Lectures on

Higher Topos Theory

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Abstract

This course surveys the theory of ∞-topoi. After introducing the basic notions (descent, univalence, modalities, ...), we discuss general constructions (limits and colimits, congruences, parametrization, Grothendieck topologies) and a range of examples, including Goodwillie calculus, finitary functors, geometric structures and schemes in Lurie's framework, and analytic stacks in the sense of Clausen–Scholze.

Preface

These are lecture notes based on the course *Higher Topos Theory* taught by Marc Hoyois at the University of Regensburg in the winter term 2025/2026, written by Bastiaan Cnossen. The presentation is at times different from the original lectures. The notations and terminology follow the preferred conventions of the note-taker, which usually match those of the lecturer. The following are exceptions:

- We say 'animae' for the plural of 'anima'.
- The presheaf category of a category C is denoted $\mathrm{PSh}(C)$ rather than $P(C)$.
- Given topoi S and T, we refer to a left exact colimit-preserving functor $f^* \colon S \to T$ as a *morphism of logoi* rather than a *geometric morphism*. We refer to the right adjoint $f_* \colon T \to S$ as a *morphism of topoi*. We write Logos and Topos for the resulting (non-full) subcategories of Cat, so that $\mathrm{Logos} \simeq \mathrm{Topos}^{\mathrm{op}}$. This terminology is due to Anel and Joyal [AJ21].

1 Introduction

Topos theory grew out of Grothendieck's work on étale cohomology. Its basic objects are categories of sheaves, which encode how local data can be glued. Higher topos theory replaces set-valued sheaves by sheaves of animae, so that both the objects and their descent data retain their full homotopy-coherent structure. Consequently, the resulting notion of topos carries an internal homotopy theory, including truncations and connected covers, loop and delooping constructions, homotopy group objects, and Eilenberg–MacLane objects. In this way, the higher-categorical notion of a topos directly captures the cohomology theory, without a need to talk about derived functors.

In these notes, we take this higher-categorical notion as the primary one. Classical topos theory appears as the 0-truncated part of the higher theory: Given a topos T, the full subcategory $T_{\leq 0}$ of 0-truncated objects is a classical topos. We accordingly use "topos" for the higher notion and "classical topos" when we specifically mean the classical one.

As a rough guide, higher topos theory may be viewed as classical topos theory with sets replaced by animae:

$$\text{Higher topos theory} \quad \sim \quad \text{Classical topos theory with sets replaced by animae.}$$

As will become clear during the course, this analogy does not capture the entire theory, and the left-hand side is a lot richer than the right-hand side suggests. For example, higher topoi support approximation procedures, such as Postnikov and Goodwillie towers, which cannot be seen within the classical topos $T_{\leq 0}$ alone.

Descent

The organizing idea in topos theory is *descent*: reconstructing global objects from local data together with gluing information.

Consider first the familiar setting of vector bundles on a manifold M. Given an open cover $\{U_i\}$ of M, a vector bundle on M is completely determined by its restrictions to each open set U_i, together with transition functions on the overlaps $U_i \cap U_j$ satisfying the usual cocycle condition on triple overlaps. In categorical terms, the category $\mathrm{Vect}(M)$ of vector bundles on M satisfies descent for open covers.

This same pattern appears in algebraic geometry. A scheme is built by gluing affine pieces $\mathrm{Spec}(R)$ along Zariski open subschemes. For an affine scheme $\mathrm{Spec}(R)$, a quasi-coherent sheaf is simply an R-module. For a general scheme X obtained by gluing affines, the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X can be defined via descent: we specify compatible R-modules for all

affine maps $\mathrm{Spec}(R) \to X$, where these modules must satisfy gluing conditions on overlaps. The assignment $X \mapsto \mathrm{QCoh}(X)$ thus satisfies descent for affine covers, by construction.

These examples motivate the classical notion of a *Grothendieck topos*. One starts with a small category C of "local models" (such as open subsets of a topological space, or affine schemes), equips it with a notion of "coverings" generalizing open covers, and forms the category of sheaves on C with respect to these coverings. This sheaf category is a left exact localization of the presheaf category of C. The presheaf category freely adds colimits to the local models, while the localization imposes the relations expressing that covering families glue effectively.

But the presentation in terms of a category C with coverings is not intrinsic to the resulting topos: different choices of $(C, \text{coverings})$ can give rise to equivalent sheaf categories. The invariant content is the descent property itself, and this can be recognized directly on the category. An object parametrized by $X \in T$ is an object of the slice category $T_{/X}$. If X is exhibited as the colimit of a diagram $X_\bullet$, such an object restricts to a compatible family of objects parametrized by the X_i; descent says that every compatible family arises uniquely in this way. Equivalently, the assignment $X \mapsto T_{/X}$ takes colimits in T to limits of categories. This leads to the definition of a topos as a presentable category satisfying descent for all colimits.

The geometric viewpoint

Even though a topos is defined as a category satisfying descent, we often think of it as a geometric object in its own right. This viewpoint is justified already by the basic example coming from topology.

Indeed, every topological space X gives rise to a topos $\mathrm{Shv}(X)$, its category of sheaves of animae. This assignment determines a functor $\mathrm{Top} \to \mathrm{Topos}$, which is fully faithful when restricted to sober spaces, including all Hausdorff spaces. Thus we may "do topology" by studying topoi rather than topological spaces. In fact, the terminology "topos" itself was chosen by the Grothendieck school to reflect the view that topology is fundamentally the study of topoi [McL90].

Classical geometric structures generalize naturally to this setting. For instance, the notion of a *locally ringed space*, a topological space equipped with a structure sheaf of rings, extends to that of a *locally ringed topos*. One can develop a theory of schemes as certain locally ringed topoi, recovering classical algebraic geometry while allowing for higher categorical phenomena.

Taking the geometric viewpoint seriously leads to a natural convention regarding morphisms. Every continuous map $f : X \to Y$ of topological spaces induces a pullback functor $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$. This functor is left exact and preserves all colimits; in other words, it preserves the descent structure on the sheaf categories. Given topoi S and T, a natural class of functors $f^* : S \to T$ between them is the class of left exact colimit-preserving functors. Following the geometric viewpoint, we treat such a functor as a morphism of topoi from T to S, rather than the other way around.

Content

The notes follow three intertwined developments. The first concerns the intrinsic structure of a single topos. In Chapter 2, we formulate descent as an exactness property of colimits, identify effective epimorphisms as the intrinsic covers, and prove the Giraud and presheaf-localization characterizations of topoi. Chapter 3 then develops the internal homotopy theory made possible by this

exactness: connected–truncated factorization systems, homotopy group objects, hypercompletion, and gerbes.

The second development concerns morphisms, quotients, and approximation procedures. In Chapter 4, the algebraic perspective of logoi leads to presentations by generators and relations and hence to limits and colimits of topoi, while the geometric perspective identifies étale topoi over T with objects of T. Chapter 5 studies fiberwise factorization systems and their relation to left exact localizations. The generalized Blakers–Massey theorem, the description of generated congruences by iterated diagonals, and products of modalities culminate in the Goodwillie tower of a topos.

The third development applies the general theory and explores further structural and geometric topics. Chapter 6 treats sites, groupoid actions and principal bundles, hypercovers, convergence and dimension, shape, coherence and pretopoi, exponentiable topoi, and parametrized objects. Finally, Chapter 7 explains how schemes and other locally defined geometric objects are encoded by structured and fractured topoi, how six-functor formalisms produce such structures, and how these ideas appear in analytic stacks and Gestalten.

Appendices. Chapter A collects background on factorization systems and saturated classes used throughout the text. Chapter B records basic facts about limits and colimits in categories of sections of cartesian and cocartesian fibrations.

2 Foundations

The definition of a topos isolates a strong exactness property of colimits. Given a diagram $X_\bullet \colon I \to T$ in a category with pullbacks, there is a natural functor

$$T_{/\mathrm{colim}_i X_i} \longrightarrow \lim_{i \in I^{\mathrm{op}}} T_{/X_i}.$$

The target describes compatible objects over the pieces X_i, while the source describes objects over the object obtained by gluing these pieces together. The category T satisfies *descent* if this functor is an equivalence for every small diagram. A topos is, by definition, a presentable category in which all colimits satisfy descent.

