition, colimits and precancellation follow from the characterization as left orthogonal maps. □

Lemma 11.5.7. *Let A be an n-connected anima. Then its suspension $\Sigma A \coloneqq * \amalg_A *$ is $(n+1)$-connected.*

Proof. We use condition 4 of Lemma 11.5.3. Let A be n-connected; we must show that for each $(n+1)$-truncated anima T the map

$$T \to \mathrm{Map}(\Sigma A, T) = \Sigma_{t_0,t_1:T} \mathrm{Map}(A, t_0 = t_1), \quad t \mapsto (t, t, a \mapsto \mathtt{refl}_t) \tag{11.23}$$

is an equivalence. This map is the composite

$$T \xrightarrow{t \mapsto (t,t,\mathtt{refl}_t)} \Sigma_{t_0,t_1}(t_0 = t_1) \xrightarrow{\Sigma_{t_0,t_1}(p \mapsto a \mapsto p)} \Sigma_{t_0,t_1} \mathrm{Map}(A, t_0 = t_1) \tag{11.24}$$

of two equivalences: the left one formally, and the right one because A is n-connected and each $t_0 = t_1$ is n-truncated. □

Corollary 11.5.8. *For each $n \geq -1$, the n-sphere S^n, defined inductively via*

$$S^{-1} \coloneqq \emptyset \quad \textit{and} \quad S^n \coloneqq \Sigma S^{n-1}, \tag{11.25}$$

is $(n-1)$-connected.

Proof. The empty set is (vacuously) (-2)-connected, thus the claim follows by induction using Lemma 11.5.7. □

Lemma 11.5.9. *Let $n \geq -1$ and A be ∞-connected. Then every map $f\colon S^n \to A$ is merely constant.*

Proof. We prove the statement by induction on n. For $n = -1$ choose $a \in A$; then every map $\emptyset = S^{-1} \to A$ is vacuously homotopic to const_a. In the induction step consider a map $f\colon S^{n+1} = \{0\} \amalg_{S^n} \{1\} \to A$, which amounts to the data of $f0, f1 \in A$ and a map $f'\colon S^n \to f0 = f1$. Since the anima $f0 = f1$ is still ∞-connected, there merely exists a trivialization $f'\colon S^n \to \{p\} \xrightarrow{p} (f0 = f1)$, which means that f factors as $f\colon S^{n+1} \to \{0\} \amalg_{\{p\}} \{1\} \to A$, thus trivializing it. □

11.6 Finite and π-finite animae

We now combine the natural numbers, covers, and truncations to formulate finiteness conditions on animae. There are two constructively different notions already for sets. A finite set comes with the mere existence of a cardinal, whereas a finitely enumerable set comes only with the mere existence of a finite surjection onto it. The distinction propagates to higher animae and must be kept visible in a theory that does not assume excluded middle.

For a natural number n, we write $\{1, \ldots, n\}$ for the standard set with n elements, with $\{1, \ldots, 0\} = \emptyset$.

11.6.1 Finite and finitely enumerable sets

Definition 11.6.1. Let S be a set. A *cardinal presentation* of S is a natural number n together with a bijection

$$\{1, \ldots, n\} \xrightarrow{\sim} S.$$

We say that S is *finite* if the anima of cardinal presentations of S is merely inhabited. We denote the resulting proposition by $\texttt{isFinite}(S)$.

Thus finiteness is a property rather than a choice of an enumeration or of a cardinal. It is often called *Bishop finiteness* when it is necessary to distinguish it from weaker constructive notions.

Definition 11.6.2. Let S be a set. A *finite enumeration* of S is a natural number n together with a cover

$$\{1, \ldots, n\} \longrightarrow S.$$

We say that S is *finitely enumerable* if the anima of its finite enumerations is merely inhabited.

Every finite set is finitely enumerable, but the converse is not constructively valid. The missing ingredient is precisely decidable equality.

Proposition 11.6.3 (Finite sets and finite enumerations)**.** *A set is finite if and only if it is finitely enumerable and has decidable equality.*

Proof. The standard sets $\{1, \ldots, n\}$ have decidable equality, by induction on n. Hence a cardinal presentation of S gives both a finite enumeration and decidable equality on S. These are properties of S, so the same conclusion follows from the mere existence of a cardinal presentation.

Conversely, assume that S has decidable equality. Given a finite enumeration $f\colon \{1, \ldots, n\} \to S$, we may remove repetitions from the list $f(1), \ldots, f(n)$: at each step, decidable equality determines

whether the next value has already occurred. This produces a natural number m, an embedding $\{1, \ldots, m\} \to S$, and the same image as f. Since f is a cover, the resulting embedding is also a cover and hence an equivalence by Proposition 11.2.10. It is therefore a cardinal presentation of S. Again, the desired conclusion is a proposition, so a merely given finite enumeration suffices. □

Example 11.6.4. A proposition P, regarded as a set, is finite if and only if it is decidable. Indeed, decidability identifies P either with the empty set or with the one-point set. Conversely, a cardinal presentation of P decides P: cardinal zero gives $\neg P$, and a positive cardinal gives a term of P. Since decidability of a proposition is itself a proposition, the mere existence of a cardinal presentation suffices.

The following elementary closure properties will be used implicitly below.

Proposition 11.6.5 (Finite sums, products, and choice)**.** *Let S be a finite set and let $T \colon S \to \mathrm{Set}$ be a family of finite sets. Then the dependent sum $\sum_{s:S} T(s)$ and the dependent product $\prod_{s:S} T(s)$ are finite. Moreover, for any family of animae $X \colon S \to \mathrm{An}$, the canonical map*

$$\left\| \prod_{s:S} X(s) \right\|_{-1} \longrightarrow \prod_{s:S} \|X(s)\|_{-1}$$

is an equivalence.

Proof. All three assertions are propositions, so we may choose a cardinal presentation of S and argue by induction on its cardinal. We first prove the final assertion. For the empty set it is immediate. Adjoining one element reduces it to the canonical equivalence

$$\|A \times B\|_{-1} \simeq \|A\|_{-1} \times \|B\|_{-1}.$$

This equivalence follows by applying the induction principle for propositional truncation twice.

Apply the resulting finite-choice statement to the anima of cardinal presentations of each $T(s)$. We may now, merely, choose cardinal presentations of all the $T(s)$ simultaneously. The assertions about the dependent sum and dependent product then reduce to the corresponding elementary cardinal calculations for standard finite sets. □

The last assertion is finite choice. It does not choose data from a merely inhabited arbitrary family. Rather, it says that finitely many merely inhabited animae have a merely inhabited product.

11.6.2 The recursive finiteness hierarchy

For an anima X, its set truncation $\|X\|_0$ is its set of connected components. Finiteness of this set controls degree zero, while finiteness of the identity animae controls all higher degrees. This gives a definition that does not require homotopy groups or chosen basepoints.

Definition 11.6.6. Let X be an anima. We define recursively what it means for X to be *n-finite*, for $n : \mathbb{N}$.

(1) The anima X is 0-finite if it is a finite set.

(2) The anima X is $(n+1)$-finite if its set of components $\|X\|_0$ is finite and, for every $x, y : X$, the identity anima $x = y$ is n-finite.

We denote the resulting proposition by $\mathtt{isFinite}_n(X)$.

The word "proposition" in the last sentence is justified inductively. At height zero, finiteness was defined using propositional truncation. At the successor step, both conditions are products of propositions.

Proposition 11.6.7 (Basic properties of n-finite animae)**.** *Let $n : \mathbb{N}$ and let X be an anima.*

(1) If X is n-finite, then it is n-truncated.

(2) If X is n-finite, then it is $(n+1)$-finite.

(3) Being n-finite is invariant under equivalence.

(4) Finite coproducts and finite products of n-finite animae are n-finite.

Proof. All four statements follow by induction on n. At height zero, the first statement is part of the definition, the second uses decidable equality of finite sets, and the remaining statements follow from Proposition 11.6.5. At the successor step, apply the induction hypothesis to the identity animae. The set of components remains finite under equivalence and under finite sums and products. For products, use

$$(x_1, x_2) = (y_1, y_2) \simeq (x_1 = y_1) \times (x_2 = y_2),$$

and for coproducts use that identity animae between distinct summands are empty. □

Example 11.6.8. A finite set is n-finite for every $n : \mathbb{N}$. If G is a finite group, its classifying anima $\mathrm{B}G$ is 1-finite: it is connected, and its identity anima at the distinguished object is the finite set G. More generally, the recursive definition recognizes an n-truncated anima as n-finite precisely when its set of components and all its iterated identity animae are finite at the appropriate heights.

Definition 11.6.9. An anima X is *π-finite* if there merely exists a natural number n for which X is n-finite. In formulas,

$$\mathtt{isPiFinite}(X) \coloneqq \left\| \sum_{n:\mathbb{N}} \mathtt{isFinite}_n(X) \right\|_{-1}.$$

The truncation in this formula matters. A proof of π-finiteness does not choose a truncation bound. By Proposition 11.6.7, every π-finite anima is merely n-truncated for some n, has finitely many components, and has finite iterated identity animae. Conversely, these conditions, with a common finite truncation bound, recover the recursive definition. For a set, π-finiteness is equivalent to finiteness. In particular, Example 11.6.4 shows that a proposition is π-finite precisely when it is decidable.

11.6.3 Finite maps

Definition 11.6.10. A map $f : X \to Y$ of animae is *n-finite* if each of its fibers is an n-finite anima. It is *π-finite* if each of its fibers is π-finite.

These are fiberwise properties, hence are stable under pullback. They are also stable under composition. We record the stronger dependent-sum formulation.

Proposition 11.6.11 (Dependent sums of finite animae)**.** *Let B be an n-finite anima and let $E : B \to$ An be a family of n-finite animae. Then its total anima*

$$\sum_{b:B} E(b)$$

is n-finite. Consequently, n-finite maps are closed under composition. The same assertions hold with "n-finite" replaced by "π-finite".

Proof. For the statement about n-finite animae, the assertion follows by induction on n. At height zero it is the dependent-sum assertion of Proposition 11.6.5. At the successor step, the identity anima between (b, e) and (b', e') is

$$\sum_{p:b=b'} \big(p_* e = e'\big).$$

It is n-finite by the induction hypothesis. The same induction also shows that decidable subanimae of n-finite animae are n-finite. To verify the remaining component condition, reduce to the finitely many components of B and merely choose a representative b of each of them, using finite choice. The components of the total anima lying over the component of b are the quotient of the finite set $\|E(b)\|_0$ by transport along loops in B at b. This orbit relation is decidable: for two components of $E(b)$, the anima of loops carrying one to the other is a decidable subanima of the n-finite anima $b = b$, so its mere inhabitation is decidable. A quotient of a finite set by a decidable equivalence relation is finite, by the same deduplication argument as in Proposition 11.6.3. Hence the total anima has finitely many components.

Now suppose that the base and fibers are π-finite. There merely exists a bound for the base. Over each of its finitely many components, there merely exists a bound for the corresponding fiber. Finite choice from Proposition 11.6.5 produces, merely, finitely many bounds; their maximum is a common bound. After increasing all bounds to this maximum by Proposition 11.6.7, the fixed-height case applies. □

11.6.4 Finite covers and iterated diagonals

There is another natural way to express finiteness using only covers and diagonals. It is particularly useful because it makes sense for maps from the outset, but its constructive meaning differs from that of the recursive definition above.

Definition 11.6.12. An anima X is *finitely covered* if there is merely a finite set S and a cover $S \to X$. A map of animae is *finitely covered* if all its fibers are finitely covered.

For a map $f : X \to Y$, put $\Delta^0(f) = f$ and define its iterated relative diagonals recursively by

$$\Delta^{k+1}(f) = \Delta\big(\Delta^k(f)\big).$$

We say that f satisfies the *Kuratowski finiteness condition* if $\Delta^k(f)$ is finitely covered for every $k : \mathbb{N}$.

For a set S, being finitely covered is exactly being finitely enumerable. It is therefore weaker than finiteness unless equality in S is decidable. The same issue occurs at every stage of the iterated-diagonal condition: the fibers detect finite enumerability of component sets, but do not supply their decidable equality.

In the ordinary category of animae, classical logic erases this distinction. Anel proves that an anima has finitely many components and finite homotopy groups in every degree if and only if its terminal map satisfies the Kuratowski finiteness condition [Ane25, Proposition 2.1.1]. He identifies these animae with the coherent animae, and identifies the π-finite animae with the truncated coherent animae. Thus, in the classical model, one may recognize π-finite animae as the truncated animae whose terminal maps and all iterated diagonals are finitely covered.

Constructively, the recursive definition in Definition 11.6.6 is the stronger one and is the convention we adopt. Every recursively n-finite anima is n-truncated and satisfies the Kuratowski finiteness condition. A converse requires decidable equality for the component sets detected at the successive stages, or a classicality principle that supplies it. Moreover, in an arbitrary model of synthetic category theory the internally defined Kuratowski condition should not be identified with categorical coherence without additional hypotheses on the ambient model.

12 Yoneda and presheaf categories

The Yoneda lemma identifies morphisms out of a presheaf represented by an object with evaluation at that object; it follows that every locally small category embeds fully faithfully into its presheaf category. A complementary result is Yoneda density: every presheaf can canonically be written as a colimit of representable presheaves. These two results together make arbitrary presheaves accessible through representable ones.

To establish these results in our synthetic setup, we must first construct the opposite category and the Hom-functor. This leads to the twisted arrow category construction, which in turn is used to establish a form of the Yoneda lemma natural in both the representing object and the presheaf.

After having established Yoneda, we collect various important consequences. We first deduce that every category C is coherently equivalent to its double opposite $(C^{\mathrm{op}})^{\mathrm{op}}$. We then study hom animae of functor categories and the density presentation of a presheaf by its category of elements. Density leads to the free-cocompletion property of presheaf categories and to a version of Yoneda with coefficients. We conclude with small presheaves on categories which need not be small.

12.1 Opposite categories

In ordinary category theory, every category C gives rise to another category C^{op}, called the *opposite category* of C. While C^{op} has the same objects as C, the morphisms in C^{op} are reversed: a morphism from x to y in C^{op} is given by a morphism from y to x in C. Our goal in this section is to provide an analogous construction when C is a *synthetic* category.

Since our synthetic language does not offer a primitive operation that simply reverses all morphisms, we proceed by giving an explicit construction of C^{op} in terms of functors $C \to \mathrm{An}$ that are corepresented by objects of C. This construction has the additional advantage that the Hom-functor is built into the construction.

In order to motivate our definition of C^{op}, we briefly recall the situation in classical category theory. Since there categories are defined analytically, we may directly construct the category C^{op} by reversing all morphisms. We may further construct the *Yoneda embedding*

$$h_C \colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}), \qquad x \mapsto \mathrm{Hom}_C(-, x),$$

which is called an embedding since it is fully faithful. By applying this construction to C^{op} in place of C, we obtain a fully faithful functor

$$h_{C^{\mathrm{op}}} \colon C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{Set}).$$

The essential image consists of precisely those functors $F\colon C \to \mathrm{Set}$ that are *corepresentable*, meaning that they are naturally isomorphic to a functor of the form $\mathrm{Hom}_C(x,-)\colon C \to \mathrm{Set}$ for some object x of C.

For our synthetic definition of the opposite category C^{op} of C, we will turn this logic on its head and *define* C^{op} as the full subcategory of $\mathrm{Fun}(C, \mathrm{An})$ spanned by the corepresentable functors. Making this precise will take up the remainder of this section.

12.1.1 Corepresentable functors and the opposite category

Throughout this discussion, we fix a universe of categories Cat with associated universe of animae An, and construct the opposite of categories that are small with respect to this universe. We explain at the end why the resulting definition of C^{op} is independent of the chosen universe level, giving rise to a well-defined opposite for an arbitrary category C.

Construction 12.1.1 (Covariant Hom-functor)**.** Let C be a small category and let x be an object of C. The slice category $C_{x/}$ fits in a pullback square of the form

$$\begin{array}{ccc} C_{x/} & \longrightarrow & \mathrm{Fun}([1], C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ C & \xrightarrow{(x, \mathrm{id}_C)} & C \times C. \end{array}$$

The left vertical map is a small left fibration by Theorem 4.3.3, hence fits into a pullback square of the form

$$\begin{array}{ccc} C_{x/} & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C & \xrightarrow{\mathrm{Hom}_C(x,-)} & \mathrm{An}. \end{array}$$

This defines the functor $\mathrm{Hom}_C(x,-)\colon C \to \mathrm{An}$.

Definition 12.1.2 (Corepresentable functor)**.** We say that a functor $F\colon C \to \mathrm{An}$ is *corepresentable* if there exists an object x of C and an identification $\mathrm{Hom}_C(x,-) = F$.

A priori the condition of a functor to be corepresentable seems to involve a choice of object x of C. It turns out that the presentation of a presheaf by a corepresentable one is unique if it exists, so that corepresentability is really a *property* of a functor F. The following is the key statement we need for this:

Proposition 12.1.3. *For a functor $F\colon C \to \mathrm{An}$, let $p_F\colon \int F \to C$ denote the small cocartesian fibration classified by F. Then F is corepresentable if and only if $\int F$ admits an initial object.*

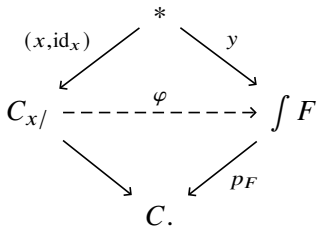
Proof. Let x be an object of C. Observe that there is an equivalence between the following three types of data:

(1) Objects y of the fiber of p_F over x.

(2) Functors $\varphi\colon C_{x/} \to \int F$ over C.

(3) Natural transformations $\mathrm{Hom}_C(x,-) \to F$ of functors $C \to \mathrm{An}$.

Directed univalence identifies the data in (2) and (3). For comparing (1) and (2), note that the slice $C_{x/}$ has an initial object (x, id_x), so that the functor $(x, \mathrm{id}_x) : * \to C_{x/}$ is initial by Proposition 9.1.13. It follows from Remark 9.1.5 that for every object y of $\int F$ satisfying $p_F(y) = x$ there exists a unique functor $\varphi : C_{x/} \to \int F$ fitting in the following commutative diagram:



We claim that the object y is initial in $\int F$ if and only if the corresponding natural transformation $\mathrm{Hom}_C(x,-) \to F$ is a natural isomorphism of functors $C \to \mathrm{An}$. Indeed, since directed univalence respects composition, the latter is equivalent to the condition that $\varphi : C_{x/} \to \int F$ is an equivalence. Since φ is a morphism between left fibrations over C, it is an equivalence if and only if it is an initial functor, see Lemma 9.1.6. By Proposition 9.1.8, this is in turn equivalent to $y : * \to \int F$ being an initial functor, which by Proposition 9.1.13 happens precisely when y is initial in $\int F$. □

As a consequence of Proposition 12.1.3, we expect to be able to form the full subcategory C^{op} of $\mathrm{Fun}(C, \mathrm{An})$ spanned by the corepresentable functors. In order to make this precise, we need to formulate the condition of being corepresentable in terms of a category. We start with a formulation of initiality.

Definition 12.1.4. For a small category C, we define

$$\texttt{hasInitial}(C) := \sum_{x : * \to C} \texttt{isEquiv}(C_{x/} \to C).$$

More precisely, we define this category via the following pullback square:

$$\begin{array}{ccc} \texttt{hasInitial}(C) & \longrightarrow & \mathrm{Iso}(\mathrm{Cat}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{Iso}}} \\ C^{\simeq} & \xrightarrow{x \mapsto (C_{x/} \to C)} & \mathrm{Fun}([1], \mathrm{Cat}), \end{array}$$

where the bottom functor is obtained by applying directed univalence to the cocartesian functor

$$\begin{array}{ccccc} \mathrm{Fun}([1], C) \times_{\mathrm{ev}_0, C} C^{\simeq} & \xrightarrow{(\mathrm{pr}_2, \mathrm{ev}_1 \circ \mathrm{pr}_1)} & C^{\simeq} \times C & & \\ & \searrow^{\mathrm{pr}_2} & & \swarrow_{\mathrm{pr}_1} & \\ & & C^{\simeq} & & \end{array}$$

over $C^{\simeq}$.

Lemma 12.1.5. *The category* $\texttt{hasInitial}(C)$ *is a proposition.*

Proof. Note that the type $\texttt{isEquiv}(C_{x/} \to C)$ is a proposition for every x. So we get an embedding $\texttt{hasInitial}(C) \to C^{\simeq}$, and thus we obtain a full subcategory $\langle \texttt{hasInitial}(C) \rangle \hookrightarrow C$ consisting of all initial objects of C. Notice that every morphism in this subcategory is automatically an isomorphism, hence it is actually equivalent to the anima $\texttt{hasInitial}(C)$. It remains to show that the functor $\texttt{hasInitial}(C) \to *$ is fully faithful, since all fully faithful functors are embeddings. The mapping animae on both sides are contractible by the defining property of initial objects. □

Construction 12.1.6. Let C be a small category. We will construct an anima $(C^{\mathrm{op}})^{\simeq}$ equipped with an embedding

$$(C^{\mathrm{op}})^{\simeq} \hookrightarrow \mathrm{Map}(C, \mathrm{An})$$

such that a functor $F\colon C \to \mathrm{An}$ is contained in $(C^{\mathrm{op}})^{\simeq}$ if and only if it is corepresentable, or equivalently if the total category $\int F$ of the left fibration $p_F\colon \int F \to C$ associated to F has an initial object.

Consider the anima $\Gamma := \mathrm{Map}(C, \mathrm{An})$. Then the inclusion $\Gamma \hookrightarrow \mathrm{Fun}(C, \mathrm{An})$ gives rise via currying to a functor $\Gamma \times C \to \mathrm{An}$, and hence to a left fibration $p\colon X \to \Gamma \times C$. Working in context Γ, we may then consider the statement $\texttt{hasInitial}(X)$ in context Γ, which by Lemma 12.1.5 defines a proposition in context Γ, which translated back to the absolute context corresponds to the desired embedding of animae of the form

$$(C^{\mathrm{op}})^{\simeq} \hookrightarrow \Gamma = \mathrm{Map}(C, \mathrm{An}).$$

For any point $\gamma \in \Gamma$, the fiber of this map is the proposition $\texttt{hasInitial}(X_\gamma)$, which is precisely what we wanted.

Definition 12.1.7 (Opposite category)**.** For a small category C, we define the category C^{op} as the full subcategory

$$C^{\mathrm{op}} \subseteq \mathrm{Fun}(C, \mathrm{An})$$

spanned by $(C^{\mathrm{op}})^{\simeq} \subseteq \mathrm{Map}(C, \mathrm{An})$, and refer to it as the *opposite category* of C. We will write

$$h_{C^{\mathrm{op}}}\colon C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{An})$$

for the inclusion functor called the *Yoneda embedding of the opposite*.

Remark 12.1.8. The definition of C^{op} does not depend on the universe for which C is small: if An' is a larger universe of animae, we will have a full embedding of An into An', hence a full embedding

$$\mathrm{Fun}(C, \mathrm{An}) \hookrightarrow \mathrm{Fun}(C, \mathrm{An}')\,,$$

and thus the definition of C^{op} as full subcategory of the domain or of the codomain will be canonically equivalent.

12.1.2 Basic properties

The opposite category of C is functorial in C:

Proposition 12.1.9. *Let $f\colon C \to D$ be a functor. Then the left Kan extension functor*

$$f_!\colon \mathrm{Fun}(C, \mathrm{An}) \to \mathrm{Fun}(D, \mathrm{An})$$

restricts to a functor

$$f^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}.$$

Proof. Consider the induced adjunction

$$f_! \colon \mathrm{Fun}(C, \mathrm{An}) \rightleftarrows \mathrm{Fun}(D, \mathrm{An}) : f^*.$$

For every object x of C, we obtain a commutative square

$$\begin{array}{ccc} C_{x/} & \xrightarrow{f_{x/}} & D_{f(x)/} \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D, \end{array}$$

and it follows from the diagram

$$\begin{array}{ccccc} & & * & & \\ & {\scriptstyle (x,\mathrm{id}_x)} \swarrow & & \searrow {\scriptstyle (f(x),\mathrm{id}_{f(x)})} & \\ C_{x/} & & \xrightarrow{f_{x/}} & & D_{f(x)/} \end{array}$$

that $f_{x/}$ is an initial functor, using the cancellation property (2) from Proposition 9.1.8. As a consequence, we obtain an identification

$$f_!(\mathrm{Hom}_C(x, -)) = \mathrm{Hom}_D(f(x), -),$$

and in particular $f_!(\mathrm{Hom}_C(x, -)) \colon D \to \mathrm{An}$ is corepresentable. □

Definition 12.1.10 (Hom-functor)**.** We define the *Hom-functor* of C

$$\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}$$

as the functor obtained by uncurrying the inclusion functor $h_{C^{\mathrm{op}}} \colon C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(C, \mathrm{An})$.

Lemma 12.1.11. *For a category C, there exists a preferred equivalence*

$$C^{\simeq} \xrightarrow{\sim} (C^{\mathrm{op}})^{\simeq}, \qquad x \mapsto \mathrm{Hom}_C(x, -).$$

Proof. By directed univalence, $(C^{\mathrm{op}})^{\simeq}$ may be described as the anima of small left fibrations $p \colon X \to C$ whose total category has an initial object. Since the category of initial objects is a proposition by Lemma 12.1.5, sending such a left fibration to the image in C of its initial object defines a canonical map

$$(C^{\mathrm{op}})^{\simeq} \to C^{\simeq}.$$

In the other direction, send an object x of C to the left fibration

$$p_x \colon C_{x/} \to C,$$

whose initial object is (x, id_x). The first composite is the identity. The other composite is the identity by Proposition 12.1.3, which identifies any small left fibration with an initial object over x with $p_x \colon C_{x/} \to C$. □

Lemma 12.1.12. *Let C be a category and let $x, y : C^{\simeq}$ be two objects. There is a preferred isomorphism*

$$C^{\mathrm{op}}(y,x) \xrightarrow{\sim} C(x,y)\,.$$

For every functor $f\colon C \to D$, the following square commutes:

$$\begin{array}{ccc} C^{\mathrm{op}}(y,x) & \xrightarrow{(f^{\mathrm{op}})_*} & D^{\mathrm{op}}(f(y),f(x)) \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ C(x,y) & \xrightarrow{f_*} & D(f(x),f(y)). \end{array}$$

Proof. By definition, there is a canonical isomorphism

$$C^{\mathrm{op}}(y,x) \xrightarrow{\sim} \mathrm{Map}_C(C_{y/}, C_{x/})\,.$$

On the other hand, since the fiber of the canonical projection $C_{x/} \to C$ at y is $C(x,y)$ by definition, Corollary 9.1.14 provides a canonical isomorphism

$$\mathrm{Map}_C(C_{y/}, C_{x/}) \xrightarrow{\sim} C(x,y)\,.$$

Composing these two isomorphisms gives the claimed identification, which sends a functor $\alpha\colon C_{y/} \to C_{x/}$ over C to its value at id_y.

To prove the compatibility, work in the anima context $C^{\mathrm{op}}(y,x)$ and let $\alpha\colon C_{y/} \to C_{x/}$ be the universal such functor. Applying the left Kan extension functor $f_!\colon \mathrm{Fun}(C,\mathrm{An}) \to \mathrm{Fun}(D,\mathrm{An})$ gives a map which under the identifications from Proposition 12.1.9 corresponds to the functor of left fibrations $f_!\alpha\colon D_{f(y)/} \to D_{f(x)/}$. The functors $f_{x/}$ and $f_{y/}$ in that proposition represent the Kan-extension units, so naturality of the unit gives a commutative square

$$\begin{array}{ccc} C_{y/} & \xrightarrow{\alpha} & C_{x/} \\ {\scriptstyle f_{y/}}\downarrow & & \downarrow{\scriptstyle f_{x/}} \\ D_{f(y)/} & \xrightarrow{f_!\alpha} & D_{f(x)/}. \end{array}$$

This square is compatible with the projections to D. Evaluating it at id_y and using $f_{y/}(\mathrm{id}_y) = \mathrm{id}_{f(y)}$ gives

$$(f_!\alpha)(\mathrm{id}_{f(y)}) \simeq f\big(\alpha(\mathrm{id}_y)\big)$$

in $D(f(x),f(y))$. Since f^{op} is the restriction of $f_!$, the two sides are precisely the two composites in the asserted square, evaluated on its universal term. This proves that the square commutes. □

Proposition 12.1.13. *For every small category C, the opposite category C^{op} is small.*

Proof. It suffices to check that $\mathrm{Map}([1], C^{\mathrm{op}})$ is small (see Definition 8.4.5). We may write this as

$$\mathrm{Map}([1], C^{\mathrm{op}}) \xrightarrow{\sim} \sum_{(x,y)\colon (C^{\mathrm{op}})^{\simeq}\times(C^{\mathrm{op}})^{\simeq}} C^{\mathrm{op}}(x,y).$$

Since $C^{\simeq}$ is small, so is $(C^{\mathrm{op}})^{\simeq}$ by Lemma 12.1.11. Therefore, it suffices to prove that each $C^{\mathrm{op}}(x,y)$ is small. But we have seen in Lemma 12.1.12 that $C^{\mathrm{op}}(x,y) \xrightarrow{\sim} C(y,x)$, and since $C(y,x)$ is small for all x and y, this completes the proof. □

Lemma 12.1.14. *If C is a category and $x : C^{\simeq}$ a terminal (initial) object of C, then interpreting x as an object of C^{op} by Lemma 12.1.11, it becomes an initial (terminal) object in the category C^{op}.*

Proof. If x is a terminal in C, then $C(y, x)$ is contractible for all $y : C^{\simeq}$. Therefore, by virtue of the preceding lemma, we see that $C^{\mathrm{op}}(x, y)$ is contractible for all $y : C^{\mathrm{op}}$, which proves that x is initial in C^{op}. The case of an initial object is done similarly. □

Lemma 12.1.15. *If $p\colon X \to C$ is a right (left) fibration, then $p^{\mathrm{op}}\colon X^{\mathrm{op}} \to C^{\mathrm{op}}$ is a left (right) fibration.*

Proof. Let us assume that p is a right fibration. We have to check that the square

$$\begin{array}{ccc} \mathrm{Map}([1], X^{\mathrm{op}}) & \xrightarrow{(p^{\mathrm{op}})_*} & \mathrm{Map}([1], C^{\mathrm{op}}) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ (X^{\mathrm{op}})^{\simeq} & \xrightarrow{(p^{\mathrm{op}})^{\simeq}} & (C^{\mathrm{op}})^{\simeq} \end{array}$$

is a pullback square. Identifying $C^{\simeq}$ and $(C^{\mathrm{op}})^{\simeq}$, this square is a diagram of the form below.

$$\begin{array}{ccc} \sum_{x,y:X^{\simeq}} X^{\mathrm{op}}(y, x) & \longrightarrow & \sum_{x,y:C^{\simeq}} C^{\mathrm{op}}(y, x) \\ \downarrow & & \downarrow \\ X^{\simeq} & \xrightarrow{p^{\simeq}} & C^{\simeq} \end{array}$$

By virtue of Lemma 12.1.12, this square is equivalent to

$$\begin{array}{ccc} \sum_{x,y:X^{\simeq}} X(x, y) & \longrightarrow & \sum_{x,y:C^{\simeq}} C(x, y) \\ \downarrow & & \downarrow \\ X^{\simeq} & \xrightarrow{p^{\simeq}} & C^{\simeq} \end{array}$$

which in turn is equivalent to the commutative square

$$\begin{array}{ccc} \mathrm{Map}([1], X) & \xrightarrow{p_*} & \mathrm{Map}([1], C) \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ X^{\simeq} & \xrightarrow{p^{\simeq}} & C^{\simeq} \end{array}$$

and the latter is a pullback square, since p is a right fibration.
The dual case is proven exactly in the same way, simply by exchanging the roles of the functor ev_0 and ev_1. □

Proposition 12.1.16. *If C is a category and x an object of C, there are canonical equivalences*

$$(C^{\mathrm{op}})_{x/} \xrightarrow{\sim} (C_{/x})^{\mathrm{op}} \quad \textit{and} \quad (C^{\mathrm{op}})_{/x} \xrightarrow{\sim} (C_{x/})^{\mathrm{op}}\,.$$

Proof. By virtue of Lemma 12.1.14, the object

$$(x, \mathrm{id}_x) \in ((C_{/x})^{\mathrm{op}})^{\simeq}$$

defines an initial object in $(C_{/x})^{\mathrm{op}}$. On the other hand, Lemma 12.1.15 ensures that the right fibration $C_{/x} \to C$ induces a left fibration of the form

$$(C_{/x})^{\mathrm{op}} \to C^{\mathrm{op}}\,.$$

The latter sends the initial object of $(C_{/x})^{\mathrm{op}}$ to x. Proposition 12.1.3 therefore gives a canonical equivalence from $(C^{\mathrm{op}})_{x/}$ to $(C_{/x})^{\mathrm{op}}$ over C^{op}. The second equivalence is proved similarly. □

Remark 12.1.17. The identifications of the previous proposition are functorial in x. Indeed, given any map $u\colon x \to y$ in C, the functor

$$C_{/x} \to C_{/y}$$

defined by composition with u is the unique functor over C that sends the terminal object of $C_{/x}$ to the object determined by the pair (x, u). Therefore, the functor

$$(C_{/x})^{\mathrm{op}} \to (C_{/y})^{\mathrm{op}}$$

is the unique functor over C^{op} sending the initial object to the object defined as the pair (x, u). A similar description is true of the functor

$$(C^{\mathrm{op}})_{x/} \to (C^{\mathrm{op}})_{y/}$$

induced by the map from y to x in C^{op} corresponding to u. The first equivalence of Proposition 12.1.16 preserves initial objects (since all equivalences have this property). This shows that post-composing with u in C corresponds through the equivalence of Lemma 12.1.12 to pre-composing with u in C^{op}. Similarly, the second equivalence of Proposition 12.1.16 shows that that pre-composing with u in C corresponds through the equivalence of Lemma 12.1.12 to post-composing with u in C^{op}.

Proposition 12.1.18. *A functor $f\colon C \to D$ has a right (left) adjoint if and only if the functor $f^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ has a right (left) adjoint.*

Proof. This follows straightaway from Proposition 6.2.3 and Lemma 12.1.12, using the conclusion of Remark 12.1.17. □

Proposition 12.1.19. *Let $f\colon C \to D$ be a functor between locally small categories. The existence of a right adjoint to f is equivalent to the existence, for each object y of D, of an object $g(y)$ of C and of a map $\varepsilon\colon f(g(y)) \to y$ such that the induced map*

$$\mathrm{Hom}_C(x, g(y)) \xrightarrow{\text{apply } f} \mathrm{Hom}_D(f(x), f(g(y))) \xrightarrow{\text{compose with } \varepsilon} \mathrm{Hom}_D(f(x), y)$$

is an isomorphism of animae.
Dually, the existence of a left adjoint to f is equivalent to the existence, for each object y of D, of an object $g(y)$ of C and of a map $\eta\colon y \to f(g(y))$ such that the induced map

$$\mathrm{Hom}_C(g(y), x) \xrightarrow{\text{apply } f} \mathrm{Hom}_D(f(g(y)), f(x)) \xrightarrow{\text{compose with } \eta} \mathrm{Hom}_D(y, f(x))$$

is an isomorphism of animae.

Proof. We prove only the first assertion (the second one follows from the first by duality, thanks to Remark 12.1.17 and Proposition 12.1.18).
By directed univalence, for any objects a and b of a locally small category X, and for any small anima K corresponding to an object $k : \mathrm{An}^{\simeq}$, there is a canonical equivalence of the form

$$\mathrm{Map}(K, X(a,b)) \xrightarrow{\sim} \mathrm{An}(k, \mathrm{Hom}_X(a,b)) .$$

This proposition is thus a translation of Proposition 6.2.3, using Remark 12.1.17. □

12.1.3 Functoriality in the universe

The preceding construction assigns a functor f^{op} to each functor f. We now assemble these assignments coherently into a functor on the universe of small categories. We fix a universe of small categories $\pi_{\mathrm{univ}} : \mathrm{Cat}_\bullet \to \mathrm{Cat}$, and assume that Cat is small with respect to some larger universe Cat'.

Construction 12.1.20. The functor $\mathrm{ev}_1 : \mathrm{Fun}([1], \mathrm{Cat}) \to \mathrm{Cat}$ is a Cat'-small cocartesian fibration, whose straightening is by definition the functor

$$\mathrm{Cat} \to \mathrm{Cat}', \qquad S \mapsto \mathrm{Cat}_{/S}.$$

Given any functor between small categories $u : S \to T$, the induced functor

$$u_! : \mathrm{Cat}_{/S} \to \mathrm{Cat}_{/T}$$

is induced by composition with u, and since Cat has pullbacks, it has a right adjoint

$$u^* : \mathrm{Cat}_{/T} \to \mathrm{Cat}_{/S}$$

assigning to each functor $p : Y \to T$ the first projection $S \times_T Y \to S$. Therefore, the functor ev_1 above is a cartesian fibration as well (see Proposition 6.5.11).

Let LFib be the full subcategory of $\mathrm{Fun}([1], \mathrm{Cat})$ whose objects are the left fibrations between small categories. Since left fibrations are closed under pullbacks, any cartesian morphism in $\mathrm{Fun}([1], \mathrm{Cat})$ whose target lies in LFib lies entirely in LFib. It follows that the evaluation map

$$\mathrm{ev}_1 : \mathrm{LFib} \to \mathrm{Cat}$$

is a cartesian fibration, and that the embedding $\mathrm{LFib} \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat})$ is a cartesian functor over Cat. On the other hand, it follows from Theorem 9.3.1, combined with the straightening equivalence $\mathrm{LFib}(S) \simeq \mathrm{Fun}(S, \mathrm{An})$, that each pullback functor $u^* : \mathrm{LFib}(T) \to \mathrm{LFib}(S)$ admits a left adjoint

$$u_! : \mathrm{LFib}(S) \to \mathrm{LFib}(T),$$

so $\mathrm{ev}_1 : \mathrm{LFib} \to \mathrm{Cat}$ is also a cocartesian fibration.

Let CoRep be the full subcategory of LFib whose objects are the left fibrations between small categories whose domain has an initial object. By Proposition 12.1.9, the functor $u_!$ preserves this property for each u. We deduce that

$$q := \mathrm{ev}_1 : \mathrm{CoRep} \longrightarrow \mathrm{Cat}$$

is a cocartesian fibration and the full embedding

$$j\colon \mathrm{CoRep} \hookrightarrow \mathrm{LFib}$$

is a cocartesian functor over Cat.

Proposition 12.1.21 (Functoriality of opposites and co-Yoneda)**.** *The assignment $C \mapsto C^{\mathrm{op}}$ extends canonically to a functor*

$$(-)^{\mathrm{op}}\colon \mathrm{Cat} \longrightarrow \mathrm{Cat}.$$

Moreover, the co-Yoneda embeddings are the fibers of a cocartesian functor

$$j\colon \mathrm{CoRep} \longrightarrow \mathrm{LFib}$$

over Cat*.*

Proof. The functor $(-)^{\mathrm{op}}$ is classified by the cocartesian fibration $q\colon \mathrm{CoRep} \to \mathrm{Cat}$. Equivalently, it is characterized by the pullback square

$$\begin{array}{ccc} \mathrm{CoRep} & \longrightarrow & \mathrm{Cat}_{\bullet} \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{(-)^{\mathrm{op}}} & \mathrm{Cat}. \end{array}$$

The cocartesian functor $j\colon \mathrm{CoRep} \hookrightarrow \mathrm{LFib}$ over Cat has already been constructed. On the fiber over C, straightening identifies it with the co-Yoneda embedding $h_{C^{\mathrm{op}}}\colon C^{\mathrm{op}} \to \mathrm{Fun}(C, \mathrm{An})$. □

Definition 12.1.22. The functor obtained in Proposition 12.1.21 is called the *opposite category functor*.

12.2 The twisted arrow category

The twisted arrow category is the unstraightening of the Hom-functor. Informally, its objects are morphisms $x \to y$ of C, and a morphism between two such objects is a twisted commutative square: it is contravariant in the source and covariant in the target. The projection $\mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ records these two endpoints. Besides packaging all morphisms of C into a single category, this construction will control the naturality of the Yoneda isomorphism and later provide the indexing category for ends and density formulas.

12.2.1 Definition and fibers

Definition 12.2.1 (Twisted arrow category)**.** For a small category C, we define the left fibration $\mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ as the one classified by the Hom-functor:

$$\begin{array}{ccc} \mathrm{Tw}(C) & \longrightarrow & \mathrm{An}_{\bullet} \\ {\scriptstyle (s,t)}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C^{\mathrm{op}} \times C & \xrightarrow{\mathrm{Hom}_C} & \mathrm{An}. \end{array}$$

We refer to its total category $\mathrm{Tw}(C)$ as the *twisted arrow category* of C. We denote the two projections to C^{op} and C by

$$s\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \qquad \text{and} \qquad t\colon \mathrm{Tw}(C) \to C.$$

Lemma 12.2.2. *If C is a small category, so is* $\mathrm{Tw}(C)$.

Proof. The left fibration $\mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ has small fibers, and its base is small by Proposition 12.1.13. Hence its total category is small. □

Proposition 12.2.3 (Functoriality of twisted arrow categories)**.** *There is a functor*

$$\mathrm{Tw}\colon \mathrm{Cat} \longrightarrow \mathrm{Cat}$$

and a natural transformation

$$(s,t)\colon \mathrm{Tw} \longrightarrow (-)^{\mathrm{op}} \times \mathrm{id}_{\mathrm{Cat}}$$

whose component at a category C is the left fibration $(s,t)\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ of Definition 12.2.1.

Proof. Recall from Construction 12.1.20 the subcategories $\mathrm{CoRep} \hookrightarrow \mathrm{LFib} \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat})$. We may think of the inclusion $\mathrm{CoRep} \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat})$ as a natural transformation from the source evaluation $\mathrm{ev}_0\colon \mathrm{CoRep} \to \mathrm{Cat}$ to the target evaluation $\mathrm{ev}_1\colon \mathrm{CoRep} \to \mathrm{Cat}$. Unstraightening gives a cocartesian functor of the form bellow.

$$\begin{array}{ccc} \displaystyle\int_{(X \to C)\colon \mathrm{CoRep}} X & \longrightarrow & \displaystyle\int_{(X \to C)\colon \mathrm{CoRep}} C \\ & \searrow \quad \swarrow & \\ & \mathrm{CoRep} & \end{array}$$

Here the two integrals denote the unstraightenings of ev_0 and ev_1, respectively. It follows from Proposition 12.1.21 that the functor $\mathrm{ev}_1\colon \mathrm{CoRep} \to \mathrm{Cat}$ is a small cocartesian fibration. By regularity of the universe, we may then compose the two small cocartesian fibrations in the previous diagram with ev_1 to give two small cocartesian fibrations over Cat. The displayed functor is then also cocartesian over Cat: a cocartesian lift for either composite is obtained by first lifting in CoRep and then in its unstraightening, and the displayed functor preserves these lifts.

The right-hand total category is the pullback $\mathrm{CoRep} \times_{\mathrm{Cat}} \mathrm{Cat}_{\bullet}$, formed using ev_1 and π_{univ}. Its projection to Cat is therefore the product of the cocartesian fibrations classifying $(-)^{\mathrm{op}}$ and $\mathrm{id}_{\mathrm{Cat}}$. Straightening the displayed functor over Cat consequently gives a natural transformation

$$(s,t)\colon \mathrm{Tw} \longrightarrow (-)^{\mathrm{op}} \times \mathrm{id}_{\mathrm{Cat}},$$

where $\mathrm{Tw}\colon \mathrm{Cat} \to \mathrm{Cat}$ classifies the left-hand composite fibration.

It remains to identify its component at C. The fiber of $\mathrm{CoRep} \to \mathrm{Cat}$ over C is C^{op}, and the inclusion into $\mathrm{LFib}(C)$ is the co-Yoneda embedding by Proposition 12.1.21. Thus the component is

$$\int_{x\colon C^{\mathrm{op}}} C_{x/} \longrightarrow C^{\mathrm{op}} \times C.$$

This is a cocartesian functor over C^{op} whose fiber at x is the left fibration $C_{x/} \to C$. Transport in C^{op} gives functors between these left fibrations over C, which preserve cocartesian morphisms. Hence Proposition 6.5.15 shows that the displayed functor is a cocartesian fibration. Its fibers are animae, so it is a left fibration, classified by the uncurrying of $h_{C^{\mathrm{op}}}$, namely Hom_C. It therefore agrees with the defining left fibration $\mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$.

For a functor $f\colon C \to D$, cocartesian transport in CoRep is given by left Kan extension. Its unit on a corepresentable corresponds to the canonical functor $C_{x/} \to D_{f(x)/}$ from Proposition 12.1.9. Consequently, $\mathrm{Tw}(f)$ restricts to this functor on the source fibers and sends id_x to $\mathrm{id}_{f(x)}$. □

Proposition 12.2.4. *The fibers of both projections $s\colon \mathrm{Tw}(C) \to C^{\mathrm{op}}$ and $t\colon \mathrm{Tw}(C) \to C$ have initial objects.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc} C^{\simeq} \times C & & \xrightarrow{\quad\quad\quad\quad\quad\quad} & & \\ {\scriptstyle\simeq}\downarrow & & & \searrow^{\sum_{x:C^{\simeq}} \mathrm{Hom}_C(x,-)} & \\ (C^{\mathrm{op}})^{\simeq} \times C & \hookrightarrow & C^{\mathrm{op}} \times C & \xrightarrow[\mathrm{Hom}_C]{} & \mathrm{An}. \end{array}$$

So we obtain pullback squares

$$\begin{array}{ccc} \sum_{x:C^{\simeq}} C_{x/} & \longrightarrow & \mathrm{Tw}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ C^{\simeq} \times C & \longrightarrow & C^{\mathrm{op}} \times C \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_1} \\ C^{\simeq} \simeq (C^{\mathrm{op}})^{\simeq} & \hookrightarrow & C^{\mathrm{op}}. \end{array}$$

In particular, the fiber of s over an object $x \in C^{\mathrm{op}}$ is the slice category $C_{x/}$, which has an initial object (x, id_x).

Now consider an object y of C and let $\mathrm{Tw}(C)_y$ denote the fiber of $t\colon \mathrm{Tw}(C) \to C$ at y:

$$\begin{array}{ccc} \mathrm{Tw}(C)_y & \longrightarrow & \mathrm{Tw}(C) \\ {\scriptstyle p_y}\downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ C^{\mathrm{op}} & \xrightarrow{(\mathrm{id},y)} & C^{\mathrm{op}} \times C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_2} \\ * & \xrightarrow{\quad y \quad} & C. \end{array}$$

Note that p_y is a left fibration whose fiber at y is given by $C^{\mathrm{op}}(y,y) \xrightarrow{\sim} C(y,y)$, hence we obtain an object (y, id_y) in this fiber, producing an object $\mathrm{id}_y \in \mathrm{Tw}(C)_y$. We then obtain a unique map

$$\begin{array}{ccccc} & & * & & \\ & \swarrow^{(y,\mathrm{id}_y)} & & \searrow^{\mathrm{id}_y} & \\ C^{\mathrm{op}}_{y/} & & \dashrightarrow & & \mathrm{Tw}(C)_y \\ & \searrow & & \swarrow_{p_y} & \\ & & C^{\mathrm{op}}. & & \end{array}$$

It will thus suffice to show that for every object x of C^{op} (i.e. every object x of C) the dashed map induces an equivalence

$$C^{\mathrm{op}}(y,x) \xrightarrow{\sim} C(x,y).$$

Under the isomorphism from Lemma 12.1.12, we have to check in fact that the dashed map induces the identity of $C(x,y)$. A term f in $C(x,y)$ corresponds to a functor

$$f\colon [1] \to C\,.$$

By functoriality of the formation of the twisted arrow category (Proposition 12.2.3), we obtain a commutative square below.

$$\begin{array}{ccc} \mathrm{Tw}([1]) & \xrightarrow{\mathrm{Tw}(f)} & \mathrm{Tw}(C) \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ [1]^{\mathrm{op}} \times [1] & \xrightarrow{f^{\mathrm{op}} \times f} & C^{\mathrm{op}} \times C \end{array}$$

Since f sends the ordered pair $0, 1$ to x, y, this induces a commutative diagram of the form

$$\begin{array}{ccccc} [1](0,1) & \xrightarrow{\sim} & [1]^{\mathrm{op}}(1,0) & \dashrightarrow & [1](0,1) \\ \downarrow & & \downarrow & & \downarrow \\ C(x,y) & \xrightarrow{\sim} & C^{\mathrm{op}}(y,x) & \dashrightarrow & C(x,y) \end{array}$$

in which the vertical maps are induced by f^{op} and f. All the maps of the upper horizontal line are isomorphisms because they are maps between terminal objects (expressing that 1 is terminal in $[1]$, say). Interpreting this proof in context $C(x,y)$ for f the universal term of $C(x,y)$, this proves that the dashed map is indeed fiber-wise the identity. □

Remark 12.2.5. Given any object x of C seen as an object of C^{op}, the proof of the preceding proposition shows that the fiber of the target functor

$$t\colon \mathrm{Tw}(C) \to C$$

at x is equivalent to $(C^{\mathrm{op}})_{x/}$. Using Proposition 12.1.16, this means that we have a canonical isomorphism over C^{op} of the form

$$(C_{/x})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Tw}(C)_x\,.$$

Corollary 12.2.6. *The maps s and t induce equivalences*

$$\Pi_\infty(C^{\mathrm{op}}) \xleftarrow{\Pi_\infty(s)} \Pi_\infty(\mathrm{Tw}(C)) \xrightarrow{\Pi_\infty(t)} \Pi_\infty(C).$$

Proof. Both s and t are cocartesian fibrations: they are the Grothendieck constructions of the functors $x \mapsto C_{x/}$ on C^{op} and $y \mapsto (C_{/y})^{\mathrm{op}}$ on C, respectively. Consequently they are proper. For either projection f, the inclusion of the fiber $X_y \to X_{/y}$ is terminal. By Proposition 12.2.4, the fiber has an initial object and is therefore weakly contractible. Thus every relative slice $X_{/y}$ is weakly contractible, and Corollary 9.5.8 implies that $\Pi_\infty(f)$ is an equivalence. □

12.2.2 Relative twisted-arrow formulas

Let $u\colon C \to D$ be a functor between small categories. We will compare the Hom-functors of C and D: left Kan extension of Hom_C along u in the second variable gives the functor $(c,d) \mapsto \mathrm{Hom}_D(u(c),d)$. The analogous extension in the first variable gives $(d,c) \mapsto \mathrm{Hom}_D(d,u(c))$. We prove these formulas by showing that the corresponding functors between twisted arrow categories are initial.

Lemma 12.2.7. *Consider the functor $\tilde{u} = ((s,t), \mathrm{Tw}(u))$ defined by*

$$\begin{array}{ccccc} \mathrm{Tw}(C) & \xrightarrow{\tilde{u}} & C^{\mathrm{op}} \times D \times_{D^{\mathrm{op}}\times D} \mathrm{Tw}(D) & \longrightarrow & \mathrm{Tw}(D) \\ {\scriptstyle (s,t)}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ C^{\mathrm{op}} \times C & \xrightarrow[\mathrm{id}_{C^{\mathrm{op}}} \times u]{} & C^{\mathrm{op}} \times D & \xrightarrow[u^{\mathrm{op}}\times \mathrm{id}_D]{} & D^{\mathrm{op}} \times D. \end{array}$$

Then the functor $\tilde{u}$ is initial.

Proof. By virtue of Lemma 9.6.8, it suffices to show that $\tilde{u}$ induces an initial functor fiberwise over each object of C^{op}. Let x be an object of C (and thus an object of C^{op}). The functor induced over x fits into a diagram of the form

$$\begin{array}{ccc} & * & \\ {\scriptstyle (x,\mathrm{id}_x)}\swarrow & & \searrow{\scriptstyle (u(x),\mathrm{id}_{u(x)})} \\ C_{x/} & \longrightarrow & D_{u(x)/}, \end{array}$$

and thus by the cancellation property it follows that the bottom map is initial, as desired. □

Proposition 12.2.8. *Let $u\colon C \to D$ be a functor between small categories. The adjunction*

$$(\mathrm{id}_{C^{\mathrm{op}}} \times u)_!\colon \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An}) \rightleftarrows \mathrm{Fun}(C^{\mathrm{op}} \times D, \mathrm{An}) : (\mathrm{id}_{C^{\mathrm{op}}} \times u)^*.$$

induces a canonical isomorphism

$$(\mathrm{id}_{C^{\mathrm{op}}} \times u)_!(\mathrm{Hom}_C) \xrightarrow{\sim} (u^{\mathrm{op}} \times \mathrm{id}_D)^*(\mathrm{Hom}_D)$$

in $\mathrm{Fun}(C^{\mathrm{op}} \times D, \mathrm{An})$.

Proof. This is a direct interpretation of the previous lemma in view of the explicit description of the functor $(\mathrm{id}_{C^{\mathrm{op}}} \times u)_!$ given by Theorem 9.3.1. □

Lemma 12.2.9. *Consider the functor $\bar{u} = ((s,t), \mathrm{Tw}(u))$ defined by*

$$\begin{array}{ccccc} \mathrm{Tw}(C) & \xrightarrow{\bar{u}} & D^{\mathrm{op}} \times C \times_{D^{\mathrm{op}}\times D} \mathrm{Tw}(D) & \longrightarrow & \mathrm{Tw}(D) \\ {\scriptstyle (s,t)}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ C^{\mathrm{op}} \times C & \xrightarrow[u^{\mathrm{op}}\times \mathrm{id}_C]{} & D^{\mathrm{op}} \times C & \xrightarrow[\mathrm{id}_{D^{\mathrm{op}}} \times u]{} & D^{\mathrm{op}} \times D. \end{array}$$

Then the functor $\bar{u}$ is initial.

Proof. As in the proof of Lemma 12.2.7, it suffices to show that $\bar{u}$ induces an initial functor fiberwise over each object of C. Let x be an object of C. By Remark 12.2.5, the functor induced on fibers over x fits into a diagram of the form

$$\begin{array}{ccc} & * & \\ {\scriptstyle (x,\mathrm{id}_x)}\swarrow & & \searrow{\scriptstyle (u(x),\mathrm{id}_{u(x)})} \\ (C^{\mathrm{op}})_{x/} & \dashrightarrow & (D^{\mathrm{op}})_{u(x)/}, \end{array}$$

and thus, by the cancellation property, is initial. □

Proposition 12.2.10. *Let $u\colon C \to D$ be a functor between small categories. The adjunction*

$$(u^{\mathrm{op}} \times \mathrm{id}_C)_!\colon \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An}) \rightleftarrows \mathrm{Fun}(D^{\mathrm{op}} \times C, \mathrm{An}) : (u^{\mathrm{op}} \times \mathrm{id}_C)^*.$$

induces a canonical isomorphism

$$(u^{\mathrm{op}} \times \mathrm{id}_C)_!(\mathrm{Hom}_C) \xrightarrow{\sim} (\mathrm{id}_{D^{\mathrm{op}}} \times u)^*(\mathrm{Hom}_D)$$

in $\mathrm{Fun}(D^{\mathrm{op}} \times C, \mathrm{An})$.

Proof. This is a reformulation of Lemma 12.2.9. □

12.2.3 Fully faithful functors

In passing, we record a criterion of fully faithful functors in terms of the twisted arrow category.

Proposition 12.2.11. *For a functor $u\colon C \to D$ between small categories, the following conditions are equivalent:*

(1) The functor u is fully faithful.

(2) The functor $u_!\colon \mathrm{Fun}(C, \mathrm{An}) \to \mathrm{Fun}(D, \mathrm{An})$ is fully faithful.

(3) The commutative square

$$\begin{array}{ccc} \mathrm{Tw}(C) & \overset{\mathrm{Tw}(u)}{\dashrightarrow} & \mathrm{Tw}(D) \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ C^{\mathrm{op}} \times C & \xrightarrow{u^{\mathrm{op}} \times u} & D^{\mathrm{op}} \times D \end{array}$$

is a pullback square.

(4) The functor $u^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is fully faithful.

Proof. The equivalence between (1) and (3) follows because both conditions ask the induced map $C(x, y) \to D(u(x), u(y))$ to be an equivalence for all objects x and y of C. The equivalence between (1) and (4) is Lemma 12.1.12, applied to all pairs of objects. The fact that (4) follows from (2) holds by definition of u^{op}. It remains to show that (1) implies (2).

Let $p\colon X \to C$ be a small left fibration, classified by a functor $F\colon C \to \mathrm{An}$. We may factor $u \circ p$ as

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{u} & D, \end{array}$$

where j is initial and q is a left fibration. Then q is a small left fibration and thus corresponds to a functor $u_!(F)\colon D \to \mathrm{An}$. The natural transformation $F \to u^* u_! F$ corresponds to the functor $(p, j)\colon X \to C \times_D Y$, and we have to show this is an equivalence. Since it is a morphism of left fibrations over C, we may equivalently show that for every object x of C, the induced map

$$X_x \to Y_{u(x)}$$

is an equivalence. Now, this map fits into the following commutative square

$$\begin{array}{ccc} X_x & \longrightarrow & Y_{u(x)} \\ \downarrow & & \downarrow \\ X_{/x} & \xrightarrow{u_{/x}} & Y_{/u(x)}, \end{array}$$

and since the vertical functors are both terminal (because left fibrations are proper, by Corollary 9.4.9), we may equivalently show that the functor $u_{/x}$ is a weak homotopy equivalence. Since u is fully faithful, we have a pullback square

$$\begin{array}{ccc} C_{/x} & \longrightarrow & D_{/u(x)} \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow{u} & D, \end{array}$$

and hence an equivalence $C_{/x} \xrightarrow{\sim} C \times_D D_{/u(x)}$. It follows that we also have an equivalence $X_{/x} \xrightarrow{\sim} X \times_D D_{/u(x)}$, and so it will suffice to show that the map $j_{/u(x)}\colon X \times_D D_{/u(x)} \to Y_{/u(x)}$ is a weak equivalence. But since this functor is a pullback of j along the smooth functor $D_{/u(x)} \to D$, $j_{/u(x)}$ is an initial functor, and hence in particular a weak equivalence. □

Corollary 12.2.12. *Consider the pushout square of categories below.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array}$$

If f is fully faithful, so is g.

Proof. We may assume that this is a commutative square of small categories (up to a choice of suitable universe). Let us assume that f is fully faithful. By virtue of Proposition 12.2.11 the functor

$$f_!\colon \mathrm{Fun}(A, \mathrm{An}) \to \mathrm{Fun}(B, \mathrm{An})$$

is fully faithful. In other words, if we consider the induced pullback square

$$\begin{array}{ccc} \mathrm{Fun}(D, \mathrm{An}) & \xrightarrow{v^*} & \mathrm{Fun}(B, \mathrm{An}) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Fun}(C, \mathrm{An}) & \xrightarrow{u^*} & \mathrm{Fun}(A, \mathrm{An}) \end{array}$$

the functor $f_!$ is a left adjoint section of f^*. Proposition 4.4.15 thus tells us that the functor g^* has a left adjoint section, i.e. the functor $g_!$ is fully faithful. Another application of Proposition 12.2.11 implies that the functor g itself is fully faithful. □

12.3 The Yoneda lemma

The goal of this section is to prove the Yoneda lemma. Let us start by introducing the key player:

Definition 12.3.1 (Presheaves)**.** A *presheaf* on a category C is a functor $F\colon C^{\mathrm{op}} \to \mathrm{An}$. We write

$$\mathrm{PSh}(C) \quad := \quad \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$$

for the category of presheaves on C.

Definition 12.3.2. A category C is *locally small* if the left fibration

$$(s,t)\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$$

has small fibers. In this case it classifies a functor $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$, whose transpose

$$h_C\colon C \to \mathrm{PSh}(C), \qquad y \mapsto \mathrm{Hom}_C(-, y),$$

is called the *Yoneda functor*. A presheaf F is *representable* if there is an object y of C and an isomorphism $h_C(y) \xrightarrow{\sim} F$.

The Yoneda lemma will identify morphisms from $h_C(x)$ to a presheaf F with objects of $F(x)$. We first fix x and prove that this identification is functorial in F. We then use twisted arrow categories to let x vary as well, and prove that h_C is fully faithful. This justifies calling it the *Yoneda embedding*. In the last subsection we assemble these results into an identification of functors on $C^{\mathrm{op}} \times \mathrm{PSh}(C)$.

12.3.1 Evaluation at a fixed object

For a fixed object x of C, evaluation at id_x should identify a morphism $h_C(x) \to F$ with an object of $F(x)$. We prove this first in the covariant form, for morphisms $\mathrm{Hom}_C(x, -) \to F$ with $F\colon C \to \mathrm{An}$. The contravariant statement follows by applying it to C^{op}.

Under straightening, a natural transformation $\Phi \to \Psi$ of functors $C \to \mathrm{An}$ corresponds to a section of the left fibration classified by Ψ after pulling back along $\int \Phi \to C$. The following theorem expresses this correspondence as a pullback square, so that it also describes how these sections vary with Ψ. For $\Phi = \mathrm{Hom}_C(x, -)$, the category $\int \Phi = C_{x/}$ has initial object id_x; evaluation there will give the desired Yoneda identification.

Theorem 12.3.3 (Parametrized evaluation)**.** *Consider a category C equipped with a functor $\Phi\colon C \to \mathrm{An}$, and let $q\colon \int \Phi \to C$ denote the associated left fibration:*

$$\begin{array}{ccc} \int \Phi & \xrightarrow{\Phi_\bullet} & \mathrm{An}_\bullet \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C & \xrightarrow{\Phi} & \mathrm{An}. \end{array}$$

Then there exists a canonical pullback square of the form

$$\begin{array}{ccc} \mathrm{Fun}(C, \mathrm{An})_{\Phi/} & \longrightarrow & \mathrm{Fun}(\int \Phi, \mathrm{An}_\bullet) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_{\mathrm{univ}})_*} \\ \mathrm{Fun}(C, \mathrm{An}) & \xrightarrow[q^*]{} & \mathrm{Fun}(\int \Phi, \mathrm{An}). \end{array}$$

Proof. Both sides of the asserted square are left fibrations over $\mathrm{Fun}(C, \mathrm{An})$. We first construct a comparison between them and then prove that it is an equivalence by identifying its fiber over each functor $\Psi\colon C \to \mathrm{An}$. The final part of the proof checks that the resulting fiberwise equivalence is the canonical one supplied by straightening.

Consider the following pullback square:

$$\begin{array}{ccc} W & \longrightarrow & \mathrm{Fun}(\int \Phi, \mathrm{An}_\bullet) \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_{\mathrm{univ}})_*} \\ \mathrm{Fun}(C, \mathrm{An}) & \xrightarrow[q^*]{} & \mathrm{Fun}(\int \Phi, \mathrm{An}). \end{array}$$

The pair $w = (q, \Phi_\bullet)$ determines an object $w : W^{\simeq}$ such that $p(w) = \Phi$. Since p is a left fibration, this determines a unique functor

$$f\colon \mathrm{Fun}(C, \mathrm{An})_{\Phi/} \to W$$

over $\mathrm{Fun}(C, \mathrm{An})$ such that $f(\Phi) = w$. To prove the theorem, it suffices to prove that f is an equivalence. Since this is a functor between two left fibrations over $\mathrm{Fun}(C, \mathrm{An})$, it suffices to check this on fibers over any object of $\mathrm{Fun}(C, \mathrm{An})$. Finally, up to replacing the underlying anima context, it suffices to consider fibers defined in the absolute context. Consider an arbitrary object Ψ of $\mathrm{Fun}(C, \mathrm{An})$ (in the absolute context). The fiber of $\mathrm{Fun}(C, \mathrm{An})_{\Phi/}$ over $\mathrm{Fun}(C, \mathrm{An})$ is by definition the mapping anima

$$\mathrm{Fun}(C, \mathrm{An})(\Phi, \Psi).$$

By straightening/unstraightening (Theorem 8.3.15), this is equivalent to the relative mapping anima $\mathrm{Map}_C(\int \Phi, \int \Psi)$ over C. Since $\int \Psi$ sits in a pullback square of the form

$$\begin{array}{ccc} \int \Psi & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C & \xrightarrow{\Psi} & \mathrm{An}, \end{array}$$

we have by adjunction the equivalence below, in which $\int \Phi$ is seen over An with structural map $\Psi \circ q$.

$$\mathrm{Map}_C\left(\int \Phi, \int \Psi\right) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{An}}\left(\int \Phi, \mathrm{An}_\bullet\right)$$

We also have the following canonical pullback squares (the first one by definition and the second one by conservativity of left fibrations).

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{An}}\left(\int \Phi, \mathrm{An}_\bullet\right) & \longrightarrow & \mathrm{Map}\left(\int \Phi, \mathrm{An}_\bullet\right) & \longrightarrow & \mathrm{Fun}\left(\int \Phi, \mathrm{An}_\bullet\right) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_{\mathrm{univ}})_*} \\ * & \xrightarrow[\Psi\circ q]{} & \mathrm{Map}\left(\int \Phi, \mathrm{An}\right) & \longrightarrow & \mathrm{Fun}\left(\int \Phi, \mathrm{An}\right) \end{array}$$

It remains to check that the map induced by f on fibers over Ψ is the composition

$$\mathrm{Fun}(C, \mathrm{An})(\Phi, \Psi) \xrightarrow{\sim} \mathrm{Map}_C\left(\int \Phi, \int \Psi\right) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{An}}\left(\int \Phi, \mathrm{An}_\bullet\right).$$

A map $u\colon \Phi \to \Psi$ can be seen as a functor $u\colon [1] \to \mathrm{Fun}(C, \mathrm{An})$. Since $f(\Phi) = w$, there is a unique functor $v\colon [1] \to W$ such that $p(v) = u$. By definition, we then have $f(u) = v$. On the other hand the straightening/unstraightening equivalence consists in interpreting u as a functor $\tilde{u}\colon [1] \times C \to \mathrm{An}$ corresponding to a left fibration $q\colon X \to [1] \times C$ and to apply Construction 8.2.1. In other words, we see $[1] \times \int \Phi$ over $[1] \times C$ by pulling back $\int \Phi$ along the projection $[1] \times C \to C$, which we interpret by transposition as a functor

$$\bar{u}\colon [1] \to \mathrm{Fun}(\int \Phi, [1] \times C)$$

and we consider the unique lifting of $\bar{u}$ of the form

$$\bar{v}\colon [1] \to \mathrm{Fun}(\int \Phi, X)$$

with source the composition of the embedding of the fiber $X_0 \hookrightarrow X$ with the identification of $\int \Phi$ with the fiber X_0. In other words, we consider the commutative square

$$\begin{array}{ccc} \Phi & = & \Phi \\ \| & & \downarrow{\scriptstyle u} \\ \Phi & \xrightarrow[u]{} & \Psi \end{array}$$

and pick the lower horizontal line in the image of this square by f. This is indeed the same as applying f directly. □

A more flexible form follows by restricting the total left fibration along an initial functor.

Corollary 12.3.4. *Consider a commutative diagram of the following form:*

$$\begin{array}{ccccc} U & \xrightarrow{i} & V & \xrightarrow{\Psi} & \mathrm{An}_\bullet \\ & \searrow^{f} & \downarrow{\scriptstyle q} & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ & & C & \xrightarrow{\Phi} & \mathrm{An}. \end{array}$$

Assume that i is an initial functor. Then the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(C, \mathrm{An})_{\Phi/} & \longrightarrow & \mathrm{Fun}(U, \mathrm{An}_\bullet) \\ \downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Fun}(C, \mathrm{An}) & \xrightarrow[f^*]{} & \mathrm{Fun}(U, \mathrm{An}) \end{array}$$

is a pullback square.

Proof. By Theorem 12.3.3 the diagram is a pullback square if we replace U by V and i by id_V. But applying part (3) of Proposition 9.1.4 to the universal left fibration $\pi_{\mathrm{univ}} : \mathrm{An}_\bullet \to \mathrm{An}$, we obtain a pullback square of the form

$$\begin{array}{ccc} \mathrm{Fun}(V, \mathrm{An}_\bullet) & \xrightarrow{i^*} & \mathrm{Fun}(U, \mathrm{An}_\bullet) \\ {\scriptstyle (\pi_{\mathrm{univ}})_*}\downarrow & & \downarrow{\scriptstyle (\pi_{\mathrm{univ}})_*} \\ \mathrm{Fun}(V, \mathrm{An}) & \xrightarrow[i^*]{} & \mathrm{Fun}(U, \mathrm{An}), \end{array}$$

and thus we may paste the two pullback squares to get the result. □

Corollary 12.3.5. *Consider a category C with an object $x : C^{\simeq}$ defined in the absolute context. We assume that, for each object y of C, $C(x, y)$ is small (i.e. that the projection of $C_{x/}$ to C is a left fibration with small fibers). We consider the functor*

$$h_{C^{\mathrm{op}}}(x) = \mathrm{Hom}_C(x, -) : C \to \mathrm{An}$$

determined by the following pullback square.

$$\begin{array}{ccc} C_{x/} & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C & \xrightarrow[h_{C^{\mathrm{op}}}(x)]{} & \mathrm{An}. \end{array}$$

Then we have a homotopy pullback square

$$\begin{array}{ccc} \mathrm{Fun}(C, \mathrm{An})_{h_{C^{\mathrm{op}}}(x)/} & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Fun}(C, \mathrm{An}) & \xrightarrow{\mathrm{ev}_x} & \mathrm{An}. \end{array}$$

In particular, we obtain an identification

$$\mathrm{ev}_x = \mathrm{Hom}_{\mathrm{Fun}(C, \mathrm{An})}(h_{C^{\mathrm{op}}}(x), -)$$

in the category $\mathrm{Fun}(\mathrm{Fun}(C, \mathrm{An}), \mathrm{An})$.

Proof. The map $U = * \xrightarrow{(x, \mathrm{id}_x)} C_{x/}$ is initial, and hence we may apply Corollary 12.3.4. □

We also record three specializations of parametrized evaluation, concerning pointed animae and morphisms out of terminal functors.

Corollary 12.3.6. *Let e be the terminal object of* An. *Then there is an equivalence* $\mathrm{An}_{e/} \xrightarrow{\sim} \mathrm{An}_\bullet$ *over* An.

Proof. We apply Theorem 12.3.3 with $C = *$ and with $\Phi = e : * \to \mathrm{An}$. Since e is terminal, it represents the left fibration $* \to *$. Using the equivalence $\mathrm{Fun}(*, \mathrm{An}) \xrightarrow{\sim} \mathrm{An}$, we thus obtain a pullback square of the form

$$\begin{array}{ccc} \mathrm{An}_{e/} & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{An} & = & \mathrm{An}, \end{array}$$

and thus the top functor is an equivalence. □

Corollary 12.3.7. *There is an identification*

$$\mathrm{Hom}_{\mathrm{An}}(e,-) = \mathrm{id}_{\mathrm{An}}$$

of functors $\mathrm{An} \to \mathrm{An}$.

Proof. The equivalence $\mathrm{An}_{e/} \xrightarrow{\sim} \mathrm{An}_\bullet$ of left fibrations over An corresponds under straightening/unstraightening to an identification $\mathrm{Hom}_{\mathrm{An}}(e,-) = \mathrm{id}_{\mathrm{An}}$. □

Corollary 12.3.8. *Consider a small category C, and let e_C denote the terminal object of* $\mathrm{Fun}(C, \mathrm{An})$ *(i.e. the constant functor with value e). Then there is a pullback square*

$$\begin{array}{ccc} \mathrm{Fun}(C, \mathrm{An}_\bullet) & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle (\pi_{\mathrm{univ}})_*}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Fun}(C, \mathrm{An}) & \xrightarrow{\mathrm{Hom}(e_C,-)} & \mathrm{An} \end{array}$$

Proof. By definition of the functor $\mathrm{Hom}(e_C, -)$, there is a pullback square if we replace the left upper corner by the slice category $\mathrm{Fun}(C, \mathrm{An})_{e_C/}$. By Lemma 4.3.2 this slice category is equivalent to $\mathrm{Fun}(C, \mathrm{An}_{e/})$, and so the claim follows from the equivalence $\mathrm{An}_{e/} \xrightarrow{\sim} \mathrm{An}_\bullet$ from Corollary 12.3.6. □

12.3.2 Naturality and the Yoneda embedding

The preceding subsection identifies morphisms out of a representable functor with evaluation, functorially in the target functor. To allow the representing object to vary, we return to presheaves on C. The two constructions we want to identify are now functors on $C^{\mathrm{op}} \times \mathrm{PSh}(C)$. We compare their associated left fibrations using $\mathrm{Tw}(C)$; this will also prove that the Yoneda functor is fully faithful.

For a small category C, choose a universe Cat' such that Cat (and hence An) is small with respect to Cat'. We write An' for the full subcategory of Cat' spanned by the animae. We then consider the functors

$$(s,t)\colon\ \mathrm{Tw}(\mathrm{PSh}(C)) \to \mathrm{PSh}(C)^{\mathrm{op}} \times \mathrm{PSh}(C)$$

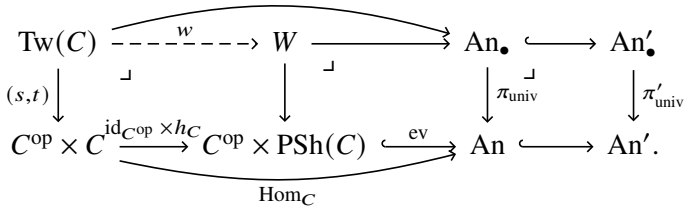
and

$$(s,t)\colon\ \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C.$$

Define a category V and a functor $v\colon\ \mathrm{Tw}(C) \to V$ by means of the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Tw}(C) & \overset{\mathrm{Tw}(h_C)}{\overset{v}{\dashrightarrow}} & V & \longrightarrow & \mathrm{Tw}(\mathrm{PSh}(C)) & \longrightarrow & \mathrm{An}'_\bullet \\ {\scriptstyle (s,t)}\downarrow & & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \pi'_{\mathrm{univ}}} \\ C^{\mathrm{op}} \times C & \xrightarrow[\mathrm{id}_{C^{\mathrm{op}}} \times h_C]{} & C^{\mathrm{op}} \times \mathrm{PSh}(C) & \xrightarrow{(h_C)^{\mathrm{op}} \times \mathrm{id}} & \mathrm{PSh}(C)^{\mathrm{op}} \times \mathrm{PSh}(C) & \xrightarrow{\mathrm{Hom}} & \mathrm{An}'. \end{array}$$

We similarly define a category W and a functor $w\colon \mathrm{Tw}(C) \to W$ by means of the following commutative diagram:



The category V records morphisms from representable presheaves to arbitrary presheaves, whereas W records evaluation of a presheaf at an object. The functors v and w compare both constructions with the universal morphism parametrized by $\mathrm{Tw}(C)$. The next lemma says that both comparisons are initial; straightening will therefore identify the functors classified by V and W.

Lemma 12.3.9. *Both functors v and w are initial.*

Proof. For v, this is a particular case of Lemma 12.2.7. The proof for w is similar. Recall from Lemma 9.6.8 that for a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ C \times D & \xrightarrow[\mathrm{id}_C \times g]{} & C \times C \end{array}$$

such that p and q are left fibrations, if f is fiberwise initial over C then f is initial. Therefore, in order to show that w is initial, it will suffice to show that for every object x of C (seen as an object of C^{op}), the induced map $w_x\colon \mathrm{Tw}(C)_x \to W_x$ is initial:

$$\begin{array}{ccccc} \mathrm{Tw}(C)_x & \xrightarrow{w_x} & W_x & \longrightarrow & \mathrm{An}_\bullet \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C & \xrightarrow[h_C]{} & \mathrm{PSh}(C) & \xrightarrow{\mathrm{ev}_x} & \mathrm{An}. \end{array}$$

But we already know from Corollary 12.3.5 that $V_x \simeq W_x$ holds fiberwise, and on fibers the maps v_x and w_x fit into a commutative diagram over $\mathrm{PSh}(C)$ as follows:

$$\begin{array}{ccc} C_{x/} & \xrightarrow{v_x} & V_x \\ {\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Tw}(C)_x & \xrightarrow{w_x} & W_x \end{array}$$

Indeed, to see the commutativity of this square, since the projection of W_x to $\mathrm{PSh}(C)$ is a left fibration, by virtue of Proposition 9.1.13, it suffices to check that the initial object of $C_{x/}$ is sent to the same object of W_x either way. This amounts to say that, in the equivalence

$$F(x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{An})}(h_{C^{\mathrm{op}}}(x), F)$$

provided by Corollary 12.3.5, in the case where $F = h_{C^{\mathrm{op}}}(x)$, the identity of x is sent to the identity of $h_{C^{\mathrm{op}}}(x)$. This is true by construction.

Now, since v_x is initial (see the proof of Lemma 12.2.7), this implies that w_x is initial as well. This finishes the proof. □

Proposition 12.3.10. *The Yoneda embedding $h_C : C \to \mathrm{PSh}(C)$ is fully faithful.*

Proof. The proof of the previous lemma shows that the square

$$\begin{array}{ccc} C_{x/} & \xrightarrow{v_x} & V_x \\ \downarrow & & \downarrow \\ C & \xrightarrow{h_C} & \mathrm{PSh}(C) \end{array}$$

is a pullback for all x, where we use the canonical identification $C_{x/} \xrightarrow{\sim} \mathrm{Tw}(C)_x$. It then follows from Proposition 12.2.11, recalling the definition of V, that the functor $h_C : C \to \mathrm{PSh}(C)$ is fully faithful. □

Corollary 12.3.11. *For every category C, there exists a preferred equivalence*

$$C \xrightarrow{\sim} (C^{\mathrm{op}})^{\mathrm{op}}.$$

Proof. For any object x of C, the functor $h_C(x)$ is corepresentable in C^{op}, because, as explained in Remark 12.2.5, the corresponding left fibration is the functor $(C^{\mathrm{op}})_{x/} \xrightarrow{\sim} (C_{/x})^{\mathrm{op}} \to C^{\mathrm{op}}$:

$$\begin{array}{ccc} (C^{\mathrm{op}})_{x/} & \longrightarrow & \mathrm{Tw}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ C^{\mathrm{op}} & \longrightarrow & C^{\mathrm{op}} \times C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_2} \\ * & \xrightarrow{x} & C \end{array}$$

It follows that we obtain a factorization

$$\begin{array}{ccc} C^{\simeq} & \dashrightarrow & ((C^{\mathrm{op}})^{\mathrm{op}})^{\simeq} \\ \downarrow & & \downarrow \\ C & \xrightarrow[h_C]{} & \mathrm{PSh}(C), \end{array}$$

and hence we obtain a functor $C \to (C^{\mathrm{op}})^{\mathrm{op}}$. The composition of this functor with the full embedding of $(C^{\mathrm{op}})^{\mathrm{op}}$ into $\mathrm{PSh}(C)$ is the Yoneda embedding that we already know to be fully faithful. It follows that the functor $C \to (C^{\mathrm{op}})^{\mathrm{op}}$ is fully faithful. But we also know by Lemma 12.1.11 that both C and $(C^{\mathrm{op}})^{\mathrm{op}}$ have the same objects, which implies (strong) essentially surjectivity. This finishes the proof. □

Remark 12.3.12. This corollary means that there exists an equivalence $\mathrm{Tw}(C^{\mathrm{op}}) \simeq \mathrm{Tw}(C)$ making the following diagram commute:

$$\begin{array}{ccccc} \mathrm{Tw}(C^{\mathrm{op}}) & & \overset{\sim}{\dashrightarrow} & & \mathrm{Tw}(C) \\ {\scriptstyle (s,t)}\downarrow & & & & \downarrow{\scriptstyle (s,t)} \\ (C^{\mathrm{op}})^{\mathrm{op}} \times C^{\mathrm{op}} & \underset{\sim}{\longleftarrow} & C \times C^{\mathrm{op}} & \xrightarrow[\sim]{\mathrm{swap}} & C^{\mathrm{op}} \times C. \end{array}$$

Remark 12.3.13. If C is locally small, $\mathrm{Hom}_C : C^{\mathrm{op}} \times C \to \mathrm{An}$ is a functor. In particular any morphism $f : x \to y$ in C, seen as functor from $[1]$ to C, induces for each object z of C a morphism

$$f_* : \mathrm{Hom}_C(z, x) \to \mathrm{Hom}_C(z, y)$$

defined as $f_* := \mathrm{Hom}_C(z, f)$. In other words, composition by f is now encoded in the functorial nature of Hom_C.

Corollary 12.3.14. *Let $f : x \to y$ be a morphism in a locally small category C. Assume that, for any object z of C, f induces an isomorphism*

$$f_* : \mathrm{Hom}_C(z, x) \xrightarrow{\sim} \mathrm{Hom}_C(z, y)\,.$$

Then f is an isomorphism in C.

Proof. Since isomorphisms in $\mathrm{PSh}(C)$ are detected object-wise, this corollary asserts that the Yoneda embedding is conservative. But any fully faithful functor is conservative, so that this corollary is a direct consequence of Proposition 12.3.10. □

Proposition 12.3.15. *Consider a pullback square of the form below.*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ C^{\mathrm{op}} & \xrightarrow{F} & \mathrm{An} \end{array}$$

Then F is representable if and only if the category X has an initial object.

Proof. In view of the preceding remark, this is a special instance of Proposition 12.1.3. □

12.3.3 A fully functorial Yoneda lemma

We now identify evaluation and morphisms out of representables as functors on $C^{\mathrm{op}} \times \mathrm{PSh}(C)$. The initial functors of the preceding subsection describe both as the same left Kan extension of Hom_C. Full faithfulness of the Yoneda embedding will moreover show that the resulting identification is determined by its restriction to representable presheaves.

Theorem 12.3.16 (Yoneda lemma)**.** *Let C be a locally small category. There is an isomorphism*

$$\mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{An})}(h_C(x), F) \xrightarrow{\sim} F(x),$$

which is functorial in both variables $F : \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ and $x : C^{\mathrm{op}}$.

It is the unique functorial identification which extends the inverse of identification

$$\mathrm{Hom}_C(x, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{An})}(h_C(x), h_C(y))$$

on representable functors expressing the full faithfulness of the Yoneda embedding h_C.

Proof. Choose a larger universe Cat' in which C and $\mathrm{PSh}(C)$ are small, and write An' for its category of animae. We have a Hom-functor of the form:

$$\begin{array}{ccc} \mathrm{Tw}(\mathrm{PSh}(C)) & \longrightarrow & \mathrm{An}'_{\bullet} \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{PSh}(C)^{\mathrm{op}} \times \mathrm{PSh}(C) & \xrightarrow{\mathrm{Hom}} & \mathrm{An}' \,. \end{array}$$

The Yoneda embedding of C being fully faithful, it follows that the functor $\mathrm{id}_{C^{\mathrm{op}}} \times h_C$ is fully faithful as well, and hence the left Kan extension functor

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_! \colon \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An}') \to \mathrm{Fun}(C^{\mathrm{op}} \times \mathrm{PSh}(C), \mathrm{An}')$$

is fully faithful, by Proposition 12.2.11. On the other hand, by virtue of Proposition 12.2.8, if

$$i \colon \mathrm{An} \to \mathrm{An}'$$

denotes the canonical full embedding, we get the identification

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i \circ \mathrm{Hom}_C) = (h_C^{\mathrm{op}} \times \mathrm{id}_{\mathrm{PSh}(C)})^*(i \circ \mathrm{Hom})$$

and, by a similar argument, the initiality of the functor w from Lemma 12.3.9 means that

$$(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i \circ \mathrm{Hom}_C) = i \circ \mathrm{ev}$$

holds. Moreover, the full faithfulness of the Yoneda embedding can be expressed by saying that there is a pullback square of the form

$$\begin{array}{ccc} \mathrm{Tw}(C) & \xrightarrow{h_C} & \mathrm{Tw}(\mathrm{PSh}(C)) \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ C^{\mathrm{op}} \times C & \xrightarrow{(h_C)^{\mathrm{op}} \times h_C} & \mathrm{PSh}(C)^{\mathrm{op}} \times \mathrm{PSh}(C) \end{array}$$

inducing a pullback square of the form

$$\begin{array}{ccc} \mathrm{Tw}(C) & \xrightarrow{v} & V \\ {\scriptstyle (s,t)}\downarrow & \lrcorner & \downarrow \\ C^{\mathrm{op}} \times C & \xrightarrow{\mathrm{id}_{C^{\mathrm{op}}} \times h_C} & C^{\mathrm{op}} \times \mathrm{PSh}(C) \,. \end{array}$$

By construction, we have a pullback square of the form

$$\begin{array}{ccc} \mathrm{Tw}(C) & \xrightarrow{w} & W \\ {\scriptstyle (s,t)}\downarrow & \lrcorner & \downarrow \\ C^{\mathrm{op}} \times C & \xrightarrow{\mathrm{id}_{C^{\mathrm{op}}} \times h_C} & C^{\mathrm{op}} \times \mathrm{PSh}(C) \,. \end{array}$$

The two identifications with $(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!(i \circ \mathrm{Hom}_C)$ give the asserted identification of functors. The last two pullback squares show that its restriction to representable presheaves expresses the full faithfulness of h_C. Finally, full faithfulness of $(\mathrm{id}_{C^{\mathrm{op}}} \times h_C)_!$ shows that this restriction determines the identification uniquely. These two descriptions identify the left fibrations classified by evaluation and by morphisms out of representable presheaves. We therefore obtain the fully functorial statement. □

Corollary 12.3.17. *Let C and D be small categories. Then the Yoneda embedding of D induces a fully faithful functor*

$$\mathrm{Fun}(C, D) \hookrightarrow \mathrm{Fun}(C, \mathrm{PSh}(D)) \simeq \mathrm{Fun}(C \times D^{\mathrm{op}}, \mathrm{An})$$

whose essential image is the collection of those functors $F\colon C \times D^{\mathrm{op}} \to \mathrm{An}$ with the property that $F(x, -)\colon D^{\mathrm{op}} \to \mathrm{An}$ is representable for any object x of C.

Proof. Apply part (2) of Proposition 12.2.11 objectwise in C. □

Proposition 12.3.18. *Let $u\colon C \to D$ be a functor between small categories. We consider the adjunction*

$$u_!\colon \mathrm{PSh}(C) \rightleftarrows \mathrm{PSh}(D) :u^*,$$

where u^* is the functor defined by the rule $F \mapsto F \circ u^{\mathrm{op}}$ and $u_!$ is the left Kan extension functor along u^{op}. Then there is a canonical commutative square of the form below.

$$\begin{array}{ccc} C & \xrightarrow{h_C} & \mathrm{PSh}(C) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u_!} \\ D & \xrightarrow{h_D} & \mathrm{PSh}(D) \end{array}$$

Proof. By Proposition 12.2.3, the functor $\mathrm{Tw}(u)$ over $u^{\mathrm{op}} \times u$ gives, after straightening, a natural transformation

$$\mathrm{Hom}_C \longrightarrow (u^{\mathrm{op}} \times u)^* \mathrm{Hom}_D\,.$$

Currying gives a natural transformation $h_C \to u^* h_D u$. Taking its adjunct under the adjunction on C-indexed functor categories from Proposition 4.4.4 gives a natural transformation

$$\theta\colon u_! h_C \longrightarrow h_D u.$$

We will show that θ is an isomorphism.

Choose a larger universe in which the two presheaf categories are small. For $x : C$ and $F : \mathrm{PSh}(D)$, the Yoneda lemma yields isomorphisms of animae

$$\mathrm{Hom}(h_D(u(x)), F) \simeq F(u(x)) = (u^* F)(x) \simeq \mathrm{Hom}(h_C(x), u^* F)\,.$$

By adjunction, Proposition 12.1.19 tells us that we also have a canonical equivalence of the form

$$\mathrm{Hom}(h_C(x), u^* F) \simeq \mathrm{Hom}(u_!(h_C(x)), F)$$

and thus an identification of the form

$$\mathrm{Hom}(h_D(u(x)), F) \simeq \mathrm{Hom}(u_!(h_C(x)), F)\,.$$

This equivalence is precomposition with θ_x. Indeed, under the adjunction, precomposition sends a natural transformation $\gamma\colon h_D(u(x)) \to F$ to the composite

$$h_C(x) \longrightarrow u^* h_D(u(x)) \xrightarrow{u^*\gamma} u^* F.$$

The first map sends id_x to $\mathrm{id}_{u(x)}$, by the construction of $\mathrm{Tw}(u)$. So evaluating this composite at id_x gives $\gamma_{u(x)}(\mathrm{id}_{u(x)})$, which is also the image of γ under the first Yoneda isomorphism above. By applying this construction to the universal object γ in its mapping anima, this identifies the two maps of animae. It follows that precomposition with θ_x is an isomorphism for every F, and hence θ_x is an isomorphism itself. Isomorphisms in a functor category are detected objectwise, so θ is the required natural isomorphism. □

Corollary 12.3.19. *For any functor $f : C \to D$, there is a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ (C^{\mathrm{op}})^{\mathrm{op}} & \xrightarrow[(f^{\mathrm{op}})^{\mathrm{op}}]{} & (D^{\mathrm{op}})^{\mathrm{op}}, \end{array}$$

(where the vertical maps are the equivalences given by Corollary 12.3.11).

Proof. This follows from Proposition 12.3.18 and Proposition 12.1.9. □

12.3.4 Consequences of Yoneda

We record various useful consequences of the Yoneda lemma: pointwise criteria for representability of presheaves and adjunctions between functors, and poset trunctions.

Theorem 12.3.20. *Let $f : C \to D$ be a functor between small categories. We consider the adjunction*

$$f_! : \mathrm{PSh}(C) \rightleftarrows \mathrm{PSh}(D) : f^*,$$

where f^* is defined by the rule $G \mapsto G \circ f^{\mathrm{op}}$. Then the following assertions are equivalent:

(1) The functor f admits a right adjoint.

(2) There is a functor $g : D \to C$ together with a functorial identification

$$\mathrm{Hom}_D(f(x), y) \xrightarrow{\sim} \mathrm{Hom}_C(x, g(y))$$

natural in $(x, y) : C^{\mathrm{op}} \times D$.

(3) For any object y in D, the functor $f^* h_D(y) = \mathrm{Hom}_D(f(-), y) : C^{\mathrm{op}} \to \mathrm{An}$ is representable.

(4) There is a functor $g : D \to C$ together with a natural identification

$$g_! \xrightarrow{\sim} f^*$$

of functors from $\mathrm{PSh}(D)$ to $\mathrm{PSh}(C)$.

Proof. Condition (2) implies (3) by evaluation at each object of D, while (3) $\implies$ (2) is Corollary 12.3.17. We will show that the following chain of implications holds.

$$(1) \implies (4) \implies (3) \quad \text{and} \quad (2) \implies (1)$$

To show that (1) implies (4), we observe first that, for any functor $g\colon D \to C$, we have an adjunction of the form

$$g_!\colon \mathrm{PSh}(D) \rightleftarrows \mathrm{PSh}(C) : g^*.$$

If g is right adjoint to f, then g^* is right adjoint to f^*, and hence $g_! \xrightarrow{\sim} f^*$ since both are left adjoints to g^*.

Finally, assume (4), and let $y : D$. By Proposition 12.3.18, we have $g_!(h_D(y)) \simeq h_C(g(y))$. Together with (4), this gives

$$f^* h_D(y) \simeq g_!(h_D(y)) \simeq h_C(g(y)),$$

so $f^* h_D(y)$ is representable, proving (3).

Let us assume finally that assertion (2) holds. Given any object y of D, there is a unique map

$$\varepsilon\colon f(g(y)) \to y$$

corresponding to the identity of $g(y)$. We then have a map for each object x of C of the form

$$\mathrm{Hom}_C(x, g(y)) \xrightarrow{\text{apply } f} \mathrm{Hom}_D(f(x), f(g(y))) \xrightarrow{\text{compose with } \varepsilon} \mathrm{Hom}_D(f(x), y)$$

that is an equivalence by assumption. It follows from Proposition 12.1.19 that f has a right adjoint. □

Lemma 12.3.21. *Let C be a locally small category, let $F\colon C^{\mathrm{op}} \to \mathrm{An}$ be a presheaf, and let y be an object of C. Suppose that the following conditions hold.*

(i) For each object x of C, there is a morphism

$$s_x\colon \mathrm{Hom}_C(x, y) \to F(x)\,.$$

(ii) For each morphism $u\colon x' \to x$ in C, the square

$$\begin{array}{ccc} \mathrm{Hom}_C(x, y) & \xrightarrow{s_x} & F(x) \\ {\scriptstyle u^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathrm{Hom}_C(x', y) & \xrightarrow{s_{x'}} & F(x') \end{array}$$

commutes in An.

Then there is a unique morphism of presheaves

$$s\colon \mathrm{Hom}_C(-, y) \to F$$

equipped with an identification $\alpha\colon s(y)(\mathrm{id}_y) \simeq s_y(\mathrm{id}_y)$. The component of s at $x : C^{\simeq}$ is s_x. In particular, if each s_x is an isomorphism, then F is representable.

Proof. By the Yoneda lemma, evaluation at id_y defines an isomorphism of animae

$$\mathrm{Hom}_{\mathrm{PSh}(C)}(\mathrm{Hom}_C(-, y), F) \xrightarrow{\sim} F(y).$$

Its fiber over $s_y(\mathrm{id}_y)$ is therefore contractible. This fiber is precisely the anima of pairs (s, α), proving the uniqueness assertion.

Given an arbitrary morphism $u\colon x \to y$, we may compute

$$s(x)(u) \simeq F(u)(s(y)(\mathrm{id}_y)) \simeq F(u)(s_y(\mathrm{id}_y)) \simeq s_x(u),$$

where we used the naturality of s, the identification α, and condition (ii) evaluated at id_y. Taking u to be the universal term of the anima $\mathrm{Hom}_C(x, y)$ gives an isomorphism of maps $s(x) \simeq s_x$.

If every s_x is an isomorphism, then s is an isomorphism by the pointwise criterion for natural isomorphisms, and it follows that F is represented by y. □

We may now show that the poset truncations from Chapter 11 exist. We construct them by applying propositional truncation to representable presheaves.

Construction 12.3.22. Let C be a category and choose a universe Cat where C is small. Consider the composite

$$h_{-1}\colon C \xrightarrow{h} \mathrm{PSh}(C) \xrightarrow{L} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Prop}) =: \mathrm{PSh}_{-1}(C) \tag{12.1}$$

of the Yoneda embedding with the functor L given by postcomposition with the propositional truncation functor $\|-\|_{-1}\colon \mathrm{An} \to \mathrm{Prop}$.

Denote by $\|C\|_{\mathrm{pos}} \subseteq \mathrm{PSh}_{-1}(C)$ the essential image of h_{-1}, which is tautologically equipped with a map $C \to \|C\|_{\mathrm{pos}}$.