We begin in Section 2.1 by unpacking this condition. Descent separates into two complementary assertions: colimits are *universal*, meaning that they are preserved by base change, and *effective*, meaning that compatible diagrams over the pieces glue to diagrams over the colimit. This formulation makes descent amenable to categorical arguments and shows, in particular, that left exact localizations of topoi are again topoi.

The basic covers in a topos are the *effective epimorphisms*. Their targets are recovered as colimits of their Čech nerves, which are groupoid objects encoding all iterated overlaps of the cover. In Section 2.2, we use the effectivity and universality of groupoid colimits to study these maps, prove the epi–mono factorization system, and establish the delooping principle for group objects. This is the first indication that the exactness axioms of a topos support an internal homotopy theory.

Finally, Section 2.3 compares the descent definition with several intrinsic and extrinsic characterizations. For a presentable category, descent is equivalent to the Giraud axioms, to the combination of universal colimits with an object classifier, and to being a left exact localization of a presheaf category. The last description is especially useful: it reduces many statements about arbitrary topoi to statements about presheaf topoi, and ultimately to the topos An of animae. We conclude by recording general exactness properties of topoi and constructing the subobject classifier as a special case of a classifying object for a local class of morphisms.

2.1 Topoi and descent

The definition of topoi relies on the concept of *descent*.

Definition 2.1. Let I be a small category, and let T be a category with pullbacks and I-indexed colimits. We say that T *satisfies descent for I-indexed colimits*, or that *I-indexed colimits are van Kampen*, if the functor

$$T^{\mathrm{op}} \to \mathrm{Cat}, \qquad X \mapsto T_{/X}$$

takes I-indexed colimit diagrams in T to limit diagrams in Cat. Explicitly, for every diagram $X_\bullet \colon I \to T$ the canonical functor

$$T_{/\operatorname{colim}_{i\in I} X_i} \longrightarrow \lim_{i\in I^{\mathrm{op}}} T_{/X_i}$$

is an equivalence.

Definition 2.2. A *topos* is a presentable category satisfying descent for all colimits. We will alternatively refer to a topos as a *logos*.

Let us unwind this definition. Given a small category I and an I-indexed diagram $X_\bullet \colon I \to T$, set $X := \operatorname{colim}_i X_i$. The objects of the limit $\lim_{i\in I^{\mathrm{op}}} T_{/X_i}$ are families of objects over the X_i equipped with compatible base-change identifications. Descent asserts that every such family is obtained uniquely from an object over X.

To understand this condition better, we provide an alternative description of this limit in terms of cartesian natural transformations.

Definition 2.3. Given two functors $F, G \colon I \to T$, a natural transformation $\alpha \colon F \to G$ is called *cartesian* if for every morphism $i \to j$ in I the naturality square

$$\begin{array}{ccc} F(i) & \xrightarrow{\alpha(i)} & G(i) \\ \downarrow & \lrcorner & \downarrow \\ F(j) & \xrightarrow{\alpha(j)} & G(j) \end{array}$$

is a pullback square. We denote by $\mathrm{Fun}^{\mathrm{cart}}(I,T) \subseteq \mathrm{Fun}(I,T)$ the wide subcategory spanned by the cartesian natural transformations.

Given a diagram $X_\bullet \colon I \to T$, evaluation at $i \in I$ determines a functor $\mathrm{ev}_i \colon \mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet} \to T_{/X_i}$. Furthermore, by definition of cartesianness these evaluation functors are compatible with base change, resulting in a comparison functor $\mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet} \to \lim_{i\in I^{\mathrm{op}}} T_{/X_i}$.

Lemma 2.4. *Let C be a category with pullbacks and let $X_\bullet \colon I \to C$ be a functor. Then the functor $\mathrm{Fun}^{\mathrm{cart}}(I,C)_{/X_\bullet} \to \lim_{i\in I^{\mathrm{op}}} C_{/X_i}$ is an equivalence.*

More informally, objects of the limit are given by I-indexed diagrams $Y_\bullet$ in C equipped with a cartesian natural transformation $Y_\bullet \to X_\bullet$.

Proof. Recall that a limit of a diagram $I^{\mathrm{op}} \to \mathrm{Cat}$ of categories may be computed as the category of cartesian sections of the cartesian unstraightening of the diagram. By definition, the functor $C_{/-} \colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is the cartesian straightening of the target map $t \colon \mathrm{Ar}(C) \to C$, so the unstraightening of $C_{/-}$ is t. It follows that the unstraightening of the functor $i \mapsto C_{/X_i}$ is the base change of t along $X_\bullet$:

$$\begin{array}{ccc} C/X_\bullet & \longrightarrow & \mathrm{Ar}(C) \\ \downarrow & \lrcorner & \downarrow t \\ I & \xrightarrow{X_\bullet} & C. \end{array}$$

The category of sections $I \to C/X_\bullet$ of this map is equivalent to the category of maps $I \to \mathrm{Ar}(C)$ whose target projection is $X_\bullet \colon I \to C$, which in turn is equivalent to the slice-category $\mathrm{Fun}(I,C)_{/X_\bullet}$.

It thus remains to show that a section $\sigma\colon I \to C/X_\bullet$ is cartesian precisely if the corresponding natural transformation $Y_\bullet \to X_\bullet$ is a cartesian natural transformation. But this is an immediate consequence of the fact that the t-cartesian morphisms in $\mathrm{Ar}(C)$ are precisely the pullback squares in C. □

With this alternative description of $\lim_i T_{/X_i}$ at hand, we can immediately deduce that the functor $T_{/X} \to \lim_i T_{/X_i}$ admits a left adjoint:

Lemma 2.5. *Let T be a category with pullbacks and I-indexed colimits, and let $X = \mathrm{colim}_i X_i$ be a colimit in T. Then the functor $T_{/X} \to \lim_i T_{/X_i}$ admits a left adjoint*

$$\mathrm{colim}\colon \lim_i T_{/X_i} \to T_{/X}$$

sending a cartesian transformation $Y_\bullet \to X_\bullet$ to its colimit $\mathrm{colim}_i Y_i \to \mathrm{colim}_i X_i$.

Proof. The colimit functor $\mathrm{colim}\colon \mathrm{Fun}(I,T) \to T$ admits a right adjoint $\mathrm{const}\colon T \to \mathrm{Fun}(I,T)$ sending an object Y to the constant functor on Y. By slicing this adjunction over an arbitrary diagram $X_\bullet \in \mathrm{Fun}(I,T)$, with colimit denoted X, we obtain another adjunction

$$\mathrm{colim}\colon \mathrm{Fun}(I,T)_{/X_\bullet} \rightleftarrows T_{/X},$$

see [Lur09, Proposition 5.2.5.1]. The right adjoint in this adjunction is given by the composite $T_{/X} \xrightarrow{\mathrm{const}} \mathrm{Fun}(I,T)_{/\mathrm{const}\, X} \to \mathrm{Fun}(I,T)_{/X_\bullet}$, with the second map given by base change along the colimit cocone $X_\bullet \to \mathrm{const}_X$. We now make the following two observations:

- The category $\mathrm{Fun}(I,T)_{/X_\bullet}$ contains the limit $\lim_{i\in I^{\mathrm{op}}} T_{/X_i} \simeq \mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet}$ as a full subcategory: given transformations $Y_\bullet \to Z_\bullet \to X_\bullet$, if both $Z_\bullet \to X_\bullet$ and $Y_\bullet \to X_\bullet$ are cartesian transformations, then so is $Y_\bullet \to Z_\bullet$ by the pasting law for pullback squares.
- Since any transformation $\mathrm{const}\, Y \to \mathrm{const}\, X$ between constant diagrams is cartesian, and cartesian transformations are closed under base change, the right adjoint $T_{/X} \to \mathrm{Fun}(I,T)_{/X_\bullet}$ lands in this full subcategory.

It follows that the adjunction restricts to an adjunction $\lim_i T_{/X_i} \rightleftarrows T_{/X}$, as desired. □

We can now characterize when the functor $T_{/X} \to \lim_i T_{/X_i}$ is an equivalence. By Lemma 2.5, this happens precisely when the unit and counit of the adjunction are natural isomorphisms. Explicitly, this means:

(1) **Counit:** For any map $Y \to X = \mathrm{colim}_i X_i$ in T, the canonical map $\mathrm{colim}_i (X_i \times_X Y) \to Y$ is an isomorphism.

(2) **Unit:** Given a cartesian transformation $Y_\bullet \to X_\bullet$ of functors $I \to T$, each of the maps $Y_i \to (\mathrm{colim}_i Y_i) \times_{\mathrm{colim}_i X_i} X_i$ is an isomorphism.

Since these conditions will frequently show up in the remainder of these notes, it is useful to give them names:

Definition 2.6 (Effective colimits)**.** Let T be a category with pullbacks and I-indexed colimits. We say that *I-indexed colimits in T are effective* if, for every cartesian transformation $Y_\bullet \to X_\bullet$ of functors $I \to T$, the extended natural transformation $\overline{Y}_\bullet \to \overline{X}_\bullet$ of colimit cocones $I^\triangleright \to T$ is again cartesian, i.e. for every $j \in I$ the commutative square

$$\begin{array}{ccc} Y_j & \longrightarrow & \operatorname{colim}_i Y_i \\ \downarrow & & \downarrow \\ X_j & \longrightarrow & \operatorname{colim}_i X_i \end{array}$$

is a pullback square.

Definition 2.7 (Universal colimits)**.** Let T be a category with pullbacks and I-indexed colimits. We say that *I-indexed colimits in T are universal* if, for every morphism $f\colon A \to B$, the pullback functor $f^*\colon T_{/B} \to T_{/A}$ preserves I-indexed colimits.

Lemma 2.8. *For a category T with pullbacks and I-indexed colimits, the following conditions are equivalent:*

(1) *I-indexed colimits are universal in T;*

(2) *For every diagram $X_\bullet\colon I \to T$, the counit of the adjunction $T_{/X} \leftrightarrows \lim_i T_{/X_i}$ is an isomorphism;*

(3) *For every diagram $X_\bullet\colon I \to T$, the functor $T_{/X} \to \lim_i T_{/X_i}$ is fully faithful.*

Proof. The equivalence between (2) and (3) is the standard characterization of a fully faithful right adjoint. If colimits are universal, then for every $Y \to X = \operatorname{colim}_i X_i$ we have

$$\operatorname{colim}_i (X_i \times_X Y) \xrightarrow{\sim} X \times_X Y \xrightarrow{\sim} Y,$$

so the counit is an isomorphism.

Conversely, assume (2), let $A \to B$ be a morphism, and let $X_\bullet$ be an I-indexed diagram in $T_{/B}$ with colimit $X \to B$. Apply (2) to the underlying diagram $X_\bullet$ and the object $A \times_B X \to X$. The resulting counit is

$$\operatorname{colim}_i \big(X_i \times_X (A \times_B X)\big) \xrightarrow{\sim} A \times_B X.$$

Since $X_i \times_X (A \times_B X) \xrightarrow{\sim} A \times_B X_i$, this says exactly that pullback along $A \to B$ preserves the given colimit. Hence I-indexed colimits are universal. □

With this terminology, we may summarize the above discussion as follows:

Corollary 2.9. *Let T be a category with pullbacks and I-indexed colimits. Then T satisfies descent for I-indexed colimits if and only if I-indexed colimits are both effective and universal.* □

In particular, we get:

Corollary 2.10. *A category T is a topos if and only if it is presentable and I-indexed colimits are both effective and universal for every small category I.* □

2.1.1 Examples of descent

Let us illustrate the descent condition by giving some examples.

Example 2.11 (Initial object)**.** Let T be a category with an initial object. Then the initial object $\emptyset$ is van Kampen if and only if it is *strictly initial*, meaning that any morphism $X \to \emptyset$ is an isomorphism.

Indeed, by definition $\emptyset$ is van Kampen if and only if the functor $T_{/\emptyset} \to *$ is an equivalence. This functor admits a fully faithful left adjoint $* \to T_{/\emptyset}$ hitting the identity $\mathrm{id}_\emptyset$. For the functor to be an equivalence, the counit $\emptyset \to X$ must be an isomorphism for every $X \in T_{/\emptyset}$, which is precisely the universality condition.

Example 2.12 (Coproducts)**.** For a category T with arbitrary coproducts, we say that *coproducts are disjoint* if for all $X, Y \in T$ the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \sqcup Y \end{array}$$

is a pullback square. We claim that arbitrary coproducts in T are van Kampen if and only if they are universal and T has disjoint coproducts.

By definition, coproducts are van Kampen if and only if they are universal and for every collection of maps $\{Y_i \to X_i\}_{i \in I}$ the commutative square

$$\begin{array}{ccc} Y_j & \longrightarrow & Y := \coprod_{i \in I} Y_i \\ \downarrow & & \downarrow \\ X_j & \longrightarrow & X := \coprod_{i \in I} X_i \end{array}$$

is a pullback square for every $j \in I$. To see that this reduces to disjointness, consider $X' := \bigsqcup_{i \in I \setminus \{j\}} X_i$ and $Y' := \bigsqcup_{i \in I \setminus \{j\}} Y_i$. We may write $X = X_j \sqcup X'$ and $Y = Y_j \sqcup Y'$. Invoking universality of coproducts once more shows that the condition reduces to disjointness of binary coproducts.

Example 2.13 (Group actions)**.** This example previews the groupoid objects introduced in Section 2.2.1. Informally, a groupoid object is a simplicial object encoding objects, invertible morphisms between them, and all the coherences for composition. A group object is the special case in which the object of objects is terminal.

Let T be a topos, and let G be a group object in T, regarded as a groupoid object $G_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to T$ satisfying $G_0 = *$. Let $\mathbf{B}G := \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} G_n$. Then there is an equivalence

$$T_{/\mathbf{B}G} \quad \simeq \quad \lim_{[n] \in \mathbf{\Delta}} T_{/G^n} \quad =: \quad \mathrm{Mod}_G(T).$$

In other words, a *module over* G is a simplicial object $X_\bullet$ with a cartesian transformation $X_\bullet \to G_\bullet$, which we can depict (suppressing degeneracies) as a simplicial diagram of face maps:

$$\begin{array}{ccccc} \cdots & \overset{d_2}{\underset{d_0}{\overset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}}} & X_1 & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & X_0 \\ \downarrow & & \downarrow & & \downarrow \\ \cdots & \overset{d_2}{\underset{d_0}{\overset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}}} & G & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & * \end{array}$$

with each square cartesian. In particular, this implies $X_1 \simeq G \times X_0$. The two maps $G \times X_0 \simeq X_1 \to X_0$ can then be interpreted as the projection map and an *action map*. The rest of the simplicial diagram $X_\bullet$ records the higher coherences of this G-action on X_0.

Example 2.14. Let T be a topos. Given an anima $A \in \mathrm{An}$, we obtain an equivalence

$$\mathrm{Fun}(A, T) \quad = \quad \lim_A T_{/*} \quad \simeq \quad T_{/\operatorname{colim}_A *}.$$

We refer to this equivalence as the *Grothendieck construction*.

Exercise 2.15. Figure out which colimits are van Kampen in the following categories:

- The category Set of sets, the category $\mathrm{An}_{\leq n}$ of n-truncated animae;
- A stable cocomplete category;
- The opposite category $\mathrm{An}^{\mathrm{op}}$.

2.1.2 Descent via pullback squares

The descent condition can be reformulated in terms of colimits in the category $\mathrm{Ar}^{\mathrm{pb}}(T)$ of pullback squares in T. This is defined as the wide subcategory of $\mathrm{Ar}(T)$ whose morphisms are the pullback squares in T.

Proposition 2.16 (Rezk)**.** *A presentable category T is a topos if and only if $\mathrm{Ar}^{\mathrm{pb}}(T)$ admits colimits and the inclusion $\mathrm{Ar}^{\mathrm{pb}}(T) \hookrightarrow \mathrm{Ar}(T)$ preserves colimits.*

Proof. First assume that T is a topos. We claim that the subcategory $\mathrm{Ar}^{\mathrm{pb}}(T) \hookrightarrow \mathrm{Ar}(T)$ is closed under colimits. To see this, note that a functor $I \to \mathrm{Ar}^{\mathrm{pb}}(T)$ corresponds under currying to a cartesian natural transformation $Y_\bullet \to X_\bullet$. Passing to colimits in T extends this to a natural transformation of colimit cocones $\overline{Y}_\bullet \to \overline{X}_\bullet$. By effectivity of I-indexed colimits, this remains a cartesian natural transformation, hence corresponds to a cocone $I^\triangleright \to \mathrm{Ar}^{\mathrm{pb}}(T)$.