Proposition 12.3.23. *The category $\|C\|_{\mathrm{pos}}$ is a poset and the functor $C \to \|C\|_{\mathrm{pos}}$ is a cover both on objects and on mapping animae.*

Proof. The category Prop is a poset by Lemma 10.1.9, hence also $\mathrm{PSh}_{-1} C = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Prop})$ by Corollary 10.3.8 and its full subcategory $\|C\|_{\mathrm{pos}}$. Moreover, the functor $C \to \|C\|_{\mathrm{pos}}$ is an objectwise cover by construction. It remains to show that for objects $c, d : C^{\simeq}$, the map

$$C(c, d) \to \|C\|_{\mathrm{pos}}([c], [d]) = \mathrm{PSh}_{-1}(C)(h_{-1}c, h_{-1}d) \tag{12.2}$$

is a cover. By Yoneda and unraveling the construction, this map is identified with the propositional truncation map

$$C(c, d) \to \|C(c, d)\|_{-1}, \tag{12.3}$$

which is a cover by Lemma 11.4.8. □

Corollary 12.3.24. *The assignment $C \mapsto \|C\|_{\mathrm{pos}}$ assembles to left adjoints to the inclusions*

$$\mathrm{Poset} \hookrightarrow \mathrm{Cat} \quad \textit{and} \quad \mathrm{Set} \hookrightarrow \mathrm{An}. \tag{12.4}$$

Proof. The first statement follows directly from Corollary 11.4.15 and Proposition 12.3.23 by the usual pointwise construction of adjoints. For the second statement, let C be an anima. Every morphism of $\|C\|_{\mathrm{pos}}$ merely lifts along the cover on mapping animae to a morphism of C, and is therefore invertible. Invertibility is a proposition, so the mere lift suffices. Thus $\|C\|_{\mathrm{pos}}$ is an anima, and, being a poset, it is a set. □

12.4 Duality for opposite categories

The preceding results identify C with its double opposite on objects and morphisms. We now upgrade this to coherent duality on the universe of categories. This permits systematic passage between the cartesian and cocartesian theories, and supplies the dual constructions whose proofs were postponed in Chapter 9.

12.4.1 Formal duality

We start by establishing that passing to opposites is an invertible operation.

Proposition 12.4.1. *The functor* $(-)^{\mathrm{op}}\colon \mathrm{Cat} \to \mathrm{Cat}$ *is an equivalence.*

Proof. It suffices to check that the functor is fully faithful and strongly essentially surjective. Strong essential surjectivity follows from Corollary 12.3.11. It remains to prove full faithfulness.

Fix categories C and D. We must show that the induced map

$$\mathrm{Cat}(C,D)^{\simeq} \to \mathrm{Cat}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\simeq}, \quad f \mapsto f^{\mathrm{op}}$$

is an equivalence. We claim that an inverse is given by sending $g\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ to the composite

$$C \xrightarrow{\sim} (C^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} (D^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} D\,.$$

This follows right away from Corollary 12.3.19 and Remark 12.3.12. □

Corollary 12.4.2. *The opposite category functor has the following properties.*

(i) It commutes with limits (in particular with pullbacks). In particular, we have $*^{\mathrm{op}} \xrightarrow{\sim} *$ *and for categories* C *and* D *there exists a canonical isomorphism*

$$(C \times D)^{\mathrm{op}} \xrightarrow{\sim} C^{\mathrm{op}} \times D^{\mathrm{op}}\,.$$

(ii) For any categories C *and* D *there exists a canonical isomorphism*

$$\mathrm{Fun}(C,D)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})\,.$$

Proof. Equivalences of categories commute with limits and colimits, and this explains assertion (i). For any categories A, C and D, the evaluation functor

$$\mathrm{ev}\colon C \times \mathrm{Fun}(C,D) \to D$$

induces thanks to the compatibility of $(-)^{\mathrm{op}}$ with cartesian products a functor

$$\mathrm{ev}^{\mathrm{op}}\colon C^{\mathrm{op}} \times \mathrm{Fun}(C,D)^{\mathrm{op}} \to D^{\mathrm{op}}$$

hence, by transposition, a comparison functor

$$\delta\colon \mathrm{Fun}(C,D)^{\mathrm{op}} \to \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})\,.$$

On the other hand, there are canonical equivalences:

$$\begin{aligned}\mathrm{Map}(A, \mathrm{Fun}(C, D)) &\simeq \mathrm{Map}(A \times C, D) \\ &\simeq \mathrm{Map}((A \times C)^{\mathrm{op}}, D^{\mathrm{op}}) \\ &\simeq \mathrm{Map}(A^{\mathrm{op}} \times C^{\mathrm{op}}, D^{\mathrm{op}}) \\ &\simeq \mathrm{Map}(A^{\mathrm{op}}, \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})) \\ &\simeq \mathrm{Map}((A^{\mathrm{op}})^{\mathrm{op}}, \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\mathrm{op}}) \\ &\simeq \mathrm{Map}(A, \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\mathrm{op}}) \,.\end{aligned}$$

where the final equivalence is induced by the equivalence $A \xrightarrow{\sim} (A^{\mathrm{op}})^{\mathrm{op}}$. Using Corollary 12.3.19, we see that the equivalence

$$\mathrm{Map}(A, \mathrm{Fun}(C, D)) \simeq \mathrm{Map}(A, \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\mathrm{op}})$$

obtained above as a composition of equivalences is induced by postcomposition with $\delta^{\mathrm{op}} \colon \mathrm{Fun}(C, D) \to \mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\mathrm{op}}$. Therefore, δ^{op}, and hence δ, is an equivalence. □

Lemma 12.4.3. *There is a unique equivalence* $[1] \xrightarrow{\sim} [1]^{\mathrm{op}}$ *exchanging* 0 *and* 1.

Proof. Since $[1](0,1) = [1]^{\mathrm{op}}(1,0)$, there is a canonical morphism from 1 to 0 in $[1]^{\mathrm{op}}$, i.e. a functor

$$u \colon [1] \to [1]^{\mathrm{op}}$$

inducing the map

$$u^{\simeq} \colon \{0,1\} \xrightarrow{\sim} \{0,1\}$$

defined by the rule $0 \mapsto 1$ and $1 \mapsto 0$. The functor u must be fully faithful because it induces the endomorphism of the empty anima or of the terminal anima

$$[1](x,y) \to [1]^{\mathrm{op}}(u(x), u(y))$$

for all $x, y : [1]^{\simeq}$. Being strongly essentially surjective and fully faithful, the functor is thus an equivalence. Any equivalence exchanging the two objects must carry the unique nonidentity morphism to the unique nonidentity morphism, so the anima of all equivalences from $[1]$ to $[1]^{\mathrm{op}}$ is contractible. □

Remark 12.4.4. The preceding lemma can also be proved by a functoriality argument. To construct an equivalence $[1]^{\mathrm{op}} \xrightarrow{\sim} [1]$, it will suffice to show that for every category C there is an equivalence $\mathrm{Map}([1], C) \simeq \mathrm{Map}([1]^{\mathrm{op}}, C)$ which is natural in C. But we have

$$\begin{aligned}\mathrm{Map}([1], C) &\simeq \sum_{(x,y) : C^{\simeq} \times C^{\simeq}} C(x,y) \\ &\simeq \sum_{(x,y) : C^{\simeq} \times C^{\simeq}} C(y,x) \\ &\simeq \mathrm{Map}([1], C^{\mathrm{op}}) \simeq \mathrm{Map}([1]^{\mathrm{op}}, C).\end{aligned}$$

where the middle equivalence is given by the flip map $C^{\simeq} \times C^{\simeq} \to C^{\simeq} \times C^{\simeq}$ sending (x, y) to (y, x). Given a functor $f\colon C \to C'$, the above equivalences fit in a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}([1], C) & \xrightarrow{\simeq} & \mathrm{Map}([1]^{\mathrm{op}}, C) \\ {\scriptstyle f\circ -}\downarrow & & \downarrow{\scriptstyle f\circ -} \\ \mathrm{Map}([1], C') & \xrightarrow{\simeq} & \mathrm{Map}([1]^{\mathrm{op}}, C'). \end{array}$$

We may now pick a map $f \in \mathrm{Map}([1], [1]^{\mathrm{op}})$ corresponding to the identity of $[1]^{\mathrm{op}}$ under the equivalence to $\mathrm{Map}([1]^{\mathrm{op}}, [1]^{\mathrm{op}})$, and similarly we may pick its inverse $g \in \mathrm{Map}([1]^{\mathrm{op}}, [1])$ as the map corresponding to the identity of $[1]$ under the equivalence to $\mathrm{Map}([1], [1])$. It follows from the above naturality that f is inverse to g. Tracing through the above equivalences shows that f exchanges the objects 0 and 1.

Proposition 12.4.5. *There is a preferred equivalence* $\mathrm{Fun}([1], C)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}([1], C^{\mathrm{op}})$ *which fits functorially in a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Fun}([1], C)^{\mathrm{op}} & & \xrightarrow{\qquad\simeq\qquad} & & \mathrm{Fun}([1], C^{\mathrm{op}}) \\ {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)^{\mathrm{op}}}\downarrow & & & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ (C \times C)^{\mathrm{op}} & \xrightarrow{\simeq} & C^{\mathrm{op}} \times C^{\mathrm{op}} & \xrightarrow{\mathrm{flip}} & C^{\mathrm{op}} \times C^{\mathrm{op}}. \end{array}$$

Proof. Using Lemma 12.4.3 together with assertion (ii) of Corollary 12.4.2, we have

$$\mathrm{Fun}([1], C)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}([1]^{\mathrm{op}}, C^{\mathrm{op}}) \xrightarrow{\sim} \mathrm{Fun}([1], C^{\mathrm{op}})\,.$$

These isomorphisms fit in the stated commutative square. □

Corollary 12.4.6. *Let $f\colon C \to D$ be a functor.*

- *If f admits a right (left) adjoint section, then f^{op} admits a left (right) adjoint section.*
- *The functor f is initial (resp. terminal) if and only if f^{op} is terminal (resp. initial).*
- *The functor f is proper (resp. smooth) if and only if f^{op} is smooth (resp. proper).*
- *If f is a cocartesian (resp. cartesian) fibration, then $f^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is a cartesian (resp. cocartesian) fibration.*
- *If f is a left (resp. right) fibration, then $f^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is a right (resp. left) fibration.*

Proof. These assertions follow from Corollary 12.4.2 and Proposition 12.4.5. For the second assertion, use the definitions of initial and terminal functors. The third then follows from the definition of proper and smooth functors, since opposite categories commute with pullbacks. □

Corollary 12.4.7 (Smooth base change in disguise)**.** *Let $p\colon A \to B$ be a smooth functor and consider a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\psi} & B \end{array}$$

where f is terminal and both φ and ψ are right fibrations. Then, for any functor $v\colon B' \to B$, the induced functor

$$B' \times_B X \longrightarrow B' \times_B Y$$

is terminal.

Proof. Pass to opposite categories. By Corollaries 12.4.2 and 12.4.6, the resulting square satisfies the proper case of Corollary 9.6.7. It follows that the opposite of the displayed functor is initial, so the displayed functor is terminal. □

12.4.2 Cartesian straightening

We may use coherent duality to dualize the straightening/unstraightening correspondence from Section 8.7.

Definition 12.4.8. Let S be a small category, corresponding to an object $s\colon * \to \mathrm{Cat}$. We let

$$\mathrm{Cart}(S) \subseteq \mathrm{Cat}_{/S} := \mathrm{Cat}_{/s}$$

denote the (non-full) subcategory of cartesian fibrations over S and cartesian functors over S (defined as an appropriate variation of Construction 8.7.3 and of Construction 8.7.4). We denote by

$$\mathrm{RFib}(S) \subseteq \mathrm{Cart}(S)$$

the full subcategory spanned by the right fibrations over S.

Lemma 12.4.9. *The equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat} \xrightarrow{\sim} \mathrm{Cat}$ induces an equivalence*

$$\mathrm{Cat}_{/S} \xrightarrow{\sim} \mathrm{Cat}_{/S^{\mathrm{op}}}.$$

This equivalence restricts to equivalences of the form

$$\mathrm{RFib}(S) \xrightarrow{\sim} \mathrm{LFib}(S^{\mathrm{op}}) \qquad \textit{and} \qquad \mathrm{Cart}(S) \xrightarrow{\sim} \mathrm{CoCart}(S^{\mathrm{op}}).$$

Proof. The equivalence $(-)^{\mathrm{op}}$ carries the slice over S to the slice over S^{op}, which gives the first assertion. Its restrictions follow from Corollary 12.4.6. □

Theorem 12.4.10 (Straightening/unstraightening)**.** *For every small category S there are canonical equivalences*

$$\mathrm{Un}\colon \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Cat}) \xrightarrow{\sim} \mathrm{Cart}(S) \qquad \textit{and} \qquad \mathrm{Un}\colon \mathrm{PSh}(S) \xrightarrow{\sim} \mathrm{RFib}(S).$$

Proof. Combine Theorem 8.7.8 for S^{op} with Lemma 12.4.9. □

Corollary 12.4.11. *For any small category S, there is a canonical equivalence of the form*

$$\mathrm{Fun}(S, \mathrm{Cat}^{\mathrm{op}})^{\simeq} \simeq \mathrm{Cart}(S)^{\simeq}.$$

Proof. By Corollary 12.4.2 and Corollary 12.3.11, there are canonical isomorphisms

$$\begin{aligned}\mathrm{Fun}(S, \mathrm{Cat}^{\mathrm{op}})^{\simeq} &\simeq \big(\mathrm{Fun}(S, \mathrm{Cat}^{\mathrm{op}})^{\mathrm{op}}\big)^{\simeq} \\ &\simeq \mathrm{Fun}(S^{\mathrm{op}}, (\mathrm{Cat}^{\mathrm{op}})^{\mathrm{op}})^{\simeq} \\ &\simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Cat})^{\simeq} \\ &\simeq \mathrm{Cart}(S)^{\simeq},\end{aligned}$$

where the final isomorphism is straightening from Theorem 12.4.10. □

Corollary 12.4.12. *Let $p\colon C \to S$ be a cartesian fibration. We assume given a subcategory $i\colon W \hookrightarrow C$ such that $p \circ i$ is a cartesian fibration and i is a cartesian functor over S. Then the functor $\eta_C\colon C \to L_S(C, W)$ exhibits $L_S(C, W)$ as the localization of C by the morphisms in W which are sent to isomorphisms in S.*

Proof. This follows from Theorem 8.8.9 for $p^{\mathrm{op}}\colon C^{\mathrm{op}} \to S^{\mathrm{op}}$ and applying the functor $(-)^{\mathrm{op}}$. □

12.4.3 Coherent involutivity

We have previously shown that every category C is equivalent to its double opposite $(C^{\mathrm{op}})^{\mathrm{op}}$, and that these equivalences are compatible with individual functors $f\colon C \to D$. We will now enhance this to a coherent identification $\mathrm{id}_{\mathrm{Cat}} = ((-)^{\mathrm{op}})^{\mathrm{op}}$ of endofunctors on Cat.

Construction 12.4.13. Let RFib be the full subcategory of $\mathrm{Fun}([1], \mathrm{Cat})$ whose objects are the right fibrations between small categories. The functor

$$\mathrm{ev}_1\colon \mathrm{Fun}([1], \mathrm{Cat}) \to \mathrm{Cat}$$

is a cocartesian fibration. Since, for any functor between small synthetic categories $u\colon A \to B$, the functor

$$u_!\colon \mathrm{Cat}_{/A} \to \mathrm{Cat}_{/B}$$

induced by composition with u has a right adjoint (defined by pullbacks along u), the functor ev_1 is a cartesian fibration as well. The property of right fibrations to be stable under pullbacks in Cat precisely means that the restriction of ev_1 to RFib remains a cartesian fibration:

$$\mathrm{ev}_1\colon \mathrm{RFib} \to \mathrm{Cat}\,.$$

For any small category C

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Cat}_{\bullet} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ * & \xrightarrow{c} & \mathrm{Cat} \end{array}$$

the fiber of the latter cartesian fibration at c is precisely $\mathrm{RFib}(C)$. But $\mathrm{ev}_1\colon \mathrm{RFib} \to \mathrm{Cat}$ is a cocartesian fibration as well, since the pullback functors

$$u^*\colon \mathrm{RFib}(B) \to \mathrm{RFib}(A)$$

have a left adjoint $u_!$ for each functor between small categories $u\colon A \to B$.

We define Rep as the full subcategory of RFib whose objects are the fibrations between small categories whose domain has a terminal object. Proposition 12.3.18 implies that taking the codomain of these right fibrations defines a functor

$$\mathrm{ev}_1 : \mathrm{Rep} \to \mathrm{Cat}$$

that is a cocartesian fibration, so that the embedding of Rep into RFib defines a cocartesian functor over Cat.

Lemma 12.4.14. *The functor* $\mathrm{ev}_1 : \mathrm{Rep} \to \mathrm{Cat}$ *is a directed univalent cocartesian fibration. More precisely, there is a canonical pullback square of the form*

$$\begin{array}{ccc} \mathrm{Rep} & \xrightarrow{\simeq} & \mathrm{Cat}_\bullet \\ {\scriptstyle \mathrm{ev}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{((-)^{\mathrm{op}})^{\mathrm{op}}} & \mathrm{Cat} \end{array}$$

Proof. By definition, there is the pullback square below.

$$\begin{array}{ccc} \mathrm{CoRep} & \xrightarrow{\simeq} & \mathrm{Cat}_\bullet \\ {\scriptstyle \mathrm{ev}_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{(-)^{\mathrm{op}}} & \mathrm{Cat} \end{array}$$

There is also an equivalence $\mathrm{RFib}(A) \xrightarrow{\sim} \mathrm{LFib}(A^{\mathrm{op}})$ functorially in A given by Lemma 12.4.9. This yields a pullback square of the form

$$\begin{array}{ccc} \mathrm{RFib} & \xrightarrow{\simeq} & \mathrm{LFib} \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathrm{Cat} & \xrightarrow{(-)^{\mathrm{op}}} & \mathrm{Cat}\,. \end{array}$$

The displayed square therefore follows from the stability of pullback squares under horizontal composition. The first assertion follows from the fact that any cocartesian fibration equivalent to a directed univalent one is directed univalent. □

Lemma 12.4.15. *Recall that there is an ambient universe of large categories* Cat'. *Consider an equivalence of the form* $\varphi' : \mathrm{Cat}' \to \mathrm{Cat}'$ *that we suppose to restrict to an equivalence of the form*

$$\varphi : \mathrm{Cat} \to \mathrm{Cat}\,.$$

We also assume that there is an equivalence $[1] \xrightarrow{\sim} \varphi([1])$ *together with commutative triangles of the form*

$$\begin{array}{ccc} \mathrm{Map}([1], C) & \xrightarrow{\simeq} & \mathrm{Map}([1], \varphi(C)) \\ & {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \searrow \quad \swarrow {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} & \\ & C \times C & \end{array}$$

where the horizontal map is induced by φ *and by the chosen equivalence* $[1] \xrightarrow{\sim} \varphi([1])$. *Then there is an isomorphism of functors* $\mathrm{id}_{\mathrm{Cat}} \xrightarrow{\sim} \varphi$.

Proof. We form the following pullback square.

$$\begin{array}{ccc} \varphi^*\mathrm{Cat}_\bullet & \xrightarrow{\simeq} & \mathrm{Cat}_\bullet \\ {\scriptstyle \varphi^*\pi_{\mathrm{univ}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{\varphi} & \mathrm{Cat} \end{array}$$

We can repeat all the constructions of straightening/unstraightening and of the opposite category functor with respect to $\varphi^*\pi_{\mathrm{univ}}$ instead of π_{univ}. Let $\psi : \mathrm{Cat} \to \mathrm{Cat}$ be the opposite category functor associated to $\varphi^*\pi_{\mathrm{univ}}$. The preceding theorem also holds replacing π_{univ} by $\varphi^*\pi_{\mathrm{univ}}$. However, our assumption on the compatibility of φ with the interval $[1]$ means that φ preserves the properties of being a right (left) fibration and of having a terminal object. In other words, the cocartesian fibration $\mathrm{ev}_1 : \mathrm{Rep} \to \mathrm{Cat}$ is independent of φ, and we get from the appropriate version of Lemma 12.4.14 a pullback square of the form:

$$\begin{array}{ccc} \mathrm{Rep} & \xrightarrow{\simeq} & \varphi^*\mathrm{Cat}_\bullet \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \varphi^*\pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{\psi\circ\psi} & \mathrm{Cat} \end{array}$$

But, since π_{univ} is directed univalent, this means that the identity of Rep determines an isomorphism of functors of the form

$$\varphi \circ \psi \circ \psi \xrightarrow{\sim} ((-)^{\mathrm{op}})^{\mathrm{op}} .$$

But we have pullback squares of the form

$$\begin{array}{ccccc} \mathrm{CoRep} & \xrightarrow{\simeq} & \varphi^*\mathrm{Cat}_\bullet & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \varphi^*\pi_{\mathrm{univ}}} & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{\psi} & \mathrm{Cat} & \xrightarrow{\varphi} & \mathrm{Cat} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{CoRep} & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{(-)^{\mathrm{op}}} & \mathrm{Cat} \end{array}$$

and by univalence of π_{univ}, the identity of CoRep determines an isomorphism of functors

$$(-)^{\mathrm{op}} \xrightarrow{\sim} \varphi \circ \psi .$$

We obtain that

$$(-)^{\mathrm{op}} \circ \psi \xrightarrow{\sim} ((-)^{\mathrm{op}})^{\mathrm{op}} .$$

Therefore, we must have

$$\psi = (-)^{\mathrm{op}} .$$

This implies in turn that $\varphi \circ ((-)^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} ((-)^{\mathrm{op}})^{\mathrm{op}}$, hence $\varphi = \mathrm{id}_{\mathrm{Cat}}$. □

Proposition 12.4.16. *There is a functorial equivalence $C \xrightarrow{\sim} (C^{\mathrm{op}})^{\mathrm{op}}$ inducing a pullback square of the form*

$$\begin{array}{ccc} \mathrm{Rep} & \xrightarrow{\simeq} & \mathrm{Cat}_\bullet \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ \mathrm{Cat} & \xrightarrow{\mathrm{id}_{\mathrm{Cat}}} & \mathrm{Cat} . \end{array}$$

Proof. It follows from Proposition 12.4.5 that we may apply Lemma 12.4.15 to $\varphi = ((-)^{\mathrm{op}})^{\mathrm{op}}$. □

Remark 12.4.17. The previous proposition allows a fairly explicit description of the objects and morphisms of $\mathrm{Cat}_\bullet$, using the equivalence with Rep. An object of Rep is a right fibration $p\colon X \to A$ together with a terminal object x of X. But then the induced functor $X_{/x} \to A_{/p(x)}$ preserves terminal objects and is a morphism of right fibrations over A. Therefore, it is an equivalence. Since the projection $X_{/x} \to X$ is an equivalence as well, this means that, up to equivalence, p is totally determined by the pair (A, a) consisting of a small synthetic category A and of an object a of A. A morphism in Rep is by definition a commutative square in Cat of the form

$$\begin{array}{ccc} A_{/a} & \xrightarrow{u} & B_{/b} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array}$$

where a is an object of A, b an object of B, and p and q are the canonical projections. The functor u is totally determined by the functor

$$\tilde{u}\colon A_{/a} \to A \times_B B_{/b}$$

over A. Since q is a right fibration, this means that this functor is totally determined by the image of the terminal object (a, id_a) in the fiber product $A \times_B B_{/b}$. This image must be a pair of the form (a, v) where $v\colon f(a) \to b$ is a morphism in B. In other words, a morphism in $\mathrm{Cat}_\bullet$ is a pair of the form

$$(f, v)\colon (A, a) \to (B, b)$$

where $f\colon A \to B$ is a functor between small categories, a is an object of A, b is an object of B, and $v\colon f(a) \to b$ is a morphism of B.

This allows to describe objects and morphisms in any Grothendieck construction (i.e. in the total category associated to a functor with values in Cat). Indeed, let $F\colon A \to \mathrm{Cat}$ be a functor from a category A to the category of small categories. We can then form the following pullback square.

$$\begin{array}{ccc} \int_{a:A} F(a) & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle p_F}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ A & \longrightarrow & \mathrm{Cat} \end{array}$$

Any object of $\int_{a:A} F(a)$ will have an image in A. If we also write $F(a)$ the category corresponding to $F(a) : \mathrm{Cat}^{\simeq}$, using the pullback square

$$\begin{array}{ccc} F(a) & \longrightarrow & \int_{a:A} F(a) \\ \downarrow & & \downarrow{\scriptstyle p_F} \\ [0] & \xrightarrow{a} & A \end{array}$$

We see that an object of $\int_{a:A} F(a)$ is a pair (a, x) where a is an object of A and x an object of $F(a)$. A morphism in $\int_{a:A} F(a)$ is a pair of the form

$$(u, v)\colon (a, x) \to (b, y)$$

where $u\colon a \to b$ is a morphism in A and $v\colon F(u)(x) \to y$ is a morphism in $F(b)$ (since one can interpret the pair $(F(u), v)$ as a morphism in $\mathrm{Cat}_\bullet$).

Remark 12.4.18. The embedding

$$\mathrm{Rep} \hookrightarrow \mathrm{RFib}$$

is a cocartesian functor over Cat by definition. The meaning of Proposition 12.4.16 is thus that the Yoneda embedding

$$h_C \colon C \hookrightarrow \mathrm{PSh}(C)$$

is a natural transformation from the identity of Cat to the functor RFib that sends a small category C to $\mathrm{RFib}(C) = \mathrm{PSh}(C)$ and a functor $u \colon C \to D$ to the induced left Kan extension

$$u_! \colon \mathrm{PSh}(C) \to \mathrm{PSh}(D)\,.$$

The associated commutative square

$$\begin{array}{ccc} C & \xrightarrow{u} & D \\ {\scriptstyle h_C}\downarrow & & \downarrow{\scriptstyle h_D} \\ \mathrm{PSh}(C) & \xrightarrow{u_!} & \mathrm{PSh}(D) \end{array}$$

is the one of Proposition 12.3.18.

12.5 Local smallness and functor categories

We begin with consequences of Yoneda for locally small categories. Recall that C is locally small if, for all objects x and y the hom anima $\mathrm{Hom}_C(x, y)$ is small. Equivalently, C is locally small if and only if the functor $(s, t) \colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ has small fibers.

Example 12.5.1. If C is small, then it is locally small.

Example 12.5.2. If C is small, then the category $\mathrm{Fun}(C, \mathrm{An})$ is locally small: given functors $F_0, F_1 \colon C \to \mathrm{An}$ corresponding to left fibrations $p_0 \colon X_0 \to C$ and $p_1 \colon X_1 \to C$, we have an equivalence

$$\mathrm{Fun}(C, \mathrm{An})(F_0, F_1) \simeq \mathrm{Map}_C(X_0, X_1),$$

and the right-hand side is small.

Theorem 12.5.3 (Small limits of animae)**.** *The category* An *has small limits. More precisely, given a small category I and a functor $F \colon I \to \mathrm{An}$, the unique morphism to the terminal functor*

$$h_{I^{\mathrm{op}}} \to (e_I)_{I^{\mathrm{op}}} \qquad \textit{in} \qquad \mathrm{Fun}(I^{\mathrm{op}}, \mathrm{Fun}(I, \mathrm{An}))$$

induces through the Yoneda lemma a cone

$$c \colon \mathrm{Hom}(e_I, F)_I \to F = \mathrm{Hom}(h_{I^{\mathrm{op}}}(-), F)$$

which exhibits $\mathrm{Hom}(e_I, F)$ *as the limit of F in* An*, i.e. the induced map*

$$\mathrm{Hom}_{\mathrm{An}}(x, \mathrm{Hom}(e_I, F)) \to \mathrm{Hom}(x_I, F)$$

is an equivalence for all objects x of An*.*

Proof. The category $\mathrm{Fun}(I, \mathrm{An})$ is locally small, so $\mathrm{Hom}(e_I, F)$ is a small anima. For every small anima x, the cartesian closed structure of An and its pointwise extension to functor categories give natural equivalences

$$\mathrm{Hom}_{\mathrm{An}}(x, \mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{An})}(e_I, F)) \simeq \mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{An})}(x \times e_I, F) \simeq \mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{An})}(x_I, F).$$

Under these isomorphisms, the displayed cone is the universal cone. Hence $\mathrm{Hom}(e_I, F)$ is the limit of F. □

Theorem 12.5.4 (Representables preserve limits)**.** *Let C be a locally small category and let x be an object of C. Then the functor*

$$\mathrm{Hom}_C(x, -) \colon C \to \mathrm{An}$$

preserves limits.

Proof. Let $F \colon I \to C$ admit a limit ℓ. The universal property of ℓ identifies morphisms $x \to \ell$ with cones from x to F. Equivalently, it gives a natural equivalence

$$\mathrm{Hom}_C(x, \ell) \simeq \mathrm{Hom}_{\mathrm{Fun}(I,C)}(x_I, F).$$

A natural transformation from the constant functor x_I to F is precisely a cone over the functor $i \mapsto \mathrm{Hom}_C(x, F(i))$. Thus there is a natural equivalence

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(x_I, F) \simeq \mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{An})}(e_I, \mathrm{Hom}_C(x, F(-))).$$

By Theorem 12.5.3, the latter is the limit of the I-indexed functor $i \mapsto \mathrm{Hom}_C(x, F(i))$. Thus the canonical map

$$\mathrm{Hom}_C(x, \ell) \longrightarrow \lim_{i:I} \mathrm{Hom}_C(x, F(i))$$

is an isomorphism. □

Theorem 12.5.5 (Hom animae of functor categories)**.** *Let C be a locally small category and let I be a small category. Then the hom animae of* $\mathrm{Fun}(I, C)$ *may be computed as*

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(F, G) \simeq \lim_{u:i\to j} \mathrm{Hom}_C(F(i), G(j)),$$

where the right-hand side is defined as the limit of the composite functor

$$\mathrm{Tw}(I) \xrightarrow{(s,t)} I^{\mathrm{op}} \times I \xrightarrow{F^{\mathrm{op}} \times G} C^{\mathrm{op}} \times C \xrightarrow{\mathrm{Hom}_C} \mathrm{An}.$$

In particular, the category $\mathrm{Fun}(I, C)$ *is again locally small.*

Proof. Choose a universe so that C is small, and consider the fully faithful functor $h_C \colon C \to \mathrm{PSh}(C)$. Then $h_C \colon \mathrm{Fun}(I, C) \to \mathrm{Fun}(I, \mathrm{PSh}(C))$ is fully faithful as well. Given $F, G \colon I \to C$, we have

$$\mathrm{Hom}(F, G) \simeq \mathrm{Hom}(h_C(F), h_C(G)).$$

Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \overset{\mathrm{Tw}(G)}{\frown} & & \\
\mathrm{Tw}(I) & \overset{w}{\dashrightarrow} & W & \longrightarrow & \mathrm{Tw}(C) \\
{\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle (s,t)} \\
I^{\mathrm{op}} \times I & \xrightarrow{\mathrm{id} \times G} & I^{\mathrm{op}} \times C & \xrightarrow{G^{\mathrm{op}} \times \mathrm{id}} & C^{\mathrm{op}} \times C.
\end{array}$$

We showed in Lemma 12.2.7 that the unique functor w obtained from this diagram is an initial functor. Let e be the terminal object of $\mathrm{Fun}(\mathrm{Tw}(I), \mathrm{An})$, so that we have an identification

$$\mathrm{Hom}_I = (s,t)_!(e)$$

of functors $I^{\mathrm{op}} \times I \to \mathrm{An}$. It follows from the diagram that there is an identification

$$\mathrm{Hom}_C(G(-),-) = (G^{\mathrm{op}} \times \mathrm{id}_C)^* \mathrm{Hom}_C(-,-) \simeq (\mathrm{id}_{I^{\mathrm{op}}} \times G)_! \mathrm{Hom}_I(-,-) \simeq (\mathrm{id}_{I^{\mathrm{op}}} \times G)_!(s,t)_!(e)$$

of functors $I^{\mathrm{op}} \times C \to \mathrm{An}$. The covariant Yoneda embedding $C^{\mathrm{op}} \to \mathrm{Fun}(C, \mathrm{An})$ is fully faithful. Applying it objectwise to F^{op} and G^{op}, and then using the preceding identification and the two adjunctions, gives

$$\begin{aligned}
\mathrm{Hom}(F,G) &\simeq \mathrm{Hom}(\mathrm{Hom}_C(G(-),-), \mathrm{Hom}_C(F(-),-)) \\
&\simeq \mathrm{Hom}((\mathrm{id}_{I^{\mathrm{op}}} \times G)_!(s,t)_!(e), \mathrm{Hom}_C(F(-),-)) \\
&\simeq \mathrm{Hom}(e, (s,t)^*(\mathrm{id}_{I^{\mathrm{op}}} \times G)^* \mathrm{Hom}_C(F(-),-)) \\
&\simeq \lim_{\mathrm{Tw}(I)} (s,t)^*((i,j) \mapsto \mathrm{Hom}_C(F(i), G(j))).
\end{aligned}$$

□

Remark 12.5.6. Consider a category I and a functor $\Phi\colon I^{\mathrm{op}} \times I \to D$. If D admits limits indexed by $\mathrm{Tw}(I)$, then we may define the *end of* Φ as the limit in D of the composite $\mathrm{Tw}(I) \xrightarrow{(s,t)} I^{\mathrm{op}} \times I \xrightarrow{\Phi} D$. This is sometimes denoted $\int_{i \in I} \Phi(i,i)$, though we will mostly avoid this notation. In this language, the previous theorem may be rephrased as

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(F,G) \simeq \int_{i \in I} \mathrm{Hom}_C(F(i), G(i)).$$

12.6 Yoneda density and categories of elements

The Yoneda lemma describes morphisms out of a single representable presheaf. Density gives the complementary global statement: every presheaf is canonically a colimit of representable ones, indexed by its category of elements. This presentation will be the main input for the free-cocompletion property below and the pointwise Kan-extension formulas of the next chapter.

12.6.1 Colimits through categories of elements

Consider a small category A and consider a functor $F\colon A^{\mathrm{op}} \to \mathrm{An}$, corresponding to a cocartesian fibration $E \to A^{\mathrm{op}}$. We write $A_{/F} := E^{\mathrm{op}}$, so that we may write E as $(A_{/F})^{\mathrm{op}}$:

$$\begin{array}{ccc}
(A_{/F})^{\mathrm{op}} & \longrightarrow & \mathrm{An}_\bullet \\
{\scriptstyle p^{\mathrm{op}}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\
A^{\mathrm{op}} & \xrightarrow{F} & \mathrm{An}.
\end{array}$$

The functor $p_!\colon \mathrm{PSh}(A_{/F}) \to \mathrm{PSh}(A)$ sends the terminal functor e to F, hence we have

$$\mathrm{colim}_{A^{\mathrm{op}}} F = \mathrm{colim}_{A^{\mathrm{op}}} p_!(e) \simeq \mathrm{colim}_{(A_{/F})^{\mathrm{op}}} e.$$

To compute this, we must factor the map $(A_{/F})^{\mathrm{op}} \to e$ into an initial functor and a left fibration:

$$\begin{array}{ccc} (A_{/F})^{\mathrm{op}} & \xrightarrow{\text{initial}} & \Pi_\infty((A_{/F})^{\mathrm{op}}) \\ \| & & \downarrow \text{left fib} \\ (A_{/F})^{\mathrm{op}} & \longrightarrow & e. \end{array}$$

Thus

$$\mathrm{colim}_{A^{\mathrm{op}}} F \simeq \Pi_\infty((A_{/F})^{\mathrm{op}}) \simeq \Pi_\infty(A_{/F}),$$

where the second isomorphism is Corollary 12.2.6.

Corollary 12.6.1. *Let $F\colon A^{\mathrm{op}} \to \mathrm{An}$ be a representable functor. Then $\mathrm{colim}_{A^{\mathrm{op}}} F$ is contractible.*

Proof. We have $\mathrm{colim}_{A^{\mathrm{op}}} F = \Pi_\infty(A_{/F})$. But since F is representable, the category $A_{/F}$ admits a terminal object, meaning that there is a terminal functor $* \to A_{/F}$. By the dual of Example 9.5.4, this implies that $\Pi_\infty(A_{/F})$ is contractible. □

12.6.2 Yoneda density

The preceding computation gives the colimit of an anima-valued presheaf as an object of An. We now consider a different colimit, taken in the presheaf category itself. For a presheaf $F\colon C^{\mathrm{op}} \to \mathrm{An}$, each object (x, s) of $C_{/F}$ gives a morphism $h_C(x) \to F$ by the Yoneda lemma. We will show that this cocone exhibits F as the colimit of these representable presheaves.

Let $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ be a directed univalent cocartesian fibration defining a universe of small categories with corresponding universe of small animae An.

Remark 12.6.2. Recall that, for any small category C, the presheaf category $\mathrm{PSh}(C)$ admits all small colimits: given a small category I, the colimit functor

$$\mathrm{colim}_I\colon \mathrm{Fun}(I, \mathrm{PSh}(C)) \to \mathrm{PSh}(C)$$

may be described as follows: given the projection $\mathrm{pr}_2\colon I \times C^{\mathrm{op}} \to C^{\mathrm{op}}$, it is given by

$$\mathrm{Fun}(I, \mathrm{PSh}(C)) \simeq \mathrm{Fun}(I \times C^{\mathrm{op}}, \mathrm{An}) \xrightarrow{(\mathrm{pr}_2)_!} \mathrm{PSh}(C).$$

A fundamental computation of colimits in a presheaf category is the following one.

Theorem 12.6.3. *Let C be a small category. Then the colimit of the Yoneda embedding*

$$h_C\colon C \to \mathrm{PSh}(C)$$

is the terminal object.

Proof. Under the equivalence $\mathrm{Fun}(C, \mathrm{PSh}(C)) \simeq \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An})$, pulling back along the projection $\mathrm{pr}_1\colon C^{\mathrm{op}} \times C \to C^{\mathrm{op}}$

$$\mathrm{pr}_1^*\colon \mathrm{PSh}(C) \to \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An})$$

corresponds to the functor producing C-indexed constant objects

$$\mathrm{PSh}(C) \to \mathrm{Fun}(C, \mathrm{PSh}(C))\,.$$

Passing to left adjoint functors we thus see that there is a canonical commutative triangle of the form

$$\begin{array}{ccc} \mathrm{Fun}(C, \mathrm{PSh}(C)) & = & \mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An}) \\ & {\scriptstyle \mathrm{colim}_C} \searrow \quad \swarrow {\scriptstyle (\mathrm{pr}_1)_!} & \\ & \mathrm{PSh}(C) & \end{array}$$

so that the horizontal equivalence sends h_C to Hom_C. In other words, we have to prove that $(\mathrm{pr}_1)_!(\mathrm{Hom}_C)$ is a terminal object. Since there is always a canonical map to the terminal object and since the property for a natural transformation to be invertible can be checked objectwise, given a small category I, the property for an object F of $\mathrm{Fun}(I, \mathrm{An})$ to be a terminal object is equivalent to the property that $F(i)$ is terminal in An for all $i\colon I^{\simeq}$. Therefore, it suffices now to prove that the evaluation of $(\mathrm{pr}_1)_!(\mathrm{Hom}_C)$ at each $x\colon C^{\simeq}$ is terminal. Since the projection map from $C^{\mathrm{op}} \times C$ to C is a cocartesian fibration, we know from Lemma 9.6.5 that we may compute the value at some object x of C as the colimit of the functor over the fiber of pr_1 at x, which in this case is just C. So it remains to show that

$$\operatorname*{colim}_{y\colon C} \mathrm{Hom}_C(x, y)$$

is contractible for each object x. This is a particular case of Corollary 12.6.1 with $A = C^{\mathrm{op}}$. □

Corollary 12.6.4 (Yoneda density)**.** *Let C be a small category and let $F\colon C^{\mathrm{op}} \to \mathrm{An}$ be a functor corresponding to a right fibration p_F:*

$$\begin{array}{ccc} C_{/F} & \longrightarrow & \mathrm{An}_\bullet^{\mathrm{op}} \\ {\scriptstyle p_F}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi^{\mathrm{op}}_{\mathrm{univ}}} \\ C & \xrightarrow[F^{\mathrm{op}}]{} & \mathrm{An}^{\mathrm{op}}. \end{array}$$

Then F is canonically a colimit of representable presheaves: there is a preferred isomorphism

$$\operatorname*{colim}_{(x,s)\colon C_{/F}} h_C(x) \xrightarrow{\sim} F$$

in $\mathrm{PSh}(C)$.

Proof. The functor $p_F\colon C_{/F} \to C$ induces an adjunction

$$(p_F)_!\colon \mathrm{PSh}(C_{/F}) \rightleftarrows \mathrm{PSh}(C) :p_F^*.$$

By Theorem 12.6.3 the colimit $\mathrm{colim}_{(x,s)\in C_{/F}} h_{C_{/F}}(x, s)$ is a terminal object $e_{C_{/F}}$ in $\mathrm{PSh}(C_{/F})$, hence it follows that

$$\begin{aligned} \mathrm{colim}_{(x,s)\in C_{/F}} h_C(x) &\simeq \mathrm{colim}(h_C \circ p_F) \simeq \mathrm{colim}((p_F)_! h_{C_{/F}}) \\ &\simeq (p_F)_!(\mathrm{colim}\, h_{C_{/F}}) \simeq (p_F)_!(e_{C_{/F}}). \end{aligned}$$

So it remains to show that there exists a canonical isomorphism $(p_F)_!(e_{C_{/F}}) \xrightarrow{\sim} F$. To see this, consider the diagram

$$\begin{array}{ccccc} C_{/F} & \longrightarrow & * & \longrightarrow & \mathrm{An}_\bullet^{\mathrm{op}} \\ \| & \lrcorner & \| & \lrcorner & \downarrow{\scriptstyle \pi^{\mathrm{op}}_{\mathrm{univ}}} \\ C_{/F} & \longrightarrow & * & \xrightarrow{e} & \mathrm{An}^{\mathrm{op}}. \end{array}$$

The bottom functor is the one corresponding to $e_{C_{/F}} : C_{/F}^{\mathrm{op}} \to \mathrm{An}$, hence its associated right fibration is the identity functor on $C_{/F}$. The left Kan extension is obtained by factoring its composite with p_F into a terminal functor followed by a right fibration. Since p_F is already a right fibration, this factorization is

$$\begin{array}{ccc} C_{/F} & = & C_{/F} \\ \| & & \downarrow{\scriptstyle p_F} \\ C_{/F} & \longrightarrow & C. \end{array}$$

Hence the functor $(p_F)_!(e_{C_{/F}}) : C^{\mathrm{op}} \to \mathrm{An}$ is the functor classifying the right fibration $p_F : C_{/F} \to C$, which is F. □

Remark 12.6.5. The reason for the notation $C_{/F}$ in the previous statement is that the category $C_{/F}$ fits into a pullback square of the following form:

$$\begin{array}{ccc} C_{/F} & \longrightarrow & \mathrm{PSh}(C)_{/F} \\ {\scriptstyle p_F}\downarrow & \lrcorner & \downarrow \\ C & \xrightarrow[h_C]{} & \mathrm{PSh}(C). \end{array}$$

So objects in $C_{/F}$ are really pairs (x, s) where x is an object of C and $s : h_C(x) \to F$ is a morphism in $\mathrm{PSh}(C)$.

Remark 12.6.6. Let I and C be categories, and let $F : I \to C$ be a functor. Assume that $\lim_I F$ exists in C. Then $\mathrm{const}_{\lim_I F}$ is the limit of F relative to the constant-diagram functor $C \to \mathrm{Fun}(I, C)$. If $\nu : \mathrm{const}_{\lim_I F} \to F$ denotes the universal cone, then the pair $(\lim_I F, \nu)$ is a terminal object of $C_{/F}$:

$$\begin{array}{ccc} C_{/F} & \longrightarrow & \mathrm{Fun}(I, C)_{/F} \\ \downarrow & \lrcorner & \downarrow \\ C & \xrightarrow[\mathrm{const}]{} & \mathrm{Fun}(I, C). \end{array}$$

12.6.3 Right fibrations and density

Proposition 12.6.7. *Let A be a small category. The full embedding*

$$\mathrm{PSh}(A) = \mathrm{RFib}(A) \hookrightarrow \mathrm{Cat}_{/A}$$

has a left adjoint L sending each functor $u : X \to A$ to the colimit of the functor

$$X \xrightarrow{u} A \xrightarrow{h_A} \mathrm{PSh}(A)\,.$$

Proof. Let $F\colon A^{\mathrm{op}} \to \mathrm{An}$ be a presheaf with corresponding right fibration $p_F\colon A_{/F} \to A$. If e denotes the constant presheaf on X with value the terminal object of An, we see that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}_{/A}}((X,u),(A_{/F},p_F)) &\simeq \mathrm{Hom}_{\mathrm{Cat}_{/X}}((X,\mathrm{id}_X),(X\times_A A_{/F},\mathrm{pr}_1))\\ &\simeq \mathrm{Hom}_{\mathrm{RFib}(X)}((X,\mathrm{id}_X),(X\times_A A_{/F},\mathrm{pr}_1))\\ &\simeq \mathrm{Hom}(e,u^*(F))\\ &\simeq \mathrm{Hom}(u_!(e),F)\,.\end{aligned}$$

It suffices to check that $u_!(e)$ is the colimit of the composition of the Yoneda embedding h_A with the functor u. By virtue of Theorem 12.6.3, the colimit of the Yoneda embedding of X is e. Since $u_!$ commutes with colimits, and since $u_! \circ h_X = h_A \circ u$, this proves that

$$\mathrm{Hom}_{\mathrm{Cat}_{/A}}((X,u),(A_{/F},p_F)) = \mathrm{Hom}(\mathrm{colim}_{x:X}\, h_A(u(x))),F)$$

holds functorially in F. We conclude by applying Theorem 12.3.20 to the embedding of the category $\mathrm{PSh}(A)^{\mathrm{op}}$ into $(\mathrm{Cat}_{/A})^{\mathrm{op}}$. □

Remark 12.6.8. The left adjoint of the proposition above is natural with respect to functors $u_!$. In other words, for any functor between small categories

$$u\colon A \to B$$

there is a canonical commutative square

$$\begin{array}{ccc}\mathrm{Cat}_{/A} & \xrightarrow{L} & \mathrm{PSh}(A)\\ {\scriptstyle u_!}\downarrow & & \downarrow{\scriptstyle u_!}\\ \mathrm{Cat}_{/B} & \xrightarrow{L} & \mathrm{PSh}(B)\end{array}$$

corresponding to the commutative square of right adjoints below.

$$\begin{array}{ccc}\mathrm{PSh}(B) & \hookrightarrow & \mathrm{Cat}_{/B}\\ {\scriptstyle u^*}\downarrow & & \downarrow{\scriptstyle u^*}\\ \mathrm{PSh}(A) & \hookrightarrow & \mathrm{Cat}_{/A}\end{array}$$

Proposition 12.6.9. *Let $u\colon A \to B$ be a right fibration between small categories, corresponding to a presheaf $F\colon B^{\mathrm{op}} \to \mathrm{An}$.*

$$\begin{array}{ccc}A^{\mathrm{op}} & \longrightarrow & \mathrm{An}_\bullet\\ {\scriptstyle u^{\mathrm{op}}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}}\\ B^{\mathrm{op}} & \xrightarrow{F} & \mathrm{An}\end{array}$$

Then the canonical commutative square

$$\begin{array}{ccc}A & \xrightarrow{h_A} & \mathrm{PSh}(A)\\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u_!}\\ B & \xrightarrow{h_B} & \mathrm{PSh}(B)\end{array}$$

is a pullback square in which all vertical maps are right fibrations with small fibers and the functor $u_!$ induces an equivalence

$$\mathrm{PSh}(A) \xrightarrow{\sim} \mathrm{PSh}(B)_{/F}$$

over $\mathrm{PSh}(B)$.

Proof. We observe that the functor $u_!\colon \mathrm{Cat}_{/A} \to \mathrm{Cat}_{/B}$ sends the full subcategory $\mathrm{RFib}(A)$ to $\mathrm{RFib}(B)$. The composite

$$\mathrm{RFib}(B) \hookrightarrow \mathrm{Cat}_{/B} \xrightarrow{L} \mathrm{RFib}(B)$$

is the identity, this means that there is the canonical commutative diagram below.

$$\begin{array}{ccccc}
\mathrm{PSh}(A) & \xrightarrow{\sim} & \mathrm{RFib}(A) & \hookrightarrow & \mathrm{Cat}_{/A} \\
\downarrow{\scriptstyle u_!} & & \downarrow{\scriptstyle u_!} & & \downarrow{\scriptstyle u_!} \\
\mathrm{PSh}(B) & \xrightarrow{\sim} & \mathrm{RFib}(B) & \hookrightarrow & \mathrm{Cat}_{/B}
\end{array}$$

The square on the right hand side is a pullback square: since the two horizontal maps are full embeddings, it suffices to check this on objects, which follows from Proposition 4.2.16. In particular, since $u_!\colon \mathrm{Cat}_{/A} \to \mathrm{Cat}_{/B}$ is a right fibration, so is the functor

$$u_!\colon \mathrm{RFib}(A) \to \mathrm{RFib}(B)\,.$$

It is now clear that the equivalence $\mathrm{Cat}_{/A} \xrightarrow{\sim} (\mathrm{Cat}_{/B})_{/(A,u)}$ canonically induced by $u_!$ restricts to an equivalence

$$\mathrm{RFib}(A) \xrightarrow{\sim} \mathrm{RFib}(B)_{/(A,u)}\,.$$

Therefore, since the functor $u_!\colon \mathrm{PSh}(A) \to \mathrm{PSh}(B)$ sends the terminal object to F, it induces an equivalence

$$\mathrm{PSh}(A) \xrightarrow{\sim} \mathrm{PSh}(B)_{/F}\,.$$

over $\mathrm{PSh}(B)$. □

12.7 Small presheaves

We now consider presheaves on categories which need not be small. By enlarging the universe, we can still form representable presheaves and their small colimits. The presheaves obtained in this way form the small-presheaf completion. For a small category, Yoneda density identifies this completion with its presheaf category.

Construction 12.7.1 (Small-presheaf completion)**.** Let C be a category. We choose a universe Cat' enlarging Cat such that both C and Cat are Cat'-small, and write An' for the full subcategory of animae in Cat'. We let C' be the image of the map

$$\sum_{X:\mathrm{Cat}^{\simeq}} \mathrm{Hom}_{\mathrm{Cat}'}(X, C) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}')\,, \quad (u\colon X \to C) \mapsto \operatorname*{colim}_{x:X} h_C(u(x))$$

where $h_C\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}')$ is the Yoneda embedding. A presheaf $F\colon C^{\mathrm{op}} \to \mathrm{An}'$ is said to be *small* if it is a term of C'. We define $\widehat{C}$ to be the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}')$ spanned by small presheaves.

For instance, any representable presheaf is small, hence we have a Yoneda embedding

$$h_C \colon C \to \widehat{C}\,.$$

Lemma 12.7.2. *A category C has small colimits if and only if the small Yoneda embedding*

$$h_C \colon C \to \widehat{C}$$

has a left adjoint

$$L \colon \widehat{C} \to C\,.$$

Proof. Assume that C has small colimits. Given any small category X and any functor $u \colon X \to C$, we let a be the colimit of u in C and F be the small presheaf defined as the colimit of $h_C \circ u$. For any $b : C$ we thus have

$$\begin{aligned}
\mathrm{Hom}_C(a,b) &\simeq \lim_{x:X^{\mathrm{op}}} \mathrm{Hom}_C(u(x),b) \\
&\simeq \lim_{x:X^{\mathrm{op}}} \mathrm{Hom}_{\widehat{C}}(h_C(u(x)), h_C(b)) \\
&\simeq \mathrm{Hom}_{\widehat{C}}(\operatorname*{colim}_{x:X} h_C(u(x)), h_C(b)) \\
&= \mathrm{Hom}_{\widehat{C}}(F, h_C(b))\,.
\end{aligned}$$

functorially in $b : C$. Applying Theorem 11.2.13 to h_C^{op} now implies that h_C has a left adjoint.

Conversely, if L is a left adjoint of the small Yoneda embedding h_C, for any small category X and any functor $u \colon X \to C$, the colimit of $h_C \circ u$ in $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}')$ yields by definition a small presheaf F. The adjunction $L \dashv h_C$ identifies maps out of $L(F)$ with cocones under u, so $L(F)$ is a colimit of u in C. □

Lemma 12.7.3. *If C is locally small, then so is $\widehat{C}$.*

Proof. Let G be a small presheaf on C. Then G is in fact a functor from C^{op} to An: given any object $a : C^{\simeq}$, $G(a)$ is a small colimit of animae of the form $\mathrm{Hom}_C(a, u(x))$, hence is small. This means that the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}')$ spanned by those F such that $\mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}')}(F, G)$ is small contains all representable presheaves and is stable under small colimits (because the embedding of An into An′ preserves the formation of small limits). Every small presheaf is a small colimit of representables, so this subcategory contains all of $\widehat{C}$. □

Lemma 12.7.4. *If C is small then $\widehat{C} = \mathrm{PSh}(C)$ holds.*

Proof. We have seen in the proof of Lemma 12.7.3 that $\widehat{C}$ is a full subcategory of $\mathrm{PSh}(C)$. The converse embedding follows from Corollary 12.6.4, since the domain of the right fibration associated to any functor from C^{op} to An is small. □

Lemma 12.7.5. *For any functor $u \colon C \to D$, the associated left Kan extension functor*

$$u_! \colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}') \to \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{An}')$$

restricts to a functor of type

$$u_! \colon \widehat{C} \to \widehat{D}\,.$$

(a) If C and D have small colimits, there is a natural transformation of the form

$$comp_{u,F} \colon L(u_!(F)) \to u(L(F))$$

for all $F : \widehat{C}$. This natural transformation is invertible if and only if the functor u commutes with small colimits.

(b) If furthermore C is small and D is locally small, then the functor $u_!$ is left adjoint to the functor

$$u^* \colon \widehat{D} \to \widehat{C} = \mathrm{PSh}(C)\,.$$

Proof. Choose a small presentation

$$F \simeq \mathrm{colim}_{i:I}\, h_C(x_i).$$

Since $u_!$ preserves small colimits and representable presheaves,

$$u_!F \simeq \mathrm{colim}_{i:I}\, h_D(u(x_i)),$$

so $u_!F$ is small.

If C and D have small colimits, this presentation identifies $L_C(F)$ with $\mathrm{colim}_I\, x_i$ and $L_D(u_!F)$ with $\mathrm{colim}_I\, u(x_i)$. The canonical comparison from the latter colimit to $u(\mathrm{colim}_I\, x_i)$ gives $comp_{u,F}$. It is natural and independent of the chosen presentation by the universal properties of L_C and L_D. It is invertible for every small presheaf F if and only if u preserves small colimits.

If C is small and D locally small, Lemma 12.7.3 and Lemma 12.7.4 imply that the functor

$$u^* \colon \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{An}') \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}')$$

restricts to a functor of the form

$$u^* \colon \widehat{D} \to \widehat{C} = \mathrm{PSh}(C)\,.$$

Therefore the canonical adjunction $(u_!, u^*)$ restricts to small presheaves. □

12.8 Presheaf categories as free cocompletions

We will now combine the previous results to show that the presheaf category $\mathrm{PSh}(A)$ of a small category A is the free completion of A under small colimits.

Theorem 12.8.1 (Kan)**.** *Let A be a small synthetic category and C any synthetic category with small colimits. For each functor $u \colon A \to C$, there is a functor that commutes with small colimits*

$$Lu_! \colon \mathrm{PSh}(A) \to C$$

equipped with a canonical identification

$$Lu_! \circ h_A \simeq u$$

of functors from A to C. Furthermore, if C is locally small, the functor $Lu_!$ is left adjoint to the functor

$$u^*h_C \colon C \to \mathrm{PSh}(A)\,, \quad y \mapsto \mathrm{Hom}_C(u(-), y)\,.$$

Proof. We choose a universe of large animae An' and we write $\mathrm{PSh}(D)' = \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{An}')$ for the category of functors from a category D with values in An'. The functor

$$u_!\colon \mathrm{PSh}(A)' \to \mathrm{PSh}(C)'$$

commutes with (small) colimits (being a left adjoint). We have a commutative square of the following form.

$$\begin{array}{ccc} \mathrm{PSh}(A) & \xrightarrow{u_!} & \widehat{C} \\ \downarrow & & \downarrow \\ \mathrm{PSh}(A)' & \xrightarrow{u_!} & \mathrm{PSh}(C)' \end{array}$$

This implies that the induced functor

$$u_!\colon \mathrm{PSh}(A) \to \widehat{C}$$

preserves colimit cocones indexed by small categories. The functor

$$L\colon \widehat{C} \to C$$

is a left adjoint by Lemma 12.7.2, hence preserves all colimits. This implies that the composite $Lu_!$ preserves small colimits.

If furthermore $C$is locally small then Lemma 12.7.5 tells us that $u_!$ is a left adjoint to u^*. Since L is a left adjoint to the small Yoneda embedding, the composite $Lu_!$ is a left adjoint to $u^* h_C$. □

Corollary 12.8.2. *Let A be small and let C with small colimits. Then a functor*

$$\Phi\colon \mathrm{PSh}(A) \to C$$

commutes with small colimits if and only if there is an isomorphism of the of the form $\Phi \simeq Lu_!$ for $u := \Phi \circ h_A$. In particular, if furthermore C is locally small, the property for a functor Φ as above to commutes with small colimits is equivalent to the property of having a right adjoint.

Proof. For $u = \Phi h_A$, the canonical adjunction

$$u_!\colon \mathrm{PSh}(A)' \rightleftarrows \mathrm{PSh}(C)' : u^*.$$

induces an adjunction of the form:

$$u_!\colon \mathrm{Fun}(\mathrm{PSh}(A), \mathrm{PSh}(A)') \rightleftarrows \mathrm{Fun}(\mathrm{PSh}(A), \mathrm{PSh}(C)') : u^*.$$

For a presheaf $F\colon A^{\mathrm{op}} \to \mathrm{An}$ we have

$$\mathrm{Hom}(h_A^{\mathrm{op}}(-), F) = F$$

by the Yoneda lemma. Then Φ induces a natural transformation

$$\mathrm{Hom}_{\mathrm{PSh}(A)}(-,-) \to \mathrm{Hom}_C(\Phi^{\mathrm{op}}(-), \Phi(-)),$$

and hence we obtain a canonical homomorphism

$$F = \mathrm{Hom}(h_A(-), F) \to \mathrm{Hom}_C(\Phi h_A(-), \Phi(F)) = \mathrm{Hom}_C(u(-), \Phi(F)) = u^* h_C \Phi(F),$$

which is functorial in F. In other words, we have constructed a natural transformation

$$\iota \to u^* h_C \Phi(-)$$

where $\iota\colon \mathrm{PSh}(A) \hookrightarrow \mathrm{PSh}(A)'$ denotes the canonical full embedding. By passing to adjoints, this yields a canonical morphism

$$u_! \to h_C\, \Phi.$$

Applying the functor L and using that $L\, h_C \simeq \mathrm{id}_C$, this yields a canonical map

$$Lu_! \to \Phi$$

inducing the identity of u when composing with h_A. Therefore, we know that the map $Lu_!(F) \to \Phi(F)$ is an isomorphism whenever F is a representable presheaf. But since both $Lu_!$ and Φ commute with colimits, the collection of objects on which the map $Lu_! \to \Phi$ is an isomorphism is closed under colimits in $\mathrm{PSh}(A)$, and so by Corollary 12.6.4 it is an equivalence for an arbitrary presheaf F.

The last assertion comes from the fact that any functor (isomorphic to a functor) of the form $Lu_!$ □

Taking $A = *$ in Theorem 12.8.1, an object y of a category C admitting small colimits gives a colimit-preserving functor $\overline{y}\colon \mathrm{An} \to C$. For a small anima s, we write

$$s \otimes y := \overline{y}(s) = \mathrm{colim}_s\, y$$

and call this the *tensor* of y by s. Thus $s \otimes y$ is the colimit of the constant diagram on y indexed by s.

Remark 12.8.3. Local smallness is necessary for the right-adjoint assertion in Corollary 12.8.2. Indeed, if every colimit-preserving functor $\mathrm{An} \to C$ has a right adjoint, then, for each object y of C, the tensor functor $s \mapsto s \otimes y$ has a right adjoint $R\colon C \to \mathrm{An}$. Hence there is a natural equivalence

$$\mathrm{Hom}_C(s \otimes y, x) \simeq \mathrm{Hom}_{\mathrm{An}}(s, R(x)).$$

In particular, taking s to be the terminal anima we see that $\mathrm{Hom}_C(y, x) \simeq \mathrm{Hom}_{\mathrm{An}}(*, R(x)) = R(x)$, and hence C is locally small.

Notation 12.8.4. Given two categories C and D, we write $\mathrm{Fun}_!(C, D)$ for the full subcategory of $\mathrm{Fun}(C, D)$ spanned by the functors from C to D that preserve small colimits. We also write $\mathrm{Fun}^L(C, D)$ for the full subcategory of $\mathrm{Fun}(C, D)$ spanned by functors that admit a right adjoint. There are canonical full embeddings of the form:

$$\mathrm{Fun}^L(C, D) \hookrightarrow \mathrm{Fun}_!(C, D) \hookrightarrow \mathrm{Fun}(C, D)\,.$$

Corollary 12.8.5. *Let A be as small synthetic category and C a synthetic category with small colimits. Then the Yoneda embedding $h_A\colon A \to \mathrm{PSh}(A)$ induces an equivalence:*

$$\mathrm{Fun}_!(\mathrm{PSh}(A), C) \xrightarrow{\sim} \mathrm{Fun}(A, C)\,.$$

In particular, the rule assigning to a functor $u\colon A \to C$ the functor $Lu_!$ is itself a functor.

Furthermore, if C is locally small, then we have

$$\mathrm{Fun}^L(\mathrm{PSh}(A), C) = \mathrm{Fun}_!(\mathrm{PSh}(A), C)\,.$$

Proof. The last assertion is a reformulation of the last assertion of Corollary 12.8.2.

We observe that there is a canonical identification of the form:

$$\mathrm{Fun}_!(\mathrm{PSh}(A), \mathrm{Fun}([1], C)) = \mathrm{Fun}([1], \mathrm{Fun}_!(\mathrm{PSh}(A), C)) .$$

By virtue of Corollary 6.3.9 it suffices thus to check that the Yoneda embedding $h_A \colon A \to \mathrm{PSh}(A)$ induces an equivalence:

$$\mathrm{Fun}_!(\mathrm{PSh}(A), C)^{\simeq} \xrightarrow{\sim} \mathrm{Fun}(A, C)^{\simeq} .$$

This follows immediately from the first parts of Theorem 12.8.1 and of Corollary 12.8.2. □

12.9 Yoneda with coefficients

The end formula for hom animae in functor categories has a relative colimit form. It expresses a diagram as a colimit of representable diagrams with coefficients and leads to a duality between colimit-preserving functors out of a presheaf category and functor-valued diagrams.

Recall that $s \otimes x = \mathrm{colim}_s x$ for a small anima s and an object x of a category admitting small colimits. These tensors are functorial in both variables by the preceding construction. For $j : I$ and $X : C$, write $j \otimes X$ for the diagram

$$i \mapsto \mathrm{Hom}_I(j, i) \otimes X.$$

The hom-anima formula for tensors and the Yoneda lemma give

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(j \otimes X, G) \simeq \mathrm{Hom}_C(X, G(j)).$$

Indeed, natural transformations on the left correspond to natural transformations $\mathrm{Hom}_I(j, -) \to \mathrm{Hom}_C(X, G(-))$; if necessary we form the latter in a larger universe.

Proposition 12.9.1. *Let I be a small category and C a category with small colimits. For any functor $F \colon I \to C$, the colimit of the functor*

$$\mathrm{Tw}(I)^{\mathrm{op}} \to \mathrm{Fun}(I, C) , \quad (j, i) \mapsto j \otimes F(i)$$

is F itself.

Proof. For any functor $G \colon I \to C$, we have

$$\mathrm{Hom}(j \otimes X, G) = \mathrm{Hom}(X, G(j))$$

for any X in C. The corollary is therefore a direct reformulation of Theorem 12.5.5. □

Construction 12.9.2. Let A be a small category and B a category with small colimits. Using the tensors $s \otimes x = \mathrm{colim}_s x$, we define the *Yoneda functor of A with coefficients in B* as the functor

$$h_{A/B} \colon A \times B \to \mathrm{Fun}(A^{\mathrm{op}}, B) , \quad (a, x) \mapsto a \otimes x := \mathrm{Hom}_A(-, a) \otimes x .$$

Given any functor of the form $\Phi \colon \mathrm{Fun}(A^{\mathrm{op}}, B) \to C$ we get a functor

$$\Phi \circ h_{A/B} \colon A \times B \to C$$

which in turn yields by transposition a functor

$$^t\Phi\colon B \to \mathrm{Fun}(A, C)\,.$$

Given $x : B$, the functor $^t\Phi(x)$ assigns to each $a : A$ the term $\Phi(a \otimes x)$. In particular, we see that, if ever Φ preserves small colimits, then so does $^t\Phi$.

Theorem 12.9.3. *Let A be a small category and B and C two categories with small colimits. The assignment $\Phi \mapsto {}^t\Phi$ defines an equivalence*

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), C) \xrightarrow{\sim} \mathrm{Fun}_!(B, \mathrm{Fun}(A, C))\,.$$

Proof. We observe that we have a canonical equivalence of the form

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), \mathrm{Fun}([1], C)) \simeq \mathrm{Fun}([1], \mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), C))\,.$$

Similarly, we have

$$\mathrm{Fun}_!(B, \mathrm{Fun}(A, \mathrm{Fun}([1], C))) \simeq \mathrm{Fun}([1], \mathrm{Fun}_!(B, \mathrm{Fun}(A, C)))\,.$$

The argument below is uniform in the target category. In view of these identifications, Corollary 6.3.9 reduces the theorem to proving, for every target C with small colimits, that the assignment $\Phi \mapsto {}^t\Phi$ induces an equivalence on groupoid cores of the form

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, B), C)^{\simeq} \xrightarrow{\sim} \mathrm{Fun}_!(B, \mathrm{Fun}(A, C))^{\simeq}\,.$$

Indeed, applying this core-level statement with target $\mathrm{Fun}([1], C)$ gives, under the preceding identifications, the equivalence on arrow animae required by the Fundamental Theorem. We first record the relative co-Yoneda formula. Applying Proposition 12.9.1 to $F\colon A^{\mathrm{op}} \to B$, and using Remark 12.3.12, gives an identification

$$F = \mathrm{colim}_{(\alpha\colon a \to a') : \mathrm{Tw}(A)^{\mathrm{op}}}\, a \otimes F(a')$$

in $\mathrm{Fun}(A^{\mathrm{op}}, B)$.

Now let $\Psi\colon B \to \mathrm{Fun}(A, C)$ preserve small colimits. For $F\colon A^{\mathrm{op}} \to B$, define

$$\Psi'(F) := \mathrm{colim}_{(\alpha\colon a \to a') : \mathrm{Tw}(A)^{\mathrm{op}}}\, \Psi(F(a'))(a).$$

This construction is functorial in both Ψ and F. Moreover, Ψ' preserves small colimits: evaluation at a' preserves colimits in $\mathrm{Fun}(A^{\mathrm{op}}, B)$, Ψ preserves them by assumption, evaluation at a preserves them in $\mathrm{Fun}(A, C)$, and the final colimit over the small category $\mathrm{Tw}(A)^{\mathrm{op}}$ commutes with small colimits.

We claim that $\Psi \mapsto \Psi'$ is inverse to $\Phi \mapsto {}^t\Phi$. If $\Phi\colon \mathrm{Fun}(A^{\mathrm{op}}, B) \to C$ preserves small colimits and $\Psi = {}^t\Phi$, then the relative co-Yoneda formula gives

$$\begin{aligned}\Psi'(F) &\simeq \mathrm{colim}_{(\alpha\colon a \to a') : \mathrm{Tw}(A)^{\mathrm{op}}}\, \Phi(a \otimes F(a')) \\ &\simeq \Phi\left(\mathrm{colim}_{(\alpha\colon a \to a') : \mathrm{Tw}(A)^{\mathrm{op}}}\, a \otimes F(a')\right) \simeq \Phi(F),\end{aligned}$$

naturally in F.

Conversely, let x be an object of B and a_0 an object of A. Since Ψ preserves tensors by small animae, we have

$$\begin{aligned}\Psi'(a_0 \otimes x) &\simeq \operatorname{colim}_{(\alpha\colon a \to a'):\mathrm{Tw}(A)^{\mathrm{op}}} \Psi\big(\mathrm{Hom}_A(a', a_0) \otimes x\big)(a) \\ &\simeq \operatorname{colim}_{(\alpha\colon a \to a'):\mathrm{Tw}(A)^{\mathrm{op}}} \mathrm{Hom}_A(a', a_0) \otimes \Psi(x)(a) \\ &\simeq \Psi(x)(a_0).\end{aligned}$$

The final isomorphism is Proposition 12.9.1, evaluated at a_0. It is natural in a_0 and x, and therefore ${}^t(\Psi') \simeq \Psi$. □

13 Kan extensions, categorical colimits, and descent

The presheaf categories of the preceding chapter allow us to construct right Kan extensions and to give pointwise formulas for Kan extensions with values in any category. Together with cofinality, these formulas lead to computations of limits and colimits of categories and of colimits indexed by colimits. We then prove descent for cocartesian fibrations and animae, and deduce Quillen's theorem B.

13.1 Cofinality and pointwise Kan extensions

Yoneda density converts the universal properties of Kan extensions into pointwise limit and colimit formulas. We first combine the base-change theorems of Chapter 9 with duality to obtain cofinality results for limits and colimits. We then derive the pointwise formulas and apply them to weakly proper and weakly smooth functors.

Proposition 13.1.1. *Let $u\colon A \to B$ be a functor between small categories. Then the pullback functor*

$$u^*\colon \mathrm{Fun}(B, \mathrm{An}) \to \mathrm{Fun}(A, \mathrm{An})$$

admits a right adjoint

$$u_*\colon \mathrm{Fun}(A, \mathrm{An}) \to \mathrm{Fun}(B, \mathrm{An}).$$

Proof. By Corollary 12.8.2 it will suffice to show that u^* preserves small colimits. But this is an instance of Lemma 7.3.1. □

13.1.1 Base change and cofinality

Let A and B be small categories and let $u\colon A \to B$ be a functor. Consider the restriction functor

$$u^*\colon \mathrm{PSh}(B) \to \mathrm{PSh}(A), \qquad F \mapsto F \circ u^{\mathrm{op}}.$$

By Proposition 13.1.1, u^* admits a right adjoint

$$u_*\colon \mathrm{PSh}(A) \to \mathrm{PSh}(B).$$

Theorem 13.1.2. *Consider a pullback square of small categories*

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[g]{} & B. \end{array}$$

If u is proper or g is smooth, then for any presheaf $F\colon A^{\mathrm{op}} \to \mathrm{An}$, the Beck–Chevalley map

$$\mathrm{BC}_*\colon g^* u_*(F) \to u'_* f^*(F)$$

is an isomorphism.

Proof. For every presheaf $\Phi\colon B'^{\mathrm{op}} \to \mathrm{An}$, we need to show that the induced map

$$\mathrm{Hom}(\Phi, g^* u_*(F)) \to \mathrm{Hom}(\Phi, u'_* f^*(F))$$

is an equivalence. By adjunction, this corresponds to a map of the form

$$\mathrm{Hom}(u^* g_! \Phi, F) \to \mathrm{Hom}(f_! u'^* \Phi, F)\,.$$

The Beck–Chevalley map $\mathrm{BC}_*\colon g^* u_* \to u'_* f^*$ is, by definition, the mate of

$$\mathrm{BC}_!\colon f_! u'^* \longrightarrow u^* g_!$$

under the adjunctions $g_! \dashv g^*$, $f_! \dashv f^*$, $u^* \dashv u_*$, and $u'^* \dashv u'_*$. Consequently, under the preceding adjunction isomorphisms, the map induced by BC_* is precomposition with $\mathrm{BC}_!$. The latter is an isomorphism by Theorem 9.6.2 if g is smooth and by Theorem 9.6.6 if u is proper. Hence BC_* induces an equivalence on hom animae out of every presheaf Φ, and is therefore an isomorphism. □

Theorem 13.1.3. *Let $u\colon A \to B$ be an initial functor between small categories. For any functor $F\colon B \to \mathrm{An}$ there is a canonical isomorphism*

$$\lim_B F \xrightarrow{\sim} \lim_A u^* F.$$

Proof. Consider the functor $p\colon B \to *$, so that $\lim_B = p_*$ and $\lim_A = (pu)_* = p_* u_*$. We need to check that the unit map $\mathrm{id} \to u_* u^*$ induces an isomorphism $p_* \to p_* u_* u^*$. By passing to left adjoints, we may equivalently show that the map $u_! u^* p^* \to p^*$ is an isomorphism. To this end, consider an anima X. We compute both sides in terms of left fibrations over B. The object $p^*(X)$ corresponds to the projection $\mathrm{pr}_2\colon X \times B \to B$. Similarly, $u^* p^*(X)$ is the projection $\mathrm{pr}_2\colon X \times A \to A$. The object $u_! u^* p^*(X)$ is computed by factoring the composite $u \circ \mathrm{pr}_2\colon X \times A \to B$ into an initial functor followed by a left fibration. We use the factorization

$$\begin{array}{ccc} X \times A & \xrightarrow{\mathrm{id}_X \times u} & X \times B \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}_2} \\ A & \xrightarrow{u} & B. \end{array}$$

So this diagram exhibits $\mathrm{pr}_2\colon X \times B \to B$ as an instance of $u_! u^* p^*(X)$. With these choices the map $u_! u^* p^*(X) \to p^*(X)$ corresponds to the identity map of $X \times B$ over B, hence is an equivalence. This finishes the proof. □

Theorem 13.1.4. *Let A and C be categories and assume that C admits A-indexed limits. Then, for any initial functor $u\colon A \to B$, C also admits B-indexed limits and we have an identification*

$$\lim_B = \lim_A u^*$$

as functors $\mathrm{Fun}(B, C) \to C$.

Proof. Choose a universe in which A, B, and C are small, and write An for its category of animae. Let $F\colon B \to C$ be a functor. We need to show that the functor $C^{\mathrm{op}} \to \mathrm{An}'$, $x \mapsto \mathrm{Hom}(x_B, F)$, is representable by some object $\lim_B F$. We compute

$$\begin{aligned}
\mathrm{Hom}(x_B, F) &\simeq \mathrm{Hom}(h_C(x)_B, h_C(F)) \\
&\simeq \mathrm{Hom}(h_C(x), \lim_{b\in B} h_C(F(b))) \\
&\simeq \lim_{b\in B} \mathrm{Hom}(h_C(x), h_C(F(b))) \\
&\simeq \lim_{b\in B} \mathrm{Hom}_C(x, F(b)) \\
&\simeq \lim_{a\in A} \mathrm{Hom}_C(x, F(u(a))) \\
&\simeq \mathrm{Hom}_C(x, \lim_{a\in A} F(u(a))).
\end{aligned}$$

Hence the functor is represented by the limit $\lim_{a\in A} F(u(a))$. The identification $\lim_B = \lim_A u^*$ then follows from the Yoneda lemma. □

Corollary 13.1.5. *Let $u\colon A \to B$ be a functor between categories. If A is small and u is initial, then* An *admits B-indexed limits, and we have an identification*

$$\lim_B = \lim_A u^*$$

as functors $\mathrm{Fun}(B, \mathrm{An}) \to \mathrm{An}$. □

Corollary 13.1.6 (Cofinal computation of individual colimits)**.** *Let $u\colon A \to B$ be a terminal functor between small categories, and let $F\colon B \to C$ be a functor. Then F admits a colimit if and only if u^*F admits a colimit. In that case, the canonical map*

$$\mathrm{colim}_A u^*F \longrightarrow \mathrm{colim}_B F$$

is an isomorphism.

Proof. For every object x of C, terminality of u and Theorem 13.1.3, applied to u^{op}, give a natural equivalence

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Fun}(B,C)}(F, x_B) &\simeq \lim_{b:B^{\mathrm{op}}} \mathrm{Hom}_C(F(b), x) \\
&\simeq \lim_{a:A^{\mathrm{op}}} \mathrm{Hom}_C(F(u(a)), x) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(A,C)}(u^*F, x_A).
\end{aligned}$$

Thus F and u^*F have naturally equivalent cocone functors. One of these functors is representable if and only if the other is, and the same object represents both. □

13.1.2 Pointwise Kan extensions

Notation 13.1.7. Given a functor $u\colon A \to B$ between small categories A and B and an object b of B, we may consider the functors

$$\begin{array}{ccc} A_{/b} & \xrightarrow{i_b} & A \\ {\scriptstyle u_{/b}}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ B_{/b} & \xrightarrow[j_b]{} & B. \end{array}$$

For a functor $F\colon A^{\mathrm{op}} \to C$, we write

$$F_{/b} := F \circ i_b^{\mathrm{op}}\colon (A_{/b})^{\mathrm{op}} \to C.$$

We first compute right Kan extensions of anima-valued presheaves. Let $F\colon A^{\mathrm{op}} \to \mathrm{An}$ and let b be an object of B. By the Yoneda lemma and the adjunction $u^* \dashv u_*$, there are natural equivalences

$$\begin{aligned} u_*F(b) &\simeq \mathrm{Hom}_{\mathrm{PSh}(B)}(h_B(b), u_*F) \\ &\simeq \mathrm{Hom}_{\mathrm{PSh}(A)}(u^*h_B(b), F). \end{aligned}$$

The right fibration classified by $u^*h_B(b)$ is the projection $A_{/b} \to A$. Hence Corollary 12.6.4 gives an identification

$$u^*h_B(b) = \mathrm{colim}_{(a,\alpha):A_{/b}}\, h_A(a).$$

It follows that

$$\begin{aligned} u_*F(b) &\simeq \mathrm{Hom}_{\mathrm{PSh}(A)}\left(\mathrm{colim}_{(a,\alpha):A_{/b}}\, h_A(a), F\right) \\ &\simeq \lim_{(a,\alpha):(A_{/b})^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{PSh}(A)}(h_A(a), F) \\ &\simeq \lim_{(A_{/b})^{\mathrm{op}}} F_{/b}. \end{aligned}$$

All these equivalences are natural in b and F.

Theorem 13.1.8 (Pointwise formula for right Kan extensions)**.** *Let $u\colon A \to B$ be a functor between small categories and let $F\colon A^{\mathrm{op}} \to C$ be a functor. Assume that, for every object b of B, the functor $F_{/b}\colon (A_{/b})^{\mathrm{op}} \to C$ admits a limit. Then F admits a right Kan extension $u_*F\colon B^{\mathrm{op}} \to C$ along u^{op}, satisfying the pointwise formula*

$$u_*F(b) \simeq \lim_{(A_{/b})^{\mathrm{op}}} F_{/b}.$$

In particular, if C admits $(A_{/b})^{\mathrm{op}}$-indexed limits for every object b of B, then restriction admits a right adjoint

$$u_*\colon \mathrm{Fun}(A^{\mathrm{op}}, C) \to \mathrm{Fun}(B^{\mathrm{op}}, C).$$

Proof. Choose a universe so that C is small. The Yoneda embedding induces fully faithful functors

$$h_C\colon \mathrm{Fun}(B^{\mathrm{op}}, C) \to \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{PSh}(C)) \simeq \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{PSh}(B)),$$

and similarly for B replaced by A. The induced restriction functor $u^*\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{PSh}(B)) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{PSh}(A))$ admits a right adjoint

$$u_*\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{PSh}(A)) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{PSh}(B)),$$

For a functor $F\colon A^{\mathrm{op}} \to C$ and an object b of B, the anima-valued pointwise formula gives

$$u_*(h_C(F))(b) = \lim_{(a,s):(A_{/b})^{\mathrm{op}}} h_C(F(a)) \simeq h_C(\lim_{(A_{/b})^{\mathrm{op}}} F(a)),$$

where the final equivalence follows from the universal property of the limit in C. Thus $u_*(h_C(F))$ is pointwise representable and therefore lies in the essential image of

$$h_C\colon \mathrm{Fun}(B^{\mathrm{op}}, C) \longrightarrow \mathrm{Fun}(B^{\mathrm{op}}, \mathrm{PSh}(C))$$

by Corollary 12.3.17. Denote its unique preimage by u_*F. For $G\colon B^{\mathrm{op}} \to C$, full faithfulness of the Yoneda embedding and the ambient adjunction give natural equivalences

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}(B^{\mathrm{op}},C)}(G, u_*F) &\simeq \mathrm{Hom}_{\mathrm{Fun}(B^{\mathrm{op}},\mathrm{PSh}(C))}(h_C(G), u_*h_C(F))\\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(A^{\mathrm{op}},\mathrm{PSh}(C))}(u^*h_C(G), h_C(F))\\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(A^{\mathrm{op}},C)}(u^*G, F).\end{aligned}$$

This proves the universal property of the right Kan extension and the asserted pointwise formula. If the required limits exist for every functor $F\colon A^{\mathrm{op}} \to C$, the pointwise criterion for adjunctions from Proposition 6.2.3 shows that these right Kan extensions assemble into a right adjoint to restriction. □

Theorem 13.1.9 (Pointwise formula for left Kan extensions)**.** *Let $u\colon A \to B$ be a functor between small categories and let $F\colon A \to C$ be a functor. Assume that, for every object b of B, the composite $F \circ i_b\colon A_{/b} \to C$ admits a colimit. Then F admits a left Kan extension $u_!F\colon B \to C$ along u, satisfying the pointwise formula*

$$u_!F(b) \simeq \mathrm{colim}_{A_{/b}} F \circ i_b.$$

In particular, if C admits $A_{/b}$-indexed colimits for every object b of B, then restriction admits a left adjoint

$$u_!\colon \mathrm{Fun}(A, C) \to \mathrm{Fun}(B, C).$$

Proof. Apply Theorem 13.1.8 to the functor $F^{\mathrm{op}}\colon A^{\mathrm{op}} \to C^{\mathrm{op}}$. For every object b of B, the required limit in C^{op} is the opposite of the assumed colimit of $F \circ i_b$. The resulting right Kan extension along u^{op} becomes, upon passing to opposite categories, a left Kan extension of F along u, with the asserted pointwise formula. □

Corollary 13.1.10. *Let $u\colon A \to B$ be a functor and let $\Phi\colon C \to D$ be a functor such that C and D both admit $(A_{/b})^{\mathrm{op}}$-indexed limits for any object b of B, which are preserved by Φ. Then for every presheaf $F\colon A^{\mathrm{op}} \to C$ the Beck–Chevalley map*

$$\Phi(u_*(F)) \to u_*(\Phi(F))$$

is a natural isomorphism.

Proof. It suffices to show for every object b of B that the induced map $\Phi(u_*(F))(b) \to u_*(\Phi(F))(b)$ is an isomorphism. Under the pointwise formula of Theorem 13.1.8, this is the comparison expressing that Φ preserves the relevant limit. □

13.1.3 Weakly proper functors

Lemma 13.1.11. *Let C be a small category. The following conditions are equivalent.*

(i) *C is weakly contractible;*

(ii) *C^{op} is weakly contractible;*

(iii) *taking the associated C-indexed constant functor defines a fully faithful functor*

$$\mathrm{An} \hookrightarrow \mathrm{Fun}(C, \mathrm{An})$$

and its essential image consists of those functors sending all maps of C to isomorphisms in An.

(iv) *for any category D, taking the associated C-indexed constant functor defines a fully faithful functor*

$$D \hookrightarrow \mathrm{Fun}(C, D)\,.$$

Proof. Corollary 12.2.6 shows that conditions (i) and (ii) are equivalent. Conditions (iii) and (iv) each imply that the constant-functor embedding $\mathrm{An} \to \mathrm{Fun}(C, \mathrm{An})$ is fully faithful, where in condition (iv) we take $D = \mathrm{An}$. We first show that condition (i) implies both conditions (iii) and (iv). If C is weakly contractible, then the unique functor $C \to [0]$ exhibits $[0]$ as the localization of C by all morphisms of C. In other words, for any category D, the associated constant functor $X \mapsto X_C$ defines a fully faithful functor of the form

$$D \hookrightarrow \mathrm{Fun}(C, D)$$

whose essential image consists of those functors $F\colon C \to D$ with the property that $F(u)\colon F(x) \to F(y)$ is an isomorphism in D for any morphism $u\colon x \to y$ in C. For $D = \mathrm{An}$, this also identifies the essential image required in condition (iii). It remains to prove that full faithfulness of $\mathrm{An} \to \mathrm{Fun}(C, \mathrm{An})$ implies condition (i). Recall from Remark 9.3.4 that, if X is a small anima and if e is the terminal object of An, then we have a canonical isomorphism

$$\mathrm{Hom}(\Pi_\infty(C), X) = \mathrm{Hom}(e_C, X_C)\,.$$

If the functor $X \mapsto X_C$ is fully faithful, the isomorphism

$$\mathrm{Hom}(e, X) = \mathrm{Hom}(e_C, X_C)$$

can thus be reinterpreted as asserting that the map $\Pi_\infty(C) \to e$ induces an isomorphism of the form

$$\mathrm{Hom}(e, X) = \mathrm{Hom}(\Pi_\infty(C), X)$$

for all X. Therefore, by virtue of Corollary 12.3.14, the map $\Pi_\infty(C) \to e$ must be an isomorphism in An, which means that C is weakly contractible by definition. □

Definition 13.1.12. A functor $f\colon A \to B$ is *weakly proper* if, for any object b of B, the induced functor $A_b \to A_{/b}$ is terminal.

A functor $f\colon A \to B$ is *weakly smooth* if, for any object b of B, the induced functor $A_b \to A_{b/}$ is initial.

Example 13.1.13. Any proper functor is weakly proper. In particular, (locally) cocartesian functors are weakly proper. Dually, smooth functors are weakly smooth. In fact, a functor $u\colon A \to B$ is weakly proper if and only if $u^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$ is weakly smooth. Examples of weakly proper functors that are not locally cocartesian will come from simplicial approximation; see Proposition 15.1.7.

Lemma 13.1.14. *Let A be a small category and X a small anima. Then $\mathrm{Hom}_{\mathrm{An}}(\Pi_\infty(A), X)$ is the limit of the A-indexed constant functor with value X.*

Proof. We know that $\Pi_\infty(A)$ is the colimit of the constant A-indexed diagram with value the terminal object e of An – see Remark 9.3.4. We thus have

$$\mathrm{Hom}_{\mathrm{An}}(\Pi_\infty(A), X) = \mathrm{Hom}(e_A, X_A)$$

where X_A denotes the constant A-indexed diagram with value X. But we also know that $\mathrm{Hom}(e_A, -)$ is the limit functor for A-indexed diagrams. □

Lemma 13.1.15. *Let C be a category and let $f\colon A \to B$ be a weakly proper functor. Assume that C admits A_b^{op}-indexed limits for every object b of B. Then the functor*

$$f^*\colon \mathrm{Fun}(B^{\mathrm{op}}, C) \to \mathrm{Fun}(A^{\mathrm{op}}, C)$$

has a right adjoint

$$f_*\colon \mathrm{Fun}(A^{\mathrm{op}}, C) \to \mathrm{Fun}(B^{\mathrm{op}}, C),$$

and, for an object b of B and any functor $F\colon A^{\mathrm{op}} \to C$, there is a canonical isomorphism

$$f_*(F)(b) \simeq \lim_{A_b^{\mathrm{op}}} F|_{A_b^{\mathrm{op}}},$$

where $F|_{A_b^{\mathrm{op}}}$ denotes restriction along $A_b^{\mathrm{op}} \hookrightarrow A^{\mathrm{op}}$.

Proof. By weak properness, the functor $A_b \to A_{/b}$ is terminal, so its opposite $A_b^{\mathrm{op}} \to (A_{/b})^{\mathrm{op}}$ is initial. Applying Theorem 13.1.4, we see that C admits $(A_{/b})^{\mathrm{op}}$-indexed limits, computed over A_b^{op}. The claim now follows from Theorem 13.1.8. □

Proposition 13.1.16. *Let $f\colon A \to B$ be a functor between categories. We suppose that the following properties hold.*

(i) The functor f is either weakly proper or weakly smooth.

(ii) For each object b of B, the fiber A_b is weakly contractible.

Then f exhibits B as the localization of A by

$$W := \sum_{b:B^{\simeq}} \mathrm{Map}([1], A_b)$$

(the family of maps in A that are sent to identities in B).

Proof. We will prove the case where f is weakly proper (the weakly smooth case follows from the weakly proper one by considering $f^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$).

The universal property of localization gives a commutative triangle of the form

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & A[W^{-1}] \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \varphi} & \\ & B & \end{array}$$

where $\gamma\colon A \to A[W^{-1}]$ is the localization of A by W. We will show that φ is an equivalence.

We may choose a universe Cat so that A and B are small. The commutative triangle above thus gives rise to a commutative triangle of the form

$$\begin{array}{ccc} \mathrm{PSh}(A) & \xleftarrow{\gamma^*} & \mathrm{PSh}(A[W^{-1}]) \\ & {\scriptstyle f^*}\nwarrow \quad \nearrow{\scriptstyle \varphi^*} & \\ & \mathrm{PSh}(B) & \end{array}$$

in which γ^* is fully faithful, with essential image those functors that send any map in W to an isomorphism of An. We claim that f^* is fully faithful. Equivalently, this means that, for any functor of the form $E\colon B^{\mathrm{op}} \to \mathrm{An}$, the unit map

$$\eta_E\colon E \to f_* f^*(E)$$

is an isomorphism. We observe that the restriction of $f^*(E)$ to the fiber A_b is the constant functor with value $E(b)$. By virtue of Lemma 13.1.14 and Lemma 13.1.15 this implies that, for each object b of B, there is a canonical isomorphism of the form:

$$\mathrm{Hom}(\Pi_\infty(A_b^{\mathrm{op}}), E(b)) \simeq f_* f^*(E)(b)\,.$$

Since A_b is weakly contractible, so is A_b^{op} and, by Corollary 12.3.7, the previous identification can be rewritten as:

$$E(b) = f_* f^*(E)(b)\,.$$

This isomorphism agrees with the evaluation of η_E at b, and this proves that f^* is fully faithful. Since γ^* is fully faithful as well, the functor φ^* must also be fully faithful.

In order to prove that φ^* is an equivalence, it suffices to check that both fully faithful functors f^* and γ^* have the same essential image in $\mathrm{PSh}(A)$. Since f^* is the composition of φ^* and γ^*, it suffices to check that an object in the essential image of γ^* is in the essential image of f^*. Let $E\colon A^{\mathrm{op}} \to \mathrm{An}$ be a functor sending any map in W to an isomorphism of An. We will prove that the co-unit map

$$\varepsilon_E\colon f^* f_*(E) \to E$$

is an isomorphism. Let a be an object of A. The restriction of E to each fiber $A^{\mathrm{op}}_{f(a)}$ sends each map to an isomorphism. Since the fiber is weakly contractible, this means that the restriction to the fiber is constant with value $E(a)$. In other words, the functor

$$a\colon [0] \to A^{\mathrm{op}}_{f(a)}$$

induces isomorphisms

$$\lim_{A^{\mathrm{op}}_{f(a)}} E|_{A^{\mathrm{op}}_{f(a)}} \simeq \mathrm{Hom}(\Pi_\infty(A^{\mathrm{op}}_{f(a)}), E(a)) \simeq E(a)\,.$$

By virtue of Lemma 13.1.15, this means that there is a canonical isomorphism of the form

$$f_*(E)(f(a)) = E(a)\,.$$

The composition of this isomorphism with the defining identification $f^* f_*(E)(a) = f_*(E)(f(a))$ coincides with the evaluation of ε_E at a. Therefore, $f^* f_*(E) = E$ and thus E lies in the essential image of f^*. Since φ^* is an equivalence, so is its left adjoint $\varphi_!$. This implies finally that the functor φ itself is fully faithful, by Proposition 12.2.11. By virtue of Proposition 11.2.11, it suffices now to prove that the map

$$\varphi^{\simeq}\colon (A[W^{-1}])^{\simeq} \to B^{\simeq}$$

is a cover. By Corollary 11.2.7, it suffices to prove that

$$f^{\simeq}\colon A^{\simeq} \to B^{\simeq}$$

is a cover. But condition (ii) tells us in particular that the projection

$$\mathrm{pr}_1\colon B^{\simeq} \times_B A \to B^{\simeq}$$

induces an equivalence after applying the functor Π_∞. In other words, it exhibits $B^{\simeq}$ as the localization of $B^{\simeq} \times_B A$ by all maps. Therefore, by virtue of Corollary 11.2.16, the induced map

$$(\mathrm{pr}_1)\colon (B^{\simeq} \times_B A)^{\simeq} \to B^{\simeq}$$

is a cover. To conclude, we observe that $(B^{\simeq} \times_B A)^{\simeq} = A^{\simeq}$ holds over $B^{\simeq}$. □

13.2 Limits and colimits of categories

We now prove that the universe Cat admits all small limits and colimits. Colimits are obtained by localizing a cocartesian unstraightening, whereas limits are computed by its category of cocartesian sections. We then record consequences for total categories of cocartesian fibrations, limits of adjunctions, functor categories, and pullbacks.

13.2.1 Completeness and cocompleteness of categories

Theorem 13.2.1 (Colimits of categories)**.** *The category* Cat *admits small colimits. More precisely, let* $F\colon I \to \mathrm{Cat}$ *be a functor from a small category, with associated cocartesian fibration* $p\colon X \to I$*. Then* $\mathrm{colim}_I F$ *is the localization of* X *at the* p*-cocartesian morphisms.*

Proof. Since $p\colon X \to I$ and $I \to *$ are small cocartesian fibrations, regularity, (Cat1), shows that X is small. Closure under localizations, (Cat7), then gives a small localization

$$\gamma\colon X \longrightarrow \widetilde{X}$$

at the p-cocartesian morphisms. For a small category B, directed univalence identifies natural transformations $F \to B_I$ with cocartesian functors $X \to I \times B$ over I. Every such functor is of the form (p, φ) for a functor $\varphi\colon X \to B$. It is cocartesian if and only if φ sends the p-cocartesian morphisms to isomorphisms, since the pr_1-cocartesian morphisms in $I \times B$ are precisely the pairs (u, v) for which v is invertible. The universal property of γ therefore gives a natural equivalence

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{Cat})}(F, B_I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(\widetilde{X}, B).$$

For $B = \widetilde{X}$, the inverse of this isomorphism sends $\mathrm{id}_{\widetilde{X}}$ to the cocone $\eta_F\colon F \to \widetilde{X}_I$ classified by the cocartesian functor $(p, \gamma)\colon X \to I \times \widetilde{X}$. Under the preceding identifications, the inverse is precisely precomposition with η_F. Thus η_F is a colimit cocone.