We must show this is a colimit cocone. In other words, we must show that a commutative square in T of the form

$$\begin{array}{ccc} \operatorname{colim}_i Y_i & \longrightarrow & A \\ \downarrow & & \downarrow \\ \operatorname{colim}_i X_i & \longrightarrow & B \end{array}$$

is a pullback square if and only if each of the composite squares

$$\begin{array}{ccccc} Y_i & \longrightarrow & \operatorname{colim}_i Y_i & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ X_i & \longrightarrow & \operatorname{colim}_i X_i & \longrightarrow & B \end{array}$$

is a pullback square. This is an immediate consequence of universality of colimits.

For the converse, assume that $\mathrm{Ar}^{\mathrm{pb}}(T)$ admits colimits and the inclusion preserves them. The fact that colimit cocones in $\mathrm{Ar}^{\mathrm{pb}}(T)$ agree with colimit cocones in $\mathrm{Ar}(T)$ immediately implies effectivity of colimits.

To prove universality, fix a map $A \to B$ and a diagram $X_\bullet$ in $T_{/B}$ with colimit $X \to B$. The pullback arrows $A \times_B X_i \to X_i$ form a diagram in $\mathrm{Ar}^{\mathrm{pb}}(T)$, and the maps to $A \to B$ form a cocone in $\mathrm{Ar}^{\mathrm{pb}}(T)$. Its colimit in $\mathrm{Ar}(T)$ is the arrow

$$\mathrm{colim}_i(A \times_B X_i) \longrightarrow X.$$

Since the inclusion $\mathrm{Ar}^{\mathrm{pb}}(T) \hookrightarrow \mathrm{Ar}(T)$ preserves this colimit, the induced morphism from this arrow to $A \to B$ is a pullback square. Consequently,

$$\mathrm{colim}_i(A \times_B X_i) \xrightarrow{\sim} A \times_B X,$$

which proves universality. □

Corollary 2.17. *Let T be a topos and let $L\colon T \to T'$ be a left exact functor localization. Then also T' is a topos.*

Proof. Let $R\colon T' \hookrightarrow T$ denote the fully faithful right adjoint to L. Then this adjunction induces an adjunction

$$L_*\colon \mathrm{Ar}(T) \rightleftarrows \mathrm{Ar}(T')\colon R_*$$

on arrow categories. Moreover, since both L and R preserve pullback squares, this adjunction restricts to an adjunction

$$\mathrm{Ar}^{\mathrm{pb}}(T) \rightleftarrows \mathrm{Ar}^{\mathrm{pb}}(T').$$

The claim now follows from Proposition 2.16, as $\mathrm{Ar}^{\mathrm{pb}}(T')$ inherits its colimits from $\mathrm{Ar}^{\mathrm{pb}}(T)$ and these are compatible with the ones in $\mathrm{Ar}(T')$. □

2.2 Effective epimorphisms

We will now introduce an important class of morphisms $X \twoheadrightarrow Y$ in a topos, known as the *effective epimorphisms*. There are various ways of thinking about such morphisms:

(1) They are higher categorical analogues of the effective epimorphisms from 1-category theory, i.e. those morphisms that exhibit Y as the quotient of X at the equivalence relation $X \times_Y X \subseteq X \times X$;

(2) They are those morphisms for which the pullback functor $T_{/Y} \to T_{/X}$ is conservative, allowing us to check various conditions of maps in $T_{/X}$ after pulling back to Y;

(3) They are the topos-theoretic analogues of the *covers* in a Grothendieck site.

Following Lurie, we take perspective (1) as our definition. To make this precise, we first recall the notion of groupoid objects.

2.2.1 Groupoid objects

Recall that a *(classical) groupoid* is a classical category $\mathcal{G}$ in which all morphisms are invertible. Concretely, it consists of a set $\mathcal{G}_0$ of objects, a set $\mathcal{G}_1$ of morphisms, source/target maps $s, t\colon \mathcal{G}_1 \to \mathcal{G}_0$, unit and inverse maps $e\colon \mathcal{G}_0 \to \mathcal{G}_1$ and $i\colon \mathcal{G}_1 \to \mathcal{G}_1$, and a composition map $m\colon \mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1 \to \mathcal{G}_1$, satisfying the expected relations.

The higher categorical analogue is the following:

Definition 2.18. Let C be a category. A simplicial object $\mathcal{G}\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is called a *groupoid object* if for every $[n] \in \mathbf{\Delta}$ and every (not necessarily order-preserving) partition

$$[n] \simeq \{i_0, \dots, i_k\} \sqcup_{\{i_k\}} \{i_k, \dots, i_n\},$$

the induced diagram

$$\begin{array}{ccc} \mathcal{G}_n & \xrightarrow{(i_0,\dots,i_k)^*} & \mathcal{G}_k \\ {\scriptstyle (i_k,\dots,i_n)^*}\downarrow & \lrcorner & \downarrow{\scriptstyle i_k^*} \\ \mathcal{G}_{n-k} & \xrightarrow{i_k^*} & \mathcal{G}_0 \end{array}$$

is a pullback square in C. We let $\mathrm{Grpd}(C) \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ denote the full subcategory spanned by the groupoid objects in C.

Exercise 2.19. Show that a groupoid object in Set is nothing but a classical groupoid.

For the purposes of topos theory, we may think of groupoid objects as higher categorical analogues of equivalence relations. Given a groupoid $\mathcal{G}$ in the category of sets, the image of the map $(s,t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is an equivalence relation on $\mathcal{G}_0$. Indeed, reflexivity follows from the identity maps, transitivity follows from composition of morphisms, and symmetry follows by inverting morphisms. Conversely, equivalence relations $R \subseteq X \times X$ on a set X correspond bijectively to groupoids in sets $\mathcal{G}$ such that $\mathcal{G}_0 = X$ and the map $(s,t)\colon \mathcal{G}_1 \hookrightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is injective. Under this correspondence, the colimit $|\mathcal{G}|$ of the groupoid corresponds to the quotient X/R.

Given a groupoid object $\mathcal{G}_\bullet$ in C with $X = \mathcal{G}_0$, we similarly think of the colimit $|\mathcal{G}_\bullet|$ as a "quotient of X by the relations contained in $\mathcal{G}$". Unlike an equivalence relation, a groupoid may have several morphisms between two objects, as well as nontrivial automorphisms. This additional structure contributes nontrivially to the quotient object.

Lemma 2.20 ([Lur09, Proposition 6.1.2.11])**.** *Let C be a category and consider an augmented simplicial object $U_\bullet^+\colon \mathbf{\Delta}_+^{\mathrm{op}} \to C$. The following conditions are equivalent:*

(1) *The diagram $U_\bullet^+$ is right Kan extended from $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}} \subseteq \mathbf{\Delta}_+^{\mathrm{op}}$;*

(2) *The simplicial object $U_\bullet := U_\bullet^+|_{\mathbf{\Delta}}$ is a groupoid object and the square*

$$\begin{array}{ccc} U_1 & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & U_{-1} \end{array}$$

is a pullback square.

Proof. Assume first that $U_\bullet^+$ is right Kan extended from $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}$. The pointwise formula for right Kan extension gives canonical isomorphisms

$$U_n \xrightarrow{\sim} \underbrace{U_0 \times_{U_{-1}} \cdots \times_{U_{-1}} U_0}_{n+1 \text{ factors}}$$

for every $n \geq 0$. Thus $U_\bullet$ is the Čech nerve of $U_0 \to U_{-1}$. The displayed square in the statement is the case $n = 1$, and the remaining pullback identities show that $U_\bullet$ is a groupoid object.

Conversely, assume that $U_\bullet$ is a groupoid object and that the displayed square is a pullback square. The latter identifies

$$U_1 \xrightarrow{\sim} U_0 \times_{U_{-1}} U_0.$$

Applying the groupoid condition successively to the ordered decompositions of $[n]$ into adjacent intervals gives

$$U_n \xrightarrow{\sim} \underbrace{U_1 \times_{U_0} \cdots \times_{U_0} U_1}_{n \text{ factors}} \xrightarrow{\sim} \underbrace{U_0 \times_{U_{-1}} \cdots \times_{U_{-1}} U_0}_{n+1 \text{ factors}}.$$

These isomorphisms are induced by the vertex maps $[0] \to [n]$ and are therefore compatible with the simplicial structure. Hence $U_\bullet^+$ agrees with the Čech nerve of $U_0 \to U_{-1}$, which is precisely the right Kan extension of its restriction to $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}$. □

The definitions of universal/effective colimits specialize to groupoids as follows:

Definition 2.21. Let C be a category with pullbacks and geometric realizations.

(1) A geometric realization $X = |X_\bullet| := \mathrm{colim}_n X_n$ is called a *groupoid colimit* if the simplicial object $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is a groupoid.

(2) We say *groupoid colimits are effective in C* if for every cartesian transformation $Y_\bullet \to X_\bullet$ of groupoid objects, the commutative square

$$\begin{array}{ccc} Y_0 & \longrightarrow & |Y_\bullet| \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \longrightarrow & |X_\bullet| \end{array}$$

is a pullback square.

(3) We say that *groupoid colimits are universal* if for every morphism $f \colon A \to B$ in C, the pullback functor $f^* \colon C_{/B} \to C_{/A}$ preserves groupoid colimits.

Lemma 2.22. *Let C be a category with pullbacks and geometric realizations. Then C satisfies descent for groupoid colimits if and only if groupoid colimits are universal and effective.*

Proof. This follows as in Corollary 2.9. □

Corollary 2.23. *Groupoid colimits are effective in a topos.* □

2.2.2 Effective epimorphisms and Čech nerves

In 1-category theory, a morphism $f \colon U \to X$ is called an *effective epimorphism* if it exhibits its target as the quotient of its source at the equivalence relation $U \times_X U$ determined by f. Equivalently, X is the coequalizer of the two projection maps $U \times_X U \rightrightarrows U$.

The definition in the higher categorical setting is similar. We replace the equivalence relation $U \times_X U$ by a certain groupoid object $\check{C}_\bullet(f)$ called the *Čech nerve of f*. In degrees ≤ 1, this groupoid is given by the two projection maps $U \times_X U \rightrightarrows U$.

Construction 2.24 (Čech nerve)**.** Let C be a category with pullbacks. For a morphism $f\colon U \to X$ in C, we define its *Čech nerve* $\check{C}_\bullet(f) \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ as the image of $f \in \mathrm{Ar}(C)$ under the following functor:

$$\mathrm{Fun}([1], C) \simeq \mathrm{Fun}((\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}, C) \xrightarrow{j_*} \mathrm{Fun}(\mathbf{\Delta}_+^{\mathrm{op}}, C) \xrightarrow{i^*} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C).$$

Here $\mathbf{\Delta}_+$ is the augmented simplex category, $i\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_+$ is the canonical inclusion, $j\colon \mathbf{\Delta}_+^{\leq 0} \hookrightarrow \mathbf{\Delta}_+$ is the inclusion of the objects $[-1]$ and $[0]$, and $[1] \simeq (\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}$ is the canonical equivalence sending 0 to $[0]$ and 1 to $[-1]$. The pointwise formula for the right Kan extension j_* shows that the Čech nerve is given by

$$\check{C}_n(f) \xrightarrow{\sim} U^{\times_X^{n+1}} = \underbrace{U \times_X U \times_X \cdots \times_X U}_{n+1 \text{ times}}.$$

This computation also shows that the right Kan extension functor j_* exists.

Notation 2.25. Given a morphism $f\colon U \to X$, we will write $\check{C}^+_\bullet(f)$ for the augmented simplicial diagram $j_*(f)\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to C$, and refer to this as the *augmented Čech nerve of* f. Under the canonical equivalence between ${\mathbf{\Delta}_+}^{\mathrm{op}}$ and $(\mathbf{\Delta}^{\mathrm{op}})^\triangleright$, we may think of $\check{C}^+_\bullet(f)$ as encoding a cocone on $\check{C}_\bullet(f)$ with cone point X.

Lemma 2.26. *Let C be a category with pullbacks. Then the Čech nerve $\check{C}_\bullet(f)$ of any morphism f is a groupoid object.*

Proof. This is immediate from Lemma 2.20, since the augmented Čech nerve $\check{C}^+_\bullet(f)$ of f is by definition right Kan extended from $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}} \subseteq \mathbf{\Delta}_+^{\mathrm{op}}$. □

Definition 2.27. Let C be a category with pullbacks. A morphism $f\colon U \to X$ is called an *effective epimorphism* if the cocone $\check{C}^+_\bullet(f)$ is a colimit diagram, exhibiting X as the geometric realization $|\check{C}_\bullet(f)|$ of its Čech nerve. We denote by

$$\mathrm{EffEpi}(C) \subseteq \mathrm{Ar}(C)$$

the full subcategory spanned by the effective epimorphisms.

Warning 2.28. This terminology is standard but rather unfortunate: not every effective epimorphism is an epimorphism in the categorical sense, see Example 5.8 below for a counterexample. The terminology is a historical accident. In a 1-category, effective epimorphisms *are* always epimorphisms, but this fails in higher categories.

Lemma 2.29. *Let C be a category with pullbacks such that groupoid colimits are universal in C (e.g. C is a topos).*

(1) *A morphism $f\colon U \to X$ in C is an effective epimorphism if and only if the functor $f^*\colon C_{/X} \to C_{/U}$ is conservative.*

(2) *Effective epimorphisms are closed under base change.*

Proof. (1) Assume first that f is an effective epimorphism. By Lemma 2.26, $X \simeq \mathrm{colim}_n \check{C}_n(f)$ is a groupoid colimit. By our assumption on C, the functor $C_{/X} \hookrightarrow \lim_n C_{/\check{C}_n(f)}$ is fully faithful, hence conservative. In particular, the functors $C_{/X} \to C_{/\check{C}_n(f)}$ for $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ are jointly conservative. Since each of these functors factors through $C_{/U}$, we conclude that f^* itself must be conservative.

Conversely, assume that f^* is conservative. We must show that the map $\operatorname{colim}_n \check{C}_n(f) \to X$ is an isomorphism in C. Since f^* is conservative and preserves groupoid colimits, it suffices to show that $\operatorname{colim}_n \check{C}_n(f) \times_X U \to U$ is an isomorphism. This follows from the fact that the simplicial diagram $[n] \mapsto U^{\times_X^{(n+1)}} \times_X U \xrightarrow{\sim} U^{\times_X^{(n+2)}}$ admits an extra degeneracy.

(2) Consider a pullback square

$$\begin{array}{ccc} U' & \longrightarrow & U \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

such that f is an effective epimorphism. Since the functor $g^*\colon C_{/X} \to C_{/X'}$ preserves pullbacks, it sends $\check{C}_\bullet(f)$ to $\check{C}_\bullet(f')$. By assumption on C, it also preserves groupoid colimits, hence we see that the map $\operatorname{colim}_n \check{C}_n(f') \simeq (\operatorname{colim}_n \check{C}_n(f)) \times_X X' \to X \times_X X' = X'$ is an equivalence, implying that also f' is an effective epimorphism. □

Corollary 2.30. *Let C be a category with pullbacks such that groupoid colimits are universal in C (e.g. C is a topos). Consider a commutative diagram in C as follows:*

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \twoheadrightarrow & Y' & \longrightarrow & Y. \end{array}$$

Assume that the map $Y'' \to Y'$ is an effective epimorphism. If both the left-hand square and the outer rectangle are pullback squares, then so is the right-hand square.