The construction and the preceding calculation are valid in an arbitrary anima context. Hence Proposition 7.2.2 shows that Cat admits I-indexed colimits. Since I was arbitrary, Cat admits all small colimits. □

Corollary 13.2.2. *Let A be a small category and $F\colon A \to \mathrm{Cat}$ a functor. The canonical functors $i_a\colon F(a) \to \mathrm{colim}_{a:A} F(a)$ induce a cover of the form*

$$\sum_{a:A^{\simeq}} F(a)^{\simeq} \to (\mathrm{colim}_{a:A} F(a))^{\simeq}\,.$$

Proof. We know that $\mathrm{colim}_{a:A} F(a)$ is a localization of $\int_{a:A} F(a)$, where the latter is defined by the pullback below.

$$\begin{array}{ccc} \int_{a:A} F(a) & \longrightarrow & \mathrm{Cat}_{\bullet} \\ {\scriptstyle p_F}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ A & \longrightarrow & \mathrm{Cat} \end{array}$$

By virtue of Corollary 11.2.16, we thus have a cover

$$(\int_{a:A} F(a))^{\simeq} \to (\mathrm{colim}_{a:A} F(a))^{\simeq}\,.$$

But, as explained in Remark 12.4.17, objects of $(\int_{a:A} F(a))^{\simeq}$ are known to be pairs of the form (a, x) where $a : A^{\simeq}$ and $x : F(a)^{\simeq}$. In other words, there is a cover (in fact, an equivalence) of the form

$$\sum_{a:A^{\simeq}} F(a)^{\simeq} \to (\int_{a:A} F(a))^{\simeq}\,.$$

This corollary thus follows from the fact that covers are stable under composition. □

Corollary 13.2.3. *Let A be a small category and $F\colon A \to \mathrm{An}$ a functor. The canonical maps $i_a\colon F(a) \to \mathrm{colim}_{a:A} F(a)$ induce a cover of the form*

$$\sum_{a:A^{\simeq}} F(a) \to \mathrm{colim}_{a:A} F(a)\,.$$

Proof. This is a particular case of the previous corollary since the embedding of An into Cat commutes with colimits (being a left adjoint). □

Theorem 13.2.4 (Limits of categories)**.** *The category* Cat *admits small limits. For a functor* $F\colon I \to \mathrm{Cat}$ *from a small category, the limit* $\lim_I F$ *is the category of cocartesian sections of the cocartesian fibration* $p\colon X \to I$ *classified by* F*.*

Proof. Let $p\colon X \to I$ classify F. For every small category Y, directed univalence identifies cones from the constant diagram Y_I to F with cocartesian functors $I \times Y \to X$ over I. The exponential law, together with the fact that preservation of cocartesian morphisms is checked objectwise in Y, gives an equivalence of categories

$$\mathrm{cocart}_I(I \times Y, X) \simeq \mathrm{Fun}(Y, \mathrm{cocart}_I(I, X)).$$

It follows that

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathrm{Cat})}(Y_I, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}(Y, \mathrm{cocart}_I(I, X)).$$

Since I and X are small, so is $\mathrm{cocart}_I(I, X)$. Thus this category is an object of Cat and represents cones over F, which proves that it is $\lim_I F$. □

13.2.2 Limits in unstraightenings

We first apply this description to limits in the total category of a cocartesian fibration.

Proposition 13.2.5 (Limits in cocartesian unstraightenings)**.** *Let A be a small anima and let $p\colon E \to B$ be a small cocartesian fibration. Assume that B admits A-indexed limits and that the functor $B \to \mathrm{Cat}$ classifying p preserves them. Then E admits A-indexed limits and p preserves them.*

Proof. Write $F\colon B \to \mathrm{Cat}$ for the functor classifying p. We prove the following more precise assertion. Let $\widetilde{X}\colon A \to E$ be a functor, write $X := p\widetilde{X}$, and choose a limit cone

$$\lambda\colon \mathrm{const}_b \longrightarrow X$$

in B. Write $E_X := A \times_B E$ for the pullback of p along X. Cocartesian transport along λ defines a comparison functor

$$\kappa_\lambda\colon F(b) \longrightarrow \mathrm{cocart}_A(A, E_X).$$

Concretely, $\kappa_\lambda(x)$ is the cocartesian section whose value at $a \in A$ is $(\lambda_a)_! x$; its structure morphisms are supplied by the functoriality of cocartesian transport and the naturality of λ. If κ_λ is an equivalence and $x \in F(b)$ corresponds to the section defined by $\widetilde{X}$, then the resulting p-cocartesian cone $\mathrm{const}_x \to \widetilde{X}$ is a limit cone in E.

Indeed, every morphism of A is invertible, so every morphism in the image of $\widetilde{X}$ is an isomorphism. Isomorphisms are p-cocartesian, and hence $\widetilde{X}$ determines a cocartesian section of $E_X \to A$. Applying an inverse of κ_λ to this section gives x and the stated p-cocartesian cone.

It remains to prove that this cone is a limit cone in the total category. Let $z \in F(c)$. Consider the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_E(z, x) & \longrightarrow & \mathrm{Hom}_{\mathrm{Fun}(A,E)}(\mathrm{const}_z, \widetilde{X}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_B(c, b) & \longrightarrow & \mathrm{Hom}_{\mathrm{Fun}(A,B)}(\mathrm{const}_c, X). \end{array}$$

The lower horizontal map is an equivalence because λ is a limit cone. We claim that the square is a pullback. Fix a morphism $f\colon c \to b$. The fiber of its left vertical map over f is

$$\mathrm{Hom}_{F(b)}(f_!z, x).$$

For every $a \in A$, functoriality of cocartesian transport gives a canonical isomorphism

$$\kappa_\lambda(f_!z)(a) = (\lambda_a)_!(f_!z) = (\lambda_a f)_!z.$$

Cocartesian factorization along the canonical cone from $f_!z$ to $\kappa_\lambda(f_!z)$ identifies the fiber of the right vertical map over λf with the mapping anima, in the category of cocartesian sections of $E_X \to A$, from $\kappa_\lambda(f_!z)$ to the section defined by $\widetilde{X}$. Using the chosen identification of this section with $\kappa_\lambda(x)$, full faithfulness of κ_λ shows that the induced map between the two fibers is an equivalence. Thus the square is a pullback. Its upper horizontal map is therefore an equivalence, which is the universal property of the asserted limit cone. This proves the more precise assertion.

By Theorem 13.2.4, the target of κ_λ is the limit of $F \circ X$, so preservation of λ says precisely that κ_λ is an equivalence. The preceding construction applies to every diagram in an arbitrary anima context. The pointwise criterion for indexed limits, dual to Proposition 7.2.2, therefore assembles the construction into a right adjoint to the constant-diagram functor. The image of the constructed cone under p is λ, so p preserves these limits. □

The comparison functor κ_λ constructed in the preceding proof also detects whether the functor classifying p preserves limits.

Proposition 13.2.6 (Recognition of continuous straightenings)**.** *Let A be a small anima, let B admit A-indexed limits, and let $p\colon E \to B$ be a small cocartesian fibration classified by $F\colon B \to \mathrm{Cat}$. Then F preserves A-indexed limits if and only if, for every A-indexed limit cone $\lambda\colon \mathrm{const}_b \to X$ in an arbitrary anima context, the following conditions hold.*

(1) The comparison functor κ_λ is strongly surjective.

(2) For every $x \in F(b)$, the canonical p-cocartesian cone from x to the section $\kappa_\lambda(x)$ is a limit cone in E.

Proof. If F preserves A-indexed limits, then κ_λ is an equivalence and both conditions follow from Proposition 13.2.5.

Conversely, fix λ and two objects $x, y \in F(b)$. Since the canonical cone from y to $\kappa_\lambda(y)$ is a limit cone, it induces an equivalence

$$\mathrm{Hom}_E(x, y) \longrightarrow \mathrm{Hom}_{\mathrm{Fun}(A,E)}(\mathrm{const}_x, \kappa_\lambda(y)).$$

This equivalence lies over the equivalence

$$\mathrm{Hom}_B(b, b) \longrightarrow \mathrm{Hom}_{\mathrm{Fun}(A,B)}(\mathrm{const}_b, X)$$

induced by λ. Cocartesian factorization along the canonical cone from x to $\kappa_\lambda(x)$ identifies the fiber on the right over λ with the mapping anima between the corresponding cocartesian sections. Taking the fibers over id_b and λ therefore gives an equivalence

$$\mathrm{Hom}_{F(b)}(x, y) \longrightarrow \mathrm{Hom}_{\mathrm{cocart}_A(A,E_X)}(\kappa_\lambda(x), \kappa_\lambda(y)).$$

Thus κ_λ is fully faithful. It is strongly surjective by condition (1), so Theorem 6.3.8 shows that it is an equivalence. By Theorem 13.2.4, this says precisely that F preserves the limit cone λ. Since λ was arbitrary in an arbitrary anima context, the dual, limit-preserving form of Proposition 7.2.4 shows that F preserves all A-indexed limits. □

Corollary 13.2.7 (Colimits in cartesian unstraightenings)**.** *Let A be a small anima and let $p\colon E \to B$ be a small cartesian fibration. If B admits A-indexed colimits and the functor $B^{\mathrm{op}} \to \mathrm{Cat}$ classifying p sends them to A^{op}-indexed limits, then E admits A-indexed colimits and p preserves them.*

Conversely, the classifying functor sends these colimits to limits if the corresponding comparison functors are strongly surjective and the canonical cartesian cocones are colimit cocones in E.

Proof. Apply Propositions 13.2.5 and 13.2.6 to the opposite cocartesian fibration $p^{\mathrm{op}}\colon E^{\mathrm{op}} \to B^{\mathrm{op}}$, using Theorem 12.4.10. □

Remark 13.2.8 (Naturality and Fubini)**.** In the notation of the proof of Proposition 13.2.5, the construction is natural in the diagram, the cocartesian fibration, and the chosen limit cone. It is also natural in the indexing anima in the following precise sense. For a functor $u\colon A' \to A$ of small animae, restriction of sections makes the square

$$\begin{array}{ccc} F(b) & \xrightarrow{\kappa_\lambda} & \mathrm{cocart}_A(A, E_X) \\ \| & & \downarrow{\scriptstyle u^*} \\ F(b) & \xrightarrow{\kappa_{u^*\lambda}} & \mathrm{cocart}_{A'}(A', E_{Xu}) \end{array}$$

commutes via its canonical identification. Here $\kappa_{u^*\lambda}$ is defined from the restricted cone, whether or not that cone remains a limit cone.

If A and A' are small animae and the relevant limits are preserved by F, then the limit in E indexed by $A \times A'$ agrees canonically with either iterated limit. Under p, this is the corresponding Fubini isomorphism in B. This follows from Lemma 7.3.2 and the uniqueness of right adjoints. All of these assertions remain valid in an arbitrary anima context. No claim of validity in an arbitrary categorical context is being made.

Remark 13.2.9. As A ranges over the finite discrete animae, including the empty anima, Proposition 13.2.5 specializes to Proposition 8.7.15.

13.2.3 Limits of categories with limits

Proposition 13.2.10 (Limits of categories with limits)**.** *Let I and K be small categories and let $C_\bullet\colon K \to \mathrm{Cat}$ be a functor. Assume that every C_k admits I-indexed limits and that every transition functor $C_k \to C_{k'}$ preserves them. Then $E := \lim_{k:K} C_k$ admits I-indexed limits, computed componentwise. In particular, the projection functors $E \to C_k$ preserve these limits, and preservation of them by a functor into E is detected by the projections. The analogous assertions hold for colimits.*

Proof. Let $X\colon I \to E$ be a diagram, and write $X_k\colon I \to C_k$ for its projection. By functoriality of functor categories and pullbacks, the categories of cones $(C_k)_{/X_k}$ form a K-indexed diagram. Since $\mathrm{Fun}(I, -)$ preserves limits and limits commute with pullbacks, the definition of cone categories gives an equivalence

$$E_{/X} \simeq \lim_{k:K} (C_k)_{/X_k}.$$

Each transition functor sends limit cones to limit cones. The full subcategories spanned by these cones therefore form a subdiagram of contractible animae. Its limit is contractible as well, so it supplies a coherent family of limit cones on the X_k. Under the displayed equivalence, this family determines a cone $\mathrm{const}_Y \to X$ in E.

We claim that this is a limit cone. Mapping animae in E are computed componentwise:

$$\mathrm{Hom}_E(Z, Y) \simeq \lim_{k:K} \mathrm{Hom}_{C_k}(Z_k, Y_k).$$

Indeed, $\mathrm{Ar}(-) = \mathrm{Fun}([1], -)$ and the groupoid-core functor preserve limits. Taking the fiber of $\mathrm{Ar}(E)^{\simeq} \to E^{\simeq} \times E^{\simeq}$ over (Z, Y) therefore gives the displayed equivalence. The componentwise limit cones and Lemma 7.3.2 now give

$$\begin{aligned}\mathrm{Hom}_E(Z, Y) &\simeq \lim_{k:K} \mathrm{Hom}_{C_k}(Z_k, Y_k) \\ &\simeq \lim_{k:K} \lim_{i:I} \mathrm{Hom}_{C_k}(Z_k, X_k(i)) \\ &\simeq \lim_{i:I} \mathrm{Hom}_E(Z, X(i)).\end{aligned}$$

This equivalence is induced by the constructed cone, proving its universal property. The argument applies in every anima context, so the dual of Proposition 7.2.2 shows that E admits I-indexed limits.

By construction, the projections preserve these limits. They jointly detect isomorphisms, and hence also detect whether the canonical map from the image of a limit to the limit of the image is an isomorphism. This proves the assertion about preservation by functors into E. The colimit statement follows by passing to opposite categories. □

13.2.4 Formal consequences

We finish the section with several formal consequences of completeness and cocompleteness of Cat.

Lemma 13.2.11. *Let C and D be categories. Then there exists a canonical isomorphism*

$$\lim_C D \simeq \mathrm{Fun}(\Pi_\infty(C), D).$$

Proof. For every other category E we have

$$\begin{aligned}\mathrm{Hom}(E, \lim_C D) &\simeq \lim_C \mathrm{Hom}(E, D) \simeq \lim_{\Pi_\infty(C)} \mathrm{Hom}(E, D) \\ &\simeq \mathrm{Hom}(\Pi_\infty(C), \mathrm{Hom}(E, D)) \simeq \mathrm{Hom}(E, \mathrm{Fun}(\Pi_\infty(C), D)),\end{aligned}$$

finishing the proof. □

Corollary 13.2.12. *Let $u\colon A \to B$ be a functor between small categories. Then the functor*

$$u^*\colon \mathrm{Fun}(B, \mathrm{Cat}) \to \mathrm{Fun}(A, \mathrm{Cat})$$

admits a left adjoint

$$u_!\colon \mathrm{Fun}(A, \mathrm{Cat}) \to \mathrm{Fun}(B, \mathrm{Cat}).$$

Proof. By Theorem 13.2.1, the category Cat admits the colimits indexed by every slice $A_{/b}$. The pointwise formula of Theorem 13.1.9 therefore constructs the left Kan extension of every functor $A \to \mathrm{Cat}$ along u, and these left Kan extensions form a left adjoint to restriction. □

Remark 13.2.13. The inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}$ admits both a left adjoint Π_∞ and a right adjoint $(-)^{\simeq} \colon \mathrm{Cat} \to \mathrm{An}$. Hence it preserves both limits and colimits.

Proposition 13.2.14 (Limits in pullbacks)**.** *Consider a pullback square of categories of the form*

$$\begin{array}{ccc} X & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow{v} & S \end{array}$$

Let I be a category and assume that both u and v preserve I-indexed limit cones. If $F \colon I \to X$ is a functor such that pF and qF have limits, then F has a limit, and its limit cone is preserved by both p and q.

Proof. Up to enlarging the universe Cat, we may assume that all categories involved here are small. We write $r \colon X \to S$ for the composite of u and p. Let $F \colon I \to X$ be a functor. Let a be the limit of $i \mapsto pF_i$, let b be the limit of $i \mapsto qF_i$, and let s be the limit of $i \mapsto rF_i$. There are canonical isomorphisms $u(a) \xrightarrow{\sim} s \xrightarrow{\sim} v(b)$, hence a unique object y of X with $p(y) = a$ and $q(y) = b$, together with the required identification over S. Since the functor

$$\lim_I \colon \mathrm{Fun}(I, \mathrm{An}) \to \mathrm{An}$$

commutes with pullbacks, being a right adjoint, we have the following computation for any object x of X.

$$\begin{aligned} \lim_i \mathrm{Hom}_X(x, F_i) &\simeq \lim_i \left(\mathrm{Hom}_A(px, pF_i) \times_{\mathrm{Hom}_S(rx, rF_i)} \mathrm{Hom}_B(qx, qF_i) \right) \\ &\simeq \lim_i \mathrm{Hom}_A(px, pF_i) \times_{\lim_i \mathrm{Hom}_S(rx, rF_i)} \lim_i \mathrm{Hom}_B(qx, qF_i) \\ &\simeq \mathrm{Hom}_A(px, \lim_i pF_i) \times_{\mathrm{Hom}_S(rx, \lim_i rF_i)} \mathrm{Hom}_B(qx, \lim_i qF_i) \\ &\simeq \mathrm{Hom}_A(px, a) \times_{\mathrm{Hom}_S(rx, s)} \mathrm{Hom}_B(qx, b) \\ &\simeq \mathrm{Hom}_X(x, y) \,. \end{aligned}$$

This proves that y is the limit of $i \mapsto F_i$ in X. □

13.3 Colimits indexed by colimits

We next combine the localization description of colimits in Cat with the pointwise formula for Kan extensions. The result is a Fubini theorem for a diagram indexed by a category which is itself presented as a colimit.

13.3.1 Fiberwise and iterated colimits

Lemma 13.3.1 (Fiberwise colimits over cocartesian fibrations)**.** *Let $p \colon E \to I$ be a cocartesian fibration between small categories and let $\Phi \colon E \to C$ be a functor. Assume that the restriction $\Phi|_{E_i} \colon E_i \to C$ admits a colimit for every object i of I. Then the left Kan extension $p_! \Phi \colon I \to C$ exists and is given pointwise by*

$$(p_! \Phi)(i) \simeq \mathrm{colim}_{E_i} \Phi|_{E_i} \,.$$

Moreover, Φ admits a colimit if and only if $p_!\Phi$ admits a colimit. In that case, there is a canonical isomorphism

$$\operatorname{colim}_E \Phi \simeq \operatorname{colim}_I p_!\Phi.$$

Dually, fiberwise limits over a cartesian fibration assemble into a right Kan extension and may be computed by first taking the fiberwise limits and then the limit over the base.

Proof. For an object i of I, let $E_{/i} := E \times_I I_{/i}$ be the relative slice. There is a pullback square

$$\begin{array}{ccc} E_i & \xrightarrow{\iota_i} & E_{/i} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p_{/i}} \\ * & \xrightarrow{(i,\mathrm{id}_i)} & I_{/i}. \end{array}$$

The functor $p_{/i}$ is a cocartesian fibration. The lower horizontal functor selects the terminal object of $I_{/i}$ and is therefore a right adjoint. It follows from Proposition 6.2.4 that ι_i is a right adjoint, and hence terminal by Proposition 9.1.12. Consequently, Corollary 13.1.6 gives a canonical isomorphism

$$\operatorname{colim}_{E_{/i}} \Phi|_{E_{/i}} \simeq \operatorname{colim}_{E_i} \Phi|_{E_i}.$$

In particular, all the colimits required by the pointwise formula for left Kan extensions exist. By Theorem 13.1.9, the left Kan extension $p_!\Phi$ exists, and the preceding isomorphism identifies its value at i with the asserted fiberwise colimit.

For any object x of C, the universal property of the left Kan extension gives natural equivalences

$$\begin{aligned} \operatorname{Hom}_{\mathrm{Fun}(I,C)}(p_!\Phi, x_I) &\simeq \operatorname{Hom}_{\mathrm{Fun}(E,C)}(\Phi, p^* x_I) \\ &= \operatorname{Hom}_{\mathrm{Fun}(E,C)}(\Phi, x_E). \end{aligned}$$

Thus $p_!\Phi$ and Φ have naturally equivalent cocone functors. One admits a colimit if and only if the other does, and their colimits are then canonically isomorphic. The statement about limits follows by passing to opposite categories. □

Corollary 13.3.2 (Colimits indexed by colimits)**.** *Let I be a small category and let $F\colon I \to \mathrm{Cat}$ be a functor. Write*

$$L := \operatorname{colim}_I F$$

and let $\lambda_i\colon F(i) \to L$ denote the components of the colimit cone.

(1) *Let C be a category that admits I-indexed colimits, and let $\psi\colon L \to C$ be a functor. Assume that $\psi \circ \lambda_i\colon F(i) \to C$ admits a colimit for every object i of I. Then these colimits assemble into a functor $\psi_!\colon I \to C$, the functor ψ admits a colimit, and there is a canonical isomorphism*

$$\operatorname{colim}_L \psi \simeq \operatorname{colim}_{i:I} \operatorname{colim}_{F(i)} \psi \circ \lambda_i.$$

(2) *Let C be a category that admits I^{op}-indexed limits, and let $\psi\colon L \to C$ be a functor. Assume that $\psi \circ \lambda_i\colon F(i) \to C$ admits a limit for every object i of I. Then these limits assemble into a functor $\psi_*\colon I^{\mathrm{op}} \to C$, the functor ψ admits a limit, and there is a canonical isomorphism*

$$\lim_L \psi \simeq \lim_{i:I^{\mathrm{op}}} \lim_{F(i)} \psi \circ \lambda_i.$$

Proof. Let $p_F \colon E := \int_{i:I} F(i) \to I$ be the cocartesian fibration classified by F. By Theorem 13.2.1, the category L is the localization of E at the p_F-cocartesian morphisms. Write

$$\gamma \colon E \longrightarrow L$$

for the localization functor. Its restriction to the fiber $E_i \simeq F(i)$ identifies with λ_i.

For part (1), apply Lemma 13.3.1 to $\psi \circ \gamma \colon E \to C$. It supplies a left Kan extension

$$(p_F)_!(\psi \circ \gamma) \colon I \longrightarrow C,$$

which is the asserted functor $\psi_!$, and identifies its value at i with $\operatorname{colim}_{F(i)} \psi \circ \lambda_i$. Since C admits I-indexed colimits, the functor $\psi_!$ admits a colimit. The same lemma therefore gives

$$\operatorname{colim}_E \psi \circ \gamma \simeq \operatorname{colim}_{i:I} \operatorname{colim}_{F(i)} \psi \circ \lambda_i.$$

The localization γ is terminal by Proposition 9.1.9. Hence Corollary 13.1.6 identifies the left-hand side with $\operatorname{colim}_L \psi$, proving part (1).

For part (2), apply part (1) to the functor $F^{\mathrm{op}} \colon I \to \mathrm{Cat}$ and the functor $\psi^{\mathrm{op}} \colon L^{\mathrm{op}} \to C^{\mathrm{op}}$. The opposite-category functor is an equivalence $\mathrm{Cat} \to \mathrm{Cat}$ and therefore preserves colimits, so that

$$L^{\mathrm{op}} \simeq \operatorname{colim}_{i:I} F(i)^{\mathrm{op}}.$$

Translating the resulting statement in C^{op} back to C gives the asserted functor $\psi_* \colon I^{\mathrm{op}} \to C$ and the formula for its limit. □

13.3.2 Colimits of right fibrations

The iterated-colimit formula has a useful application to right fibrations. We show that unstraightening preserves colimits: the colimit of a diagram of right fibrations is again a right fibration, classified by the pointwise colimit of the corresponding presheaves. This is the input for the descent arguments of the next section.

Lemma 13.3.3. *Let I be small and $X \colon I \to \mathrm{PSh}(C)$. Then the canonical functor*

$$p \colon \int_{i:I} C_{/X_i} \to C$$

is a cartesian fibration.

Proof. Given a presheaf $X \colon C^{\mathrm{op}} \to \mathrm{An}$, we will see $C_{/X}$ through the following pullback square

$$\begin{array}{ccc} C_{/X} & \longrightarrow & \mathrm{PSh}(C)_{/X} \\ \downarrow & & \downarrow \\ C & \xrightarrow{h_C} & \mathrm{PSh}(C) \end{array}$$

where h_C is the Yoneda embedding.

We construct cartesian lifts explicitly. Let $u \colon i \to j$ be a morphism in I, with induced morphism $u_! \colon X_i \to X_j$. The explicit description of the Grothendieck construction given in Remark 12.4.17

means that a morphism in $\int_{i:I} C_{/X_i}$ over $f\colon a \to b$ in C amounts to a commutative square of the form

$$\begin{array}{ccc} h_C(a) & \xrightarrow{h_C(f)} & h_C(b) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X_i & \xrightarrow{u_!} & X_j \end{array}$$

in $\mathrm{PSh}(C)$, where $u\colon i \to j$ is a morphism in I. Given an object (j, b, t) and a morphism $f\colon a \to b$ in C, take $u = \mathrm{id}_j$ and the source $(j, a, t \circ h_C(f))$. Any morphism to (j, b, t) lying over a composite $f \circ g$ factors uniquely through this morphism by the same commutative triangle in $\mathrm{PSh}(C)$. It is therefore p-cartesian. These chosen lifts compose, so p is a cartesian fibration. □

Lemma 13.3.4. *Let I be small and $X\colon I \to \mathrm{PSh}(C)$. Consider the pullback*

$$\begin{array}{ccc} C_{/X_i} & \longrightarrow & \mathrm{An}_\bullet^{\mathrm{op}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi^{\mathrm{op}}_{\mathrm{univ}}} \\ C & \xrightarrow{X_i^{\mathrm{op}}} & \mathrm{An}^{\mathrm{op}}, \end{array}$$

which provides a functor $I \to \mathrm{Cat}$, $i \mapsto C_{/X_i}$. Then there is a canonical equivalence

$$\mathrm{colim}_{i\in I}\, C_{/X_i} \xrightarrow{\sim} C_{/(\mathrm{colim}_{i\in I}\, X_i)}.$$

Proof. We start with some observations:

- The functor $\mathrm{Cat}_{/C} \to \mathrm{Cat}, (X, p) \mapsto X$ preserves colimits, since it admits a right adjoint $C \times -\colon \mathrm{Cat} \to \mathrm{Cat}_{/C}$. Furthermore this functor is conservative. Hence we may compute this colimit either in $\mathrm{Cat}_{/C}$ or Cat.
- The category $\mathrm{PSh}(C)$ is equivalent to the full subcategory $\mathrm{RFib}(C) \subseteq \mathrm{Cat}_{/C}$ consisting of the right fibrations. This equivalence is given by sending X to $(C_{/X}, p_X\colon C_{/X} \to C)$.
- So it will suffice to show that $\mathrm{colim}_{i\in I}\, C_{/X_i}$, computed in $\mathrm{Cat}_{/C}$, is still contained in $\mathrm{RFib}(C)$, since then it must also be a colimit in $\mathrm{RFib}(C)$ and hence correspond to the colimit of the X_i in $\mathrm{PSh}(C)$.

Put $E := \int_{i\in I} C_{/X_i}$, and write $p\colon E \to C$ and $r\colon E \to I$ for the two projections. By Lemma 13.3.3, the functor p is a cartesian fibration. By Theorem 13.2.1, the colimit $\mathrm{colim}_i\, C_{/X_i}$ is obtained from E by inverting the r-cocartesian morphisms. Every such morphism maps to an isomorphism in C. Conversely, a morphism of E whose image in C is an isomorphism factors as an r-cocartesian morphism followed by a morphism in some $C_{/X_j}$ lying over an isomorphism in C; the latter is an equivalence because $C_{/X_j} \to C$ is a right fibration. Thus this colimit is also the localization of E at all morphisms mapped by p to isomorphisms. Applying Corollary 12.4.12 with the subcategory $W = E$ shows that the induced functor

$$\mathrm{colim}_i\, C_{/X_i} \longrightarrow C$$

is a cartesian fibration. Its fiber over $c\colon * \to C$ is the localization of $\int_i X_i(c)$ at all its morphisms. Every such morphism factors as a cocartesian morphism followed by a vertical morphism, and the

vertical morphism is an equivalence because each $X_i(c)$ is an anima. This is therefore the localization at the cocartesian morphisms, hence the anima $\operatorname{colim}_i X_i(c)$. Thus all its fibers are animae, so it is a right fibration. Under straightening, its fiber at c is $\operatorname{colim}_i X_i(c)$, and it therefore classifies the pointwise colimit $\operatorname{colim}_i X_i$. □

Proposition 13.3.5. *Let C be a small category. The embedding functor*

$$\mathrm{PSh}(C) \simeq \mathrm{RFib}(C) \hookrightarrow \mathrm{Cat}_{/C}$$

has a right adjoint defined by assigning to each $f\colon X \to C$ the presheaf

$$c \mapsto \mathrm{Hom}_{\mathrm{Cat}_{/C}}(C_{/c}, X)\,.$$

Proof. Existence of the right adjoint follows from Corollary 12.8.2 and Lemma 13.3.4. If R denotes it, then the Yoneda lemma gives

$$R(X)(c) \simeq \mathrm{Hom}_{\mathrm{PSh}(C)}(h_C(c), R(X)) \simeq \mathrm{Hom}_{\mathrm{Cat}_{/C}}(C_{/c}, X),$$

which is the asserted formula. □

13.4 Descent and Quillen's Theorem B

We now use the preceding colimit calculations to prove descent results for cocartesian fibrations and then Quillen's Theorem B. To formulate the universal cocartesian fibration as an object of a category, we temporarily work in a larger universe.

13.4.1 The universal Grothendieck construction

Straightening identifies a functor $X \to \mathrm{Cat}$ with its cocartesian unstraightening, and naturality by pullback identifies morphisms over Cat with pullback squares of cocartesian fibrations. The following construction packages this correspondence as an equivalence of categories.

Consider a bigger universe Cat' so that Cat is Cat'-small and there is a fully faithful functor $\mathrm{Cat} \hookrightarrow \mathrm{Cat}'$. Let An' denote the corresponding universe of animae.

$$\begin{array}{ccc} \mathrm{Cat}_{/\mathrm{Cat}} & \longrightarrow & \mathrm{Cat}'_{/\mathrm{Cat}} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Cat} & \hookrightarrow & \mathrm{Cat}'. \end{array}$$

The objects of the category $\mathrm{Cat}_{/\mathrm{Cat}}$ are pairs (X, F) where X is a small category and $F\colon X \to \mathrm{Cat}$ is a functor. The category $\mathrm{Fun}([1], \mathrm{Cat}')$ has a non-full subcategory $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')$ whose objects are cocartesian fibrations in Cat' and whose morphisms are the pullback squares in Cat',

$$\begin{array}{ccc} x & \xrightarrow{a} & x' \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f'} \\ y & \xrightarrow{b} & y' \end{array}$$

with f and f' cocartesian fibrations. Let univ $= (\mathrm{Cat}_\bullet, \pi_{\mathrm{univ}})$, where $\pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ is the universal cocartesian fibration with small fibers, seen as an object of $\mathrm{Fun}([1], \mathrm{Cat}')$.

There is a canonical full embedding of the form

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}) \hookrightarrow \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')$$

and we can form the relative slice category

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})_{/\mathrm{univ}} = \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}) \times_{\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}')_{/\mathrm{univ}} .$$

We observe that the forgetful functor

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})_{/\mathrm{univ}} \to \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$$

is an equivalence. Indeed, it suffices to check that it induces an equivalence of animae of the form

$$\mathrm{Map}([1], \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})_{/\mathrm{univ}}) \to \mathrm{Map}([1], \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat}))$$

which amounts to see that any pullback square of small categories

$$\begin{array}{ccc} X & \xrightarrow{a} & X' \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{b} & Y' \end{array}$$

extends uniquely into pullback squares of the form

$$\begin{array}{ccccc} X & \xrightarrow{a} & X' & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ Y & \xrightarrow{b} & Y' & \longrightarrow & \mathrm{Cat}. \end{array}$$

Evaluating at 1 induces a canonical functor

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})_{/\mathrm{univ}} \to \mathrm{Cat}_{/\mathrm{Cat}} .$$

Theorem 13.4.1 (Universal Grothendieck construction)**.** *There is a canonical equivalence*

$$\mathrm{Cat}_{/\mathrm{Cat}} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$$

assigning to each pullback square of the form

$$\begin{array}{ccc} \int F & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle p_F}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ X & \xrightarrow{F} & \mathrm{Cat}. \end{array}$$

the functor p_F, seen as a term of $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$.

Proof. We have a functor

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat})_{/\mathrm{univ}} \to \mathrm{Cat}_{/\mathrm{Cat}}\,.$$

We observe that the induced map

$$\mathrm{Map}([1],\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat})) \to \mathrm{Map}([1],\mathrm{Cat}_{/\mathrm{Cat}})$$

is an equivalence: this means that pullback squares of the form

$$\begin{array}{ccccc} X & \xrightarrow{a} & X' & \longrightarrow & \mathrm{Cat}_\bullet \\ {\scriptstyle f}\downarrow & \lrcorner & {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ Y & \xrightarrow{b} & Y' & \longrightarrow & \mathrm{Cat} \end{array}$$

with Y and Y' small categories are uniquely determined by diagrams of the form

$$\begin{array}{ccccc} & & & & \mathrm{Cat}_\bullet \\ & & & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ Y & \xrightarrow{b} & Y' & \longrightarrow & \mathrm{Cat}. \end{array}$$

The inverse is given by the rule $(X, F) \mapsto p_F$. □

13.4.2 Descent for cocartesian fibrations

The preceding theorem provides a convenient way of computing colimits in $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat})$:

Theorem 13.4.2. *The category*

$$\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat})$$

admits small colimits, and its inclusion into $\mathrm{Fun}([1],\mathrm{Cat})$ *preserves them.*

Proof. Since the functor $\mathrm{Cat} \hookrightarrow \mathrm{Cat}'$ preserves the formation of small colimits and since the functor $\mathrm{Cat}'_{/\mathrm{Cat}} \to \mathrm{Cat}'$ commutes with large colimits (hence with small ones as well), the pullback $\mathrm{Cat}_{/\mathrm{Cat}}$ therefore has small colimits and the forgetful functor

$$\mathrm{Cat}_{/\mathrm{Cat}} \to \mathrm{Cat}$$

commutes with colimits. The fact that $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat})$ admits small colimits therefore follows from Theorem 13.4.1. To see that they are preserved by the inclusion, notice that the functors $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}([1],\mathrm{Cat}) \to \mathrm{Cat}$ preserve colimits and are jointly conservative. So it suffices to show that the functors $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat}) \to \mathrm{Cat}$ preserve small colimits. This follows from the observation that the composite

$$\mathrm{Cat}_{/\mathrm{Cat}} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat}) \xrightarrow{\mathrm{ev}_1} \mathrm{Cat}$$

agrees with the forgetful functor $(X, F \colon X \to \mathrm{Cat}) \mapsto X$, which preserves colimits, and that the composite

$$\mathrm{Cat}_{/\mathrm{Cat}} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1],\mathrm{Cat}) \xrightarrow{\mathrm{ev}_0} \mathrm{Cat}$$

agrees with the functor $(X, F \colon X \to \mathrm{Cat}) \mapsto \int F = X \times_{\mathrm{Cat}} \mathrm{Cat}_\bullet$, which also commutes with colimits (since pulling back along an exponentiable functor commutes with colimits). □

Corollary 13.4.3. *Let A be a small category and $X, Y\colon A \to \mathrm{Cat}$ be two functors and $u\colon X \to Y$ a natural transformation. We assume that the following two conditions are satisfied.*

(i) For each object $a : A^{\simeq}$, the induced functor $u_a\colon X_a \to Y_a$ is a cocartesian fibration (a cartesian fibration, a right fibration, or a left fibration).

(ii) For each morphism $\alpha\colon a_0 \to a_1$ in A, the induced commutative square

$$\begin{array}{ccc} X_{a_0} & \xrightarrow{\alpha_!} & X_{a_1} \\ {\scriptstyle u_{a_0}}\downarrow & & \downarrow{\scriptstyle u_{a_1}} \\ Y_{a_0} & \xrightarrow{\alpha_!} & Y_{a_1} \end{array}$$

is a pullback square in Cat.

Then the induced functor $\mathrm{colim}_{a:A} X_a \to \mathrm{colim}_{a:A} Y_a$ *is a cocartesian fibration (a cartesian fibration, a right fibration, a left fibration, respectively) and the canonical commutative squares*

$$\begin{array}{ccc} X_a & \longrightarrow & \mathrm{colim}_{a:A} X_a \\ {\scriptstyle u_a}\downarrow & & \downarrow \\ Y_a & \longrightarrow & \mathrm{colim}_{a:A} Y_a \end{array}$$

are pullback squares for any object $a : A^{\simeq}$.

Proof. Let us discuss the case of cocartesian fibrations. The natural transformation u can be interpreted as an A-indexed diagram with values in $\mathrm{Fun}^{\mathrm{coCart}}_{\mathrm{Cart}}([1], \mathrm{Cat})$. Taking the colimit there and the preceding theorem yields the conclusions of the corollary.

Assuming that all functors u_a are left fibrations, since there is a full embedding

$$\mathrm{Fun}(A, \mathrm{An}) \hookrightarrow \mathrm{Fun}(A, \mathrm{Cat})$$

whose essential image consists of those functors taking values in An object-wise, we can see the natural transformation from Y to the constant A-indexed functor with value Cat classifying the left fibrations u_a as natural transformation from Y to the constant functor with value An. In other words, we get a commutative diagram of the form

$$\begin{array}{ccccc} \mathrm{colim}_{a:A} X_a & \dashrightarrow & \mathrm{An}_\bullet & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{colim}_{a:A} Y_a & \longrightarrow & \mathrm{An} & \hookrightarrow & \mathrm{Cat} \end{array}$$

in which all squares are pullback squares, proving that the functor induced on colimits is a left fibration.

The remaining cases of cartesian fibrations and of right fibrations are proved similarly by duality, or by applying the preceding cases to the diagram induced by composing with the equivalence $(-)^{\mathrm{op}}$, and then applying $(-)^{\mathrm{op}}$ again, using that the latter preserves both pullbacks and colimits. □

13.4.3 Descent for animae

Lemma 13.4.4. *Let A be a small category and $X, Y\colon A \to \mathrm{An}$ be two functors, with $u\colon X \to Y$ a natural transformation. We assume that, for each morphism $\alpha\colon a_0 \to a_1$ in A, the induced commutative square*

$$\begin{array}{ccc} X_{a_0} & \xrightarrow{\alpha_!} & X_{a_1} \\ {\scriptstyle u_{a_0}}\downarrow & & \downarrow{\scriptstyle u_{a_1}} \\ Y_{a_0} & \xrightarrow{\alpha_!} & Y_{a_1} \end{array}$$

is a pullback square in An*. Then the canonical commutative squares*

$$\begin{array}{ccc} X_a & \longrightarrow & \mathrm{colim}_{a:A}\, X_a \\ {\scriptstyle u_a}\downarrow & & \downarrow \\ Y_a & \longrightarrow & \mathrm{colim}_{a:A}\, Y_a \end{array}$$

are pullback squares in An *for any object $a : A^{\simeq}$.*

Proof. Since the embedding $\mathrm{An} \hookrightarrow \mathrm{Cat}$ commutes with colimits, so does the induced embedding of categories of arrows

$$\mathrm{Fun}([1], \mathrm{An}) \hookrightarrow \mathrm{Fun}([1], \mathrm{Cat})\,.$$

We apply Corollary 13.4.3, observing that any functor between animae is both a left and a right fibration. □

13.4.4 Quillen's Theorem B

Theorem 13.4.5 (Quillen's Theorem B)**.** *Let $u\colon A \to B$ be a functor between small categories. The following conditions are equivalent.*

(i) For any morphism $\beta\colon b \to b'$ in B, the induced functor

$$\beta_!\colon A_{/b} \to A_{/b'}$$

induces an equivalence $\Pi_\infty(A_{/b}) \xrightarrow{\sim} \Pi_\infty(A_{/b'})$.

(ii) For any object b of B, the square

$$\begin{array}{ccc} \Pi_\infty(A_{/b}) & \longrightarrow & \Pi_\infty(A) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Pi_\infty(u)} \\ e = \Pi_\infty(B_{/b}) & \longrightarrow & \Pi_\infty(B) \end{array}$$

is a pullback square in An*.*

(iii) Any pullback square of the form

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{g} & B \end{array}$$

in which g is a right fibration induces a pullback square

$$\begin{array}{ccc} \Pi_\infty(A') & \xrightarrow{\Pi_\infty(f)} & \Pi_\infty(A) \\ {\scriptstyle \Pi_\infty(u')}\downarrow & \lrcorner & \downarrow{\scriptstyle \Pi_\infty(u)} \\ \Pi_\infty(B') & \xrightarrow{\Pi_\infty(g)} & \Pi_\infty(B) \end{array}$$

in An.

(iv) Any pullback square of the form

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{g} & B \end{array}$$

in which g is a smooth functor induces a pullback square

$$\begin{array}{ccc} \Pi_\infty(A') & \xrightarrow{\Pi_\infty(f)} & \Pi_\infty(A) \\ {\scriptstyle \Pi_\infty(u')}\downarrow & \lrcorner & \downarrow{\scriptstyle \Pi_\infty(u)} \\ \Pi_\infty(B') & \xrightarrow{\Pi_\infty(g)} & \Pi_\infty(B) \end{array}$$

in An.

There is a dual set of equivalent conditions obtained by simultaneously replacing relative slices $A_{/b}$ and $B_{/b}$ by relative coslices $A_{b/}$ and $B_{b/}$, replacing $\beta_!$ by β^*, replacing right fibrations by left fibrations, and replacing smooth functors by proper functors.

Proof. We prove the displayed formulation in terms of slices, right fibrations, and smooth functors. The implications are organized as

$$(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii), \qquad (iii) \Longleftrightarrow (iv).$$

The dual formulation follows by applying this result to u^{op} and using Corollary 12.4.6.

Condition (ii) follows from condition (iii) since the canonical projection $B_{/b} \to B$ is a right fibration. Similarly, assuming condition (ii), any morphism $\beta\colon b \to b'$ induces a pullback square of the form

$$\begin{array}{ccc} \Pi_\infty(A_{/b}) & \longrightarrow & \Pi_\infty(A_{/b'}) \\ \downarrow & \lrcorner & \downarrow \\ \Pi_\infty(B_{/b}) & \xrightarrow{\simeq} & \Pi_\infty(B_{/b'}) \end{array}$$

in which the lower horizontal map is an endomorphism of the terminal object, so that the upper horizontal map must be invertible. It remains to prove that condition (i) implies condition (ii).

Let e be the terminal object of An. We will also write e for the constant presheaf with value e. We have $A = A_{/e}$ and $A_{/b} = A \times_B B_{/b} = A_{/u^*(\mathrm{Hom}_B(-,b))}$, where

$$\begin{array}{ccc} \mathrm{PSh}(B) & \xrightarrow{u^*} & \mathrm{PSh}(A) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{RFib}(B) & \xrightarrow{u^*} & \mathrm{RFib}(A). \end{array}$$

Since Π_∞ commutes with colimits (being a left adjoint), Lemma 13.3.4 implies that

$$X \mapsto \Pi_\infty(A_{/X})$$

preserves colimits. But the functor u^* also has a left adjoint and thus commutes with colimits. This proves that the assignment $Y \mapsto \Pi_\infty(A_{/u^*Y})$ commutes with colimits. Since we have

$$\operatorname{colim}_B h_B = e \qquad \text{and} \qquad u^*(e) = e\,,$$

by Theorem 12.6.3, we see that $\operatorname{colim}_{b:B} \Pi_\infty(A_{/b}) \simeq \Pi_\infty(A_{/e}) \simeq \Pi_\infty(A)$. Since for every morphism $\beta\colon b \to b'$ we have a pullback square

$$\begin{array}{ccc} \Pi_\infty(A_{/b}) & \xrightarrow{\simeq} & \Pi_\infty(A_{/b'}) \\ \downarrow & \lrcorner & \downarrow \\ \Pi_\infty(B_{/b}) & \xrightarrow{\simeq} & \Pi_\infty(B_{/b'}) \end{array}$$

it follows from Lemma 13.4.4 that the commutative square

$$\begin{array}{ccc} \Pi_\infty(A_{/b}) & \longrightarrow & \Pi_\infty(A) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \Pi_\infty(u)} \\ e = \Pi_\infty(B_{/b}) & \longrightarrow & \Pi_\infty(B) \end{array}$$

induced by passing to colimits is a pullback square, as desired.

We will finally prove that condition (ii) implies condition (iii). For any object b' of B', if we put $b = g(b')$, the induced functor

$$B'_{/b'} \to B_{/b}$$

is an equivalence over B (it is a terminal functor between right fibrations). The horizontal arrows in the commutative square

$$\begin{array}{ccc} A'_{/b'} & \longrightarrow & A_{/b} \\ \downarrow & & \downarrow \\ B'_{/b'} & \longrightarrow & B_{/b} \end{array}$$

are equivalences, and we thus obtain from condition (ii) a pullback square of the form

$$\begin{array}{ccc} \Pi_\infty(A'_{/b'}) & \longrightarrow & \Pi_\infty(A) \\ \downarrow & & \downarrow{\scriptstyle \Pi_\infty(u)} \\ \Pi_\infty(B'_{/b'}) & \longrightarrow & \Pi_\infty(B) \end{array}$$

for each object b' in B'. The transition maps in the two B'-indexed diagrams are equivalences, and hence their naturality squares are pullbacks. Applying Lemma 13.4.4 and using

$$\operatorname{colim}_{b':B'} \Pi_\infty(A'_{/b'}) \xrightarrow{\sim} \Pi_\infty(A') \quad \text{and} \quad \operatorname{colim}_{b':B'} \Pi_\infty(B'_{/b'}) \xrightarrow{\sim} \Pi_\infty(B')$$

shows that condition (iii) follows from condition (ii).

Since every right fibration is smooth, condition (iv) implies condition (iii). For the converse, given a pullback square of the form

$$\begin{array}{ccc} A' & \xrightarrow{f} & A \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{g} & B \end{array}$$

in which g is a smooth functor, we choose a factorization of g into $g = q \circ j$ with a terminal functor $j\colon B' \to B''$ and a right fibration $q\colon B'' \to B$. We define $A'' = B'' \times_B A$. The functor j induces a functor $i\colon A' \to A''$ that is terminal (we see this by applying Corollary 12.4.7 in the case where φ is the identity). If $p\colon A'' \to A$ denotes the base change of q, and $u''\colon A'' \to B''$ the canonical projection, we obtain a commutative diagram of the form

$$\begin{array}{ccccc} \Pi_\infty(A') & \xrightarrow{\Pi_\infty(i)} & \Pi_\infty(A'') & \xrightarrow{\Pi_\infty(p)} & \Pi_\infty(A) \\ {\scriptstyle \Pi_\infty(u')}\downarrow & & {\scriptstyle \Pi_\infty(u'')}\downarrow & \lrcorner & \downarrow{\scriptstyle \Pi_\infty(u)} \\ \Pi_\infty(B') & \xrightarrow{\Pi_\infty(j)} & \Pi_\infty(B'') & \xrightarrow{\Pi_\infty(q)} & \Pi_\infty(B) \end{array}$$

in which the two horizontal maps of the square at the left hand side are equivalences. Therefore, since the square at the right hand side is a pullback square, so is the composite. □

14 Localizations

The universal property of a localization characterizes the functors out of it, but applications also require control over its categorical structure and its mapping animae. We address these two problems in turn. We first determine when terminal objects and cartesian products survive localization. We then study localizations admitting adjoints, whose fully faithful adjoints identify the localized category with a category of local objects. Finally, a calculus of fractions gives explicit formulas for mapping animae and pointwise Kan extensions.

We fix a universe of small categories Cat. Given a category C and a subcategory $W \hookrightarrow C$, we say *localization of C at W* for localization at the collection $\mathrm{Map}([1], W)$ of morphisms in W. We usually denote it by $L(C, W)$, and by $L(C)$ when W is understood. The notation $C[W^{-1}]$ from Definition 3.3.6 emphasizes the universal target in which the morphisms of W have become invertible. The notation $L(C, W)$ instead keeps both inputs visible and is convenient when treating localization as a construction functorial in the pair (C, W); the two notations refer to the same category.

The first two sections extract structural consequences from the universal property of localization and from the theory of Bousfield localizations in Definition 4.4.9. The final section uses Yoneda density and pointwise Kan extensions from the preceding chapters to pass from the universal property to explicit calculations.

14.1 Finite products in localizations

Terminal objects require no compatibility condition on the morphisms being inverted. For binary products, we first show that localization commutes with products of categories. The unitality hypothesis below makes inversion in the product category detectable separately in each variable. We then descend the diagonal/product adjunction through localization to obtain cartesian products in the localized category.

14.1.1 Products of localized categories

Proposition 14.1.1 (Terminal objects)**.** *Let C be a category and $W \hookrightarrow C$ a subcategory. If C has a terminal object then so does the localization $L(C, W)$ and the localization functor $\gamma\colon C \to L(C, W)$ preserves terminal objects.*

Proof. Let $t\colon * \to C$ be a terminal object. The functor t is terminal by Proposition 9.1.13, and the localization functor γ is terminal by Proposition 9.1.9. Since terminal functors are stable under

composition, $\gamma t : * \to L(C, W)$ is terminal. Another application of Proposition 9.1.13 shows that $\gamma(t)$ is a terminal object. □

Definition 14.1.2 (Wide subcategory)**.** A subcategory W of a category C is *wide* if it contains every object of C.

Proposition 14.1.3 (Products of localizations)**.** *Let C and C' be two categories, with wide subcategories*

$$W \hookrightarrow C \quad \textit{and} \quad W' \hookrightarrow C'$$

respectively, and associated localization functors

$$\gamma : C \to L(C, W) \quad \textit{and} \quad \gamma' : C' \to L(C', W') \,.$$

Then the functor

$$\gamma \times \gamma' : C \times C' \to L(C, W) \times L(C', W')$$

induces an equivalence

$$L(C \times C', W \times W') \xrightarrow{\sim} L(C, W) \times L(C', W') \,.$$

Proof. Let $\varphi : C \times C' \to D$ be any functor. Given morphisms $f : x \to y$ and $f' : x' \to y'$ in C and C' respectively, we have a canonical commutative square of the form below.

$$\begin{array}{ccc} (x, x') & \xrightarrow{(\mathrm{id}_x, f')} & (x, y') \\ {\scriptstyle (f, \mathrm{id}_{x'})}\downarrow & & \downarrow{\scriptstyle (f, \mathrm{id}_{y'})} \\ (y, x') & \xrightarrow{(\mathrm{id}_y, f')} & (y, y') \end{array}$$

Since both W and W' are wide, all identities of C belong to W and all identities of C' belong to W'. Consequently, the property for the functor φ to send all morphisms in $W \times W'$ to isomorphisms is equivalent to the conjunction of the following two properties:

- For any morphism $f : x \to y$ in W and any object x' of C', the morphism $\varphi(f, \mathrm{id}_{x'})$ is an isomorphism in D.
- For any morphism $f' : x' \to y'$ in W' and any object x of C, the morphism $\varphi(\mathrm{id}_x, f')$ is an isomorphism in D.

In other words, the currying equivalence

$$\mathrm{Fun}(C \times C', D) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Fun}(C', D))$$

restricts to an equivalence

$$\mathrm{Fun}^{W \times W'}(C \times C', D) \xrightarrow{\sim} \mathrm{Fun}^{W}(C, \mathrm{Fun}^{W'}(C', D))$$

functorially in D. This implies that there is a canonical equivalence

$$\mathrm{Fun}^{W \times W'}(C \times C', D) \xrightarrow{\sim} \mathrm{Fun}(L(C, W), \mathrm{Fun}(L(C', W'), D)) \,.$$

Using the equivalence

$$\mathrm{Fun}(L(C,W)\times L(C',W'),D)\xrightarrow{\sim}\mathrm{Fun}(L(C,W),\mathrm{Fun}(L(C',W'),D))$$

it follows that the functor $\gamma\times\gamma'$ exhibits $L(C,W)\times L(C',W')$ as a localization of $C\times C'$ at $W\times W'$. □

Corollary 14.1.4 (Finite products of geometric realizations)**.** *The geometric realization functor*

$$\Pi_\infty\colon \mathrm{Cat}\to\mathrm{An}$$

preserves finite products.

Proof. The terminal category e is already a terminal anima, so $\Pi_\infty(e)\xrightarrow{\sim} e$. For binary products, apply Proposition 14.1.3 with $W=C$ and $W'=C'$. Localization at all morphisms is geometric realization, and hence the resulting equivalence is

$$\Pi_\infty(C\times C')\xrightarrow{\sim}\Pi_\infty(C)\times\Pi_\infty(C').$$ □

14.1.2 Cartesian localizations

To construct products inside a single localization, we transport the diagonal/product adjunction. We first record the general fact that an adjunction descends whenever both adjoints preserve the chosen subcategories.

Lemma 14.1.5 (Adjunctions descend to localizations)**.** *Let $F\colon C\rightleftarrows D\colon G$ be an adjunction. We consider two subcategories*

$$V\hookrightarrow C\quad\text{and}\quad W\hookrightarrow D$$

respectively, and we assume that F sends every morphism in V to a morphism in W, and, conversely, that G sends every morphism in W to a morphism in V. Then there is an induced adjunction of the form

$$\bar F\colon L(C,V)\rightleftarrows L(D,W)\colon\bar G$$

with canonical commutative squares

$$\begin{array}{ccc} C & \xrightarrow{F} & D \\ {\scriptstyle\gamma_C}\downarrow & & \downarrow{\scriptstyle\gamma_D} \\ L(C,V) & \xrightarrow{\bar F} & L(D,W) \end{array}\qquad\text{and}\qquad \begin{array}{ccc} D & \xrightarrow{G} & C \\ {\scriptstyle\gamma_D}\downarrow & & \downarrow{\scriptstyle\gamma_C} \\ L(D,W) & \xrightarrow{\bar G} & L(C,V) \end{array}$$

Furthermore, if $\eta\colon \mathrm{id}_C\to GF$ and $\varepsilon\colon FG\to\mathrm{id}_D$ are the unit and counit of the original adjunction, then the unit and counit of the induced adjunction satisfy

$$\bar\eta_{\gamma_C(x)}=\gamma_C(\eta_x)\qquad\text{and}\qquad\bar\varepsilon_{\gamma_D(y)}=\gamma_D(\varepsilon_y)$$

under the displayed identifications, for objects x of C and y of D.

Proof. Write $\gamma_C \colon C \to L(C,V)$ and $\gamma_D \colon D \to L(D,W)$ for the localization functors. Since F sends V into W and G sends W into V, functoriality of localization gives functors $\bar{F}$ and $\bar{G}$ together with identifications

$$\alpha \colon \bar{F}\gamma_C \xrightarrow{\sim} \gamma_D F \qquad \text{and} \qquad \beta \colon \bar{G}\gamma_D \xrightarrow{\sim} \gamma_C G.$$

After these identifications, applying γ_C to the unit $\eta \colon \mathrm{id}_C \to GF$ gives a natural transformation

$$\gamma_C \longrightarrow \bar{G}\bar{F}\gamma_C.$$

The universal property of γ_C identifies this with a unique natural transformation

$$\bar{\eta} \colon \mathrm{id}_{L(C,V)} \longrightarrow \bar{G}\bar{F}.$$

Similarly, applying γ_D to the counit $\varepsilon \colon FG \to \mathrm{id}_D$ produces a unique natural transformation

$$\bar{\varepsilon} \colon \bar{F}\bar{G} \longrightarrow \mathrm{id}_{L(D,W)}\,.$$

The two triangle identities for $\bar{\eta}$ and $\bar{\varepsilon}$ can be checked after precomposition with γ_C and γ_D, respectively, because precomposition with a localization functor is fully faithful. After precomposition and transport along α and β, the first triangle is γ_D applied to the triangle for F, while the second is γ_C applied to the triangle for G. Thus $\bar{F}$ is left adjoint to $\bar{G}$. The assertions about the components of the unit and counit follow from their construction. □

Proposition 14.1.6 (Cartesian localizations)**.** *Let C be a category and $W \hookrightarrow C$ a wide subcategory. We assume that C has binary cartesian products and that W is stable under cartesian product in the sense that, for any morphisms f and g in W, the induced morphism $f \times g$ belongs to W. Then the localization $L(C,W)$ has binary cartesian products and the localization functor $\gamma \colon C \to L(C,W)$ commutes with them.*

Proof. There is an adjunction

$$\Delta \colon C \rightleftarrows C \times C \colon (-) \times (-)$$

where Δ is the functor defined by the rule $x \mapsto (x,x)$. The property that W is stable under binary cartesian product precisely means that Lemma 14.1.5 applies in this situation, yielding an adjunction of the form

$$\bar{\Delta} \colon L(C,W) \rightleftarrows L(C \times C, W \times W) \colon \overline{(-) \times (-)}\,.$$

But, since W is wide, Proposition 14.1.3 provides a canonical equivalence

$$L(C \times C, W \times W) \xrightarrow{\sim} L(C,W) \times L(C,W)$$

whose composition with $\bar{\Delta}$ is the diagonal functor

$$L(C,W) \to L(C,W) \times L(C,W)\,, \qquad x \mapsto (x,x)\,.$$

This proves the existence of cartesian products in $L(C,W)$. Under the equivalence above, the compatibility square for the right adjoint from Lemma 14.1.5 identifies $\overline{(-) \times (-)}(\gamma(x), \gamma(y))$ with $\gamma(x \times y)$. Thus γ preserves binary cartesian products. □

The universal property has now supplied finite products in the localization. We next study localization functors that themselves participate in an adjunction and identify their local objects.

14.2 Adjoint localizations and local objects

There are two complementary directions. If a localization functor admits an adjoint, that adjoint is fully faithful. Conversely, a functor with a fully faithful adjoint is a Bousfield localization at the morphisms it inverts. Its essential image consists of the local objects. This description also explains how limits and colimits are computed in a reflective localization.

Proposition 14.2.1 (Adjoints of localizations)**.** *Let C be a category. If $f\colon C \to D$ is a localization of C at a subcategory $W \hookrightarrow C$ and admits a right (resp. left) adjoint $g\colon D \to C$, then g is fully faithful.*

Proof. We prove the case of right adjoints (the case of left adjoints is analogous, or follows from the case of right adjoints by applying the operator $(-)^{\mathrm{op}}$). We choose a universe Cat so that C and D are small. The universal property of a localization is invariant under passage to opposite categories, so f^{op} is a localization at the opposite subcategory W^{op}. In particular, restriction

$$f^*\colon \mathrm{PSh}(D) \longrightarrow \mathrm{PSh}(C)$$

is fully faithful. By Theorem 12.3.20, the adjunction $f \dashv g$ gives an identification

$$f^* \xrightarrow{\sim} g_!\,.$$

Hence $g_!$ is fully faithful. Applying Proposition 12.2.11 to g^{op} shows that g^{op}, and therefore g, is fully faithful. □

Corollary 14.2.2 (Recognition of Bousfield localizations)**.** *Let $f\colon C \to D$ have a fully faithful right adjoint $g\colon D \to C$, and let W be the subcategory of morphisms inverted by f. Then f is a localization at W.*

Proof. By Proposition 6.4.10, the adjunction exhibits g as a right adjoint section. Hence Proposition 4.4.16 shows that f is a localization at W. □

Under the hypotheses of the corollary, we call an object x of C *W-local* if the presheaf $\mathrm{Hom}_C(-,x)$ sends every morphism in W to an isomorphism.

Proposition 14.2.3 (Local-object criterion)**.** *Let $f\colon C \to D$ have a fully faithful right adjoint $g\colon D \to C$, let W be the subcategory of morphisms inverted by f, and let $\eta\colon \mathrm{id}_C \to gf$ be the unit. An object x of C is W-local if and only if $\eta_x\colon x \to gf(x)$ is an isomorphism. Consequently, g identifies D with the full subcategory of C spanned by the W-local objects.*

Proof. By Corollary 14.2.2, the functor f is a localization at W. Consider the adjunction

$$f_!\colon \mathrm{PSh}(C) \rightleftarrows \mathrm{PSh}(D)\colon f^*.$$

Passing this localization to opposite categories shows that the right adjoint f^* is fully faithful, and its essential image consists of those presheaves on C that send every morphism in W to an isomorphism. Consequently, for a presheaf $F\colon C^{\mathrm{op}} \to \mathrm{An}$, the unit

$$F \longrightarrow f^* f_!(F)$$

is invertible precisely when F sends every morphism inverted by f to an isomorphism. For the representable presheaf $F = \mathrm{Hom}_C(-, x)$, Proposition 12.3.18 and Theorem 12.3.20 identify this unit with the map

$$\mathrm{Hom}_C(-, x) \longrightarrow \mathrm{Hom}_D(f(-), f(x)) \xrightarrow{\sim} \mathrm{Hom}_C(-, gf(x))$$

induced by the unit $\eta_x \colon x \to gfx$. The full faithfulness of the Yoneda embedding, Proposition 12.3.10, therefore shows that the property that $\mathrm{Hom}_C(-, x)$ sends every morphism inverted by f to an isomorphism is equivalent to the invertibility of $\eta_x \colon x \to gfx$. Since the unit is invertible on every object in the image of g, and every object on which the unit is invertible belongs to that image, the final assertion follows. □

Proposition 14.2.4 (Limits and colimits in reflective localizations)**.** *Let $f \colon C \to D$ be a functor with a fully faithful right adjoint $g \colon D \to C$. Let I be a category and $F \colon I \to D$ a functor. If $c = \mathrm{colim}_{i:I}\, gF(i)$ exists in C, then*

$$\mathrm{colim}_{i:I}\, F(i) \simeq f(c).$$

If $\ell = \lim_{i:I} gF(i)$ exists in C, then ℓ is local with respect to the morphisms inverted by f, and the unit $\ell \to gf(\ell)$ is invertible. In particular, the limit of F exists and

$$g\,(\lim_{i:I} F(i)) \simeq \ell\,.$$

Equivalently, $f(\ell)$ is a limit of F.

Proof. We may assume that there is a universe Cat so that C, D as well as I are all small. Assume first that the colimit $c = \mathrm{colim}_{i:I}\, gF(i)$ exists in C. For any object d of D, the adjunction and the full faithfulness of g give

$$\begin{aligned}\mathrm{Hom}_D(f(c), d) &\simeq \mathrm{Hom}_C(c, g(d)) \\ &\simeq \lim_{i:I^{\mathrm{op}}} \mathrm{Hom}_C(gF(i), g(d)) \\ &\simeq \lim_{i:I^{\mathrm{op}}} \mathrm{Hom}_D(F(i), d).\end{aligned}$$

Thus $f(c)$ is a colimit of F.

Assume now that the limit $\ell = \lim_{i:I} gF(i)$ exists in C. For every i, the presheaf $\mathrm{Hom}_C(-, gF(i))$ sends the morphisms inverted by f to isomorphisms, by Proposition 14.2.3. The same is therefore true of its limit

$$\mathrm{Hom}_C(-, \ell) \simeq \lim_{i:I} \mathrm{Hom}_C(-, gF(i)).$$

Another application of that remark shows that the unit at ℓ is invertible, and hence gives an isomorphism

$$\ell \xrightarrow{\sim} gf(\ell).$$

Consequently, for every object d of D,

$$\begin{aligned}\mathrm{Hom}_D(d, f(\ell)) &\simeq \mathrm{Hom}_C(g(d), gf(\ell)) \\ &\simeq \mathrm{Hom}_C(g(d), \ell) \\ &\simeq \lim_{i:I} \mathrm{Hom}_C(g(d), gF(i)) \\ &\simeq \lim_{i:I} \mathrm{Hom}_D(d, F(i)).\end{aligned}$$

Thus $f(\ell)$ is a limit of F. □

Corollary 14.2.5 (Existence of limits and colimits)**.** *Let $f : C \to D$ be a functor with a fully faithful right adjoint $g : D \to C$. Let I be a category. If C admits I-indexed colimits, then so does D. If C admits I-indexed limits, then so does D.*

Thus colimits are formed in the ambient category and then reflected, whereas limits of local objects are already local. We now turn from such formal consequences of localization to an explicit description of its mapping animae.

14.3 Calculus of fractions

Fix a universe Cat and a small category C equipped with a subcategory

$$W \hookrightarrow C\,.$$

We denote by $\gamma : C \to L(C)$ the localization of C at W. Our aim is to recognize systems of denominators for which there is a canonical isomorphism

$$\operatorname*{colim}_{w : W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \xrightarrow{\sim} \mathrm{Hom}_{L(C)}(\gamma(x), \gamma(y)).$$

A term on the left is represented by a span

$$x \xleftarrow{s_w} z_w \xrightarrow{f} y$$

in which s_w becomes invertible in $L(C)$; it therefore determines the morphism $\gamma(f) \circ \gamma(s_w)^{-1}$ from $\gamma(x)$ to $\gamma(y)$.

Rather than impose a particular list of Ore conditions, we begin with a weakly contractible category $W(x)$ of denominators equipped with a distinguished identity denominator. We call such data *putative*, and then recognize when the resulting candidate Hom-functor descends through the localization. This separates the formal fraction formula from any particular method for constructing the denominator categories.

Definition 14.3.1 (Putative right calculus of fractions)**.** A *putative right calculus of fractions* at an object $x : C^{\simeq}$ is a functor of the form

$$p : W(x) \to C_{/x}$$

with the following properties:

(i) For any object $w : W(x)^{\simeq}$, if we denote by $s_w : z_w \to x$ the morphism corresponding to $p(w)$, so that

$$p(w) = (z_w, s_w)\,,$$

the morphism s_w is sent to an isomorphism in $L(C)$.

(ii) There is an object $i_x : W(x)^{\simeq}$ equipped with an identification $p(i_x) = (x, \mathrm{id}_x)$.

(iii) The category $W(x)$ is weakly contractible.

Dually, a *putative left calculus of fractions* at $x : C^{\simeq}$ is a functor of the form $p\colon W(x) \to C_{x/}$ such that the induced functor

$$p^{\mathrm{op}}\colon W(x)^{\mathrm{op}} \to (C_{x/})^{\mathrm{op}} = (C^{\mathrm{op}})_{/x}$$

is a putative right calculus of fractions at $x : (C^{\mathrm{op}})^{\simeq}$.

Notation 14.3.2 (Hom in the localization)**.** If $x : C^{\mathrm{op}}$ and $y : C$, we define

$$\mathrm{Hom}_{L(C)}(x, y) := \mathrm{Hom}_{L(C)}(\gamma^{\mathrm{op}}(x), \gamma(y))\,.$$

Construction 14.3.3 (Canonical fraction map)**.** There is a map

$$u\colon \mathrm{Hom}_C(x, y) \to \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)$$

functorial in y, induced by $i_x : W(x)^{\simeq}$ and the identification $p(i_x) = (x, \mathrm{id}_x)$. Similarly, there is a map of the form

$$v\colon \mathrm{Hom}_{L(C)}(x, y) \to \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(z_w^{\mathrm{op}}, y)$$

functorially in $y : C$. The maps $\gamma(s_w)\colon \gamma(z_w) \to \gamma(x)$ assemble into a natural transformation and are all isomorphisms. Consequently, for every y the diagram

$$w \mapsto \mathrm{Hom}_{L(C)}(z_w^{\mathrm{op}}, y)$$

is naturally isomorphic to the constant diagram with value $\mathrm{Hom}_{L(C)}(x, y)$. Since $W(x)^{\mathrm{op}}$ is weakly contractible, condition (iii) of Lemma 13.1.11 says that the constant-diagram functor is fully faithful. Since colimit is its left adjoint, the colimit of this constant diagram is its value. It follows that v is an isomorphism. Composing the map induced by the localization functor

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \to \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(z_w^{\mathrm{op}}, y)$$

with the inverse of v yields a map of the form

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \to \mathrm{Hom}_{L(C)}(x, y)$$

that is functorial in y. The latter map will be referred to as the *canonical map*.

The putative calculus therefore supplies a candidate functor

$$H_x\colon C \to \mathrm{An}, \qquad H_x(y) = \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y).$$

The extra condition in the following definition says precisely that H_x is local with respect to W, and hence descends to $L(C)$.

Definition 14.3.4 (Right calculus of fractions)**.** A *right calculus of fractions* at x is a putative right calculus of fractions $p\colon W(x) \to C_{/x}$ such that the functor from C to An defined by the rule

$$y \mapsto \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)$$

sends every morphism in W to an isomorphism in An.

Dually a *left calculus of fractions* at x is a putative left calculus of fractions at x that becomes a right calculus of fractions at x after applying the operator $(-)^{\mathrm{op}}$.

Proposition 14.3.5 (Recognition of a calculus of fractions)**.** *Let $p\colon W(x) \to C_{/x}$ be a putative right calculus of fractions at an object x of C. A necessary and sufficient condition for this putative right calculus of fractions at x to be a right calculus of fractions at x is that the canonical map*

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \longrightarrow \mathrm{Hom}_{L(C)}(x, y)$$

is invertible for all y. Furthermore, when this condition holds, the resulting isomorphism is uniquely characterized by its functoriality in y and by the property that it sends the identity of x, viewed as a term of the colimit through i_x, to the identity morphism of $\gamma(x)$ in $L(C)$.

Proof. We identify the condition that H_x descend through γ with invertibility of its localization unit. We then compare this unit with the canonical fraction map and finally use the Yoneda lemma to prove the asserted uniqueness.

We have an adjunction of the form

$$\gamma_!\colon \mathrm{Fun}(C, \mathrm{An}) \rightleftarrows \mathrm{Fun}(L(C), \mathrm{An})\colon \gamma^*$$

and the right adjoint γ^* is fully faithful, with essential image those functors from C to An sending every morphism in W to an isomorphism. The functor $\gamma_!$ preserves small colimits because it is a left adjoint, while γ^* preserves them because colimits in functor categories are computed pointwise, by Lemma 7.3.1. Hence the unit

$$\eta_F\colon F \to \gamma^*\gamma_!(F)$$

is compatible with small colimits in F. Since γ^* is fully faithful, η_F is an isomorphism if and only if F sends every morphism in W to an isomorphism. For $F = \mathrm{Hom}_C(x, -)$, the unit map coincides with the map

$$\eta\colon \mathrm{Hom}_C(x, -) \to \mathrm{Hom}_{L(C)}(\gamma(x), \gamma(-))$$

induced by γ. Set

$$H_x := \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, -).$$

Applying the naturality of Yoneda from Proposition 12.3.18 to γ^{op}, the unit at H_x identifies with the colimit of the units for the representable functors $\mathrm{Hom}_C(z_w, -)$:

$$\eta'\colon H_x \longrightarrow \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(\gamma^{\mathrm{op}}(z_w^{\mathrm{op}}), \gamma(-))$$

is an isomorphism if and only if H_x sends every morphism in W to an isomorphism. This is precisely the condition that p be a right calculus of fractions.

For every $y : C$, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_C(x, y) & \xrightarrow{u} & \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\ \mathrm{Hom}_{L(C)}(x, y) & \xrightarrow{v} & \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(z_w^{\mathrm{op}}, y) \end{array}$$

Here the vertical maps are unit maps, while u and v are the maps from Construction 14.3.3. The map v is an isomorphism, and the canonical map is $v^{-1}\eta'$. This proves the first assertion and also

shows that the canonical isomorphism sends the distinguished identity term to the identity of $\gamma(x)$ in $L(C)$.

Finally, when p is a right calculus of fractions, both H_x and $\mathrm{Hom}_{L(C)}(x, \gamma(-))$ descend uniquely to functors on $L(C)$. Any two isomorphisms with the stated property differ by an automorphism of the representable functor $\mathrm{Hom}_{L(C)}(x, -)$ that fixes the identity of x. The Yoneda lemma for $L(C)^{\mathrm{op}}$ implies that this automorphism is the identity, proving uniqueness. □

Example 14.3.6 (The trivial calculus of fractions)**.** Take $W = C^{\simeq}$ and

$$W(x) = (C^{\simeq})_{/x},$$

with its canonical functor to $C_{/x}$. The identity of x is terminal in $W(x)$, so $W(x)$ is weakly contractible. Since every denominator is already an isomorphism in C, the functor H_x is naturally isomorphic to $\mathrm{Hom}_C(x, -)$ and therefore sends W to isomorphisms. Thus this is a right calculus of fractions, and its mapping-anima formula reduces to the identity isomorphism for $\mathrm{Hom}_C(x, y)$.

The mapping-anima formula is the representable case of a pointwise formula for left Kan extension along the localization. Yoneda density extends it from representables to arbitrary presheaves.

Corollary 14.3.7 (Pointwise localization by fractions)**.** *Let $p\colon W(x) \to C_{/x}$ be a right calculus of fractions at an object x. We consider the adjunction on presheaf categories*

$$\gamma_!\colon \mathrm{PSh}(C) \rightleftarrows \mathrm{PSh}(L(C))\colon \gamma^*$$

induced by the localization functor $\gamma\colon C \to L(C)$. Then, for any presheaf $F\colon C^{\mathrm{op}} \to \mathrm{An}$ there is a canonical isomorphism of the form

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} F(z_w^{\mathrm{op}}) \xrightarrow{\sim} \gamma_!(F)(\gamma(x))\,.$$

Proof. We first construct the canonical comparison below. Both its source and target preserve small colimits in F, so Yoneda density reduces its invertibility to the representable case, which is precisely Proposition 14.3.5.

For any presheaf G on $L(C)$, the isomorphisms $\gamma(s_w)$ induce an identification from the constant diagram with value $G(\gamma(x))$ to the diagram $w \mapsto G(\gamma^{\mathrm{op}}(z_w^{\mathrm{op}}))$. Since $W(x)^{\mathrm{op}}$ is weakly contractible, the argument from Construction 14.3.3 shows that taking its colimit gives a natural equivalence

$$G(\gamma(x)) \xrightarrow{\sim} \mathrm{colim}_{w:W(x)^{\mathrm{op}}} G(\gamma^{\mathrm{op}}(z_w^{\mathrm{op}})).$$

Take $G = \gamma_!(F)$. The unit $F \to \gamma^* G$ also induces a map

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} F(z_w^{\mathrm{op}}) \longrightarrow \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} G(\gamma^{\mathrm{op}}(z_w^{\mathrm{op}}))\,.$$

Composing with the inverse of the preceding isomorphism yields a canonical map

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} F(z_w^{\mathrm{op}}) \longrightarrow \gamma_!(F)(\gamma(x)).$$

Both sides preserve small colimits in F. Therefore, to prove that this map is invertible, Corollary 12.6.4 tells us that it suffices to prove that it is invertible for $F = \mathrm{Hom}_C(-, y)$ for some object y of C. In that case, $\gamma_!(F)$ is $\mathrm{Hom}_{L(C)}(-, \gamma(y))$ by Proposition 12.3.18. The displayed map is then the canonical fraction map, which is invertible by Proposition 14.3.5. □

Remark 14.3.8 (Fractions in the presheaf category)**.** For a right calculus of fractions at x as in the previous corollary, the Yoneda lemma applied for C and for $L(C)$ respectively allows us to write

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{PSh}(C)}(h_C(z_w), F) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}(L(C))}(h_{L(C)}(\gamma(x)), \gamma_!(F))$$

for any presheaf F on C. Using Proposition 12.3.18, one can recast this formula by saying that any right calculus of fractions at x of the form

$$p\colon W(x) \to C_{/x}$$

induces through the Yoneda embedding of C a right calculus of fractions at $h_C(x)$

$$W(x) \xrightarrow{p} C_{/x} \hookrightarrow \mathrm{PSh}(C)_{/h_C(x)}$$

with respect to the localization of $\mathrm{PSh}(C)$ at the subcategory of morphisms that become invertible under $\gamma_!$.

The colimit in the fraction formula also has a total-category description. An object of the category below consists of an object w of $W(x)$ and a morphism $f\colon z_w \to y$, hence of a span

$$x \xleftarrow{s_w} z_w \xrightarrow{f} y.$$

Informally, its morphisms encode coherent refinements of such spans.

Proposition 14.3.9 (The category of fractions)**.** *Let $p\colon W(x) \to C_{/x}$ be a putative right calculus of fractions at $x : C^{\simeq}$. For any other object $y : C^{\simeq}$, consider the associated right fibration*

$$C_{/y} \to C$$

and form the pullback

$$W(x)_{/y} = W(x) \times_C C_{/y}\,.$$

There is a canonical isomorphism

$$\Pi_\infty(W(x)_{/y}) \xrightarrow{\sim} \operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)\,.$$

Moreover, p is a right calculus of fractions if and only if the functor

$$C \to \mathrm{An}, \qquad y \longmapsto \Pi_\infty(W(x)_{/y})$$

sends every morphism in W to an isomorphism. When these equivalent conditions hold, there is a canonical isomorphism

$$\Pi_\infty(W(x)_{/y}) \xrightarrow{\sim} \mathrm{Hom}_{L(C)}(x, y)\,.$$

Proof. The right fibration $W(x)_{/y} \to W(x)$ corresponds to the functor obtained by composing $\mathrm{Hom}_C(-, y)$ with the functor from $W(x)^{\mathrm{op}}$ to C^{op} induced by p^{op}. The cocartesian unstraightening of this functor is the left fibration $(W(x)_{/y})^{\mathrm{op}} \to W(x)^{\mathrm{op}}$. Therefore Remark 9.3.4 gives

$$\operatorname*{colim}_{w:W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) \xrightarrow{\sim} \Pi_\infty((W(x)_{/y})^{\mathrm{op}})\,.$$

Since there is a canonical equivalence

$$\Pi_\infty(W(x)_{/y}) \xrightarrow{\sim} \Pi_\infty((W(x)_{/y})^{\mathrm{op}})$$

by Corollary 12.2.6, this proves the assertion. The remaining statements now follow from the definition of a right calculus of fractions and Proposition 14.3.5. □

Corollary 14.3.10. *Let C be a small category equipped with a subcategory $W \hookrightarrow C$, and write $\gamma\colon C \to L(C)$ for the localization at W. Suppose that $p\colon W(x) \to C_{/x}$ is a right calculus of fractions at $x : C^{\simeq}$. For every object $y : C^{\simeq}$, form*

$$W(x)_{/y} := W(x) \times_C C_{/y}.$$

Then there is a canonical cover

$$(W(x)_{/y})^{\simeq} \to \mathrm{Hom}_{L(C)}(x, y)$$

sending a pair (w, f), with $w : W(x)^{\simeq}$ and $f\colon z_w \to y$ in C, to the morphism $\gamma(f) \circ \gamma(s_w)^{-1}$ in $L(C)$.

Proof. By Corollary 11.2.17, the canonical map

$$(W(x)_{/y})^{\simeq} \to \Pi_\infty(W(x)_{/y})$$

is a cover. Compose it with the canonical isomorphism

$$\Pi_\infty(W(x)_{/y}) \xrightarrow{\sim} \mathrm{Hom}_{L(C)}(x, y)$$

from Proposition 14.3.9. The description on terms follows from the construction of the canonical fraction map. □

We have obtained two complementary forms of control over localization. Its universal property shows that suitable finite products descend and that reflective localizations are governed by their local objects. A calculus of fractions then turns the same universal property into explicit formulas for mapping animae and pointwise Kan extensions. Finally, the category $W(x)_{/y}$ realizes the hom anima as the anima of spans and their refinements. Later chapters use this description both to produce covers of mapping animae and to analyze the category associated to a univalent Segal anima.

15 Rezk nerves and univalent Segal animae

A category C determines a simplicial anima

$$N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}, \qquad [n] \longmapsto \mathrm{Map}([n], C),$$

called its *Rezk nerve*. Its first three terms record the objects, morphisms, and composition in C. More generally, $\mathrm{Map}([n], C)$ is the anima of strings of n composable morphisms, and the simplicial structure records their restrictions and composites. In this chapter we show that this information determines C completely, and characterize the simplicial animae that arise in this way.

The main result is that the Rezk nerve is fully faithful, with essential image the univalent Segal animae (Corollaries 15.5.5 and 15.1.13). The Segal condition expresses the existence and uniqueness of composition, while univalence identifies isomorphisms with identifications of objects. Together these conditions give a description of categories entirely in terms of animae and maps between them.

We first prove full faithfulness using an abstract nerve theorem, which recovers a category as a localization of its category of simplices. We then study Segal animae and their mapping animae. The final step is Rezk completion: every Segal anima has an associated category whose nerve has the same mapping animae, and every object of that category is isomorphic to one coming from the original Segal anima. This also identifies the category of small categories as the localization of small Segal animae at Dwyer–Kan equivalences (Corollary 15.5.7).

Definition 15.0.1 (Rezk nerve)**.** Let $i\colon \mathbf{\Delta} \hookrightarrow \mathrm{Cat}$ denote the inclusion. We define the *Rezk nerve functor* as the composite functor

$$N\colon \mathrm{Cat} \xrightarrow{h_{\mathrm{Cat}}} \mathrm{PSh}(\mathrm{Cat}) \xrightarrow{i^*} \mathrm{PSh}(\mathbf{\Delta}).$$

By Theorem 12.8.1, this functor admits a left adjoint

$$\mathrm{ac} = Li_!\colon \mathrm{PSh}(\mathbf{\Delta}) \to \mathrm{Cat}$$

satisfying $\mathrm{ac}\,\Delta^n = \mathrm{ac}\,h_{\mathbf{\Delta}}([n]) = [n]$, functorially in $[n] \in \mathbf{\Delta}$. We refer to ac as the *associated category* functor.

15.1 An abstract nerve theorem

The full faithfulness of the Rezk nerve is a special case of Theorem 15.1.11 below. It gives a criterion for a small full subcategory $i\colon A \hookrightarrow \mathrm{Cat}$ to determine every small category C from the presheaf

$$a \longmapsto \mathrm{Hom}_{\mathrm{Cat}}(i(a), C).$$

The criterion asks that the categories $i(a)$ have terminal objects and that the family admit a *décalage structure*, an operation of appending a new terminal vertex. For the simplex category this is $[n] \mapsto [n] \star [0] = [n+1]$, and the theorem gives the full faithfulness of the Rezk nerve in Corollary 15.1.13.

The proof compares two ways of reconstructing C from its category of A-simplices. Evaluation at the last vertex exhibits C as a localization of that category. We show that the left adjoint to the nerve performs the same localization, so its counit is an equivalence. We begin by constructing the last-vertex evaluation and studying its fibers.

Construction 15.1.1 (Evaluation at the last vertex)**.** Let A be a small category and $i\colon A \to \mathrm{Cat}$ a functor. For each object a of A, choose a terminal object e_a of $i(a)$. For a small category C, define its category of A-simplices by the pullback

$$\begin{array}{ccc} A_{/C} & \xrightarrow{i_{/C}} & \mathrm{Cat}_{/C} \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\ i\ } & \mathrm{Cat}. \end{array}$$

Thus an object of $A_{/C}$ is a functor $\gamma\colon i(a) \to C$, and $A_{/C} \to A$ is the right fibration classified by $a \mapsto \mathrm{Hom}_{\mathrm{Cat}}(i(a), C)$. We will construct a functor

$$\tau_C\colon A_{/C} \to C, \qquad (a, \gamma) \longmapsto \gamma(e_a).$$

Recall from Proposition 12.6.7 that the inclusion $\mathrm{RFib}(C) \hookrightarrow \mathrm{Cat}_{/C}$ has a left adjoint L. The composite

$$A_{/C} \xrightarrow{i_{/C}} \mathrm{Cat}_{/C} \xrightarrow{L} \mathrm{RFib}(C)$$

sends $\gamma\colon i(a) \to C$ to the representable right fibration $C_{/\gamma(e_a)} \to C$. Indeed, factor γ as a terminal functor $u\colon i(a) \to X$ followed by a right fibration $X \to C$. The object $u(e_a)$ is terminal in X, so it identifies X with $C_{/\gamma(e_a)}$ over C. The Yoneda embedding identifies the full subcategory of representable right fibrations with C, giving the desired functor τ_C.

On a morphism represented by a commutative triangle

$$\begin{array}{ccc} i(a) & \xrightarrow{\ i(u)\ } & i(b) \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & C & \end{array} \qquad u\colon a \to b,$$

it gives the morphism $\beta(e_a) \simeq \gamma(i(u)(e_a)) \to \gamma(e_b)$ induced by terminality of e_b.

The compatibility of L with composition over the base, Remark 12.6.8, makes this construction natural in C. For every $f\colon C \to D$, we have a commutative square

$$\begin{array}{ccc} A_{/C} & \xrightarrow{\tau_C} & C \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f} \\ A_{/D} & \xrightarrow{\tau_D} & D, \end{array}$$

where $f_!$ is composition with f.

Lemma 15.1.2. *Let $f\colon C \to D$ be a right fibration. Then*

$$\begin{array}{ccc} A_{/C} & \xrightarrow{\tau_C} & C \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f} \\ A_{/D} & \xrightarrow{\tau_D} & D \end{array}$$

is a pullback square.

Proof. Both $f_!\colon A_{/C} \to A_{/D}$ and the pullback of f are right fibrations over $A_{/D}$. It therefore suffices to check their fibers. Over $\delta\colon i(a) \to D$, the comparison is evaluation at e_a: it sends a lift of δ to its value over $\delta(e_a)$. Since e_a is terminal and f is a right fibration, the arrows $z \to e_a$ and cartesian lifting reconstruct the lift uniquely from this value. Thus the comparison is an equivalence on every fiber. □

To show that last-vertex evaluation recovers C, we need to extend an A-simplex by appending a terminal vertex. The following structure makes this operation available within the indexing category A.

Definition 15.1.3. Let A be a small category and $i\colon A \to \mathrm{Cat}$ be a functor. A *décalage structure* on the pair (A, i) consists of the following data.

(i) For each object a of A, there is a terminal object e_a in $i(a)$.

(ii) There is an object ω of A satisfying $i(\omega) \simeq [0]$.

(iii) There is an endofunctor $D\colon A \to A$ equipped with a natural equivalence

$$i \circ D \xrightarrow{\sim} i(-) \star [0].$$

(iv) There are natural transformations

$$d\colon \mathrm{id}_A \to D \qquad \text{and} \qquad b\colon \omega_A \to D$$

(where ω_A is the constant A-indexed functor with value ω) and a commutative diagram in $\mathrm{Fun}(A, \mathrm{Cat})$, whose value at a is

$$\begin{array}{ccccc} i(a) & \xrightarrow{i(d_a)} & i(D(a)) & \xleftarrow{i(b_a)} & i(\omega) \\ \| & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ i(a) & \hookrightarrow & i(a) \star [0] & \hookleftarrow & [0]. \end{array}$$

Here the vertical arrows use the equivalences in (ii) and (iii), and the horizontal maps of the bottom line are the structural embeddings.

Remark 15.1.4 (Objectwise verification of décalage)**.** In Definition 15.1.3, it suffices to give the equivalences in (iii) and the compatibility diagrams in (iv) objectwise, provided that d and b are natural transformations. The natural equivalence and its compatibilities can then be constructed as follows.

Adjoining a terminal object is left adjoint to the forgetful functor from small categories with a specified terminal object and functors preserving it. Indeed, for a category E with terminal object t, extending $C \to E$ to $C \star [0] \to E$ with cone value t is the same as lifting it to $E_{/t}$, by Lemma 3.7.3. Since $E_{/t} \to E$ is an equivalence, this extension is unique. The identification is natural in C and in functors preserving t, and therefore gives the asserted adjunction.

The points selected by $i(b_a)$ are terminal by the objectwise compatibility diagrams. Naturality of b thus makes $i \circ D$ a diagram of categories with specified terminal objects and functors preserving them. Applying the preceding adjunction to A-indexed diagrams, using Proposition 4.4.4, the transformation $i(d)$ extends to a natural comparison

$$i(-) \star [0] \longrightarrow i \circ D$$

compatible with d and b. At each a, the inverse of the given objectwise equivalence is another such extension of $i(d_a)$. Uniqueness of extension identifies it with the component of this comparison, which is consequently an equivalence. Inverting the natural comparison gives (iii) together with the commutative diagram in $\mathrm{Fun}(A, \mathrm{Cat})$ required by (iv).

When C has a terminal object t, an A-simplex extends uniquely along d_a by giving the appended vertex the value t. This construction lifts the décalage structure to the category of A-simplices over C.

Lemma 15.1.5 (Décalage on categories of simplices)**.** *Let (A, i) be equipped with a décalage structure, and let C be a small category with terminal object t. Write $p\colon A_{/C} \to A$ for the projection and $A^t_{/C}$ for the fiber of τ_C at t. Then $(A_{/C}, i \circ p)$ inherits a décalage structure with distinguished object (ω, t). Its endofunctor D_C takes values in $A^t_{/C}$, and the structure restricts to this full subcategory.*

Proof. We construct D_C by extending each simplex $\gamma\colon i(a) \to C$ across $i(d_a)$ and sending the appended vertex to t. Write $p\colon A_{/C} \to A$ for the projection and form the pullbacks

$$\begin{array}{ccccc}
\tilde{A} & \longrightarrow & & & [0] \\
\downarrow & \lrcorner & & & \downarrow{\scriptstyle t} \\
D^*A_{/C} & \longrightarrow & A_{/C} & \xrightarrow{\tau_C} & C \\
{\scriptstyle D^*p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} & & \\
A & \xrightarrow{D} & A & & .
\end{array}$$

An object of $\tilde{A}$ is a pair (a, γ) with $\gamma\colon i(D(a)) \to C$ preserving terminal objects. By naturality of b and condition (iv) of Definition 15.1.3, every $i(D(u))$ preserves the appended terminal vertex. Thus this full subcategory of $D^*A_{/C}$ is stable under right-fibration transport, and its projection to A is a right fibration.

Restriction along $i(d_a)$ gives a morphism of right fibrations

$$\varphi\colon \tilde{A} \to A_{/C}$$

over A. Its fiber over $\gamma\colon i(a) \to C$ is the anima of extensions $i(a) \star [0] \to C$ with cone value t. Such extensions are lifts of γ to $C_{/t}$. Since $C_{/t} \to C$ is an equivalence, the fiber is contractible. Hence φ is an equivalence.

Choose an inverse to φ over A and define D_C by the composite

$$A_{/C} \xrightarrow{\varphi^{-1}} \tilde{A} \hookrightarrow D^* A_{/C} \longrightarrow A_{/C}.$$

The maps d_a and b_a give natural transformations

$$\mathrm{id}_{A_{/C}} \xrightarrow{d_C} D_C \xleftarrow{b_C} \omega_C,$$

where ω_C is the constant functor at the simplex $i(\omega) \simeq [0] \xrightarrow{t} C$. The first transformation is induced by restriction along $i(d_a)$; the second uses the identification of the appended vertex with t.

By construction, $pD_C \simeq Dp$, and d_C and b_C project to d and b. We therefore have equivalences

$$i(p(D_C(\xi))) \simeq i(D(p(\xi))) \simeq i(p(\xi)) \star [0],$$

natural in $\xi : A_{/C}$. Their compatibility with d_C and b_C is inherited from the compatibility diagram for the original décalage structure. Together with the terminal objects e_a of $i(a)$ and the distinguished simplex (ω, t), these data give the required décalage structure on $(A_{/C}, i \circ p)$.

Every appended vertex has value t, so D_C takes values in $A^t_{/C}$, and (ω, t) also belongs to this full subcategory. All the data consequently restrict to a décalage structure there. □

The lifted structure gives the two contractions needed to study last-vertex evaluation.

Lemma 15.1.6. *Let $i\colon A \to \mathrm{Cat}$ be a functor indexed by a small category and equipped with a décalage structure. We consider a small category C with a terminal object t, and let $A^t_{/C}$ be the fiber of the functor $\tau\colon A_{/C} \to C$ at t. Then $A^t_{/C}$ is weakly contractible and the full embedding $A^t_{/C} \to A_{/C}$ is a terminal functor.*

Proof. By Lemma 15.1.5, there are natural transformations

$$\mathrm{id}_{A_{/C}} \xrightarrow{d_C} D_C \longleftarrow \omega_C,$$

where D_C takes values in $A^t_{/C}$ and ω_C is constant at (ω, t). After restricting to $A^t_{/C}$ and taking geometric realizations, these transformations give homotopies between the identity and a constant map. Hence $A^t_{/C}$ is weakly contractible.

To prove that its inclusion is terminal, fix $\xi : A_{/C}$. On the category $(A^t_{/C})_{\xi/}$, consider the endofunctor

$$(\xi \to \zeta) \longmapsto (\xi \to \zeta \xrightarrow{d_C(\zeta)} D_C(\zeta)).$$

The transformation d_C gives a map from the identity endofunctor to this endofunctor. Naturality of d_C gives a map to it from the constant object $d_C(\xi)\colon \xi \to D_C(\xi)$, using $D_C(\xi \to \zeta)$. Thus every such coslice is weakly contractible, which proves terminality by Corollary 9.5.7 and duality. □

Proposition 15.1.7. *Let $i\colon A \to \mathrm{Cat}$ be a functor indexed by a small category and equipped with a décalage structure. For any small category C, the evaluation at the terminal object of $i(a)$ defines a weakly proper functor with weakly contractible fibers*

$$\tau_C\colon A_{/C} \to C\,.$$

In particular, the category C is the localization of $A_{/C}$ by the class of morphisms that are sent to identities by τ_C in C.

Proof. For $x : C$, the projection $C_{/x} \to C$ is a right fibration. By Lemma 15.1.2,

$$A_{/(C_{/x})} \simeq A_{/C} \times_C C_{/x}.$$

Under this identification, the fiber of τ_C at x is the fiber of $\tau_{C_{/x}}$ at the terminal object id_x. By Lemma 15.1.6, it is weakly contractible and its inclusion into $A_{/C}\times_C C_{/x}$ is terminal. This is precisely the weak properness condition. The localization assertion now follows from Proposition 13.1.16. □

We now compare this localization with the left adjoint to the nerve. From here on, assume that i is fully faithful. For a presheaf X, the left adjoint $Li_!(X)$ is obtained by gluing the categories $i(a)$ indexed by the elements of X. The next construction compares this colimit with the category of elements $A_{/X}$.

Construction 15.1.8. Let $i\colon A \to \mathrm{Cat}$ be fully faithful and equipped with a décalage structure. Pull back the universal cocartesian fibration:

$$\begin{array}{ccc} A_\bullet & \xrightarrow{i_A} & \mathrm{Cat}_\bullet \\ {\scriptstyle \pi_A}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ A & \xrightarrow{i} & \mathrm{Cat} \end{array}$$

The functor π_A is a cocartesian fibration whose fiber over each object a is the category with terminal object $i(a)$. In particular, it has a fully faithful right adjoint

$$e\colon A \to A_\bullet$$

defined by $e(a) = e_a$ on objects.