Proof. We need to show that the map $X' \to X \times_Y Y'$ is an isomorphism. By the previous lemma, this may be tested after pullback along the effective epimorphism $Y'' \twoheadrightarrow Y'$. The claim now follows by applying the 2-out-of-3 property to the following commutative diagram:

$$\begin{array}{ccc} X'' & \xrightarrow{\simeq} & X' \times_{Y'} Y'' \\ {\scriptstyle\simeq}\downarrow & & \downarrow \\ X \times_Y Y'' & \xrightarrow{\simeq} & (X \times_Y Y') \times_{Y'} Y''. \end{array}$$

□

We now show that the category of effective epimorphisms in a topos is equivalent to the category of groupoids, via passage to the Čech nerve. We start with the following simple observation:

Lemma 2.31. *Let C be a category with pullbacks and geometric realizations. Then the functor $\check{C}_\bullet\colon \operatorname{Ar}(C) \to \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ admits a left adjoint*

$$\operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \xrightarrow{i_!} \operatorname{Fun}(\mathbf{\Delta}_+^{\mathrm{op}}, C) \xrightarrow{j^*} \operatorname{Fun}((\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}, C) \simeq \operatorname{Fun}([1], C),$$

sending a simplicial object $X_\bullet$ to the map $X_0 \to \operatorname{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} X_n$.

Proof. We have $\check{C}_\bullet(-) = i^* j_*$. The functor i^* has a left adjoint given by left Kan extension along $i\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_+^{\mathrm{op}}$, and the functor j_* has a left adjoint given by the restriction functor j^*. □

We may use this adjunction to reformulate the condition of effectivity of groupoids from Definition 2.21:

Lemma 2.32. *Let C be a category with pullbacks and geometric realizations, and assume that groupoid colimits are universal. Then the following three conditions are equivalent:*

(1) *The category C satisfies descent for groupoid colimits;*

(2) *Groupoid colimits are effective in C;*

(3) *For every groupoid object $\mathcal{G}$ the unit $\mathcal{G} \to \check{C}_\bullet(\mathcal{G}_0 \to |\mathcal{G}|)$ of the adjunction from Lemma 2.31 is an isomorphism.*

Proof. The equivalence between (1) and (2) is Lemma 2.22.

From effectivity to the Čech nerve condition. Assume (2), and consider a groupoid object $\mathcal{G}$. By setting $\mathcal{G}^+_{-1} := |\mathcal{G}| = \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}_n$, we extend $\mathcal{G}$ to an augmented simplicial object $\mathcal{G}^+_\bullet$. More precisely, this is the left Kan extension of $\mathcal{G}$ along $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}_+$. The first face maps define a transformation $\mathcal{G}^+_{\bullet+1} \to \mathcal{G}^+_\bullet$ whose restriction to $\mathbf{\Delta}^{\mathrm{op}}$ is cartesian by the groupoid condition. Effectivity therefore implies that the extended transformation is cartesian at the cone point. In particular, we obtain a pullback square

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}_{n+1} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{G}_0 & \longrightarrow & \mathcal{G}^+_{-1} = |\mathcal{G}|. \end{array}$$

The top right corner is isomorphic to $\mathcal{G}_0$, since the augmented simplicial object $[n] \mapsto \mathcal{G}_{n+1}$ over $\mathcal{G}_0$ admits extra degeneracies. By Lemma 2.20, the augmented object $\mathcal{G}^+_\bullet$ is therefore the Čech nerve of $\mathcal{G}_0 \to |\mathcal{G}|$, proving (3).

From the Čech nerve condition to effectivity. Assume (3). The map $\mathcal{G}_0 \to |\mathcal{G}|$ is then an effective epimorphism for every groupoid object $\mathcal{G}$, since its Čech nerve is $\mathcal{G}$ and its geometric realization is $|\mathcal{G}|$.

Consider a cartesian transformation $X_\bullet \to Y_\bullet$ of groupoid objects, and set $A := |X_\bullet|$ and $B := |Y_\bullet|$. We must prove that the canonical map

$$u\colon X_0 \longrightarrow A \times_B Y_0$$

is an isomorphism. The map $X_0 \to A$ is an effective epimorphism, so pullback along it is conservative by Lemma 2.29. After this base change, the map u becomes

$$X_0 \times_A X_0 \longrightarrow X_0 \times_B Y_0.$$

Condition (3) identifies the source with X_1 and identifies Y_1 with $Y_0 \times_B Y_0$. Cartesianness of $X_\bullet \to Y_\bullet$ then gives

$$X_1 \xrightarrow{\sim} X_0 \times_{Y_0} Y_1 \xrightarrow{\sim} X_0 \times_B Y_0.$$

Thus the base change of u is an isomorphism, and conservativity implies that u is an isomorphism. This proves effectivity of groupoid colimits. □

Lemma 2.33. *Let C be a category satisfying descent for groupoid colimits (e.g. a topos). Then the Čech nerve functor restricts to an equivalence*

$$\check{C}_\bullet\colon \operatorname{EffEpi}(C) \xrightarrow{\sim} \operatorname{Grpd}(C)$$

The inverse sends $\mathcal{G}$ to the map $\mathcal{G}_0 \to |\mathcal{G}|$.

Proof. Consider the adjunction $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \rightleftarrows \mathrm{Ar}(C)$ from Lemma 2.31. By Lemma 2.32, the unit transformation is an isomorphism on groupoid objects. Consequently, the restriction of the realization functor to $\mathrm{Grpd}(C)$ is fully faithful.

Its essential image consists precisely of those morphisms $f\colon U \to X$ for which the counit of the adjunction is an isomorphism. This counit takes the form of a commutative square

$$\begin{array}{ccc} U & = & U \\ \downarrow & & \downarrow f \\ \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{C}_n(f) & \longrightarrow & X, \end{array}$$

and by definition this is an isomorphism if and only if f is an effective epimorphism in C. □

Definition 2.34. Let T be a topos. A *pointed connected object* is an object X equipped with an effective epimorphism $* \twoheadrightarrow X$. We denote by $T_*^{\geq 1} \subseteq T_*$ the subcategory of pointed connected objects.

Proposition 2.35 (Delooping principle)**.** *Let T be a topos. Then there is an equivalence of categories*

$$\mathbf{B}\colon \mathrm{Grp}(T) \quad \rightleftarrows \quad T_*^{\geq 1} : \mathbf{\Omega}.$$

Proof. The equivalence $\mathrm{Grpd}(T) \simeq \mathrm{EffEpi}(T)$ from Lemma 2.33 sits in a commutative triangle

$$\begin{array}{ccc} \mathrm{Grpd}(T) & \xrightarrow{\simeq} & \mathrm{EffEpi}(T) \\ & {\scriptstyle G\mapsto G_0} \searrow \quad \swarrow {\scriptstyle (U\to X)\mapsto U} & \\ & T. & \end{array}$$

Taking fibers over $* \in T$ gives the claim. □

2.2.3 The epi-mono factorization system

In a topos, every morphism can be factored into an effective epimorphism followed by a monomorphism. This factorization is in fact unique. We formulate this precisely using *factorization systems*, which are recalled in Chapter A.

Proposition 2.36. *Let C be a category, and assume that groupoids are effective and universal. Then C admits a factorization system (E, M) with E given by the effective epimorphisms and M given by the monomorphisms in C.*

Proof. Existence of factorizations. Consider an arbitrary morphism $f\colon U \to X$. Define $V := \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{C}_n(f)$. The augmentation of the Čech nerve gives a factorization

$$U \xrightarrow{e} V \xrightarrow{m} X.$$

By Lemma 2.32, the groupoid object $\check{C}_\bullet(f)$ is the Čech nerve of its realization map e. Hence e is an effective epimorphism and $\check{C}_\bullet(e) \xrightarrow{\sim} \check{C}_\bullet(f)$.

The second map is a monomorphism. It suffices to show that the first projection $p_1\colon V \times_X V \to V$ is an isomorphism, since the diagonal is a section of p_1. Pullback along the effective epimorphism

$e\colon U \to V$ is conservative by Lemma 2.29(1). We may therefore test p_1 after this pullback. Universality of groupoid colimits gives

$$V \times_X U \xrightarrow{\sim} \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{C}_{n+1}(f).$$

The augmented simplicial object $[n] \mapsto \check{C}_{n+1}(f)$ over U admits extra degeneracies, starting with the diagonal $U \to U \times_X U = \check{C}_1(f)$. Its realization is therefore U, so $V \times_X U \to U$ is an isomorphism. Conservativity now implies that p_1 is an isomorphism and hence that m is a monomorphism.

Orthogonality. Consider a solid commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & A \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ X & \longrightarrow & B \end{array}$$

where f is an effective epimorphism and $i\colon A \hookrightarrow B$ is a monomorphism. Because i is a monomorphism, the anima of fillers is (-1)-truncated. It remains only to prove that it is nonempty.

Work in the slice $C_{/B}$. Composition with i defines a fully faithful functor $C_{/A} \hookrightarrow C_{/B}$, and this functor preserves colimits. The given map $U \to A$ shows that every term $\check{C}_n(f) = U^{\times_X^{n+1}}$ of the Čech nerve belongs to its essential image. Since X is the colimit of this Čech nerve in $C_{/B}$, the object $X \to B$ also belongs to the essential image. Thus there exists a map $X \to A$ giving a filler. The anima of fillers is consequently contractible, proving orthogonality. □

2.2.4 Properties of effective epimorphisms

We now prove some basic properties of effective epimorphisms. Throughout this subsection, we fix a category C in which groupoids are effective and universal.

Lemma 2.37. *Effective epimorphisms are closed under composition.*

Proof. The effective epimorphisms are the left class of a factorization system, which is closed under composition by Proposition A.3. □

Lemma 2.38. *Any morphism which has a section is an effective epimorphism.*

Proof. Assume $f\colon X \to Y$ admits a section $s\colon Y \to X$. Then the augmented Čech nerve $\check{C}^+_\bullet(f)\colon \mathbf{\Delta}^{\mathrm{op}}_+ \to C$ admits an extra degeneracy, hence is a colimit diagram. □

Corollary 2.39. *Any morphism which is both a monomorphism and an effective epimorphism is an isomorphism. In particular, every monomorphism with a section is an isomorphism.*

Proof. For the first claim, recall that if a category C is equipped with a factorization system (L, R), then any morphism $f\colon X \to Y$ in both L and R is an isomorphism (see Lemma A.4). The second claim follows immediately from the first combined with Lemma 2.38. □

Lemma 2.40. *Given a commutative triangle*

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ X & & \xrightarrow{gf} & & Z, \end{array}$$

if gf is an effective epimorphism then so is g.

Proof. We may factor the map g as

$$Y \xhookrightarrow{i_Y} Y \sqcup_X Z \xrightarrow{\langle g, \mathrm{id}_Z \rangle} Z,$$

where the first map is the inclusion of the Y-component and the second map is induced by the maps $g\colon Y \to Z$ and $\mathrm{id}_Z\colon Z \to Z$. The first map is an effective epimorphism, since it is a cobase change of the effective epimorphism $gf\colon X \to Z$. The second map is an effective epimorphism by Lemma 2.38 since the canonical map $Z \to Y \sqcup_X Z$ is a section. Their composite is g, so g is an effective epimorphism by Lemma 2.37. □

2.3 Characterizations and consequences

The descent definition is intrinsic, but it is not always the most convenient way to recognize a topos or to prove exactness results inside one. We first compare it with the Giraud axioms, object classifiers, and presentations by presheaves. We then extract several exactness properties that are useful in practice and study the classification of local classes of morphisms.

2.3.1 Characterizations of topoi

We now provide various alternative characterizations of topoi. Throughout this section, we fix a presentable category T.

Definition 2.41. An *object classifier* is an object $U \in \widehat{T} := \mathrm{Fun}^{\lim}(T^{\mathrm{op}}, \widehat{\mathrm{An}})$ such that $\mathrm{Hom}(X, U) \simeq (T_{/X})^{\simeq}$. Note that T admits an object classifier if and only if the assignment $X \mapsto (T_{/X})^{\simeq}$ preserves limits.

Theorem 2.42. *Let T be a presentable category. Then the following are equivalent:*

(1) *The category T is a topos.*

(2) *Colimits are universal and there exists an object classifier.*

(3) *The Giraud axioms are satisfied:*

- *Colimits are universal (Definition 2.7);*
- *Coproducts are disjoint (Example 2.12);*
- *Groupoids are effective (Definition 2.21, Lemma 2.32);*

(4) *There exists a small category C and a left exact localization* $\mathrm{PSh}(C) \to T$.

Proof. (1) $\implies$ (2): Clear, since $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{An}$ preserves limits.

(2) $\implies$ (1): Assume (2), and let $X_\bullet\colon I \to T$ be a diagram with colimit X. By universality of colimits, the comparison functor

$$T_{/X} \longrightarrow \lim_{i \in I^{\mathrm{op}}} T_{/X_i}$$

is fully faithful by Lemma 2.8. To prove essential surjectivity, let U be an object classifier and choose an object of the limit on the right. Passing to cores and using the classifying property of U, this object determines a point of

$$\lim_{i \in I^{\mathrm{op}}} (T_{/X_i})^{\simeq} \xrightarrow{\sim} \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}(X_i, U) \xrightarrow{\sim} \mathrm{Hom}(X, U).$$

The final isomorphism follows because $U \colon T^{\mathrm{op}} \to \widehat{\mathrm{An}}$ preserves limits. The resulting map $X \to U$ classifies an object over X whose pullback to every X_i is isomorphic to the prescribed object. Thus the comparison functor is essentially surjective and hence an equivalence.

(1) $\implies$ (3): We verify the three conditions:

- Universality of colimits holds by definition of a topos.
- Coproducts are disjoint by Example 2.12.
- All groupoids are effective by Corollary 2.23.

(4) $\implies$ (1): Since An is a topos, it follows that $\mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ is a topos, and hence so is any left exact localization of $\mathrm{PSh}(C)$ by Corollary 2.17.

(3) $\implies$ (4): By Proposition C.5, there is an essentially small full subcategory $C \subseteq T$ which is closed under finite limits and dense in the sense of Definition C.1. The inclusion $C \hookrightarrow T$ is then left exact, so the proposition below shows that its colimit-preserving extension

$$F \colon \mathrm{PSh}(C) \longrightarrow T$$

is left exact. Its right adjoint is the restricted Yoneda functor $X \mapsto \mathrm{Hom}_T(-, X)|_C$, which is fully faithful because C is dense. Thus F is a left exact localization, giving (4); see also Corollary C.6. □

Proposition 2.43. *Let C be a category which has finite limits, and let T be a presentable category satisfying the Giraud axioms. Let $f \colon C \to T$ be a left exact functor. Then the unique colimit-preserving extension $F \colon \mathrm{PSh}(C) \to T$ of f is left exact.*

Proof. Let 1_C be terminal in C. The representable h_{1_C} is terminal in $\mathrm{PSh}(C)$, and $F(h_{1_C}) \simeq f(1_C)$ is terminal in T. It remains to show that F preserves pullbacks.

We say that a presheaf $X \in \mathrm{PSh}(C)$ is *good* if the functor F preserves every pullback square of the form

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ Z & \longrightarrow & X \end{array}$$

By universality of colimits, X is good if and only if F preserves all such pullback squares where Y and Z are representable. In particular, since f preserves pullbacks, every representable presheaf is good. Since coproducts are universal and disjoint in T, they are van Kampen by Example 2.12; it follows that coproducts of good objects are good.

Every presheaf admits an effective epimorphism from a coproduct of representables. Indeed, one may choose a representative of every connected component of every anima $X(c)$, and the resulting map

$$\coprod_{(c,x)} h_c \longrightarrow X$$

is pointwise surjective on π_0. It therefore remains to prove the following descent statement: if $X_0 \twoheadrightarrow X$ is an effective epimorphism and X_0 is good, then X is good.

Let $X_\bullet \to X$ be the augmented Čech nerve of $p\colon X_0 \to X$. The pullback identities expressing that $X_\bullet$ is a groupoid object are pullback squares over X_0, so goodness of X_0 implies that $F(X_\bullet)$ is a groupoid object. Since F preserves colimits, $F(X_\bullet) \to F(X)$ is a colimit diagram. Effectivity of groupoids in T therefore identifies it with the augmented Čech nerve of $F(p)\colon F(X_0) \to F(X)$. In particular, $F(p)$ is an effective epimorphism.