The nerve and its left adjoint form an adjunction

$$Li_!\colon \mathrm{PSh}(A) \rightleftarrows \mathrm{Cat}\colon i^*$$

Given a presheaf $X\colon A^{\mathrm{op}} \to \mathrm{An}$, we form the pullback squares below.

$$\begin{array}{ccccc} A_{\bullet/X} & \xrightarrow{\pi_{A_{/X}}} & A_{/X} & \longrightarrow & \mathrm{An}_\bullet^{\mathrm{op}} \\ {\scriptstyle p_{\bullet/X}}\downarrow & \lrcorner & {\scriptstyle p_X}\downarrow & \lrcorner & \downarrow \\ A_\bullet & \xrightarrow{\pi_A} & A & \xrightarrow{X^{\mathrm{op}}} & \mathrm{An}^{\mathrm{op}} \end{array}$$

In other words, we have:

$$A_{\bullet/X} = \int_{(a,\gamma):A_{/X}} i(a)\,.$$

Observe that the functor $\pi_{A_{/X}}$ has a right adjoint section, being the pullback of a functor with this property. The assumption of full faithfulness on i implies that there are canonical isomorphisms of the form

$$A_{/a} \xrightarrow{\sim} A_{/i(a)}$$

defined by sending $(a', u\colon a' \to a)$ to $(a', i(u)\colon i(a') \to i(a))$. The canonical isomorphism deduced from Lemma 13.3.4

$$\mathrm{colim}_{(a,\gamma):A_{/X}} A_{/a} \xrightarrow{\sim} A_{/X}$$

can thus be rewritten as

$$\mathrm{colim}_{(a,\gamma):A_{/X}} A_{/i(a)} \xrightarrow{\sim} A_{/X}\,.$$

Naturality of last-vertex evaluation gives functors $\tau_{i(a)}\colon A_{/i(a)} \to i(a)$. Since

$$\mathrm{colim}_{(a,\gamma):A_{/X}} i(a) = Li_!(X)$$

by construction, we obtain the commutative diagram

$$\begin{array}{ccc} \int_{(a,\gamma):A_{/X}} A_{/i(a)} & \longrightarrow & A_{/X} \\ \downarrow & & \downarrow \\ \int_{(a,\gamma):A_{/X}} i(a) & \longrightarrow & Li_!(X) \end{array}$$

in which both horizontal maps are localizations at cocartesian morphisms with respect to the projection to $A_{/X}$. On the other hand, the vertical arrow on the left is fiberwise a localization as well as a cocartesian functor over $A_{/X}$. Therefore, it is also a localization by Theorem 8.8.9. Furthermore, for each object (a, γ) of $A_{/X}$, $\gamma\colon h_A(a) \to X$, there are functorial equivalences of the form

$$(A_{/X})_{/(a,\gamma)} \simeq A_{/a} \simeq A_{/i(a)}$$

and we have an equivalence

$$\mathrm{Fun}([1], A_{/X}) \simeq \int_{(a,\gamma):A_{/X}} (A_{/X})_{/(a,\gamma)}$$

over $A_{/X}$, with $\mathrm{ev}_1\colon \mathrm{Fun}([1], A_{/X}) \to A_{/X}$ as structural map. These identifications give the commutative square

$$\begin{array}{ccc} \mathrm{Fun}([1], A_{/X}) & \xrightarrow{\mathrm{ev}_0} & A_{/X} \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle \tau'_X} \\ A_{\bullet/X} & \xrightarrow[j_X]{} & Li_!(X) \end{array}$$

Since i is fully faithful, the terminal object of $i(a)$ is determined by a unique map $e_a\colon \omega \to a$ in A. We will say that a map $u\colon a \to b$ in A is a *last vertex morphism* if $u \circ e_a \simeq e_b$ holds (such an isomorphism, if it exists, is necessarily unique).

Lemma 15.1.9. *Let $i\colon A \to \mathrm{Cat}$ be fully faithful and suppose that (A, i) is equipped with a décalage structure. For every small category C, the functor*

$$\tau_C\colon A_{/C} \longrightarrow C$$

exhibits C as the localization of $A_{/C}$ at the morphisms whose image in A is a last vertex morphism.

Proof. By Proposition 15.1.7, the functor τ_C is the localization at the family V of morphisms that it sends to identities. It sends every last vertex morphism to an isomorphism. It therefore suffices to show that every member of V becomes invertible after localizing at the last vertex morphisms.

Let u be a morphism in $A_{/C}$ that is sent to an identity by τ_C. Its data can be displayed as

$$\begin{array}{ccccc} \omega & \xrightarrow{d_\omega} & D(\omega) & \xleftarrow{b_\omega} & \omega \\ {\scriptstyle e_a}\downarrow & & \downarrow{\scriptstyle v} & \swarrow{\scriptstyle e_b} & \\ a & \xrightarrow{u} & b & & \end{array}$$

together with a functor $\gamma\colon i(b) \to C$. The assumption on $\tau_C(u)$ says that $\gamma i(v)\colon [1] \to C$ is constant. Consequently, all the arrows in the displayed diagram define morphisms in $A_{/C}$ with the indicated constant labels where appropriate. The morphisms e_a, e_b, and b_ω are last vertex morphisms.

Since i is fully faithful and $i(\omega) = [0]$, the object ω is terminal in A. Let $p\colon D(\omega) \to \omega$ be the unique morphism. For the constant labels just described, it defines a morphism in $A_{/C}$ and satisfies

$$pb_\omega = \mathrm{id}_\omega = pd_\omega.$$

After inverting b_ω, the first equality identifies p with b_ω^{-1}, and the second then identifies d_ω with b_ω. Thus d_ω is invertible. The relations

$$vb_\omega = e_b \qquad \text{and} \qquad vd_\omega = ue_a$$

now show successively that v and u are invertible. Hence every member of V is inverted by the localization at last vertex morphisms. The universal property of the localization at V now proves the claim. □

For the nerve of a category, we have identified the morphisms that must be inverted. The next lemma gives the same description for an arbitrary presheaf: $Li_!(X)$ is the localization of $A_{/X}$ at last vertex morphisms. The arrow category in Construction 15.1.8 allows us to compare the two localizations.

Lemma 15.1.10. *Let W be the family of morphisms in $A_{/X}$ whose image in A is a last vertex morphism. We also denote by $\tilde{W}$ the family of morphisms in* $\mathrm{Fun}([1], A_{/X})$ *whose image by the functor* ev_0 *is in W. Then the functor* ev_0 *induces an equivalence of localizations*

$$\mathrm{Fun}([1], A_{/X})[\tilde{W}^{-1}] \xrightarrow{\sim} (A_{/X})[W^{-1}]\,.$$

Furthermore, the functor q_X sends morphisms in $\tilde{W}$ to cocartesian morphisms relatively to the projection of $A_{\bullet/X}$ to $A_{/X}$, and the composite $j_X q_X$ induces an equivalence

$$\mathrm{Fun}([1], A_{/X})[\tilde{W}^{-1}] \xrightarrow{\sim} Li_!(X)\,.$$

Proof. The functor ev_0 has a fully faithful left adjoint sending an object of $A_{/X}$ to its identity (the latter being seen as an object of $\mathrm{Fun}([1], A_{/X})$). The counit of this adjunction consists of squares of the form below.

$$\begin{array}{ccc} a & = & a \\ \| & & \downarrow{\scriptstyle f} \\ a & \xrightarrow{f} & b \end{array}$$

In particular, the counits of this adjunction all belong to $\tilde{W}$. The units belong to W because they are isomorphisms. The functor ev_0 sends $\tilde{W}$ into W, and its left adjoint sends W into $\tilde{W}$. Thus the adjunction descends to the two localizations by Lemma 14.1.5, with invertible unit and counit. This proves the first equivalence.

Using Remark 12.4.17, we can describe objects of $A_{\bullet/X}$ as triples (a, γ, x) with a an object of A, $\gamma : X_a$, and $x : i(a)^{\simeq}$. Fix a morphism $(a, \gamma) \to (b, \delta)$ in $A_{/X}$ with image $f\colon a \to b$. A morphism from (a, γ, x) to (b, δ, y) over this base morphism is specified by $u\colon i(f)(x) \to y$ in $i(b)$; write it as (f, u). Factoring (f, u) as the canonical cocartesian lift $(f, \mathrm{id}_{i(f)(x)})$ followed by the vertical morphism (id_b, u) shows that it is cocartesian if and only if u is an isomorphism in $i(b)$, by Lemma 4.5.12. We also know that the localization of $A_{\bullet/X}$ by cocartesian morphisms precisely is $Li_!(X)$. An object of $\mathrm{Fun}([1], A_{/X})$, equivalently a morphism in $A_{/X}$, essentially is a pair (u, γ) with $u\colon a \to b$ a morphism in A and $\gamma : X_b$. The functor q_X sends such a pair (u, γ) to the triple (b, γ, x), where x is the image of the terminal object $e_a : i(a)^{\simeq}$ in $i(b)$ through u. For a morphism in the arrow category, its fiber component under q_X is obtained by applying the target arrow to the canonical comparison $i(f)(e_a) \to e_{a'}$, where f is the image of that morphism under ev_0. In particular, if f is a last vertex morphism, this comparison is an isomorphism. Thus q_X sends every member of $\tilde{W}$ to a cocartesian morphism.

Under the identifications in Construction 15.1.8, the functor q_X is the Grothendieck construction of the fiberwise localization functors $\tau_{i(a)}\colon A_{/i(a)} \to i(a)$. It is therefore a localization by Theorem 8.8.9. Let K be the inverse image under q_X of the cocartesian morphisms in $A_{\bullet/X}$. Since q_X is a localization and every cocartesian morphism has a preimage under q_X, the composite $j_X q_X$ is the localization at K. For the last assertion, represent an object $x : i(a)$ by the corresponding morphism $x\colon \omega \to a$ in A. A morphism $i(u)(x) \to y$ in $i(b)$ is represented, using $i(D(\omega)) = [1]$, by a commutative diagram

$$\begin{array}{ccccc} \omega & \xrightarrow{d_\omega} & D(\omega) & \xleftarrow{b_\omega} & \omega \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle v} & \swarrow{\scriptstyle y} & \\ a & \xrightarrow{u} & b & & \end{array} .$$

The square with horizontal maps x and v and vertical maps d_ω and u is a morphism in $\mathrm{Fun}([1], A_{/X})$ mapping to the prescribed morphism in $A_{\bullet/X}$.

We claim that localization at K is the same as localization at $\tilde{W}$. One inclusion follows from the preceding calculation: $\tilde{W} \subseteq K$. Conversely, let h belong to K, and write $f = \mathrm{ev}_0(h)$. If the target of the square defining h lies over $(b, \gamma) : A_{/X}$, then f may be viewed as a morphism in $A_{/i(b)}$. Its image under $\tau_{i(b)}\colon A_{/i(b)} \to i(b)$ is precisely the fiber component of $q_X(h)$, hence is an isomorphism because $q_X(h)$ is cocartesian. By Lemma 15.1.9, f becomes invertible after localizing $A_{/i(b)}$ at its last vertex morphisms. The domain functor from $A_{/i(b)}$, equivalently from the corresponding slice of $A_{/X}$, to $A_{/X}$ preserves last vertex morphisms. It follows that f becomes invertible in $(A_{/X})[W^{-1}]$. The first equivalence of this lemma now shows that h becomes invertible in $\mathrm{Fun}([1], A_{/X})[\tilde{W}^{-1}]$. Thus every member of K is already inverted by localization at $\tilde{W}$.

The two universal properties therefore identify the localizations at K and at $\tilde{W}$. Consequently, the composite $j_X q_X$ gives

$$\mathrm{Fun}([1], A_{/X})[\tilde{W}^{-1}] \xrightarrow{\sim} A_{\bullet/X}[\mathrm{Cocart}^{-1}] \xrightarrow{\sim} Li_!(X),$$

as required. □

Theorem 15.1.11 (Abstract nerve theorem)**.** *Let A be small and let $i\colon A \to \mathrm{Cat}$ be fully faithful. If (A, i) admits a décalage structure, then the functor*

$$i^*\colon \mathrm{Cat} \to \mathrm{PSh}(A)$$

defined by the rule $C \mapsto (a \mapsto \mathrm{Hom}_{\mathrm{Cat}}(i(a), C))$ is fully faithful.

Proof. For a small category C, it suffices to show that the counit $\varepsilon_C\colon Li_!(i^*(C)) \to C$ is an equivalence. Identify $A_{/i^*(C)}$ with $A_{/C}$. The counit is induced by the evaluation maps $\gamma\colon i(a) \to C$, so naturality of τ in Construction 15.1.8 gives the commutative triangle

$$\begin{array}{ccc} A_{/C} & & \\ {\scriptstyle \tau'_C}\downarrow & \searrow^{\tau_C} & \\ Li_!(i^*(C)) & \xrightarrow[\varepsilon_C]{} & C. \end{array}$$

By Lemma 15.1.9, τ_C is the localization at last vertex morphisms. By Lemma 15.1.10, so is τ'_C. The triangle therefore identifies ε_C with the equivalence between these two localizations. □

For the simplex category, décalage is simply the operation of adjoining a new last vertex.

Proposition 15.1.12. *The canonical embedding $i\colon \mathbf{\Delta} \hookrightarrow \mathrm{Cat}$ admits a décalage structure.*

Proof. The terminal object of $i([n]) = [n]$ is its last vertex n. Let $\omega = [0]$, and define

$$D([n]) = [n+1] = [n] \star [0].$$

For a morphism $u\colon [m] \to [n]$, set $D(u) = u \star \mathrm{id}_{[0]}$. The natural transformation $d\colon \mathrm{id}_{\mathbf{\Delta}} \to D$ is given by the inclusion of the first summand $[n] \hookrightarrow [n] \star [0]$, and $b\colon \omega_{\mathbf{\Delta}} \to D$ is given by the inclusion of the last vertex $[0] \hookrightarrow [n] \star [0]$. These maps are natural because the join is functorial. For every $[n]$, the resulting diagram is

$$\begin{array}{ccccc} [n] & \xrightarrow{d_{[n]}} & [n+1] & \xleftarrow{b_{[n]}} & [0] \\ \| & & \| & & \| \\ [n] & \hookrightarrow & [n] \star [0] & \hookleftarrow & [0] \end{array}$$

and hence satisfies condition *(iv)* of Definition 15.1.3. □

Corollary 15.1.13. *The Rezk nerve functor $N\colon \mathrm{Cat} \to \mathrm{PSh}(\mathbf{\Delta})$ is fully faithful.*

Proof. We apply Theorem 15.1.11 in the case $A = \mathbf{\Delta}$ with the canonical embedding $i\colon \mathbf{\Delta} \hookrightarrow \mathrm{Cat}$. The required décalage structure is supplied by Proposition 15.1.12. □

15.2 Univalent Segal animae

Having proved that the Rezk nerve is fully faithful, we turn to the conditions that characterize its essential image. The Segal condition expresses that a simplex is determined by its consecutive edges: every string of composable morphisms extends uniquely to a simplex. This gives a simplicial anima a composition law, with associativity and unit coherences supplied by its higher simplices. Univalence is the corresponding Rezk condition, relating identifications of objects to isomorphisms.

We construct the *spine* Seg^n by gluing n edges end to end, and use it to formulate the Segal condition directly. We then define Segal animae and univalence, and verify that every Rezk nerve satisfies both conditions. The section ends with Corollary 15.2.13: a morphism of univalent Segal animae is an equivalence precisely when it is fully faithful and essentially surjective. The converse characterization of Rezk nerves will follow from the construction of Rezk completion later in the chapter.

Construction 15.2.1 (The spine of a simplex)**.** The *spine* Seg^n is the simplicial anima obtained by gluing n copies of Δ^1 end to end. More precisely, set $\mathrm{Seg}^0 = \Delta^0$, with its unique vertex as last vertex, and define Seg^{n+1} by the pushout square

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{0} & \Delta^1 \\ {\scriptstyle \text{last vertex}}\downarrow & \ulcorner & \downarrow \\ \mathrm{Seg}^n & \longrightarrow & \mathrm{Seg}^{n+1}\,. \end{array}$$

The last vertex of Seg^{n+1} is the image of the vertex 1 of the new edge. All pushouts are taken in $\mathrm{PSh}(\mathbf{\Delta})$.

There is a canonical map $i_n\colon \mathrm{Seg}^n \to \Delta^n$, sending each edge to the corresponding consecutive edge of Δ^n. Inductively, i_0 is the identity, and i_{n+1} is induced by the composite $\mathrm{Seg}^n \to \Delta^n \to \Delta^{n+1}$ and the last edge $\Delta^1 \to \Delta^{n+1}$. The latter is induced by

$$[1] = [0] \star [0] \longrightarrow [n] \star [0] = [n+1],$$

using the last vertex of $[n]$ on the first factor and the identity on the second. The two maps agree on the gluing vertex, and i_{n+1} preserves the last vertex.

Definition 15.2.2. A *Segal anima* is a functor $C\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ such that, for each $n : \mathbb{N}$, the map $i_n\colon \mathrm{Seg}^n \to \Delta^n$ induces an equivalence

$$\mathrm{Hom}(\Delta^n, C) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Seg}^n, C)\,.$$

We write $\mathrm{Seg}(\mathrm{An}) \subseteq \mathrm{PSh}(\mathbf{\Delta})$ for the full subcategory spanned by the Segal animae.

Lemma 15.2.3. *For every simplicial anima X and $n : \mathbb{N}$, there is a canonical pullback square*

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{Seg}^{n+1}, X) & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle d^1} \\ \mathrm{Hom}(\mathrm{Seg}^n, X) & \xrightarrow[\textit{last vertex}]{} & X_0. \end{array}$$

Proof. Apply $\mathrm{Hom}(-, X)$ to the pushout defining Seg^{n+1} and use Yoneda. □

Remark 15.2.4. It follows by induction that

$$\mathrm{Hom}(\mathrm{Seg}^n, X) \simeq \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}.$$

Here the product with no factors is X_0, and each new factor is attached by identifying the target of the preceding edge with its source. Thus the Segal condition says precisely that the consecutive edges induce an equivalence from X_n to the anima of composable strings of n morphisms.

Construction 15.2.5. Let C be a Segal anima. An *object* of C is a term of C_0. A *morphism* of C simply is a term of C_1. We define the composition law of morphisms in C as the composition of the inverse of the equivalence

$$C_2 = \mathrm{Hom}(\Delta^2, C) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Seg}^2, C) \simeq C_1 \times_{C_0} C_1$$

with the map

$$d^1 \colon C_2 \to C_1 \,.$$

This defines a morphism

$$C_1 \times_{C_0} C_1 \to C_1 \,, \quad (f, g) \mapsto g \circ f \,.$$

Thus, if $\sigma : C_2$ denotes the unique 2-simplex whose spine is (f, g), we have $g \circ f = d_1(\sigma)$.

A morphism $f \colon x \to y$ in C is a term $f : C_1$ with identifications of the form $s : d^1(f) = x$ and $t : d^0(f) = y$. For instance, given any object x of C, there is the morphism $\mathrm{id}_x \colon x \to x$ defined as the image of $x : C_0$ through the map $C_0 \to C_1$ induced by $\Delta^1 \to \Delta^0$. The Segal equivalence in degree 3 associates to any composable triple

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t$$

a unique 3-simplex. Applying its face maps in the two possible orders identifies the two composites and gives

$$a_{f,g,h} \colon (h \circ g) \circ f = h \circ (g \circ f).$$

Likewise, the degenerate 2-simplices give the left and right unit identifications $\mathrm{id}_y \circ f = f$ and $f \circ \mathrm{id}_x = f$. The unique 4-simplex associated to four composable morphisms gives the pentagon coherence for a, while the degenerate 3-simplices give the triangle coherences involving the unit identifications. We will use these identifications and coherences without further mention.

A morphism $f \colon x \to y$ in C is an *isomorphism* if there are morphisms $g \colon y \to x$ and $h \colon y \to x$ together with identifications of the form

$$l : g \circ f = \mathrm{id}_x \quad \text{and} \quad r : f \circ h = \mathrm{id}_y \,.$$

Isomorphisms form an anima $\mathrm{Iso}(C)$ equipped with a forgetful map

$$\mathrm{Iso}(C) \to C_1 \,, \quad (f, g, h, l, r) \mapsto f$$

and there is a canonical proof that id_x is an isomorphism in x, yielding a canonical map

$$\mathrm{id} \colon C_0 \to \mathrm{Iso}(C) \,.$$

More precisely, $\mathrm{Iso}(C)$ is defined through the following diagram.

$$\begin{array}{ccccc}
\mathrm{Iso}(C) & \longrightarrow & C_2 \times_{d^0, C_1, d^2} C_2 & \longrightarrow & C_1 \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\
 & & C_2 \times C_2 & \xrightarrow{d^0 \times d^2} & C_1 \times C_1 \\
 & & \downarrow{\scriptstyle d^1 \times d^1} & & \\
C_0 \times C_0 & \xrightarrow{(x,y) \mapsto (\mathrm{id}_y, \mathrm{id}_x)} & C_1 \times C_1. & &
\end{array}$$

Given $x, y : C_0$, we denote by $x \xrightarrow{\sim} y$ the fiber of the map $\mathrm{Iso}(C) \to C_0 \times C_0$ at (x, y).

Lemma 15.2.6. *Let C be a Segal anima, let $u \colon x \to y$ be an isomorphism, and let z be an object of C. Composition with u induces equivalences*

$$u^* \colon C(y, z) \xrightarrow{\sim} C(x, z), \qquad v \longmapsto v \circ u,$$

and

$$u_* \colon C(z, x) \xrightarrow{\sim} C(z, y), \qquad v \longmapsto u \circ v.$$

Proof. Choose a left inverse $v \colon y \to x$ and a right inverse $w \colon y \to x$ for u. The associativity and unit identifications give

$$v = v \circ \mathrm{id}_y = v \circ (u \circ w) = (v \circ u) \circ w = \mathrm{id}_x \circ w = w.$$

Thus the left and right inverses agree; denote their common value by u^{-1}. Precomposition with u^{-1} is inverse to u^*, and postcomposition with u^{-1} is inverse to u_*. The required identifications again follow from associativity and the unit laws. □

Definition 15.2.7. A Segal anima C is *univalent* (or *complete*) if the map $\mathrm{id} \colon C_0 \to \mathrm{Iso}(C)$ is an equivalence. We write $\mathrm{USeg}(\mathrm{An}) \subseteq \mathrm{Seg}(\mathrm{An})$ for the full subcategory spanned by the univalent Segal animae.

Equivalently, C is univalent if, for every $x, y : C_0$, the canonical map

$$(x = y) \longrightarrow (x \xrightarrow{\sim} y)$$

is an equivalence. Indeed, these maps are the fibers of $\mathrm{id} \colon C_0 \to \mathrm{Iso}(C)$ over $C_0 \times C_0$, where C_0 maps to the product by the diagonal and $\mathrm{Iso}(C)$ by source and target. This explains the term univalence; the traditional name for the same condition is *Rezk completeness*.

Example 15.2.8. For any small category $C : \mathrm{Cat}^{\simeq}$, the Rezk nerve $N(C)$ is a univalent Segal anima.

Proof. Since n is terminal in $[n]$, a cocone on $F \colon [n] \to C$ with vertex y is determined by a morphism $F(n) \to y$, by Proposition 4.4.4. The mapping property of the join, Remark 3.6.1, therefore gives the pushout square

$$\begin{array}{ccc}
[0] & \xrightarrow{0} & [1] \\
{\scriptstyle n}\downarrow & \ulcorner & \downarrow \\
[n] & \longrightarrow & [n+1] = [n] \star [0].
\end{array}$$

Applying $\mathrm{Map}(-, C)$ gives

$$N(C)_{n+1} \simeq N(C)_n \times_{N(C)_0} N(C)_1.$$

Induction now proves the Segal condition. There are canonical equivalences

$$N(C)_0 \simeq C^{\simeq} \qquad \text{and} \qquad \mathrm{Iso}(N(C)) \simeq \mathrm{Iso}(C)^{\simeq}.$$

Under these equivalences, the identity map $N(C)_0 \to \mathrm{Iso}(N(C))$ is the groupoid core of the functor $C \to \mathrm{Iso}(C)$ that sends an object to its identity morphism. This functor is an equivalence by the Rezk axiom, Axiom F, so its groupoid core is an equivalence as well. □

Definition 15.2.9. A morphism of simplicial animae $f\colon C \to D$ is *fully faithful* if the induced square

$$\begin{array}{ccc} C_1 & \xrightarrow{f_1} & D_1 \\ {\scriptstyle (d^1,d^0)}\downarrow & & \downarrow{\scriptstyle (d^1,d^0)} \\ C_0 \times C_0 & \xrightarrow{f_0 \times f_0} & D_0 \times D_0 \end{array}$$

is cartesian. A morphism of simplicial animae $f\colon C \to D$ is *essentially surjective* if the collection of pairs of the form (x, u) with $x : C_0$ and $u\colon f(x) \xrightarrow{\sim} y$ covers D_0 through the map $(x, u) \mapsto y$.

A morphism of Segal animae is a *Dwyer–Kan equivalence* if it is both fully faithful and essentially surjective.

Remark 15.2.10. If C is a Segal anima and $x, y : C$ are two objects, the collection $C(x, y)$ of morphisms from x to y in C fits in the pullback square below.

$$\begin{array}{ccc} C(x,y) & \longrightarrow & C_1 \\ \downarrow & & \downarrow{\scriptstyle (d^1,d^0)} \\ * & \xrightarrow{(x,y)} & C_0 \times C_0 \end{array}$$

A morphism of Segal animae $f\colon C \to D$ is fully faithful if and only if, for any objects x, y of C (in some anima context Γ), the induced map

$$C(x, y) \to D(f(x), f(y))$$

is an equivalence (in context Γ).

Lemma 15.2.11. *If $f\colon C \to D$ is a fully faithful morphism of Segal animae, then there is a canonical pullback square of the form:*

$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\mathrm{Iso}(f)} & \mathrm{Iso}(D) \\ \downarrow & & \downarrow \\ C_1 & \xrightarrow{f_1} & D_1 \end{array}$$

Proof. Since all four terms in the square are animae, the vertical maps are right fibrations. By Corollary 6.1.13, it therefore suffices to check the square on fibers. Let $u\colon x \to y$ be a morphism in C, in an arbitrary anima context. By the construction of $\mathrm{Iso}(C)$, the fiber of $\mathrm{Iso}(C) \to C_1$

over u is the anima of choices of a left and a right inverse for u, together with the corresponding identifications:

$$\left(\sum_{g:C(y,x)} g \circ u = \mathrm{id}_x\right) \times \left(\sum_{h:C(y,x)} u \circ h = \mathrm{id}_y\right).$$

Since f is fully faithful, it induces equivalences on all the hom animae appearing in this expression. These equivalences preserve identities and composition, and hence identify the displayed anima with the corresponding anima of inverse data for $f(u)$ in D. Thus the square induces an equivalence on every fiber and is a pullback square. □

Proposition 15.2.12. *A morphism of Segal animae $f\colon X \to Y$ is an equivalence if and only if it is fully faithful and induces an equivalence $f_0\colon X_0 \xrightarrow{\sim} Y_0$.*

Proof. An equivalence is levelwise an equivalence, so the condition is necessary. Conversely, if $f\colon X \to Y$ is fully faithful and induces an equivalence $f_0\colon X_0 \xrightarrow{\sim} Y_0$, then it induces an equivalence

$$f_1\colon X_1 \xrightarrow{\sim} Y_1$$

as well: f_1 is then a pullback of the equivalence $f_0 \times f_0$. This implies right away that f induces an equivalence of the form

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}} \xrightarrow{\sim} \underbrace{Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1}_{n \text{ times}}.$$

The commutative square

$$\begin{array}{ccc} \mathrm{Hom}(\Delta^n, X) & \longrightarrow & \mathrm{Hom}(\Delta^n, Y) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{Hom}(\mathrm{Seg}^n, X) & \xrightarrow{\simeq} & \mathrm{Hom}(\mathrm{Seg}^n, Y) \end{array}$$

thus implies that $f_n\colon X_n \xrightarrow{\sim} Y_n$ is an equivalence for all $n : \mathbb{N}$. □

Corollary 15.2.13. *A morphism of univalent Segal animae is an equivalence if and only if it is a Dwyer–Kan equivalence.*

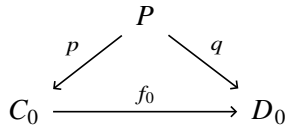
Proof. An equivalence is fully faithful and essentially surjective, so the condition is necessary. Let $f\colon C \to D$ be a Dwyer–Kan equivalence between univalent Segal animae. By virtue of Proposition 15.2.12, it suffices to prove that $f_0\colon C_0 \to D_0$ is an equivalence. It suffices to check that f_0 is both a cover and an embedding (see Proposition 11.2.10). We consider the following commutative diagram.

$$\begin{array}{ccc} C_0 & \xrightarrow{f_0} & D_0 \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{Iso}(C) & \xrightarrow{\mathrm{Iso}(f)} & \mathrm{Iso}(D) \\ \downarrow & & \downarrow \\ C_1 & \xrightarrow{f_1} & D_1 \\ \downarrow & & \downarrow \\ C_0 \times C_0 & \xrightarrow{f_0 \times f_0} & D_0 \times D_0 \end{array}$$

The upper square is cartesian because its two vertical maps are equivalences (since C and D are univalent). The lower square is cartesian because f is fully faithful. The middle square is a pullback square as well by Lemma 15.2.11. Therefore, the outer square is cartesian, which proves that f_0 is an embedding. It remains to show that f_0 is a cover. Let

$$P := C_0 \times_{f_0, D_0, s} \mathrm{Iso}(D),$$

where $s\colon \mathrm{Iso}(D) \to D_0$ sends an isomorphism to its source. Thus an object of P is a pair (x, u) consisting of an object $x : C_0$ and an isomorphism $u\colon f(x) \xrightarrow{\sim} y$ in D. We have a triangle



with $p(x, u) = x$ and $q(x, u) = y$. Since D is univalent, its source and target maps $s, t\colon \mathrm{Iso}(D) \to D_0$ are both inverse to the equivalence $\mathrm{id}\colon D_0 \to \mathrm{Iso}(D)$, and are therefore isomorphic. Consequently, $q = f_0 p$. The map q is a cover by the essential surjectivity of f, so $f_0 p$ is a cover as well. It follows from the cancellation property of covers, Corollary 11.2.7, that f_0 is a cover. Hence f_0 is an equivalence by Proposition 11.2.10, which finishes the proof by Proposition 15.2.12. □

15.3 Digression: unions of subpresheaves

The spine can also be viewed as the union of the consecutive edges and vertices inside a simplex. We justify this description by first discussing unions in a category of presheaves. The comparison at the end identifies this union with the recursive pushout construction of Construction 15.2.1.

Construction 15.3.1 (Unions in categories of presheaves)**.** Let C be a small category and $X\colon C^{\mathrm{op}} \to$ An a presheaf. Let $I^{\simeq}$ be a small subanima of $(\mathrm{PSh}(C)_{/X})^{\simeq}$ consisting of embeddings

$$U \hookrightarrow X$$

and we assume that the following conditions hold: for any $U \hookrightarrow X$ and $V \hookrightarrow X$ in I, the pullback $U \times_X V \hookrightarrow X$ is either in $I^{\simeq}$ or is empty (i.e. $\emptyset \simeq U \times_X V$).
We form the full subcategory I of $\mathrm{PSh}(C)_{/X}$ spanned by $I^{\simeq}$. We thus have a canonical functor

$$\varphi\colon I \to \mathrm{PSh}(C)$$

defined by $\varphi(U \hookrightarrow X) = U$ and a cocone $c\colon \varphi \to \mathrm{const}_I(X)$ induced by the embeddings $U \hookrightarrow X$, hence a canonical map

$$j\colon \operatorname*{colim}_{U \hookrightarrow X : I} U \to X\,.$$

The presheaf category and its slice over X are locally small. Hence I is small by Proposition 11.2.18, applied to its small anima of objects. The colimit of the U's above is called the *union of the family I in X*. This terminology is justified by the following lemma.

Lemma 15.3.2. *The map $j\colon \mathrm{colim}_I \varphi \to X$ constructed above is an embedding. More precisely, it induces an equivalence of the form*

$$\operatorname*{colim}_{U \hookrightarrow X : I} U \simeq \operatorname*{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \check{C}(T/X)$$

with $T = \sum_{U \hookrightarrow X : I^{\simeq}} U \to X$ *the canonical map and* $\check{C}(T/X)$ *defined as* $\check{C}(T/X)_c = \check{C}(T_c/X_c)$ *for all* $c : C$.

Proof. Let $Z = \operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} \check{C}(T/X)$. For each object c of C, since the evaluation at c commutes with small limits and with small colimits, the evaluation of Z at c is the colimit of $\check{C}(T_c/X_c)$, hence the canonical map $Z_c \to X_c$ is an embedding, by Corollary 11.1.12. Since evaluation at $C^{\simeq}$ is conservative, this shows that we have a canonical embedding $Z \hookrightarrow X$. Replacing X by Z, we may assume, without loss of generality, that $Z = X$ (or, equivalently, that the map $T \to X$ is a cover fiberwise).

Let c be an object of C. We want to prove that the colimit of the $U(c)$'s is $X(c)$. We consider then the following pullback squares.

$$\begin{array}{ccc} J & \longrightarrow & \mathrm{Cat}_\bullet \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ I & \xrightarrow{\varphi(c)} & \mathrm{Cat} \end{array}$$

We now have

$$\operatorname*{colim}_{U \hookrightarrow X : I} U(c) \simeq \|J\| .$$

The embeddings $U(c) \hookrightarrow X(c)$ induce a functor $J \to I \times X(c)$ hence a projection $\mathrm{pr} \colon J \to X(c)$. It suffices to check that J is weakly contractible in context $X(c)$. Since $T(c) \to X(c)$ is a cover, it is in fact sufficient to prove it in context $T(c)$. It is thus sufficient to prove this in each context $U(c)$ for each embedding $U(c) \hookrightarrow X(c)$. The fiber F of the projection pr over $U(c)$ is the full subcategory of J spanned by those pairs consisting of an embedding $V \hookrightarrow X$ together with a term of $U(c) \times_{X(c)} V(c)$ in context $U(c)$, i.e. a proof that the first projection $U(c) \times_{X(c)} V(c) \xrightarrow{\sim} U(c)$ is an equivalence. We thus have a functor

$$\varphi \colon F \to F , \quad V \longmapsto U \times_X V$$

and the second projection $U \times_X V \to V$ defines a map from φ to id_F. On the other hand, the first projection $U \times_X V \to U$ defines a map from φ to the constant functor with value U (together with the proof that $U(c) \times_{X(c)} U(c) \xrightarrow{\sim} U(c)$), proving that F is weakly contractible. □

Example 15.3.3. Let C be a small category and $X \colon C^{\mathrm{op}} \to \mathrm{An}$ a presheaf. We consider two embeddings $U \hookrightarrow X$ and $V \hookrightarrow X$. We denote by

$$U \cap V := U \times_X V \hookrightarrow X$$

the embedding of the fiber product of U and V over X. The *union of U and V in X*, denoted by $U \cup V$, is the colimit of the diagram of embeddings

$$U \hookrightarrow X , \; V \hookrightarrow X \text{ and } U \cap V \hookrightarrow X .$$

The previous lemma tells us that we have a canonical embedding

$$U \cup V \hookrightarrow X .$$

In fact, one can check that the induced commutative square

$$\begin{array}{ccc} U \cap V & \hookrightarrow & V \\ \downarrow & & \downarrow \\ U & \hookrightarrow & U \cup V \end{array}$$

is both cartesian and cocartesian. Indeed, since $U \cup V$ embeds in X, to check that it is a pullback square, it suffices to do it after replacing $U \cup V$ by X, and this holds then by definition. To prove that it is cocartesian, let I be the full subcategory of all embeddings with codomain X spanned by the embeddings with domain U, V or $U \cap V$. For every presheaf Y, a cocone from the diagram indexed by I to Y is precisely a pair of maps $U \to Y$ and $V \to Y$ whose restrictions to $U \cap V$ agree. Indeed, any additional morphism in I comes from a containment among U and V, and its compatibility is already forced by this agreement. Thus the presentation

$$U \cup V = \operatorname*{colim}_{W \hookrightarrow X : I} W$$

has the universal property of the displayed pushout square.

Proposition 15.3.4 (The spine as a union)**.** *The canonical map* $i_n \colon \mathrm{Seg}^n \to \Delta^n$ *is an embedding. It identifies* Seg^n *with the union of the vertices and consecutive edges of* Δ^n*.*

Proof. We argue by induction on n, the case $n = 0$ being immediate. Inside Δ^{n+1}, the initial face Δ^n and the last edge Δ^1 meet in their common vertex n. Indeed, a simplex factors through both precisely when it is constant at that vertex. By induction, Seg^n embeds in this initial face and contains its last vertex. Hence its intersection with the last edge is also Δ^0. By Example 15.3.3, their union is therefore the pushout

$$\mathrm{Seg}^n \amalg_{\Delta^0} \Delta^1 = \mathrm{Seg}^{n+1} .$$

Its map into Δ^{n+1} is i_{n+1}, so this map is an embedding with the asserted image. □

15.4 Restriction to a fixed anima of objects

We now study how to change the anima of objects of a Segal anima. Given X and a map $e \colon E \to X_0$, we construct a Segal anima $e^*(X)$ whose objects are parametrized by E and whose morphisms from x to y are the morphisms in X from $e(x)$ to $e(y)$. The construction is defined for arbitrary simplicial animae by pulling back their vertex data, and its universal property is expressed by a right adjoint.

Applied to the map on objects of $f \colon X \to Y$, this gives a factorization of f into a map that is an equivalence on objects followed by a fully faithful map. For Segal animae, f is fully faithful precisely when the first map is an equivalence. We also prove the two-out-of-three property for Dwyer–Kan equivalences, which will be used in Corollary 15.5.6 to characterize the maps inverted by Rezk completion.

Construction 15.4.1. Let $E : \mathrm{An}^{\simeq}$ be a small anima. We form the following pullback square.

$$\begin{array}{ccc} \mathrm{PSh}(\mathbf{\Delta})_E & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle E} \\ \mathrm{PSh}(\mathbf{\Delta}) & \xrightarrow{\mathrm{ev}_0} & \mathrm{An} \end{array}$$

Alternatively, one can see E as a constant presheaf on $\mathbf{\Delta}$ and form the co-slice category $\mathrm{PSh}(\mathbf{\Delta})_{E/}$. The category $\mathrm{PSh}(\mathbf{\Delta})_E$ is the full subcategory of $\mathrm{PSh}(\mathbf{\Delta})_{E/}$ whose objects are pairs (X, e), with X a presheaf on $\mathbf{\Delta}$ and $e \colon E \xrightarrow{\sim} X_0$ an equivalence.

Proposition 15.4.2. *The full embedding* $i_E \colon \mathrm{PSh}(\mathbf{\Delta})_E \hookrightarrow \mathrm{PSh}(\mathbf{\Delta})_{E/}$ *has a right adjoint* $(X, e) \mapsto (e^*(X), e_X)$.

Proof. Right Kan extension along $\{[0]\} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ gives an adjunction

$$\mathrm{ev}_0 \colon \mathrm{PSh}(\mathbf{\Delta}) \rightleftarrows \mathrm{An} \,:\! \mathrm{cosk}_0 .$$

For a simplicial anima X equipped with $e \colon E \to X_0$, define $e^*(X)$ by the pullback square

$$\begin{array}{ccc} e^*(X) & \longrightarrow & \mathrm{cosk}_0(E) \\ {\scriptstyle \mathrm{pr}_{E,X}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{cosk}_0(e)} \\ X & \xrightarrow[\mathrm{unit}]{} & \mathrm{cosk}_0(X_0). \end{array}$$

This is a pullback in $\mathrm{PSh}(\mathbf{\Delta})_{E/}$, with the maps from the constant simplicial anima E induced by e and the adjunction.

The pointwise formula for right Kan extensions, Theorem 13.1.8, gives

$$\mathrm{cosk}_0(Y)_n \simeq \prod_{v : \mathrm{Hom}_{\mathbf{\Delta}}([0],[n])} Y \simeq Y^{n+1},$$

since the indexing category is the discrete category of vertices of $[n]$. The unit $X \to \mathrm{cosk}_0(X_0)$ is therefore the map recording all vertices, so

$$e^*(X)_n \simeq X_n \times_{X_0^{n+1}} E^{n+1}.$$

In particular, the structure map $e_X \colon E \to e^*(X)_0$ is an equivalence, and $\mathrm{pr}_{E,X}$ defines a morphism $(e^*(X), e_X) \to (X, e)$.

Let Y be a simplicial anima equipped with an equivalence $f \colon E \xrightarrow{\sim} Y_0$. For an anima Z under E, the adjunction identifies maps $Y \to \mathrm{cosk}_0(Z)$ under E with maps $Y_0 \to Z$ under E. Since f is an equivalence, the anima of such maps is contractible. Apply this with $Z = E$ and $Z = X_0$. Mapping (Y, f) into the defining pullback thus gives a natural equivalence

$$\mathrm{Hom}((Y, f), (e^*(X), e_X)) \simeq \mathrm{Hom}((Y, f), (X, e)).$$

This proves the required adjunction. □

Remark 15.4.3. In the construction of the right adjoint in the proof of Proposition 15.4.2, we see that if X is a Segal anima, then so is $e^*(X)$. In degree n, the Segal map for $e^*(X)$ is obtained from the Segal map for X by pulling back along $E^{n+1} \to X_0^{n+1}$. It is therefore an equivalence.

Remark 15.4.4. Let $f \colon X \to Y$ be a morphism of simplicial animae. We have a unique factorization of f of the form

$$X \xrightarrow{\tilde{f}} f_0^*(Y) \xrightarrow{\mathrm{pr}_{X_0,Y}} Y$$

where $\tilde{f}$ is a morphism in $\mathrm{PSh}(\mathbf{\Delta})_{X_0}$. Furthermore, the projection $\mathrm{pr}_{X_0,Y}$ is in particular fully faithful. Indeed, its degree-one endpoint square is the defining pullback

$$f_0^*(Y)_1 = Y_1 \times_{Y_0 \times Y_0} (X_0 \times X_0).$$

Assuming that both X and Y are Segal animae, we observe the following properties:

(i) The morphism f is fully faithful if and only if $\tilde{f}$ is an equivalence.

(ii) The morphism f is essentially surjective if and only if the projection $\mathrm{pr}_{X_0,Y}$ is a Dwyer–Kan equivalence.

Indeed, property (i) is a direct consequence of Proposition 15.2.12 and property (ii) follows immediately from the definition of essential surjectivity.

Lemma 15.4.5. *Let us consider two morphisms of Segal animae $f\colon C \to D$ and $g\colon D \to E$. If two out of the three morphisms f, g or $g \circ f$ are Dwyer–Kan equivalences, so is the third.*

Proof. Since one can compose pullback squares as well as covers, it is clear that Dwyer–Kan equivalences are stable under composition.

Let us assume that f and $g \circ f$ are Dwyer–Kan equivalences. Since covers are stable under pullbacks and compositions, it follows from Corollary 11.2.7 that g is essentially surjective. We have to prove that the map

$$D_1 \to (D_0 \times D_0) \times_{(E_0 \times E_0)} E_1$$

induced by g_1 is an equivalence over $D_0 \times D_0$. Let $X = C_0 \times_{D_0} \mathrm{Iso}(D)$ and $q\colon X \to D_0$ be the map picking the codomain of isomorphisms. Since $q \times q$ is a cover, it suffices to prove that g_1 is an equivalence after pulling back along $q \times q$ by Proposition 11.2.6. But after such a pullback, $q \times q$ has a section. Furthermore, given $u\colon a \xrightarrow{\sim} b$ in a Segal anima C, we get equivalences of the form

$$u^*\colon C(b,t) \to C(a,t) \quad \text{and} \quad u_*\colon C(t,a) \to C(t,b)$$

for any $t : C_0$ by Lemma 15.2.6. Therefore, it suffices to prove that g induces an equivalence

$$D(f(x), f(y)) \xrightarrow{\sim} E(g(f(x)), g(f(y)))$$

for any terms $x, y : C_0$. This follows from the commutative triangles

$$\begin{array}{ccc} & C(x,y) & \\ {\scriptstyle\simeq}\swarrow & & \searrow{\scriptstyle\simeq} \\ D(f(x),f(y)) & \longrightarrow & E(g(f(x)),g(f(y))). \end{array}$$

Finally, assume that g and $g \circ f$ are Dwyer–Kan equivalences. The pasting lemma for pullback squares (Lemma 1.6.11) shows that f is fully faithful. Consider the cover

$$q\colon P := C_0 \times_{gf,E_0,s} \mathrm{Iso}(E) \longrightarrow E_0, \qquad (x,u) \longmapsto t(u),$$

which expresses the essential surjectivity of $g \circ f$. Pulling it back along $g_0\colon D_0 \to E_0$ gives a cover $P_D \to D_0$. A term of P_D over $y : D_0$ consists of an object $x : C_0$ and an isomorphism $u\colon g(f(x)) \xrightarrow{\sim} g(y)$. Full faithfulness of g, together with Lemma 15.2.11, identifies P_D with

$$C_0 \times_{f,D_0,s} \mathrm{Iso}(D),$$

by lifting u uniquely to an isomorphism $\widetilde{u}\colon f(x) \xrightarrow{\sim} y$. Consequently, the target map from the latter anima to D_0 is a cover. This is precisely the essential surjectivity of f, so f is a Dwyer–Kan equivalence. □

15.5 Rezk completion

It remains to show that every univalent Segal anima comes from a category. More generally, $N(\mathrm{ac}(X))$ provides a *Rezk completion* of any Segal anima X. The main step is to compute the hom animae of $\mathrm{ac}(X)$ using the calculus of last vertex fractions, showing that the unit preserves the hom animae of X.

Theorem 15.5.1. *For any Segal anima X, the unit map*

$$\eta_X \colon X \to N(\mathrm{ac}(X))$$

is a Dwyer–Kan equivalence.

The proof of Theorem 15.5.1 requires some preparation. We fix a simplicial anima X.

Construction 15.5.2. Let W be the subcategory of last vertex maps in $\mathbf{\Delta}$ (see the end of Construction 15.1.8). In other words, W is the synthetic subcategory of $\mathbf{\Delta}$ defined by $W^{\simeq} = \mathbf{\Delta}^{\simeq}$ with morphisms those morphisms

$$f \colon [m] \to [n]$$

such that $f(m) = n$. We form the pullback square below.

$$\begin{array}{ccc} W_X & \longrightarrow & W \\ \downarrow & & \downarrow \\ \mathbf{\Delta}_{/X} & \longrightarrow & \mathbf{\Delta} \end{array}$$

Inverting W_X in $\mathbf{\Delta}_{/X}$ yields $\mathrm{ac}(X)$: by virtue of Lemma 15.1.10, there is a canonical equivalence

$$\mathbf{\Delta}_{/X}[W_X^{-1}] \xrightarrow{\sim} \mathrm{ac}(X)\,.$$

We observe that $W_X^{\simeq} = (\mathbf{\Delta}_{/X})^{\simeq}$. Given an object d of $\mathbf{\Delta}_{/X}$, we write $W(d)$ for the co-slice of d, seen as an object of W_X. There is thus a canonical functor

$$p_d \colon W(d) \to (\mathbf{\Delta}_{/X})_{d/}\,.$$

By construction, this is a putative left calculus of fractions at d. For $c : \mathbf{\Delta}_{/X}$ we write

$$W(d)_{c/} := (\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}_{/X}} W(d)\,.$$

The dual version of Proposition 14.3.9 tells us that p_d is a left calculus of fractions if and only if the functor

$$(\mathbf{\Delta}_{/X})^{\mathrm{op}} \to \mathrm{An}\,, \quad c \mapsto \Pi_\infty(W(d)_{c/})$$

sends all morphisms in W_X to isomorphisms.

From now on, we fix a term $y : X_0$. We interpret the latter as the object $d = ([0], y)$ in $\mathbf{\Delta}_{/X}$.

An arbitrary object c of $\mathbf{\Delta}_{/X}$ is a pair of the form $c = ([n], x)$ for some $n : \mathbb{N}$ and $x : X_n$. We define $P_X(x, y)$ through the pullback square

$$\begin{array}{ccc} P_X(x,y) & \longrightarrow & X_{n+1} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d^{n+1}, \mathrm{ev}_{n+1})} \\ * & \longrightarrow & X_n \times X_0 \end{array}$$

where d^{n+1} is the operator induced by the canonical embedding $[n] \hookrightarrow [n] \star [0] = [n+1]$ and ev_{n+1} the operator induced by the canonical embedding $[0] \hookrightarrow [n] \star [0] = [n+1]$.

The projection $\mathbf{\Delta}_{/X} \to \mathbf{\Delta}$ induces a functor

$$(\mathbf{\Delta}_{/X})_{c/} \to \mathbf{\Delta}_{[n]/}$$

inducing with the projection $W_X \to W$ a canonical functor

$$q \colon W(d)_{c/} \to W_{[n]/} = \mathbf{\Delta}_{[n]/} \times_{\mathbf{\Delta}} W \,.$$

Let i_n be the object of $W_{[n]/}$ corresponding to the diagram

$$[n] \hookrightarrow [n] \star [0] \,.$$

Lemma 15.5.3 (Hebestreit and Steinebrunner [HS25])**.** *Under the assumptions of Construction 15.5.2, the functor $q \colon W(d)_{c/} \to W_{[n]/}$ is a right fibration with fiber at i_n canonically isomorphic to $P_X(x,y)$. Furthermore, the embedding of $P_X(x,y)$ into $W(d)_{c/}$ as fiber of q is an initial functor, inducing a canonical equivalence*

$$P_X(x,y) \xrightarrow{\sim} \Pi_\infty(W(d)_{c/}) \,.$$

If furthermore X is a Segal anima, then the presheaf

$$c \mapsto \Pi_\infty(W(d)_{c/})$$

sends all morphisms in W_X to isomorphisms, so that

$$p_d \colon W(d) \to (\mathbf{\Delta}_{/X})_{d/}$$

is a left calculus of fractions at $d = ([0], y)$ in $\mathbf{\Delta}_{/X}$ with respect to W_X.

Proof. Since $\mathbf{\Delta}_{/X} \to \mathbf{\Delta}$ is a right fibration, so is the induced functor

$$(\mathbf{\Delta}_{/X})_{c/} \to \mathbf{\Delta}_{[n]/}$$

by Proposition 6.5.7. Right fibrations being stable under pullbacks, this yields a right fibration of the form

$$(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W \to \mathbf{\Delta}_{[n]/} \times_{\mathbf{\Delta}} W \,.$$

We observe that $[0]$ is an initial object of W and that $W_X \to W$ is a right fibration (as pullback of the right fibration $\mathbf{\Delta}_{/X} \to \mathbf{\Delta}$ along the embedding $W \hookrightarrow \mathbf{\Delta}$), hence we get another right fibration through another application of Proposition 6.5.7:

$$W(d) = (W_X)_{d/} \to W_{[0]/} \simeq W \,.$$

Therefore, we get by pullback of the latter along the second projection

$$(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W \to W$$

a right fibration of the form

$$(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} (W_X)_{d/} = ((\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W) \times_W (W_X)_{d/} \to (\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W \,.$$

Composing with the right fibration $(\mathbf{\Delta}_{/X})_{c/} \times_{\mathbf{\Delta}} W \to \mathbf{\Delta}_{[n]/} \times_{\mathbf{\Delta}} W$, this proves that the functor

$$q : W(d)_{c/} \to W_{[n]/}$$

is a right fibration as composition of two right fibrations.

We observe that $i_n : (W_{[n]/})^{\simeq}$ is an initial object. Indeed, an object in $W_{[n]/}$ is a morphism $u : [n] \to [m]$. A morphism from $u : [n] \to [m]$ to $u' : [n] \to [m']$ is a morphism $f : [m] \to [m']$ such that $f \circ u = u'$ and $f(m) = m'$. For $u : [n] \to [m]$, there is a unique map $\tilde{u} : [n+1] \to [m]$ such that $\tilde{u}(i) = u(i)$ for $i : [n]^{\simeq}$ and $\tilde{u}(n+1) = m$: since objects of $\mathbf{\Delta}$ are posets (Example 10.3.5), this follows from the last assertion of Lemma 10.3.7. The morphism $\tilde{u}$ is the unique morphism from i_n to the object determined by u. Right fibrations being smooth, since $i_n : * \to W_{[n]/}$ is an initial functor, the embedding of the fiber of q at i_n is an initial functor.

Since q is a right fibration, it is conservative and its fiber at i_n is an anima. The terms of this anima tautologically consist of those of $P_X(x, y)$.

It remains to prove the last assertion of the lemma, assuming furthermore that X is a Segal anima. In this case, we observe, using Lemma 15.2.3, that there is a pullback square of the following form.

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle d^1} \\ X_n & \xrightarrow{\mathrm{ev}_n} & X_0 \end{array}$$

The latter is the outer square in the commutative diagram

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_1 \\ {\scriptstyle (d^{n+1},\mathrm{ev}_{n+1})}\downarrow & & \downarrow{\scriptstyle (d^1,d^0)} \\ X_n \times X_0 & \xrightarrow[\mathrm{ev}_n \times \mathrm{id}_{X_0}]{} & X_0 \times X_0 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ X_n & \xrightarrow{\mathrm{ev}_n} & X_0 \end{array}$$

in which the lower square is cartesian. In other words, we have a pullback square of the form

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & X_1 \\ {\scriptstyle (d^{n+1},\mathrm{ev}_{n+1})}\downarrow & & \downarrow{\scriptstyle (d^1,d^0)} \\ X_n \times X_0 & \xrightarrow[\mathrm{ev}_n \times \mathrm{id}_{X_0}]{} & X_0 \times X_0. \end{array}$$

This means that there is a canonical isomorphism of the form

$$P_X(x, y) \simeq P_X(x_n, y)$$

where $x_n : X_0$ is the evaluation of $x : X_n$ at $n : [n]^{\simeq}$. Let $c = ([n], x)$ and $c' = ([n'], x')$ be two objects of $\mathbf{\Delta}_{/X}$. A morphism $u : c \to c'$ in $\mathbf{\Delta}_{/X}$, is a morphism $u : [n] \to [n']$ in $\mathbf{\Delta}$ together with an identification $s : u^*(x') = x$. saying that u belongs to W_X means that $u(n) = n'$, which induces through s an identification $\sigma : x_n = x'_{n'}$. Let $c_0 = ([0], x_n)$ and $c'_0 = ([0], x'_{n'})$. We have a commutative square in $\mathbf{\Delta}_{/X}$ of the form

$$\begin{array}{ccc} c_0 & = & c'_0 \\ \downarrow & & \downarrow \\ c & \xrightarrow{u} & c' \end{array}$$

inducing a commutative diagram of the following form

$$\begin{array}{ccccccc} P_X(x',y) & \xrightarrow{\simeq} & \Pi_\infty(W(d)_{c'/}) & \longrightarrow & \Pi_\infty(W(d)_{c/}) & \xleftarrow{\simeq} & P_X(x,y) \\ \| & & \downarrow & & \downarrow & & \| \\ P_X(x'_{n'},y) & \xrightarrow{\simeq} & \Pi_\infty(W(d)_{c'_0/}) & = & \Pi_\infty(W(d)_{c_0/}) & \xleftarrow{\simeq} & P_X(x_n,y). \end{array}$$

This achieves the proof of the lemma. □

Exercise 15.5.4. Under the conventions and definitions of Construction 15.5.2, check that the proof of Lemma 15.5.3 shows in fact that the property for X to be a Segal anima is equivalent to the property that, for any $y : X_0$,

$$p_d \colon W(d) \to (\mathbf{\Delta}_{/X})_{d/}$$

is a left calculus of fractions at $d = ([0], y)$ in $\mathbf{\Delta}_{/X}$.

Proof of Theorem 15.5.1. Let X be a Segal anima. As explained in Construction 15.1.8, we have canonical commutative square

$$\begin{array}{ccc} \mathrm{Fun}([1], \mathbf{\Delta}_{/X}) & \xrightarrow{\mathrm{ev}_0} & \mathbf{\Delta}_{/X} \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle \tau'_X} \\ \mathbf{\Delta}_{\bullet/X} & \xrightarrow[j_X]{} & \mathrm{ac}(X) \end{array}$$

and Lemma 15.1.10 tells us that τ'_X exhibits $\mathrm{ac}(X)$ as the localization of $\mathbf{\Delta}_{/X}$ by last vertex maps. Given $n : \mathbb{N}$, the map

$$(\eta_X)_n \colon X_n \to N(\mathrm{ac}(X))_n$$

sends $\gamma : X_n$ to the composition of the functor j_X with the embedding

$$[n] \hookrightarrow \mathbf{\Delta}_{\bullet/X}$$

corresponding to the fiber of the cocartesian fibration

$$\mathbf{\Delta}_{\bullet/X} \to \mathbf{\Delta}_{/X}$$

at the object $([n], \gamma)$.

An object of $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$ essentially is a pair (u, γ) with $u \colon [m] \to [n]$ a morphism of $\mathbf{\Delta}$ and $\gamma : X_n$. For any $i : [n]^\simeq$, the projection

$$[n]_{/i} \to [n]$$

is a full embedding and induces an equivalence $[n]_{/i} \xrightarrow{\sim} [i]$. From there, we observe that, since m is a terminal object of $[m]$, the property for u to be a right fibration is equivalent to the property that u induces an equivalence $[m] \xrightarrow{\sim} [n]_{/u(m)}$. Using Remark 12.4.17, this implies that one can describe $\mathbf{\Delta}_{\bullet/X}$ as the full subcategory of $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$ whose objects are pairs (u, γ) as above, such that u is a right fibration. Let

$$r \colon \mathbf{\Delta}_{\bullet/X} \to \mathrm{Fun}([1], \mathbf{\Delta}_{/X})$$

be the resulting full embedding. The functor q_X assigns to each pair (u, γ) in $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$ the right fibration $[u(m)] = [n]_{/u(m)} \hookrightarrow [n]$ together with the same $\gamma : X_n$. The canonical commutative square

$$\begin{array}{ccc} [m]_{/m} & \longrightarrow & [n]_{/u(m)} \\ \| & & \| \\ [m] & \longrightarrow & [u(m)] \\ {\scriptstyle u}\downarrow & & \downarrow \\ [n] & = & [n] \end{array}$$

defines a map $\eta_{(u,\gamma)} \colon (u, \gamma) \to r(q_X(u, \gamma))$ in $\mathrm{Fun}([1], \mathbf{\Delta}_{/X})$.

We verify that these maps exhibit q_X as left adjoint to r. First, if u is a right fibration, then $[m] \to [n]_{/u(m)} = [u(m)]$ is an equivalence. Consequently $q_X r \xrightarrow{\sim} \mathrm{id}_{\mathbf{\Delta}_{\bullet/X}}$. It remains to verify the universal property of the unit. Let $(v \colon [p] \hookrightarrow [q], \delta)$ be an object of $\mathbf{\Delta}_{\bullet/X}$, written as an initial-segment inclusion. A morphism from (u, γ) to its image under r contains a commutative square

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [p] \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ [n] & \xrightarrow{\beta} & [q]. \end{array}$$

Since $\beta(u(m)) = v(\alpha(m))$ lies in the initial segment $v([p])$, monotonicity implies that $\beta(j)$ lies in $v([p])$ for every $j \leq u(m)$. Hence $\beta|_{[u(m)]}$ factors uniquely through v. Writing the factorization as $\bar{\alpha} \colon [u(m)] \to [p]$, the equality $v\bar{\alpha}u = v\alpha$ and the fact that v is an embedding give $\bar{\alpha}u = \alpha$. We have therefore obtained a unique factorization of the displayed square through $\eta_{(u,\gamma)}$. The compatibility with the labels in X is inherited from the original morphism. Thus precomposition with $\eta_{(u,\gamma)}$ induces the required equivalence on hom animae, proving $q_X \dashv r$ with r a right adjoint section.

The functor $j_X = j_X \circ q_X \circ r$ is isomorphic to the functor $\tau'_X \circ \mathrm{ev}_0 \circ r$.

This implies that the map

$$(\eta_X)_0 \colon X_0 \to N(\mathrm{ac}(X))_0 = \mathrm{ac}(X)^{\simeq}$$

is a cover. Indeed, $\tau'_X \colon \mathbf{\Delta}_{/X} \to \mathrm{ac}(X)$ is a localization, so Corollary 11.2.16 gives a cover

$$(\mathbf{\Delta}_{/X})^{\simeq} = \sum_{n:\mathbb{N}} X_n \longrightarrow \mathrm{ac}(X)^{\simeq}.$$

For any $\gamma : X_n$, since

$$n \colon [0] \hookrightarrow [n]$$

is a last vertex map, it induces an isomorphism from the image of $([0], \gamma(n))$ to the image of $([n], \gamma)$ in $\mathrm{ac}(X)$. Hence the preceding cover factors, up to this canonical isomorphism, through the restricted map $X_0 \to \mathrm{ac}(X)^{\simeq}$. By cancellation for covers, this restricted map is already a cover.

It remains to prove that the morphism $\eta_X \colon X \to N(\mathrm{ac}(X))$ is fully faithful. Let us consider given $x, y : X_0$. We have to prove that the induced map

$$(\eta_X)_{x,y} \colon X(x, y) \to N(\mathrm{ac}(X))(\eta_X(x), \eta_X(y)) = \mathrm{ac}(X)(\eta_X(x), \eta_X(y))$$

is an equivalence. Thanks to the last assertion of Lemma 15.5.3 together with the dual version of Proposition 14.3.9, there is a canonical isomorphism of the form

$$\Pi_\infty(W(d)_{c/}) \xrightarrow{\sim} \mathrm{ac}(X)(\eta_X(x), \eta_X(y))$$

with $c = ([0], x)$ and $d = ([0], y)$. Under the canonical identification $P_X(x, y) = X(x, y)$, these maps fit into the commutative square

$$\begin{array}{ccc} P_X(x,y) & \xrightarrow{\simeq} & \Pi_\infty(W(d)_{c/}) \\ \| & & \downarrow{\scriptstyle \simeq} \\ X(x,y) & \xrightarrow{(\eta_X)_{x,y}} & \mathrm{ac}(X)(\eta_X(x), \eta_X(y)). \end{array}$$

Here the upper map is induced by the initial embedding of the fiber from Lemma 15.5.3, and the right-hand map is the canonical fraction equivalence. To check commutativity, it suffices to apply the square to the universal term $u : X(x, y)$. Its image in $W(d)_{c/}$ is the fraction

$$([0], x) \xrightarrow{0} ([1], u) \xleftarrow{1} ([0], y),$$

where the right-hand map is a last vertex map. The canonical fraction map sends this term to the composite

$$\tau'_X([0], x) \longrightarrow \tau'_X([1], u) \xrightarrow{\simeq} \tau'_X([0], y),$$

where the second arrow is the inverse of the image of the last vertex map. On the other hand, the description of η_X above and the isomorphism $j_X \simeq \tau'_X \circ \mathrm{ev}_0 \circ r$ identify $\eta_X(u)$ with precisely this composite. The square therefore commutes. Both vertical maps and the upper map are equivalences, so $(\eta_X)_{x,y}$ is an equivalence. This completes the proof. □

Corollary 15.5.5. *A simplicial anima X is a univalent Segal anima if and only if the unit map $\eta_X \colon X \to N(\mathrm{ac}(X))$ is invertible. Consequently, the Rezk nerve induces an equivalence*

$$N \colon \mathrm{Cat} \xrightarrow{\sim} \mathrm{USeg}(\mathrm{An})$$

with inverse given by the restriction of ac.

Proof. If η_X is invertible, then X is univalent because $N(\mathrm{ac}(X))$ is. Conversely, if X is a univalent Segal anima, then by Theorem 15.5.1, the map $\eta_X \colon X \to N(\mathrm{ac}(X))$ is a Dwyer–Kan equivalence between univalent Segal animae. Corollary 15.2.13 proves the first assertion. The final assertion follows since the counit $\mathrm{ac}(N(C)) \to C$ is invertible by the full faithfulness of N, Corollary 15.1.13. □

Corollary 15.5.6. *A morphism of Segal animae $f \colon C \to D$ is a Dwyer–Kan equivalence if and only if* $\mathrm{ac}(f) \colon \mathrm{ac}(C) \to \mathrm{ac}(D)$ *is an equivalence.*

Proof. The Rezk nerve functor being fully faithful, it is conservative. Therefore, we might as well prove that Dwyer–Kan equivalences are exactly those morphisms sent by the functor $N \circ \mathrm{ac}$ to equivalences. For any morphism of Segal animae $f \colon C \to D$ we have a commutative square

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & N(\mathrm{ac}(C)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle N(\mathrm{ac}(f))} \\ D & \xrightarrow{\eta_D} & N(\mathrm{ac}(D)) \end{array}$$

in which the two horizontal maps are Dwyer–Kan equivalences by Theorem 15.5.1. Since the right vertical map is a morphism of univalent Segal animae, we see that this corollary follows immediately from the combination of Corollary 15.2.13 and Lemma 15.4.5. □

Corollary 15.5.7. *The functor* $\mathrm{ac}\colon \mathrm{Seg}(\mathrm{An}) \to \mathrm{Cat}$ *exhibits* Cat *as the localization of* Seg(An) *by Dwyer–Kan equivalences.*

Proof. The adjunction between associated categories and Rezk nerves restricts to

$$\mathrm{ac}\colon \mathrm{Seg}(\mathrm{An}) \rightleftarrows \mathrm{Cat}\colon N$$

with N fully faithful by Corollary 15.1.13. This corollary is now a direct consequence of Corollary 15.5.6 and of Proposition 4.4.16. □

Corollary 15.5.8. *Let C be a small category. Then $\Pi_\infty(C)$ is the colimit of $N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$.*

Proof. Identifying the universe Cat with the category of univalent Segal animae, this follows immediately from the fact that composing the full embedding

$$\mathrm{An} \hookrightarrow \mathrm{Cat}$$

with the nerve functor yields the functor sending each anima X to the constant functor indexed by $\mathbf{\Delta}^{\mathrm{op}}$ with value X. □

16 Filtered colimits and finite categories

16.1 The poset of natural numbers

In Chapter 10, we defined the *set* of natural numbers. We now augment the set $\mathbb{N}$ with an order to define the ordinal ω, i.e., the *poset* of natural numbers.

Construction 16.1.1. We define the function

$$(- \leq -) \colon \mathbb{N} \times \mathbb{N} \to \mathrm{Prop} \tag{16.1}$$

by double induction as follows:

$$(0 \leq m) \coloneqq * \quad \text{and} \quad (n+1 \leq 0) \coloneqq \emptyset \quad \text{and} \quad (n+1 \leq m+1) \coloneqq (n \leq m) \tag{16.2}$$

Furthermore, we define $(n < m) \coloneqq (n \leq m) \times (n \neq m)$

Exercise 16.1.2. Show that for all $n, m \in \mathbb{N}$ we have

$$(n \leq m) = (x : \mathbb{N}) \times (n + x = m). \tag{16.3}$$

Lemma 16.1.3. *The relation $\leq$ is a partial order on the set $\mathbb{N}$ which is trichotomous, i.e., for all $n, m \in \mathbb{N}$ we have*

$$((n < m) \amalg (n = m) \amalg (n > m)) \; = \; * \tag{16.4}$$

In particular $\leq$ is decidable.

Proof. All statements are straightforward to prove by induction using observational equality of natural numbers (Proposition 10.2.19). □

Definition 16.1.4. We define ω to be the unique poset with $\omega^{\simeq} = \mathbb{N}$ and $\omega(-,-) = (- \leq -)$. In other words, ω is the poset associated to the partially ordered set $(\mathbb{N}, \leq)$ through the equivalence Theorem 10.3.10.

For each $n, m \in \mathbb{N}$ we denote by $[n, m] \subseteq \omega$ the full subcategory spanned by $\{x \in \mathbb{N} \mid n \leq x \leq m\}$.

Remark 16.1.5. Note that for each $n \leq m$ we have $[n, m] \star [0] = [n, m+1]$ because we have $[n, m+1]^{\simeq} = [n, m]^{\simeq} \amalg \{m+1\}$, where the object $m+1$ is terminal in $[n, m+1]$, the inclusion $[n, m] \subseteq [n, m+1]$ is fully faithful and there are no arrows going from $m+1$ to any object in $[n, m]$. In particular, for each $n \in \mathbb{N}$ there is an isomorphism $[n] \cong [0, n]$ (which is unique by Theorem 10.4.3).

Corollary 16.1.6. *The object* $0 \in \omega$ *is initial, hence* ω *is weakly contractible.*

Proof. Immediate, because $0 \leq n$ for all $n \in \mathbb{N}$. □

Lemma 16.1.7. *The successor function* $\mathrm{succ} \colon \mathbb{N} \to \mathbb{N}$ *defines a poset map* $\omega \to \omega$.

Proof. By definition. □

Lemma 16.1.8 (Binary coproducts in ω)**.** *The poset* ω *has binary products and coproducts, namely the minimum and maximum.*

Proof. This follows from the fact that $(\mathbb{N}, \leq)$ is trichotomous. Explicitly,

$$\min(n, m) \coloneqq \begin{cases} n, \text{ if } n \leq m \\ m, \text{ if } n > m \end{cases} \tag{16.5}$$

and the maximum is defined analogously. □

Lemma 16.1.9. *For any* $n : \mathbb{N}$ *the functor*

$$\mathrm{succ}^n \colon \omega \to \omega, \quad x \mapsto x + n$$

has a left adjoint given by

$$\mathrm{pred}^n \colon x \mapsto \begin{cases} x - n, & \text{if } x \geq n \\ 0, & \text{if } x \leq n \end{cases}$$

respectively. In particular, it is terminal.

Proof. To construct pred^n as a map of posets, it suffices to show the following to statements, both of which are straightforward by induction:

- For each $x \geq n$ there is a unique $k \in \mathbb{N}$ with $n + k = x$, call it $x - n \coloneqq k$. This shows that pred^n is well-defined as a set map.
- If $x' \geq x \geq n$ then $x' - n \geq x - n$ hence pred^n preserves the order (there is nothing to check in the case $x \leq n$ because $0 \leq -$ is always true).

To establish the adjunction one verifies the inequalities $x \leq \mathrm{pred}^n(x) + n$ (with equality precisely when $x \geq n$) and $\mathrm{pred}^n(x + n) = x$. These are the unit and counit of the adjunction; the triangle identities are vacuous because ω is a poset. □

Lemma 16.1.10. *For any* $n : \mathbb{N}$*, the functor*

$$\mathrm{succ}^n \colon \omega \to \omega$$

is a fully faithful left fibration and induces an equivalence

$$\omega \xrightarrow{\sim} \omega_{\geq n} = \omega_{n/},$$

Proof. Since the counit of the adjunction of Lemma 16.1.9 is the identity, succ^n is fully faithful and the adjunction restricts to the equivalence

$$\mathrm{pred}^n : \omega_{\geq n} \xleftrightarrow{\simeq} \omega : \mathrm{succ}^n \tag{16.6}$$

because $\omega_{\geq n} \subseteq \omega$ is exactly the full subposet where the unit is invertible. Since $\mathrm{succ}^n : \omega \to \omega$ is thereby identified with the slice projection $\omega_{\geq n} = \omega_{n/} \to \omega$, it is a left fibration. □

Theorem 16.1.11 (Milnor presentation of ω)**.** *The canonical square of posets*

$$\begin{array}{ccc} \coprod_{n\in\mathbb{N}}\{n, n+1\} & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow \\ \coprod_{n\in\mathbb{N}}[n, n+1] & \longrightarrow & \omega \end{array} \tag{16.7}$$

(defined by sending each element to the corresponding element with the same name) is a pushout square in Cat.

To prove Theorem 16.1.11 we make repeated use of the following general technique:

Lemma 16.1.12 (one-point sink/source attachment)**.** *Let C be a category and $c \in C$ an object. Consider the pushout*

$$\begin{array}{ccc} \{0\} & \xrightarrow{c} & C \\ \downarrow & & \downarrow \\ [1] & \xrightarrow{\alpha} & D \end{array} \tag{16.8}$$

We say that D arises from C by a one-point sink attachment *(at $c \in C$). Then taking cores yields $C^{\simeq} \amalg \{1\} \simeq D^{\simeq}$ and the hom-animae in D are characterized under this identification by*

(1) *$C \to D$ is fully faithful, i.e. $C(x, y) \xrightarrow{\simeq} D(x, y)$ for $x, y \in C$;*

(2) *$D(1, 1) = *$;*

(3) *$C(x, c) \xrightarrow[=]{\alpha\circ -} D(x, 1)$ for $x \in C$;*

(4) *$D(1, x) = \emptyset$ for $x \in C$.*

In particular, if C is a poset, then so is D.

The dual construction is called a one-point source attachment.

Proof. Define the functor $D \to [1]$ which is constantly 0 on C and the identity on $[1]$. Since the two inclusions $0, 1 : * \to [1]$ are exponentiable (because they are a right and left fibration, respectively), the square (16.8) remains a pushout when passing to fibers over 0 and 1. In either case, the pushout degenerates and yields

$$C \xrightarrow{\sim} D_0 \quad \text{and} \quad \{1\} \xrightarrow{\sim} D_1, \tag{16.9}$$

respectively. Since $[1]^{\simeq} = \{0\} \amalg \{1\}$, we obtain an identification $D^{\simeq} \simeq C^{\simeq} \amalg \{1\}$, and the statements 1 and 2; we also get 4 because there are no arrows $1 \to 0$ in $[1]$.

Finally, note that $\{0\} \to [1]$ has a right adjoint retraction that sends $1 \mapsto 0$, hence $C \to D$ has a right adjoint retraction (by Proposition 4.4.17) that sends $1 \mapsto c$ and whose counit at $1 \in D$ is $\alpha\colon c \to 1$ (by direct inspection of the construction). The equivalence 3 is precisely the equivalence on hom-animae induced by this adjunction. □

Proof of Theorem 16.1.11. It is easy to show that the square (16.7) is a pushout *of partially ordered sets*:

- The square is tautologically a pushout on cores (a.k.a., underlying sets)
- For any partially ordered set $(S, \leq)$, a set map $\mathbb{N} \to S$ preserves the order if and only if it preserves the order when restricted to each $\{n, n+1\} \subseteq \mathbb{N}$.

In other words, if we form the actual pushout

$$\begin{array}{ccc} \coprod_{n\in\mathbb{N}}\{n, n+1\} & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow \\ \coprod_{n\in\mathbb{N}}[n, n+1] & \longrightarrow & P \end{array} \tag{16.10}$$

of categories, then we have $\|P\|_{\mathrm{pos}} = \omega$ after poset truncation; it remains to show that P is already a poset.

For each $a \in \mathbb{N}$ and $b \in \mathbb{N}\star\{\infty\}$, denote by $P[a, b]$ the same pushout but retaining only those coproduct components that are indexed by $a \leq n < b$ on the left side and $a \leq n \leq b$ in the top right corner [1]. For each finite b, the analogous pushouts formed in posets yield the posets $[a, b]$; hence we have truncation maps $P[a, b] \to [a, b]$ which are natural with respect to inclusions of intervals.

Observe that for $b = a$ the pushout (16.10) degenerates to

$$P[a, a] = \{a\} \amalg_{\emptyset} \emptyset = [0]$$

and for $b = a + 1$ the pushout (16.10) degenerates to

$$P[a, a+1] = [a, a+1] \amalg_{\{a,a+1\}} (\{a\} \amalg \{a+1\}) = [1].$$

An elementary rearranging of pushouts and coproducts shows that for each $a \leq b < \infty$ and $b \leq c \leq \infty$ we have a pushout squares of the following form

$$\begin{array}{ccc} P[b, b] & \longrightarrow & P[b, c] \\ \downarrow & & \downarrow \\ P[a, b] & \longrightarrow & P[a, c] \end{array} \tag{16.11}$$

In particular:

- Choosing $c = b + 1$ exhibits $P[a, b+1]$ as a one-point sink attachment of $P[a, b]$ at the last vertex b. Hence, inductively for all $a \leq b < \infty$, the category $P[a, b]$ is a poset, hence $P[a, b] = [a, b]$.

[1] Note that despite the notation suggesting a closed interval, there is no object named ∞ appearing anywhere in $P[a, \infty]$.

- Choosing $a = 0$, $b = 1$ and $c = \infty$ exhibits P a one-point source attachment of $P[1, \infty]$. In particular, the inclusion $[0] \to P$ is fully faithful.
- For each $n \in \mathbb{N}$, choosing $a = 0$, $b = n$ and $c = \infty$ yields the pushout

$$\begin{array}{ccc} \{n\} & \xrightarrow{n} & P[n, \infty] \\ \downarrow & & \downarrow \\ [n] & \longrightarrow & P \end{array} \tag{16.12}$$

  The top map is identified with $\{0\} \to P[0, \infty]$ by relabeling all coproduct components along the bijection $\mathrm{succ}^n \colon \mathbb{N} \to \mathbb{N}_{\geq n}$. Since the top map is fully faithful by the previous step, so is the bottom one by Corollary 12.2.12.

If we take cores of the defining pushout (16.10), then the left vertical map becomes an equivalence. Thus it follows from Corollary 13.2.2 that the map $\mathbb{N} \to P^{\simeq}$ is a cover.

We can now show that P is a poset. For any $x, y \in P$ we have to show that $P(x, y)$ is a proposition. This statement itself is a proposition, so we may choose $i, j \in \mathbb{N}$ with $i \mapsto x$ and $j \mapsto y$ (because $\mathbb{N} \to P^{\simeq}$ is a cover). Now take $n \coloneqq max(i, j)$ so that x, y are both in the image of the map $[n] \to P$. Since we have shown this map to be fully faithful, we conclude that $P(x, y)$ is identified with $[n](i, j)$ which is a proposition. □

Remark 16.1.13. There are various other equivalent ways of writing the pushout (16.7) by removing the redundant copies of 0 in the top row and shuffling the pieces around. For example, we have pushouts of the form

$$\begin{array}{ccc} \mathbb{N}_{\geq 1} \amalg \mathbb{N}_{\geq 1} & \xrightarrow{\nabla} & \mathbb{N}_{\geq 1} \\ \downarrow f & & \downarrow \\ \coprod_{n \in \mathbb{N}} [n, n+1] & \longrightarrow & \omega \end{array} \tag{16.13}$$

(where f sends the two copies of some number $n + 1 \in \mathbb{N}_{\geq 1}$ to $(n, n + 1)$ and $(n + 1, n + 1)$, respectively) and

$$\begin{array}{ccc} \mathbb{N}_{\geq 1} & \longrightarrow & \coprod_{k \in \mathbb{N}} [2k, 2k+1] \\ \downarrow & & \downarrow \\ \coprod_{k \in \mathbb{N}} [2k+1, 2k+2] & \longrightarrow & \omega \end{array} \tag{16.14}$$

Definition 16.1.14. Let C be a category. We define the category $\mathrm{Seq}_{\omega}(C)$ of *ω-indexed sequences in C* as the following pullback:

$$\begin{array}{ccc} \mathrm{Seq}_{\omega}(C) & \longrightarrow & \prod_{n \in \mathbb{N}} \mathrm{Fun}([1], C) \\ \downarrow & & \downarrow p \\ \prod_{n \in \mathbb{N}} C & \xrightarrow{\Delta} & \prod_{n \in \mathbb{N}} (C \times C), \end{array} \tag{16.15}$$

where the functor p is defined by $p((f_n)_{n \in \mathbb{N}})_i \coloneqq (t(f_i), s(f_{i+1}))$. Dually, we define the category $\mathrm{Seq}_{\omega^{\mathrm{op}}}(C)$ of *ω^{op}-indexed sequences in C* as

$$\mathrm{Seq}_{\omega^{\mathrm{op}}}(C) \coloneqq \mathrm{Seq}_{\omega}(C^{\mathrm{op}})^{\mathrm{op}}.$$

Explicitly, an object of $\mathrm{Seq}_\omega(C)$ consists of a collection of objects $(X_n)_{n\in\mathbb{N}}$ of C together with a sequence of morphisms $f_n\colon X_n \to X_{n+1}$ between them:

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

We will denote such an object by $(X_n, f_n)_{n\in\mathbb{N}}$. Dually, an object of $\mathrm{Seq}_{\omega^{\mathrm{op}}}(C)$ consists of a collection of objects $(X_n)_{n\in\mathbb{N}}$ with morphisms $f_n\colon X_{n+1} \to X_n$ going in the other direction.

Corollary 16.1.15. *For each category C, the left pushout square of Remark 16.1.13 induces equivalences*

$$\mathrm{Seq}_\omega(C) \simeq \mathrm{Fun}(\omega, C) \quad \textit{and} \quad \mathrm{Seq}_{\omega^{\mathrm{op}}}(C) \simeq \mathrm{Fun}(\omega^{\mathrm{op}}, C) \tag{16.16}$$

of categories.

Proof. The square (16.15) arises from Milnor presentation (16.13) of ω by applying $\mathrm{Fun}(-, C)$ which sends pushouts to pullbacks. □

Definition 16.1.16 (Milnor (co)limit)**.** Let $(X_n, f_n)_{n\in\mathbb{N}^{\mathrm{op}}}$ be an ω^{op}-indexed sequence in a category C with pullbacks as well as countable products (i.e. $\mathbb{N}$-indexed limits). We define the *Milnor limit* "$\lim_n$" X_n of the sequence above as the pullback

$$\begin{array}{ccc} \text{“}\lim_n\text{”}\, X_n & \longrightarrow & \prod_n X_n \\ \downarrow & \lrcorner & \downarrow q \\ \prod_n X_n & \xrightarrow{\Delta} & \prod_n (X_n \times X_n), \end{array}$$

where for $i \in \mathbb{N}$ the i-th component of the map q is given by the map $(\mathrm{pr}_i, f_i \circ \mathrm{pr}_{i+1})\colon \prod_n X_n \to X_i \times X_i$.

Dually, if $(X_n, f_n)_{n\in\mathbb{N}}$ is an ω-indexed sequence in a category C with colimits, we define its *Milnor colimit* "colim_n" X_n as the Milnor limit of the associated sequence in C^{op}.

Corollary 16.1.17. *Under the identification of Corollary 16.1.15, the Milnor colimit (or limit) of an ω (or ω^{op}) sequence agrees with its usual colimit (or limit).*

Proof. This follows from the Milnor presentation of ω and the computation of (co)limits indexed by colimits (Corollary 13.3.2). □

Remark 16.1.18. Under the identification of Corollary 16.1.15, the functor $\mathrm{succ}^*\colon \mathrm{Fun}(\omega, C) \to \mathrm{Fun}(\omega, C)$ has the following explicit description. It sends a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

to the sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \to X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to \cdots$$

By Lemma 16.1.9, the functor succ^* above has a right adjoint

$$\mathrm{succ}_*\colon \mathrm{Seq}_\omega(C) \to \mathrm{Seq}_\omega(C)$$

which explicitly sends a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

to the sequence

$$X_0 \xrightarrow{\mathrm{id}_{X_0}} X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

The counit of this adjunction is the identity $\mathrm{succ}^* \mathrm{succ}_* = \mathrm{id}$, so that, in particular, the functor succ_* is fully faithful.

Proposition 16.1.19. *For any $m, n : \mathbb{N}$ with maximum $k = \max(m, n)$, the functor succ^k induces an equivalence*

$$\omega \xrightarrow{\sim} \omega_{m/} \times_\omega \omega_{n/} .$$

Proof. There is a canonical equivalence of full subcategories of ω:

$$\omega_{k/} \simeq \omega_{m/} \times_\omega \omega_{n/} .$$

We conclude with Lemma 16.1.10. □

16.2 Filtered categories

Definition 16.2.1. A category A is *confluent* if, for any morphism $u : x \to y$ in A, the functor $u^* : A_{y/} \to A_{x/}$ defined by composition with u is a terminal functor.

Remark 16.2.2. The property for A to be confluent is equivalent to the following one: given any morphisms in A of the form $u : x \to y$ and $v : x \to z$, the category $A_{y/} \times_{A_{x/}} A_{z/}$ is weakly contractible.

$$\begin{array}{ccc} A_{y/} \times_{A_{x/}} A_{z/} & \longrightarrow & A_{z/} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle v^*} \\ A_{y/} & \xrightarrow{u^*} & A_{x/} \end{array}$$

Indeed, an object of $A_{x/}$ is a pair (z, v) with $v : x \to z$ a morphism in A and since $v^* : A_{z/} \to A_{x/}$ is a left fibration whose domain has an initial object sent to (z, v), there is a canonical equivalence of the form

$$(A_{x/})_{(z,v)/} \xrightarrow{\sim} A_{z/}$$

over $A_{x/}$, so that we have a canonical equivalence

$$(A_{y/})_{(z,v)/} \xrightarrow{\sim} A_{y/} \times_{A_{x/}} A_{z/} .$$

Our claim now follows by applying Corollary 9.5.8 with $C = (A_{x/})^{\mathrm{op}}$.

Example 16.2.3. If A is a small category with pushouts, then it is confluent. Indeed, since the functor $\mathrm{Hom}_A(-, a) : A^{\mathrm{op}} \to \mathrm{An}$ preserves pullbacks, for any morphisms in A of the form $u : x \to y$ and $v : x \to z$, if $p = y \sqcup_x z$ denotes the corresponding pushout, then

$$A_{p/} \xrightarrow{\sim} A_{y/} \times_{A_{x/}} A_{z/}$$

has an initial object and is thus weakly contractible.

Proposition 16.2.4. *Let $p\colon X \to A$ be a left fibration. If A is confluent, then so is X.*

Proof. If p is a left fibration, then, for any object x of X, the induced functor

$$X_{x/} \to A_{p(x)/}$$

is an equivalence (this is a morphism of left fibrations over A and it preserves the initial objects). Therefore, for any morphism $u\colon x \to y$ in X, we obtain a commutative square of the form

$$\begin{array}{ccc} X_{y/} & \xrightarrow{u^*} & X_{x/} \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\ A_{p(y)/} & \xrightarrow{p(u)^*} & A_{p(x)/} \end{array}$$

in which the vertical maps are equivalences. Since $p(u)^*$ is a terminal functor, the same is true about u^*. □

Theorem 16.2.5 (Rezk, Sattler–Wärn)**.** *A small category A is confluent if and only if the colimit functor*

$$\mathrm{colim}_A\colon \mathrm{Fun}(A, \mathrm{An}) \to \mathrm{An}$$

commutes with pullbacks.

Proof. Assume that A is confluent. Under the unstraightening equivalence

$$\mathrm{Un}\colon \mathrm{Fun}(A, \mathrm{An}) \xrightarrow{\sim} \mathrm{LFib}(A)\,,$$

the colimit functor colim_A corresponds to the functor

$$\mathrm{LFib}(A) \to \mathrm{An}$$

defined by assigning to each left fibration $p\colon X \to A$ the anima $\Pi_\infty(X)$. In other words, it suffices to prove that, for any pullback square of small categories of the form

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

in which all functors are left fibrations, and such that there is a left fibration $p\colon Y \to A$, the induced commutative square

$$\begin{array}{ccc} \Pi_\infty(X') & \xrightarrow{\Pi_\infty(u)} & \Pi_\infty(X) \\ {\scriptstyle \Pi_\infty(f')}\downarrow & & \downarrow{\scriptstyle \Pi_\infty(f)} \\ \Pi_\infty(Y') & \xrightarrow{\Pi_\infty(v)} & \Pi_\infty(Y) \end{array}$$

is a pullback square. In order to prove this, we start by observing that, by virtue of Proposition 16.2.4, Y itself is confluent. For each morphism $w\colon y_0 \to y_1$ in Y, the induced functor

$$w^*\colon Y_{y_1/} \to Y_{y_0/}$$

is thus terminal. Since f is a left fibration, it is in particular proper. Therefore, the pullback squares

$$\begin{array}{ccccc} X_{y_1/} & \longrightarrow & X_{y_0/} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ Y_{y_1/} & \xrightarrow{w^*} & Y_{y_0/} & \longrightarrow & Y \end{array}$$

show that w^* induces a terminal functor $X_{y_1/} \to X_{y_0/}$. Since terminal functors are weak homotopy equivalences, the equivalence of properties (i) and (iii) in Quillen's Theorem B (Theorem 13.4.5) implies that we have a canonical equivalence

$$\Pi_\infty(X') \xrightarrow{\sim} \Pi_\infty(Y') \times_{\Pi_\infty(Y)} \Pi_\infty(X)\,,$$

which proves that the functor colim_A preserves pullbacks of animae.
Conversely, let us assume that the functor colim_A preserves pullbacks in An. Observe that, for any pullback square of the form

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{An}_\bullet \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{univ}}} \\ A & \xrightarrow{F} & \mathrm{An} \end{array}$$

if we see $\Pi_\infty(X)$ as an object of An, we have a canonical equivalence

$$\mathrm{colim}_A F = \Pi_\infty(X)$$

in An. Given two morphisms $u\colon x \to y$ and $v\colon x \to z$ in A, we thus have

$$\mathrm{colim}_{a:A}(\mathrm{Hom}_A(y,a) \times_{\mathrm{Hom}_A(x,a)} \mathrm{Hom}_A(z,a)) = \Pi_\infty(A_{y/} \times_{A_{x/}} A_{z/})\,.$$

Similarly we have:

$$\mathrm{colim}_{a:A} \mathrm{Hom}_A(y,a) \times_{\mathrm{colim}_{a:A} \mathrm{Hom}_A(x,a)} \mathrm{colim}_{a:A} \mathrm{Hom}_A(z,a) = \Pi_\infty(A_{y/}) \times_{\Pi_\infty(A_{x/})} \Pi_\infty(A_{z/})$$

and since any category with an initial object is weakly contractible, this yields

$$\mathrm{colim}_{a:A} \mathrm{Hom}_A(y,a) \times_{\mathrm{colim}_{a:A} \mathrm{Hom}_A(x,a)} \mathrm{colim}_{a:A} \mathrm{Hom}_A(z,a) = e$$

where e is the terminal anima. The property that colim_A commutes with pullbacks of animae thus implies that

$$\Pi_\infty(A_{y/} \times_{A_{x/}} A_{z/}) = e\,.$$

As explained in Remark 16.2.2, this proves that A is confluent. □

Definition 16.2.6. A functor is *left exact* if it preserves terminal objects as well as pullback squares.

A functor is *right exact* if it preserves initial objects as well as pushout squares.

A functor is *exact* if it is both left exact and right exact.

Definition 16.2.7. A category is *filtered* if it is weakly contractible and confluent. A category A is *cofiltered* if A^{op} is filtered.

Example 16.2.8. If A is confluent, then, for any object x of A, the co-slice $A_{x/}$ is filtered: the projection $A_{x/} \to A$ being a left fibration, we know that $A_{x/}$ is confluent as well, by Proposition 16.2.4, and any category with an initial object is weakly contractible.

Theorem 16.2.9. *A small category A is filtered if and only if the colimit functor*

$$\operatorname{colim}_A \colon \operatorname{Fun}(A, \mathrm{An}) \to \mathrm{An}$$

is left exact.

Proof. The functor $\operatorname{colim}_A \colon \operatorname{Fun}(A, \mathrm{An}) \to \mathrm{An}$ sends the terminal object of $\operatorname{Fun}(A, \mathrm{An})$ to $\Pi_\infty(A)$. Therefore the property that the A-indexed colimit functor preserves the terminal object is equivalent to the property that A is weakly contractible. Theorem 16.2.5 thus ends the proof. □

Definition 16.2.10. A category A is *sifted* if it is weakly contractible and, for any objects $x, y : A^{\simeq}$, the category $A_{x/} \times_A A_{y/}$ is weakly contractible.

Proposition 16.2.11. *Let A be a weakly contractible small category. The following conditions are equivalent.*

(i) The category A is sifted.

(ii) For any object $x : A^{\simeq}$, the projection

$$A_{x/} \to A$$

is a terminal functor.

(iii) The diagonal functor $\Delta \colon A \to A \times A$ is a terminal functor.

(iv) For any functors $X, Y \colon A \to \mathrm{An}$, the canonical map

$$\operatorname*{colim}_{a:A} X_a \times Y_a \xrightarrow{\sim} \operatorname*{colim}_{a:A} X_a \times \operatorname*{colim}_{a:A} Y_a$$

is an equivalence.

(v) For any objects $x, y, z : A^{\simeq}$, the category $A_{x/} \times_A A_{y/} \times_A A_{z/}$ is weakly contractible.

(vi) For any objects $x, y : A^{\simeq}$, the projection

$$A_{x/} \times_A A_{y/} \to A$$

is a terminal functor.

Proof. Quillen's Theorem A (more precisely, the dual version of Corollary 9.5.8) implies that conditions (i) and (ii) are equivalent. Similarly, it shows that conditions (v) and (vi) are equivalent. Given any object (x, y) in $A \times A$ we have

$$A \times_{A\times A} (A \times A)_{(x,y)/} = A_{x/} \times_A A_{y/} \,.$$

Quillen's Theorem A thus proves that conditions (i) and (iii) are equivalent.

Since A is weakly contractible, it is clear that condition (i) follows from condition (vi).

By Theorem 13.1.4 for $C = \mathrm{An}^{\mathrm{op}}$, we see that condition (iii) implies that, for any functors $X, Y\colon A \to \mathrm{An}$, we have:

$$\operatorname*{colim}_{a:A} X_a \times Y_a \xrightarrow{\sim} \operatorname*{colim}_{(a,b):A\times A} X_a \times Y_b\,.$$

There is furthermore a canonical equivalence

$$\operatorname*{colim}_{(a,b):A\times A} F(a,b) \xrightarrow{\sim} \operatorname*{colim}_{a:A}\operatorname*{colim}_{b:A} F(a,b)$$

for any functor $F\colon A \times A \to \mathrm{An}$. Since the functor $T \times (-)$ commutes with small colimits in An for any small anima T, we also have

$$\operatorname*{colim}_{a:A}\operatorname*{colim}_{b:A} X_a \times Y_b \xrightarrow{\sim} \operatorname*{colim}_{a:A} X_a \times \operatorname*{colim}_{b:A} Y_b\,.$$

Putting all this together shows that condition (iv) follows from condition (iii).

It suffices now to prove that condition (iv) implies condition (v).

Condition (iv) means that the functor

$$\mathrm{colim}_A\colon \mathrm{Fun}(A, \mathrm{An}) \to \mathrm{An}$$

commutes with binary cartesian products. This implies that it commutes with ternary cartesian products. For $x, y, z : A^{\simeq}$, we have

$$\operatorname*{colim}_{a:A} \mathrm{Hom}_A(x,a) \times \mathrm{Hom}_A(y,a) \times \mathrm{Hom}_A(z,a) = \Pi_\infty(A_{x/} \times_A A_{y/} \times_A A_{z/})\,.$$

We also have

$$\operatorname*{colim}_{a:A} \mathrm{Hom}_A(t,a) = \Pi_\infty(A_{t/}) = *$$

for any $t : A^{\simeq}$. Therefore,

$$\operatorname*{colim}_{a:A} \mathrm{Hom}_A(x,a) \times \operatorname*{colim}_{a:A} \mathrm{Hom}_A(y,a) \times \operatorname*{colim}_{a:A} \mathrm{Hom}_A(z,a) = *$$

and thus condition (iv) implies that $A_{x/} \times_A A_{y/} \times_A A_{z/}$ is weakly contractible. □

Example 16.2.12. The category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted, by Corollary 10.4.7.

Corollary 16.2.13. *A category is filtered if and only if it is sifted and confluent.*

Proof. Any left exact functor preserves terminal objects as well as binary cartesian products. Theorem 16.2.5 and Proposition 16.2.11 thus imply that any filtered category is sifted. The converse implication holds for free, since any sifted category is weakly contractible, by definition. □

Proposition 16.2.14. *Let A be a category. Then A is sifted if and only if the following two conditions hold.*

(i) The map $A^{\simeq} \to [0]$ is a cover.

(ii) For any object $x : A^{\simeq}$, the projection $A_{x/} \to A$ is terminal.

Proof. We observe that the canonical functor $A^{\simeq} \to \Pi_\infty(A)$ is always a cover by Corollary 11.2.16. Since any equivalence is a cover, Proposition 16.2.11 shows that these are necessary conditions. Conversely, let us assume that conditions (i) and (ii) above hold. In order to prove that A is sifted, it suffices to prove that A is weakly contractible. For any anima Γ, the localization of A by all its maps in context Γ simply is $\Pi_\infty(A) \times \Gamma$ (with structural map to Γ the second projection). But since $A^{\simeq} \to [0]$ is a cover, the functor $(-) \times A^{\simeq}$ is conservative. Since weak contractibility is a proposition, it therefore suffices to prove it after base change along the cover $A^{\simeq} \to [0]$. In that context, there is a canonical object $x : A^{\simeq}$. The co-slice $A_{x/}$ being weakly contractible, since it has an initial object, and terminal functors being weak equivalences, we observe that A must be weakly contractible. □

Proposition 16.2.15. *A poset P is filtered if and only if it is sifted.*

Proof. Corollary 16.2.13 shows that this is a necessary condition. For the converse, assuming P to be sifted, in order to prove that it is filtered, it suffices to check that it is confluent. Let us consider $x, y, z : P^{\simeq}$ such that $x \leq y$ and $x \leq z$. Since the projection $P_{x/} \to P$ is an embedding by Lemma 10.3.7, the canonical map

$$P_{y/} \times_{P_{x/}} P_{z/} \to P_{y/} \times_P P_{z/}$$

is an equivalence. This proves that P is confluent whenever it is sifted, by Remark 16.2.2. □

Proposition 16.2.16. *The poset ω is filtered.*

Proof. This follows from Corollary 16.1.6, Proposition 16.1.19 and Proposition 16.2.15. □

Corollary 16.2.17. *The formation of sequential colimits is compatible with terminal objects and pullbacks of animae. In other words, the functor*

$$\mathrm{colim}_\omega \colon \mathrm{Fun}(\omega, \mathrm{An}) \to \mathrm{An}$$

is left exact.

Proof. This follows from Theorem 16.2.9 and from the previous proposition. □

Proposition 16.2.18. *Let $u \colon A \to B$ be a terminal functor with A filtered (sifted, respectively). Then B is filtered (sifted, respectively).*

Proof. The functor

$$u^* \colon \mathrm{Fun}(B, \mathrm{An}) \to \mathrm{Fun}(A, \mathrm{An})$$

commutes with small limits and, for any functor $F \colon B \to \mathrm{An}$ there is a functorial isomorphism

$$\mathrm{colim}_A u^*(F) \xrightarrow{\sim} \mathrm{colim}_B F\,.$$

Therefore, any exactness property of colim_A induces a similar property for colim_B. □

Proposition 16.2.19. *Let $p \colon A \to B$ be a functor between small categories. We assume that the following conditions hold:*

(i) p is proper (e.g. a cocartesian fibration);

(ii) B is filtered (sifted, respectively);

(iii) for any $b : B^{\simeq}$, the corresponding fiber A_b is filtered (sifted, respectively).

Then A is filtered (sifted, respectively).

Proof. By Lemma 9.6.5, properness of p gives

$$\operatorname{colim}_{a:A} F(a) \simeq \operatorname{colim}_{b:B} \operatorname{colim}_{x:A_b} F(x)$$

for every $F\colon A \to \mathrm{An}$. Restriction along $A_b \to A$ preserves limits. The hypotheses on B and the fibers of p imply that colim_A is left exact (preserves finite products, respectively). Apply Theorem 16.2.9 (Proposition 16.2.11, respectively). □

Corollary 16.2.20. *Let B be a small filtered (sifted) category and $A\colon B \to \mathrm{Cat}$ a functor such that A_b is filtered (sifted, resp.) for any $b : B^{\simeq}$. Then $\operatorname{colim}_B A$ is filtered (sifted, resp.).*

Proof. We know that $\operatorname{colim}_B A$ is a localization of $\int_B A$. Since any localization is a terminal functor, it suffices to check that $\int_B A$ is filtered (sifted), by Proposition 16.2.18. This follows by applying Proposition 16.2.19 with p the projection of $\int_B A$ to B. □

16.3 Filtered colimits of categories

We fix a universe Cat.

Definition 16.3.1. A functor is said to *commute with small filtered colimits* if it commutes with colimits of type I for any filtered category $I : \mathrm{Cat}^{\simeq}$.

A full subcategory C of a category D is said to be stable under small filtered colimits if the embedding $C \hookrightarrow D$ commutes with filtered colimits.

Example 16.3.2. The functor

$$\lim_{\lrcorner}\colon \mathrm{Fun}(\lrcorner, \mathrm{An}) \to \mathrm{An}\,, \quad (X \to Z \leftarrow Y) \mapsto X \times_Z Y$$

commutes with small filtered colimits, since colim_I is left exact for every small filtered category I.

Lemma 16.3.3. *The full subcategory of synthetic Segal animae is stable under small filtered colimits in* $\mathrm{PSh}(\mathbf{\Delta})$.

Proof. For each $n : \mathbb{N}$, the functor

$$\mathrm{PSh}(\mathbf{\Delta}) \to \mathrm{An}\,, \quad X \mapsto X_n = \mathrm{Hom}(\Delta^n, X)$$

commutes with colimits (being a left adjoint) hence with small filtered ones. This implies that the functor

$$X \mapsto \mathrm{Hom}(\mathrm{Seg}^n, X) = \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$$

commutes with small filtered colimits: by induction on n, this amounts to the property that the formation of pullbacks commutes with small filtered colimits in An (Example 16.3.2). Given a small filtered category I and a functor

$$I \to \mathrm{PSh}(\mathbf{\Delta}) \,, \quad i \mapsto X_i \,,$$

and assuming that each X_i is a synthetic Segal anima, we obtain for each $n : \mathbb{N}$ a commutative square

$$\begin{array}{ccc} \mathrm{colim}_{i:I} \mathrm{Hom}(\Delta^n, X_i) & \xrightarrow{\simeq} & \mathrm{colim}_{i:I} \mathrm{Hom}(\mathrm{Seg}^n, X_i) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Hom}(\Delta^n, \mathrm{colim}_{i:I} X_i) & \longrightarrow & \mathrm{Hom}(\mathrm{Seg}^n, \mathrm{colim}_{i:I} X_i) \end{array}$$

in which the vertical maps are isomorphisms because of the arguments above, whereas the upper horizontal one is an isomorphism as a colimit of isomorphisms. This proves that $\mathrm{colim}_{i:I} X_i$ is a synthetic Segal anima. □

Lemma 16.3.4. *The full subcategory of univalent synthetic Segal animae is stable under small filtered colimits in* $\mathrm{PSh}(\mathbf{\Delta})$.

Proof. The diagram of pullback squares

$$\begin{array}{ccccc} \mathrm{Iso}(X) & \longrightarrow & X_2 \times_{X_1} X_2 & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ & & X_2 \times X_2 & \longrightarrow & X_1 \times X_1 \\ & & \downarrow & & \\ X_0 \times X_0 & \longrightarrow & X_1 \times X_1 & & \end{array}$$

shows that the assignment $X \mapsto \mathrm{Iso}(X)$ may be seen as a functor from $\mathrm{PSh}(\mathbf{\Delta})$ to An that commutes with small filtered colimits. The map of type $X_0 \to \mathrm{Iso}(X)$ sending each $x : X_0$ to (x, x, id_x) is functorial and the full subcategory of those X such that $X_0 \xrightarrow{\sim} \mathrm{Iso}(X)$ through this map is clearly stable under small filtered colimits. Therefore, we conclude the proof with Lemma 16.3.3. □

Proposition 16.3.5. *The Rezk nerve functor* $N \colon \mathrm{Cat} \to \mathrm{PSh}(\mathbf{\Delta})$ *commutes with small filtered colimits.*

Proof. By Corollaries 15.5.5 and 15.1.13, the nerve identifies Cat with the full subcategory of univalent synthetic Segal animae. This subcategory is closed under small filtered colimits by Lemma 16.3.4. □

Definition 16.3.6. Let C be a locally small category with small filtered colimits. An object X of C is *compact* if the functor

$$C \to \mathrm{An} \,, \quad Y \mapsto \mathrm{Hom}_C(X, Y)$$

commutes with small filtered colimits.

Example 16.3.7. Let A be a small category. For any $a : A^{\simeq}$, the associated representable presheaf $h_A(a) = \mathrm{Hom}_A(-, a)$ is compact in $\mathrm{PSh}(A)$: the functor

$$Y \mapsto \mathrm{Hom}(h_A(a), Y) \simeq Y(a)$$

commutes with small colimits, since colimits of presheaves are computed objectwise. Hence it commutes with filtered colimits in particular.

Proposition 16.3.8. *If C is a locally small category with small filtered colimits and pushouts, then the full subcategory of compact objects is stable under pushouts.*

Proof. Mapping out of a pushout gives a pullback of hom animae. Since filtered colimits commute with pullbacks in An, a pushout of compact objects is compact. □

Theorem 16.3.9. *For any $n : \mathbb{N}$, the category $[n]$ is a compact object of* Cat.

Proof. The nerve $N([n]) = \Delta^n$ is a representable presheaf, hence compact by Example 16.3.7. Since N is fully faithful and preserves small filtered colimits (Corollary 15.1.13 and Proposition 16.3.5), $[n]$ is compact in Cat. □

Corollary 16.3.10. *The functor*

$$\mathrm{Cat} \to \mathrm{An}\,, \quad C \mapsto C^{\simeq}$$

commutes with small filtered colimits.

Proof. The preceding theorem tells us that $[0]$ is compact and $C^{\simeq} = \mathrm{Hom}([0], C)$. □

Theorem 16.3.11. *For any small filtered category A, the functor*

$$\mathrm{colim}_A \colon \mathrm{Fun}(A, \mathrm{Cat}) \to \mathrm{Cat}$$

is left exact.

Proof. The nerve is fully faithful and preserves limits and small filtered colimits (Corollary 15.1.13 and Proposition 16.3.5). It therefore suffices to prove the assertion in $\mathrm{PSh}(\mathbf{\Delta})$. There, limits and colimits are computed objectwise, so the assertion follows from the left exactness of filtered colimits in An. □

Corollary 16.3.12. *Let A be a small filtered category, and $C, D \colon A \to \mathrm{Cat}$ two functors. We assume given a natural transformation $u_a \colon C_a \to D_a$, $a : A$, such that u_a is an embedding (a full embedding) for all $a : A^{\simeq}$. Then the induced functor*

$$\mathrm{colim}_A C \to \mathrm{colim}_A D$$

is an embedding (a full embedding, respectively).

Proof. By Theorem 16.3.11, the functor colim_A is left exact. Any left exact functor preserves pullback squares of the form

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow f \\ X & \xrightarrow{f} & Y, \end{array}$$

hence preserves embeddings. For full embeddings, use the characterization of fully faithful functors in Remark 6.3.2. Its defining square is preserved because colim_A is left exact and, by Theorem 16.3.9, the functors $\mathrm{Hom}_{\mathrm{Cat}}([1], -)$ and $(-)^{\simeq} = \mathrm{Hom}_{\mathrm{Cat}}([0], -)$ commute with filtered colimits. □

Remark 16.3.13 (Morphisms in a filtered colimit of categories)**.** Let A be a small filtered category and $C\colon A \to \mathrm{Cat}$ a functor. One can understand the colimit of C as follows. Objects are easy to understand by Corollary 16.3.10:

$$\operatorname*{colim}_{a:A} C_a^{\simeq} = \Big(\operatorname*{colim}_{a:A} C_a\Big)^{\simeq}.$$

Assume given $a_0, a_1 : A^{\simeq}$ and two objects $x_{a_0} : C_{a_0}^{\simeq}$ and $y_{a_1} : C_{a_1}^{\simeq}$. We will write x and y for the object of $\mathrm{colim}_A C$ obtained as the images of x_{a_0} and y_{a_1} through the canonical functors

$$C_{a_0} \to \mathrm{colim}_A C \quad \text{and} \quad C_{a_1} \to \mathrm{colim}_A C$$

respectively.

For any map $u\colon a_0 \to a$ in A, we will write $x_a : C_a^{\simeq}$ for the image of x_{a_0} through the functor induced by u. Similarly, for any map $v\colon a_1 \to a$ in A, we will write $y_a : C_a^{\simeq}$ for the image of y_{a_1} through the functor induced by v.

We observe that the projection $A_{a_0/} \times_A A_{a_1/} \to A$ is terminal (since A is sifted, using Proposition 16.2.11). The terms of $A_{a_0/} \times_A A_{a_1/}$ literally are of the form (a, u, v) with $a : A$ together with $u\colon a_0 \to a$ and $v\colon a_1 \to a$ in A. In other words, we have an equivalence

$$\operatorname*{colim}_{(a,u,v):A_{a_0/}\times_A A_{a_1/}} C_a \xrightarrow{\sim} \operatorname*{colim}_{a:A} C_a\,.$$

The category $A_{a_0/} \times_A A_{a_1/}$ is filtered as well: it is weakly contractible, and it is confluent, since the projection to A is a left fibration, so that we can apply Proposition 16.2.4. Therefore, the colimit of the $A_{a_0/} \times_A A_{a_1/}$-indexed diagram of cartesian squares

$$(a,u,v) \mapsto \begin{array}{ccc} \mathrm{Hom}_{C_a}(x_a, y_a) & \longrightarrow & \mathrm{Hom}([1], C_a) \\ \downarrow & \lrcorner & \downarrow \\ [0] & \longrightarrow & C_a^{\simeq} \times C_a^{\simeq} \end{array}$$

yields a cartesian square after applying the colimit functor. Using Theorem 16.3.9 for $n = 1$ and $n = 0$, this produces a canonical equivalence of the form

$$\operatorname*{colim}_{(a,u,v):A_{a_0/}\times_A A_{a_1/}} \mathrm{Hom}_{C_a}(x_a, y_a) = \mathrm{Hom}_{\mathrm{colim}_A C}(x, y)\,,$$

expressing morphisms from x to y as a filtered colimit.

16.4 Finite categories

We will construct the universe of finite categories. We fix a universe Cat.

Construction 16.4.1. Given a full embedding $i\colon C \hookrightarrow \mathrm{Cat}$, we consider the functor

$$\mathrm{colim}_{\ulcorner} :\ \mathrm{Fun}(\ulcorner, \mathrm{Cat}) \to \mathrm{Cat}$$

and define $E(C)^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}$ as the image of the induced map

$$\mathrm{Fun}(\ulcorner, C)^{\simeq} \xrightarrow{i^{\simeq}} \mathrm{Fun}(\ulcorner, \mathrm{Cat})^{\simeq} \xrightarrow{\mathrm{colim}^{\simeq}_{\ulcorner}} \mathrm{Cat}^{\simeq}\,.$$

We define $E(C)$ as the full subcategory of Cat with collection of objects $E(C)^{\simeq}$, with corresponding embedding

$$i_{E(C)}\colon E(C) \hookrightarrow \mathrm{Cat}\,.$$

Since the pushout of any diagram of the form $x \xleftarrow{\mathrm{id}_x} x \xrightarrow{\mathrm{id}_x} x$ is x itself, the functor $\mathrm{const}_{\ulcorner} : C \to \mathrm{Fun}(\ulcorner, C)$ induces a canonical functor

$$e_C\colon C \to E(C)$$

such that $i_{E(C)} \circ e_C \simeq i$. We observe that, by virtue of Proposition 11.2.18, if C is small, then so is $E(C)$ (as full subcategory of a locally small category with a small collection of objects).

We define C_0 as the full subcategory of Cat with objects $\emptyset$, $[0]$ and $[1]$, with

$$i_0\colon C_0 \hookrightarrow \mathrm{Cat}$$

the embedding. We define for any $n : \mathbb{N}$

$$e_n\colon C_n \hookrightarrow C_{n+1}$$

through $C_{n+1} = E(C_n)$ and $e_n = e_{C_n}$. There is also a full embedding

$$i_{n+1}\colon C_{n+1} \to \mathrm{Cat}$$

defined by $i_{n+1} = i_{E(C_n)}$. The sequence of maps e_n, $n : \mathbb{N}$, define a functor $C_\bullet\colon \omega \to \mathrm{Cat}$.

Definition 16.4.2. We define the category FinCat of *finite categories* as the colimit

$$\mathrm{FinCat} = \operatorname*{colim}_{n:\omega} C_n\,.$$

Remark 16.4.3. It follows from Corollary 16.3.12 that FinCat is the full subcategory of Cat induced by the embedding

$$\operatorname*{colim}_{n:\omega} C_n^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}\,.$$

Since ω as well as each C_n are small, so is FinCat.

Lemma 16.4.4. *The category $\ulcorner$ is a finite category.*

Proof. We know that $[n]$ belongs to C_0 for $n = 0$ and $n = 1$ and we have a pushout square of the form below.

$$\begin{array}{ccc} [0] & \longrightarrow & [1] \\ \downarrow & & \downarrow \\ [1] & \longrightarrow & \ulcorner \end{array}$$

This shows that $\ulcorner$ belongs to C_1. □

Proposition 16.4.5. *Any finite category is compact in* Cat.

Proof. The generators $[0]$ and $[1]$ are compact by Theorem 16.3.9. The initial category is compact because filtered categories are weakly contractible. By Proposition 16.3.8, induction shows that every object of C_n is compact. Any object of FinCat belongs to some C_n after base change along a cover (Corollary 13.2.2). Compactness is a proposition, so it descends along this cover. □

Lemma 16.4.6. *The category* FinCat *is the smallest full subcategory of* Cat *stable under pushouts containing the objects* $\emptyset$, $[0]$, *and* $[1]$. *In other words, given a full subcategory C of* Cat, *if the embedding $C \hookrightarrow$ Cat is a right exact functor, then the following conditions are equivalent.*

(i) The embedding FinCat $\hookrightarrow$ Cat *factors through the embedding $C \hookrightarrow$ Cat.*

(ii) The embedding $\mathrm{FinCat}^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}$ *factors through the embedding* $C^{\simeq} \hookrightarrow \mathrm{Cat}^{\simeq}$.

(iii) The objects $[0]$ *and* $[1]$ *belong to C.*

Proof. By Lemma 16.4.4 and Proposition 16.4.5, $\ulcorner$ is compact in Cat. Hence there is an equivalence

$$\operatorname*{colim}_{n:\omega} \mathrm{Fun}(\ulcorner, C_n)^{\simeq} \xrightarrow{\sim} \mathrm{Fun}(\ulcorner, \mathrm{FinCat})^{\simeq}.$$

Thus, after base change along the canonical cover, any functor $F\colon \ulcorner \to \mathrm{FinCat}$ factors through some C_n. In that context, the pushout of F in Cat belongs to C_{n+1}, hence to FinCat. Since belonging to the full subcategory FinCat $\hookrightarrow$ Cat is a proposition, this cover-local conclusion descends. Therefore, FinCat is stable under pushouts in Cat.

Conditions (i) and (ii) are equivalent by the universal property of full embeddings, and (i) implies (iii). Conversely, right exactness and (iii) ensure that C contains C_0. Since C is closed under pushouts, induction gives $C_n \subset C$ for all n. Taking the colimit of these full embeddings gives FinCat $\hookrightarrow C$ by Corollary 16.3.12 because $\mathrm{colim}_{\omega} C = C$ by weak contractibility of ω (Corollary 16.1.6). □

Proposition 16.4.7. *The opposite of a finite category is finite: for any C :* FinCat, *we have* C^{op} : FinCat.

Proof. The functor $(-)^{\mathrm{op}}$ preserves pushouts and carries each of $\emptyset$, $[0]$, and $[1]$ to an equivalent category. Hence the full subcategory of categories whose opposites are finite contains these generators and is closed under pushouts. Apply Lemma 16.4.6. □

Proposition 16.4.8. *If A and B are finite categories, then so is $A \times B$.*

Proof. Products preserve colimits in each variable. Applying Lemma 16.4.6 successively in the two variables therefore reduces the assertion to products of $\emptyset$, $[0]$, and $[1]$ with one another. Only $[1] \times [1]$ needs consideration: it is the pushout

$$[2] \sqcup_{[1]} [2],$$

where both maps $[1] \to [2]$ select the long edge. Moreover, $[2]$ is finite because $[2] = [1] \sqcup_{[0]} [1]$. Thus all products of generators are finite, as required. □

Corollary 16.4.9. *For any finite category A, the functor*

$$\mathrm{Fun}(A, -) \colon \mathrm{Cat} \to \mathrm{Cat}$$

commutes with small filtered colimits.

Proof. Let I be a small filtered category and $C \colon I \to \mathrm{Cat}$ a functor. In order to prove that the canonical functor

$$\operatorname*{colim}_{i:I} \mathrm{Fun}(A, C_i) \to \mathrm{Fun}(A, \operatorname*{colim}_{i:I} C_i)$$

is an equivalence, it suffices to check that the induced map

$$\mathrm{Hom}_{\mathrm{Cat}}([1], \operatorname*{colim}_{i:I} \mathrm{Fun}(A, C_i)) \to \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Fun}(A, \operatorname*{colim}_{i:I} C_i))$$

is an equivalence (by Corollary 6.3.9, taking into account the correspondence between Map and $\mathrm{Hom}_{\mathrm{Cat}}$ through directed univalence). Since $[1]$ and $[1] \times A$ are compact, we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}}([1], \operatorname*{colim}_{i:I} \mathrm{Fun}(A, C_i)) &= \operatorname*{colim}_{i:I} \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Fun}(A, C_i)) \\
&= \operatorname*{colim}_{i:I} \mathrm{Hom}_{\mathrm{Cat}}([1] \times A, C_i) \\
&= \mathrm{Hom}_{\mathrm{Cat}}([1] \times A, \operatorname*{colim}_{i:I} C_i) \\
&= \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Fun}(A, \operatorname*{colim}_{i:I} C_i)) \,.
\end{aligned}$$

□

Proposition 16.4.10. *The embedding* $\mathrm{FinCat} \hookrightarrow \mathrm{Cat}$ *commutes with pushouts.*

Proof. By Lemma 16.4.6, a pushout in Cat of finite categories is finite. Since FinCat is a full subcategory, this is also a pushout in FinCat. □

Corollary 16.4.11. *If A and B are finite categories, then so is $A \star B$.*

Proof. There is a pushout of the form below.

$$\begin{array}{ccc}
A \times [1]^{\simeq} \times B & \hookrightarrow & A \times [1] \times B \\
\downarrow & & \downarrow \\
A \sqcup B & \hookrightarrow & A \star B
\end{array}$$

The previous proposition together with Proposition 16.4.8 thus imply that $A \star B$ is finite. □

Proposition 16.4.12. *Let C be a category. If C has an initial (terminal) object as well as pushouts (pullbacks), then C has colimits (limits, resp.) of type A for any finite category A.*

Proof. Since C has an initial object, it has colimits of type $\emptyset$. It also has colimits of type $[0]$ and $[1]$ (more generally, of type A for any A with a terminal object). The full subcategory of Cat spanned by categories A such that C has A-indexed colimits is stable under pushouts, by Corollary 13.3.2. This full subcategory also contains C_0. Therefore, it contains FinCat as well, by Lemma 16.4.6. □

Proposition 16.4.13. *Let $f \colon C \to D$ be a functor. We assume that C and D both have initial (terminal) objects and pushouts (pullbacks). If f is right (left) exact then it commutes with colimits (limits) of type A for any finite category A.*

Proof. We will consider the case of finite colimits (the case of finite limits will follow by replacing C by C^{op}).

The full subcategory T of Cat spanned by categories A such that f commutes with A-indexed colimits is stable under pushouts. Indeed, if $A = A_1 \sqcup_{A_0} A_2$ and f commutes with colimits indexed by A_0, A_1 or A_2, for any functor $X\colon A \to C$, if we denote by $X_{|E}$ the composition of X with the structural functor $E \to A$ for $E = A_0, A_1, A_2$, then we have canonical isomorphisms

$$\mathrm{colim}_E \ fX_{|E} \xrightarrow{\sim} f(\mathrm{colim}_E \ X_{|E})\,.$$

If f is right exact, a double application of Corollary 13.3.2 yields isomorphisms

$$\begin{aligned}
\mathrm{colim}_A \, fX &\simeq (\mathrm{colim}_{A_1} \ fX_{|A_1}) \sqcup_{(\mathrm{colim}_{A_0} \ fX_{|A_0})} (\mathrm{colim}_{A_2} \ fX_{|A_2}) \\
&\simeq f(\mathrm{colim}_{A_1} \ X_{|A_1}) \sqcup_{f(\mathrm{colim}_{A_0} \ X_{|A_0})} f(\mathrm{colim}_{A_2} \ X_{|A_2}) \\
&\simeq f\big((\mathrm{colim}_{A_1} \ X_{|A_1}) \sqcup_{(\mathrm{colim}_{A_0} \ X_{|A_0})} (\mathrm{colim}_{A_2} \ X_{|A_2})\big) \\
&\simeq f(\mathrm{colim}_A \ X)\,.
\end{aligned}$$

If f preserves initial objects, this full subcategory T contains $\emptyset$. It clearly contains $[0]$ and $[1]$. Therefore, by Lemma 16.4.6, T contains all finite categories. □

Recall that an anima X is finitely covered if there merely exists a natural number n and a cover $\{1, \dots, n\} \to X$. If X is a set we also also call such choice a *finite enumeration* (see Definition 11.6.2).

Definition 16.4.14. Starting from $A_0 \subseteq \mathrm{An}$ the full subcategory spanned by $\emptyset$ and $[0]$, define by $A_{n+1} \subseteq \mathrm{An}$ to be full subcategory of those animae that can merely be obtained by forming a pushout in A_n (analogously to Construction 16.4.1).

An anima A is called *finite* if lies in the full subcategory $\mathrm{FinAn} \coloneqq \mathrm{colim}_{n:\omega} A_n$.

Lemma 16.4.15. *Let C be a finite category.*

(1) *The anima $\Pi_\infty A$ is a finite anima.*

(2) *The anima $C^{\simeq}$ is finitely covered.*

Warning 16.4.16. For a finite category C, the anima $C^{\simeq}$ need not be finite again. Here is the sketch of an example.

Consider a finitely presented group $G = \langle g_1, \dots, g_p \mid r_1, \dots, r_q \rangle$ with a finitely generated but not finitely presented subgroup $H = \langle h_1, \dots, h_n \rangle \triangleleft G$. From this data, build the following finite category.

- Start with the walking retract $R = \{s : 0 \triangleleft 1 : r\}$, i.e., the pushout

$$\begin{array}{ccc}
[1] & \xrightarrow{02} & [2] \\
\downarrow & \ulcorner & \downarrow \\
[0] & \longrightarrow & R
\end{array} \tag{16.17}$$

  (where s is the image of $0 \to 1$ and r the image of $1 \to 2 = 0$) Denote by $e \coloneqq sr\colon 1 \to 1$ the nontrivial idempotent.

- For each generator g_i, attach an automorphism $g_i \colon 1 \xrightarrow{\sim} 1$, and for each relation r_j an identification between the corresponding composites.
- For each generator h_k of H attach an identification $u_k \colon h_k e = e h_k$.

The resulting category C is finite by construction. However, the core $C^{\simeq}$ is not a finite anima. Indeed, every finite anima has finitely presented fundamental groups, but one can compute that $\pi_1(C^{\simeq}, 0) \cong H$, which was not finitely presented by assumption.

Proof. (1) By induction on n we see that the geometric realization functor $\Pi_\infty \colon \mathrm{Cat} \to \mathrm{An}$ restricts to a functor $\Pi_\infty \colon C_n \to A_n$. Indeed, it fixes $\emptyset$ and $[0]$ and sends $[1] \mapsto [0]$ and preserves pushouts. Taking colimits, it restricts to a functor $\Pi_\infty \colon \mathrm{FinCat} \to \mathrm{FinAn}$.

(2) Clearly the cores of $\emptyset$, $[0]$ and $[1]$ are finitely covered (even finite) sets. Hence by Lemma 16.4.6 it suffices to show that categories with finitely covered cores are closed under pushouts. Consider a functor $X \colon \ulcorner \to \mathrm{Cat}$ corresponding to a diagram $C \leftarrow B \to D$ of categories and assume that we have finite covers $S \to B$, $T \to C$ and $U \to D$ (we may choose them because we are proving a mere existence statement). By the computation of colimits of categories (Theorem 13.2.1), we can compute the pushout $C \amalg_B D$ as a localization of the category $\int_{\ulcorner} X$ whose core is covered by the finite set $S \amalg T \amalg D$. Since the core of a category always covers any of its localizations (Corollary 11.2.16), we see that $C \amalg_B D$ is finitely covered.

□

Corollary 16.4.17. *An anima A is finite (as an anima) if and only if it is a finite category, i.e., we have* $\mathrm{FinAn} = \mathrm{FinCat} \cap \mathrm{An}$ *as full subcategories of* Cat.

Proof. "Only if" is clear because $A_n \subseteq C_n$ for each n, and "if" follows from Lemma 16.4.15 because $A = \Pi_\infty A$. □

Definition 16.4.18. Let P be a poset. We call P a *finite poset* if the following hold:

(1) There merely exists a finite enumeration of the set $P^{\simeq}$;

(2) Its order is decidable, in the sense that there is a term of

$$\prod_{x,y:P^{\simeq}} \big(P(x,y) \sqcup \neg P(x,y)\big) .$$

Remark 16.4.19. By Proposition 11.6.3, a set S is finite (in the sense of Definition 11.6.1) if and only if S is finite as a poset.

Lemma 16.4.20. *If P is a finite poset, then its core $P^{\simeq}$ is a finite set.*

Proof. The order on P being decidable, equality in $P^{\simeq}$ is decidable as well: antisymmetry identifies $x = y$ with $P(x,y) \times P(y,x)$; in other words, $P^{\simeq}$ is a finite poset. The claim thus follows from Remark 16.4.19 applied to the set $P^{\simeq}$. □

Lemma 16.4.21. *Finite posets are closed under pushouts of posets.*

Proof. Consider a diagram $Q \xleftarrow{f} P \xrightarrow{g} R$ of finite posets. We aim to show that the pushout $Q \amalg_P R$ (in posets) is finite again.

Let us temporarily fix cardinal presentations of $P^{\simeq}, Q^{\simeq}, R^{\simeq}$ of size $l, m, n : \mathbb{N}$, respectively. Using these presentations, we perform the following construction steps.

- Define a reflexive and transitive relation $\leq$ (preorder) on the set $X \coloneqq P^{\simeq} \amalg Q^{\simeq} \amalg R^{\simeq}$ by starting with the coproduct order inherited from P, Q and R and imposing

  $$p \leq f(p) \leq g(p) \leq p$$

  for each $p : P^{\simeq}$.

- Define the equivalence relation on X generated by $(x \sim y) \coloneqq (x \leq y) \times (y \leq x)$.

- Both the preorder $\leq$ and the equivalence relation $\sim$ can be decided by a bounded number of equalities and/or order relations in P, Q and R, each of which is decidable; hence both $\leq$ and $\sim$ are decidable.

- Using the chosen cardinal presentations we can construct a decidable subset $Y \subseteq X$ which intersects each $\sim$-equivalence class exactly once. For example, one could use the bijection $X \cong \{1, \dots, l\} \amalg \{1, \dots m\} \amalg \{1, \dots n\} \cong \{1, \dots l + m + n\}$ to define a decidable linear order o on X and then define Y to consist of those elements of X that are o-minimal in their $\sim$-equivalence class.

- Denote by $u \colon X \to Y$ the map of sets that sends each $x \in X$ to the unique $y \in Y$ with $x \sim y$. Moreover, we have a square

  $$\begin{array}{ccc} P^{\simeq} & \xrightarrow{f} & Q^{\simeq} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u|_{Q^{\simeq}}} \\ R^{\simeq} & \xrightarrow{u|_{R^{\simeq}}} & Y \end{array} \tag{16.18}$$

  of sets which commutes because $f(p) \sim p \sim g(p)$ for all $p \in P$ and u identifies $\sim$-related terms by construction.

- By construction, the preorder $\leq$ is antisymmetric when restricted to Y; hence we have a partial order on Y. Passing through Theorem 10.3.10, this yields a poset T with $(T^{\simeq}, \leq) = (Y, \leq)$.

- By construction of the order and Lemma 10.3.7, the commutative square above has a unique upgrade to a commutative square of posets

  $$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u|_{Q}} \\ R & \xrightarrow{u|_{R}} & T \end{array} \tag{16.19}$$

  Showing that this square is a pushout of posets is a straightforward computation involving maps of underlying sets and the order relation.

- By construction, the poset T has again a finite set of objects (Y was defined as a decidable subset of $P^{\simeq} \amalg Q^{\simeq} \amalg R^{\simeq}$) and decidable order.

- Since the pushout $Q \amalg_P R$ (in posets) is isomorphic to T, it too is a finite poset.

Of course the construction of T itself heavily depends on the chosen cardinal presentations of $P^{\simeq}$, $Q^{\simeq}$ and $R^{\simeq}$. But the final conclusion, namely that $Q \amalg_P R$ is finite, is a proposition; hence we may discharge the propositional truncation, and conclude the proof. □

Corollary 16.4.22.

- *The poset truncation $\|C\|_{\mathrm{pos}}$ of a finite category C is a finite poset.*
- *The set truncation $\|A\|_0$ of a finite anima A is a finite set.*

Proof. The first statement follows from Lemma 16.4.21 because $\emptyset$, $[0]$ and $[1]$ are finite posets and $\|-\|_{\mathrm{pos}}$ sends pushouts of categories to pushouts of posets.

The second statement follows from the first, because the poset truncation of an anima is its set truncation. □

Proposition 16.4.23. *Let P be a poset. Then P is finite as a poset if and only if it is a finite category.*

Proof. The statement "if" follows directly from Corollary 16.4.22, so it remains to show "only if". Since the assertion that P is a finite category is a proposition, we may fix an isomorphism $\{1, \dots, n\} \cong P^{\simeq}$ to do so.

We first observe, by induction on n, that every nonempty poset with decidable order and n objects has a maximal object. Indeed, remove one object p. If no object remains, then p is maximal. Otherwise, choose by induction a maximal object m of the remaining poset and decide whether $m \leq p$. If $m \leq p$, then p is maximal; if not, then m remains maximal after adjoining p. For the first assertion, an object of the remaining poset above p would also be above m, and hence would equal m; antisymmetry would then identify it with p. The second assertion follows because the only new possible object above m is p.

We now prove the proposition by strong induction on n. The case $n = 0$ is clear. Suppose that P has $n + 1$ objects and choose a maximal object m. Let Q be the full subposet on the objects distinct from m, let $P_{\leq m}$ be the full subposet on the objects $x \leq m$, and put $P_{<m} = Q \cap P_{\leq m}$. The poset Q has n objects. Since the predicate $x \leq m$ is decidable, filtering a cardinal presentation of Q gives a cardinal presentation of $P_{<m}$ with at most n objects. Both Q and $P_{<m}$ are therefore finite categories by the induction hypothesis. Moreover,

$$P_{\leq m} = P_{<m} \star [0]\,,$$

so $P_{\leq m}$ is finite by Corollary 16.4.11.

It remains to verify that the square

$$\begin{array}{ccc} P_{<m} & \hookrightarrow & P_{\leq m} \\ \downarrow & & \downarrow \\ Q & \hookrightarrow & P \end{array}$$

is a pushout in Cat. Every chain $[k] \to P$ lies in Q or in $P_{\leq m}$: if it contains m, maximality forces all its vertices to lie below m. A chain lies in both subposets precisely when it lies in $P_{<m}$. Since equality with m is decidable, this exhibits each nerve level as a union of two subsets with the stated

intersection. Applying the Rezk nerve therefore gives a levelwise pushout square. Mapping this levelwise pushout into the nerve of an arbitrary category, and using the full faithfulness of the nerve from Corollary 15.1.13, gives the pushout universal property in Cat. Since finite categories are closed under pushouts by Lemma 16.4.6, P is finite, as required. □

17 Small subuniverses and presentability

In this chapter, we fix once and for all a universe Cat of small categories as well as a universe of large categories Cat′ (so that any small category is large and Cat itself is large).

17.1 Left exact localizations

We consider a category C.

Definition 17.1.1. We say that a family of morphisms $F \hookrightarrow \mathrm{Map}([1], C)$ is *closed under pullbacks* if the following property is verified: for any map $f : x \to y$ in F, and for any map $v : y' \to y$ in C, there exists a cartesian square of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y \end{array}$$

such that f' belongs to F.

If $W \hookrightarrow C$ is a subcategory of C, we say that W is closed under pullbacks in C if the associated family of morphisms $\mathrm{Map}([1], W)$ is closed under pullbacks in C.

Lemma 17.1.2. *Assume that F is a family of morphisms in C that is* closed under pullbacks. *For any map $v : y' \to y$ in F, and any object x of C, the induced functor*

$$v_! \times_C \mathrm{id}_{C_{/x}} : C_{/y'} \times_C C_{/x} \to C_{/y} \times_C C_{/x}$$

has a right adjoint, which we can describe as follows. It sends an object (z, s, f) of $C_{/y} \times_C C_{/x}$, seen as a diagram

$$\begin{array}{ccc} & z & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\ x & & y \end{array}$$

to the object (z', s', f') obtained by forming the cartesian square

$$\begin{array}{ccc} z' & \xrightarrow{w} & z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y \end{array}$$

and by choosing a composition s' of s and w.

Proof. Since the Yoneda embedding is fully faithful, continuous, and compatible with slices, we see that proving the lemma for $\mathrm{PSh}(C)$ implies that it holds for C itself. In other words, we may assume, without loss of generality, that C has finite products and pullbacks. But then we have

$$C_{/y} \times_C C_{/x} = C_{/y \times x}$$

and we observe that pulling back along $v \times \mathrm{id}_x$ in C produces the expected right adjoint. □

Proposition 17.1.3. *If the subcategory $W \hookrightarrow C$ is closed under pullbacks in C, for any object $x : C^{\simeq}$, forming the pullback square*

$$\begin{array}{ccc} W(x)^{\simeq} & \xhookrightarrow{p^{\simeq}} & (C_{/x})^{\simeq} \\ \downarrow & & \downarrow \\ \mathrm{Fun}([1], W)^{\simeq} & \hookrightarrow & \mathrm{Fun}([1], C)^{\simeq} \end{array}$$

defines a full embedding $p \colon W(x) \to C_{/x}$ that is a right calculus of fractions at x.

Proof. Let $v \colon y' \to y$ be a map in W. With the notation of the previous lemma, we have a functor

$$v_! \colon C_{/y'} \to C_{/y}$$

which encodes the operation of composing with v, and we know that the induced functor

$$v_! \times_C \mathrm{id}_{C_{/x}} \colon C_{/y'} \times_C C_{/x} \to C_{/y} \times_C C_{/x}$$

has a right adjoint by Lemma 17.1.2. The explicit description of the latter shows that, when we restrict ourselves to the subcategories in the diagram

$$v_! \times_C \mathrm{id}_{W(x)} \colon C_{/y'} \times_C W(x) \to C_{/y} \times_C W(x)\,,$$

we still get a functor with a right adjoint. But functors with right adjoints are initial, hence are weak homotopy equivalences, by Quillen's Theorem A. Applying Π_∞ yields the map induced by applying the functor

$$y \mapsto \operatorname*{colim}_{w : W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, -)$$

to v. □

Definition 17.1.4. We say that $W \hookrightarrow C$ has the 2-*out-of*-3 *property* if, for any commutative triangle in C of the form

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & z & \end{array}$$

if two out of f, g and h are in W, then so is the third.

Theorem 17.1.5. *Let C be a category and $W \hookrightarrow C$ a subcategory. We denote by $\gamma \colon C \to L(C)$ the localization of C by W. We assume that the category C has pullbacks and that the subcategory W is closed under pullbacks and has the* 2-*out-of*-3 *property. Then the following properties hold.*

(i) For any object x of C, the full subcategory $W(x)$ of $C_{/x}$ spanned by pairs (y, f) such that $f\colon y \to x$ is a morphism in W is a right calculus of fractions at x.

(ii) For any object x of C, the category $W(x)^{\mathrm{op}}$ is filtered.

(iii) The category $L(C)$ has pullbacks and the functor γ commutes with pullbacks.

(iv) Given any category D with pullbacks and any functor $f\colon C \to D$ sending any map in W to an isomorphism in D, the induced functor $\bar{f}\colon L(C) \to D$ commutes with pullbacks if and only if the functor f itself has the same property.

(v) For any object x of C, the functor γ induces a functor

$$\gamma_{/x}\colon C_{/x} \to L(C)_{/\gamma(x)}$$

that exhibits $L(C)_{/\gamma(x)}$ as the localization of $C_{/x}$ at the subcategory $W(x) \hookrightarrow C_{/x}$.

Proof. Property (i) is a particular case of Proposition 17.1.3. Let x be an object of C. We check immediately that the collection of objects in $W(x)$ is stable under pullbacks in $C_{/x}$. Therefore, $W(x)$ has pullbacks, which implies right away that $W(x)^{\mathrm{op}}$ is confluent (see Example 16.2.3). But $W(x)^{\mathrm{op}}$ has an initial object, hence is weakly contractible, which shows property (ii).

In order to establish (iii), we want to prove that, for each pair of morphisms

$$x \xrightarrow{f} y \xleftarrow{v} y'$$

in $L(C)$, there exists a pullback square in $L(C)$ of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y \end{array}$$

in $L(C)$. It suffices to prove such a statement in context

$$\Gamma = \sum_{x,y,y' : L(C)^{\simeq}} L(C)(x,y) \times L(C)(y',y)$$

for the universal term $\gamma : \Gamma$ (in context Γ). Since

$$\gamma^{\simeq}\colon C^{\simeq} \to L(C)^{\simeq}$$

is a cover by Corollary 11.2.16, so is

$$\gamma^{\simeq} \times \gamma^{\simeq} \times \gamma^{\simeq}\colon C^{\simeq} \times C^{\simeq} \times C^{\simeq} \to L(C)^{\simeq} \times L(C)^{\simeq} \times L(C)^{\simeq}\,.$$

We thus have a cover

$$\sum_{x,y,y' : C^{\simeq}} L(C)(\gamma(x),\gamma(y)) \times L(C)(\gamma(y'),\gamma(y)) \to \Gamma\,.$$

In other words, it suffices to prove the existence of pullbacks in the case where all objects x, y, y' come from C. Furthermore, for all objects x and y in C, by virtue of Corollary 11.2.16, we have a cover of the form

$$(W(x)_{/y})^{\simeq} \to \Pi_\infty(W(x)_{/y}) = L(C)(\gamma(x), \gamma(y)) .$$

This yields a cover

$$\sum_{x,y,y' : C^{\simeq}} (W(x)_{/y})^{\simeq} \times (W(y')_{/y})^{\simeq} \to \Gamma .$$

This means that it suffices to prove that the formation of pullbacks from diagrams as above is possible in $L(C)$ in the case where f and v are of the form

$$f = \gamma(s)^{-1} \circ \gamma(p) \quad \text{and} \quad v = \gamma(t)^{-1} \circ \gamma(q) \quad \text{respectively,}$$

with $p : x_0 \to y$ and $q : y'_0 \to y$ any maps in C, and $s : x_0 \to x$ and $t : y'_0 \to y'$ any maps in W. Therefore, it suffices to check that any pullback square in C of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y \end{array}$$

in C yields a pullback square of the form

$$\begin{array}{ccc} \gamma(x') & \xrightarrow{\gamma(u)} & \gamma(x) \\ \downarrow{\scriptstyle \gamma(f')} & & \downarrow{\scriptstyle \gamma(f)} \\ \gamma(y') & \xrightarrow{\gamma(v)} & \gamma(y) \end{array}$$

in $L(C)$. For this, it suffices to prove that, given any object t in $L(C)$, we have a pullback square of the form

$$\begin{array}{ccc} \mathrm{Hom}_{L(C)}(t, x') & \xrightarrow{u_*} & \mathrm{Hom}_{L(C)}(t, x) \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Hom}_{L(C)}(t, y') & \xrightarrow{v_*} & \mathrm{Hom}_{L(C)}(t, y) \end{array}$$

in An. It suffices to prove this for the universal term t in context $L(C)^{\simeq}$. Using the cover $C^{\simeq} \to L(C)^{\simeq}$, we may assume, without loss of generality, that t comes from C. In other words, in order to prove property (iii), it suffices to prove that any functor of type $C \to \mathrm{An}$ given by the rule

$$x \mapsto \mathrm{Hom}_{L(C)}(t, x)$$

commutes with pullbacks for any $t : C^{\simeq}$. But property (i) ensures that such a functor can be computed through the rule of the form

$$x \mapsto \operatorname*{colim}_{w : W(t)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, x) .$$

Since each functor $\mathrm{Hom}_C(z_w, -)$ commutes with pullbacks, and since $W(t)^{\mathrm{op}}$ is filtered, we see that, for any finite category A and any A-indexed functor with values in C

$$a \mapsto x_a$$

the canonical map

$$\operatorname*{colim}_{w:W(t)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, \lim_{a:A} x_a) = \operatorname*{colim}_{w:W(t)^{\mathrm{op}}} \lim_{a:A} \mathrm{Hom}_C(z_w^{\mathrm{op}}, x_a) \to \lim_{a:A} \operatorname*{colim}_{w:W(t)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, x_a)$$

is an equivalence, by Theorem 16.2.9 and Proposition 16.4.13.

The proof of property (iv) is similar. First, if $\bar{f}$ preserves pullback squares, since γ preserves pullback squares as well, so does the composite $f = \bar{f} \circ \gamma$. Conversely, assuming that f preserves pullbacks, if Γ denotes the collection of all pullback squares in $L(C)$, it suffices to prove that the pullback square in $L(C)$ corresponding to the universal term of Γ yields a pullback square in D in context Γ. Since, by right calculus of fractions, as seen in the proof of property (iii), the collection of pullback squares in C covers Γ, it suffices to prove that the functor $\bar{f}$ sends the image of any pullback square of C to a pullback square of D. Since $\bar{f} \circ \gamma = f$, this amounts to the property that f preserves pullback squares.

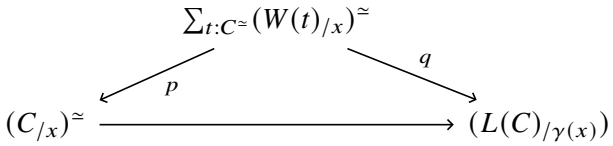
Finally, let us prove property (v). We consider an object $x : C^{\simeq}$ and we want to prove that the induced functor

$$\bar{\gamma}_{/x} \colon L(C_{/x}) \to L(C)_{/\gamma(x)}$$

is an equivalence. It suffices to check that the functor $\bar{\gamma}_{/x}$ is fully faithful and induces a cover

$$(\gamma_{/x})^{\simeq} \colon (C_{/x})^{\simeq} \to (L(C)_{/\gamma(x)})^{\simeq}\,.$$

The latter property comes from the fact that there is a commutative diagram of the form



with p and q defined explicitly as follows. A term in the top vertex of the triangle above is a triple of the form $t \xleftarrow{s} z \xrightarrow{g} x$ in C, with s in W. The map p sends such a triple to the pair (z, g) whereas the map q sends it to $(t, \gamma(s)^{-1} \circ \gamma(g))$. The commutativity of this triangle is provided by the identification in $L(C)_{/\gamma(x)}$ provided by the invertible morphism induced termwise by s. Therefore, in order to prove essential surjectivity, it suffices to check that the map q is a cover, which follows right away from the fact that it is the composition of the cover

$$\sum_{t:C^{\simeq}} (W(t)_{/x})^{\simeq} \to \sum_{t:C^{\simeq}} L(C)(\gamma(t), \gamma(x))$$

with the cover

$$\sum_{t:C^{\simeq}} L(C)(\gamma(t), \gamma(x)) \to \sum_{t:L(C)^{\simeq}} L(C)(t, \gamma(x))\,.$$

It remains to prove that, for any maps $u \colon y \to x$ and $v \colon z \to x$ in C, the induced map

$$L(C_{/x})((y,u),(z,v)) \to L(C)_{/\gamma(x)}((\gamma(y),\gamma(u)),(\gamma(z),\gamma(v)))$$

is an equivalence. We observe that property (i) applies to $C_{/x}$ with respect to the full subcategory $W(x) \hookrightarrow C_{/x}$. We thus have a right calculus of fractions at (y, u): taking into account the

identification of $C_{/y}$ with the slice of $C_{/x}$ over (y,u), we see that the collection of morphisms in $L(C_{/x})$ from (y,u) to (z,v) is thus the colimit

$$\operatorname*{colim}_{w:W(y)^{\mathrm{op}}} \mathrm{Hom}_{C_{/x}}((t_w,u_w)^{\mathrm{op}},(z,v))\,,$$

where we see each term $w : W(y)$ as a pair consisting of $s_w\colon t_w \to y$ in W and define $u_w\colon t_w \to x$ as the composition of s_w and u. In other words, we can consider the diagram assigning to each term w in $W(y)^{\mathrm{op}}$ the diagram

$$[0] \xrightarrow{u_w} \mathrm{Hom}_C(t_w,x) \xleftarrow{v_*} \mathrm{Hom}_C(t_w,z)\,.$$

Forming first the pullback and then the $W(y)^{\mathrm{op}}$-indexed colimit yields $L(C_{/x})((y,u),(z,v))$, whereas, by right calculus of fractions at y in C, we see that taking the $W(y)^{\mathrm{op}}$-indexed colimit first and then forming the pullback produces the collection of morphisms from $\gamma(y)$ to $\gamma(z)$ over $\gamma(x)$ in $L(C)$. Therefore, the property of fully faithfulness of the functor $\bar{\gamma}_{/x}$ follows from property (ii) and another application of Theorem 16.2.9. □

Corollary 17.1.6. *Let C be a small category with finite limits, and $W \hookrightarrow C$ a subcategory closed under pullbacks, with associated localization functor $\gamma\colon C \to L(C)$. Then $L(C)$ has finite limits and γ is left exact.*

Proof. This follows right away from Theorem 17.1.5 (iii) and from Theorem 17.1.5 (v) in the case where x is the terminal object of C. □

Definition 17.1.7. Let C be a category with pushouts. A subcategory $W \hookrightarrow C$ is closed under pushouts in C if $W^{\mathrm{op}} \hookrightarrow C^{\mathrm{op}}$ is closed under pullbacks.

Corollary 17.1.8. *Let C be a category and $W \hookrightarrow C$ a subcategory. We denote by $\gamma\colon C \to L(C)$ the localization of C by W. We assume that the category C has pushouts and that the subcategory W is closed under pushouts and has the 2-out-of-3 property. Then the following properties hold.*

(i) For any object x of C, the full subcategory $W(x)$ of $C_{x/}$ spanned by pairs (y,f) such that $f\colon x \to y$ is a morphism in W is a left calculus of fractions at x.

(ii) For any object x of C, the category $W(x)$ is filtered.

(iii) The category $L(C)$ has pushouts and the functor γ commutes with them.

(iv) Given any category D with pushouts and any functor $f\colon C \to D$ sending any map in W to an isomorphism in D, the induced functor $\bar{f}\colon L(C) \to D$ commutes with pushouts if and only if the functor f itself has the same property.

(v) For any object x of C, the functor γ induces a functor

$$\gamma_{x/}\colon C_{x/} \to L(C)_{\gamma(x)/}$$

that exhibits $L(C)_{\gamma(x)/}$ as the localization of $C_{x/}$ at the subcategory $W(x) \hookrightarrow C_{x/}$.

Proof. Apply Theorem 17.1.5 to C^{op}. □

Corollary 17.1.9. *Let C be a small category with finite colimits, and $W \hookrightarrow C$ a subcategory closed under pushouts, with associated localization functor $\gamma\colon C \to L(C)$. Then $L(C)$ has finite colimits and γ is right exact.*

Proof. Apply Corollary 17.1.6 to C^{op}. □

Definition 17.1.10. A subcategory $W \hookrightarrow C$ is *saturated* if the obvious commutative square

$$\begin{array}{ccc} W & \hookrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ L(C)^{\simeq} & \hookrightarrow & L(C) \end{array}$$

is cartesian.

The next proposition shows that left exact localizations can always be described by localizing a saturated subcategory.

Proposition 17.1.11. *Let C be a category and $W \hookrightarrow C$ a subcategory. We denote by $\gamma\colon C \to L(C)$ the localization of C by W. We assume that the category C has pullbacks and that the subcategory W is closed under pullbacks and has the 2-out-of-3 property. We define $\bar{W}$ through the following pullback square.*

$$\begin{array}{ccc} \bar{W} & \hookrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ L(C)^{\simeq} & \hookrightarrow & L(C) \end{array}$$

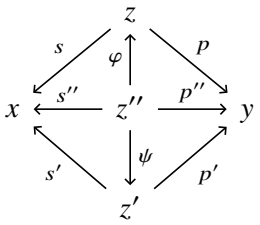
Then $\bar{W}$ is closed under pullbacks, has the 2-out-of-3 property, and the functor γ exhibits $L(C)$ as the localization of C by $\bar{W}$.

Proof. Assertion (iii) of Theorem 17.1.5 implies that $\bar{W}$ is closed under pullbacks. The rest of the proposition is obvious. □

Lemma 17.1.12. *Let $W \hookrightarrow C$ be a subcategory closed under pullbacks and with the 2-out-of-3 property. Given two objects x and y in C and two diagrams of the form*

$$x \xleftarrow{s} z \xrightarrow{p} y \quad \text{and} \quad x \xleftarrow{s'} z' \xrightarrow{p'} y$$

with s and s' in W, the anima $\gamma(s)^{-1} \circ \gamma(p) = \gamma(s')^{-1} \circ \gamma(p')$ is covered by the collection of commutative diagrams in C of the form



with s'' in W.

In particular, the collection of morphisms of the form $f : x \to y$ in C inducing an isomorphism in $L(C)$ is covered by the collection of diagrams of the form

$$x'' \xrightarrow{p'} x' \xrightarrow{p} x \xrightarrow{f} y$$

with the property that each composite $f \circ p$ and $p \circ p'$ is in W.

Proof. We observe that $W(x)_{/y}$ always has pullbacks (for any objects x and y) and thus that the localization $\Pi_\infty(W(x)_{/y})$ is obtained by a right calculus of fractions at any object of $W(x)_{/y}$. This implies the first assertion right away. The last assertion follows from the first by interpreting the fact that a map in $L(C)$ is invertible if and only if it has a right inverse which itself has a right inverse. □

Proposition 17.1.13. *Let $W \hookrightarrow C$ be a subcategory closed under pullbacks and with the* 2*-out-of-*3 *property. Then W is saturated if and only if the collection of diagrams of the form*

$$x''' \xrightarrow{p''} x'' \xrightarrow{p'} x' \xrightarrow{p} x$$

with the property that each composite $p' \circ p''$ and $p \circ p'$ is in W coincides with the collection of diagrams of the same shape with the property that all three morphisms p, p' and p'' are in W.

Proof. It suffices to prove that the collection of diagrams of the form

$$x''' \xrightarrow{p''} x'' \xrightarrow{p'} x' \xrightarrow{p} x$$

with the property that each composite $p' \circ p''$ and $p \circ p'$ is in W is covered by the collection of diagrams of the same shape with the property that all three morphisms p, p' and p'' are in W. This follows right away from Lemma 17.1.12. □

Proposition 17.1.14. *Let $W \hookrightarrow C$ be a saturated subcategory closed under pullbacks. Then the localization functor $C \to L(C)$ restricted to W exhibits $L(C)^{\simeq}$ as the localization of W by all its maps:*

$$\Pi_\infty(W) \simeq L(C)^{\simeq} .$$

Proof. Let $x : C^{\simeq}$. Since $W^{\simeq} = C^{\simeq}$, we may see x as an object of W. We observe that W is closed under pullbacks with the 2-out-of-3 property as a subcategory of W. If we define $W(x)$ as the full subcategory of $C_{/x}$ with objects those pairs (y, s) with $s \colon y \to x$ a map in W, we thus have a right calculus of fractions at x

$$W(x) \hookrightarrow C_{/x} \quad \textit{in } C$$

and a right calculus of fractions at x

$$W(x) = W_{/x} \quad \textit{in } W.$$

Given any other object y, the map

$$\Pi_\infty(W)(x, y) \to L(C)(x, y)$$

may thus be described as the canonical map

$$\operatorname*{colim}_{w : W(x)^{\mathrm{op}}} \mathrm{Hom}_W(z_w^{\mathrm{op}}, y) \to \operatorname*{colim}_{w : W(x)^{\mathrm{op}}} \mathrm{Hom}_C(z_w^{\mathrm{op}}, y) .$$

It suffices now to check that the map above is an embedding with image $L(C)^{\simeq}(x,y)$. We observe right away that $W(x)$ has pullbacks, hence that $W(x)^{\mathrm{op}}$ is confluent. Since the functor

$$\operatorname*{colim}_{W(x)^{\mathrm{op}}} : \mathrm{PSh}(W(x)) \to \mathrm{An}$$

preserves pullbacks, it preserves embeddings. Therefore, since each map

$$\mathrm{Hom}_W(z_w^{\mathrm{op}}, y) \hookrightarrow \mathrm{Hom}_C(z_w^{\mathrm{op}}, y)$$

is an embedding, this proves that $\Pi_\infty(W)(x,y)$ embeds into $L(C)(x,y)$. In order to prove that the image consists exactly of isomorphisms from x to y in the localization $L(C)$, it suffices to work in the context of such equivalences that come from an object of $W(x)_{/y}$. But if $s : z \to x$ is in W and $p : z \to y$ is a map in C such that $\gamma(s)^{-1} \circ \gamma(p)$ is invertible in $L(C)$, then so is $\gamma(p)$ itself. Since W is saturated, this implies right away that p is in W as well. □

17.2 Small subuniverses

Lemma 17.2.1. *Let I be a small filtered category any $i \mapsto C_i$ an I-indexed functor with values in* Cat *whose colimit will be denoted by C. We consider a functor $u : B \to C$ and define $B_i = B \times_C C_i$ for each $i : I$. Then the canonical functor*

$$\operatorname*{colim}_{i\,:\,I} B_i \to B$$

is an equivalence of categories.

Proof. Since the nerve functor preserves finite limits as well as small filtered colimits, this follows from the fact that colimits are universal in $\mathrm{PSh}(\Delta)$, which follows itself from the analogous property in An. □

Lemma 17.2.2. *Let $F \hookrightarrow \mathrm{Cat}^{\simeq}$ be a small family of categories. We consider a locally small category with small colimits and a small full subcategory A of C. The smallest full subcategory of C containing A and stable under colimits indexed by any $I : F$ exists and is small as well.*

Proof. It suffices to prove that there exists at least one full subcategory of C that is small, contains A, and is stable under any I-indexed colimit for any $I : F$. It suffices to prove this after replacing F by any larger small family of categories. Therefore, we may assume that, for any $B : F$ and any full embedding $A \hookrightarrow B$, the category A is a term of F as well. We may also assume that $[0] : F$. We introduce the following operator on the collection of all full subcategories of C. Given $B \hookrightarrow C$, we define $U(B)^{\simeq}$ as the collection of all colimits in C of all I-indexed diagrams with values in B for all $I : F$. We define $U(B)$ as the full subcategory of C spanned by $U(B)^{\simeq}$. Since $[0] : F$, we must have B as full subcategory of $U(B)$, so that we get functors

$$B \hookrightarrow U(B) \hookrightarrow C\,.$$

We define $U^n(B)$ for each $n : \mathbb{N}$ through $U^0(B) = B$ and $U^{n+1}(B) = U(U^n(B))$. The Milnor colimit of the induced diagram of categories

$$B \hookrightarrow U(B) \hookrightarrow \cdots \hookrightarrow U^n(B) \hookrightarrow U^{n+1}(B) \hookrightarrow \cdots$$

defines a full subcategory B' of C. It is clear that the operator U preserves the property of being small. Therefore, if B is small, so is B'. We claim that B' is stable under I-indexed colimits for any $I : F$. Indeed, let

$$\varphi : I \to B'$$

be any functor with $I : F$. We form the pullback square of categories

$$\begin{array}{ccc} I_n & \xrightarrow{\varphi_n} & B_n \\ \downarrow & \lrcorner & \downarrow \\ I & \xrightarrow{\varphi} & B' \end{array}$$

for each $n : \mathbb{N}$, with $B_n \hookrightarrow B'$ the canonical embedding. Since ω is filtered, Lemma 17.2.1 ensures that I is the Milnor colimit of the I_n's. Therefore, by virtue of Corollary 13.3.2, it suffices to prove that each functor

$$\varphi_n : I_n \hookrightarrow B_n \hookrightarrow B'$$

has a colimit in B'. Since $I_n : F$, it has a colimit in C that belong to B_{n+1}. Therefore, it has a colimit in B'. Applying what precedes to $B = A$ ends the proof. □

Definition 17.2.3. A *family of small categories* is an embedding of the form $\kappa \hookrightarrow \mathrm{Cat}^{\simeq}$. We denote by Cat_κ the full subcategory of Cat with collection of objects the anima κ. The objects of Cat_κ will be said to be *strictly κ-small*.

A small category A is said to be *weakly κ-filtered* if for any strictly κ-small category K, the constant diagram functor $A \to \mathrm{Fun}(K, A)$ is terminal.

A small category A is said to be *κ-filtered* if, for any strictly κ-small category K, the functor

$$\operatorname*{colim}_A : \mathrm{Fun}(A, \mathrm{An}) \to \mathrm{An}$$

commutes with K^{op}-indexed limits.

A small category K is *κ-small* if, for any small κ-filtered category A, the functor

$$\operatorname*{colim}_A : \mathrm{Fun}(A, \mathrm{An}) \to \mathrm{An}$$

commutes with K^{op}-indexed limits. We denote by Cat_κ the full subcategory of Cat with objects the κ-small categories.

A family of small categories κ is *localizing* if, for any a small category C with strictly κ-small colimits and W any family of morphisms in C that is stable under κ-small colimits in $\mathrm{Fun}([1], C)$, with corresponding localization functor

$$\gamma : C \to L(C),$$

the following properties hold.

(a) The category $L(C)$ has strictly κ-small colimits and the localization functor $\gamma : C \to L(C)$ commutes with them.

(b) For any strictly κ-small category K, the canonical functor

$$\gamma_* \colon L(\mathrm{Fun}(K, C)) \to \mathrm{Fun}(K, L(C))$$

is an equivalence, where $L(\mathrm{Fun}(K, C))$ is the localization of $\mathrm{Fun}(K, C)$ by maps that are objectwise in W.

for any small category C with strictly κ-small colimits and any family W of morphisms in C, if W is stable under strictly κ-small colimits in $\mathrm{Fun}([1], C)$, then the localization $L(C)$ of C by W has strictly κ-small colimits and the localization functor

$$\gamma \colon C \to L(C)$$

commutes with them.

A family of categories κ is *elementary* if it has the following properties:

(i) it contains the finite category $\ulcorner = [0] \star ([1]^{\simeq})$;

(ii) for any strictly κ-small category K the opposite of the twisted arrow category $\mathrm{Tw}(K)^{\mathrm{op}}$ is strictly κ-small;

(iii) for any strictly κ-small category K and any objects $x, y \colon K$, the anima $K(x, y)$ is strictly κ-small;

Remark 17.2.4. Any κ-filtered category is weakly κ-filtered. However, the converse is not true in general. For a family of categories to be well behaved a (partial) converse is usually needed.

Example 17.2.5. The family of finite posets is elementary.

Proof. The pushout category $\ulcorner$ is a finite poset. Opposites preserve finite posets. For a finite poset P, the objects of $\mathrm{Tw}(P)$ form a decidable subobject of $P^{\simeq} \times P^{\simeq}$, and the order on $\mathrm{Tw}(P)$ is decidable. Filtering the product of two finite enumerations therefore shows that $\mathrm{Tw}(P)$ is a finite poset. Finally, each $P(x, y)$ is a decidable proposition, hence is either empty or terminal and so is a finite poset. □

Definition 17.2.6. Let $\kappa \hookrightarrow \mathrm{Cat}^{\simeq}$ be a family of small categories. We say that κ is a *small subuniverse* of Cat if the following properties are satisfied.

SU1. The anima κ is small.

SU2. Any small category with strictly κ-small colimits is κ-filtered.

Example 17.2.7. The empty family $\emptyset$ is a localizing small subuniverse.

Example 17.2.8. The family $\{\emptyset\}$ that consists solely of the empty category is a small localizing subuniverse (axiom SU2 holds obviously once we identify the A-indexed colimit of the point with the localization of A by all its maps $\Pi_\infty(A)$) and the localizing property is given by Proposition 14.1.1). The notion of small $\{\emptyset\}$-filtered category simply is the notion of small weakly contractible category.

Example 17.2.9. Finite sets form a small subuniverse and the corresponding notion of $\mathrm{Fin}^{\simeq}$-filtered category is the notion of of sifted category (as a combination of the previous example with Proposition 16.2.11). It is also localizing by Proposition 14.1.1 and Proposition 14.1.6.

Example 17.2.10. The family given by the image of the functor $[0] \to \mathrm{Cat}^{\simeq}$ with value $\ulcorner$ is a subuniverse by Theorem 16.2.5. The notion of $\ulcorner$-filtered category corresponds to the notion of distilled category.

Example 17.2.11. Finite categories form an small subuniverse (as combination of Example 17.2.8 and Example 17.2.10). The corresponding notion of $\mathrm{FinCat}^{\simeq}$-filtered categories simply is the usual notion of filtered category.

Example 17.2.12. Finite posets form an elementary small subuniverse ω. The notion of ω-filtered category coincides with the one of filtered category.

Proof. Elementarity is Example 17.2.5. The family is small: for each $n : \mathbb{N}$, the decidable partial orders on $\{1, \dots, n\}$ form a small family, and the family of finite posets is the image in $\mathrm{Cat}^{\simeq}$ of the coproduct of these families over $n : \mathbb{N}$.

A category with colimits indexed by all finite posets has an initial object and pushouts, since both $\emptyset$ and $\ulcorner$ are finite posets. It is confluent by Example 16.2.3, and it is weakly contractible because it has an initial object. It is therefore filtered. Its colimit functor on animae is left exact by Theorem 16.2.9, and hence commutes with limits indexed by finite posets by Proposition 16.4.13. This proves axiom *SU2*. The same argument shows that every filtered category is ω-filtered. Conversely, if a category is ω-filtered, then its colimit functor preserves terminal objects and pullbacks, since $\emptyset$ and $\ulcorner$ belong to ω. It is therefore filtered by Theorem 16.2.9. □

Proposition 17.2.13. *Let κ be an elementary family of small categories such that any small category with strictly κ-small colimits is κ-filtered, then strictly κ-small categories. Then κ is localizing.*

Proof. We consider a small category C with strictly κ-small colimits and W a family of morphisms in C that is stable under κ-small colimits in $\mathrm{Fun}([1], C)$, with corresponding localization functor

$$\gamma \colon C \to L(C)\,.$$

We already know that $L(C)$ has finite colimits and that γ is right exact. Since the finite category $\ulcorner$ is κ-small, the family of maps W is stable under pushouts. Therefore, for any object y of C, the full subcategory $W(y)$ of the slice $C_{y/}$ whose objects are the pairs (z, s) with $s \colon y \to z$ a map in W defines a left calculus of fractions at y with respect to W. We thus have an explicit formula for the mapping space of $L(C)$, namely

$$\operatorname*{colim}_{(z,s)\colon W(y)} \mathrm{Hom}_C(x, z) \simeq \mathrm{Hom}_{L(C)}(x, y)$$

for any objects $x, y \colon C^{\simeq}$. Moreover, it is easily seen that $W(y)$ is stable under κ-small colimits in $C_{y/}$. In particular, $W(y)$ is κ-filtered. Therefore, for any κ-small category K and any functor

$$\varphi \colon K \to C$$

with colimit ξ in C and any object y of C we have:

$$\begin{aligned}\lim_{x\colon K^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(\varphi^{\mathrm{op}}(x), y) &= \lim_{x\colon K^{\mathrm{op}}} \mathrm{Hom}_{L(C)}(\varphi^{\mathrm{op}}(x), y)\\ &= \lim_{x\colon K^{\mathrm{op}}} \operatorname*{colim}_{(z,s)\colon W(y)} \mathrm{Hom}_C(\varphi^{\mathrm{op}}(x), z)\\ &= \operatorname*{colim}_{(z,s)\colon W(y)} \lim_{x\colon K^{\mathrm{op}}} \mathrm{Hom}_C(\varphi^{\mathrm{op}}(x), z)\\ &= \operatorname*{colim}_{(z,s)\colon W(y)} \mathrm{Hom}_C(\xi, z)\\ &= \mathrm{Hom}_{L(C)}(\xi, y)\,.\end{aligned}$$

In other words, the colimit of $\gamma \circ \varphi$ exists in $L(C)$ and is nothing else than $\gamma(\xi)$. We will now prove that the canonical functor

$$\gamma_*\colon L(\mathrm{Fun}(K, C)) \to \mathrm{Fun}(K, L(C))$$

is an equivalence for any strictly κ-small category K, where $L(\mathrm{Fun}(K, C))$ is the localization of $\mathrm{Fun}(K, C)$ by maps that are objectwise in W. Since $\mathrm{Tw}(K)^{\mathrm{op}}$ is strictly κ-small, any object of $\mathrm{Fun}(K, C)$ is a strictly κ-small colimit of objects of the form $x \otimes a$ with $a\colon C$ and $x\colon K$ (this follows from Proposition 12.9.1 applied to $\mathrm{Ind}(C)$ through the full embedding of $\mathrm{Fun}(K, C)$ into $\mathrm{Fun}(K, \mathrm{Ind}(C))$). Observe that the evaluation of $x \otimes a$ at $y\colon K^{\simeq}$ is the $K(x, y)$-indexed coproduct of copies of a. The image of $x \otimes a$ in $\mathrm{Fun}(K, C)$ is thus $x \otimes \gamma(a)$ since the localization functor

$$\mathrm{Fun}(K, C) \to L(\mathrm{Fun}(K, C))$$

commutes with strictly κ-small colimits. Therefore, in order to prove that γ_* is fully faithful, it suffices to prove that we have an equivalence of the form

$$\mathrm{Hom}(F, G) \to \mathrm{Hom}(\gamma_*(F), \gamma_*(G))$$

with F of the form $F = x \otimes a$. Since the adjunction

$$x \otimes (-)\colon C \rightleftarrows \mathrm{Fun}(K, C)\colon \mathrm{ev}_x$$

is compatible with maps that are objectwise in W, it induces an adjunction of the form

$$x \otimes (-)\colon L(C) \rightleftarrows L(\mathrm{Fun}(K, C))\colon \mathrm{ev}_x$$

by Lemma 14.1.5. Therefore, we have

$$\mathrm{Hom}_{L(\mathrm{Fun}(K,C))}(F, G) = \mathrm{Hom}_{L(C)}(\gamma(a), \gamma(G(x)))\,.$$

Since $\gamma_*(F) = x \otimes \gamma(a)$, we obiously have

$$\mathrm{Hom}_{\mathrm{Fun}(K,L(C))}(\gamma_*(F), \gamma_*(G)) = \mathrm{Hom}_{L(C)}(\gamma(a), \gamma(G(x)))\,,$$

which proves that γ_* is fully faithful. Since γ_* commutes with K-indexed colimits, it suffices to check that objects of the form $x \otimes a$ with $x\colon K$ and $a\colon L(C)^{\simeq}$ are in the image of

$$(\gamma_*)^{\simeq}\colon L(\mathrm{Fun}(K, C))^{\simeq} \to \mathrm{Fun}(K, L(C))^{\simeq}\,.$$

This follows right away from the fact that

$$K \times C^{\simeq} \to K \times L(C)^{\simeq}$$

is a cover and that γ commutes with L-indexed colimits for $L = K(x, y)$ for any $y \colon K^{\simeq}$. This proves that γ_* is an equivalence of categories.

Now, the adjunction

$$\operatorname*{colim}_K : \operatorname{Fun}(K, C) \rightleftarrows C \colon \operatorname{const}_K$$

induces through Lemma 14.1.5 an adjunction of the form

$$\operatorname*{hocolim}_K : L(\operatorname{Fun}(K, C)) \rightleftarrows L(C) \colon \operatorname{const}_K .$$

Identifying $L(\operatorname{Fun}(K, C))$ and $\operatorname{Fun}(K, L(C))$, the functor $\operatorname{hocolim}_K$ yields the existence of the functor

$$\operatorname*{colim}_K : \operatorname{Fun}(K, L(C)) \to L(C) .$$

This achieves the proof that strictly κ-small categories form a localizing family of small categories. □

Corollary 17.2.14. *Elementary small subuniverses are localizing.*

From now on, in this section, we fix a small subuniverse $\kappa \hookrightarrow \mathrm{Cat}^{\simeq}$.

Notation 17.2.15. Given a category C, we let $\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$ be the full subcategory of $\mathrm{PSh}(C)$ of presheaves that preserve K^{op}-indexed limits for any strictly κ-small category K. It is clear that $\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$ is stable by limits in $\mathrm{PSh}(C)$.

Lemma 17.2.16. *The Yoneda embedding $h \colon C \to \mathrm{PSh}(C)$ takes its values in $\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$ and the induced functor*

$$C \hookrightarrow \operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$$

preserves colimit cocones indexed by strictly κ-small categories.

Proof. It is clear that h takes values in $\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$ since the Yoneda embedding always preserve limit cones (i.e. preserve the limits that exist). Let K by a strictly κ-small category and $\varphi \colon K \to C$ any functor with colimit y. For any F in $\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$, we have:

$$\begin{aligned} \operatorname{Hom}(h(y), F) &= F(y) \\ &= \lim_{x \colon K^{\mathrm{op}}} F(\varphi^{\mathrm{op}}(x)) \\ &= \lim_{x \colon K^{\mathrm{op}}} \operatorname{Hom}(h^{\mathrm{op}}\varphi^{\mathrm{op}}(x), F) \\ &= \operatorname{Hom}(\operatorname*{colim}_{x \colon K} h(\varphi(x)), F) . \end{aligned}$$

This shows that the colimit cocone $\varphi(x) \to y$ is sent by h to a colimit cocone in $\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$. □

Lemma 17.2.17. *Let C be a small category with strictly κ-small colimits. The category $\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$ has small κ-filtered colimits and the full embedding*

$$\operatorname{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An}) \hookrightarrow \mathrm{PSh}(C)$$

commute with them.

Proof. Let I be a small κ-filtered category and $i \mapsto F_i$ an I-indexed functor with values in $\mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$. We denote by F the colimit of the diagram $i \mapsto F_i$ in $\mathrm{PSh}(C)$. Then, for any functor $\varphi\colon K \to C$ with K a strictly κ-small category, we have:

$$\begin{aligned} F\Big(\operatorname*{colim}_{x\colon K} \varphi(x)\Big) &= \operatorname*{colim}_{i\colon I} F_i\Big(\operatorname*{colim}_{x\colon K} \varphi(x)\Big) \\ &= \operatorname*{colim}_{i\colon I} \lim_{x\colon K^{\mathrm{op}}} F_i(\varphi^{\mathrm{op}}(x)) \\ &= \lim_{x\colon K^{\mathrm{op}}} \operatorname*{colim}_{i\colon I} F_i(\varphi^{\mathrm{op}}(x)) \\ &= \lim_{x\colon K^{\mathrm{op}}} F(\varphi^{\mathrm{op}}(x))\,. \end{aligned}$$

In other words, the functor F commutes with K^{op}-indexed limits. □

Notation 17.2.18. Given a small category C, we denote by $\mathrm{Ind}_{\kappa}(C)$ the full subcategory of $\mathrm{PSh}(C)$ with objects those presheaves $F\colon C^{\mathrm{op}} \to \mathrm{An}$ such that there exists a small κ-filtered category I, a functor $\varphi\colon I \to C$, and an isomorphism of the form

$$\operatorname*{colim}_{x\colon I} \varphi(x) = F\,.$$

Proposition 17.2.19. *Let Cbe a small category with strictly κ-small colimits. Given a presheaf*

$$F\colon C^{\mathrm{op}} \to \mathrm{An}\,,$$

the following assertions are equivalent.

(i) The presheaf F belongs to $\mathrm{Ind}_{\kappa}(C)$.

(ii) The functor $F\colon C^{\mathrm{op}} \to \mathrm{An}$ commutes with K^{op}-indexed limits for any strictly κ-small category K.

(iii) The category $C_{/F}$ has strictly κ-small colimits.

(iv) The category $C_{/F}$ is κ-filtered.

Proof. Condition (iv) follows from condition (iii) by definition of subuniverses.
Since representable presheaves commute with limits, Lemma 17.2.17 implies that condition (ii) follows from condition (i).
Let F be a presheaf on C that commutes with K^{op}-indexed limits for any strictly κ-small category K. We then have the following pullback square, in which the lower horizontal map is the Yoneda embedding.

$$\begin{array}{ccc} C_{/F} & \longrightarrow & \mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})_{/F} \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An}) \end{array}$$

By virtue of Lemma 17.2.16, the Yoneda embedding above preserve strictly κ-small colimits. The vertical functor on the right side has a right adjoint (since $\mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$ is stable by pullbacks in $\mathrm{PSh}(C)$) hence commutes with all colimits that exist. More precisely this functor creates colimits:

since it is also conservative (being right fibration), any cocone in $\mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})_{/F}$ that become a colimit cocone in $\mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$ is a colimit cocone. The dual version of Proposition 13.2.14 together with Lemma 17.2.16 show that $C_{/F}$ has strictly κ-small colimits and that the projection to C preserves and detects them. In other words, condition (iii) follows from condition (ii).
By virtue of Corollary 12.6.4, (i) follows from (iv). □

Corollary 17.2.20. *If C is a small category with strictly κ-small colimits, then we have*

$$\mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An}) = \mathrm{Ind}_{\kappa}(C)\,.$$

Proposition 17.2.21. *Let $u\colon C \to D$ be a strictly κ-small colimits preserving functor between small categories with strictly κ-small colimits. Then the adjunction*

$$u_!\colon \mathrm{PSh}(C) \rightleftarrows \mathrm{PSh}(D)\colon u^*$$

restricts to an adjunction of the form

$$u_!\colon \mathrm{Ind}_{\kappa} C \rightleftarrows \mathrm{Ind}_{\kappa} D\colon u^*$$

such that the right adjoint u^* commutes with κ-small colimits.

Proof. Since the property for a functor to preserve K^{op}-indexed limits for any strictly κ-small category K is compatible with composition of functors, it is clear that u^* sends $\mathrm{Fun}^{\kappa}(D^{\mathrm{op}}, \mathrm{An})$ to $\mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, \mathrm{An})$. The previous corollary thus tells us that u^* sends $\mathrm{Ind}_{\kappa}(D)$ to $\mathrm{Ind}_{\kappa}(C)$. The fact that $u_!$ sends $\mathrm{Ind}_{\kappa}(C)$ to $\mathrm{Ind}_{\kappa}(D)$ follows from the fact that $u_! \circ h_C = h_D \circ u$. □

Definition 17.2.22. Led D be a locally small category with small colimits. An object x of D is *κ-compact* is the functor

$$\mathrm{Hom}_D(x, -)\colon D \to \mathrm{An}$$

commute with strictly κ-small colimits. We denote by D^{κ} the full subcategory of κ-compact objects of D.

A *κ-presentable category* is a locally small category with small colimits D such that there exists a small family E of κ-compact objects in D that generates D in the sense that a map $u\colon y \to z$ in D is an isomorphism if and only in the induced map

$$u_*\colon \mathrm{Hom}_D(x, y) \to \mathrm{Hom}_D(x, z)$$

for any $x\colon E$.

Remark 17.2.23. By definition, κ-compact objects are stable under strictly κ-small colimits. One checks that κ-compact objects are stable under retracts as well. Indeed, if y is κ-compact and if there are maps $s\colon x \to y$ and $p\colon y \to x$ such that $p \circ s = \mathrm{id}_x$, for any κ-filtered category I any I-indexed functor $i \mapsto z_i$, the canonical map

$$\operatorname*{colim}_{i\colon I} \mathrm{Hom}(x, z_i) \to \mathrm{Hom}(x, \operatorname*{colim}_{i\colon I} z_i)$$

is a retract of the isomorphism

$$\operatorname*{colim}_{i\colon I} \mathrm{Hom}(y, z_i) \to \mathrm{Hom}(y, \operatorname*{colim}_{i\colon I} z_i)\,.$$

Theorem 17.2.24. *Let D be a κ-presentable category and let us choose a small generating family E of κ-compact objects in D. We denote by C the smallest full subcategory of D stable under strictly κ-small colimits contining E. Then C has the following properties:*

(i) C is small;

(ii) all the objects of C as κ-compact;

(iii) there is a canonical identification of the form $\mathrm{Ind}_\kappa(C) = D$.

Proof. The first assertion is a particular case of Lemma 17.2.2 The fact that D^κ is stable under strictly κ-small colimits follows right away from the definitions of κ-filtered categories and of κ-compact objects. By construction, C is a full subcategory of D^κ, which proves assertion (ii).

Let $i\colon C \to D$ be the embedding of C into D. Since D is locally small with small colimits, the functor (defined a restricted Yoneda embedding)

$$i^*\colon D \to \mathrm{PSh}(C)\,, \quad y \mapsto \mathrm{Hom}_D(i^{\mathrm{op}}(-), y)$$

has a left adjoint

$$i_!\colon \mathrm{PSh}(C) \to D$$

equipped with an identification

$$i_!(h(x)) = i(x)$$

functorially in $x\colon C$, with $h\colon C \to \mathrm{PSh}(C)$ the Yoneda embedding of C. We define a functor

$$\varphi\colon \mathrm{Ind}_\kappa(C) \to D$$

as the composition of $i_!$ with the full embedding of $\mathrm{Ind}_\kappa(C)$ into $\mathrm{PSh}(C)$. We observe that, for any object y of D, the presheaf $i^*(y)$ belongs to $\mathrm{Ind}_\kappa(C)$. Indeed, by virtue of Proposition 17.2.19, it suffices to check that it preserves K^{op}-indexed limits for any strictly κ-small category K. This amounts to say that the functor i preserves strictly κ-small colimits, which holds by definition of C. We thus get an adjunction of the form

$$\varphi\colon \mathrm{Ind}_\kappa(C) \rightleftarrows D\colon \psi$$

where ψ induced by i^* through the previous observation. The functor ψ is conservative since E generates D and since isomorphisms can be detected objectwise in a functor category. Therefore, it suffices to prove that the functor φ is fully faithful. In other words, it suffices to prove that, for any object F of $\mathrm{Ind}_\kappa(C)$, the unit map

$$\eta_F\colon F \to \psi\varphi(F)$$

is an isomorphism. We observe that the functor ψ commutes with κ-filtered colimits since all objects of C are κ-compact in D. Since the functor φ preserves colimits (being a left adjoint), this shows that the functor $\psi\varphi$ preserves κ-filtered colimits. By definition of $\mathrm{Ind}_\kappa(C)$, we know that F is a κ-filtered colimit of representable presheaves. Therefore, it suffices to prove that the unit map η_F is invertible when $F = h(y)$ for some object y of C. But then, for any object x of C, we have

$$\begin{aligned} F(x) &= \mathrm{Hom}_C(x, y) \\ &= \mathrm{Hom}_D(i(x), i(y)) \\ &= \psi\varphi(F)(x)\,. \end{aligned}$$

□

Construction 17.2.25. Let C be a small category. We can apply the previous theorem to $D = \mathrm{PSh}(C)$ with E the family of representable presheaves on C. We define $\langle C \rangle_\kappa$ as the smallest full subcategory of the category of presheaves on C containing representable presheaves and stable under strictly κ-small colimits. We then have a canonical equivalence

$$\mathrm{Ind}_\kappa(\langle C \rangle_\kappa) = \mathrm{PSh}(C)\,.$$

Assume now that C has strictly κ-small colimits. We claim that the Yoneda embedding

$$h\colon C \hookrightarrow \langle C \rangle_\kappa$$

has a left adjoint

$$\lambda\colon \langle C \rangle_\kappa \to C\,.$$

Indeed, given an object X of $\langle C \rangle_\kappa$, seen as a presheaf on C, we have an associated right fibration $p\colon C_{/X} \to C$, and for any term y of C we have

$$\begin{aligned}
\mathrm{Hom}(X, h(y)) &= \mathrm{Hom}(\operatorname*{colim}_{C_{/X}} h \circ p, h(y)) \\
&= \lim_{(C_{/X})^{\mathrm{op}}} \mathrm{Hom}(h^{\mathrm{op}} \circ p^{\mathrm{op}}, h(y)) \\
&= \lim_{(C_{/X})^{\mathrm{op}}} \mathrm{Hom}_C(p^{\mathrm{op}}, y) \\
&= \mathrm{Hom}_C(\operatorname*{colim}_{C_{/X}} p, y)
\end{aligned}$$

functorially in y. This induces an adjunction

$$\lambda_!\colon \mathrm{PSh}(\langle C \rangle_\kappa) \rightleftarrows \mathrm{PSh}(C)\colon h_!$$

or equivalently an identification $h_! = \lambda^*$. Since h is fully faithful, its left adjoint λ is a localization (by Proposition 4.4.16) and thus the functor $\lambda^* = h_!$ is fully faithful. By virtue of Proposition 17.2.21, the adjoints $\lambda_!$ and λ^* restrict to an adjunction of the form

$$\lambda_!\colon \mathrm{Ind}_\kappa(\langle C \rangle_\kappa) \rightleftarrows \mathrm{Ind}_\kappa(C)\colon \lambda^*\,.$$

In other words, we have an adjunction of the form

$$\lambda_!\colon \mathrm{PSh}(C) \rightleftarrows \mathrm{Ind}_\kappa(C)\colon h_!\,.$$

This explains the following corollary.

Corollary 17.2.26. *Let C be small category with κ-small colimits. Then $\mathrm{Ind}_\kappa(C)$ has small colimits as well as small limits.*

Proof. We have seen above that the full embedding

$$h_!\colon \mathrm{Ind}_\kappa(C) \hookrightarrow \mathrm{PSh}(C)$$

has a left adjoint. Since $\mathrm{PSh}(C)$ has small (co)limits, we conclude with Corollary 14.2.5. □

Remark 17.2.27. If C is a small category with strictly κ-small colimits, then $\mathrm{Ind}_\kappa(C)$ is κ-presentable. Indeed, it is locally small (as full subcategory of a locally small category) and has small colimits by the previous corollary. We know that

$$\mathrm{PSh}(C) \to \mathrm{An}\,, \quad F \mapsto \mathrm{Hom}(X, F)$$

commutes with small colimits whenever X is a representable presheaf. Therefore, by Lemma 17.2.17, representable presheaves are κ-compact as objects of $\mathrm{Ind}_\kappa(C)$. Since representable presheaves form a generating family of $\mathrm{Ind}_\kappa(C)$ (because this is the case in $\mathrm{PSh}(C)$ already), this proves that $\mathrm{Ind}_\kappa(C)$ is κ-presentable. Therefore, Theorem 17.2.24 may be rephrased by claiming that a category is κ-presentable if and only if it is equivalent to a category of the form $\mathrm{Ind}_\kappa(C)$ for some small category C with strictly κ-small colimits.

Corollary 17.2.28. *Any κ-presentable category has small limits.*

Proof. By virtue of Theorem 17.2.24, it suffices to prove that $\mathrm{Ind}_\kappa(C)$ has small limits for any small category C with strictly κ-small colimits, which is documented in Corollary 17.2.26. □

Definition 17.2.29. A category is *κ-small* if it belongs to the smallest full subcategory of Cat containing $[n]$ for all $n\colon \mathbb{N}$ and stable under strictly κ-small colimits.

Corollary 17.2.30. *The category of κ-small categories is small. Any small category is a small κ-filtered colimit of κ-small categories.*

Proof. The first assertion follows from Lemma 17.2.2. For the second one, we consider the equivalence

$$\mathrm{PSh}(\Delta) = \mathrm{Ind}_\kappa(\langle\Delta\rangle_\kappa)$$

obtained by applying Theorem 17.2.24 with $D = \mathrm{PSh}(\Delta)$ and E the family of representable presheaves. Therefore, given any small category C, its nerve $N(C)$ canonically is a small κ-filtered colimit of objects of $\langle\Delta\rangle_\kappa$. Since the nerve is fully faithful, this implies that C is the colimit of the functor

$$\langle\Delta\rangle_\kappa \hookrightarrow \mathrm{PSh}(\Delta) \xrightarrow{\mathrm{ac}} \mathrm{Cat}\,.$$

To conclude the proof, it suffices to observe the obvious fact that the latter takes values in the full subcategory of κ-small categories. □

Proposition 17.2.31. *Any category with strictly κ-small colimits has κ-small colimits.*

Proof. Let C be category. Let $C' = \mathrm{Fun}(C, \mathrm{An})^{\mathrm{op}}$ and $h\colon C \to C'$ be the Yoneda embedding. Let E be the full subcategory of Cat with objects those small categories A such that A-indexed colimits exist in C. Equivalently, a small category A belongs to E if and only if C is stable under A-indexed colimits in C'. A direct application of Lemma 13.3.1 shows that E is stable under strictly κ-small colimits in Cat. Since E contains all strictly small colimits, it contains all κ-small categories. □

Lemma 13.3.1 also yields right away:

Proposition 17.2.32. *Any functor between categories with κ-small colimits that commutes with strictly κ-small colimits commutes with κ-small colimits.*

Proposition 17.2.33. *Any category with strictly κ-small colimits as well as small κ-filtered colimits has small colimits.*

Proof. It follows from Corollary 17.2.30 and from Lemma 13.3.1 that any category with κ-small colimits as well as small κ-filtered colimits has small colimits. Therefore, it suffices to prove that any category with strictly κ-small colimits has κ-small colimits. Applying Lemma 13.3.1 once again, in order to conclude the proof, it suffices to observe finally that, since $[n]$ has a terminal object, any category has colimits indexed by $[n]$ for all $n\colon \mathbb{N}$. □

Proposition 17.2.34. *If A is κ-small then so is A^{op}.*

Proof. The assignment $A \mapsto A^{\mathrm{op}}$ is an equivalence of categories. Therefore, the full subcategory E of Cat spanned by those small categories A such that A^{op} is κ-small is stable under κ-small colimits in Cat. It contains $[n]$ for all n since $[n] = [n]^{\mathrm{op}}$. Therefore, any κ-small category belongs to E. □

Corollary 17.2.35. *If C has strictly κ-small colimits, then* $\mathrm{Ind}_\kappa(C)$ *is the full subcategory of* $\mathrm{PSh}(C)$ *spanned by presheaves $F\colon C^{\mathrm{op}} \to \mathrm{An}$ that preserve κ-small limits.*

17.3 Presentable categories with respect to a subuniverse

We fix once and for all in this section a small subuniverse $\kappa \hookrightarrow \mathrm{Cat}^{\simeq}$.

Notation 17.3.1. We denote by RexCat^κ the subcategory of Cat with morphisms the functors between small categories with κ-small colimits that commute with κ-small colimits. Given two small categories with κ-small colimits A and B, we write $\mathrm{Rex}^\kappa(A, B)$ for the full subcategory of $\mathrm{Fun}(A, B)$ spanned by functors that commute with κ-small colimits.

Recall from Construction 17.2.25 that we can assign to each small category C a small category $\langle C\rangle_\kappa$ with κ-small colimits equipped with a canonical fully faithful functor

$$h\colon C \to \langle C\rangle_\kappa\,.$$

It is an easy exercise to prove the following lemma using Proposition 13.2.14.

Lemma 17.3.2. *The category* RexCat_κ *has small limits and the embedding* $\mathrm{RexCat}_\kappa \hookrightarrow \mathrm{Cat}$ *commutes with them.*

Theorem 17.3.3. *Given a small category C and and a category D with κ-small colimits, the functor $h\colon C \to \langle C\rangle_\kappa$ induces an equivalence of categories of the form*

$$\mathrm{Rex}^\kappa(\langle C\rangle_\kappa, D) = \mathrm{Fun}(C, D)\,.$$

In particular, the embedding

$$\mathrm{RexCat}^\kappa \hookrightarrow \mathrm{Cat}$$

has a left adjoint given objectwise by $C \mapsto \langle C\rangle_\kappa$.

Proof. Up to enlarging the universe, we may assume that D is small as well. We will now prove the theorem under this extra assumption. It suffices to prove that h induces an equivalence of the form

$$\mathrm{Map}(A, \mathrm{Rex}^{\kappa}(\langle C\rangle_{\kappa}, D)) = \mathrm{Map}(A, \mathrm{Fun}(C, D))$$

for all A and D (we can even restrict to the case $A = [1]$). Since we have

$$\mathrm{Map}(A, \mathrm{Rex}^{\kappa}(\langle C\rangle_{\kappa}, D)) = \mathrm{Rex}^{\kappa}(\langle C\rangle_{\kappa}, \mathrm{Fun}(A, D))^{\simeq}$$

and

$$\mathrm{Map}(A, \mathrm{Fun}(C, D)) = \mathrm{Fun}(C, \mathrm{Fun}(A, D))^{\simeq}$$

it suffices to prove that h induces an equivalence of the form

$$\mathrm{Rex}^{\kappa}(\langle C\rangle_{\kappa}, D)^{\simeq} = \mathrm{Fun}(C, D)^{\simeq}\,.$$

Since $\mathrm{Ind}_{\kappa}(D)$ has small colimits and is locally small the Yoneda embedding

$$h\colon C \to \mathrm{PSh}(C)$$

induces an equivalence of the form

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{PSh}(C), \mathrm{Ind}_{\kappa}(D)) = \mathrm{Fun}(C, \mathrm{Ind}_{\kappa}(D))\,.$$

We can thus identify $\mathrm{Fun}(C, D)$ with the full subcategory of $\mathrm{Fun}^{\mathrm{L}}(\mathrm{PSh}(C), \mathrm{Ind}_{\kappa}(D))$ spanned by those small colimits preserving functors that preserve representable presheaves. Furthermore, we have a canonical equivalence

$$\mathrm{Ind}_{\kappa}(\langle C\rangle_{\kappa})) = \mathrm{PSh}(C)\,.$$

Therefore, each functor $\varphi\colon \langle C\rangle_{\kappa} \to D$ that preserves κ-small colimits will induce by Proposition 17.2.21 a left adjoint

$$\varphi_!\colon \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_{\kappa}(\langle C\rangle_{\kappa})), \mathrm{Ind}_{\kappa}(D))$$

that is totally determined by its restriction to C. More precisely, since

$$\mathrm{Ind}_{\kappa}(\mathrm{Iso}(D)) = \mathrm{Iso}(\mathrm{Ind}_{\kappa}(D))$$

we see that for two functor

$$\varphi, \varphi'\colon \langle C\rangle_{\kappa} \to D$$

there is a canonical equivalence of the form

$$(\varphi = \varphi') \xrightarrow{\sim} (\varphi_! = \varphi'_!)\,.$$

This shows that composing with h induces a map

$$\mathrm{Rex}^{\kappa}(\langle C\rangle_{\kappa}, D)^{\simeq} \to \mathrm{Fun}(C, D)^{\simeq}$$

that is an embedding. It suffices now to check that it has a section. If we start from a functor $\psi\colon C \to D$, we may consider the small colimits preserving functor

$$\psi_!\colon \mathrm{PSh}(C) \to \mathrm{Ind}_{\kappa}(D)$$

associated to

$$C \xrightarrow{\psi} D \hookrightarrow \mathrm{Ind}_\kappa(D)\,.$$

We observe that the restriction of $\psi_!$ to $\langle C\rangle_\kappa$ takes its values in representable presheaves on D: since D is stable under κ-small colimits in $\mathrm{Ind}_\kappa(D)$, the full subcategory of $\mathrm{PSh}(C)$ spanned by those presheaves whose image through $\psi_!$ is in D is stable by κ-small colimits and contains representable presheaves on C. The restriction of $\psi_!$ to $\langle C\rangle_\kappa$ thus yields a functor

$$\varphi\colon \langle C\rangle_\kappa \to D$$

such that $\varphi \circ h = \psi$. □

Lemma 17.3.4. *We consider given a small category with κ-small colimits and a family of maps $W \hookrightarrow \mathrm{Map}([1], C)$, with $\gamma\colon C \to L(C)$ the associated localization functor. We assume furthermore that γ has a right adjoint. If D is an other category with κ-small colimits, a functor $f\colon L(C) \to D$ commutes with κ-small colimits if and only if $f \circ \gamma\colon C \to D$ has the same property.*

Proof. Since the right adjoint δ of γ is fully faithful, we know that $L(C)$ has κ-small colimits and that they are computed by applying the right adjoint of γ, taking the colimit in C, and then applying γ to the result. This implies right away that, if f commutes with κ-small colimits, so does $f \circ \gamma$. Conversely, if $f \circ \gamma$ preserves colimit cocones indexed by κ-small categories, given any κ-small category I and any I-indexed functor $i \mapsto x_i$ with values in C, we have

$$\begin{aligned}
\mathrm{colim}\, f(x_i) &= \mathrm{colim}\,(f \circ \gamma)(\delta(x_i)) \\
&= (f \circ \gamma)(\mathrm{colim}\,\delta(x_i)) \\
&= f(\gamma(\mathrm{colim}\,\delta(x_i))) \\
&= f(\mathrm{colim}\,\gamma(\delta(x_i))) \\
&= f(\mathrm{colim}\, x_i)\,.
\end{aligned}$$

Therefore, the functor f commutes with κ-small colimits. □

Proposition 17.3.5. *Given any small category with κ-small colimits C, given a κ-presentable category D, the Yoneda embedding*

$$h\colon C \hookrightarrow \mathrm{Ind}_\kappa(C)$$

induces an equivalence of the form

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(C), D) \simeq \mathrm{RexCat}_\kappa(C, D)\,.$$

Proof. Let $\gamma\colon \langle C\rangle_\kappa \to C$ be the κ-small colimits preserving functor asociated to the identity of C (left adjoint to the Yoneda embedding of C into $\langle C\rangle_\kappa$). The Yoneda embedding of C induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{PSh}(C), D) \simeq \mathrm{Fun}(C, D)\,.$$

Using the equivalence

$$\mathrm{PSh}(C) = \mathrm{Ind}_\kappa(\langle C\rangle_\kappa)\,,$$

it than can be recasted as an equivalence of the form

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(\langle C\rangle_\kappa), D) \simeq \mathrm{RexCat}_\kappa(\langle C\rangle_\kappa, D)$$

by Theorem 17.3.3. On the other hand, there is a canonical commutative square of the form

$$\begin{array}{ccc} \mathrm{RexCat}_\kappa(C,D) & \hookrightarrow & \mathrm{Fun}(C,D) \\ {\scriptstyle \gamma^*}\downarrow & & \downarrow{\scriptstyle \gamma^*} \\ \mathrm{RexCat}_\kappa(\langle C\rangle_\kappa, D) & \hookrightarrow & \mathrm{Fun}(\langle C\rangle_\kappa, D) \end{array}$$

in which the two horizontal functors as well as the vertical one on the right are fully faithful, thus proving that the one on the left is fully faithful as well. This means that Lemma 17.3.4 may be reinterpreted by saying that the commutative square above is cartesian. Let

$$\gamma_!\colon \mathrm{Ind}_\kappa(\langle C\rangle_\kappa) \to \mathrm{Ind}_\kappa(C)$$

be the left Kan extension of γ (left adjoint of the full embedding γ^* of $\mathrm{Ind}_\kappa(C)$ into $\mathrm{PSh}(C)$). Precomposing with the localization $\gamma_!$ induces a fully faithful functor

$$\mathrm{Fun}(\mathrm{Ind}_\kappa(C), D) \hookrightarrow \mathrm{Fun}(\mathrm{Ind}_\kappa(\langle C\rangle_\kappa), D)$$

and thus a fully faithful functor of the form

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(C), D) \hookrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(\langle C\rangle_\kappa), D)\,.$$

The discussion above tells us in particular that the functor

$$h^*\colon \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_\kappa(C), D) \to \mathrm{RexCat}_\kappa(C,D)$$

is fully faithful, so that it only remains to prove its essential surjectivity. Let $\varphi\colon C \to D$ be a functor that preserves κ-small colimits. We get an adjunction

$$\varphi_!\colon \mathrm{PSh}(C) \rightleftarrows D\colon \varphi^*\,.$$

For any object y of D, the functor

$$\varphi^*(y) = \mathrm{Hom}_D(-, y)$$

preserves κ-small limits and thus belongs to $\mathrm{Ind}_\kappa(C)$ by Corollary 17.2.35. In other words, we have a unit-isomorphism of the form

$$\varphi^* = \gamma^*\,\gamma_!\,\varphi^*\,.$$

Equivalently, for any presheaf $F\colon C^{\mathrm{op}} \to \mathrm{An}$, the unit $\eta_F\colon F \to \gamma^*\,\gamma_!(F)$ induces an isomorphism of the form

$$\mathrm{Hom}(\gamma^*\,\gamma_!(F), \varphi^*(y)) = \mathrm{Hom}(F, \varphi^*(y))\,.$$

Let us consider the functor

$$\Phi\colon \mathrm{Ind}_\kappa(C) \to D$$

defined as $\Phi(F) = \varphi_!(\gamma^*(F))$. We observe that Φ is right adjoint to the functor

$$\Psi\colon D \to \mathrm{Ind}_\kappa(C)$$

defined through $\gamma^*\Psi(y) = \varphi^*(y)$. It is clear that $\Phi \circ h = \varphi$ by construction. □

Lemma 17.3.6. *Let us assume that the subuniverse κ is localizing. We consider given a small category with κ-small colimits and a family of maps $W \hookrightarrow \mathrm{Map}([1], C)$ stable under κ-small colimits, with $\gamma\colon C \to L(C)$ the associated localization functor. If D is an other category with κ-small colimits, a functor $f\colon L(C) \to D$ commutes with κ-small colimits if and only if $f \circ \gamma\colon C \to D$ has the same property.*

Proof. We will use Proposition 17.2.13. Let K be a strictly κ-small category. The equivalence

$$L(\mathrm{Fun}(K, C)) = \mathrm{Fun}(K, L(C))$$

yields a cover of the form

$$\mathrm{Fun}(K, C)^{\simeq} \to \mathrm{Fun}(K, L(C))^{\simeq}\,.$$

Since the localization functor $\gamma\colon C \to L(C)$ commutes with strictly κ-small colimits, this implies that f commutes with strictly κ-small colimits if and only if $f \circ \gamma$ has the same property. □

Corollary 17.3.7 (Right exact functors out of a localization)**.** *Let C and L be small categories with finite colimits, and let $\gamma\colon C \to L$ be a right exact localization at a family of morphisms W. For every category D with finite colimits, precomposition with γ induces an equivalence*

$$\mathrm{Rex}(L, D) \simeq \mathrm{Rex}^{W}(C, D),$$

where $\mathrm{Rex}^{W}(C, D)$ is the full subcategory of $\mathrm{Rex}(C, D)$ spanned by the functors sending every morphism in W to an isomorphism. Dually, for a left exact localization between small categories with finite limits, the corresponding assertion holds for left exact functors into any category with finite limits.

Proof. The family of all morphisms inverted by γ is closed under finite colimits in $\mathrm{Fun}([1], C)$, since γ is right exact, and defines the same localization as W. The finite posets form a localizing subuniverse by Example 17.2.12 and Corollary 17.2.14. Applying Lemma 17.3.6 to this family shows that a functor $f\colon L \to D$ is right exact if and only if $f\gamma$ is right exact. Thus the universal property of localization restricts to the displayed equivalence. The left exact version follows by passing to opposite categories. □

Proposition 17.3.8. *If the small subuniverse κ is localizing then RexCat_{κ} has small colimits.*

Proof. Given small categories with κ-small colimits A and B and a family W of maps in A, we denote by $\mathrm{RexCat}^{\kappa}_{W}(A, B)$ the full subcategory of $\mathrm{RexCat}_{\kappa}(A, B)$ spanned by those functors that send maps in W to isomorphisms of B.
Let I be a small category and

$$i \mapsto C_i$$

an I-indexed diagram with values in RexCat_{κ}. We let C' be the colimit of the aforementioned diagram in Cat. Since left adjoints preserve colimits, we have

$$\operatorname*{colim}_{i\colon I} \langle C_i \rangle_{\kappa} = \langle C' \rangle_{\kappa}$$

in RexCat_{κ}. For each $i\colon I^{\simeq}$, the canonical full embedding $C_i \hookrightarrow \langle C_i \rangle_{\kappa}$ has a left adjoint $\gamma_i\colon \langle C_i \rangle_{\kappa} \to C_i$. We denote by W_i the family of maps in $\langle C_i \rangle_{\kappa}$ that become invertible after applying γ_i. We denote by W'_i the image of W_i through the canonical functor

$$can_i\colon \langle C_i \rangle_{\kappa} \to \langle C' \rangle_{\kappa}\,.$$

We define W as the union of all the W_i''s, and $\bar{W}$ as the closure of W under κ-small colimits. We define finally C as the localization of $\langle C' \rangle_\kappa$ by $\bar{W}$.

Given a small category D with κ-small colimits, the functor γ_i being a localization with a right adjoint, Lemma 17.3.4 implies that we have a canonical equivalence of the form

$$\gamma_i^* \colon \mathrm{Rex}^\kappa(C_i, D) \xrightarrow{\sim} \mathrm{Rex}^\kappa_{W_i}(\langle C_i \rangle_\kappa, D)$$

for all $i \colon I^\simeq$. We also observe that the equivalence

$$\mathrm{Rex}^\kappa(\langle C' \rangle_\kappa, D)^\simeq = \lim_{i \colon I^{\mathrm{op}}} \mathrm{Rex}^\kappa(\langle C_i \rangle_\kappa, D)^\simeq$$

restricts to an equivalence of the form

$$\mathrm{Rex}^\kappa_W(\langle C' \rangle_\kappa, D)^\simeq = \lim_{i \colon I^{\mathrm{op}}} \mathrm{Rex}^\kappa_{W_i}(\langle C_i \rangle_\kappa, D)^\simeq .$$

Using Lemma 17.3.6 for the first line, this yields the following identifications

$$\begin{aligned} \mathrm{Rex}^\kappa(C, D)^\simeq &= \mathrm{Rex}^\kappa_{\bar{W}}(\langle C' \rangle_\kappa, D)^\simeq \\ &= \mathrm{Rex}^\kappa_W(\langle C' \rangle_\kappa, D)^\simeq \\ &= \lim_{i \colon I^{\mathrm{op}}} \mathrm{Rex}^\kappa_{W_i}(\langle C_i \rangle_\kappa, D)^\simeq \\ &= \lim_{i \colon I^{\mathrm{op}}} \mathrm{Rex}^\kappa(C_i, D)^\simeq \end{aligned}$$

thus proving that C it the colimit of $i \mapsto C_i$ in RexCat_κ. □

Proposition 17.3.9. *If κ is elementary then κ-small categories are stable under binary products in* Cat.

Proof. Since $[1] \times [1]$ is a pushout of a diagram of the form

$$[2] \leftarrow [1] \rightarrow [2] ,$$

and since $[n]$ is κ-small for any $n \colon \mathbb{N}$, the product $[1] \times [1]$ is κ-small. Since the functor $(-) \times [1]$ commutes with pushouts, one deduces that $[m] \times [1]$ is κ-small by induction on $m \colon \mathbb{N}$. Similarly, one deduces then that $[m] \times [n]$ is κ-small for all $m, n \colon \mathbb{N}$. The category of those small categories A such that $A \times [n]$ is κ-small for all n is stable under κ-small colimits and contains $[m]$ for all m. therefore, $A \times [n]$ is κ-small for all A κ-small and all $n \colon \mathbb{N}$. A similar argument shows that $A \times B$ is κ-small for any A and B κ-small. □

Corollary 17.3.10. *If κ is elementary, then κ-small categories are κ-compact in* Cat *and, for any κ-small category A, the functor*

$$\mathrm{Fun}(A, -) \colon \mathrm{Cat} \rightarrow \mathrm{Cat}$$

commutes with small κ-filtered colimits.

Proof. Since $\ulcorner$ is κ-compact in Cat and since κ-compact objects are stable under retracts, we know that $[n]$ is κ-compact for $n = 0$ and $n = 1$. Since κ-compact objects are stable under κ-small colimits, they are in particular stable under pushouts, so that $[n]$ is compact for all n. This implies that κ-small

categories are κ-compact in Cat. If A is a κ-compact category, then, in order to prove that $\mathrm{Fun}(A, -)$ commutes with small κ-filtered colimits, it suffices to check that the functor

$$\mathrm{Map}([1], \mathrm{Fun}(A, -)) = \mathrm{Map}([1] \times A, -)$$

commutes with small κ-filtered colimits. In other words, it suffices to know that $[1] \times A$ is κ-compact in Cat. Since it is κ-small by the preceding proposition, this achieves the proof. □

Corollary 17.3.11. *If κ is elementary, the property of having a right (left) adjoint is stable under κ-small colimits in* $\mathrm{Fun}([1], \mathrm{Cat})$.

Proof. Since $A \mapsto A^{\mathrm{op}}$ commute with κ-small colimits (being an equivalence), it suffices to ficus on the property of having a right adjoint section. But the functor $\mathrm{Fun}([1], -)$ commutes with κ-filtered colimits by the previous corollary and κ-filtered categories are distilled (since κ is elementary), so that κ-filtered colimits of categories commute with pullbacks (using the fact that the Rezk nerve is fully faithful and has a left adjoint, it suffices to prove that distilled colimits commute with pullbacks in $\mathrm{PSh}(\Delta)$). This implies right away that the property of having a right adjoint section is compatible with the formation of κ-filtered colimits. □

Corollary 17.3.12. *If the small subuniverse κ is elementary then* RexCat_κ *is κ-presentable. Furthermore, the forgetful functor*

$$\mathrm{RexCat}_\kappa \hookrightarrow \mathrm{Cat}$$

commutes with small κ-filtered colimits.

Proof. The forgetful functor $\mathrm{RexCat}_\kappa \hookrightarrow \mathrm{Cat}$ is conservative so that Theorem 17.3.3 implies that $\langle [1] \rangle_\kappa$ is a generator of RexCat_κ. Since $[1]$ is κ-compact in Cat, it suffices to prove that the forgetful functor

$$\mathrm{RexCat}_\kappa \hookrightarrow \mathrm{Cat}$$

commutes with small κ-filtered colimits. Let I be a small κ-filtered category and $i \mapsto C_i$ an I-indexed diagram with values in RexCat_κ. We denote by C the colimit of this diagram in Cat. For any κ-small category A, we have by Corollary 17.3.10

$$\operatorname*{colim}_{i\colon I} \mathrm{Fun}(A, C_i) = \mathrm{Fun}(A, C)\,.$$

Therefore, it follows right away from Corollary 17.3.11 that C has κ-small colimits. If D is any small category with κ-small colimits, and if $f\colon C \to D$ is a functor such that

$$f_i\colon C_i \xrightarrow{can_i} C \xrightarrow{f} D$$

commutes with κ-small colimits for all $i\colon I^{\simeq}$, we then have commutative squares of the form

$$\begin{array}{ccc} \mathrm{Fun}(A, C_i) & \xrightarrow{\mathrm{colim}_A} & C_i \\ \downarrow{\scriptstyle f_{i*}} & & \downarrow{\scriptstyle f_i} \\ \mathrm{Fun}(A, D) & \xrightarrow{\mathrm{colim}_A} & D \end{array}$$

forming an I-indexed cocone in $\mathrm{Fun}([1], \mathrm{Cat})$ hence yielding a commutative square of the form

$$\begin{array}{ccc} \mathrm{Fun}(A, C) & \xrightarrow{\mathrm{colim}_A} & C_i \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_i} \\ \mathrm{Fun}(A, D) & \xrightarrow{\mathrm{colim}_A} & D \end{array}$$

thus proving that f commutes with κ-small colimits. □

17.4 Idempotent completion

Definition 17.4.1. Let x and y be two objects of a category C. Recall that we say that x is a *retract* of y if there is a morphism of the form $q : y \to x$ that admits a section.

A *projector* is a morphism of the form $p : y \to y$ for some object y in a category C equipped with an identification $\pi : p = p^2$. We say that such a projector *has an image* if there exists an object x together with morphisms $q : y \to x$ and $s : x \to y$ such that s is a section of q, $p = s \circ q$ and $\pi : p = p^2$ is isomorphic to the identification coming from

$$\begin{aligned} p &= s \circ q \\ &= s \circ \mathrm{id}_x \circ q \\ &= s \circ q \circ s \circ q \\ &= p \circ p \,. \end{aligned}$$

Lemma 17.4.2. *If C is a category with ω-indexed colimits, then any projector has an image in C.*

Proof. Let $p : y \to y$ be a projector. We denote by Y the ω-indexed diagram corresponding to the sequence

$$y \xrightarrow{\mathrm{id}_y} y \xrightarrow{\mathrm{id}_y} y \to \cdots \to y \xrightarrow{\mathrm{id}_y} y \xrightarrow{\mathrm{id}_y} y \to \cdots$$

We then define an ω-indexed diagram X by forming the sequence of morphisms below.

$$y \xrightarrow{p} y \xrightarrow{p} y \to \cdots \to y \xrightarrow{p} y \xrightarrow{p} y \to \cdots$$

We clearly have

$$y = \operatorname*{colim}_{\omega} Y$$

and we define

$$x := \operatorname*{colim}_{\omega} X \,.$$

The sequence of commutative squares

$$\begin{array}{ccccccccc} y & \xrightarrow{\mathrm{id}_y} & y & \xrightarrow{\mathrm{id}_y} & y & \xrightarrow{\mathrm{id}_y} & y & \longrightarrow & \cdots \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \end{array}$$

defines a map $Q\colon Y \to X$ and similarly,

$$\begin{array}{ccccccccc} y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ y & \xrightarrow{\mathrm{id}_y} & y & \xrightarrow{\mathrm{id}_y} & y & \xrightarrow{\mathrm{id}_y} & y & \longrightarrow & \cdots \end{array}$$

defines a map $S\colon X \to Y$. This yields two maps

$$q\colon y \to x \quad \text{and} \quad s\colon x \to y$$

by passing to colimits. It remains to prove that s is a section of q. The composition $q \circ s$ is obtained from $Q \circ S$ by taking the colimit. But $Q \circ S$ simply is the sequence of maps

$$\begin{array}{ccccccccc} y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \\ \downarrow{\scriptstyle p^2} & & \downarrow{\scriptstyle p^2} & & \downarrow{\scriptstyle p^2} & & \downarrow{\scriptstyle p^2} & & \\ y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \end{array}$$

and $\pi\colon p = p^2$ yields an isomorphism from $Q \circ S$ to the sequence of maps

$$\begin{array}{ccccccccc} y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \end{array}$$

But the colimit of the latter sequence clearly is the identity of x, since $\omega_{1/} \to \omega$ is a terminal functor. □

Lemma 17.4.3. *Let C be a category and $q\colon y \to x$ a map in C equipped with a section $s\colon x \to y$. If we define $p := s \circ q$, then x canonically is the colimit of the diagram*

$$y \xrightarrow{p} y \xrightarrow{p} y \to \cdots \to y \xrightarrow{p} y \xrightarrow{p} y \to \cdots$$

Proof. We may assume that C is small. It suffices to prove the lemma for the image of the diagram through the Yoneda embedding of C: since the Yoneda embedding is fully faithful, if an object of C happens to be a colimit in $\mathrm{PSh}(C)$ of a diagram taking values in C, it must be a colimit in C as well. Therefore, we may assume that C has ω-indexed colimits. Let x' be the colimit of the diagram as stated in the lemma. The sequence of commutative squares defined by $q \circ p = q \circ s \circ q = q$

$$\begin{array}{ccccccccc} y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \\ x & \xrightarrow{\mathrm{id}_x} & x & \xrightarrow{\mathrm{id}_x} & x & \xrightarrow{\mathrm{id}_x} & x & \longrightarrow & \cdots \end{array}$$

induces a map $u\colon x' \to x$. Similarly, the sequence of commutative squares associated to the relation $p \circ s = s \circ q \circ s = s$

$$\begin{array}{ccccccccc} x & \xrightarrow{\mathrm{id}_x} & x & \xrightarrow{\mathrm{id}_x} & x & \xrightarrow{\mathrm{id}_x} & x & \longrightarrow & \cdots \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\ y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \end{array}$$

induces a map $v\colon x \to x'$. It is clear that $u \circ v = \mathrm{id}_x$ by functoriality, since $q \circ s = \mathrm{id}_x$. Finally, the composition $v \circ u$ is the colimit of the diagram

$$\begin{array}{ccccccccc} y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ y & \xrightarrow{p} & y & \xrightarrow{p} & y & \xrightarrow{p} & y & \longrightarrow & \cdots \end{array}$$

that is the identity of x', by a cofinality argument as at the end of the proof of Lemma 17.4.2. □

Proposition 17.4.4. *Let $\kappa \hookrightarrow \mathrm{Cat}$ be a small subuniverse. We consider a small category with κ-small colimits. Any retract of a representable presheaf on C is a κ-compact object in $\mathrm{Ind}_\kappa(C)$. Conversely, the collection of retracts of representable presheaves covers the collection of κ-compact objects in $\mathrm{Ind}_\kappa(C)$.*

Proof. Let y be an object of C and $q\colon \mathrm{Hom}_C(-, y) \to F$ a map in $\mathrm{PSh}(C)$ equipped with a section $s\colon F \to \mathrm{Hom}_C(-, y)$. The the property of comuting with κ-small limits is stable under retracts so that F belongs to $\mathrm{Ind}_\kappa(C)$. We already know that the property of being κ-compact is stable under retracts.
Conversely, let F be a compact object of $\mathrm{Ind}_\kappa(C)$. There is a small κ-filtered category I, a functor $\varphi\colon I \to C$ and an isomorphism of the form

$$\operatorname*{colim}_{i\colon I} h_C(\varphi(i)) = F\,.$$

Since F is κ-compact, we have an equivalence:

$$\operatorname*{colim}_{i\colon I} \mathrm{Hom}(F, h_C(\varphi(i))) \simeq \mathrm{Hom}(F, F)\,.$$

Since we have a canonical cover of the form

$$\sum_{i\colon I^\simeq} \mathrm{Hom}(F, h_C(\varphi(i))) \to \operatorname*{colim}_{i\colon I} \mathrm{Hom}(F, h_C(\varphi(i)))$$

this means that there exists $i\colon I^\simeq$ and a map $q\colon F \to h_C(\varphi(i))$ such that its composition with the structural map

$$h_C(\varphi(i)) \to \operatorname*{colim}_{i\colon I} h_C(\varphi(i)) \xrightarrow{\sim} F$$

is the identity of F. □

Corollary 17.4.5. *Let $\kappa \hookrightarrow \mathrm{Cat}$ be a small subuniverse and D be κ-presentable category. Then the full subcategory D^κ of κ-compact objects of D is small.*

Proof. By virtue of Theorem 17.2.24, we may assume that $D = \mathrm{Ind}_\kappa(C)$ for some small category C with κ-small colimits. Since the collection of retracts of representable presheaves forms a small family, we conclude with Proposition 17.4.4 and Proposition 11.2.18. □

Definition 17.4.6. A category C is called *idempotent complete* if any projector in C has an image.

Proposition 17.4.7. *Let $\kappa \hookrightarrow \mathrm{Cat}$ be a small subuniverse. A small category with κ-small colimits C is idempotent complete if and only if any κ-compact object of $\mathrm{Ind}_\kappa(C)$ is representable.*

Proof. If C is idempotent complete, then it follows from Proposition 17.4.4 that any κ-compact object of $\mathrm{Ind}_\kappa(C)$ is representable. Conversely, any projector in C has an image in $\mathrm{Ind}_\kappa(C)$ by Lemma 17.4.2, and such an image will be a κ-compact object of $\mathrm{Ind}_\kappa(C)$ by Proposition 17.4.4. Therefore, if any κ-compact object of $\mathrm{Ind}_\kappa(C)$ is representable, any projector of C has an image in C. □

Definition 17.4.8. Let $\kappa \hookrightarrow \mathrm{Cat}$ be a small subuniverse. Let us choose a universe of large categories Cat' (so that Cat is both an object and a full subcategory of Cat'). We define $\mathrm{Pr}^{\mathrm{L}}_\kappa$ as the subcategory of Cat' whose objects are the κ-presentable categories and whose morphisms are the functors that commute with small colimits and that preserve κ-compact objects.

Given A and B two κ-presentable categories, we denote by $\mathrm{Fun}^{\mathrm{L}}_\kappa(A,B)$ the full subcategory of $\mathrm{Fun}(A,B)$ spanned by the functors that preserve small colimits as well as κ-compact objects. In particular, we have:

$$\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_\kappa}(A,B) = \mathrm{Fun}^{\mathrm{L}}_\kappa(A,B)^{\simeq}\,.$$

Theorem 17.4.9. *Let $\kappa \hookrightarrow \mathrm{Cat}$ be a small subuniverse. Given a small category with κ-small colimits C and a κ-presentable category D, there is a canonical equivalence of categories of the form*

$$\mathrm{Rex}(C, D^\kappa) \simeq \mathrm{Fun}^{\mathrm{L}}_\kappa(\mathrm{Ind}(C), D)\,.$$

In particular, assigning the full subcategory of κ-compact objects defines a fully faithful functor

$$\mathrm{Pr}^{\mathrm{L}}_\kappa \to \mathrm{RexCat}^\kappa\,, \quad D \mapsto D^\kappa$$

that is right adjoint to the functor

$$\mathrm{Ind}_\kappa \colon \mathrm{RexCat} \to \mathrm{Pr}^{\mathrm{L}}_\kappa\,, \quad C \mapsto \mathrm{Ind}_\kappa(C)\,.$$

Proof. Corollary 17.4.5 and Theorem 17.2.24 ensure that

$$\mathrm{Ind}_\kappa(D^\kappa) = D\,.$$

By definition, there is pullback square of the form

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{L}}_\kappa(\mathrm{Ind}(C), D) & \longrightarrow & \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}(C), D) \\ \downarrow & \lrcorner & \downarrow \simeq \\ \mathrm{Rex}(C, D^\kappa) & \hookrightarrow & \mathrm{Rex}(C, D) \end{array}$$

in which, by virtue of Proposition 17.3.8, the vertical arrow on the right side is an equivalence. This proves the first assertion of the theorem. The second one follows from the first by applying the operator $(-)^{\simeq}$. □

Corollary 17.4.10. *For any small subuniverse $\kappa \hookrightarrow \mathrm{Cat}$, the category $\mathrm{Pr}^{\mathrm{L}}_\kappa$ is locally small and has small limits. If furthermore the subuniverse κ is localizing (e.g. if it is elementary) then $\mathrm{Pr}^{\mathrm{L}}_\kappa$ has small colimits as well.*

Proof. We know that RexCat_κ has small limits by Lemma 17.3.2 and that, if κ is furthermore localizing, it also has small colimits by Proposition 17.3.8. Since $\mathrm{Pr}^{\mathrm{L}}_\kappa$ is a full subcategory of RexCat_κ through a right adjoint, all kinds of (co-)limits that exist in RexCat_κ also exist in $\mathrm{Pr}^{\mathrm{L}}_\kappa$ by Corollary 14.2.5. □

Lemma 17.4.11. *If the subuniverse κ is elementary, then the full embedding $\mathrm{Pr}^{\mathrm{L}}_{\kappa} \hookrightarrow \mathrm{RexCat}^{\kappa}$ commutes with small κ-filtered colimits.*

Proof. Let I be a small κ-filtered category and $D\colon I \to \mathrm{Pr}^{\mathrm{L}}_{\kappa}$ a functor. We define

$$C := \operatorname*{colim}_{i\colon I} D(i)^{\kappa}\,.$$

For any κ-compactly generated category E we have:

$$\begin{aligned}\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ind}_{\kappa}(C), E) &= \mathrm{Rex}^{\kappa}(C, E)\\ &= \mathrm{Rex}^{\kappa}(\operatorname*{colim}_{i\colon I} D(i)^{\kappa}, E)\\ &= \lim_{i\colon I^{\mathrm{op}}} \mathrm{Rex}(D(i)^{\kappa}, E)\\ &= \lim_{i\colon I^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{L}}(D(i), E)\end{aligned}$$

obviously inducing an equivalence

$$\mathrm{Fun}^{\mathrm{L}}_{\kappa}(\mathrm{Ind}(C), E) \simeq \lim_{i\colon I^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{L}}_{\kappa}(D(i), E)$$

In particular, the colimit of $i \mapsto D(i)$ in $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ simply is $\mathrm{Ind}_{\kappa}(C)$. The lemma thus simply means that C is idempotent complete by Proposition 17.4.7. But, using that $[2]$ is compact in Cat, since I is filtered, we see that the collection of projectors in C is the colimit of the diagram assigning to each $i\colon I$ the collection of projectors in $D(i)$. The fact that each projector has an image in each $D(i)$ thus implies that each projector in C has an image in C as well. □

Theorem 17.4.12. *If the subuniverse κ is localizing (e.g. if it is elementary), then the category $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ is κ-presentable:*

$$\mathrm{Pr}^{\mathrm{L}}_{\kappa}\colon (\mathrm{Pr}^{\mathrm{L}}_{\kappa})^{\simeq}\,.$$

Proof. The preceding lemma allows us to apply Proposition 18.3.12. □

Remark 17.4.13. The functor $C \mapsto \mathrm{Ind}_{\kappa}(C)$ restricts to an equivalence from the full subcategory of RexCat^{κ} spanned by small idempotent complete categories with κ-small colimits to $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$: this is what the combination of Theorem 18.6.10 and of Proposition 17.4.7 means. In other words, for any small category with κ-small colimits C, the Yoneda embedding induces a fully faithful right exact functor $C \hookrightarrow \mathrm{Ind}_{\kappa}(C)^{\kappa}$ that is the universal κ-small colimits preserving functor from C to a small idempotent complete category with κ-small colimits. We call $\mathrm{Ind}_{\kappa}(C)^{\kappa}$ the *idempotent completion* of C.

18 Ind-objects

18.1 Compactly generated left Bousfield localizations

Recall the notion of compact object (Definition 16.3.6).

Definition 18.1.1. Let C be a locally small category with small filtered colimits. An object $x : C^{\simeq}$ is *compact* if the functor

$$\mathrm{Hom}_C(x, -) \colon C \to \mathrm{An}$$

commutes with small filtered colimits.

We denote by C_ω the full subcategory of C whose collection of objects consists of the compact objects of C.

Example 18.1.2. If A is a small category and $a : A^{\simeq}$, then $\mathrm{Hom}_A(-, a)$ is a compact object of $C = \mathrm{PSh}(A)$. Indeed, by virtue of the Yoneda Lemma, this amounts to claiming that the functor $F \mapsto F(a)$ commutes with filtered colimits, which is due to the fact that it commutes with all colimits.

Lemma 18.1.3. *Let C be a locally small category with small colimits. The full subcategory C_ω has finite colimits and the full embedding $C_\omega \hookrightarrow C$ is a right exact functor.*

Proof. By virtue of Proposition 16.4.12 and Proposition 16.4.13, this amounts to saying that the initial object is compact and that, for any pushout square of C of the form

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \xrightarrow{u'} & b' \end{array}$$

with a, b and a' compact, the object b is compact. Indeed, if $c : C$ is initial, then the functor $\mathrm{Hom}_C(c, -)$ is the constant functor with value the terminal object of An. Claiming that c is compact amounts to claiming that, for any small filtered category I, the I-indexed colimit functor preserves terminal objects in An, which follows from the fact that, by definition, filtered categories are weakly contractible. As for the second assertion, we observe that, for any small filtered category I, and for any functor $i \mapsto x_i$ with values in C, we have the following immediate computations, using the very

definition of compactness together with Theorem 16.2.9.

$$\begin{aligned}\operatorname*{colim}_{i:I} \mathrm{Hom}_C(b', x_i) &= \operatorname*{colim}_{i:I}\Big(\mathrm{Hom}_C(a', x_i) \times_{\mathrm{Hom}_C(a,x_i)} \mathrm{Hom}_C(b, x_i)\Big) \\ &= \operatorname*{colim}_{i:I} \mathrm{Hom}_C(a', x_i) \times_{\operatorname{colim}_{i:I} \mathrm{Hom}_C(a,x_i)} \operatorname*{colim}_{i:I} \mathrm{Hom}_C(b, x_i) \\ &= \mathrm{Hom}_C(a', \operatorname*{colim}_{i:I} x_i) \times_{\mathrm{Hom}_C(a, \operatorname{colim}_{i:I} x_i)} \mathrm{Hom}_C(b, \operatorname*{colim}_{i:I} x_i) \\ &= \mathrm{Hom}_C(b', \operatorname*{colim}_{i:I} x_i) \end{aligned}$$

□

Definition 18.1.4. Let C be a locally small category and $S \hookrightarrow \mathrm{Map}([1], C)$ a family of morphisms in C. An object $x : C^{\simeq}$ is said to be *S-local* if, for any map $f\colon a \to b$ belonging to S, the induced map

$$f^*\colon C(b, x) \to C(a, x)$$

is an equivalence.

We denote by $L_S(C)$ the full subcategory of C spanned by S-local objects.

A morphism $f\colon a \to b$ in C is called an *S-equivalence* if, for any S-local object x, the induced map

$$f^*\colon C(b, x) \to C(a, x)$$

is an equivalence.

Proposition 18.1.5. *Let C be a locally small category with small filtered colimits. If S is a family of morphisms between compact objects of C, then $L_S(C)$ has small filtered colimits and the full embedding $L_S(C) \hookrightarrow C$ commutes with small filtered colimits. In other words: any small filtered colimit of S-local objects in C is an S-local object.*

Proof. This is straightforward. □

The main goal of this section is to prove the theorem below.

Theorem 18.1.6. *Let C be a locally small category with small colimits and $S \hookrightarrow \mathrm{Map}([1], C_\omega)$ a small family of maps between compact objects of C. Then the full embedding $L_S(C) \hookrightarrow C$ has a left adjoint*

$$loc_S\colon C \to L_S(C)$$

exhibiting $L_S(C)$ as the localization of C by the family of S-equivalences. Furthermore, the functor loc_S preserves compact objects.

The proof of this theorem will need a little bit of preparation. Fix C locally small with filtered colimits.

Definition 18.1.7. A family of morphisms $S \hookrightarrow \mathrm{Map}([1], C)$ is *codiagonal-stable* if, for any map $u\colon a \to b$ in S, if we form the pushout square

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ {\scriptstyle u}\downarrow & & \downarrow \\ b & \longrightarrow & b \amalg_a b \end{array}$$

then the codiagonal map $v\colon b \amalg_a b \to b$ induced by the obvious commutative square

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ {\scriptstyle u}\downarrow & & \| \\ b & = & b \end{array}$$

belongs to S.

Lemma 18.1.8. *If S is codiagonal-stable, then an object $x : C^{\simeq}$ is S-local if and only if, for any map $u\colon a \to b$ in S, the induced map*

$$u^*\colon \operatorname{Hom}_C(b,x) \to \operatorname{Hom}_C(a,x)$$

has a section.

Proof. This is clearly a necessary condition. Conversely, let us consider an object x with the property that u^* has a section for any $u : S$. We observe that, for any map $u\colon a \to b$ in S, the codiagonal map $v\colon b \amalg_a b \to b$ has a section, hence

$$v^*\colon \operatorname{Hom}_C(b,x) \to \operatorname{Hom}_C(b \amalg_a b, x)$$

is a section. In other terms, v^* has a left inverse as well as a right inverse and is thus an equivalence. In other words, for any $u\colon a \to b$ in S, the induced commutative square

$$\begin{array}{ccc} \operatorname{Hom}_C(b,x) & = & \operatorname{Hom}_C(b,x) \\ \| & & \downarrow{\scriptstyle u^*} \\ \operatorname{Hom}_C(b,x) & \xrightarrow{u^*} & \operatorname{Hom}_C(a,x) \end{array}$$

is a pullback square. This means that the map $u^*\colon \operatorname{Hom}_C(b,x) \to \operatorname{Hom}_C(a,x)$ is an embedding. Since u^* also is a cover, it must be an isomorphism, by Proposition 11.2.10. □

Lemma 18.1.9. *The family of S-equivalences has the following stability properties.*

(i) It has the 2-out-of-3 property.

(ii) It is stable under small colimits: for any small category I and any functors $a, b\colon I \to C$ and any natural transformation

$$u\colon a \to b$$

if $u_i\colon a_i \to b_i$ is an S-equivalence for any object $i : I^{\simeq}$, then the induced morphism

$$\operatorname*{colim}_{i:I} u_i\colon \operatorname*{colim}_{i:I} a_i \to \operatorname*{colim}_{i:I} b_i$$

is an S-equivalence.

(iii) For any pushout square in C of the form

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \xrightarrow{u'} & b' \end{array}$$

with u an S-equivalence, the morphism u' is an S-equivalence.

Proof. Properties (i) and (ii) are straightforward (they follow right away from the analogous properties for the family of isomorphisms in An, using the fact that the functor $\mathrm{Hom}_C(x, -)$ commutes with small limits for any object x). As for property (iii), we simply observe that we have an induced pullback square of the form

$$\begin{array}{ccc} \mathrm{Hom}_C(b', x) & \xrightarrow{g^*} & \mathrm{Hom}_C(b, x) \\ {\scriptstyle (u')^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathrm{Hom}_C(a', x) & \xrightarrow{f^*} & \mathrm{Hom}_C(a, x) \end{array}$$

proving that u' is an S-equivalence whenever u has this property. □

Construction 18.1.10. Let $S \hookrightarrow \mathrm{Map}([1], C)$ be a family of maps in C. We define $\nabla(S)$ as the union in $\mathrm{Map}([1], C)$ of S with the image of the map

$$\kappa \colon S \to \mathrm{Map}([1], C)$$

that assigns to each $u \colon a \to b$ in S the associated codiagonal $\kappa(u) = v \colon b \amalg_a b \to b$. We define a sequence of embeddings of anima

$$S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow S_{n+1} \hookrightarrow \cdots$$

by defining $S_0 = S$ and $S_{n+1} = \nabla(S_n)$ for all $n : \mathbb{N}$. We put

$$\bar{S} = \operatorname*{colim}_{n:\omega} S_n \,.$$

Lemma 18.1.11. *Assume that $S \hookrightarrow \mathrm{Map}([1], C)$ is a small family of maps between compact objects. Then the following properties hold.*

(i) The family $\bar{S}$ is a small family of maps between compact objects.

(ii) The family $\bar{S}$ is codiagonal-stable.

(iii) An object $x : C^{\simeq}$ is S-local if and only if it is $\bar{S}$-local.

Proof. It is clear that $\nabla(S)$ is small whenever S is small. Therefore, each S_n is small. Since small animae are stable under ω-indexed colimits, this proves that $\bar{S}$ is small. The second part of property (i) follows right away from Lemma 18.1.3. The property for $\bar{S}$ to be codiagonal-stable amounts to the property that $\bar{S}$ contains $\nabla(\bar{S})$, which holds by construction. Finally, it is clear that any $\bar{S}$-local object is S-local because S is a subfamily of $\bar{S}$ by construction. Conversely, it suffices to prove that any S-local object x is also $\nabla(S)$-local, which follows right away from properties (i) and (iii) of Lemma 18.1.9. □

Proof of Theorem 18.1.6 through the small object argument. By virtue of Lemma 18.1.11, we may assume that S is codiagonal-stable. We define a functor

$$L \colon C \to C$$

as well as a natural transformation $l \colon \mathrm{id}_C \to L$ as follows. There is a functor

$$\mathrm{An} \times C \to C \,, \quad (a, x) \mapsto a \otimes x$$

defined by the functorial formula

$$\mathrm{Hom}_C((a \otimes x)^{\mathrm{op}}, y) = \mathrm{Hom}_{\mathrm{An}}(a^{\mathrm{op}}, \mathrm{Hom}_C(x^{\mathrm{op}}, y))$$

for $a : \mathrm{An}$ and $x, y : C$. More precisely, if a is an anima and x an object of C, then $a \otimes x$ is the a-indexed colimit of the constant diagram with value x.

Given $x : C$ and an object a of C, there is thus a canonical map

$$c_{a,x} \colon \mathrm{Hom}_C(a, x) \otimes a \to x$$

corresponding to the identity of $\mathrm{Hom}_C(a, x)$, functorially in x. We define $A(x)$ as the colimit indexed by S of the diagram that assigns $\mathrm{Hom}_C(a, x) \otimes a$ for each term $u \colon a \to b$ of S. Similarly, we define $B(x)$ as the colimit indexed by S of the diagram that assigns $\mathrm{Hom}_C(a, x) \otimes b$ for each term $u \colon a \to b$ of S. We define

$$\alpha_x \colon A(x) \to x$$

as the map induced by the family $c_{a,x}$ indexed by each term $u \colon a \to b$ of S. We form the pushout square

$$\begin{array}{ccc} A(x) & \xrightarrow{\alpha_x} & x \\ {\scriptstyle i_x}\downarrow & & \downarrow{\scriptstyle l_x} \\ B(x) & \xrightarrow{\beta_x} & L(x) \end{array}$$

where i_x is the S-indexed colimit of the maps

$$\mathrm{id} \otimes u \colon \mathrm{Hom}_C(a, x) \otimes a \to \mathrm{Hom}_C(a, x) \otimes b\,.$$

We now define a sequence of functors

$$L^0 \xrightarrow{l^1} L^1 \xrightarrow{l^2} \cdots \to L^n \xrightarrow{l^{n+1}} L^{n+1} \to \cdots$$

by induction through the formulas

$$L^0 = \mathrm{id}_C \quad \text{and} \quad L^{n+1} = L \circ L^n\,, \quad l^{n+1} = l_{L^n}\,.$$

We define

$$\lambda_x \colon x \to loc_S(x)$$

functorially in x by the formula

$$loc_S(x) = \operatorname*{colim}_{n:\omega} L^n(x)$$

with λ_x the map induced by the component of the colimit cocone at $0 : \mathbb{N}$. It follows from Lemma 18.1.9 that l_x is an S-equivalence for all x. We can also see the map λ_x as the ω-indexed colimit of the natural transformation

$$\begin{array}{ccccccccccc} x & = & x & = & \cdots & = & x & = & x & = & \cdots \\ \| & & {\scriptstyle l_x}\downarrow & & & & \downarrow & & \downarrow & & \\ x & \xrightarrow{l_x} & L(x) & \longrightarrow & \cdots & \longrightarrow & L^n(x) & \xrightarrow{l_{L^n(x)}} & L^{n+1}(x) & \longrightarrow & \cdots \end{array}$$

in which the vertical maps are S-equivalences (this is proven by induction, as finite compositions of maps of the form l_y for various y's). Therefore, Lemma 18.1.9 (ii) implies that λ_x is an S-equivalence for any x.

We claim that $loc_S(x)$ is S-local. Indeed, for any $u\colon a \to b$ in S, and for any $n : \mathbb{N}$, there is a canonical commutative diagram of the form

$$\begin{array}{ccccc}
\mathrm{Hom}_C(b, L^n(x)) & \xrightarrow{(l_x^{n+1})_*} & \mathrm{Hom}_C(b, L^{n+1}(x)) & \longrightarrow & \mathrm{Hom}_C(b, loc_S(x)) \\
\downarrow{\scriptstyle u^*} & \overset{\lambda_n}{\nearrow} & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle u^*} \\
\mathrm{Hom}_C(a, L^n(x)) & \xrightarrow{(l_x^{n+1})_*} & \mathrm{Hom}_C(a, L^{n+1}(x)) & \longrightarrow & \mathrm{Hom}_C(a, loc_S(x))
\end{array}$$

in which the dashed arrow λ_n is induced by the map $\beta_{L^n(x)}$. We can think of the collection of the maps λ_n as a morphism of sequences of morphisms of anima:

$$\begin{array}{ccccccccc}
\mathrm{Hom}_C(a, x) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_C(a, L^n(x)) & \xrightarrow{(l_x^{n+1})_*} & \mathrm{Hom}_C(a, L^{n+1}(x)) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \lambda_0} & & & & \downarrow{\scriptstyle \lambda_n} & & \downarrow{\scriptstyle \lambda_{n+1}} & & \\
\mathrm{Hom}_C(b, L(x)) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_C(b, L^{n+1}(x)) & \longrightarrow & \mathrm{Hom}_C(b, L^{n+2}(x)) & \longrightarrow & \cdots
\end{array}$$

Taking the associated Milnor colimits yields a morphism

$$\lambda'\colon \operatorname*{colim}_{n:\omega} \mathrm{Hom}_C(a, L^n(x)) \to \operatorname*{colim}_{n:\omega} \mathrm{Hom}_C(b, L^{n+1}(x))\,.$$

Since ω is filtered, the embedding $\omega_{1/} \to \omega$ is terminal, hence we have

$$\operatorname*{colim}_{n:\omega} \mathrm{Hom}_C(b, L^{n+1}(x)) = \operatorname*{colim}_{n:\omega} \mathrm{Hom}_C(b, L^n(x))\,.$$

The map λ' thus yields a map

$$\lambda\colon \operatorname*{colim}_{n:\omega} \mathrm{Hom}_C(a, L^n(x)) \to \operatorname*{colim}_{n:\omega} \mathrm{Hom}_C(b, L^n(x))\,.$$

Since $\mathrm{Hom}_C(t, loc_S(x))$ canonically is the colimit of the $\mathrm{Hom}_C(t, L^n(x))$'s for any compact object t, the map λ determines a map

$$\bar{\lambda}\colon\ \mathrm{Hom}_C(a, loc_S(x)) \to \mathrm{Hom}_C(b, loc_S(x))$$

that is a section of the map

$$u^*\colon\ \mathrm{Hom}_C(b, loc_S(x)) \to \mathrm{Hom}_C(a, loc_S(x))$$

by construction. Lemma 18.1.8 thus proves that $loc_S(x)$ is S-local.

Consider now an S-local object y in C. Then, for any object $x : C^{\simeq}$, the S-equivalence λ_x induces an isomorphism

$$\lambda_x^*\colon\ \mathrm{Hom}_C(loc_S(x), y) \xrightarrow{\simeq} \mathrm{Hom}_C(x, y)\,.$$

Since this equivalence is obviously functorial in y, this shows that the full embedding of $L_S(C)$ into C is a right adjoint by Theorem 12.3.20.

The characterization of $L_S(C)$ as localization of C by S-equivalences follows from Proposition 4.4.16 and from Corollary 12.3.14 applied to $L_S(C)$.

It remains to prove that the left adjoint loc_S preserves compact objects. Let $x : C^{\simeq}$ be a compact object. We consider a small filtered category I as well as an I-indexed functor $i \mapsto y_i$ with values

in $L_S(C)$. We know from Proposition 18.1.5 that the colimit of the diagram of the y_i's can be computed independently in $L_S(C)$ or in C itself. In other words, we have

$$\begin{aligned}\operatorname*{colim}_{i:I} \mathrm{Hom}_{L_S(C)}(loc_S(x), y_i) &= \operatorname*{colim}_{i:I} \mathrm{Hom}_C(x, y_i) \\ &= \mathrm{Hom}_C(x, \operatorname*{colim}_{i:I} y_i) \\ &= \mathrm{Hom}_{L_S(C)}(loc_S(x), \operatorname*{colim}_{i:I} y_i)\,,\end{aligned}$$

thus proving that $loc_S(x)$ is a compact object of $L_S(C)$. □

Recall that $\mathrm{Fun}_!(C, D)$ is the full subcategory $\mathrm{Fun}(C, D)$ spanned by functors commuting with small colimits.

Corollary 18.1.12. *Let C be a locally small category with small colimits and S a small family of morphisms between compact objects of C, with associated localization functor*

$$loc_S \colon C \to L_S(C)\,.$$

Given any category D with small colimits, we denote by $\mathrm{Fun}_!^S(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ spanned by colimit preserving functors sending any morphism in S to an isomorphism of D. Then pre-composing with the functor loc_S induces an equivalence

$$\mathrm{Fun}_!(L_S(C), D) \simeq \mathrm{Fun}_!^S(C, D)$$

for any category D with small colimits.

Proof. By virtue of Theorem 18.1.6 pre-composing with loc_S induces a full embedding of the form

$$\mathrm{Fun}_!(L_S(C), D) \hookrightarrow \mathrm{Fun}_!^S(C, D)$$

It suffices to prove that the induced map

$$\mathrm{Fun}_!(L_S(C), D)^{\simeq} \hookrightarrow \mathrm{Fun}_!^S(C, D)^{\simeq}$$

is a cover. Let $F \colon C \to D$ be a small-colimit-preserving functor sending all maps in S to isomorphisms in D. We define $F_S \colon L_S(C) \to D$ as the following composite:

$$F_S \colon L_S(C) \hookrightarrow C \xrightarrow{F} D\,.$$

It suffices to prove two properties:

(i) $F_S \circ loc_S = F$ holds (i.e. is inhabited);

(ii) F_S commutes with small colimits.

The second property follows from the first: we know that small colimits in $L_S(C)$ can always be computed by forming first the colimits in C and then applying loc_S. Let W be the family of morphisms of C that are sent to isomorphisms of D. Keeping track of the explicit construction of the functor loc_S through the small object argument as in the proof of Theorem 18.1.6, we observe that, for any object x of C, the unit map $x \to loc_S(x)$ belongs to W. This implies right away that $F_S \circ loc_S = F$ holds true. □

18.2 Left exact presheaves

Notation 18.2.1. Let C and D be two categories. We denote by $\mathrm{Lex}(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ whose objects are the left exact functors from C to D. In other words,

$$\mathrm{Lex}(C, D) = \mathrm{Fun}^{\kappa}(C^{\mathrm{op}}, D^{\mathrm{op}})^{\mathrm{op}}$$

with κ the subuniverse of finite posets.

Proposition 18.2.2. *Let C be a small category with finite colimits. The full embedding*

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \hookrightarrow \mathrm{PSh}(C)$$

commutes with filtered colimits and has a left adjoint.

Proof. The first assertion is a particular case of Lemma 17.2.17 for κ the subuniverse of finite posets. The construction of the left adjoint is a particular case of Construction 17.2.25.
The proposition can also be proved with the small object argument as follows. Let $h\colon C \to \mathrm{PSh}(C)$ be the Yoneda embedding. If $\emptyset_C$ denotes the initial object of C, there is a canonical comparison map

$$i\colon \emptyset \to h(\emptyset_C)\,.$$

Similarly, given any pushout square in C of the form

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \xrightarrow{u'} & b' \end{array}$$

there is a canonical comparison map

$$comp\colon h(a') \amalg_{h(a)} h(b) \to h(b')\,.$$

This defines a map from the collection of pushout squares of C into $\mathrm{Map}([1], \mathrm{PSh}(C))$, the image of which we will denote by S'. We write S for the union of S' with the image of the map from $[0]$ to $\mathrm{Map}([1], \mathrm{PSh}(C))$ corresponding to i above. We observe the following properties:

(a) An object $F\colon C^{\mathrm{op}} \to \mathrm{An}$ of $\mathrm{PSh}(C)$ is S-local if and only if it is a left exact functor;

(b) The anima S is small and consists of morphisms between compact objects of $\mathrm{PSh}(C)$.

Indeed, property (a) is a direct translation of the definitions through the Yoneda lemma (under the form of Corollary 12.3.14). The smallness of S follows from Proposition 11.2.18. The last assertion is a direct consequence of Example 18.1.2 and Lemma 18.1.3.
The proposition is now a particular case of Proposition 18.1.5 and Theorem 18.1.6. □

Corollary 18.2.3. *If C is a small category with finite colimits, then the category* $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$ *of left exact functors from C^{op} to* An *has small colimits as well as small limits.*

Proof. This is a particular case of Corollary 17.2.26. □

Proposition 18.2.4. *Let C be a small category with finite colimits. For any object $x : C^{\simeq}$, the representable presheaf* $\mathrm{Hom}_C(-, x)$ *is a left exact functor from* C^{op} *to* An*. Furthermore, the Yoneda embedding*

$$h\colon C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$$

is a right exact functor.

Proof. This is a particular case of Lemma 17.2.16. □

Corollary 18.2.5. *If C is a small category with finite colimits, then, for any object $x : C^{\simeq}$, the associated representable presheaf $h_C(x) = \mathrm{Hom}_C(-, x)$ is a compact object of* $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$.

Proof. For any left exact functor $F\colon C^{\mathrm{op}} \to \mathrm{An}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Lex}(C^{\mathrm{op}},\mathrm{An})}(h_C(x), F) &= \mathrm{Hom}_{\mathrm{PSh}(C)}(h_C(x), F)\\ &= F(x)\,.\end{aligned}$$

Since the functor

$$\mathrm{PSh}(C) \to \mathrm{An}\,, \quad F \mapsto F(x)$$

commutes with colimits and since the embedding of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$ into $\mathrm{PSh}(C)$ commutes with filtered colimits (see Proposition 18.2.2), this implies that $h_C(x)$ is a compact object of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$. □

Definition 18.2.6. Let C be a category. We let Γ be the collection of all functors of the form $p\colon X \to C$ where X is a small filtered category. The category of (small) *ind-objects* in C is the full subcategory of the category of presheaves on C (with values in the category of large animae for a large enough universe) spanned by the image of the map

$$p\colon X \to C \quad \mapsto \quad \operatorname*{colim}_{x:X} h_C(p(x))\,.$$

The Yoneda embedding of C induces a full embedding of the form

$$C \hookrightarrow \mathrm{Ind}(C)\,.$$

Remark 18.2.7. Another way to rephrase the definition is

$$\mathrm{Ind}(C) = \mathrm{Ind}_\kappa(C)$$

with κ the subuniverse of finite posets. Therefore, a particular case of Corollary 17.2.20 tells us that, for any small category with finite colimits, we have:

$$\mathrm{Ind}(C) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})\,.$$

Theorem 18.2.8. *Let $u\colon C \to D$ be a right exact functor between small categories with finite colimits. Then the adjunction*

$$u_!\colon \mathrm{PSh}(C) \rightleftarrows \mathrm{PSh}(D)\colon u^*$$

restricts to an adjunction of the form

$$u_!\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \rightleftarrows \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An})\colon u^*\,.$$

Proof. This isa particular case of Proposition 17.2.21. □

Notation 18.2.9. Given two categories with finite colimits C and D, we denote by $\mathrm{Rex}(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ spanned by finite colimit preserving functors.

Theorem 18.2.10. *Let C be a small category with finite colimits. For any category D with small colimits, precomposing with the Yoneda embedding*

$$h_C : C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$$

induces an equivalence of the form

$$\mathrm{Fun}_!(\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}), D) \simeq \mathrm{Rex}(C, D)\,.$$

Proof. This is a particular case of Proposition 17.3.5. □

18.3 Compactly generated categories

Construction 18.3.1. Let C be a category with finite colimits, and let $i : E \hookrightarrow C$ be a full subcategory. We consider the functor

$$\mathrm{colim}_{\ulcorner} : \mathrm{Fun}(\ulcorner, E) \to C$$

and define $U(E)^{\simeq} \hookrightarrow C^{\simeq}$ as the image of the induced map

$$\mathrm{Fun}(\ulcorner, E)^{\simeq} \xrightarrow{i^{\simeq}} \mathrm{Fun}(\ulcorner, C)^{\simeq} \xrightarrow{\mathrm{colim}_{\ulcorner}^{\simeq}} C^{\simeq}\,.$$

We define $U(E)$ as the full subcategory of C with collection of objects $U(E)^{\simeq}$, with corresponding embedding

$$i_{U(E)} : U(E) \hookrightarrow C\,.$$

Since the pushout of any diagram of the form $x \xleftarrow{\mathrm{id}_x} x \xrightarrow{\mathrm{id}_x} x$ is x itself, the functor $\mathrm{const}_{\ulcorner} : E \to \mathrm{Fun}(\ulcorner, E)$ induces a canonical functor

$$u_E : E \to U(E)$$

such that $i_{U(E)} \circ u_E \simeq i$. We observe that, by virtue of Proposition 11.2.18, if C is locally small and if E is small, then $U(E)$ is small as well (as full subcategory of a locally small category with a small collection of objects).

We define $U_0(E)$ as the full subcategory of C with objects the initial object as well as the terms of $E^{\simeq}$. We denote by

$$i_0 : U_0(E) \hookrightarrow C$$

the embedding. We define for any $n : \mathbb{N}$

$$u_n(E) : U_n(E) \hookrightarrow U_{n+1}(E)$$

through $U_{n+1}(E) = U(U_n(E))$ and $u_n(E) = u_{U_n(E)}$. There is also a full embedding

$$i_{n+1} : U_{n+1}(E) \to C$$

defined by $i_{n+1} = i_{U(U_n(E))}$. The sequence of maps $u_n(E), n : \mathbb{N}$, defines a functor $U_\bullet(E) : \omega \to \mathrm{Cat}$. We denote by $\langle E \rangle_{\mathrm{fp}}$ the colimit of $U_\bullet(E)$. The family of full embeddings $i_n : U_n(E) \hookrightarrow C$ induces a canonical full embedding $\langle E \rangle_{\mathrm{fp}} \hookrightarrow C$ by Corollary 16.3.12. It follows from Lemma 16.4.4 that any functor from $\ulcorner$ to $\langle E \rangle_{\mathrm{fp}}$ factors through $U_n(E)$ for some n, hence that the colimit of such a diagram belongs to $U_{n+1}(E)$, hence to $\langle E \rangle_{\mathrm{fp}}$. This implies that $\langle E \rangle_{\mathrm{fp}}$ has finite colimits and that its embedding into C is right exact.

Definition 18.3.2. Given a full subcategory E of a category C with finite colimits, we call $\langle E \rangle_{\mathrm{fp}}$ the *smallest finitely cocomplete full subcategory of C generated by E*.

Remark 18.3.3. This terminology is justified by the fact that, given any full subcategory D of C, if D is closed under finite colimits in C (i.e. D has finite colimits and the embedding of D into C is right exact), the property that all objects of E are objects of D is equivalent to the property that all objects of $\langle E \rangle_{\mathrm{fp}}$ are objects of D (exercise left to the reader). We also observe that, if C is locally small and if $E^{\simeq}$ is small, then $\langle E \rangle_{\mathrm{fp}}$ is small.

Definition 18.3.4. A *compactly generated category* is a category D with the following properties:

(i) D is locally small;

(ii) D has small colimits;

(iii) there is given a small family of compact objects $E \hookrightarrow D^{\simeq}$ that generates D in the sense that, for any morphism $f : x \to y$ in D, if the induced map

$$f_* : \mathrm{Hom}_D(e, x) \to \mathrm{Hom}_D(e, y)$$

is an isomorphism for any $e : E$, then f is an isomorphism in D.

Example 18.3.5. The category An is compactly generated with generating family of compact objects the terminal objects of An.

Example 18.3.6. The category Cat of small categories is compactly generated: a family of generators is given by the single object $[1]$ by Corollary 6.3.9 and Theorem 16.3.9.

Example 18.3.7. If C is a small category with finite colimits, then $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$ is compactly generated, the representable presheaves forming a generating family of compact objects by Corollary 18.2.5. This class of examples covers all cases, as expressed by the following theorem.

Theorem 18.3.8. *Let D be a compactly generated category. We consider a small family of compact objects $E \hookrightarrow D^{\simeq}$ that generates D and form the smallest finitely cocomplete full subcategory $C = \langle E \rangle_{\mathrm{fp}}$ of D generated by E. Then there is a canonical equivalence*

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \simeq D$$

whose composition with the Yoneda embedding yields the embedding $C \hookrightarrow D$.

Proof. This is a particular case of Theorem 17.2.24. □

Corollary 18.3.9. *If D is compactly generated, then D has small limits.*

Proof. This follows from Corollary 18.2.3 and Theorem 18.3.8. □

Corollary 18.3.10. *Any small category is a small filtered colimit of finite categories. More precisely, there is a canonical equivalence* $\mathrm{Lex}(\mathrm{FinCat}^{\mathrm{op}}, \mathrm{An}) = \mathrm{Cat}$.

Proof. We know that Cat is compactly generated by virtue of Corollary 6.3.9 and Theorem 16.3.9. This yields the equivalence $\mathrm{Lex}(\mathrm{FinCat}^{\mathrm{op}}, \mathrm{An}) = \mathrm{Cat}$ by Theorem 18.3.8. The first assertion is then a particular case of Corollary 17.2.35 with $C = \mathrm{FinCat}$ and κ the subuniverse of finite posets, using the definition of $\mathrm{Ind}_\kappa(C)$. □

Corollary 18.3.11. *A category has small colimits if and only if it has finite colimits as well as small filtered colimits. Similarly, if C and D are categories with small colimits, a functor $f : C \to D$ commutes with small colimits if and only if it commutes with finite colimits as well as with small filtered colimits.*

Proof. This is a direct application of the first assertion of the previous corollary and of Lemma 13.3.1. □

Proposition 18.3.12. *Let D be a compactly generated category and E a locally small category with small colimits. We consider a functor*

$$f^* : D \to E$$

admitting a conservative right adjoint

$$f_* : E \to D\,.$$

Then the following conditions are equivalent.

(i) The left adjoint functor f^* preserves compact objects.

(ii) There is a generating family of compact objects of D that are sent to compact objects of E through f^*.

(iii) The right adjoint functor f_* commutes with small filtered colimits.

In particular, if one of the equivalent conditions above is fulfilled, then E is compactly generated.

Proof. It is clear that (ii) follows from (i). Let $X \hookrightarrow D^{\simeq}$ be a small family of compact objects generating D. Let us assume furthermore that $f^*(x)$ is compact in E for all $x : X$. We consider a small filtered category I and an I-indexed diagram $i \mapsto y_i$ taking values in E. We want to prove that the canonical map

$$\operatorname*{colim}_{i:I} f_*(y_i) \to f_*(\operatorname*{colim}_{i:I} y_i)$$

is an isomorphism. This follows from the computations below, for each $x : X$.

$$\begin{aligned}
\mathrm{Hom}_D(x, \operatorname*{colim}_{i:I} f_*(y_i)) &= \operatorname*{colim}_{i:I} \mathrm{Hom}_D(x, f_*(y_i)) \\
&= \operatorname*{colim}_{i:I} \mathrm{Hom}_E(f^*(x), y_i) \\
&= \mathrm{Hom}_E(f^*(x), \operatorname*{colim}_{i:I} y_i) \\
&= \mathrm{Hom}_D(x, f_*(\operatorname*{colim}_{i:I} y_i))\,.
\end{aligned}$$

This implies right away that assertion (iii) is a consequence of assertion (ii). Finally, let us assume that assertion (iii) holds true, and let $x : D^{\simeq}$ be a compact object of D. We want to prove that $f^*(x)$ is compact. We consider again a small filtered category I and an I-indexed diagram $i \mapsto y_i$ taking values in E and compute:

$$\begin{aligned}\operatorname*{colim}_{i:I} \mathrm{Hom}_E(f^*(x), y_i) &= \operatorname*{colim}_{i:I} \mathrm{Hom}_D(x, f_*(y_i)) \\ &= \mathrm{Hom}_D(x, \operatorname*{colim}_{i:I} f_*(y_i)) \\ &= \mathrm{Hom}_D(x, f_*(\operatorname*{colim}_{i:I} y_i)) \\ &= \mathrm{Hom}_E(f^*(x), \operatorname*{colim}_{i:I} y_i)\,.\end{aligned}$$

This completes the proof. □

18.4 Left Bousfield localization of compactly generated categories

Proposition 18.4.1. *Let D be a compactly generated category and $S \hookrightarrow \mathrm{Map}([1], D)$ a small family of morphisms between compact objects. Then the category of S-local objects $L_S(D)$ is compactly generated.*

Proof. By virtue of Theorem 18.1.6, the full embedding

$$L_S(D) \hookrightarrow D$$

has a left adjoint

$$loc_S \colon D \to L_S(D)$$

that preserves compact objects. Let $E \hookrightarrow D^{\simeq}$ be a small family of compact objects in D. We denote by $E_S \hookrightarrow L_S(D)^{\simeq}$ the image of the composed map

$$E \hookrightarrow D^{\simeq} \xrightarrow{loc_S^{\simeq}} L_S(D)^{\simeq}\,.$$

Proposition 11.2.18 ensures that E_S is small and all of its terms are compact objects of $L_S(D)$. By construction, the map $p \colon E \to E_S$ is a cover. It suffices now to check that the family of functors of the form

$$\mathrm{Hom}_{L_S(D)}(y, -) \quad \text{for all } y : E_S$$

is conservative. Let $f \colon a \to b$ be a morphism in $L_S(D)$ such that

$$f_* \colon \mathrm{Hom}_{L_S(D)}(y, a) \xrightarrow{\simeq} \mathrm{Hom}_{L_S(D)}(y, b)$$

is an isomorphism for all $y : E_S^{\simeq}$. This implies that

$$f_* \colon \mathrm{Hom}_{L_S(D)}(p(x), a) \xrightarrow{\simeq} \mathrm{Hom}_{L_S(D)}(p(x), b)$$

is an isomorphism for all $x : E^{\simeq}$. But $p(x) = loc_S(x)$ and this means that

$$f_* \colon \mathrm{Hom}_D(x, a) \xrightarrow{\simeq} \mathrm{Hom}_D(x, b)$$

is an isomorphism for all $x : E^{\simeq}$. In other words, the image of f through the full embedding of $L_S(D)$ into D is an isomorphism. Since full embeddings are conservative, this proves that f is an isomorphism in $L_S(D)$. □

Proposition 18.4.2. *Let D be a compactly generated category and $C \hookrightarrow D$ a small full subcategory such that $C^{\simeq}$ is a generating family of compact objects in D. We consider a family of morphisms*

$$S \hookrightarrow \mathrm{Map}([1], C)$$

with associated localization functor by S-equivalences

$$loc_S \colon D \to L_S(D)$$

We let $W \hookrightarrow C$ be the subcategory of morphisms in C that become invertible in $L_S(D)$ through the functor loc_S, with associated localization functor $\gamma \colon C \to L(C)$, and we define C_S as the full subcategory of $L_S(D)$ spanned by the image of the map

$$C^{\simeq} \hookrightarrow D^{\simeq} \xrightarrow{loc_S^{\simeq}} L_S(D)^{\simeq}\,.$$

Then there is a canonical equivalence

$$L(C) \simeq C_S\,.$$

Proof. The functor loc_S obviously restricts to a right exact functor

$$l \colon C \to C_S$$

that sends maps in W to isomorphisms. Corollary 17.1.8 (iv) implies that there is a unique right exact functor

$$\lambda \colon L(C) \to C_S$$

equipped with an identification $\lambda \circ \gamma = l$. We will prove that λ is an equivalence. Theorem 18.3.8 provides a canonical equivalence

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \simeq D$$

extending the embedding $C \hookrightarrow D$. The proof of Proposition 18.4.1 allows us to apply Theorem 18.3.8 once again to provide an equivalence of the form

$$\mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{An}) \simeq L_S(D)$$

extending $C_S \hookrightarrow L_S(D)$. Under these equivalences, the functor

$$l^* \colon \mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$$

corresponds to the full embedding $L_S(D) \hookrightarrow D$.

On the other hand the functor

$$\gamma^* \colon \mathrm{PSh}(L(C)) \to \mathrm{PSh}(C)$$

is fully faithful with essential image consisting of those presheaves on C which send maps in W to isomorphisms of An. Corollary 17.1.8 (iii) means that there is a canonical pullback square of the form below.

$$\begin{array}{ccc} \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{An}) & \xrightarrow{\gamma^*} & \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{PSh}(L(C)) & \xrightarrow{\gamma^*} & \mathrm{PSh}(C) \end{array}$$

In particular the upper horizontal arrow above is a full embedding whose essential image consists of those left exact presheaves on C sending maps in W to isomorphisms in An. Under the equivalence $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \simeq D$, those correspond to T-local objects where

$$T = \mathrm{Map}([1], W) \hookrightarrow \mathrm{Map}([1], C) \hookrightarrow \mathrm{Map}([1], D)\,.$$

Since $S \hookrightarrow T$, it is clear that any T-local object is S-local. Conversely, any S-local object y is T-local: indeed, for any map $u\colon a \to b$ in W, since $loc_S(u)$ is an isomorphism by definition of W, the map

$$u^*\colon \mathrm{Hom}_D(b, y) \to \mathrm{Hom}_D(a, y)$$

is isomorphic to the isomorphism

$$u^*\colon \mathrm{Hom}_D(loc_S(b), y) \xrightarrow{\simeq} \mathrm{Hom}_D(loc_S(a), y)\,.$$

We obtain a commutative square of the form

$$\begin{array}{ccc} \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{An}) & \xrightarrow{\lambda_!} & \mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{An}) \\ \simeq\downarrow & & \downarrow\simeq \\ L_T(D) & = & L_S(D) \end{array}$$

thus proving that the upper horizontal map is an equivalence. In other words, the adjunction provided by Theorem 18.2.8

$$\lambda_!\colon \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{An}) \rightleftarrows \mathrm{Lex}(C_S^{\mathrm{op}}, \mathrm{An})\colon \lambda^*$$

consists of equivalences. Since the restriction of $\lambda_!$ to representable presheaves is isomorphic to the composition of λ with the Yoneda embedding of C_S, this proves that the functor λ is fully faithful. Since

$$\lambda^{\simeq}\colon L(C)^{\simeq} \to C_S^{\simeq}$$

is a cover by construction, this proves that λ is an equivalence. □

Notation 18.4.3. Recall that, for any category D with small colimits and any functor $f\colon A \to B$ between small categories, the functor

$$f^*\colon \mathrm{Fun}(B, D) \to \mathrm{Fun}(A, D)$$

has a left adjoint

$$f_!\colon \mathrm{Fun}(A, D) \to \mathrm{Fun}(B, D)\,.$$

In particular, given any small category A and object $a : A^{\simeq}$ the evaluation functor $F \mapsto F_a$ has a left adjoint

$$D \to \mathrm{Fun}(A, D)\,, \quad x \mapsto a \otimes x$$

defined by $a \otimes x = a_!(x)$, where we see a as an A-valued functor. If D is locally small, we thus have the formula

$$\mathrm{Hom}(x, F_a) = \mathrm{Hom}(a \otimes x, F)$$

functorially in x and F, which determines $a \otimes x$. Given a family of objects $E \hookrightarrow D^{\simeq}$, we define E_A as the image of the map

$$A^{\simeq} \times E \to \mathrm{Map}(A, D)\,, \quad (a, x) \mapsto a \otimes x\,.$$

Proposition 18.4.4. *Let D be a compactly generated category. For any small category A, the functor category* $\mathrm{Fun}(A, D)$ *is compactly generated. More precisely, given a small family of compact objects $E \hookrightarrow D^{\simeq}$ that generates D, the family E_A is a family of compact objects that generates* $\mathrm{Fun}(A, D)$.

Proof. Given $a : A^{\simeq}$, the assignment $F \mapsto F_a$ commutes with colimits and therefore, for any compact object x of D, the object $a \otimes x$ is compact: the functor $\mathrm{Hom}(a \otimes x, -)$ is the composition of the evaluation at a with $\mathrm{Hom}(x, -)$. One can detect isomorphisms in $\mathrm{Fun}(A, D)$ object-wise, and can detect isomorphisms in D by evaluating on terms of E. This shows that the family of functors of the form $\mathrm{Hom}(a \otimes x, -)$ is jointly conservative. □

Proposition 18.4.5. *Let D be a compactly generated category. For any finite category A, any functor $F\colon A \to D$ taking values (objectwise) in compact objects of D is compact as an object of* $\mathrm{Fun}(A, D)$.

Proof. The assertion is clearly true if $A = \emptyset$ or $A = [0]$. Let us prove it for $A = [1]$. Given an object x of D, and a morphism $u\colon a \to b$ in D, there is a correspondence of the form

$$\begin{array}{ccc} x & \xrightarrow{f} & a \\ \| & & \downarrow u \\ x & \xrightarrow{u\circ f} & b \end{array} \quad \Leftrightarrow \quad f\colon x \to a\,.$$

In other words, the object $0 \otimes x$ can be described as the morphism

$$\mathrm{id}_x\colon x \to x$$

seen as a functor from $[1]$ to D. Similarly, if $\emptyset$ denotes the initial object of D, we see that the object $1 \otimes x$ corresponds to the unique morphism of the form

$$\emptyset \to x\,.$$

Given any morphism $u\colon a \to b$ in D there is a functor $\varphi\colon \ulcorner \to \mathrm{Fun}([1], D)$ yielding a diagram of the form

$$0 \otimes a \leftarrow 1 \otimes a \to 1 \otimes b$$

corresponding to the commutative diagram below.

$$\begin{array}{ccccc} a & \longleftarrow & \emptyset & \xrightarrow{\mathrm{id}_\emptyset} & \emptyset \\ {\scriptstyle \mathrm{id}_a}\downarrow & & \downarrow & & \downarrow \\ a & \xleftarrow{\mathrm{id}_a} & a & \xrightarrow{u} & b \end{array}$$

A simple inspection of universal properties shows that the colimit of this functor φ is nothing else than $u\colon a \to b$ (seen as a functor from $[1]$ to D). By virtue of Lemma 18.1.3, this proves that, if ever both a and b are compact in D, so is u in $\mathrm{Fun}([1], D)$.

Let $C \hookrightarrow \mathrm{Cat}$ be the full subcategory of Cat whose objects are those small categories such that any A-indexed functor with values in D_ω is a compact object of $\mathrm{Fun}(A, D)$. We know that C contains $\emptyset$, $[0]$ as well as $[1]$. By virtue of Lemma 16.4.6, it suffices now to prove that C is stable under pushouts in Cat.

Let us consider a pushout square in Cat of the form

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{u'} & B' \end{array}$$

with A, A' as well as B in C. Given a functor $F\colon B' \to D$, we write $F_V\colon V \to D$ for the composition of F with the functor $v\colon V \to B'$, for v being either u, a or u'. We observe that the assignment $F \mapsto F_V$ commutes with colimits (in particular with small filtered ones). For any functors $F, G\colon B' \to D$, since we have a pullback square of the form

$$\begin{array}{ccc} \mathrm{Fun}(B', D) & \xrightarrow{b^*} & \mathrm{Fun}(B, D) \\ {\scriptstyle (u')^*}\downarrow & & \downarrow{\scriptstyle u^*} \\ \mathrm{Fun}(A', D) & \xrightarrow{a^*} & \mathrm{Fun}(A, D) \end{array}$$

we have a canonical identification of the form

$$\mathrm{Hom}(F, G) = \mathrm{Hom}(F_{A'}, G_{A'}) \times_{\mathrm{Hom}(F_A, G_A)} \mathrm{Hom}(F_B, G_B)\,.$$

Since filtered colimits commute with pullbacks in An, this shows that F is compact whenever F_V is compact in $\mathrm{Fun}(V, D)$ for $V = A$, $V = A'$, and $V = B$. This implies right away that C is stable under pushouts in Cat and completes the proof of the proposition. □

Lemma 18.4.6. *Let P be a finite poset and C a small category with finite colimits. The restriction of the functor*

$$P \times \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}))\,, \quad (a, x) \mapsto a \otimes x$$

to $P \times C$ via the Yoneda embedding of C takes values in the full subcategory $\mathrm{Fun}(P^{\mathrm{op}}, C)$. This induces an equivalence $\langle P \times C\rangle_{\mathrm{fp}} = \mathrm{Fun}(P^{\mathrm{op}}, C)$ (in other words, $\mathrm{Fun}(P^{\mathrm{op}}, C)$ is the smallest finitely cocomplete full subcategory of $\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}))$ generated by objects of the form $a \otimes x$ for $a : P^{\simeq}$ and $x : C^{\simeq}$). In particular, we have

$$\mathrm{Ind}(\mathrm{Fun}(P^{\mathrm{op}}, C)) = \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C))\,.$$

Proof. The evaluation of $a \otimes x$ at an object e of P is

$$\sum_{u:\mathrm{Hom}_P(e,a)} x\,.$$

Since the order on P is decidable, the proposition $\mathrm{Hom}_P(e, a)$ is decidable and hence is a finite set. This implies the first claim.

Therefore, in order to prove the second assertion, it suffices to know that any object of $\mathrm{Fun}(P^{\mathrm{op}}, C)$ is a finite colimit of objects of the form $a \otimes x$. The opposite P^{op} is again a finite poset. The objects of $\mathrm{Tw}(P^{\mathrm{op}})$ form a decidable subobject of $(P^{\mathrm{op}})^{\simeq} \times (P^{\mathrm{op}})^{\simeq}$, and its order is decidable. Filtering a finite enumeration of the product therefore shows that $\mathrm{Tw}(P^{\mathrm{op}})$ is a finite poset. In particular, a new application of Proposition 16.4.23 tells us that $\mathrm{Tw}(P^{\mathrm{op}})^{\mathrm{op}}$ is a finite category. Therefore, Proposition 12.9.1 concludes the proof of the second assertion. The third one follows right away from the second one by Theorem 18.3.8. □

Theorem 18.4.7. *Let C be a small category with finite colimits and $W \hookrightarrow C$ a subcategory closed under pushouts and with the 2-out-of-3 property, with associated localization functor $\gamma : C \to L(C)$. Given a small category A, we denote by $L(\mathrm{Fun}(A, C))$ the localization of $\mathrm{Fun}(A, C)$ by the family of natural transformations $u : F \to G$ such that $u_a : F_a \to G_a$ is a map of W for any object $a : A^{\simeq}$. Then, for any finite poset P, the functor*

$$\gamma_* : \mathrm{Fun}(P, C) \to \mathrm{Fun}(P, L(C))$$

induces an equivalence

$$L(\mathrm{Fun}(P, C)) \simeq \mathrm{Fun}(P, L(C)) .$$

Proof. Recall that any finite poset is finite as a category (Proposition 16.4.23). Let us define $D = \mathrm{Fun}(P, \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}))$ and let $S \hookrightarrow \mathrm{Map}([1], D)$ be the family of morphisms in $\mathrm{Fun}(P, C)$ that are objectwise in W. We see right away that S-local objects simply are left exact functors from $L(C)^{\mathrm{op}}$ to $\mathrm{Fun}(P, \mathrm{An})$, hence that

$$D_S = \mathrm{Fun}(P, \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{An})) .$$

Since, by virtue of Proposition 18.4.5 the Yoneda embedding

$$h_C : C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$$

induces a full embedding

$$\mathrm{Fun}(P, C) \hookrightarrow D$$

taking values in compact objects, Proposition 18.4.2 implies that the functor

$$loc_S = (\gamma_!)_* : D \to D_S$$

induces an equivalence

$$L(\mathrm{Fun}(P, C)) = \mathrm{Fun}(P, C)_S$$

where $\mathrm{Fun}(P, C)_S$ is the full subcategory of $\mathrm{Fun}(P, \mathrm{Lex}(L(C)^{\mathrm{op}}, \mathrm{An}))$ spanned by the essential image of the functor

$$\mathrm{Fun}(P, C) \xrightarrow{\gamma_*} \mathrm{Fun}(P, L(C)) \hookrightarrow D_S .$$

This implies that the induced functor

$$L(\mathrm{Fun}(P, C)) \to \mathrm{Fun}(P, L(C))$$

is right exact and fully faithful (right exactness was already known by Corollary 17.1.8 anyway). For any $a : P^{\simeq}$ and any object x of C, we see that the formation of $a \otimes x$ commutes with γ by right exactness. Lemma 18.4.6 thus implies that the comparison functor is essentially surjective: its essential image is stable under finite colimits and contains all objects of the form $a \otimes x$, hence everything. □

Theorem 18.4.8. *Let P be a finite poset. For any categories C and D with finite colimits, the canonical equivalence from Theorem 12.9.3*

$$\mathrm{Fun}_!(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)), \mathrm{Ind}(D)) \simeq \mathrm{Fun}_!(\mathrm{Ind}(C), \mathrm{Fun}(P, \mathrm{Ind}(D)))$$

restricts to a functorial equivalence

$$\mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \simeq \mathrm{Rex}(C, \mathrm{Fun}(P, D)) .$$

Proof. The equivalence is defined by sending any functor $\Phi\colon \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)) \to \mathrm{Ind}(D)$ to the transposition of the functor obtained by composing Φ with the functor

$$P \times \mathrm{Ind}(C) \to \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)) \,, \quad (a, x) \mapsto a \otimes x \,.$$

On the other hand, since $\mathrm{Fun}(P^{\mathrm{op}}, C)$ is the smallest full subcategory of $\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C))$ stable under finite colimits and containing objects of the form $a \otimes x$ for $a : P^{\simeq}$ and $x : C^{\simeq}$, yielding the equivalence

$$\mathrm{Ind}(\mathrm{Fun}(P^{\mathrm{op}}, C)) = \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)) \,,$$

by Lemma 18.4.6, we may see $\mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D)$ as the full subcategory of the category

$$\mathrm{Fun}_!(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)), \mathrm{Ind}(D))$$

spanned by those functors that send objects of the form $a \otimes x$ to representable presheaves on D by Theorem 18.3.8. We also have similarly

$$\mathrm{Fun}_!(\mathrm{Ind}(C), \mathrm{Fun}(P, \mathrm{Ind}(D))) = \mathrm{Rex}(C, \mathrm{Fun}(P, \mathrm{Ind}(D)))$$

by another application of Theorem 18.3.8.

In other words, we have a pullback square of the form:

$$\begin{array}{ccc} \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) & \longrightarrow & \mathrm{Rex}(C, \mathrm{Fun}(P, D)) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_!(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Ind}(C)), \mathrm{Ind}(D)) & \xrightarrow{\simeq} & \mathrm{Fun}_!(\mathrm{Ind}(C), \mathrm{Fun}(P, \mathrm{Ind}(D))) \end{array}$$

in which the upper horizontal map must be an equivalence. □

Remark 18.4.9. We can also give an alternative proof of Theorem 18.4.7 using Theorem 18.4.8 instead of Proposition 18.4.2 (exercise left to the reader).

Remark 18.4.10. One can understand the functoriality of the equivalence provided by the previous theorem. Given a functor between finite posets $u\colon P \to Q$, if C has finite colimits, if $u^*(F) = F \circ u$, there is an adjunction

$$u_!\colon \mathrm{Fun}(P, \mathrm{Ind}(C)) \rightleftarrows \mathrm{Fun}(Q, \mathrm{Ind}(C))\colon u^*$$

and the formula

$$u_!(F)(y) = \operatorname*{colim}_{x : P \,,\, u(x) \leq y} F(x)$$

The indexing poset is finite: the predicate $u(x) \leq y$ is decidable, so it filters a finite enumeration of P to a finite enumeration of this full subposet. Consequently, this formula shows that the adjunction restricts to one of the form

$$u_!\colon \mathrm{Fun}(P, C) \rightleftarrows \mathrm{Fun}(Q, C)\colon u^* \,.$$

There is thus a functor

$$(u^{\mathrm{op}})_!\colon \mathrm{Fun}(P^{\mathrm{op}}, C) \to \mathrm{Fun}(Q^{\mathrm{op}}, C)$$

inducing a functor

$$((u^{\mathrm{op}})_!)^*\colon \mathrm{Rex}(\mathrm{Fun}(Q^{\mathrm{op}}, C), D) \to \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \,.$$

For $a : P$ and $x : C$, we have

$$(u^{\mathrm{op}})_!(a \otimes x) = u(a) \otimes x\,.$$

This means that the functor $((u^{\mathrm{op}})_!)^*$ above corresponds under the correspondence of the previous theorem to the functor of type

$$\mathrm{Rex}(C, \mathrm{Fun}(Q, D)) \to \mathrm{Rex}(C, \mathrm{Fun}(P, D))$$

defined by $\Psi \mapsto u^*\Psi$.

On the other hand, since both $(u^{\mathrm{op}})_!$ and its right adjoint $(u^{\mathrm{op}})^*$ are right exact, the induced adjunction

$$((u^{\mathrm{op}})^*)^* \colon \mathrm{Fun}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \rightleftarrows \mathrm{Fun}(\mathrm{Fun}(Q^{\mathrm{op}}, C), D) \colon ((u^{\mathrm{op}})_!)^*$$

restricts to an adjunction

$$((u^{\mathrm{op}})^*)^* \colon \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \rightleftarrows \mathrm{Rex}(\mathrm{Fun}(Q^{\mathrm{op}}, C), D) \colon ((u^{\mathrm{op}})_!)^*\,.$$

Therefore, $((u^{\mathrm{op}})^*)^*$ corresponds to the functor of type

$$\mathrm{Rex}(C, \mathrm{Fun}(P, D)) \to \mathrm{Rex}(C, \mathrm{Fun}(Q, D))$$

defined by $\Phi \mapsto u_!\Phi$.

If furthermore u has a right adjoint $v \colon Q \to P$, then

$$(v^{\mathrm{op}})_! = (u^{\mathrm{op}})^*\,.$$

This means that $(u^{\mathrm{op}})^*$ has a further right adjoint, namely $(v^{\mathrm{op}})^*$, that one can interpret through the correspondence of Theorem 18.4.8 as well.

18.5 Free completion by finite colimits

Definition 18.5.1. Let C be a small category. We define $\langle C \rangle_\omega$ as the smallest finitely cocomplete full subcategory of $\mathrm{PSh}(C)$ generated by the family of representable presheaves. The Yoneda embedding thus yields a canonical fully faithful functor $h_C \colon C \to \langle C \rangle_\omega$.

Theorem 18.5.2. *For any category D with finite colimits, pre-composing with the Yoneda embedding yields an equivalence of the form*

$$\mathrm{Rex}(\langle C \rangle_\omega, D) \simeq \mathrm{Fun}(C, D)\,.$$

Proof. This is a particular case of Theorem 17.3.3. □

Notation 18.5.3. We define RexCat as the subcategory of Cat whose objects are the small categories with finite colimits and whose morphisms are the right exact functors. Dually, we define LexCat as the subcategory of Cat whose objects are the small categories with finite limits and whose morphisms are the left exact functors. There is a canonical equivalence

$$\mathrm{RexCat} \simeq \mathrm{LexCat}\,, \quad C \mapsto C^{\mathrm{op}}\,.$$

Corollary 18.5.4. *The canonical embedding* RexCat $\hookrightarrow$ Cat *has a left adjoint given by the rule* $C \mapsto \langle C \rangle_\omega$.

Theorem 18.5.5. *The category* RexCat *of small categories with finite colimits is compactly generated (in particular, it has both small colimits and small limits).*

Proof. This is a particular case of Corollary 17.3.12. □

Remark 18.5.6. Let Cat′ be a universe of large categories, so that every small category is large and Cat itself is large. Let RexCat′ be the category of large categories with finite colimits. Then the embedding

$$\mathrm{RexCat} \hookrightarrow \mathrm{RexCat}'$$

commutes with small colimits as well as with small limits. Recall that the embedding Cat $\hookrightarrow$ Cat′ already has these properties. The case of small colimits follows from the explicit description of colimits in RexCat given in the proof of Proposition 17.3.8. the case of small limits follows right away from Lemma 17.3.2.

Remark 18.5.7. The functor

$$\mathrm{Rex} \colon \mathrm{RexCat}^{\mathrm{op}} \times \mathrm{RexCat} \to \mathrm{Cat}$$

preserves small limits in each variable. To see this, we use the fact that, for any small category A and any small categories with finite colimits B and C, we have

$$\mathrm{Map}(A, \mathrm{RexCat}(B, C)) = \mathrm{Rex}(B, \mathrm{Fun}(A, C))^{\simeq}\,.$$

It thus suffices to recall that

$$\mathrm{Hom}_{\mathrm{RexCat}}(-,-) = \mathrm{RexCat}(-,-)^{\simeq} \colon \mathrm{RexCat}^{\mathrm{op}} \times \mathrm{RexCat} \to \mathrm{An}$$

preserves small limits in each variable and to convince ourselves that

$$\mathrm{Fun}(A,-) \colon \mathrm{RexCat} \to \mathrm{RexCat}$$

commutes with limits. For the latter, this follows from the fact that the embedding

$$\mathrm{RexCat} \hookrightarrow \mathrm{Cat}$$

is conservative and commutes with small limits as well as with the operator $\mathrm{Fun}(A,-)$. Therefore, it suffices to check that

$$\mathrm{Fun}(A,-) \colon \mathrm{Cat} \to \mathrm{Cat}$$

commutes with small limits, which holds true since it is right adjoint to the functor

$$B \mapsto A \times B\,.$$

18.6 Tensor product of finitely cocomplete categories

Construction 18.6.1. Let C and D be two small categories with finite colimits. We have a Yoneda embedding:

$$h\colon C \times D \to \langle C \times D \rangle_\omega \,.$$

We let W be the smallest family of morphisms in $\langle C \times D \rangle_\omega$ stable under pushouts and satisfying the 2-out-of-3 property containing the following maps, where 0 denotes an initial object of $\langle C \times D \rangle_\omega$:

(i) $0 \to h(x, y)$ if x is an initial object of C;

(ii) $0 \to h(x, y)$ if y is an initial object of D;

(iii) the canonical comparison map $h(x_1, y) \amalg_{h(x_0,y)} h(x_2, y) \to h(x, y)$ for any pushout square

$$\begin{array}{ccc} x_0 & \longrightarrow & x_2 \\ \downarrow & \ulcorner & \downarrow \\ x_1 & \longrightarrow & x \end{array}$$

in C and any object y in D;

(iv) the canonical comparison map $h(x, y_1) \amalg_{h(x,y_0)} h(x, y_2) \to h(x, y)$ for any pushout square

$$\begin{array}{ccc} y_0 & \longrightarrow & y_2 \\ \downarrow & \ulcorner & \downarrow \\ y_1 & \longrightarrow & y \end{array}$$

in D and any object x in C.

We define finally

$$C \otimes D := L(\langle C \times D \rangle_\omega, W)$$

to be the localization of $\langle C \times D \rangle_\omega$ by W, with localization functor

$$\gamma\colon \langle C \times D \rangle_\omega \to C \otimes D \,.$$

By construction, there is a canonical functor

$$\otimes\colon C \times D \to C \otimes D \,, \quad (x, y) \mapsto x \otimes y := \gamma(h_{C\times D}(x, y)) \,.$$

Definition 18.6.2. Let C, D and E be categories with finite colimits. A functor

$$\varphi\colon C \times D \to E$$

is *right exact in each variable* if, for each object $x : C^\simeq$, the functor

$$\varphi(x, -)\colon D \to E$$

is right exact, and, for each object $y : D^\simeq$, the functor

$$\varphi(-, y)\colon C \to E$$

is right exact.

We denote by $\mathrm{Fun}^{rex,rex}(C, D; E)$ the full subcategory of $\mathrm{Fun}(C \times D, E)$ spanned by functors that are right exact in each variable.

Example 18.6.3. By construction, for any two categories with finite colimits C and D, the canonical functor

$$\otimes \colon C \times D \to C \otimes D \ , \quad (x, y) \mapsto x \otimes y$$

is right exact in each variable.

Remark 18.6.4. For any B, C and D with finite colimits the identification

$$\mathrm{Fun}(B, \mathrm{Fun}(C, D)) = \mathrm{Fun}(B \times C, D)$$

restricts to:

$$\mathrm{Rex}(B, \mathrm{Rex}(C, D)) = \mathrm{Fun}^{rex,rex}(B, C; D) \ .$$

Theorem 18.6.5. *Given C and D two categories with finite colimits, the assignment*

$$\varphi \mapsto ((x, y) \mapsto \varphi(x \otimes y))$$

defines, for any category E with finite colimits, an equivalence

$$\mathrm{RexCat}(C \otimes D, E) \simeq \mathrm{Fun}^{rex,rex}(C, D; E)$$

Proof. The localization functor $\gamma \colon \langle C \times D \rangle_\omega \to C \otimes D$ is right exact by Corollary 17.1.9. By Corollary 17.3.7, there is therefore an equivalence:

$$\mathrm{Rex}(C \otimes D, E) \simeq \mathrm{RexCat}^W(\langle C \times D \rangle_\omega, E) \ .$$

On the other hand, we have

$$\mathrm{RexCat}(\langle C \times D \rangle_\omega, E) = \mathrm{Fun}(C \times D, E)$$

by Theorem 18.5.2. This produces a canonical full embedding

$$\mathrm{Rex}(C \otimes D, E) \hookrightarrow \mathrm{Fun}(C \times D, E)$$

whose image at the level of objects obviously is $\mathrm{Fun}^{rex,rex}(C, D; E)$, by definition of W. □

Corollary 18.6.6. *Let $u \colon A \to B$ be a right exact functor between small categories with finite colimits. Then, for any compactly generated category C, the functor*

$$u^* \colon \mathrm{Lex}(B^{\mathrm{op}}, C) \to \mathrm{Lex}(A^{\mathrm{op}}, C)$$

defined by composition with u^{op} has a left adjoint

$$u_! \colon \mathrm{Lex}(A^{\mathrm{op}}, C) \to \mathrm{Lex}(B^{\mathrm{op}}, C) \ .$$

Proof. We may assume that $C = \mathrm{Lex}(E^{\mathrm{op}}, \mathrm{An})$ for some small category with finite colimits E by Theorem 18.3.8. We observe that, for any category D with finite limits, we have

$$\mathrm{Lex}(A^{\mathrm{op}}, D) = \mathrm{Rex}(A, D^{\mathrm{op}})^{\mathrm{op}}$$

which allows us to reinterpret Theorem 18.6.5 through

$$\mathrm{Lex}(A^{\mathrm{op}}, C) = \mathrm{Lex}((A \otimes E)^{\mathrm{op}}, \mathrm{An}) \ .$$

Doing the same for B, replacing u by the induced functor

$$u \otimes \mathrm{id}_E \colon A \otimes E \to B \otimes E$$

we see that it suffices to consider the case where $C = \mathrm{An}$. This is then a particular case of Theorem 18.2.8. □

Example 18.6.7. We define $\mathrm{FinAn} = \langle [0] \rangle_\omega$. We thus have

$$\mathrm{RexCat}(\mathrm{FinAn}, C) = C$$

for any category C with finite colimits. For any finite poset P we have

$$\begin{aligned} \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}), C) &= \mathrm{Rex}(\mathrm{FinAn}, \mathrm{Fun}(P, C)) \\ &= \mathrm{Fun}(P, C) \\ &= \mathrm{Fun}(\langle P \rangle_\omega, C)\,. \end{aligned}$$

In other words, we have:

$$\langle P \rangle_\omega = \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn})$$

In general, for any categories with finite colimits C and D, we have:

$$\begin{aligned} \mathrm{Rex}(\langle P \rangle_\omega \otimes C, D) &= \mathrm{Rex}(\langle P \rangle_\omega, \mathrm{Rex}(C, D)) \\ &= \mathrm{Fun}(P, \mathrm{Rex}(C, D)) \\ &= \mathrm{Rex}(C, \mathrm{Fun}(P, D)) \\ &= \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D)\,. \end{aligned}$$

In other words, we have

$$\langle P \rangle_\omega \otimes C = \mathrm{Fun}(P^{\mathrm{op}}, C)$$

for all C with finite colimits.

Remark 18.6.8. If A and B are pointed right exact categories, the functor

$$A \times B \to B \times A\,, \quad (a, b) \mapsto (b, a)$$

induces a unique right exact functor

$$\theta_{A,B} \colon A \otimes B \to B \otimes A$$

such that $\theta_{A,B}(a \otimes b) = b \otimes a$ for all terms $a : A$ and $b : B$. This comes from the fact that the canonical equivalence

$$\mathrm{Fun}(A \times B, C) = \mathrm{Fun}(B \times A, C)$$

restricts to an equivalence of the form

$$\mathrm{Fun}^{rex,rex}(A, B; C) = \mathrm{Fun}^{rex,rex}(B, A; C)\,.$$

The unicity of the functor $\theta_{A,B}$ above implies that it must be an equivalence with inverse $\theta_{B,A}$.

Similarly, for A, B and C pointed right exact categories, the identification

$$(A \times B) \times C = A \times (B \times C)\,, \quad ((x, y), z) \mapsto (x, (y, z))$$

yields a unique identification of the form

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)\,, \quad (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)\,.$$

Definition 18.6.9. Let us choose a universe of large categories Cat' (so that Cat is both an object and a full subcategory of Cat'). We define $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ as the subcategory of Cat' whose objects are the compactly generated categories and whose morphisms are the functors that commute with small colimits and that preserve compact objects.

Given A and B two compactly generated categories, we denote by $\mathrm{Fun}^{\mathrm{L}}_{\omega}(A, B)$ the full subcategory of $\mathrm{Fun}(A, B)$ spanned by the functors that preserve small colimits as well as compact objects. In particular, we have:

$$\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_{\omega}}(A, B) = \mathrm{Fun}^{\mathrm{L}}_{\omega}(A, B)^{\simeq} .$$

Theorem 18.6.10. *Given a small category with finite colimits C and a compactly generated category D, there is a canonical equivalence of categories of the form*

$$\mathrm{Rex}(C, D^{\omega}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\omega}(\mathrm{Ind}(C), D) .$$

In particular, assigning the full subcategory of compact objects defines a fully faithful functor

$$\mathrm{Pr}^{\mathrm{L}}_{\omega} \to \mathrm{RexCat} , \quad D \mapsto D_{\omega}$$

that is right adjoint to the functor

$$\mathrm{Ind} \colon \mathrm{RexCat} \to \mathrm{Pr}^{\mathrm{L}}_{\omega} , \quad C \mapsto \mathrm{Ind}(C) .$$

Proof. This is a particular case of Theorem 17.4.9. □

Proposition 18.6.11. *The category $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ is compactly generated:*

$$\mathrm{Pr}^{\mathrm{L}}_{\omega} : (\mathrm{Pr}^{\mathrm{L}}_{\omega})^{\simeq} .$$

Proof. This is a particular case of Theorem 17.4.12. □

Lemma 18.6.12. *For any finite poset P, the category $\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn})$ is a compact object of* RexCat.

Proof. For any small category with finite colimits C, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{RexCat}}(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}), C) &= \mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}), C)^{\simeq} \\ &= \mathrm{Rex}(\mathrm{FinAn}, \mathrm{Fun}(P, C))^{\simeq} \\ &= \mathrm{Fun}(P, C)^{\simeq} \\ &= \mathrm{Hom}_{\mathrm{Cat}}(P, C) . \end{aligned}$$

Since the forgetful functor $\mathrm{RexCat} \hookrightarrow \mathrm{Cat}$ commutes with small filtered colimits, the lemma follows from the fact that finite posets are compact objects of Cat (since they are finite categories). □

Proposition 18.6.13. *If A and B are compact in* RexCat*, then so is $A \otimes B$.*

Proof. If there are finite posets P and Q such that $A = \langle P \rangle_{\omega}$ and $B = \langle Q \rangle_{\omega}$ respectively, then we have

$$A \otimes B = \langle P \times Q \rangle_{\omega}$$

The product of finite enumerations of P and Q gives a finite enumeration of $P \times Q$, and its product order is decidable. Thus $P \times Q$ is a finite poset, so this particular case follows from the previous lemma.

On the other hand, we know that $\langle [1] \rangle_\omega$ is a compact generator of RexCat. Therefore, applying Proposition 17.4.4 to the finite-colimit closure of $\langle [1] \rangle_\omega$ in RexCat, with κ the subuniverse of finite posets, we see that for a full subcategory of RexCat to contain all compact objects of RexCat, it suffices that it is stable under finite colimits, under the formation of images of projectors, and contains $\langle [1] \rangle_\omega$.

Given A in RexCat, since $A \otimes (-)$ commutes with small colimits, the full subcategory X_A of RexCat spanned by those small categories with finite colimits B such that $A \otimes B$ is compact is stable under finite colimits as well as under the formation of images of projectors (these being a particular kind of colimits, as explained in the proof of Lemma 17.4.2). If $A = \langle P \rangle_\omega$ for some finite poset P, we deduce that any compact object of RexCat belongs to X_A. Since $A \otimes B = B \otimes A$, this means that, for A an arbitrary compact object of RexCat, X_A contains all categories of the form $\langle P \rangle_\omega$ for any finite poset. Therefore, for any compact object A of RexCat, X_A contains all compact objects of RexCat, and this proves the proposition. □

Corollary 18.6.14. *A small category with finite colimits A is compact in* RexCat *if and only if the functor*

$$\mathrm{Rex}(A, -) \colon \mathrm{RexCat} \to \mathrm{Cat}$$

commutes with small filtered colimits.

Proof. Since RexCat is compactly generated, this follows straight away from the formula

$$\mathrm{Hom}_{\mathrm{RexCat}}(A \otimes B, C) = \mathrm{Hom}_{\mathrm{RexCat}}(A, \mathrm{Rex}(B, C))$$

and from Proposition 18.6.13. □

18.7 Pointed categories with pushouts

Definition 18.7.1. A small *pointed right exact category* is a small category C with pushouts as well as with an initial object that is also terminal. We denote by $\mathrm{RexCat}_\bullet$ the full subcategory of RexCat spanned by small pointed right exact categories.

Construction 18.7.2. Let $*$ be the terminal object of An. We know that $\mathrm{An}_\bullet = \mathrm{An}_{*/}$ and that forgetting the base point is the universal left fibration with small fibers:

$$\pi_{\mathrm{univ}} \colon \mathrm{An}_\bullet \to \mathrm{An}\,.$$

This functor has a left adjoint

$$\mathrm{An} \to \mathrm{An}_\bullet\,, \quad x \mapsto x_+$$

where $x_+ = x \amalg *$ is equipped with the canonical embedding $* \to x \amalg *$. We define the *pointed 0-sphere* as $S^0 := *_+$. We observe that the functor π_{univ} commutes with small filtered colimits and is conservative (being a left fibration). Since An is compactly generated by the point $*$, this shows that $\mathrm{An}_\bullet$ is compactly generated by S^0, by virtue of Proposition 18.3.12.

Let C be a small category with finite colimits. We let $*$ be the terminal object of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$ and observe that the canonical equivalence

$$\mathrm{An}_{*/} = \mathrm{An}_\bullet$$

induces an equivalence:

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})_{*/} = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) .$$

In particular, $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$ is compactly generated by Proposition 18.4.4. The forgetful functor

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$$

has a left adjoint

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) , \quad x \mapsto x_+ .$$

We let $C_\bullet^{\simeq}$ be the image of the map

$$C^{\simeq} \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)^{\simeq} , \quad x \mapsto h_C(x)_+$$

and we define $C_\bullet$ as the smallest full subcategory of $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$ stable under finite colimits containing the terms of $C_\bullet^{\simeq}$. There is a canonical right exact functor

$$C \to C_\bullet , \quad x \mapsto h_C(x)_+$$

with $C_\bullet$ pointed.

Proposition 18.7.3. *Given a small category with finite colimits C and a pointed right exact category D, pre-composing with the canonical right exact functor $C \to C_\bullet$ induces an equivalence of the form*

$$\mathrm{Rex}(C_\bullet, D) \simeq \mathrm{Rex}(C, D) .$$

In particular, the full embedding $\mathrm{RexCat}_\bullet \hookrightarrow \mathrm{RexCat}$ *has a left adjoint*

$$C \mapsto C_\bullet .$$

Proof. It suffices to prove that, for any category A, we have an equivalence of the form

$$\mathrm{Map}(A, \mathrm{Rex}(C_\bullet, D)) \simeq \mathrm{Map}(A, \mathrm{Rex}(C, D)) .$$

Using the identification

$$\mathrm{Map}(A, \mathrm{Rex}(C, D)) = \mathrm{Rex}(C, \mathrm{Fun}(A, D))^{\simeq} ,$$

we see that it suffices to prove that we get an equivalence of the form:

$$\mathrm{Rex}(C_\bullet, D)^{\simeq} \simeq \mathrm{Rex}(C, D)^{\simeq} .$$

In other words, it suffices to prove the last assertion of the proposition.

We may assume that D is small (up to enlarging the ambient universe). Since D is also pointed, we observe that the canonical right exact functor from D to $D_\bullet$ is an equivalence. Indeed, there is an equivalence

$$\mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}_\bullet) \simeq \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An})$$

due to the fact that the Yoneda embedding is exact, hence sends the initial object of D to an object $*$ that is both initial and terminal, which allows us to use the canonical identification

$$\mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An})_{*/} = \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}_\bullet)\,.$$

This implies that the canonical functor $D \to D_\bullet$ is fully faithful (being the restriction of an equivalence of categories). On the other hand, we know that its essential image is stable under finite colimits and contains the image of any object of D, which proves the essential surjectivity, by definition of $D_\bullet$. Any right exact functor $f\colon C \to D$ extends to small colimit preserving functors

$$f_!\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}) \text{ and } f_!\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}_\bullet)$$

by Corollary 18.6.6. Since the functor $x \mapsto x_+$ commutes with colimits, we have a commutative square of the form

$$\begin{array}{ccc} \mathrm{PSh}(C) & \longrightarrow & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ \mathrm{PSh}(D) & \longrightarrow & \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{An}_\bullet) \end{array}$$

by Corollary 13.1.10 and this restricts to a commutative square of the form

$$\begin{array}{ccc} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) & \longrightarrow & \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}) & = & \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}_\bullet) \end{array}$$

through an easy application of Theorem 18.2.8. By definition of $C_\bullet$, this induces a commutative square of the form

$$\begin{array}{ccc} C & \longrightarrow & C_\bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\bullet} \\ D & = & D_\bullet \end{array}$$

This shows that the map

$$\mathrm{Rex}(C_\bullet, D)^{\simeq} \to \mathrm{Rex}(C, D)^{\simeq}$$

has a section. We will now prove that this is an embedding. Observe first that, by virtue of Theorem 18.2.10, the factorization of f through $C_\bullet$ is totally determined by the factorization of

$$f_!\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}) = \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}_\bullet)$$

through $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$. The right adjoint of the latter is the composition of the functor

$$f^*\colon \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}) = \mathrm{Lex}(D^{\mathrm{op}}, \mathrm{An}_\bullet) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$$

with the forgetful functor

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})\,.$$

The uniqueness of left adjoints and the compatibility of the formation of adjoints with respect to composition shows that the factorization through $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$ is unique. This implies that the factorization of f through $C_\bullet$ is unique. □

Corollary 18.7.4. *The category* RexCat$_\bullet$ *has small colimits as well as small limits.*

Proof. This follows from the preceding proposition and from Corollary 14.2.5. □

Proposition 18.7.5. *The full embedding* RexCat$_\bullet \hookrightarrow$ RexCat *commutes with small filtered colimits.*

Proof. Let I be a small filtered category and $i \mapsto C_i$ an I-indexed diagram with values in RexCat. We assume that, for any object i of I, the initial object $*$ of C_i is also a terminal object of C_i. We thus have pullback squares of the form

$$\begin{array}{ccc} C_i & \xrightarrow{x \mapsto \mathrm{id}_x} & \mathrm{Fun}([1], C_i) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ [0] & \xrightarrow{*} & C_i \end{array}$$

functorially in i. Since the functor $\mathrm{Fun}([1], -)$ commutes with small filtered colimits in Cat and since small filtered colimits are compatible with the formation of pullbacks in Cat, Theorem 18.5.5 implies that the colimit C of the diagram $i \mapsto C_i$ in RexCat yields a pullback square in Cat of the form

$$\begin{array}{ccc} C & \xrightarrow{x \mapsto \mathrm{id}_x} & \mathrm{Fun}([1], C) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ [0] & \xrightarrow{*} & C \end{array}$$

thus proving that the initial object $*$ of C induces an equivalence $C_{/*} \simeq C$. This proves that C belongs to RexCat$_\bullet$ and implies that C is a colimit of diagram $i \mapsto C_i$ in RexCat$_\bullet$ as well. □

Corollary 18.7.6. *The category* RexCat$_\bullet$ *is compactly generated.*

Proof. This follows from Proposition 18.3.12 and from the preceding proposition. □

Proposition 18.7.7. *Let C be a small category. For any pointed right exact category D, there is a canonical equivalence*

$$\mathrm{Rex}(\langle C \rangle_{\omega\bullet}, D) \simeq \mathrm{Fun}(C, D)\,.$$

Proof. This is the conjunction of Theorem 18.5.2 and of Proposition 18.7.3. □

Remark 18.7.8. Let C be a category with pushouts. We denote by

$$1^* = \mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$$

the functor assigning to a map of C its codomain. The existence of an initial object $*$ in C amounts to the existence of a left adjoint

$$1_! \colon C \to \mathrm{Fun}([1], C)$$

of 1^*. Indeed, if $*$ is initial in C, given a morphism $u \colon x \to y$ in C and a term a in C, there is an obvious correspondence of the form

$$\begin{array}{ccc} * & \longrightarrow & x \\ \downarrow & & \downarrow{\scriptstyle u} \\ a & \xrightarrow{f} & y \end{array} \quad \Leftrightarrow \quad f \colon a \to y\,.$$

Conversely, a correspondence as above for some object $* : C^{\simeq}$ implies in the particular case $a := y$ and $u := \mathrm{id}_y$ that there is a unique map from $*$ to y for any $y : C^{\simeq}$. This computation also shows that the left adjoint $1_!$ sends an object x to the unique map of type $* \to x$.

The property that the initial object $*$ of C is also terminal is equivalent to the property that the functor $1_!$ itself has a left adjoint

$$1^? \colon \mathrm{Fun}([1], C) \to C\,.$$

Indeed, if $*$ is also terminal, the dual version of the preceding observation tells us that the functor

$$0^* = \mathrm{ev}_0 \colon \mathrm{Fun}([1], C) \to C$$

has a right adjoint

$$0_* \colon C \to \mathrm{Fun}([1], C)$$

assigning to each term x of C the unique map of type $x \to *$. The identity of $*$ being both an initial object and a terminal object of $\mathrm{Fun}([1], C)$, the functor

$$0_* \colon \mathrm{Fun}([1], C) \to \mathrm{Fun}([1], \mathrm{Fun}([1], C))$$

yields the assignment

$$\begin{array}{ccc} x & & x & \longrightarrow & * \\ {\scriptstyle u}\downarrow & \mapsto & {\scriptstyle u}\downarrow & & \| \\ y & & y & \longrightarrow & * \end{array}$$

Since C has pushouts, the full embedding $i \colon \ulcorner \hookrightarrow \square$ induces an adjunction

$$i_! \colon \mathrm{Fun}(\ulcorner, C) \rightleftarrows \mathrm{Fun}(\square, C) \colon i^*$$

where $i_!$ sends a diagram of the form $b \leftarrow a \to c$ in C to the corresponding pushout square

$$\begin{array}{ccc} a & \longrightarrow & c \\ \downarrow & \ulcorner & \downarrow \\ b & \longrightarrow & b \amalg_a c\,. \end{array}$$

With these conventions, the functor $1^?$ is constructed as $i_! \circ i^* \circ 0_*$. In other words, it sends a map $u \colon x \to y$ in C to the pushout square

$$\begin{array}{ccc} x & \longrightarrow & * \\ {\scriptstyle u}\downarrow & \ulcorner & \downarrow \\ y & \longrightarrow & \mathrm{cofib}(u)\,. \end{array}$$

The object $1^?(u) = \mathrm{cofib}(u)$, also known as the *cofiber* of the morphism u, is thus the evaluation at the terminal object of $\square = [1] \times [1]$ of $i_!(0_*(u))$.

Conversely, if C has an initial object and if $1_!$ has a further left adjoint $1^?$, since $1_!$ is fully faithful with image those maps of the form $* \to x$, applying 0^* to the unit map

$$\eta_u \colon u \to 1_! 1^?(u)$$

yields for $u = \mathrm{id}_x$ a unique map $x \to *$ for all $x : C^{\simeq}$, hence $*$ is a terminal object of C.

Definition 18.7.9. Let C be a pointed right exact category with initial object $*$. The *suspension functor*

$$\Sigma\colon C \to C$$

is defined by forming for each term $x : C$ the following pushout square

$$\begin{array}{ccc} x & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \Sigma(x)\,. \end{array}$$

We warn the reader that, despite what the notation suggests, although the two maps from $*$ to $\Sigma(x)$ are the same (since $*$ is initial), the commutativity constraint making this square commutative is not the identity in general.

Proposition 18.7.10. *The suspension functor $\Sigma\colon C \to C$ is right exact.*

Proof. We adopt the notation of Remark 18.7.8. The functor 0_* is right exact because the functor $x \mapsto *$ is a right exact functor and so is the identity functor $x \mapsto x$. The cofiber functor $1^? = \mathrm{cofib}$ is right exact because it is a left adjoint. Therefore, $\Sigma = 1^? \circ 0_*$ is right exact. □

Proposition 18.7.11. *For any right exact functor between pointed right exact categories $f\colon C \to D$, there is a canonical identification $\Sigma f(x) = f(\Sigma(x))$ functorially in $x : C$.*

Proof. The functor f preserves initial objects hence preserves terminal objects as well and thus commutes with the operator 0_*. Since it is right exact, it is compatible with the formation of pushouts, hence of cofibers. □

Remark 18.7.12. We define $\mathrm{FinAn} := \langle [0] \rangle_\omega$, so that

$$\mathrm{Ind}(\mathrm{FinAn}) = \mathrm{An}\,.$$

For any category C with finite colimits, we thus have

$$\mathrm{Rex}(\mathrm{FinAn}, C) = C$$

by Theorem 18.5.2. If C is pointed, we even get by Proposition 18.7.7:

$$\mathrm{Rex}(\mathrm{FinAn}_\bullet, C) = C\,.$$

The suspension functor

$$\Sigma\colon \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$$

thus induces a functor

$$\Sigma^*\colon \mathrm{Rex}(\mathrm{FinAn}_\bullet, C) \to \mathrm{Rex}(\mathrm{FinAn}_\bullet, C)\,.$$

Lemma 18.7.13. *Given a fixed pointed right exact category C, under the identification $\mathrm{Rex}(\mathrm{FinAn}_\bullet, C) = C$, the two functors*

$$\Sigma^*\colon \mathrm{Rex}(\mathrm{FinAn}_\bullet, C) \to \mathrm{Rex}(\mathrm{FinAn}_\bullet, C) \quad \textit{and} \quad \Sigma\colon C \to C$$

agree. More generally, for any pointed right exact category A, the suspension functor

$$\Sigma\colon \mathrm{Rex}(A, C) \to \mathrm{Rex}(A, C)$$

agrees with the operation of pre-composition with the suspension functor $\Sigma\colon A \to A$.

Proof. The last assertion means that, for any right exact functor $\varphi\colon A \to C$, we have

$$\Sigma(\varphi(a)) = \varphi(\Sigma(a))$$

functorially in $a : A$. This is precisely what the preceding proposition claims to be true. □

Proposition 18.7.14. *If A and B are small right exact categories, and if either A or B is pointed, then $A \otimes B$ is pointed.*

Proof. Being pointed for a small right exact category C is equivalent to the property that the canonical functor $C \to C_\bullet$ is fully faithful. Indeed, if this is the case, then its essential image is the smallest full subcategory of $C_\bullet$ stable under pushouts containing objects coming from C, which is $C_\bullet$ by definition. Equivalently, we deduce right away from Theorem 18.3.8 that this means that the functor $x \mapsto x_+$

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$$

is an equivalence. If B is pointed, we have

$$\begin{aligned}\mathrm{Lex}((A \otimes B)^{\mathrm{op}}, \mathrm{An}) &= \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{An})) \\ &= \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{An}_\bullet)) \\ &= \mathrm{Lex}((A \otimes B)^{\mathrm{op}}, \mathrm{An}_\bullet)\,.\end{aligned}$$

Therefore, $A \otimes B$ is pointed whenever B is pointed. The case where A is pointed is either proved similarly or deduced from what precedes using Remark 18.6.8. □

Proposition 18.7.15. *Let A and B be two small categories with finite colimits. The canonical right exact functor $(-)_+\colon B \to B_\bullet$ induces a functor*

$$A \otimes B \to A \otimes B_\bullet\,, \quad x \otimes y \mapsto x \otimes y_+$$

that exhibits $A \otimes B_\bullet$ as the free pointed right exact category associated to $A \otimes B$ and thus induces a canonical equivalence

$$(A \otimes B)_\bullet \simeq A \otimes B_\bullet\,.$$

Proof. We have

$$\begin{aligned}\mathrm{Rex}(A \otimes B, (A \otimes B)_\bullet) &= \mathrm{Rex}(A, \mathrm{Rex}(B, (A \otimes B)_\bullet)) \\ &= \mathrm{Rex}(A, \mathrm{Rex}(B_\bullet, (A \otimes B)_\bullet)) \\ &= \mathrm{Rex}(A \otimes B_\bullet, (A \otimes B)_\bullet)\,.\end{aligned}$$

There is thus a unique right exact functor $u\colon A \otimes B_\bullet \to (A \otimes B)_\bullet$ sending $x \otimes y_+$ to $(x \otimes y)_+$ for all $x : A$ and $y : B$. Similarly, the universal property of $(A \otimes B)_\bullet$ and Proposition 18.7.14 provide a functor $v\colon (A \otimes B)_\bullet \to A \otimes B_\bullet$ uniquely determined by the property that it sends $(x \otimes y)_+$ to $x \otimes y_+$ for all $x : A$ and $y : B$. Therefore u and v must be inverse to each other. □

Remark 18.7.16. Given a pointed right exact category C, the suspension functor

$$\Sigma\colon C \to C$$

agrees with the functor

$$\Sigma \otimes \mathrm{id}_C \colon \mathrm{FinAn}_\bullet \otimes C \to \mathrm{FinAn}_\bullet \otimes C$$

induced by the suspension $\Sigma \colon \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$ up to the canonical equivalences

$$C \simeq \mathrm{FinAn} \otimes C \simeq \mathrm{FinAn}_\bullet \otimes C\,.$$

Indeed, for $S^0 := *_+$ and $S^1 := \Sigma(S^0)$, we see with Proposition 18.7.11 that

$$S^1 \otimes x = S^0 \otimes \Sigma(x)$$

for all $x : C$. Since $\mathrm{Rex}(\mathrm{FinAn}_\bullet, C) = C$ through evaluation at S^0, this proves the claim. This identification is useful to prove the following proposition.

Proposition 18.7.17. *The suspension functor can be promoted to an endomorphism of the identity of* $\mathrm{RexCat}_\bullet$ *in the category* $\mathrm{Fun}(\mathrm{RexCat}_\bullet, \mathrm{RexCat}_\bullet)$. *In other words, we get a commutativity constraint*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\Sigma} \\ C & \xrightarrow{f} & D \end{array} \qquad \kappa_f \colon \Sigma \circ f \xrightarrow{\simeq} f \circ \Sigma$$

for any right exact functor f*, functorially in* f.

Proof. There is a functorial identification

$$a_C \colon \mathrm{FinAn}_\bullet \otimes C \xrightarrow{\sim} C$$

and the suspension $\Sigma \colon \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$. We will see that the assignment

$$C \mapsto a_C \circ \Sigma \otimes \mathrm{id}_C \circ a_C^{-1}$$

provides a natural transformation with the expected properties. □

Exercise 18.7.18. Prove that, for any finite poset P, one has

$$\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}_\bullet) = \langle P \rangle_{\omega\bullet}\,.$$

Deduce that, for any C pointed and right exact, the suspension functor

$$\Sigma \colon \mathrm{Fun}(P, C) \to \mathrm{Fun}(P, C)$$

is induced through the identification above by pre-composition with the suspension functor

$$\Sigma \colon \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}_\bullet) \to \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{FinAn}_\bullet)\,.$$

19 Elements of stable category theory

19.1 Stable categories

Definition 19.1.1. A *stable category* is a pointed right exact category such that the suspension functor

$$\Sigma\colon C \to C$$

is an equivalence.

We denote by StabCat the full subcategory of RexCat spanned by small stable categories.

Lemma 19.1.2. *Let S be the image of the map $[0] \to \mathrm{Map}([1], \mathrm{RexCat})$ corresponding to the right exact functor*

$$\Sigma\colon \langle [1]\rangle_{\omega\bullet} \to \langle [1]\rangle_{\omega\bullet}\,.$$

Then S is a collection of morphisms between compact objects and the S-local objects in $\mathrm{RexCat}_\bullet$ *precisely are the small stable categories. Moreover, for any small pointed right exact category C, the suspension functor $\Sigma\colon C \to C$ is an S-equivalence.*

Proof. The suspension functor $\Sigma\colon C \to C$ is an equivalence if and only if the induced map

$$\Sigma_*\colon \mathrm{Map}([1], C) \to \mathrm{Map}([1], C)$$

is an equivalence. We have:

$$\begin{aligned}\mathrm{Map}([1], C) &= \mathrm{Fun}([1], C)^{\simeq}\\ &= \mathrm{Rex}(\langle [1]\rangle_{\omega\bullet}, C)^{\simeq}\end{aligned}$$

The map Σ_* coincides with the map defined by pre-composition with $\Sigma\colon \langle [1]\rangle_{\omega\bullet} \to \langle [1]\rangle_{\omega\bullet}$ by the first part of Lemma 18.7.13. This proves the first assertion. The last one follows right away from the last part of Lemma 18.7.13. □

Proposition 19.1.3. *The category* StabCat *is compactly generated and the full embedding*

$$\mathrm{StabCat} \hookrightarrow \mathrm{RexCat}_\bullet$$

commutes with filtered colimits and has a left adjoint.

Proof. Since $\mathrm{RexCat}_\bullet$ is compactly generated by Corollary 18.7.6, the preceding lemma says that we can apply Proposition 18.4.1. □

Remark 19.1.4. One can prove directly that (small) stable categories are stable under (small) colimits in pointed right exact categories. Indeed, by virtue of Proposition 18.7.17, if $i \mapsto C_i$ is a diagram in StabCat indexed by a small category I, with colimit C in $\mathrm{RexCat}_\bullet$, the suspension $\Sigma\colon C \to C$ is the colimit of the suspension functors $\Sigma\colon C_i \to C_i$, hence is an equivalence as well. One can construct the left adjoint of the embedding of StabCat into $\mathrm{RexCat}_\bullet$ explicitly as follows.

Lemma 19.1.5. *The exchange isomorphism between Σ^2 and Σ, obtained from Proposition 18.7.17 with $f = \Sigma^2$, is the associativity constraint*

$$\Sigma \circ \Sigma^2 = \Sigma^2 \circ \Sigma$$

(in any pointed category with finite colimits).

Proof. Let C be a pointed category with finite colimits. Under the equivalence

$$\mathrm{FinAn}_\bullet \otimes C = C$$

we can identify $\Sigma\colon C \to C$ with the functor $\Sigma \otimes \mathrm{id}_C$. In other words, it suffices to prove the lemma in the case where $C = \mathrm{FinAn}_\bullet$. Composing the equivalence $\mathrm{FinAn}_\bullet \otimes \mathrm{FinAn}_\bullet = \mathrm{FinAn}_\bullet$ with the canonical functor

$$\mathrm{FinAn}_\bullet \times \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet \otimes \mathrm{FinAn}_\bullet\,, \quad (x, y) \mapsto x \otimes y$$

defines a functor

$$\mathrm{FinAn}_\bullet \times \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet\,, \quad (x, y) \mapsto x \wedge y$$

that commutes with finite colimits in each variables. The universal property of $\mathrm{FinAn}_\bullet$ implies that there are functorial equivalences

$$x \wedge S^0 = x$$

extending the identification $S^0 \wedge S^0 = S^0$. The equivalence $S^0 \wedge (S^0 \wedge S^0) = (S^0 \wedge S^0) \wedge S^0$ extends uniquely to a functorial equivalence of the form

$$\wedge(y \wedge z) = (x \wedge y) \wedge z\,.$$

Finally, there is a unique functorial equivalence of the form

$$\tau_{x,y}\colon x \wedge y = y \wedge x$$

inducing the identity of $S^0 \wedge S^0$.

We observe that, for any maps u and v in $\mathrm{FinAn}_\bullet$ we have identities of the form

$$\begin{aligned}(u \wedge \mathrm{id}) \circ (\mathrm{id} \wedge v) &= u \wedge v \\ &= (\mathrm{id} \wedge v) \circ (u \wedge \mathrm{id})\,.\end{aligned}$$

Furthermore, we can identify

$$\Sigma\colon \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$$

with the functor

$$S^1 \wedge (-)\colon \mathrm{FinAn}_\bullet \to \mathrm{FinAn}_\bullet$$

where $S^1 := \Sigma(S^0)$. The exchange isomorphism

$$\Sigma(\Sigma^2(x)) = \Sigma^2(\Sigma(x))$$

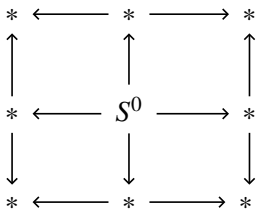
can be interpreted as the map

$$\tau_{S^1,S^2} \wedge \mathrm{id}_x \colon S^1 \wedge S^2 \wedge x = S^2 \wedge S^1 \wedge x\,.$$

Therefore, in order to prove this lemma, it suffices to prove that the map τ_{S^1,S^2} is the identity. One can see S^1 as the colimit of the diagram

$$* \leftarrow S^0 \rightarrow *$$

(where $*$ is the terminal anima). And S^2 is the colimit of the diagram bellow.

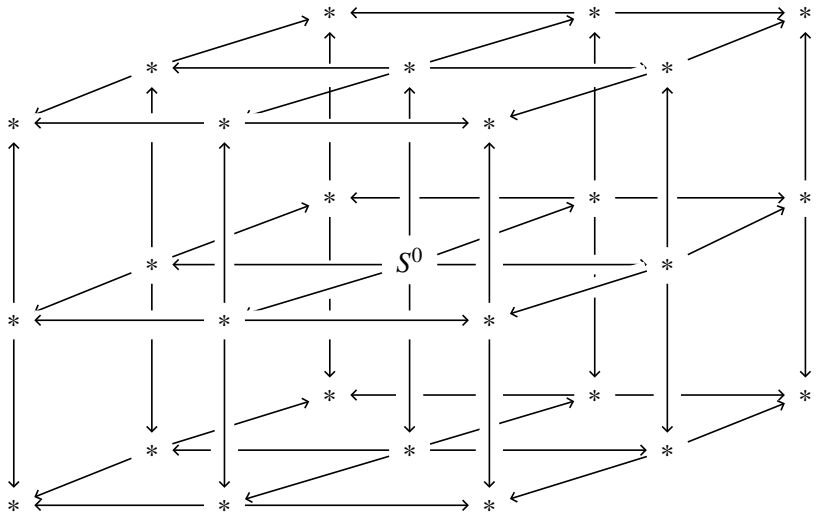


Wrting $S^2 := S^1 \wedge S^1$, the isomorphism

$$\tau_{S^1,S^1} \colon S^1 \wedge S^1 \xrightarrow{\sim} S^1 \wedge S^1$$

is the map induced by applying a a quarter turn (in any direction) to the commutative diagram above.

Let us consider the commutative diagram bellow.



Its colimit will be denoted by S^3. There are canonical identifications

$$S^1 \wedge S^2 = S^3 \quad \text{and} \quad S^2 \wedge S^1 = S^3\,.$$

Picking a direction of rotation for each axis going through S^0 in the cube above, making a quarter turn around each of them defines three automorphisms P, Q and R of S^3. Up to reindexing, we may assume that $R\colon S^3 \to S^3$ correspond to $\tau_{S^1,S^1} \wedge \mathrm{id}_{S^1}$ and Q to $\mathrm{id}_{S^1} \wedge \tau_{S^1,S^1}$, so that τ_{S^1,S^2} corresponds to $Q \circ R$. This shows that

$$Q \circ R = R \circ Q\,.$$

Therefore, this also shows that R and P commute with each other, as well as do Q and P. But we can choose P, Q and R so that $P \circ R = R \circ Q$ holds on the nose (at the level of rotations). This implies that $P = Q$. Therefore, we have $Q \circ R = R^2$. Since $\tau^2_{x,x} = \mathrm{id}_{x\wedge x}$ for all x. we see that R^2 is the identity, which proves the lemma. □

Remark 19.1.6. Let C be a small pointed right exact category. One constructs a sequence of right exact functors

$$C_0 \xrightarrow{f_1} C_1 \to \cdots \to C_n \xrightarrow{f_{n+1}} C_{n+1} \to \cdots$$

by $C_n = C$ for all $n : \mathbb{N}$ and $f_{n+1} = \Sigma$ for all $n : \mathbb{N}$. We define the *Spanier-Whitehead stabilization of C* as the Milnor colimit in $\mathrm{RexCat}_\bullet$ of this sequence of functors (one can compute this Milnor colimit as an ω-indexed colimit, hence as an ω-indexed colimit in Cat itself):

$$\mathrm{SW}(C) := \text{“colim”}_{n:\mathbb{N}}\ C_n\,.$$

By construction there is a cover

$$\sum_{n:\mathbb{N}} C_n^{\simeq} \to \mathrm{SW}(C)^{\simeq}\,.$$

For x an object of C and $n : \mathbb{N}$, we write $\Sigma^{\infty-n}(x)$ for the image of x in $\mathrm{SW}(C)$, where x is seen as a term of C_n. Given another object y of C and a natural number $p : \mathbb{N}$, the general explicit description of mapping animae in filtered colimits of categories given in Remark 16.3.13 yields for $m := \max\{n, p\}$:

$$\operatorname*{colim}_{k:\omega_{m/}} \mathrm{Hom}_C(\Sigma^{k-n}(x), \Sigma^{k-p}(y)) = \mathrm{Hom}_{\mathrm{SW}(C)}(\Sigma^{\infty-n}(x), \Sigma^{\infty-p}(y))\,.$$

The component $C_0 = C$ yields a canonical right exact functor

$$\Sigma^\infty\colon C \to \mathrm{SW}(C)\,, \quad x \mapsto \Sigma^{\infty-0}(x)\,.$$

We have a morphism of sequences of the form

$$\begin{array}{ccccccccccc}
C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots & \longrightarrow & C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots \\
{\scriptstyle\Sigma^2}\downarrow & & {\scriptstyle\Sigma^2}\downarrow & & & & {\scriptstyle\Sigma^2}\downarrow & & {\scriptstyle\Sigma^2}\downarrow & & \\
C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots & \longrightarrow & C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots
\end{array}$$

in which each commutative square is given by the commutativity constraint of Proposition 18.7.17 for $f = \Sigma^2$. However, by virtue of Lemma 19.1.5, this commutativity constraint simply is the associativity constraint.

The induced functor on the Milnor colimits simply is the double suspension functor

$$\Sigma^2\colon \mathrm{SW}(C) \to \mathrm{SW}(C)\,.$$

Theorem 19.1.7. *For any pointed right exact category C, the Spanier-Whitehead stabilization* $\mathrm{SW}(C)$ *is a stable category.*

Proof. It suffices to prove that the functor

$$\Sigma^2 \colon \mathrm{SW}(C) \to \mathrm{SW}(C)$$

is an equivalence. As explained in the previous remark, this functor is the one induced by applying the Milnor colimit construction to the morphism of sequences

$$\begin{array}{ccccccccccc}
C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots & \longrightarrow & C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots \\
\downarrow{\scriptstyle \Sigma^2} & & \downarrow{\scriptstyle \Sigma^2} & & & & \downarrow{\scriptstyle \Sigma^2} & & \downarrow{\scriptstyle \Sigma^2} & & \\
C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots & \longrightarrow & C & \xrightarrow{\Sigma} & C & \xrightarrow{\Sigma} & \cdots
\end{array}$$

in which the commutativity constraints are the identities. Using the identification $\omega_{2/} = \omega$ and the fact that the successor operation on ω defines a terminal functor, we see immediately that Σ^2 is an equivalence whose inverse is the diagram □

Remark 19.1.8. Using the last assertion of Lemma 19.1.2, we see that, for any pointed right exact category C and any stable category D, the canonical functor

$$\Sigma^\infty \colon C \to \mathrm{SW}(C)$$

induces an equivalence

$$\mathrm{RexCat}(\mathrm{SW}(C), D)^\simeq \simeq \mathrm{Rex}(C, D)^\simeq .$$

In particular, Theorem 19.1.7 yields another proof that the full embedding $\mathrm{StabCat} \hookrightarrow \mathrm{RexCat}_\bullet$ has a left adjoint given on objects by the Spanier-Whitehead construction. Applying Proposition 18.3.12, we see that StabCat is compactly generated. Overall, this yields an alternative proof of Proposition 19.1.3, arguably more explicit.

19.2 Spectra

Lemma 19.2.1. *Let A and B be pointed right exact categories. If either A or B is stable, then both $A \otimes B$ and* $\mathrm{Rex}(A, B)$ *are stable.*

Proof. Assume that A is stable. The suspension functor $\Sigma \colon A \to A$ induces a functor

$$\Sigma \otimes \mathrm{id}_B \colon A \otimes B \to A \otimes B$$

that is isomorphic to the suspension of $A \otimes B$. To see this, one can for instance use the canonical equivalence

$$(\mathrm{FinAn}_\bullet \otimes A) \otimes B \simeq \mathrm{FinAn}_\bullet \otimes (A \otimes B)$$

and conclude with Remark 18.7.16. This implies that $A \otimes B$ is stable. The case where B is stable is deduced from the equivalence $A \otimes B \simeq B \otimes A$.

Similarly, if A is stable, Lemma 18.7.13 implies that the suspension on $\mathrm{Rex}(A, B)$ is induced from the one on A and thus that it is an equivalence. The fact that $\mathrm{Rex}(A, B)$ is stable whenever B has this property is even more tautological. □

Theorem 19.2.2. *The embedding* $\mathrm{StabCat} \hookrightarrow \mathrm{RexCat}_\bullet$ *has a right adjoint given by the assignment*

$$B \mapsto \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet), B)\,.$$

In particular, it commutes with small colimits.

Proof. Given A a small stable category and B a small pointed right exact category, since $(-) \otimes A$ commutes with colimits in $\mathrm{RexCat}_\bullet$ (being a left adjoint, by definition in RexCat, but then also in $\mathrm{RexCat}_\bullet$, by Proposition 18.7.15), and since the category ω is weakly contractible, we see that

$$\begin{aligned} A &= \mathrm{FinAn} \otimes A \\ &= \mathrm{FinAn}_\bullet \otimes A \\ &= \mathrm{SW}(\mathrm{FinAn}_\bullet) \otimes A\,. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \mathrm{Rex}(A, B) &= \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet) \otimes A, B) \\ &= \mathrm{Rex}(A, \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet), B))\,. \end{aligned}$$

Since, by virtue of Lemma 19.2.1 and Theorem 19.1.7, $\mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet), B)$ is stable, this implies the theorem right away. □

Construction 19.2.3. Let C be a pointed left exact category. We denote by $\mathrm{Sp}(C)$ the Milnor limit of the diagram

$$\cdots \to C \xrightarrow{\Omega} C \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C\,.$$

Corollary 19.2.4. *For any pointed left exact category* C, $\mathrm{Sp}(C)$ *is a stable category.*

Proof. We have $\mathrm{Sp}(C) = \mathrm{Rex}(\mathrm{SW}(\mathrm{FinAn}_\bullet), C^{\mathrm{op}})^{\mathrm{op}}$. We conclude with Lemma 19.2.1 and Theorem 19.1.7. □

Definition 19.2.5. Let $\mathrm{An}_\bullet = \mathrm{An}_{*/}$ be the category of small pointed animae. The stable category Sp of (small) *spectra* is defined as

$$\mathrm{Sp} := \mathrm{Sp}(\mathrm{An}_\bullet)\,.$$

Lemma 19.2.6. *Let* C *be a pointed right exact category with pullbacks. Then* C *is a stable category if and only if* C^{op} *is a stable category.*

Proof. Since C has pullbacks with a terminal object that is also initial, one can apply Remark 18.7.8 to C^{op} and translate it to properties of C: this means that the functor

$$0_* \colon C \to \mathrm{Fun}([1], C)$$

has a right adjoint

$$0^! \colon \mathrm{Fun}([1], C) \to C$$

that takes a map $u \colon a \to b$ in C to its fiber $0^!(u) = \mathrm{fib}(u)$ appearing in the pullback below.

$$\begin{array}{ccc} \mathrm{fib}(u) & \longrightarrow & a \\ \downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ * & \longrightarrow & b \end{array}$$

We define the loop functor

$$\Omega\colon C \to C$$

with the formula $\Omega := 0^! \circ 1_!$. Recall that $\Sigma = 1^? \circ 0_*$ so that Ω obviously is a right adjoint functor to Σ. On the other hand, we see through the equivalence $[1]^{\mathrm{op}} = [1]$ that

$$\Omega^{\mathrm{op}}\colon C^{\mathrm{op}} \to C^{\mathrm{op}}$$

coincides with the suspension functor on C^{op}. Since a left adjoint functor is an equivalence if and only if its right adjoint functor is an equivalence, this shows that C is stable if and only if C^{op} has the same property. □

Proposition 19.2.7. *For any (small) pointed right exact category C, we have*

$$\mathrm{Lex}(\mathrm{SW}(C)^{\mathrm{op}}, \mathrm{An}) \simeq \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})\,.$$

Proof. Since C is pointed we have

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$$

functorially in C. It is clear from Lemma 18.7.13 that the functor $\mathrm{Lex}((-)^{\mathrm{op}}, \mathrm{An}_\bullet)$ takes the diagram

$$C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} \cdots \to C \xrightarrow{\Sigma} C \xrightarrow{\Sigma} \cdots$$

to the diagram

$$\cdots \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \xrightarrow{\Omega} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet) \xrightarrow{\Omega} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$$

itself isomorphic to the image of

$$\cdots \to \mathrm{An}_\bullet \xrightarrow{\Omega} \mathrm{An}_\bullet \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathrm{An}_\bullet \xrightarrow{\Omega} \mathrm{An}_\bullet$$

through the functor $\mathrm{Lex}(C^{\mathrm{op}}, -)$. We conclude by observing that the functor $\mathrm{Lex}(C^{\mathrm{op}}, -)$ preserves limits. □

Corollary 19.2.8. *For any small stable category C, there is a canonical equivalence*

$$\mathrm{Ind}(C) \simeq \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})\,.$$

Proof. Since C is stable, the canonical functor $C \to \mathrm{SW}(C)$ is an equivalence. We conclude with Proposition 19.2.7 above and Remark 18.2.7. □

Corollary 19.2.9. *The idempotent completion of a small stable category is stable.*

Proof. The idempotent completion of a small category C with finite colimits is obtained by taking the full subcategory of compact objects in $\mathrm{Ind}(C)$. If C is stable, then so is $\mathrm{Ind}(C)$ (by Lemma 19.2.1, Lemma 19.2.6 and Corollary 19.2.8), and the suspension functor on the idempotent completion of C is obtained from the one on $\mathrm{Ind}(C)$ by functoriality, hence is an equivalence. □

Corollary 19.2.10. *The category of spectra is stable and compactly generated. More precisely, there is a canonical equivalence*

$$\mathrm{Lex}(\mathrm{SW}(\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{An}) \simeq \mathrm{Sp}\,.$$

Proof. We already know that Sp is stable by Corollary 19.2.4. On the other hand, Proposition 19.2.7 for $C = [0]$ yields

$$\mathrm{Lex}(\mathrm{SW}(\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{An}) = \mathrm{Lex}((\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{Sp})\,.$$

But since Sp is pointed, we also have

$$\mathrm{Lex}((\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{Sp}) = \mathrm{Lex}((\mathrm{FinAn})^{\mathrm{op}}, \mathrm{Sp}) = \mathrm{Sp}\,. \qquad \square$$

Notation 19.2.11. The preceding corollary together with Theorem 18.2.8 implies that the functor

$$\mathrm{FinAn}_\bullet \to \mathrm{SW}(\mathrm{FinAn}_\bullet)\,, \quad x \mapsto \Sigma^\infty(x)$$

extends to a small colimit preserving functor

$$\Sigma^\infty \colon \mathrm{An}_\bullet \to \mathrm{Sp}$$

that has a right adjoint

$$\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}_\bullet\,.$$

We observe that if $S^0 = *_+$, then $\Omega^\infty = \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty(S^0), -)$ (see Remark 12.8.3).

Proposition 19.2.12. *A morphism of spectra $u \colon E \to F$ is an isomorphism if and only if, for any integer $n : \mathbb{N}$, the induced map*

$$\Omega^\infty(\Sigma^n(u)) \colon \Omega^\infty(\Sigma^n(E)) \to \Omega^\infty(\Sigma^n(F))$$

is an equivalence.

Proof. Let C be the full subcategory of Sp whose objects are those spectra X such that

$$u_* \colon \mathrm{Hom}_{\mathrm{Sp}}(X, E) \to \mathrm{Hom}_{\mathrm{Sp}}(X, F)$$

is an isomorphism. Assuming that $\Omega^\infty(\Sigma^n(u))$ is an equivalence for all n means that C contains $\Omega^n(\Sigma^\infty(S^0))$ for all $n : \mathbb{N}$. On the other hand, C clearly is stable under small colimits in Sp. There is a unique right exact functor

$$\mathrm{FinAn}_\bullet \to C$$

sending the point to $\Sigma^\infty(S^0)$. Since C is stable, this extends to a unique right exact functor

$$i \colon \mathrm{SW}(\mathrm{FinAn}_\bullet) \to C$$

sending $\Sigma^\infty(S^0)$ to itself. Theorem 18.2.10 implies that there is a unique small colimits preserving functor

$$i_! \colon \mathrm{Lex}(\mathrm{SW}(\mathrm{FinAn}_\bullet)^{\mathrm{op}}, \mathrm{An}) \to C$$

extending i. Corollary 19.2.10 together with a new application of Theorem 18.2.10 tell us that composing $i_!$ with the full embedding $C \hookrightarrow \mathrm{Sp}$ is the identity. Therefore, $C = \mathrm{Sp}$. This proves that u is an isomorphism. $\square$

19.3 Exact functors between stable categories

Definition 19.3.1. Let C be a pointed right exact category with initial object 0. A *cofiber sequence* in C is a pushout square of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & \ulcorner & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & x'' \end{array} .$$

Let C be a pointed left exact category (i.e. such that C^{op} is a pointed right exact category). A *fiber sequence* in C is a pullback square of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & x'' \end{array} .$$

Lemma 19.3.2. *Let C be a category with a terminal object 0 that is also an initial object. We assume that C has both pullbacks and pushouts. The following assertions are equivalent.*

(i) The category C is stable.

(ii) The category C^{op} is stable.

(iii) A commutative square of C is a pullback square if and only if it is a pushout square.

(iv) A commutative square of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & x'' \end{array}$$

in C is a fiber sequence if and only if it is a cofiber sequence.

Proof. We already know that assertions (i) and (ii) are equivalent by Lemma 19.2.6. Let 0 be the initial object of C. It is clear that condition (iv) follows from condition (iii). Let us show that condition (i) follows from condition (iv). For any object $x : C$, the cofiber sequence

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(x) \end{array}$$

is a fiber sequence as well, which means that there is a functorial identification

$$x = \Omega(\Sigma(x)) .$$

The same argument applies to C^{op} and thus shows that there is a functorial identification of the form

$$\Sigma(\Omega(x)) = x .$$

This shows that both Σ and Ω are equivalences of categories.

Let us assume now that assertion (iv) holds and consider a pullback square in C of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y\,. \end{array}$$

We complete this diagram by forming further pullback squares below

$$\begin{array}{ccccccc} \Omega(x') & \xrightarrow{\Omega(f')} & \Omega(y') & \longrightarrow & 0 & & \\ {\scriptstyle \Omega(u)}\downarrow & & \downarrow{\scriptstyle \Omega(v)} & & \downarrow & & \\ \Omega(x) & \xrightarrow{\Omega(f)} & \Omega(y) & \longrightarrow & \mathrm{fib}(v) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{fib}(f) & \longrightarrow & x' & \xrightarrow{u} & x \\ & & \downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ & & 0 & \longrightarrow & y' & \xrightarrow{v} & y \end{array}$$

and observe that the upper left square is the image of the lower left one through the functor Ω up to a flip along the diagonal. Since fiber sequences are cofiber sequences as well, the two squares

$$\begin{array}{ccc} \Omega(x') & \longrightarrow & 0 \\ {\scriptstyle \Omega(u)}\downarrow & & \downarrow \\ \Omega(x) & \longrightarrow & \mathrm{fib}(v) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Omega(y') & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Omega(v)} & & \downarrow \\ \Omega(y) & \longrightarrow & \mathrm{fib}(v) \end{array}$$

are pushout squares, hence that the image of

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y\,. \end{array}$$

by the functor Ω is a pushout square. Since Ω is an equivalence, this says that the latter square is a pushout square as well. Applying the same argument to C^{op} shows that condition (iii) follows from condition (iv).

It remains to check that condition (iv) follows from condition (i). Let

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & x'' \end{array}$$

be a fiber sequence. We also form the following cofiber sequence.

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ {\scriptstyle v}\downarrow & & \downarrow \\ x'' & \xrightarrow{w} & y \end{array}$$

Composing the squares

$$\begin{array}{ccccc} x' & \xrightarrow{u} & x & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle v} & & \downarrow \\ 0 & \longrightarrow & x'' & \xrightarrow{w} & y \end{array}$$

defines canonical a map $a\colon x' \to \Omega(y)$. On the other hand, there is a canonical pushout square of the form

$$\begin{array}{ccc} y & \longrightarrow & \Sigma(x) \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \Sigma(v)} \\ 0 & \longrightarrow & \Sigma(x'') \end{array}$$

hence also a pushout of the form

$$\begin{array}{ccc} \Omega(y) & \longrightarrow & x \\ \downarrow & \ulcorner & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & x'' \end{array}$$

inducing a canonical map $b\colon \Omega(y) \to x'$. We see that $b \circ a$ is the identity of x' because its image through Σ is part of an endomorphism of the pushout square

$$\begin{array}{ccc} x' & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma(x') \end{array}$$

that is the identity on the shape $\ulcorner$. The same observation in C^{op} shows that $a \circ b$ is the identity. □

Theorem 19.3.3. *If C is a stable category then so is C^{op}. In other words, any stable category has pullbacks.*

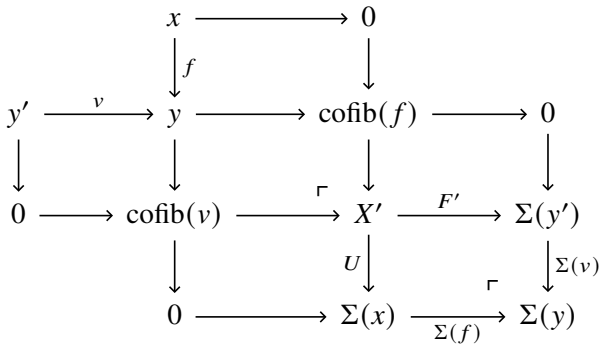
Proof. Let C be a stable category. The Yoneda embedding is a fully faithful right exact functor

$$h_C\colon C \hookrightarrow \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) \xrightarrow{\sim} \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$$

into a stable compactly generated category. Since those have small limits, Lemma 19.3.2 applies to them. In other words, the family of pullback squares in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$ agrees with the family of pushout squares. This will be used to prove the existence of pullbacks in C as follows. Let us consider a diagram of the form

$$\begin{array}{ccc} & & x \\ & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y \end{array}$$

in C. We complete this diagram through further pushout squares as follows.



We define x' through $\Sigma(x') := X'$ and similarly f' through $\Sigma(f') := F'$ and u through $\Sigma(u) := U$. We obtain a pushout square of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y \end{array}$$

whose image through Σ is the square in the lower right of the big diagram above. The image of the latter square through h_C is thus a pushout in $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An})$, hence a pullback square as well. The functor h_C being fully faithful, this implies right away that this is a pullback square in C. □

Theorem 19.3.4. *Let A and B be two stable categories, and $\varphi\colon A \to B$ be a functor. The following assertions are equivalent.*

(i) The functor φ is right exact.

(ii) The functor φ is left exact.

(iii) The functor φ preserves initial objects and cofiber sequences.

(iv) The functor φ preserves terminal objects and fiber sequences.

Proof. It is clear that any left exact functor preserves fiber sequences by definition. Let us assume condition (iv) to hold and prove that φ is left exact. We just have to prove that it preserves pullbacks. Let

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{v} & y \end{array}$$

be a pullback square in A. We extend it by forming the fiber of f'.

$$\begin{array}{ccc} \mathrm{fib}(f') & \xrightarrow{i} & x' \\ \downarrow & & \downarrow{\scriptstyle f'} \\ 0 & \longrightarrow & y' \end{array}$$

We then get a commutative diagram in B of the form

$$\begin{array}{ccccc} \varphi(\mathrm{fib}(f')) & \xrightarrow{\varphi(i)} & \varphi(x') & \xrightarrow{\varphi(u)} & \varphi(x) \\ \downarrow & & \downarrow{\scriptstyle \varphi(f')} & & \downarrow{\scriptstyle \varphi(f)} \\ 0 & \longrightarrow & \varphi(y') & \xrightarrow{\varphi(v)} & \varphi(y) \end{array}$$

in which the outer square as well as the square on the left both are pullback squares. But B being stable these are pushout squares as well. This implies that the square on the right is a pushout square, hence a pullback square as well. In other words conditions (i) and (iii) are equivalent and conditions (ii) and (iv) are equivalent.

It suffices now to prove that conditions (iii) and (iv) are equivalent. This follows from the fact that in any stable category, the collection of initial objects coincides with the collection of terminal objects, and that the collection of cofiber sequences agrees with the collection of fiber sequences. □

Proposition 19.3.5. *Let $u\colon A \to B$ be an exact functor between small stable categories. Then the induced functor*

$$u^*\colon \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Sp})$$

has a right adjoint

$$u_*\colon \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Sp})\,.$$

Proof. Since we have

$$\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$$

for any small stable category C by Proposition 19.2.7, Theorem 18.2.10 tells us that it suffices to prove that the functor u^* commutes with small colimits. Since it is left exact (being a right adjoint by Theorem 18.2.8), it is an exact functor, by Theorem 19.3.4. Therefore, by virtue of Corollary 18.3.11, it suffices to prove that u^* commutes with small filtered colimits. But Theorem 18.2.8 tells us that there is a commutative square of the form

$$\begin{array}{ccc} \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{An}) & \xrightarrow{u^*} & \mathrm{Lex}(A^{\mathrm{op}}, \mathrm{An}) \\ \downarrow & & \downarrow \\ \mathrm{PSh}(B) & \xrightarrow{u^*} & \mathrm{PSh}(A) \end{array}$$

in which the two vertical maps are fully faithful embeddings that commute with small filtered colimits by Proposition 18.2.2. Since the lower horizontal map in the square above commutes with small colimits, this implies that the upper horizontal one commutes with small filtered colimits. □

19.4 Representable presheaves on stable categories

Construction 19.4.1. Let C be a small stable category. Recall from Corollary 19.2.8 that

$$\mathrm{Ind}(C) = \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})\,.$$

The Yoneda embedding $C \hookrightarrow \mathrm{Ind}(C)$ thus yields a fully faithful exact functor

$$h_C^{\mathrm{Sp}}\colon C \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$$

hence an exact functor of the form

$$(h_C^{\mathrm{Sp}})^{\mathrm{op}} \colon C^{\mathrm{op}} \to \mathrm{Rex}(C, \mathrm{Sp}^{\mathrm{op}})$$

corresponding to a functor

$$\hom_C \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}$$

that is (right) exact in each variable. For x in C^{op} and y in C there is a functorial identification

$$\Omega^\infty(\hom_C(x, y)) = \mathrm{Hom}_C(x, y)\,.$$

Therefore, if C is locally small, $\hom_C(x, y)$ belongs to the category Sp of small spectra.

Lemma 19.4.2. *If C is a stable locally small category, then* $\hom_C$ *takes its values in* Sp. *Moreover, the induced functor*

$$\hom_C \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}$$

preserves small limits in each variable.

Proof. We can choose a larger universe for which C is small, with corresponding category of large spectra Sp' and thus a functor

$$\hom_C \colon C^{\mathrm{op}} \times C \to \mathrm{Sp}'\,.$$

An object of Sp' is by definition an infinite sequence of possibly large pointed animae x_n equipped with isomorphisms $x_n \simeq \Omega(x_{n+1})$ for all $n : \mathbb{N}$. In the case of $\hom_C(a, b)$ with a and b objects of C, by construction, this sequence may be obtained through

$$x_n = \mathrm{Hom}_C(a, \Sigma^n(b))$$

together with the isomorphisms

$$\begin{aligned}\mathrm{Hom}_C(a, \Sigma^n(b)) &= \mathrm{Hom}_C(a, \Omega(\Sigma^{n+1}(b))) \\ &= \Omega(\mathrm{Hom}_C(a, \Sigma^{n+1}(b)))\,.\end{aligned}$$

The last assertion of the lemma follows from the analogous property for the ordinary Hom functor Hom_C. □

Theorem 19.4.3. *Let $f \colon C \to D$ be an exact functor between locally small stable categories. Then f has a right adjoint $g \colon D \to C$ if and only if there is a functorial isomorphism of spectra*

$$\hom_D(f(x), y) = \hom_C(x, g(y))$$

for any $x : C$ and any $y : D$.

Proof. This is clearly a sufficient condition since one can recover Hom from hom by applying Ω^∞.

For the converse, we may choose to work in a universe for which both C and D are small. The property that g is right adjoint to f is that $g_! = f^*$, where

$$f^* \colon \mathrm{PSh}(D) \to \mathrm{PSh}(C)$$

is the functor defined by precomposition by f^{op} whereas

$$g_!\colon \mathrm{PSh}(D) \to \mathrm{PSh}(C)$$

extends g by small colimits along the Yoneda embedding. By virtue of Theorem 18.2.8, these identifications remain true when we restrict ourselves to ind-objects. We thus get identifications of the form $g_! = f^*$ where this time we consider functors

$$f^*, g_!\colon \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Sp})\,.$$

Since we have a canonical commutative square of the form

$$\begin{array}{ccccc} D & \hookrightarrow & \mathrm{Ind}(D) & \hookrightarrow & \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Sp}) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_!} & & \downarrow{\scriptstyle g_!} \\ C & \hookrightarrow & \mathrm{Ind}(C) & \hookrightarrow & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Sp}) \end{array}$$

we must have

$$g_!(\hom_D(-, y)) = \hom_C(-, g(y))$$

for all $y : D$. Since

$$f^*(\hom_D(-, y)) = \hom_D(f(-), y)$$

holds by definition, this proves the theorem. □

Proposition 19.4.4. *Let C be a locally small stable category with small colimits. Given an object x of C, the following conditions are equivalent.*

(i) The object x is compact in C.

(ii) The functor $\hom_C(x, -)\colon C \to \mathrm{Sp}$ commutes with filtered colimits.

(iii) The functor $\hom_C(x, -)\colon C \to \mathrm{Sp}$ commutes with small colimits.

Proof. Corollary 18.3.11 implies that conditions (ii) and (iii) are equivalent. We have an adjunction

$$\Sigma^\infty((-)_+)\colon \mathrm{An} \rightleftarrows \mathrm{Sp}\colon \Omega^\infty$$

with the left adjoint preserving compact objects, hence the right adjoint Ω^∞ commutes with small filtered colimits. We know from Proposition 19.2.12 that the collection of functors $\Omega^\infty \circ \Sigma^n$ indexed by $n : \mathbb{N}$ is jointly conservative. We observe that

$$\Omega^\infty(\Sigma^n(\hom_C(x, y))) = \mathrm{Hom}_C(x, \Sigma^n(y))$$

for all y. Therefore, condition (ii) simply means that each functor

$$y \mapsto \mathrm{Hom}_C(x, \Sigma^n(y))$$

commutes with filtered colimits. Since the functors Ω^n commute with all colimits, this is clearly equivalent to the property of compactness for x. □

Corollary 19.4.5. *Let C be a compactly generated stable category and let $C^\simeq$ be the full subcategory of its compact objects. Then $C^\simeq$ is stable and the embedding $C^\simeq \hookrightarrow C$ is exact.*

Proof. We know that $C^{\simeq}$ is stable under finite colimits in C. The preceding proposition implies that it is also stable under the functor Ω since functors of the form $\hom(x, -)$ commute with Ω and that the latter commute with filtered colimits – being an equivalence. This implies this corollary right away. □

Proposition 19.4.6 (Compact generation criterion)**.** *Let C be a locally small stable category with small colimits, and let $\mathcal{G}$ be a small family of compact objects. The following conditions are equivalent:*

(1) The smallest full stable subcategory of C that contains $\mathcal{G}$ and is closed under small colimits is C;

(2) The functors

$$\mathrm{Hom}_C(\Sigma^n G, -)\colon C \to \mathrm{An}, \qquad G \in \mathcal{G},\ n \in \mathbb{Z},$$

are jointly conservative;

(3) The functors

$$\mathrm{hom}_C(\Sigma^n G, -)\colon C \to \mathrm{An}, \qquad G \in \mathcal{G},$$

(4) Every object of C is a filtered colimit of objects in the smallest full subcategory of C containing $\mathcal{G}$, closed under shifts and under finite colimits.

If these conditions hold, then C is compactly generated, and its compact objects are precisely the objects that are retracts of objects in the subcategory described in (3).

Proof. For any objects X and Y of C and any integer n, we have a functorial identification of the form:

$$\Omega^\infty(\Sigma^n(\mathrm{hom}_C(X, Y))) = \mathrm{Hom}_C(X, \Sigma^n(Y))\,.$$

It follows thus from Proposition 19.2.12 that assertions (3) and (4) are equivalent. Write $C^{\mathcal{G}\text{-fin}}$ for the smallest full subcategory containing $\mathcal{G}$ and closed under shifts, finite colimits. This subcategory is small. Its objects are compact, since filtered colimits of animae commute with finite limits, and compactness is preserved under shifts and retracts.

Condition (4) implies (1), because every full stable subcategory closed under small colimits and containing $\mathcal{G}$ contains $C^{\mathcal{G}\text{-fin}}$ and all its filtered colimits. Suppose next that (1) holds. To prove (3), it is enough, by stability, to show that the displayed functors jointly detect zero objects. Let X satisfy

$$\mathrm{hom}_C(G, X) \simeq 0$$

for every $G \in \mathcal{G}$. It follows from Proposition 19.4.4 that each functor $\mathrm{hom}_C(G, -)$, with G in $\mathcal{G}$, commutes with small colimits. Therefore, the full subcategory of objects Y for which $\mathrm{hom}_C(G, X)$ vanishes for all G in $\mathcal{G}$ is stable and closed under small colimits. It contains $\mathcal{G}$, and hence equals C by (1). Taking $Y = X$ shows that $X = 0$.

It remains to prove that (2) implies (4). But condition (2) implies by Remark 18.2.7 and Theorem 18.3.8 that $\mathrm{Ind}(C^{\mathcal{G}\text{-fin}}) = C$. This proves (4), obviously. This also implies the last assertion since the full subcategory of compact objects of C precisely is then the idempotent completion of $C^{\mathcal{G}\text{-fin}}$ by Proposition 17.4.4. □

19.5 ω-Presentable stable categories

Definition 19.5.1. We define $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ as the full subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ whose objects are the stable compactly generated categories.

Remark 19.5.2. If D and E both are compactly generated, and if E is stable, then $\mathrm{Fun}^{\mathrm{L}}(D, E)$ is stable as well.

Proposition 19.5.3. *A compactly generated category D is stable if and only if there is a small stable category C and an equivalence of the form* $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp}) \simeq D$.

Proof. It is clear that $\mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp})$ is stable and compactly generated. Conversely, we can consider a small category C such that

$$\mathrm{Ind}(C) = D\,.$$

We observe that, for any compact object x of D, $\Omega(x)$ is still compact: since Ω is left adjoint to the equivalence Σ, and since Σ commutes with small colimits, it is clear that the functor

$$\mathrm{Hom}_D(\Omega(x), -) = \mathrm{Hom}_D(x, \Sigma(-))$$

preserves small filtered colimits. Let $\bar{C}$ be the smallest full subcategory of D stable under pushouts and containing any object of the form $\Omega^n(h_C(x))$ for $x : C^{\simeq}$ and $n : \mathbb{N}$. It is clear that $\bar{C}$ is small and it follows from Theorem 18.3.8 that

$$\mathrm{Ind}(\bar{C}) = D\,.$$

But then it is clear that $\Sigma\colon \bar{C} \to \bar{C}$ is an equivalence with inverse the restriction to $\bar{C}$ of the loop functor of D. Therefore, $\bar{C}$ is stable. We conclude the proof with Corollary 19.2.8. □

Theorem 19.5.4. *The functor*

$$\mathrm{StabCat} \to \mathrm{StabPr}^{\mathrm{L}}_{\omega}\,, C \mapsto \mathrm{Ind}(C)$$

is left adjoint to the fully faithful functor defined by the rule $D \mapsto D_{\omega}$.

Proof. This follows right away from Theorem 18.6.10 since both functors $\mathrm{Ind}(-)$ and $(-)_{\omega}$ preserve the property of being stable. □

Corollary 19.5.5. *The category* $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ *is compactly generated.*

Proof. We know that $D \mapsto D_{\omega}$ defines a fully faithful functor from $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ to RexCat that commutes with filtered colimits. Since the full embedding of StabCat into RexCat also commutes with small filtered colimits, this implies that both full embeddings

$$\mathrm{StabPr}^{\mathrm{L}}_{\omega} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}_{\omega} \quad \text{and} \quad \mathrm{StabPr}^{\mathrm{L}}_{\omega} \hookrightarrow \mathrm{StabCat}$$

commute with small filtered colimits. Proposition 18.3.12 completes the proof. □

19.6 Verdier quotients

Proposition 19.6.1. *Let C be a stable category and W a family of maps in C with the* 2*-out-of-*3 *property and stable under pullbacks. We also assume that one of the following conditions is verified:*

(i) W is also stable under pushouts;

(ii) W is stable under the suspension functor $\Sigma\colon C \to C$.

The localization $L(C)$ of C by W is then stable and the localization functor $\gamma\colon C \to L(C)$ is exact.

Proof. We know that $L(C)$ is pointed left exact and that the localization functor

$$\gamma\colon C \to L(C)$$

is left exact by Theorem 17.1.5.

Under condition (i), we can apply dually Corollary 17.1.8 and see that $L(C)$ has pushouts and that γ is right exact (Proposition 14.1.1 and its dual version ensure that $L(C)$ has a terminal object that is also initial and that γ preserves those). The proof of Theorem 17.1.5 (iv) tells us that the collection of pullback squares of C covers the collection of pullback squares of $L(C)$. Since all pullback squares of C also are pushout squares, and since γ is exact, this implies that all pullback squares of $L(C)$ are pushouts. Applying the previous argument to C^{op} shows that all pushout squares of $L(C)$ are pullback squares as well.

Under condition (ii), Lemma 14.1.5 tells us that the adjunction equivalence

$$\Sigma\colon C \rightleftarrows C\colon \Omega$$

induces an adjunction equivalence after localization, which implies right away that $\Omega\colon L(C) \to L(C)$ is an equivalence of categories. Therefore, $L(C)^{\mathrm{op}}$ is stable, and thus so is $L(C)$ itself. □

Definition 19.6.2. Let $i\colon A \to B$ be an exact full embedding between stable categories. The *Verdier quotient* of B by A is the stable category B/A defined by forming the pushout square

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & B/A \end{array}$$

in the category StabCat of stable categories (that are small for a big enough universe), where 0 denotes the terminal stable category.

Remark 19.6.3. In the commutative square defining B/A above, the commutativity constraint expressing that $p \circ i$ is the constant functor taking the value 0 in B/A is unique: the 0 functor is a terminal object in the category of exact functors from A to B/A.

Lemma 19.6.4. *In a stable category C, for any morphism $u\colon x \to y$ in C, the cofiber of u is canonically isomorphic to the suspension of its fiber.*

Proof. This follows right away from the fact that in the commutative diagram

$$\begin{array}{ccccc} \mathrm{fib}(u) & \longrightarrow & x & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle u} & & \downarrow \\ 0 & \longrightarrow & y & \longrightarrow & \mathrm{cofib}(u) \end{array}$$

both squares are pullbacks as well as pushouts. □

Lemma 19.6.5. *Let C be a stable category and*

$$\begin{array}{ccc} x' & \xrightarrow{f} & x \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ y' & \xrightarrow{g} & y \end{array}$$

a pullback (hence also a pushout) square in C. Then u is an isomorphism if and only if u' is an isomorphism.

Proof. In a pullback square as in the lemma, if u is invertible, so is u'. The dual statement for pushouts implies the converse. □

Lemma 19.6.6. *Let $u\colon x \to y$ be a morphism in a stable category C. The following assertions are equivalent.*

(i) The morphism u is an isomorphism.

(ii) The fiber of u is isomorphic to 0.

(iii) The cofiber of u is isomorphic to 0.

Proof. It follows immediately from Lemma 19.6.4 that conditions (ii) and (iii) are equivalent. We deduce right away from Lemma 19.6.5 that conditions (i) and (ii) are equivalent. □

Proposition 19.6.7. *Let $i\colon A \to B$ be a fully faithful exact functor between stable categories. The family W of morphisms of B with fibers in A agrees with the family of morphisms of B with cofibers in A. Moreover W is stable under pullbacks as well as under pushouts, and the resulting localization of B by W is nothing else than the Verdier quotient B/A.*

Proof. The first assertion follows right away from Lemma 19.6.4. The stability of W under pullbacks is obvious, since fibers remain constant through pullbacks. Stability under pushouts follows by a dual argument. For composable morphisms f and g, the cofiber sequence $\mathrm{cofib}(f) \to \mathrm{cofib}(gf) \to \mathrm{cofib}(g)$ shows that W has the 2-out-of-3 property, since A is a full stable subcategory. Thus Proposition 19.6.1 shows that $L(B)$ is stable and the localization functor $\gamma\colon B \to L(B)$ is exact. By Corollary 17.3.7, precomposition with γ therefore identifies exact functors $L(B) \to C$ with exact functors $B \to C$ inverting W, for every stable category C.

Given an exact functor $\varphi\colon B \to C$, using Lemma 19.6.6, we see that the property for φ to send any map in W to an isomorphism in C is equivalent to the property of sending any object of A to 0 in C. Consequently, there is a pullback square of the form

$$\begin{array}{ccc} \mathrm{RexCat}^W(B,C) & \longrightarrow & \mathrm{RexCat}(B,C) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ 0 & \longrightarrow & \mathrm{RexCat}(A,C) \end{array}$$

which identifies $L(B)$ with the pushout defining B/A. □

Corollary 19.6.8. *Let $i\colon A \to B$ be an exact full embedding between small stable categories with corresponding Verdier quotient $p\colon B \to B/A$. The induced functor*

$$p^*\colon \mathrm{Lex}((B/A)^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Sp})$$

is fully faithful. Given an exact functor of the form

$$F\colon B^{\mathrm{op}} \to \mathrm{Sp}\,,$$

the following assertions are equivalent.

(i) The functor F belongs to the essential image of p^*.

(ii) The functor $i^*(F) = F \circ i^{\mathrm{op}}$ is constant with value 0.

(iii) We have $F(i(a)) = 0$ in Sp for any object a of A.

(iv) We have $\Omega^\infty(F(i(a))) \simeq *$ in $\mathrm{An}_\bullet$ for any object a of A.

Proof. The fact that the functor p^* is fully faithful with essential image described by condition (ii) is a direct consequence of the description of the universal property exhibiting B/A as a localization provided by Proposition 19.6.7, using Remark 18.5.6 and the left exact version of Corollary 17.3.7. Conditions (ii) and (iii) obviously are equivalent. Since

$$\Omega^\infty\colon \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Lex}(C^{\mathrm{op}}, \mathrm{An}_\bullet)$$

is an equivalence for any small stable category C, all of the above may be repeated replacing Sp by $\mathrm{An}_\bullet$ and this implies that conditions (iii) and (iv) are equivalent. □

Remark 19.6.9. Let $i\colon A \to B$ be an exact fully faithful functor between small stable categories, with Verdier quotient. Let W be the family of maps with fibers in A.

Using the Yoneda embedding

$$B \hookrightarrow \mathrm{Ind}(B) = \mathrm{Lex}(B^{\mathrm{op}}, \mathrm{Sp})$$

we may see W as a small family of maps in $\mathrm{Ind}(B)$. The W-local objects are thus those presheaves satisfying condition (iii) of Corollary 19.6.8. More precisely, if $L_W\,\mathrm{Ind}(B)$ denotes the full subcategory of W-local objects in $\mathrm{Ind}(B)$, the functor p^* induces a fully faithful and essentially surjective functor from $\mathrm{Ind}(B/A)$ to $L_W\,\mathrm{Ind}(B)$. Since p^* corresponds to the full embedding of $L_W\,\mathrm{Ind}(B)$ to $\mathrm{Ind}(B)$, its left adjoint

$$p_!\colon\ \mathrm{Ind}(B/A) \to \mathrm{Ind}(B)$$

correspond to the W-localization functor

$$loc_W \colon L_W \operatorname{Ind}(B) \to \operatorname{Ind}(B)\,.$$

One can then use Proposition 18.4.2 to identify the localization of B by W with the smallest full subcategory of $\operatorname{Ind}(B/A)$ stable under pushouts and containing objects of the form $p(b)$ for $b : B^{\simeq}$. Let $\gamma \colon B \to L(B)$ be this localization.

We also get a Verdier quotient of the form

$$\begin{array}{ccc} \operatorname{Ind}(B/A) & \xrightarrow{p^*} & \operatorname{Ind}(B) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ 0 & \longrightarrow & \operatorname{Ind}(A) \end{array}$$

since i^* is a localization (having a fully faithful left adjoint $i_!$) and that the objects of $\operatorname{Ind}(B)$ that are sent to 0 through i^* precisely are the objects in the essential image of p^* by Corollary 19.6.8, so that we can apply Proposition 19.6.7 (for large stable categories, i.e. seeing $\operatorname{Ind}(B)$ as a stable category in larger universe).

Definition 19.6.10. A *six gluing functors datum* is a diagram of stable categories of the form

$$U \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{j^*}}}} X \underset{i_*}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{}}}} Z$$

in which all functors are exact, with the following properties.

(i) We have $i^* \circ j_! = 0$.

(ii) Both functors $j_!$ and i_* are fully faithful.

(iii) The functor j^* is right adjoint to $j_!$ and left adjoint to j_*.

(iv) The functor i^* is left adjoint to i_*.

(v) The pair (j^*, i^*) is jointly conservative: for any object $x : X^{\simeq}$, we have $x = 0$ if and only if $j^*(x) = 0$ and $i^*(x) = 0$.

Remark 19.6.11. If we have a six gluing functors datum as above, then, for any object x of X the following square induced by the co-unit of the adjunction $(j_!, j^*)$ and by the unit of the adjunction (i^*, i_*)

$$\begin{array}{ccc} j_!(j^*(x)) & \longrightarrow & x \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_*(i^*(x)) \end{array}$$

is a cofiber sequence in X (there is a unique way to make this square commutative by condition (i)). Indeed, it suffices to prove that the induced map from the cofiber of the co-unit map $j_!(j^*(x)) \to x$

to $i_*(i^*(x))$ is an isomorphism. By condition (v) of the definition, this means that it suffices to check that the two commutative squares

$$\begin{array}{ccc} j^*(j_!(j^*(x))) & \longrightarrow & j^*(x) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & j^*(i_*(i^*(x))) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} i^*(j_!(j^*(x))) & \longrightarrow & i^*(x) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i^*(i_*(i^*(x))) \end{array}$$

are cofiber sequences. But they are isomorphic to

$$\begin{array}{ccc} j^*(x) & = & j^*(x) \\ \downarrow & & \downarrow \\ 0 & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 0 & \longrightarrow & i^*(x) \\ \| & & \| \\ 0 & \longrightarrow & i^*(x) \end{array}$$

respectively, which are obviously cocartesian squares.

We define then a functor

$$i^!\colon X \to Z$$

by assigning to each $x : X$ a fiber sequence of the form

$$\begin{array}{ccc} i^!(x) & \longrightarrow & i^*(x) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i^*(j_*(j^*(x))) \end{array}$$

where the vertical map on the right side is the image by i^* of the unit map of the adjunction (j^*, j_*).

We can define $I(x)$ through the pullback square

$$\begin{array}{ccc} I(x) & \longrightarrow & x \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & j_*(j^*(x)) \end{array}$$

and observe that since j^* sends the vertical map on the right side to an isomorphism, we have $j^*(I(x)) = 0$. In other words the cofiber sequence

$$\begin{array}{ccc} j_!(j^*(I(x))) & \longrightarrow & I(x) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_*(i^*(I(x))) \end{array}$$

shows that the unit map $I(x) \to i_*(i^*(I(x)))$ is an isomorphism. Since $i^*(I(x)) = i^!(x)$ essentially by definition, we thus have

$$I(x) = i_*(i^!(x))$$

and this yields a canonical fiber sequence of the form below.

$$\begin{array}{ccc} i_*(i^!(x)) & \longrightarrow & x \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & j_*(j^*(x)) \end{array}$$

Remark that since i_* is fully faithful, $i^* \circ i_*$ is the identity and thus the fiber sequence above may be taken as a definition of $i^!(x)$ as well. The following lemma explains why we speak of six gluing functors.

Lemma 19.6.12. *With a six gluing functors datum as above, the functor $i^!\colon X \to Z$ introduced in the previous remark is right adjoint to i_*.*

Proof. Given x in X and z in Z, since i_* is fully faithful, it suffices to prove that

$$\mathrm{Hom}(i_*(z), i_*i^!(x)) = \mathrm{Hom}(i_*(z), x)$$

This will follow from an identification of the form

$$\mathrm{hom}(i_*(z), i_*i^!(x)) = \mathrm{hom}(i_*(z), x)$$

by applying Ω^∞. By construction of $i^!$, for any a and x in X, we have a pullback square of the form below.

$$\begin{array}{ccc} \mathrm{hom}(a, i_*i^!(x)) & \longrightarrow & \mathrm{hom}(a, x) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{hom}(a, j_*j^*(x)) \end{array}$$

We want to prove that the upper horizontal map is invertible whenever a is of the form $i_*(z)$. By virtue of Lemma 19.6.6 it suffices to prove that

$$\mathrm{hom}(i_*(z), j_*(u)) = 0$$

for all z in Z and all u in U. But j^* is left adjoint to j_* so that this amounts to proving the relation $j^* \circ i_* = 0$. This follows from the relation $i^* \circ j_! = 0$ and from Theorem 19.4.3 by transposition:

$$\mathrm{hom}(a, j^*\, i_*(b)) = \mathrm{hom}(i^*\, j_!(a), b)\,. \qquad \square$$

Proposition 19.6.13. *We consider a six gluing functors datum of the form below, with X, U and Z small.*

$$U \underset{j_*}{\overset{j_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{j^*}}}} X \underset{i^!}{\overset{i^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{i_*}}}} Z$$

Then all commutative squares below are both pullbacks and pushouts in StabCat.

$$\begin{array}{ccc} U & \xrightarrow{j_!} & X \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ 0 & \longrightarrow & Z \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{i_*} & X \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ 0 & \longrightarrow & U \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{j_*} & X \\ \downarrow & & \downarrow{\scriptstyle i^!} \\ 0 & \longrightarrow & Z \end{array}$$

Proof. We first prove that each square is a pullback square of stable categories. Since all functors $j_!$, j_* and i_* are fully faithful, it suffices to check this on objects. For the first two squares, this follows right away from the pushout squares of the form

$$\begin{array}{ccc} j_!(j^*(x)) & \longrightarrow & x \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_*(i^*(x)) \end{array}$$

for each $x : X^{\simeq}$. For the third one, this follows from the pullback squares of the form

$$\begin{array}{ccc} i_*(i^!(x)) & \longrightarrow & x \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & j_*(j^*(x)) \end{array}$$

for each $x : X^{\simeq}$.

Finally, each functor with a fully faithful left or right adjoint is a localization. In the case of exact functors between stable categories, an exact localization is always obtained by inverting the morphisms whose fiber becomes zero in the localization. Therefore, the description of Verdier quotients as localizations provided by Proposition 19.6.7 ends the proof. □

Construction 19.6.14. Given a small stable category C, we will write $C' = \mathrm{Ind}(C)_\omega$ for its idempotent completion. Each exact functor $u\colon C \to D$ between small stable categories induces an exact functor

$$u_!\colon \mathrm{Ind}(C) \to \mathrm{Ind}(D)$$

that preserves compact objects and thus restricts to an exact functor after idempotent completions:

$$u'\colon C' \to D'\,.$$

The naturality of the Yoneda embedding yields a commutative square of the form

$$\begin{array}{ccc} C & \longrightarrow & D \\ \downarrow & & \downarrow \\ C' & \xrightarrow{u'} & D' \end{array}$$

in which the two vertical maps are fully faithful exact functors.

Theorem 19.6.15 (Thomason)**.** *Let $i\colon A \to B$ be a fully faithful exact functor between small stable categories, and let B/A be the associated Verdier quotient.*

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & B/A \end{array}$$

By passing to idempotent completions, we obtain a cartesian square in StabCat *of the form*

$$\begin{array}{ccc} A' & \xhookrightarrow{i'} & B' \\ \downarrow & & \downarrow{\scriptstyle p'} \\ 0 & \longrightarrow & (B/A)' \end{array}$$

with i' fully faithful, and the induced functor

$$B'/A' \to (B/A)'$$

is an exact fully faithful functor that induces an equivalence after idempotent completion

$$(B'/A')' \simeq (B/A)'\,.$$

Moreover, the induced commutative square of (large) stable categories

$$\begin{array}{ccc} \mathrm{Ind}(A) & \xhookrightarrow{i_!} & \mathrm{Ind}(B) \\ \downarrow & & \downarrow{\scriptstyle p_!} \\ 0 & \longrightarrow & \mathrm{Ind}(B/A) \end{array}$$

has the property that $i_!$ is fully faithful and that $p_!$ induces an equivalence

$$\mathrm{Ind}(B)/\mathrm{Ind}(A) \simeq \mathrm{Ind}(B/A)\,.$$

Proof. Since i is fully faithful, the induced functor

$$i_!\colon \mathrm{PSh}(A) \to \mathrm{PSh}(B)$$

is fully faithful, hence its restriction

$$i_!\colon \mathrm{Ind}(A) \to \mathrm{Ind}(B)$$

is fully faithful as well. Restricting further to compact objects thus yields an exact fully faithful functor

$$i'\colon A' \to B'\,.$$

Since $\mathrm{Ind}(C) = \mathrm{Ind}(C')$ for any small stable category C, Remark 19.6.9 implies right away that the comparison functor

$$B/A \to B'/A'$$

is fully faithful and induces an equivalence

$$\mathrm{Ind}(B/A) = \mathrm{Ind}(B'/A')\,.$$

Passing to compact objects, this proves that B/A and B'/A' have the same idempotent completion.

Let us consider finally the commutative square in $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ given by

$$\begin{array}{ccc} \mathrm{Ind}(A) & \xhookrightarrow{i_!} & \mathrm{Ind}(B) \\ \downarrow & & \downarrow{\scriptstyle p_!} \\ 0 & \longrightarrow & \mathrm{Ind}(B/A) \end{array}$$

In order to prove that this is both a pullback square and a pushout square of (large) stable categories, we will apply Proposition 19.6.13. In other words, it suffices to check that we have a six gluing functors datum of the form

$$\mathrm{Ind}(A) \underset{i_*}{\overset{i_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{i^*}}}} \mathrm{Ind}(B) \underset{p_*}{\overset{p_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{p^*}}}} \mathrm{Ind}(B/A)$$

The existence of adjoint functors follows from Theorem 18.2.8 and Proposition 19.3.5. The only non-trivial property to be checked is the fact that i^* and $p_!$ form a jointly conservative family of functors. In other words, we have to check that, for an object X of $\mathrm{Ind}(B)$ with $i^*(X) = 0$ and $p_!(X) = 0$ we must have $X = 0$. But by Corollary 19.6.8, the condition that $i^*(X) = 0$ is equivalent to the condition that X is in the essential image of p^*. If $X = p^*(Y)$ for some Y, since p^* is fully faithful, we then have $p_!(X) = Y$. Therefore, if $p_!(X) = 0$ we must have $Y = 0$ hence $X = 0$ as well. Since $(-)_\omega$ preserves pullback squares (being a right adjoint), this proves the theorem. □

Definition 19.6.16. Let B be a stable category. A *thick subcategory* of B is a fully faithful exact functor $i\colon A \to B$ with the property that, for any projector of the form $p\colon x \to x$ in A, if $i(p)$ has an image in B, then p has an image in A. We will then say by a slight abuse that A is a thick subcategory of B.

Proposition 19.6.17. *Let $i\colon A \to B$ be a fully faithful exact functor between stable categories. We denote by $\gamma\colon B \to B/A$ the associated Verdier quotient. The following conditions are equivalent.*

(i) The commutative square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & B/A \end{array}$$

is a pullback square.

(ii) A is a thick subcategory of B.

Proof. If (i) holds and $p\colon x \to x$ is a projector in A that has an image y in B, then $\gamma(y) = 0$ because $\gamma(y)$ is the image of a projector on $\gamma(i(p)) = 0$. Therefore, y is in the essential image of i. Conversely, if we denote by C' the idempotent completion of any stable category C, we know from the previous theorem that we have a pullback square of the form

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B'/A' \end{array}$$

Since $B/A \to B'/A'$ is fully faithful, this readily implies that the full subcategory of B whose objects are sent to 0 in B/A is precisely the intersection of A' with B. The explicit description of idempotent completions obtained from Proposition 17.4.4 shows that condition (ii) follows from condition (i). □

20 K-Theory, after Quillen, Waldhausen and Thomason

20.1 Completion by small sifted colimits

Notation 20.1.1. We let FinSet be the full subcategory of An spanned by finite sets.

Given a small category with finite coproducts C we denote by $P_\Sigma(C)$ the full subcategory of $\mathrm{PSh}(C)$ spanned by finite products preserving functors. In other words, $P_\Sigma(C) = \mathrm{Ind}_\kappa(C)$ for κ the localizing subuniverse of finite sets. In this particular case, κ-filtered categories correspond to sifted categories.

Proposition 20.1.2. *Let C be a small category with finite coproducts. The category $P_\Sigma(C)$ is compactly generated. Furthermore, the embedding*

$$P_\Sigma(C) \hookrightarrow \mathrm{PSh}(C)$$

commutes small sifted colimits.

Proof. It is even a κ-presentable categories by virtue of Remark 17.2.27. □

Remark 20.1.3. If C is small with finite coproducts, the Yoneda embedding of C takes values in $P_\Sigma(C)$ and the induced functor

$$h_C \colon C \hookrightarrow P_\Sigma(C)$$

commutes with finite coproducts. Indeed, for any object x of C, the functor $\mathrm{Hom}_C(-, x)$ takes coproducts in C to products in An.

Lemma 20.1.4. *Any small category with finite coproducts is sifted.*

Proof. Let A be a small category with an initial object $*$ as well as binary coproducts. Since $A = A_{*/}$, A is weakly contractible. We also have

$$A_{(x \amalg y)/} = A_{x/} \times_A A_{y/}$$

for all $x, y : A^{\simeq}$, proving that $A_{x/} \to A$ is a terminal functor for all x. □

Lemma 20.1.5. *Let C be a small category with finite coproducts. Given a presheaf $F \colon C^{\mathrm{op}} \to \mathrm{An}$, the following assertions are equivalent.*

(i) The functor F preserves finite products.

(ii) *The category $C_{/F}$ has finite coproducts.*

(iii) *The category $C_{/F}$ is sifted.*

(iv) *There is a small sifted category A, a functor $\varphi\colon A \to C$ and an isomorphism of type*

$$\operatorname*{colim}_{a:A} h_C(\varphi(a)) = F$$

in $\mathrm{PSh}(C)$.

Proof. This is a particular case of Proposition 17.2.19. □

Proposition 20.1.6. *Let $u\colon C \to D$ be a finite coproducts preserving functor between small categories with finite coproducts. Then the adjunction*

$$u_!\colon \mathrm{PSh}(C) \rightleftarrows \mathrm{PSh}(D)\colon u^*$$

restricts to an adjunction of the form

$$u_!\colon P_\Sigma(C) \rightleftarrows P_\Sigma(D)\colon u^*\,.$$

Proof. This is a particular case of Proposition 17.2.21. □

Remark 20.1.7. The previous proposition is also a way to interpret localizations that are compatible with finite coproducts. Given C a small category with finite coproducts and $W \hookrightarrow \mathrm{Map}([1], C)$ a family of morphisms containing all identities and stable under binary coproducts, let us denote by

$$\gamma\colon C \to L(C)$$

the localization of C by W. By virtue of Proposition 14.1.1 and Proposition 14.1.6, the category $L(C)$ has an initial object as well as binary coproducts and the functor γ commutes with them. The preceding proposition thus provides an adjunction of the form

$$\gamma_!\colon P_\Sigma(C) \rightleftarrows P_\Sigma(L(C))\colon \gamma^*$$

with γ^* full faithful. The essential image of γ^* consists precisely of the functors that preserve finite products and send maps in W to isomorphisms. One ca identify $L(C)$ as the full subcategory of $P_\Sigma(()C)$ with objects the terms of the image of the map

$$C^{\simeq} \to P_\Sigma(C) \quad x \mapsto \gamma_!(\mathrm{Hom}_C(-, x))\,.$$

Proposition 20.1.8. *Let C be a small category with finite coproducts and D a category with small colimits. The equivalence*

$$\mathrm{Fun}_!(\mathrm{PSh}(C), D) = \mathrm{Fun}(C, D)$$

restricts to an equivalence from the category of small colimits preserving functors from $P_\Sigma(()C)$ to D and the category of finite coproducts preserving functors from C to D.

Proof. This is a particular case of Proposition 17.3.5. □

Proposition 20.1.9. *Let D be a locally small category with small colimits. We assume that there is a small family of objects E in D with the following properties.*

(i) For any $x : E$, the functor

$$\mathrm{Hom}_D(x, -) \colon D \to \mathrm{An}$$

commutes with small sifted colimits.

(ii) The family E generates D.

We let C be the smallest full subcategory of D containing the initial object as well as all the terms of E closed under binary coproducts. Then C is small and the full embedding

$$i \colon C \hookrightarrow D$$

extends to an equivalence

$$P_\Sigma(C) = D\,.$$

In particular, any object of D is canonically a small sifted colimit of a functor with values in C.

Proof. This is a particular case of Theorem 17.2.24. □

The preceding definition applies to $D = \mathrm{An}$ with E the collection of terminal objects. Since FinSet is the smallest full subcategory of An containing the empty anima as well as the terminal one and stable under binary coproducts, we get the identification

$$P_\Sigma(\mathrm{FinSet}) = \mathrm{An}\,.$$

20.2 Additive categories

Lemma 20.2.1 (the Eckmann-Hilton argument, le lemme de Godement)**.** *Let A be an anima together with a term $e : A$ as well as two maps*

$$\circ \colon A \times A \to A\,, \quad (a, b) \mapsto a \circ b \quad \textit{and} \quad \bullet \colon A \times A \to A\,, \quad (a, b) \mapsto a \bullet b\,.$$

We assume that the following rules hold.

(i) For every term $a : A$, there are terms of the following types:

$$\begin{aligned} u^l_a &: a = e \circ a \\ u^r_a &: a \circ e = a \\ v^l_a &: a = e \bullet a \\ v^r_a &: a \bullet e = a\,. \end{aligned}$$

(ii) For every term $a, b, c, d : X$, there is a term of type

$$\gamma_{a,b,c,d} : (a \bullet b) \circ (c \bullet d) = (a \circ c) \bullet (b \circ d)\,.$$

Then, for any terms $a, b : A$, there are terms

$$\kappa_{a,b} : a \bullet b = a \circ b \quad \textit{and} \quad \kappa'_{a,b} : a \circ b = b \bullet a\,.$$

Proof. The term $\kappa_{a,b}$ is obtained as follows.

$$\begin{aligned} a \bullet b &= (a \circ e) \bullet (e \circ b) \\ &= (a \bullet e) \circ (e \bullet b) \\ &= a \circ b \,, \end{aligned}$$

Similarly, the sequence

$$\begin{aligned} a \circ b &= (e \bullet a) \circ (b \bullet e) \\ &= (e \circ b) \bullet (a \circ e) \\ &= b \bullet a \end{aligned}$$

yields $\kappa'_{a,b}$. □

Example 20.2.2. Given an anima X and a term $x : X$, we can form

$$\Omega(X, x) := \mathrm{Hom}_X(x, x) \,.$$

there is thus a composition law

$$c_{X,x} \colon \Omega(X, x) \times \Omega(X, x) \to \Omega(X, x) \,.$$

This operation defines a functor

$$\Omega \colon \mathrm{An}_\bullet \to \mathrm{An}_\bullet$$

that commutes with products. We define

$$\bullet \colon \Omega^2(X, x) \times \Omega^2(X, x) = \Omega(\Omega(X, \mathrm{id}_x) \times \Omega(X, \mathrm{id}_x)) \xrightarrow{\Omega(c_{X,x})} \Omega^2(X, x)$$

where $\Omega^2(X, x) := \Omega(\Omega(X, x), \mathrm{id}_x)$. There is also the composition law

$$\circ := c_{\Omega(X,x),\mathrm{id}_x} \colon \Omega^2(X, x) \times \Omega^2(X, x) \to \Omega^2(X, x) \,.$$

One checks immediately that Lemma 20.2.1 applies to the anima $A = \Omega^2(X, x)$ with these definitions of $\circ$ and $\bullet$.

Definition 20.2.3. A category C is *semiadditive* if its has an initial object 0 as well as binary coproducts with the following properties:

(i) The initial object 0 turns out to be a terminal object as well. In particular, for any object x of C, there is a map $p_x \colon x \to 0$.

(ii) For any objects x and y, the maps

$$p_x \amalg \mathrm{id}_y \colon x \amalg y \to x \amalg 0 = x \quad \text{and} \quad \mathrm{id}_x \amalg p_y \colon x \amalg y \to 0 \amalg y = y$$

exhibit $x \amalg y$ as a product of x and y in C.

Construction 20.2.4. Let C be a semiadditive category. For any objects $x, y : C^{\simeq}$, there is an operation

$$\mathrm{Hom}_C(x,y) \times \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_C(x,y)\,, \quad \mapsto (f,g) \mapsto f+g$$

defined by the following composition

$$\mathrm{Hom}_C(x,y) \times \mathrm{Hom}_C(x,y) = \mathrm{Hom}_C(x \amalg x, y) \xrightarrow{\Delta^*} \mathrm{Hom}_C(x,y)$$

with $\Delta\colon x \to x \amalg x$ the map defined by the pair $(\mathrm{id}_X, \mathrm{id}_x)$ using the description of $x \amalg x$ as a product. This construction is functorial in the sense that for any functor between semiadditive categories

$$f\colon C \to D$$

that preserves finite coproducts (hence finite products as well), we have a canonical commutative square of the form

$$\begin{array}{ccc} \mathrm{Hom}_C(x,y) \times \mathrm{Hom}_C(x,y) & \xrightarrow{\quad + \quad} & \mathrm{Hom}_C(x,y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_D(f(x),f(y)) \times \mathrm{Hom}_D(f(x),f(y)) & \xrightarrow{+} & \mathrm{Hom}_D(f(x),f(y)) \end{array}$$

We observe furthermore that $\mathrm{Fun}(I, C)$ is semiadditive for any category I whenever C is pre-additive. This expresses how functorial is this construction in the variables $x, y : C$.

Example 20.2.5. The category RexCat is semiadditive. The initial object is the category $[0]$ and the coproduct is given by the cartesian product of the underlying categories.

Construction 20.2.6 (2×2 matrices)**.** In a semiadditive category C, given for objects x, y, z, w a map

$$\varphi\colon x \amalg y \to z \amalg w$$

amounts to a pair of maps of type $u\colon x \amalg y \to z$ and $v\colon x \amalg y \to w$ (seeing $z \amalg w$ as a product) and seeing $x \amalg y$ as a coproduct, u correspond to a pair of maps of the form

$$a\colon x \to z \quad \text{and} \quad b\colon y \to z$$

whereas v corresponds to a pair of maps of the form

$$c\colon x \to w \quad \text{and} \quad d\colon y \to w\,.$$

In other words, defining a map φ as above is equivalent to defining a 2×2-matrix:

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The map φ is defined in formula by $\varphi(x,y) = (a(x) + b(y), c(x) + d(y))$.

Definition 20.2.7. An *additive* category is a semiadditive category C with the property that, for any object x of C, the matrix

$$\begin{pmatrix} \mathrm{id}_x & \mathrm{id}_x \\ \mathrm{id}_x & 0 \end{pmatrix}$$

(with $0\colon x \to x$ the unique map factorizing through 0) defines an automorphism of $x \amalg x$.

Proposition 20.2.8. *Any stable category is additive.*

Proof. This means that, for any objects x and y in a stable category C, for any maps $f, g : x \to y$ in C, one can reconstruct canonically the pair (f, g) from the pair $(f + g, f)$. For this, it suffices to produce a map $-g : x \to y$ such that $(f + g) + (-g) = f$ holds true. Since we have

$$\mathrm{Hom}_C(x, y) = \Omega^2(\mathrm{Hom}_C(x, \Sigma^2(y)), 0)\,,$$

Lemma 20.2.1 together with Example 20.2.2 tell us that one can see the operation $(f, g) \mapsto f + g$ as the composition law in the anima $\Omega(\mathrm{Hom}_C(x, \Sigma(y)), 0)$. Therefore, inverses for this composition law exist and this proves the proposition. □

Corollary 20.2.9. *Any full subcategory closed under finite (co)products of a stable category is additive.*

20.3 Waldhausen categories

In this chapter, we fix once and for all a universe of small categories Cat.

Definition 20.3.1. A *pre-Waldhausen* category is a small category A equipped with two subcategories

$$wA \hookrightarrow A \quad \text{and} \quad cA \hookrightarrow A$$

with the following properties, where morphisms of A that belong to wA are called *weak equivalences*, whereas those belonging to cA are called *cofibrations*.

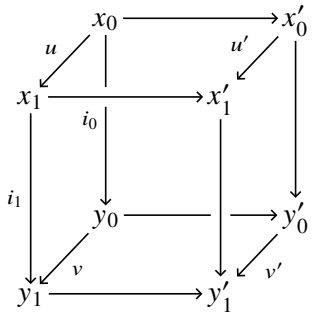
W0. The category A has an initial object $*$. Furthermore, for any object x of A, the map $* \to a$ is a cofibration.

W1. For any object x of A, the identity of a is both a weak equivalence and a cofibration.

W2. The family of cofibrations is stable under pushouts in A.

W3. The family of trivial cofibrations (i.e. of maps that are both weak equivalences and cofibrations) is stable under pushouts.

W4 For any commutative cube in A of the form



with both maps i_0 and i_1 cofibrations and both the front face and the back face cocartesian, if the three maps u, u' and v are weak equivalences, then so is v'.

A Waldhausen category is a pre-Waldhausen such that its initial object happens to be a terminal object as well.

Pre-Waldhausen categories naturally form a subanima of $\mathrm{Fun}(\ulcorner, \mathrm{Cat})^{\simeq}$. A morphism of pre-Waldhausen categories is a morphism in $\mathrm{Fun}(\ulcorner, \mathrm{Cat})$, or equivalently, a functor

$$f : A \to B$$

preserving weak equivalences as well as cofibrations, with the additional property that, for any pushout square in A the form

$$\begin{array}{ccc} x & \xrightarrow{u} & x' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ y & \xrightarrow{v} & y' \end{array}$$

with i (hence also i') a cofibration, the associated commutative square

$$\begin{array}{ccc} f(x) & \xrightarrow{f(u)} & f(x') \\ {\scriptstyle f(i)}\downarrow & & \downarrow{\scriptstyle f(i')} \\ f(y) & \xrightarrow{f(v)} & f(y') \end{array}$$

is a pushout square in B.

Pre-Waldhausen categories naturally form a category $\mathrm{WaldCat}^{\mathrm{pre}}$, or rather a subcategory of $\mathrm{Fun}(\ulcorner, \mathrm{Cat})$.

Lemma 20.3.2. *Let A be a category equipped with a subcategory $cA \hookrightarrow A$ containing all identities and corresponding to a family of morphisms stable under pushouts in A. Define a cA-*cofibrant square *as a commutative square in A of the form*

$$\begin{array}{ccc} x & \xrightarrow{u} & x' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ y & \xrightarrow{v} & y' \end{array}$$

in which i (hence i' as well) is a map in cA and such that the induced map $y \amalg_x x' \to y'$ is a map in cA as well. Then the family of cA-cofibrant squares forms a family of maps in $\mathrm{Fun}([1], A)$ *that is stable under composition as well as under pushouts.*

Proof. Let us consider a composable pair of squares of the form

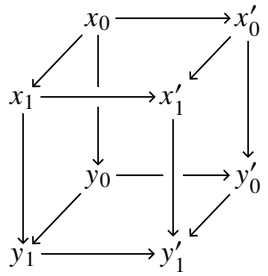
$$\begin{array}{ccccc} x & \xrightarrow{u} & x' & \xrightarrow{u'} & x'' \\ {\scriptstyle i}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i''} \\ y & \xrightarrow{v} & y' & \xrightarrow{v'} & y'' \end{array}$$

each of then being cA-cofibrant. In the induced commutative diagram

$$\begin{array}{ccc} x' & \longrightarrow & x'' \\ \downarrow & & \downarrow \\ y \amalg_x x' & \longrightarrow & y \amalg_x x'' \\ \downarrow & & \downarrow \\ y' & \longrightarrow & y' \amalg_{x'} x'' \end{array}$$

the upper square is cocartesian and so is the outer square, so that the lower square is cocartesian as well. In particular, the vertical map at the lowest right corner is in cA. This implies that the map $y \amalg_x x'' \to y''$ is in cA, since it is the composition of two maps in cA.

Let us consider a commutative cube of the form



in which the back faces is cA-cofibrant and the two vertical faces on the left and right hand side are cocartesian. We observe that the induced commutative square

$$\begin{array}{ccc} y_0 \amalg_{x_0} x'_0 & \longrightarrow & y_1 \amalg_{x_1} x'_1 \\ \downarrow & & \downarrow \\ y'_0 & \longrightarrow & y'_1 \end{array}$$

is cocartesian. This implies right away that the front face is cA-cofibrant. □

Construction 20.3.3. Given a pre-Waldhausen categories A and B, we write $\mathrm{Wald}(A, B)$ for the full subcategory of $\mathrm{Fun}(A, B)$ whose objects are morphisms of pre-Waldhausen categories. In fact $\mathrm{Wald}(A, B)$ is a Waldhausen category itself: one defines weak equivalences as the natural transformations between morphisms of Waldhausen categories

$$\alpha : f \to g$$

such that $\alpha_x : f(x) \to g(x)$ is a weak equivalence for any object $x : A^{\simeq}$. Cofibrations are those natural transformations between morphisms of Waldhausen categories

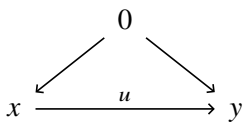
$$\alpha : f \to g$$

with the property that, for any cofibration $i : x \to y$ in A, the induced commutative square

$$\begin{array}{ccc} f(x) & \xrightarrow{\alpha_x} & g(x) \\ {\scriptstyle f(i)}\downarrow & & \downarrow{\scriptstyle g(i)} \\ f(y) & \xrightarrow{\alpha_y} & g(y) \end{array}$$

induces a cofibration of type $f(y) \amalg_{f(x)} g(x) \to g(y)$ (the stability of these cofibrations under pushouts follows straight away from the preceding lemma).

A particular case of interest is $A = [0] \star [1]$ for some $n : \mathbb{N}$. A morphism $[0] \star [1] \to B$ literally is a commutative triangle of the form



with u a cofibration. There is thus an equivalence from $\mathrm{Wald}([0] \star [1], B)$ to the full subcategory of $\mathrm{Fun}([1], B)$ whose objects are the cofibrations of B. More generally, given $n : \mathbb{N}$, we define $B^{[n]}$ as the full subcategory of $\mathrm{Fun}([n], B)$ whose objects are sequences of cofibrations of the form

$$x_0 \to \cdots \to x_n$$

in B, the assignment

$$(x_0 \to \cdots \to x_n) \mapsto (0 \to x_0 \to \cdots \to x_n)$$

defines an equivalence

$$B^{[n]} \simeq \mathrm{Wald}([0] \star [n], B) .$$

Example 20.3.4. Any pointed small category with finite colimits is a Waldhausen category in which the weak equivalences are the isomorphisms and all morphisms are cofibrations. This observation defines a full embedding of the form

$$\mathrm{RexCat}_\bullet \hookrightarrow \mathrm{WaldCat} .$$

Particular cases of interest are stable categories.

Example 20.3.5. If A is a Waldhausen category the so is $\mathrm{Fun}(I, A)$ for any small category I: we define weak equivalences (cofibration, resp.) as those natural transformations $u \colon F \to G$ such that $u_i \colon F_i \to G_i$ is a weak equivalence (a cofibration, resp.) for all objects $i : I^{\simeq}$. We define $w(I, A)$ as the full subcategory of $\mathrm{Fun}(I, A)$ whose objects are the functors sending all maps in I to weak equivalences in A. We define weak equivalences and cofibrations in $w(I, A)$ as those maps that weka equivalences and cofibrations, respectively, in $\mathrm{Fun}(I, A)$. This turns $w(I, A)$ into a Waldhausen category.

Notation 20.3.6. For A : WaldCat and $[n] : \mathbb{N}$ we write

$$w_n(A) := w([n], A) .$$

20.4 Additive functors

Definition 20.4.1. A *short exact sequence* in a Waldhausen category A is a pushout square of the form

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & & \downarrow{\scriptstyle v} \\ * & \longrightarrow & x'' \end{array}$$

in which u is a cofibration. We denote by $\mathrm{ExSeq}(A)$ the full subcategory of $\mathrm{Fun}(\square, A)$ whose objects are short exact sequences. The assignment

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & & \downarrow{\scriptstyle v} \\ * & \longrightarrow & x'' \end{array} \qquad \mapsto (u \colon x' \to x)$$

defines an equivalence

$$\mathrm{ExSeq}(A) \simeq \mathrm{Wald}([0] \star [1], A) .$$

Since $\mathrm{Wald}([0] \star [1], A)$ is a Waldhausen category (as a particular case of Construction 20.3.3), this turns $\mathrm{ExSeq}(A)$ into a Waldhausen category.

The proofs of the following lemma are obvious and they are only stated for the record.

Lemma 20.4.2. *We denote by* 0 *the Waldhausen category with underlying category* $[0]$*, Then* 0 *is both an initial and a terminal object of* WaldCat

Lemma 20.4.3. *If A and B are Waldhausen categories, then the cartesian product of the underlying small categories $A \times B$ is canonically endowed with the structure of a Waldhausen category through*

$$w(A \times B) = wA \times wB \quad \textit{and} \quad c(A \times B) = cA \times cB$$

The projections $\mathrm{pr}_1 \colon A \times B \to A$ *and* $\mathrm{pr}_2 \colon A \times B \to B$ *are morphisms of Waldhausen categories and exhibit $A \times B$ as the cartesian product of A and B in* WaldCat*. Dually, the two functors*

$$i_1 \colon A \to A \times B\,, \quad x \mapsto (x, *) \quad \textit{and} \quad i_2 \colon B \to A \times B\,, \quad y \mapsto (*, y)$$

are morphisms of Waldhausen categories and exhibit $A \times B$ as the coproduct of A and B in WaldCat*.*

In other words:

Proposition 20.4.4. *The category* WaldCat *is semiadditive.*

In particular, any full subcategory of WaldCat stable under finite coproducts is semiadditive as well.

Definition 20.4.5. An *admissible class of Waldhausen categories* is a family of Waldhausen categories stable under finite products as well as under the operators $\mathrm{Wald}([n], -)$ and $w([n], -)$ for all $n : \mathbb{N}$. Given such a class C, we define $\mathrm{WaldCat}_C$ as the full subcategory of WaldCat with objects the terms of C.

Example 20.4.6. Stable categories form an admissible class. In this case, we simply have $\mathrm{StabCat} = \mathrm{WaldCat}_C$. Therefore, any family of stable categories closed under finite products and under the operator $\mathrm{Fun}([n], -)$ is admissible. For instance, compact objects in $\mathrm{StabPr}^{\mathrm{L}}_{\omega}$ for a small admissible class.

Definition 20.4.7. Let C be an admissible class of Waldhausen categories and D a pointed category with finite limits. A functor $E \colon \mathrm{WaldCat}_C \to D$ is *additive* it satisfies the following properties.

(1) It preserves finite products.

(2) For any Waldhausen category A, the functor

$$A \to w([1], A)\,, \quad x \mapsto \mathrm{id}_x$$

induces an equivalence of type

$$E(A) \cong E(w([1], A))\,.$$

(3) For any Waldhausen category A the functor

$$(\ker, \mathrm{coker}) \colon \mathrm{ExSeq}(A) \to A \times A$$

defined by the assignment

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & & \downarrow{\scriptstyle v} \\ * & \longrightarrow & x'' \end{array} \quad \mapsto (x', x'')$$

induces an equivalence of type

$$E(\mathrm{ExSeq}(A)) \cong E(A) \times E(A)\,.$$

if it preserves finite products and if, for any Waldhausen category A the functor

$$(\ker, \mathrm{coker}) \colon \mathrm{ExSeq}(A) \to A \times A$$

defined by the assignment

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ \downarrow & & \downarrow{\scriptstyle v} \\ * & \longrightarrow & x'' \end{array} \quad \mapsto (x', x'')$$

induces an equivalence of type

$$E(\mathrm{ExSeq}(A)) \cong E(A) \times E(A)\,.$$

Proposition 20.4.8. *The property of being additive is preserved by small sifted colimits in the category of functors from* $\mathrm{WaldCat}_C$ *to* $\mathrm{An}_\bullet$.

Proof. Since colimits in $\mathrm{Fun}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$ are computed pointwise and since the property of being additive is expressed by the invertibility of functorially given maps between finite products of evaluations, this follows right away from the fact that small sifted colimits commute with finite products in $\mathrm{An}_\bullet$. □

Lemma 20.4.9. *Let C be an admissible class of Waldhausen categories. For any finite products preserving functor*

$$F \colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet\,,$$

if F is additive then the functor

$$A \mapsto \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} F(S_n(A))$$

is additive as well.

Proof. This follows immediately from Proposition 20.4.8, once we have noticed that Δ^{op} if sifted, and that the operators S_n and w_m commute with finite products as well as with the operator ExSeq. □

Proposition 20.4.10. *Let $F \colon \mathrm{WaldCat}_C \to D$ be an additive functor. We consider two morphisms*

$$p \colon A \to A' \quad \textit{and} \quad q \colon A \to A''$$

in $\mathrm{WaldCat}_C$ *together with the following extra data.*

(i) There are given maps $i\colon A' \to A$ and $j\colon A'' \to A$ such that i is a section of p and j a section of q.

(ii) Each composition $p \circ j$ and $q \circ i$ vanishes (i.e. is isomorphic to the constant functor with value the terminal object).

(iii) There is given a map

$$\varphi\colon A \to \mathrm{ExSeq}(A)$$

so that $\varphi(x)$ is an exact sequence of the form

$$\begin{array}{ccc} i(p(x)) & \longrightarrow & x \\ \downarrow & & \downarrow \\ * & \longrightarrow & j(q(x)) \end{array}$$

for all $x : A$. Then p and q together induce an equivalence of type

$$F(A) = F(A') \times F(A'')\,.$$

Proof. Since p and q commute with finite coproducts, condition (ii) ensures that the functor

$$u\colon A' \times A'' \to A\,, \quad (x, y) \mapsto i(x) \amalg j(y)$$

defines a section of the functor

$$(p, q)\colon A \to A' \times A''\,.$$

It suffices to check that the composition of $u \circ (p, q)$ becomes the identity of $F(A)$ after applying F. We define a functor

$$\psi\colon A \to \mathrm{ExSeq}(A)$$

by the rule below.

$$\psi(x) = \begin{array}{ccc} i(p(x)) & \longrightarrow & i(p(x)) \amalg j(q(x)) \\ \downarrow & & \downarrow \\ * & \longrightarrow & j(q(x)) \end{array}$$

Since postcomposing with (ker, coker) turns φ and ψ equal, we have an equality of type $F(\varphi) = F(\psi)$. Let $p\colon \mathrm{ExSeq}(A) \to A$ be the functor defined by the rule

$$\begin{array}{ccc} x' & \longrightarrow & x \\ \downarrow & & \downarrow \\ * & \longrightarrow & x'' \end{array} \mapsto x\,.$$

Since $p \circ \varphi = \mathrm{id}_A$ and $p \circ \psi = u \circ (p, q)$, this shows that $F(p, q)$ is invertible. □

Corollary 20.4.11. *We consider an additive functor of the form*

$$F\colon \mathrm{WaldCat}_C\colon \to D\,.$$

Let $n : \mathbb{N}$ and $i_n \colon [n] \hookrightarrow [n+1]$ the map defined by $i_n(k) = k+1$. Then, for any Waldhausen category A, the induced functor

$$i_n^* \colon F(S_{n+1}(A)) \to F(S_n(A))$$

is equivalent over $F(S_n(A))$ to the projection

$$\mathrm{pr}_1 \colon F(S_n(A)) \times F(S_1(A)) \to F(S_n(A))\,.$$

Proof. The case $n = 0$ is clear so that we may assume $n > 0$. If we identify $S_n(A)$ with $A^{[n-1]}$, the functor i_n^* simply restricts from $[n]$ to $[n-1]$ in the following way: to each sequence of cofibrations

$$x_1 \to x_2 \to \cdots \to x_{n+1}$$

we form the sequence of cofibrations

$$x_2/x_1 \to \cdots \to x_{n+1}/x_1$$

where we have formed the pushout squares

$$\begin{array}{ccc} x_1 & \longrightarrow & x_k \\ \downarrow & & \downarrow \\ * & \longrightarrow & x_k/x_1 \end{array}$$

in A for all $k \geq 1$. There is also a functor from $A^{[n]}$ to A sending a sequence of shape

$$x_1 \to x_2 \to \cdots \to x_{n+1}$$

to x_1. The corollary will follow from the fact that the induced functor

$$A^{[n]} \to A^{[n-1]} \times A$$

is sent to an equivalence through any additive functor F.

There is an obvious functor

$$A^{[n-1]} \to A^{[n]}$$

sending a sequence

$$y_1 \to \cdots \to y_n$$

to the sequence

$$0 \to y_1 \to \cdots \to y_n$$

and a functor

$$A \to A^{[n]}$$

sending a term $z : A$ to the sequence

$$z \xrightarrow{\mathrm{id}_z} \cdots \xrightarrow{\mathrm{id}_z} z\,.$$

One checks immediately that the assumptions of Proposition 20.4.10 are fulfilled and this implies this corollary. □

Lemma 20.4.12. *Let X be a simplicial anima. Then X_0 is canonically isomorphic to the colimit of the composition*

$$\Delta^{\mathrm{op}} \xrightarrow{dec^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X} \mathrm{An}$$

where $dec\colon \Delta \to \Delta$ is the functor $[n] \mapsto [0] \star [n]$.

Proof. We form the following pullback square.

$$\begin{array}{ccc} C & \longrightarrow & \Delta_{/X} \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{dec} & \Delta \end{array}$$

The colimit we want to compute is $\Pi_\infty(C)$. Sending a pair $([n], x)$, with $x\colon [n+1] \to X$, to x_0 yields a functor

$$\pi\colon C \to X_0$$

(the evaluation at the first vertex of $[n+1]$). We will prove that this turns C into a contractible category in context X_0. There is indeed a section

$$s\colon X_0 \to C$$

of the functor π: sending any term $x : X_0$ to the pair $([0], \mathrm{id}_x)$ where $\mathrm{id}_x\colon [1] \to X$ is the composition of $x\colon [0] \to X$ with the unique map $[1] \to [0]$. There is a functorial map

$$r_n\colon [n+2] \to [n+1]$$

induced by $[1] \to [0]$ and the presentations $[n+2] = [1] \star [n]$ and $[n+1] = [0] \star [n]$. There is similarly a map $i_n\colon [n+1] \to [n+2]$ induced by the map $1\colon [0] \to [1]$. We associate to each pair $([n], x)$ in C the pair $([n+1], x \circ r_n)$, which defines a functor

$$d\colon C \to C$$

over X_0. The embeddings i_n define a natural transformation $\mathrm{id}_C \to d$ and the embeddings $[1] \hookrightarrow [1] \star [n]$ a natural transformation from s to d, all of this in context X_0. This proves that C is weakly contractible in context X_0. □

Proposition 20.4.13. *Let C be an admissible class of Waldhausen categories. We consider a finite products preserving functor*

$$F\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet\,.$$

There is a canonical commutative square

$$\begin{array}{ccc} F(A) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} F(S_n(A)) \end{array}$$

functorially in A and F, that is cartesian whenever F is additive.

Proof. We observe that $F(S_0(A))$ is contractible for all A and thus, by virtue of Lemma 20.4.12 there is a canonical equivalence of the form

$$\operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} F(S_{n+1}(A)) = *$$

The functorial embedding $i_n\colon [n] \hookrightarrow [n+1]$ defined through the identification of $[n+1]$ with $[0] \star [n]$ defines a functor

$$i_n^*\colon S_{n+1}(A) \to S_n(A)$$

functorially in $[n]$ and in A. We obtain a canonical commutative square of the form

$$\begin{array}{ccc} F(A) & \longrightarrow & \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} F(S_{n+1}(A)) \\ \downarrow & & \downarrow \\ F(S_0(A)) & \longrightarrow & \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} F(S_n(A)) \end{array}$$

where the vertical maps are induced by the functors i_n^* above and the horizontal maps are instances of the canonical map

$$t_0 \to \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} t_n$$

for any functor $t\colon \Delta^{\mathrm{op}} \to \mathrm{An}_\bullet$.

It suffices now to prove that the commutative square above is cartesian whenever F is additive. By virtue of Lemma 13.4.4, it suffices to prove that, for any additive functor $F\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet$, and any map $u\colon [m] \to [n]$ in Δ, the functor F sends the commutative square of Waldhausen categories

$$\begin{array}{ccc} S_{n+1}(A) & \longrightarrow & S_{m+1}(A) \\ {\scriptstyle i_n^*}\downarrow & & \downarrow{\scriptstyle i_m^*} \\ S_n(A) & \longrightarrow & S_m(A) \end{array}$$

to a pullback square. We conclude this proof by observing thanks to Corollary 20.4.11 that this commutative square is equivalent to the commutative square

$$\begin{array}{ccc} F(S_n(A)) \times F(S_1(A)) & \xrightarrow{\mathrm{id} \times u^*} & F(S_m(A)) \times F(S_1(A)) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ F(S_n(A)) & \xrightarrow{u^*} & F(S_m(A)) \end{array}$$

which is obviously cartesian. □

Corollary 20.4.14. *For any admissible class of Waldhausen categories, the category of additive functors from* $\mathrm{WaldCat}_C$ *to* $\mathrm{An}_\bullet$ *is additive.*

Proof. Since any object of $\mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$ is of the form $\Omega^2(F)$ by the preceding proposition, we can repeat the proof of Corollary 20.2.9. □

20.5 Construction of K-theory

Definition 20.5.1. A *region* is a finite poset R with pushouts.

If R is a region, we can define the *length* of a map $x \leq y$ in R: this is the maximal $n : \mathbb{N}$ such that there exists an embedding $u\colon [n] \hookrightarrow R$ with $u(0) = x$ and $u(n) = y$. Indeed, the decidable order on R gives decidable equality. After base change to a cardinal presentation of $R^{\simeq}$, any such embedding satisfies $n + 1 \leq |R^{\simeq}|$. The possible values of n and the corresponding embeddings therefore form a finite decidable collection, so this maximum exists. Its uniqueness ensures that it descends from the cover of cardinal presentations. An *elementary square* in R is a pushout square in R in which all maps are of length 1. An *edge map* in R is a map of length at least 2 that does not appear as the diagonal of an elementary square. An *edge object* of R is an object that is the source or the target of an edge map.

A morphism of regions is a functor (between regions) of the form $f : R \to S$ that preserves pushout squares as well as edge objects.

A *Waldhausen region* is a region equipped with a family of maps closed under composition and under pushouts, called cofibrations (in a pedantic way, we could say that $[0] \star R$ is a pre-Waldhausen category). A morphism of Waldhausen regions is a morphism of regions that preserves cofibrations.

Example 20.5.2. For any $m, n : \mathbb{N}$ the product $[m] \times [n]$ is a region with no edge object. On the other hand, $[0] \star ([m] \times [n])$ is a region whose unique edge object is the initial object. The elementary squares in $[m] \times [n]$ are those of the form

$$\begin{array}{ccc} (i,j) & \longrightarrow & (i,j+1) \\ \downarrow & & \downarrow \\ (i+1,j) & \longrightarrow & (i+1,j+1) \end{array}$$

for all $0 \leq i < m$ and $0 \leq j < n$. Unless stated otherwise, we will consider $[m] \times [n]$ as a Waldhausen region with cofibrations those maps of the form $(i, j) \leq (i, k)$.

Example 20.5.3. For $n : \mathbb{N}$, the category $\mathrm{Fun}([1], [n])$ may be identified with the full subposet of $[n] \times [n]$ whose objects are terms of the form (i, j) with $i \leq j$. This description turns $\mathrm{Fun}([1], [n])$ into a region whose edge objects are the identities, i.e. the pairs of the form (i, i) for any $i : [n]^{\simeq}$. The elementary squares in $\mathrm{Fun}([1], [n])$ are those of the form

$$\begin{array}{ccc} (i,j) & \longrightarrow & (i,j+1) \\ \downarrow & & \downarrow \\ (i+1,j) & \longrightarrow & (i+1,j+1) \end{array}$$

for all $0 \leq i < j < n$. Unless stated explicitly otherwise, we will consider $\mathrm{Fun}([1], [n])$ as a Waldhausen region with cofibrations those maps of the form $(i, j) \leq (i, k)$. This defines a functor

$$[n] \mapsto \mathrm{Fun}([1], [n])$$

from Δ to the category of Waldhausen regions.

Definition 20.5.4. Let A be a Waldhausen category and R a Waldhausen region. We denote by $S(R, A)$ the full subcategory of $\mathrm{Fun}(R, A)$ whose objects are those functors of the form $x\colon R \to A$ with the following properties.

(i) The functor x preserves cofibrations.

(ii) The functor x preserves pushout squares.

(iii) The functor x sends edge objects to initial objects.

Remark 20.5.5. Any morphism of regions $f\colon R \to Q$ induces a right exact functor

$$f^*\colon S(Q, A) \to S(R, A)$$

obtained as the restriction of $f^*\colon \mathrm{Fun}(Q, A) \to \mathrm{Fun}(R, A)$.

Construction 20.5.6. For any Waldhausen category A and any Waldhausen region R, the category $S(R, A)$ has a natural structure of Waldhausen category. More precisely, the canonical embedding $R \hookrightarrow [0] \star R$ induces an equivalence

$$S([0] \star R, A) \simeq S(R, A)$$

and $S([0] \star R, A)$ may be seen as a full subcategory of $\mathrm{Wald}([0] \star R, A)$ whose objects are the functors sending edge objects to initial objects. We define weak equivalences and cofibrations in $S(R, A)$ as those maps that have these properties when seen in $\mathrm{Wald}([0] \star R, A)$.

Definition 20.5.7. We define a functor

$$S\colon \mathrm{WaldCat} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{WaldCat})$$

assigning to each $A : \mathrm{WaldCat}$ and $[n] : \Delta$ the Waldhausen category

$$S_n(A) := S(\mathrm{Fun}([1], [n]), A)\,.$$

We call $S(A) : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{WaldCat})$ *the Waldhausen S-construction* associated with A.

Lemma 20.5.8. *Let A be a Waldhausen category and $n : \mathbb{N}$. The fully faithful right exact functor $u\colon [n] \hookrightarrow \mathrm{Fun}([1], [n])$ defined by $u(i) = (0, i)$ induces an equivalence*

$$u^*\colon S_n(A) \xrightarrow{\sim} \mathrm{Wald}([n], A)\,.$$

Proof. It suffices to check that

$$u^*\colon \mathrm{Fun}([1], S_n(A))^{\simeq} \to \mathrm{Fun}([1], \mathrm{Wald}([n], A))^{\simeq}$$

is an equivalence. Given any Waldhausen category B, we can equip $\mathrm{Fun}([1], B)$ with a Waldhausen structure in which the weak equivalences and the cofibrations are defined termwise. We then observe that

$$\mathrm{Fun}([1], \mathrm{Wald}([n], A)) = \mathrm{Wald}([n], \mathrm{Fun}([1], A)) \quad \text{and}\, \mathrm{Fun}([1], S_n(A)) = S_n(\mathrm{Fun}([1], A))\,.$$

Therefore, we see that it suffices to prove that

$$u^*\colon S_n(A)^{\simeq} \to \mathrm{Wald}([n], A)^{\simeq}$$

is an equivalence. This is done by induction on n. For $n = 0$, this is clear: both sides are contractible. For $n = 1$, both sides are equivalent to $A^{\simeq}$ and u^* is equivalent to the identity of $A^{\simeq}$. Given $n > 0$ for which the statement is known to be true, we deduce the case of $n + 1$ as follows. Let R_n be the full subcategory of $\mathrm{Fun}([1], [n + 1])$ whose objects are the pairs of the form (i, j) with $j \leq n$ or $i = 0$. The embedding

$$v\colon R_n \hookrightarrow \mathrm{Fun}([1], [n + 1])$$

induces an equivalence

$$v^*\colon S(\mathrm{Fun}([1], [n + 1]), A)^{\simeq} \xrightarrow{\sim} S(R_n, A)^{\simeq}$$

whose inverse is given by completing each diagram of the form

$$\begin{array}{ccccccccc}
* & \longrightarrow & x_{0,1} & \longrightarrow & \cdots & \longrightarrow & x_{0,n} & \longrightarrow & x_{0,n+1} \\
 & & \downarrow & & & & \downarrow & & \\
 & & * & \longrightarrow & \cdots & \longrightarrow & x_{1,n} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \vdots & & \\
 & & & & & & \downarrow & & \\
 & & & & & & * & &
\end{array}$$

(in which $x_{0,n} \to x_{0,n+1}$ is a cofibration) through pushouts (and a map to $*$).

$$\begin{array}{ccccccccc}
* & \longrightarrow & x_{0,1} & \longrightarrow & \cdots & \longrightarrow & x_{0,n} & \longrightarrow & x_{0,n+1} \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \cdots & \longrightarrow & x_{1,n} & \longrightarrow & x_{1,n+1} \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & * & \longrightarrow & x_{n,n+1} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & *
\end{array}$$

We see that there is a pullback square of the form

$$\begin{array}{ccc}
S(R_n, A) & \longrightarrow & S_n(A) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{Wald}([2], A) & \longrightarrow & A
\end{array}$$

induced by the obvious embedding $\mathrm{Fun}([1],[n]) \hookrightarrow R_n$ and the map $w\colon [2] \hookrightarrow R_n$ defined by $w(0) = (0,0)$, $w(1) = (0,n)$ and $w(2) = (0,n+1)$. Since there is a pullback of the form

$$\begin{array}{ccc} \mathrm{Wald}([n+1],A) & \longrightarrow & \mathrm{Wald}([n],A) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Wald}([2],A) & \longrightarrow & A \end{array}$$

the induction hypothesis implies that there is a canonical equivalence

$$S(R_n,A) \simeq \mathrm{Wald}([n+1],A)\,. \qquad \square$$

Remark 20.5.9. We may see the assignment $A \mapsto A^{\simeq}$ as a functor of type

$$\mathrm{WaldCat} \to \mathrm{An}_\bullet\,.$$

Using the canonical equivalence of type $[m] \times \mathrm{Fun}([1],[n]) = \mathrm{Fun}([1],[n]) \times [m]$, we get a canonical equivalence of type

$$S_m(w_m(A)) = w_m(S_n(A))$$

for all A : WaldCat. We will simply write

$$w_m S_n(A) := w_m(S_n(A))$$

and denote

$$wS(A)\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{An}_\bullet\,.$$

We thus get a functor

$$wS\colon \mathrm{WaldCat} \to \mathrm{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathrm{An}_\bullet)\,.$$

Recall that the universal left fibration $\mathrm{An}_\bullet \to \mathrm{An}$ commutes with colimits indexed by small weakly contractible categories, in particular with small sifted colimits.

Definition 20.5.10. Given a Waldhausen category A, we define the *K-theory* of A as the pointed anima

$$K(A) := \Omega\Big(\operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} w_m S_n(A)^{\simeq}\Big)\,.$$

20.6 The additivity theorem

Recall that $\operatorname*{colim}_{\Delta^{\mathrm{op}}}\colon \mathrm{PSh}(\Delta) \to \mathrm{An}$ is called the geometric realization functor.

Lemma 20.6.1. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\;u\;} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative triangle of simplicial animae. We assume that, in any anima context, for any map $\sigma\colon \Delta^n \to S$, the induced map of type

$$\Delta^n \times_S X \to \Delta^n \times_S Y$$

induces an isomorphism in An *after geometric realization. Then the geometric realization of u is an isomorphism.*

Proof. Our assumptions mean that, for any small anima Γ and any term $n : \mathbb{N}$, for any map of the form $\sigma\colon \Gamma \times \Delta^n \to S$, the induced map of type

$$\sigma^*(u)\colon (\Gamma \times \Delta^n) \times_S X \to (\Gamma \times \Delta^n) \times_S Y$$

induces an isomorphism in An after geometric realization. But any simplicial anima is canonically a colimit of simplicial animae of the form $\Gamma \times \Delta^n$. Furthermore, for any morphism $f\colon A \to S$ in $\mathrm{PSh}(\Delta)$, the pullback

$$f^*\colon \mathrm{PSh}(\Delta)_{/S} \to \mathrm{PSh}(\Delta)_{/A}$$

has a right adjoint: we can identify

$$\mathrm{PSh}(\Delta)_{/S} = \mathrm{PSh}(\Delta_{/S})$$

so that f^* is the pullback functor along the functor

$$f_!\colon \Delta_{/A} \to \Delta_{/S}\,.$$

In particular, the functor f^* commutes with colimits. Therefore, the map u is canonically a colimit of morphisms of the form $\sigma^*(u)$ induced by pullbacks from maps σ as above. Since the realization functor commutes with colimits (this is a colimit functor itself by definition), the geometric realization of u is the colimit of the geometric realization of each of the $\sigma^*(u)$'s. We conclude by the obvious observation that colimits preserve isomorphisms. □

Remark 20.6.2. Let $n : \mathbb{N}$ and A a small category. Let $h\colon \Delta^1 \times \Delta^n \to \Delta^n$ be the map induced by the functor expressing the fact that n is a terminal object of $[n]$.

The map h induces a functor

$$\eta\colon \Delta_{/[1]} \times_\Delta \Delta_{/[n]} \to \Delta_{/\mathrm{Fun}([1],[n])}$$

sending $u\colon [n] \to [m]$ and $v\colon [m] \to [1]$ to the unique natural transformation

$$\kappa_{u,v}\colon u \to \bar{u}$$

where $\bar{u}\colon [m] \to [n]$ is defined as the composition of (v, u) with h. Since $u(i) \leq \bar{u}(i)$, this defines $\kappa_{u,v}$.

The evaluation functor

$$\mathrm{ev}\colon [1] \times \mathrm{Fun}([1],[m]) \to [m]$$

is functorial in $[m]$ hence so is its product with the identity of $[1]$

$$\mathrm{id}_{[1]} \times \mathrm{ev}\colon [1] \times [1] \times \mathrm{Fun}([1],[m]) \to [1] \times [m]\,.$$

This defines a functor

$$c_{[m]}\colon [1] \times \mathrm{Fun}([1],[m]) \to \mathrm{Fun}([1], [1] \times [m])$$

functorially in $[m]$. With the identification

$$\mathrm{Fun}([1] \times \mathrm{Fun}([1],[m]), A) = \mathrm{Fun}(\mathrm{Fun}([1],[m]), \mathrm{Fun}([1], A))$$

this defines a functor

$$c^*_{[m]}\colon \mathrm{Fun}(\mathrm{Fun}([1],[1]\times[m]),A)\to \mathrm{Fun}(\mathrm{Fun}([1],[m]),\mathrm{Fun}([1],A))$$

functorially in $[m]$ and A. Assigning each $u\colon [m]\to \mathrm{Fun}([1],[n])$ to the corresponding map $[1]\times[m]\to[n]$ defines a functor

$$u^*\colon \mathrm{Fun}(\mathrm{Fun}([1],[n]),A)\to \mathrm{Fun}(\mathrm{Fun}([1],[1]\times[m]),A)$$

functorially in u, seen as a term of $\Delta_{/\mathrm{Fun}([1],[n])}$. Composing with the functor η above, this means that, for any functor

$$x\colon \mathrm{Fun}([1],[n])\to A$$

if, for any map $w\colon [m]\to[n]$ one writes

$$w^*(x)\colon \mathrm{Fun}([1],[n])\to A$$

for the composition of x with the functor induced by w by applying $\mathrm{Fun}([1],-)$, then for any maps $u\colon [m]\to[n]$ and $v\colon [m]\to[1]$, one has a canonical map

$$\kappa_{u,v,x}\colon u^*(x)\to \bar{u}^*(x)$$

functorially in (u,v).

Lemma 20.6.3. *Let A be a Waldhausen category in which weak equivalences all are isomorphisms. The functor*

$$(\ker,\mathrm{coker})\colon \mathrm{ExSeq}(A)\to A\times A$$

induces a morphism of simplicial animae

$$S(\mathrm{ExSeq}(A))^{\simeq}\to S(A)^{\simeq}\times S(A)^{\simeq}$$

that yields an isomorphism after geometric realization.

Proof. We have a commutative triangle of the form

$$\begin{array}{ccc} S(\mathrm{ExSeq}(A))^{\simeq} & \longrightarrow & S(A)^{\simeq}\times S(A)^{\simeq} \\ & {\scriptstyle S(\ker)^{\simeq}}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & S(A)^{\simeq} & \end{array}$$

and we will apply Lemma 20.6.1 to it. We consider a map

$$x\colon \Delta^n\to S(A)$$

i.e. a functor $x\colon \mathrm{Fun}([1],[n])\to A$ satisfying the conditions of Definition 20.5.4. We have to prove that the induced map

$$\Delta^n\times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq}\to \Delta^n\times_{S(A)^{\simeq}}(S(A)^{\simeq}\times S(A)^{\simeq})$$

induces an isomorphism after geometric realization. We will consider first the case where $n=0$. In that case x simply is the initial object $*$ in A. It suffices to check that the comparison map between

pullbacks above is an equivalence of simplicial animae, which can be checked objectwise. If we evaluate at $[p]$, this is the same as evaluating at 0, up to replacing A by $\mathrm{Wald}([p+1], A)$. Therefore, it suffices to check that we have an equivalence when we evaluate at $[0]$. The left hand side is then the collection of pushout squares of the form

$$\begin{array}{ccc} * & \longrightarrow & x \\ \downarrow & & \downarrow v \\ * & \longrightarrow & x'' \end{array}$$

whereas the right hand side is the collection of pairs of objects of the form $(*, x'')$. It is clear that assigning x'' to each pushout square as above is an equivalence. Let us consider the general case. Let us write $x_0 \colon \Delta^0 \to S(A)$ for the restriction of x along $\{0\} \hookrightarrow \Delta^n$. This induces a pullback square of the form

$$\begin{array}{ccc} \Delta^0 \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} & \xrightarrow{\simeq} & \Delta^0 \times_{S(A)^{\simeq}} (S(A)^{\simeq} \times S(A)^{\simeq}) \\ \downarrow i & & \downarrow j \\ \Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} & \longrightarrow & \Delta^n \times_{S(A)^{\simeq}} (S(A)^{\simeq} \times S(A)^{\simeq}) \end{array}$$

in which the upper horizontal map is an equivalence (by the preceding discussion). Therefore, it suffices to check that the two vertical maps induce an isomorphism after geometric realization. But the vertical map j on the right is equivalent to

$$S(A)^{\simeq} \xrightarrow{\sim} \{0\} \times S(A)^{\simeq} \hookrightarrow \Delta^n \times S(A)^{\simeq}.$$

Since the geometric realization functor commutes with finite products (because Δ^{op} is sifted), it suffices to check that any map of the form $\Delta^0 \to \Delta^n$ induces an isomorphism after geometric realization. This follows from the fact that the geometric realization of Δ^n is contractible (as is the colimit of any representable presheaf on a small category).

What precedes tells us that

$$\Delta^0 \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} \simeq S(A)^{\simeq}.$$

There is a canonical map

$$p \colon \Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} \to S(A)^{\simeq})$$

obtained through the second projection

$$\Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq} \to S(\mathrm{ExSeq}(A))^{\simeq}$$

followed from the map

$$S(\mathrm{ExSeq}(A))^{\simeq} \to S(A)^{\simeq}$$

induced by the functor

$$\mathrm{coker} \colon \mathrm{ExSeq}(A) \to A.$$

It is immediate to see that the vertical map i on the left hand side of the square above is a section of p. It suffices now to prove that the other composite $i \circ p$ simplicially homotopic to the identity.

Let $h\colon \Delta^1 \times \Delta^n \to \Delta^n$ be the map induced by the functor expressing the fact that n is a terminal object of $[n]$. Keeping track of the notations of Remark 20.6.2, for any maps $u\colon [m] \to [n]$ and $v\colon [m] \to [1]$ to the unique natural transformation

$$\kappa_{u,v}\colon u \to \bar{u}$$

produces a canonical map

$$\kappa_{u,v,x}\colon u^*(x) \to \bar{u}^*(x)$$

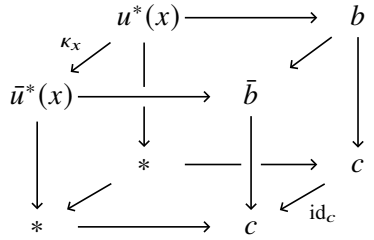
functorially in (u, v). A term of the category $\Delta_{/X}$ for

$$X = \Delta^1 \times \left(\Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq}\right)$$

literally is a tuple of the form (u, v, ξ) with (u, v) as above any ξ a tern of the category of pushout square in A of the form below.

$$\xi \quad = \quad \begin{array}{ccc} u^*(x) & \longrightarrow & b \\ \downarrow & & \downarrow \\ * & \longrightarrow & c \end{array}$$

Such a term ξ is thus canonically the back face of a term of the category of commutative cubes in A of the form



in which the back face, the front face, as well as the top face are pushout squares. We define

$$\bar{\xi} \quad := \quad \begin{array}{ccc} \bar{u}^*(x) & \longrightarrow & b \\ \downarrow & & \downarrow \\ * & \longrightarrow & c \end{array}$$

and the rule

$$H(u, v, \xi) = (\bar{u}, \bar{\xi})$$

produces a well-defined functor over Δ, hence a morphism of simplicial animae

$$H\colon \Delta^1 \times \left(\Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq}\right) \to \Delta^n \times_{S(A)^{\simeq}} S(\mathrm{ExSeq}(A))^{\simeq}$$

which is a homotopy between the identity and $i \circ p$. □

Lemma 20.6.4. *The functor* $\operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} w_m S_n(-)^{\simeq}\colon \mathrm{WaldCat} \to \mathrm{An}_\bullet$ *is additive.*

Proof. We observe that

$$w_m(\mathrm{ExSeq}(A)) = \mathrm{ExSeq}(w_m(A))$$

functorially in $[m]$ and A. Therefore, it follows from Lemma 20.6.3, it suffices to prove that the functor

$$\Phi\colon A \mapsto \operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} w_m(A)^{\simeq}$$

sends the functor

$$i\colon A \to w_1(A)\,, \quad x \mapsto \mathrm{id}_x$$

to an equivalence. Let us consider the functor

$$\mathrm{ev}_0\colon w_1(A) \to A\,.$$

It suffices to check that $i \circ \mathrm{ev}_0$ becomes simplicially homotopic to the identity after applying the functor

$$w(-)^{\simeq}\colon \mathrm{WaldCat} \to \mathrm{PSh}(\Delta)\,.$$

The map $\min\colon [1]\times[1] \to [1]$ induces a functor

$$h\colon [1] \times \mathrm{Fun}([1], A) \to \mathrm{Fun}([1], A)$$

corresponding by transposition to the functor

$$\mathrm{ev}_0^*\colon \mathrm{Fun}([1], A) \to \mathrm{Fun}([1]\times[1], A)\,.$$

Taking the Rezk nerve yields a map

$$N(h)\colon \Delta^1 \times N(\mathrm{Fun}([1], A)) \to N(\mathrm{Fun}([1], A))\,.$$

We observe that there is a canonical embedding

$$w(A)^{\simeq} \hookrightarrow N(\mathrm{Fun}([1], A))$$

so that the map $N(h)$ restricts to a map

$$H\colon \Delta^1 \times w(A)^{\simeq} \to w(A)^{\simeq}$$

that is the homotopy we were looking for. □

Theorem 20.6.5 (Waldhausen)**.** *The K-theory functor* $K\colon \mathrm{WaldCat} \to \mathrm{An}_\bullet$ *is additive.*

Proof. This follows from Lemma 20.6.4 since the loop functor commutes with finite products. □

20.7 The universal property of Waldhausen construction

Notation 20.7.1. Given an admissible class of Waldhausen categories C, and D a pointed category with finite limits, we denote by

$$\mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, D)$$

the full subcategory of $\mathrm{Fun}(\mathrm{WaldCat}_C, D)$ spanned by finite product preserving functors and we write

$$\mathrm{Add}(\mathrm{WaldCat}_C, D)$$

for the full subcategory of $\mathrm{Fun}(\mathrm{WaldCat}_C, D)$ spanned by additive functors. Up to size issues one can enforce the additivity property for formal reasons, as explained in the next proposition.

Proposition 20.7.2. *If D is a pointed compactly generated category, then so is* $\mathrm{Add}(\mathrm{WaldCat}_C, D)$ *for any small admissible class of Waldhausen categories C. More precisely,* $\mathrm{Add}(\mathrm{WaldCat}_C, D)$ *is a left Bousfield localization of the compactly generated category* $\mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, D)$ *by a small family of maps between compact objects. In particular, the full embedding*

$$\mathrm{Add}(\mathrm{WaldCat}_C, D) \hookrightarrow \mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, D)$$

commutes with filtered colimits and has a left adjoint. Furthermore, in the case where $D = \mathrm{An}_\bullet$, the full embedding

$$\mathrm{WaldCat}_C^{\mathrm{op}} \hookrightarrow \mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$$

commutes small with sifted colimits.

Proof. The is a Yoneda embedding

$$h\colon \mathrm{WaldCat}_C^{\mathrm{op}} \to \mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$$

We define the family of maps S in $\mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$ explicitly as follows:

(i) For any $A : \mathrm{WaldCat}^{\simeq}$, the image of

$$A \mapsto w([1], A)\ , \quad x \mapsto \mathrm{id}_x$$

through h is in S;

(ii) for any $A : \mathrm{WaldCat}^{\simeq}$, the image of

$$(\ker, \mathrm{coker})\colon \mathrm{ExSeq}(A) \to A \times A$$

through h is in S.

It follows right away from the Yoneda lemma applied to $\mathrm{WaldCat}_C^{\mathrm{op}}$ that the S-local objects in the compactly generated category $\mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$ precisely are the additive functors. The first part of the proposition thus follows from Proposition 18.4.1 and Proposition 18.1.5. Proposition 20.4.8 ensures the last part to be true. □

Remark 20.7.3. Given any category C with an initial object $*$, for any integer $n : \mathbb{N}$, let us denote by $i\colon [n] \to [n+1]$ the upper inclusion $k \mapsto k+1$. The functor

$$i^*\colon \mathrm{Fun}([n+1], C) \to \mathrm{Fun}([n], C)$$

has a right adjoint i_* ad a left adjoint $i_!$ sending a sequence of the form

$$a_1 \to \cdots \to a_n$$

to

$$a_1 \xrightarrow{\mathrm{id}_{a_1}} a_1 \to \cdots \to a_n \quad \text{and} \quad * \to a_1 \to \cdots \to a_n \quad \text{respectively.}$$

Both functors $i_!$ and i_* are fully faithful. There is an obvious natural transformation $i_! \to i_*$ corresponding to the inverse of $i^* i_* = \mathrm{id}$. Let

$$\kappa\colon \mathrm{Fun}([n], C) \to \mathrm{Fun}([n], C)$$

be the functor sending

$$a_0 \to \cdots \to a_n$$

to

$$a_0 \xrightarrow{\mathrm{id}} a_0 \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} a_0$$

with the obvious natural transformation $\kappa \to \mathrm{id}$. Given any $a : \mathrm{Fun}([n], C)$, there is a canonical (and functorial) pushout square of the form below.

$$\begin{array}{ccc} i_! i^* \kappa(a) & \longrightarrow & i_! i^*(a) \\ \downarrow & & \downarrow \\ i_* i^* \kappa(a) & \longrightarrow & a \end{array}$$

Lemma 20.7.4. *For each $n : \mathbb{N}$, there is an equivalence*

$$\mathrm{Wald}(S_n(C), D) = \mathrm{Wald}(C, S_n(D))$$

functorially in C and D.

Proof. We proceed by induction on n. The case $n = 0$ being obvious.

If we assume C and D to have finite colimits, we know that there is an explicit equivalence of the form

$$\mathrm{Rex}(\mathrm{Fun}(P^{\mathrm{op}}, C), D) \simeq \mathrm{Rex}(C, \mathrm{Fun}(P, D))$$

for any finite poset P by Theorem 18.4.8, which is functorial in a way that is explained in Remark 18.4.10. Given a general Waldhausen category C, with Yoneda embedding

$$h \colon C \to \mathrm{PSh}(C)$$

we can consider the left Bousfield localization of $\mathrm{PSh}(C)$ by the family of maps S of the form:

(i) $h(*) \to *$;

(ii) $h(y) \amalg_{h(x)} h(z) \to h(y \amalg_x z)$ for any cofibration $x \to y$ and any map $x \to z$;

There is a further left Bousfield localization that furthermore inverts all maps of the form $h(u)$ for u a weak equivalence of C.

We define weak equivalences of $C' := L_S \mathrm{PSh}(C)$ to be the maps that become isomorphisms after applying the left adjoint to the further left Bousfield localization alluded to above. Cofibrations in C' simply are defined to be all maps. The Yoneda embedding of C defines a fully faithful functor that is also a morphism of Waldhausen categories $C \to C'$.

The functor $u^* \colon D' \to C'$ defined by $F \mapsto F \circ^{\mathrm{op}} u$ has a left adjoint $u_!$ defined by applying the left adjoint of u^* at the level of ordinary presheaves and then applying the S-localization functor. Since representable presheaves are S-local and since $u_!$ preserves representable presheaves, the commutative square

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{PSh}(C) \\ \downarrow & & \downarrow{\scriptstyle u_!} \\ D & \longrightarrow & \mathrm{PSh}(D) \end{array}$$

restricts to a commutative square of the form below.

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \downarrow & & \downarrow{\scriptstyle u_!} \\ D & \longrightarrow & D' \end{array}$$

Using the equivalence $[n] \simeq [n]^{\mathrm{op}}$, we have a canonical equivalence of the form:

$$\mathrm{Rex}(\mathrm{Fun}([n], C'), D') \simeq \mathrm{Rex}(C', \mathrm{Fun}([n], D'))\,.$$

We want to show that this restricts to an equivalence

$$\mathrm{Wald}(C^{[n]}, D) \simeq \mathrm{Wald}(C, D^{(n)})\,.$$

This is the precise form of the induction we want to establish. This is easily seen to be implied by the preceding remark together with Remark 18.4.10. □

Essentially the same argument proves the following.

Lemma 20.7.5. *For each $n : \mathbb{N}$, there is an equivalence*

$$\mathrm{Wald}(w_n(C), D) = \mathrm{Wald}(C, w_n(D))$$

functorially in C and D.

Definition 20.7.6. Let C be a small admissible class of Waldhausen categories. We denote by

$$i\colon \mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet) \hookrightarrow \mathrm{Fun}^\times(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$$

the canonical embedding and by

$$\alpha\colon \mathrm{Fun}^\times(\mathrm{WaldCat}_C, \mathrm{An}_\bullet) \to \mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$$

its left adjoint. A morphism of finite products preserving presheaves $u\colon F \to G$ is said to be an *additive equivalence* if $\alpha(u)\colon \alpha(F) \to \alpha(G)$ is an isomorphism.

Lemma 20.7.7. *Let C be a small admissible class of Waldhausen categories. The subcategory of additive equivalences has the following stability properties.*

(i) Additive equivalences are stable under small sifted colimits: for any small sifted category and any natural transformation $u\colon F \to G$ between I-indexed functors with values in $\mathrm{Fun}^\times(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$, if $u_i\colon F_i \to G_i$ is an additive equivalence for all $i : I^\simeq$, then the induced map

$$\operatorname*{colim}_{i:I} u_i\colon \operatorname*{colim}_{i:I} F_i \to \operatorname*{colim}_{i:I} G_i$$

is an additive equivalence.

(ii) The functor $F \mapsto F^n$ preserves additive equivalences for all $n\colon \mathbb{N}$.

(iii) The assignment $F \mapsto F \circ S_n$ preserves additive equivalences for all $n : \mathbb{N}$.

(iv) The assignment $F \mapsto F \circ w_n$ preserves additive equivalences for all $n : \mathbb{N}$.

Proof. any morphism $u\colon F \to G$ in $\mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$ fits functorially in a commutative square of the form

$$\begin{array}{ccc} F & \longrightarrow & i\alpha(F) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle i\alpha(u)} \\ G & \longrightarrow & i\alpha(G) \end{array}$$

in which the two horizontal maps are additive equivalences, being the unit the adjunction (α, i) with i fully faithful. Since u is an additive equivalence if and only if $i\alpha(u)$ is an isomorphism, property (i) follows from the fact that the functor $i\alpha$ commutes with small sifted colimits (since i has this property whereas α commutes with all colimits).

Similarly, in order to prove assertion (ii), it suffices to check that, for any functor

$$F\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet$$

that commutes with finite products and any $n : \mathbb{N}$, the canonical map

$$F^n \to i\alpha(F)^n$$

is an additive equivalence. Equivalently, we have to see that for any additive functor

$$E\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet$$

the map above induces an identification of type

$$\mathrm{Hom}(i\alpha(F)^n, E) = \mathrm{Hom}(F^n, E)\,.$$

Since both assignments

$$F \mapsto F^n \quad \text{and} \quad F \mapsto i\alpha(F)^n$$

commute with small sifted colimits and since any such F is a small sifted colimit of corepresentable functors, we may assume that there is a Waldhausen category A in the class C such that $F = \mathrm{Wald}(A, -)^{\simeq}$. Since the category $\mathrm{WaldCat}_C$ is semi-additive, we then have

$$F^n = \mathrm{Wald}(A^n, -)^{\simeq}\,.$$

Since E commutes with finite products then have by the Yoneda lemma:

$$\mathrm{Hom}(F^n, E) = E(A^n) = E(A)^n\,.$$

But since the category $\mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$ is semi-additive as well, we also have

$$\mathrm{Hom}(i\alpha(F)^n, E) = \mathrm{Hom}(i\alpha(F), E)^n\,.$$

Since $\mathrm{Hom}(i\alpha(F), E) = \mathrm{Hom}(F, E)$, this proves property (ii).

The proof of assertion (iii) is similar. This is obvious for $n \leq 1$ so that we may assume $n > 1$. Evaluating at each vertex of shape $(0, i)$ for $1 < i \leq n$, yields a natural transformation of the form

$$S_n(A) \to A^{n-1}$$

hence a functorial morphism of type

$$F(S_n(A)) \to F(A^{n-1}) = F(A)^{n-1}$$

for any functor F that commutes with finite products. Since $F \mapsto F^{n-1}$ preserves additive equivalences, it suffices to prove that the map

$$F \circ S_n \to F^{n-1}$$

is an additive equivalence for all F. Since this is a map between constructions that are compatible with small sifted colimits, it suffices to prove this in the case where $F = \mathrm{Wald}(A, -)^{\simeq}$ for some Waldhausen category A in the class C. By virtue of Lemma 20.7.4, there is then a canonical equivalence of type

$$F \circ S_n = \mathrm{Wald}(S_n(A), -)^{\simeq}.$$

Therefore, for any additive functor E we have

$$\mathrm{Hom}(F, E) = E(S_n(A)) = E(A)^{n-1}$$

by Corollary 20.4.11 (and an easy induction). We also have $F^{n-1} = \mathrm{Wald}(A^{n-1}, -)$ by semi-additivity of $\mathrm{WaldCat}_C$, hence $\mathrm{Hom}(F^{n-1}, E) = E(A)^{n-1}$. This implies that $\mathrm{Hom}(-, E)$ turns the map from $F \circ S_n$ to F^{n-1} to an equivalence and thus proves assertion (iii).

Finally, the natural map $A \to w_n(A)$ induces a map

$$F \to F \circ w_n$$

and it suffices to prove that the latter is an additive equivalence. Since this is a map between functors that preserve small sifted colimits and since F is a small sifted colimit of corepresentable functors, it suffices to prove this for F of the form $\mathrm{Wald}(A, -)^{\simeq}$ for some A in the class C. By virtue of Lemma 20.7.5, we then have

$$F \circ w_n = \mathrm{Wald}(w_n(A), -)^{\simeq}.$$

Therefore, for any additive functor E, the map $A \to w_n(A)$ induces an equivalence

$$E(A) \cong E(w_n(A))$$

which is equivalent to $\mathrm{Hom}(F \circ w_n, E) \to \mathrm{Hom}(F, E)$. □

Theorem 20.7.8. *Let C be any admissible class of Waldhausen categories. Then the full embedding*

$$\mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet) \hookrightarrow \mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet)$$

has a left adjoint defined explicitly by assigning to each finite-product-preserving functor

$$F\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet$$

the functor defined by the rule

$$A \mapsto \Omega\Big(\operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} F(w_m S_n(A))\Big)$$

Proof. Given a finite products functor $F\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet$, we define, for $A : \mathrm{WaldCat}_C$,

$$PF(A) := \operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} F(w_m S_n(A))\,.$$

We will establish first that PF is additive. For this, we observe that the formation of PF commutes with the operation of restriction to $\mathrm{WaldCat}_{C'}$ for any embedding $C' \hookrightarrow C$, with C' an admissible class of Waldhausen categories. Furthermore, a finite-product-preserving functor from $\mathrm{WaldCat}_C$ to $\mathrm{An}_\bullet$ is additive if and only if its restriction to $\mathrm{WaldCat}_{C'}$ for any small C' is additive. Therefore, in order to prove that PF is additive, without any loss of generality, we may assume that C is small. Since F commutes with finite products, F is then a small sifted colimit of corepresentable functors by Lemma 20.1.5. By virtue of the last assertion of Proposition 20.1.2, we see that the functor $F \mapsto PF$ commutes with sifted colimits (as a small sifted colimit of functors that preserves sifted colimits). Therefore, we deduce from Proposition 20.4.8 that it suffices to prove that PF is additive whenever F is corepresentable by a Waldhausen category B in the class C. But then, we observe that

$$w_m S_n \mathrm{WaldCat}(B, A)^{\simeq} = \mathrm{WaldCat}(B, w_m S_n(A))^{\simeq}\,.$$

for all m, n and A functorially. In other words we then have

$$PF(A) = \operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} w_m S_n \mathrm{WaldCat}(B, A)^{\simeq}\,.$$

Using the fact that

$$\mathrm{WaldCat}(B, \mathrm{ExSeq}(A)) = \mathrm{ExSeq}(\mathrm{WaldCat}(B, A))$$

$$\text{and} \quad w([1], \mathrm{Wald}(B, A)) = \mathrm{Wald}(B, w([1], A))\,,$$

we conclude from Lemma 20.6.4 that PF is additive.

To prove the second part, we also reduce to the case where C is small: we choose a larger universe Cat' containing Cat so that C is small. If An' is the full subcategory of animae in Cat', we know that the embedding

$$\mathrm{An} \hookrightarrow \mathrm{An}'$$

commutes with small colimits and with small limits. This means that the induced full embedding

$$\mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet) \hookrightarrow \mathrm{Fun}^{\times}(\mathrm{WaldCat}_C, \mathrm{An}'_\bullet)$$

also commutes with small limits and with small colimits. In particular, it commutes with the loop functor Ω as well as with the operator P defined above. This means that it suffices to prove the universal property of the construction ΩP in the larger universe Cat' for which C is small.

From now on, we focus on the case where C is small and we keep track of the notations of Definition 20.7.6.

Now, Proposition 20.4.13 tells us that the operator P comes equipped with a natural map of the form

$$c_F\colon F \to \Omega(PF)$$

that is invertible whenever F is additive. Furthermore, since the category Δ^{op} is small and sifted, Lemma 20.7.7 tells us that the operator P preserves additive equivalences. Let $G = i\alpha(F)$ be the

universal additive functor associated to F. The unit map $\eta\colon F \to G$ is then an additive equivalence with additive codomain. We obtain a commutative square

$$\begin{array}{ccc} F & \xrightarrow{\eta} & G \\ {\scriptstyle c_F}\downarrow & & \downarrow{\scriptstyle c_G} \\ \Omega(PF) & \xrightarrow{\Omega P\eta} & \Omega(PG) \end{array}$$

with $P\eta$ an additive equivalence between additive functors, hence an equivalence. Since c_G is an equivalence, this proves that c_F is an additive equivalence with additive codomain, which implies the theorem right away. □

Remark 20.7.9. In the case where the weak equivalences of the Waldhausen category A all are isomorphisms, the simplicial object

$$[m] \mapsto w_m(S_n(A))$$

is constant for all n and thus have a simplification:

$$\operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(S_n(A)) = \operatorname*{colim}_{[m]\colon \Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]\colon \Delta^{\mathrm{op}}} F(w_m S_n(A))$$

Corollary 20.7.10. *Let C be any admissible class of Waldhausen categories. Then the universal additive functor associated to the functor*

$$\mathrm{WaldCat}_C \to \mathrm{An}_\bullet\,, \quad A \mapsto A^{\simeq}$$

is the additive functor

$$\mathrm{WaldCat}_C \to \mathrm{An}_\bullet\,, \quad A \mapsto K(A)\,.$$

Corollary 20.7.11. *Let C be any admissible class of Waldhausen categories and A a Waldhausen category that belongs to C. Then the universal additive functor associated to the functor*

$$\mathrm{WaldCat}_C \to \mathrm{An}_\bullet\,, \quad B \mapsto \mathrm{Wald}(A,B)^{\simeq}$$

is the additive functor

$$\mathrm{WaldCat}_C \to \mathrm{An}_\bullet\,, \quad B \mapsto K(\mathrm{Wald}(A,B))\,.$$

Examples of admissible classes of Waldhausen categories are the one of small stable categories as well as the one of small pointed categories with finite colimits. We might in either cases above restrict ourselves to the idempotent-complete categories.

Remark 20.7.12. Let C be an admissible class of Waldhausen categories. We let W be the smallest family of morphisms in $\mathrm{WaldCat}_C$ stable under finite products containing any map of the form:

1. $A \to w_1(A)$, defined by $x \mapsto \mathrm{id}_x$, for any A in C;
2. $(\ker, \mathrm{coker})\colon \mathrm{ExSeq}(A) \to A \times A$ for any A in C.

We observe that W contains all identities since the identity of A is a retract of the map from A to $w_1(A)$ above. We denote by

$$\gamma\colon \mathrm{WaldCat}_C \to L_{\mathrm{Add}}(\mathrm{WaldCat}_C)$$

the localization of $\mathrm{WaldCat}_C$ by W. We know that this localization has finite products and that the functor γ preserves them by Proposition 14.1.1 and Proposition 14.1.6. By virtue of Proposition 20.1.6, there is thus an adjunction

$$\gamma_!\colon \mathrm{Fun}^\times(\mathrm{WaldCat}_C, \mathrm{An}) \rightleftarrows \mathrm{Fun}^\times(L_{\mathrm{Add}}(\mathrm{WaldCat}_C), \mathrm{An})\colon \gamma^*$$

with γ^* fully faithful. The previous corollary means that, for any A and B in C, the map

$$\mathrm{Wald}(A, -)^\simeq \to \gamma^*\gamma_!(\mathrm{Wald}(A, -)^\simeq)$$

evaluated at B literally is the canonical map

$$\mathrm{Wald}(A, B)^\simeq \to K(\mathrm{Wald}(A, B))\,.$$

In other words we have a canonical identification

$$\mathrm{Hom}_{L_{\mathrm{Add}}(\mathrm{WaldCat}_C)}(\gamma(A), \gamma(B)) = K(\mathrm{Wald}(A, B))$$

functorially in (A, B) seen in $\mathrm{WaldCat}_C^{\mathrm{op}} \times \mathrm{WaldCat}_C$.

We also have a canonical identification

$$\mathrm{Add}(\mathrm{WaldCat}_C, \mathrm{An}_\bullet) = \mathrm{Fun}^\times(L_{\mathrm{Add}}(\mathrm{WaldCat}_C), \mathrm{An})\,.$$

In particular, if C is small, any additive functor $E\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet$ is canonically a small sifted colimit of functors of the form $K(\mathrm{Wald}(A, -))$ with A in C.

20.8 The localization theorem

Construction 20.8.1. Let $f\colon A \to B$ be a morphism of Waldhausen categories. Recall that $i_n\colon [n] \to [n+1]$ is the upper embedding defined through $i \mapsto i+1$. We form the pullback square below.

$$\begin{array}{ccc} S_n(f\colon A \to B) & \xrightarrow{q_n} & S_{n+1}(B) \\ {\scriptstyle p_n}\downarrow & \lrcorner & \downarrow{\scriptstyle i_n^*} \\ S_n(A) & \xrightarrow{S_n(f)} & S_n(B) \end{array}$$

This construction clearly is functorial in $[n] : \Delta^{\mathrm{op}}$.

Proposition 20.8.2 (Waldhausen)**.** *Let C be an admissible class of Waldhausen categories. We consider a morphism $f\colon A \to B$ in $\mathrm{WaldCat}_C$. For any additive functor $E\colon \mathrm{WaldCat}_C \to \mathrm{An}_\bullet$, there is a canonical pullback square of the form below.*

$$\begin{array}{ccc} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} E(S_n(f\colon A \to B)) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} E(S_n(A)) & \longrightarrow & \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} E(S_n(B)) \end{array}$$

In particular, there is a canonical fiber sequence of the form below.

$$\begin{array}{ccc} E(A) & \xrightarrow{E(f)} & E(B) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} E(S_n(f\colon A \to B)) \end{array}$$

Proof. The last assertion follows from the first by Proposition 20.4.13. The proof of this proposition is in fact a variation on the proof of the latter. Since

$$\operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} E(S_{n+1}(B)) = 0\,,$$

by Lemma 13.4.4, it suffices to prove that, for each $n : \mathbb{N}$, the induced commutative square

$$\begin{array}{ccc} E(S_n(f\colon A \to B)) & \longrightarrow & E(S_{n+1}(B)) \\ \downarrow & & \downarrow \\ E(S_n(A)) & \longrightarrow & E(S_n(B)) \end{array}$$

is a pullback square. Through the identifications $S_n(A) = A^{[n-1]}$ for $n > 0$, one can identify $S_n(f\colon A \to B)$ as a category with terms the pairs (x, y) where x is a term in $A^{[n-1]}$ and y a commutative diagram in B of the form

$$\begin{array}{ccccccc} y_1 & \longrightarrow & y_2 & \longrightarrow & \cdots & \longrightarrow & y_n \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \longrightarrow & f(x_1) & \longrightarrow & \cdots & \longrightarrow & f(x_{n-1}) \end{array}$$

in which all squares are pushout squares.

Let $v\colon S_n(B) \hookrightarrow S_{n+1}(B)$ be the full embedding defined by sending a sequence

$$x_1 \to \cdots \to x_n$$

to the sequence

$$0 \to x_1 \to \cdots \to x_n\,.$$

Since u is a section of i_n^*, there is a canonical pullback square of the form below

$$\begin{array}{ccc} S_n(A) & \longrightarrow & S_n(B) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ S_n(f\colon A \to B) & \longrightarrow & S_{n+1}(B) \end{array}$$

(in particular u is fully faithful and a section of p_n). There is also a full embedding

$$j\colon B \hookrightarrow S_n(f\colon A \to B)$$

defined by sending x to the pair consisting of the sequence of length n

$$x \xrightarrow{\mathrm{id}_x} \cdots \xrightarrow{\mathrm{id}_x} x$$

and of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & f(x) & \longrightarrow & \cdots & \longrightarrow & f(x) \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \longrightarrow & f(x) & \longrightarrow & \cdots & \longrightarrow & f(x) \end{array}$$

in which all the vertical maps are identities. The functor j is a section of the functor π sending a pair (x, y) to the object y_1. There is an obvious functorial exact sequence of the form

$$\begin{array}{ccc} j(\pi(x,y)) & \longrightarrow & (x,y) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & v(p_n(x,y)) \end{array}$$

and therefore a direct application of Proposition 20.4.10 shows that the pair of functors

$$(p_n, \pi) \colon S_n(f \colon A \to B) \to S_n(A) \times B$$

induces an equivalence

$$E(S_n(f \colon A \to B)) = E(S_n(A)) \times E(B)\,.$$

It follows that the squares

$$\begin{array}{ccc} E(S_n(f \colon A \to B)) & \longrightarrow & E(S_{n+1}(B)) \\ \downarrow & & \downarrow \\ E(S_n(A)) & \longrightarrow & E(S_n(B)) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E(S_n(A)) \times E(B) & \xrightarrow{S_n(f)\times \mathrm{id}} & E(S_n(B)) \times E(B) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ E(S_n(A)) & \xrightarrow{S_n(f)} & E(S_n(B)) \end{array}$$

are equivalent, hence are both cartesian, which achieves this proof. □

Theorem 20.8.3 (Thomason-Trobaugh)**.** *Let B be a small stable category and A a thick subcategory of B. Then the pullback square*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B/A \end{array}$$

induces a fiber sequence

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(B/A) \end{array}$$

in $\mathrm{An}_\bullet$.

Proof. We let C be the Waldhausen category defined by $C = B$ as categories, with cofibrations being all maps but with weak equivalences being the maps whose cofiber lies in A. We observe then that the equivalence

$$S_{n+1}(B) = B^{[n]}$$

induces an equivalence of the form

$$S_n(f\colon A \to B) = w_n(C)\,.$$

Applying Proposition 20.8.2 to the admissible class of Waldhausen categories given by small stable categories, with $E = K$, and taking into account Remark 20.7.9, we see that there is a canonical fiber sequence of the form below.

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(C) \end{array}$$

It suffices now to see that the morphism of Waldhausen categories $C \to B/A$ induced by the functor $B \to B/A$ yields an equivalence of the form

$$K(C) = K(B/A)\,.$$

We want to understand the map

$$\operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} w_m(S_n(C))^{\simeq} \to \operatorname*{colim}_{[m]:\Delta^{\mathrm{op}}} \operatorname*{colim}_{[n]:\Delta^{\mathrm{op}}} \mathrm{Iso}_m(S_n(B/A))$$

where $\mathrm{Iso}_n = \mathrm{Fun}([n], (-)^{\simeq})$. The assignment

$$[m] \mapsto w_m(S_n(C))^{\simeq}$$

is the nerve of the category of weak equivalences $w(S_n(C))$, whereas

$$[m] \mapsto \mathrm{Fun}([m], S_n(B/A)^{\simeq})$$

is the nerve of the anima $S_n(B/A)^{\simeq}$. Taking the colimit in the variable $[m]$ amounts to consider the map

$$\Pi_\infty(w(S_n(C))) \to S_n(B/A)^{\simeq}\,.$$

It suffices to check that the latter is an equivalence for all n. The case $n = 0$ is obvious (we are comparing two terminal categories). For $n > 0$, identifying S_n with $(-)^{[n-1]}$, taking into account Theorem 18.4.7 with $P = [n-1]$, this is a particular case of Proposition 17.1.14. □

21 Presentable categories

21.1 Accessibility axiom

Definition 21.1.1. A universe Cat is *accessible* if, for any small anima Γ and any family of small categories indexed by Γ

$$x\colon \Gamma \mapsto C_x\colon \mathrm{Cat}$$

there is an elementary small subuniverse κ in Cat such that each C_x belongs to κ.

We formulate now an extra axiom for synthetic category theory.

Axiom O (accessibility axiom)**.** Any universe is contained in an accessible universe.

Remark 21.1.2. Many theorems of synthetic category theory do not need this axiom. We could compare it as a weak form of the axiom of choice in set theory (it is a good replacement of transfinite induction). The comparison holds in many ways: using the accessibility axiom is arguably less constructive then the others. Moreover, we will see later that the ordinary axiom of choice implies the accessibility axiom (this explains why the theory of accessible categories and the theory of presentable categories are formulated in classical category theory through the language of cardinals). However, the accessibility axiom is much more robust than the axiom of choice. More precisely, it allows to develop the theory of presentable categories in full, then the theory of Grothendieck-Lurie topoi. For each topos X, we will see that univalent Segal objects of X yield a semantic interpretation of synthetic category theory in which the accessibility axiom holds as well.

From now on, we fix an accessible universe Cat, the objects of which will be called *small categories* and a larger universe Cat′, whose objects will be called *large categories*, such that Cat is large and any small category is large.

Remark 21.1.3. We can see Cat as an elementary subuniverse of Cat′. Indeed, for any elementary subuniverse κ in Cat, we can see κ as an elementary subuniverse of Cat′. Therefore, any (large) category with Cat-small colimits having κ-small ones, it is κ-filtered for any elementary subuniverse κ of Cat. This implies that such a category is Cat-filtered. Since $\mathbb{N}$ is small, any such category is idempotent complete. We can thus identify the category of large categories with small colimits with the one of Cat-presentable categories. This means that we can see the category of large categories with small filtered colimits (and small colimits preserving functors between them) as a (very large) Cat-presentable category – this is a particular case of Theorem 17.2.24 with $D = \mathrm{Cat}'$ and $\kappa = \mathrm{Cat}$. If An′ denotes the category of large animae, we can thus see large categories as those presheaves of

the form

$$\mathrm{Cat}^{\mathrm{op}} \to \mathrm{An}'$$

that preserve small limits. A large category is Cat-compact if and only if it is small.

We denote by $\mathrm{Cat}'_{\mathrm{colim}}$ the category of large categories with small colimits and small colimits preserving functors between them.

21.2 The category of presentable categories

Definition 21.2.1. A (large) category C is *presentable* if there exists an elementary subuniverse κ in Cat such that C is κ-presentable.

We define the category Pr^{L} as the full subcategory of $\mathrm{Cat}'_{\mathrm{colim}}$ whose objects are the presentable categories, with morphisms the functors that preserve small colimits.

Lemma 21.2.2. *Let κ be an elementary subuniverse of* Cat *and C a κ-presentable category. For any elementary subuniverse λ in* Cat *containing κ (i.e. such that any strictly κ-small category if λ-small), the category C is λ-small. Furthermore, given any small family F of objects of C, there exists an elementary subuniverse of* Cat *such that all objects in S are λ-small.*

Proof. The first assertion is obvious. We can choose an elementary universe λ such that each $C^{\kappa}_{/x}$ for $x\colon F$ is λ-small. Then each object x in F is λ-compact. □

Definition 21.2.3. Let C be a locally small category and S a family of morphisms in C. We denote by $L_S(C)$ the full subcategory of *S-local objects* of C, i.e. of those objects $x\colon C^{\simeq}$ such that the presheaf

$$\mathrm{Hom}_C(-,x)\colon C^{\mathrm{op}} \to \mathrm{An}$$

takes any term of S to an isomorphism of An. We say that the *left Bousfield localization of C exists* if the full embedding

$$L_S(C) \hookrightarrow C$$

has a left adjoint.

Example 21.2.4. If C has small colimits and if S is a small family of maps between compact objects, then the left Bousfield localization exists, by Theorem 18.1.6.

Proposition 21.2.5. *Let κ be a localizing subunivserse of* Cat*. If C is κ-presentable, then, for any small family S of morphisms of C between κ-small objects, the left Bousfield localization exists. Furthermore, the category $L_S(C)$ of S-local objects is κ-presentable and the full embedding*

$$L_S(C) \hookrightarrow C$$

is a right adjoint that commutes with κ-filtered colimits.

Proof. Let C^{κ} be the full subcategory of κ-compact objects of C. We may see S as a small family of morphisms in C^{κ}, hence as a functor

$$\coprod_{f\colon S} \Delta^1 \to C^{\kappa}\,.$$

Recall that, for any small category C, there is the free small category with κ-small colimits $\langle C \rangle_\kappa$ associated to C by Theorem 17.3.3. Furthermore, since the subuniverse κ is localizing, the category RexCat^κ of small categories with κ-small colimits has small colimits by Proposition 17.3.8.

The family of maps S thus defines functor preserving κ-small colimits of the form

$$\left\langle \coprod_{f\colon S} \Delta^1 \right\rangle_\kappa \to C^\kappa \,.$$

We can then form the pushout in RexCat^κ of the form bellow.

$$\begin{array}{ccc} \left\langle \coprod_{f\colon S} \Delta^1 \right\rangle_\kappa & \longrightarrow & C^\kappa \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ \langle S \rangle_\kappa & \longrightarrow & S^{-1}C^\kappa \end{array}$$

This commutative square remains a pushout in the category of large categories with κ-small colimits. We will prove that

$$L_S(C) = \mathrm{Ind}_\kappa(S^{-1}C^\kappa)\,.$$

The functor γ induces an adjunction

$$\gamma_!\colon \mathrm{PSh}(C^\kappa) \rightleftarrows \mathrm{PSh}(S^{-1}C^\kappa)\colon \gamma^*$$

with γ^* fully faithful that restrict to an adjunction of the form

$$\gamma_!\colon \mathrm{Ind}_\kappa(C^\kappa) \rightleftarrows \mathrm{Ind}_\kappa(S^{-1}C^\kappa)\colon \gamma^*$$

by Proposition 17.2.21. Moreover, we have

$$\mathrm{Ind}_\kappa(C^\kappa) = C$$

by Theorem 17.2.24 and Corollary 17.4.5. By construction, the full subcategory $\mathrm{Ind}_\kappa(S^{-1}C^\kappa)$ is contained in the full subcategory $L_S(C)$ of S-local objects. Conversely, any S-local object belongs to the essential image of γ^*. Indeed, since $C^\kappa \hookrightarrow C$ preserves κ-small colimits, given an S-local object x in C, the restricted Hom functor

$$\Phi := \mathrm{Hom}_C(-,x)^{\mathrm{op}}_{|C^\kappa}\colon C^\kappa \to \mathrm{An}^{\mathrm{op}}$$

preserves κ-small colimits and sends maps of S to isomorphisms, hence determines a commutative square of the form

$$\begin{array}{ccc} \left\langle \coprod_{f\colon S} \Delta^1 \right\rangle_\kappa & \longrightarrow & C^\kappa \\ \downarrow & & \downarrow{\scriptstyle \Phi} \\ \langle S \rangle_\kappa & \longrightarrow & \mathrm{An}^{\mathrm{op}} \end{array}$$

and thus a unique functor

$$\Psi\colon S^{-1}C^\kappa \to \mathrm{An}^{\mathrm{op}}$$

such that $\gamma \circ \Psi = \Phi$. In other words, if $F = \Psi^{\mathrm{op}}$, we see that x is isomorphic to $\gamma^*(F)$ through the identification of C with $\mathrm{Ind}_\kappa(C^\kappa)$. We have thus proved that

$$\mathrm{Ind}_\kappa(S^{-1}C^\kappa) = L_S(C)$$

so that the embedding $L_S(C) \hookrightarrow C$ corresponds to γ^*. The left adjoint of γ^* thus proves that the left Bousfield localization of C by S exists. The fact that S-local objects are stable under κ-filtered colimits can be checked directly or can be seen as a particular case of the last assertion of Proposition 17.2.21. Finally, Remark 17.2.27 proves that $L_S(C)$ is κ-presentable. □

Remark 21.2.6. The proof of the preceding proposition does not use the accessibility axiom. However, the latter is used to obtain the following corollary.

Corollary 21.2.7. *If C is presentable, then, for any small family of maps S in C, the left Bousfield localization of C by S exists. Furthermore, the category $L_S(C)$ of S-local objects is presentable.*

Proof. We may assume that there is an elementary subuniverse κ in Cat such that C is κ-presentable (more precisely, it suffices to prove the corollary in the context whose terms are the elementary subuniverses κ such that C is κ-presentable). We may choose κ such that S consists of maps between κ-small objects by Lemma 21.2.2. We can now apply Proposition 21.2.5 directly and this achieves the proof. □

Lemma 21.2.8. *For any presentable category C, there is a small category A and a small family of maps S in* $\mathrm{PSh}(A)$ *such that C is the left Bousfield localization of* $\mathrm{PSh}(A)$ *by S.*

Proof. Let κ be an elementary subuniverse of Cat such that

$$C = \mathrm{Ind}_\kappa(C^\kappa)\,.$$

We define $A = C^\kappa$ and let S be the family of maps obtained throught the following procedure. Given any strictly κ-small category E and any functor

$$F\colon E \to A$$

we consider the comparison map

$$\mathrm{colim}\ h \circ F \to h(\mathrm{colim}\ F)$$

where $h\colon C^\kappa \hookrightarrow \mathrm{PSh}(A)$ is the Yoneda embedding. A direct application of the Yoneda lemma shows that the S-local objects precisely are the functors from A^{op} to An that preserve κ-small limits. In other words, we have

$$L_S(\mathrm{PSh}(A)) = \mathrm{Ind}_\kappa(C^\kappa)$$

by Corollary 17.2.20. □

Lemma 21.2.9. *The functor*

$$\mathrm{Cat} \to \mathrm{Cat}'_{\mathrm{colim}}\,, \quad A \mapsto \mathrm{PSh}(A)$$

commutes with small colimits and takes values in Pr^{L}.

Proof. The forgetful functor

$$\mathrm{Cat}'_{\mathrm{colim}} \hookrightarrow \mathrm{Cat}'$$

has a left adjoint

$$C \mapsto \langle C \rangle_{\mathrm{Cat}}$$

by Corollary 17.2.35 for $\kappa = \mathrm{Cat}$. In particular, the latter preserves small colimits. On the other hand, for C small, Corollary 12.8.5 tells us that

$$\langle C \rangle_{\mathrm{Cat}} = \mathrm{PSh}(C) .$$

Since the embedding $\mathrm{Cat} \hookrightarrow \mathrm{Cat}'$ preserves small colimits, and since small colimits preserving functors are stable under composition, this proves the lemma. □

Lemma 21.2.10. *For any elementary subuniverse κ and finite poset P, the category* $\mathrm{Fun}(P^{\mathrm{op}}, \mathrm{An}^{\kappa})$ *agrees with the category of κ-compact objects of* $\mathrm{PSh}(P)$ *and we habe*

$$\langle P \rangle_{\mathrm{Cat}} = \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{An}^{\kappa}) .$$

Proof. The first assertion is an easy consequence of Lemma 18.6.12. The second assertion follows from the first and from the very construction of $\langle P \rangle_{\mathrm{Cat}}$. □

Lemma 21.2.11. *The category* $\mathrm{PSh}(A)$ *is* Cat*-compact in* $\mathrm{Cat}'_{\mathrm{colim}}$ *for any small category A.*

Proof. The forgetful functor

$$\mathrm{Cat}'_{\mathrm{colim}} \hookrightarrow \mathrm{Cat}'$$

commutes with Cat-filtered colimits and any finite poset is compact in Cat'. Therefore, the previous lemma for $\kappa = \mathrm{Cat}$ implies that $\mathrm{PSh}(P)$ is a Cat-compact object of $\mathrm{Cat}'_{\mathrm{colim}}$ for any finite poset P. Since any small category is a small colimit of finite posets, Lemma 21.2.9 and Remark 17.2.23 achieve the proof. □

Lemma 21.2.12. *The category* Pr^{L} *is stable under small colimits in* $\mathrm{Cat}'_{\mathrm{colim}}$.

Proof. Let $i \mapsto C_i$ a functor indexed by a small category I and taking values in Pr^{L}. For each $i \colon I^{\simeq}$, there is an elementary subuniverse $\kappa(i)$ in Cat such that C_i is $\kappa(i)$-presentable. We may assume given an elementary subuniverse κ of Cat such that I as well as each $(C_i)^{\kappa(i)}$ are κ-small. We may also choose κ so that any small category that is $\kappa(i)$-small for some i is κ-small. In this case, C_i is κ-presentable for all i. Possibly after enlarging κ, we may assume that the diagram $i \mapsto C_i$ takes its values in $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$. Let C be the colimit of the diagram $i \mapsto C_i$ in $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$. We may also assume that $\mathbb{N}$ is κ-small, so that any category with κ-small colimits is idempotent-complete as well:

$$\mathrm{RexCat}_{\kappa} = \mathrm{Pr}^{\mathrm{L}}_{\kappa} .$$

Let $\mathrm{RexCat}'_{\kappa}$ be the category of large categories with κ-small colimits. The explicit description of colimits in $\mathrm{RexCat}'_{\kappa}$ given in the proof of Lemma 17.3.6 show that both embeddings

$$\mathrm{RexCat}_{\kappa} \hookrightarrow \mathrm{RexCat}'_{\kappa} \quad \text{and} \quad \mathrm{RexCat}'_{\kappa} \hookrightarrow \mathrm{Cat}'_{\mathrm{colim}}$$

commute with small colimits. Therefore C is the colimit of $i \mapsto C_i$ in $\mathrm{Cat}'_{\mathrm{colim}}$. □

Theorem 21.2.13. *Let C be a (large) category. The following conditions are equivalent.*

(i) The category C is presentable.

(ii) There is a small category A as well as a small family of maps S between presheaves on A with values in small animae and an equivalence between C and the left Bousfield localization of $\mathrm{PSh}(A)$ *by S.*

(iii) The category C is Cat*-compact as an object of* $\mathrm{Cat}'_{\mathrm{colim}}$.

Proof. The equivalence between assertions (i) and (ii) follows from Corollary 21.2.7 and Lemma 21.2.8. Since Cat-compact objects ar stable under small colimits, in particular under pushouts, it follows from Lemma 21.2.11 and Lemma 21.2.12 that any left Bousfield localization of a category of presheaves on a small category is Cat-compact. Therefore, any presentable category is Cat-compact. The full subcategory of Cat-compact objects of the Cat-presentable (very large) category $\mathrm{Cat}'_{\mathrm{colim}}$ is the smallest full subcategory containing $\mathrm{PSh}([1]) = \langle [1] \rangle_{\mathrm{Cat}}$ stable under small colimits. Therefore, Lemma 21.2.12 and Lemma 21.2.11 also imply that any Cat-compact object of $\mathrm{Cat}'_{\mathrm{colim}}$ is a presentable category. This shows that assertions (ii) and (iii) are equivalent and thus ends the proof of the theorem. □

Corollary 21.2.14. *The category* Pr^{L} *is the full subcategory of* Cat*-compact objects in the* Cat*-presentable category* $\mathrm{Cat}'_{\mathrm{colim}}$ *of large categories with small colimits and small colimits preserving functors between them. In particular,* Pr^{L} *has small colimits and the embedding functor*

$$\mathrm{Pr}^{\mathrm{L}} \hookrightarrow \mathrm{Cat}'_{\mathrm{colim}}$$

commutes with them.

Corollary 21.2.15. *Any large category with small colimits is a large* Cat*-filtered colimit of presentable categories with cocontinuous transition maps.*

Proposition 21.2.16. *Let C be a presentable category and D a locally small category with small colimits. A functor $\varphi\colon C \to D$ has a right adjoint if and only if it commutes with small colimits.*

Proof. It is clear that φ commutes with colimits if it has a right adjoint. Conversely, let us assume that φ commutes with small colimits. By virtue of Theorem 11.2.13 and of Lemma 21.2.8 it suffices to prove the proposition in a context where we can choose a small category A and a small family of morphisms S in $\mathrm{PSh}(A)$ such that C is the left Bousfield localization of the category of presheaves on A by S. We identify C with the full subcategory of S-local objects in the category of presheaves on A. We denote by

$$\gamma\colon \mathrm{PSh}(A) \to C$$

be the left adjoint of the obvious full embedding. A particular case of Lemma 17.3.6 with $\kappa = \mathrm{Cat}$ tells us that the functor $f := \varphi \circ \gamma$ commutes with small colimits. Therefore, by virtue of Corollary 12.8.5, the functor f has a right adjoint g. We claim that the functor g takes values in S-local objects: for any map $u\colon x \to y$ in S and any object z of D we have the map

$$u^*\colon \mathrm{Hom}(y, g(z)) \to \mathrm{Hom}(x, g(z))$$

is isomorphic to the map

$$f(u)^*\colon\ \mathrm{Hom}(f(y), z) \to \mathrm{Hom}(f(x), z)$$

hence is an isomorphism since f factorizes through the functor γ and thus sends terms of S to isomorphisms of C. Therefore, there is a unique functor

$$\psi\colon D \to C$$

whose composition with the full embedding $i\colon C \hookrightarrow \mathrm{PSh}(A)$ yields g. For x in C and y in D we thus have

$$\begin{aligned}\mathrm{Hom}(\varphi(x), y) &= \mathrm{Hom}(f(i(x)), y)\\ &= \mathrm{Hom}(i(x), g(y))\\ &= \mathrm{Hom}(i(x), i(\psi(y)))\\ &= \mathrm{Hom}(x, \psi(y))\end{aligned}$$

□

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