We record one consequence. Suppose that a map $W \to X$ admits a lift $W \to X_0$. Using either projection $X_0 \times_X X_0 \to X_0$ as appropriate, there is a pullback square over X_0 identifying

$$W \times_X X_0 \xrightarrow{\sim} W \times_{X_0} (X_0 \times_X X_0).$$

Goodness of X_0, together with the degree-one identification

$$F(X_0 \times_X X_0) \xrightarrow{\sim} F(X_0) \times_{F(X)} F(X_0),$$

therefore gives

$$F(W \times_X X_0) \xrightarrow{\sim} F(W) \times_{F(X)} F(X_0). \tag{2.1}$$

To test that X is good, we may take Y and Z to be representable. Since p is pointwise surjective on π_0, the maps $Y \to X$ and $Z \to X$ admit lifts through X_0 in the corresponding mapping animae. The same is then true of $Y \times_X Z \to X$. Applying Equation (2.1) to these three objects, and using goodness of X_0 for the pullback square

$$(Y \times_X Z) \times_X X_0 \xrightarrow{\sim} (Y \times_X X_0) \times_{X_0} (Z \times_X X_0),$$

shows that the comparison map

$$F(Y \times_X Z) \longrightarrow F(Y) \times_{F(X)} F(Z)$$

becomes an isomorphism after pullback along $F(X_0) \to F(X)$. This map is an effective epimorphism, so the corresponding pullback functor is conservative. The comparison map is therefore itself an isomorphism. Thus X is good, completing the proof. □

Remark 2.44. Marc mentioned he did not yet manage to find a direct proof that (3) implies (1) that does not go through the presentation of a topos as a left exact localization of a presheaf topos.

2.3.2 Exactness in topoi

Presenting a topos as a left exact localization of a presheaf topos allows many exactness questions to be reduced to the corresponding statements in An. We record four useful instances. The last one is more specialized, but it is a convenient criterion in calculations with simplicial objects.

Proposition 2.45 (cf. [Lur09, Proposition 6.4.5.9])**.** *The following statements hold true in a topos:*

(1) *Filtered colimits commute with finite limits.*

(2) *Sifted colimits commute with finite products.*

(3) *For every anima A, the colimit functor* $\mathrm{colim}_A\colon \mathrm{Fun}(A,T) \to T$ *preserves limits indexed by weakly contractible categories. For example, every G-equivariant pullback diagram induces an isomorphism*

$$(X \times_Z Y)/G \xrightarrow{\sim} X/G \times_{Z/G} Y/G.$$

(4) *Given a diagram of simplicial objects*

$$\begin{array}{ccc} & & Y_\bullet \\ & & \downarrow \\ X_\bullet & \longrightarrow & Z_\bullet \end{array}$$

such that $\tau_0 Z_\bullet$ is constant, we get an isomorphism

$$\mathrm{colim}_n(X_n \times_{Z_n} Y_n) \xrightarrow{\sim} (\mathrm{colim}_n X_n) \times_{\mathrm{colim}_n Z_n} (\mathrm{colim}_n Y_n).$$

Proof. (1) Writing T as a left exact localization of a presheaf topos, this follows from the analogous statement in An, where it is a classical fact. (A model-independent argument for this may be found in [SW25].)

(2) Given two diagrams $X_\bullet, Y_\bullet\colon I \to T$, with I sifted, we need to show that the canonical map

$$\mathrm{colim}_{i\in I}(X_i \times Y_i) \to (\mathrm{colim}_{i\in I} X_i) \times (\mathrm{colim}_{i\in I} Y_i)$$

induced by the projections is an isomorphism. We may factor this map as a composite

$$\mathrm{colim}_{i\in I}(X_i \times Y_i) \to \mathrm{colim}_{i\in I}\,\mathrm{colim}_{j\in I}(X_i \times Y_j) \to (\mathrm{colim}_{i\in I} X_i) \times (\mathrm{colim}_{j\in I} Y_j),$$

where the first map is induced by the diagonal functor $I \to I \times I$. Since I is sifted, this map is final by definition, so the first map is an isomorphism. Moreover, it follows from descent that the functor $- \times -\colon T \times T \to T$ preserves colimits in both variables separately, so that also the second map is an isomorphism.

(3) Given $A \in \mathrm{An}$, descent gives an equivalence

$$\mathrm{Fun}(A,T) \simeq T_{/\,\mathrm{colim}_A *}.$$

Under this equivalence, the colimit functor $\mathrm{colim}_A\colon \mathrm{Fun}(A,T) \to T$ corresponds to the forgetful functor $T_{/\,\mathrm{colim}_A *} \to T$. Limits in a slice are computed in the underlying category whenever the indexing category is nonempty. In particular, this forgetful functor preserves limits indexed by weakly contractible categories, which proves the claim.

(4) Let P denote the constant value of $\tau_0 Z_\bullet$. The comparison map in the statement is a morphism over P. Since $Z_0 \to P$ is an effective epimorphism, it is enough by Lemma 2.29 to prove that this map becomes an isomorphism after base change along $Z_0 \to P$.

We therefore work in the slice $T_{/Z_0}$ and suppress this base change from the notation. Each Z_n is now connected over Z_0, and the degeneracy map $Z_0 \to Z_n$ supplies a point. Thus $Z_\bullet$ is a simplicial object in the category of pointed connected objects of $T_{/Z_0}$. Applying the delooping equivalence from Proposition 2.35 degreewise, and using its naturality, we may write

$$Z_\bullet \simeq \mathbf{B}G_\bullet$$

for a simplicial group object $G_\bullet$ in $T_{/Z_0}$. Under the equivalence between objects over $\mathbf{B}G_n$ and G_n-objects from Example 2.13, the simplicial objects $X_\bullet$ and $Y_\bullet$ correspond to simplicial $G_\bullet$-objects $X'_\bullet$ and $Y'_\bullet$. Degreewise, we then have

$$X_n \times_{\mathbf{B}G_n} Y_n \xrightarrow{\sim} (X'_n \times Y'_n)/G_n.$$

The quotient on the right is itself a simplicial colimit, namely the realization of the action groupoid. The desired comparison therefore compares the two possible orders of realization of the resulting bisimplicial object. These orders agree by the Fubini theorem for colimits. Moreover, finite products commute with simplicial colimits by part (2), so the products occurring in the action groupoids may also be formed before or after realization. Both sides are consequently identified with the same total colimit, proving the claim after base change and hence in T. □

2.3.3 The subobject classifier

The object classifier of a topos is generally a large object. Nevertheless, many suitably small families of morphisms are represented by objects of the topos itself. The relevant hypothesis is locality: membership in the family must be stable under base change and detectable after passing to a cover. We develop this general classification result before specializing it to monomorphisms and the subobject classifier.

Definition 2.46. Let Σ be a class of morphisms in a topos T. We say that Σ is a *local class* if the following conditions are satisfied:

(1) It is closed under base change;

(2) It is closed under small coproducts;

(3) Given a pullback square in T of the form

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

in which $f' \in \Sigma$ and g is an effective epimorphism, we also have $f \in \Sigma$.

Proposition 2.47. *Let Σ be a class of morphisms in a topos T which is stable under base change. The following conditions are equivalent:*

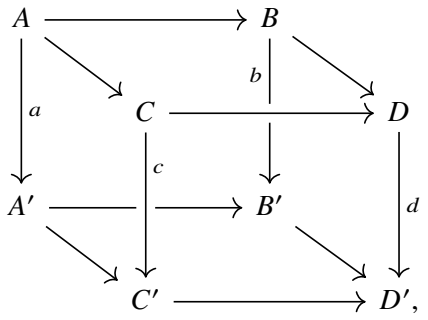
(1) *The class Σ is a local class;*

(2) *The functor $T^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ given by sending $X \in T$ to the full subcategory of $T_{/X}$ spanned by the morphisms in Σ preserves limits;*

(3) *The functor $T^{\mathrm{op}} \to \widehat{\mathrm{An}}$ given by sending $X \in T$ to the full subanima of $(T_{/X})^{\simeq}$ spanned by the morphisms in Σ preserves limits;*

(4) *The full subcategory of $\mathrm{Ar}^{\mathrm{pb}}(T)$ spanned by the morphisms in Σ is closed under colimits;*

(5) *The class Σ is closed under coproducts, and for every commutative cube in T of the form*



if the top and bottom squares are pushout squares, the left and back squares are pullback squares, and $a, b, c \in \Sigma$, then also $d \in \Sigma$.

Proof. We first reformulate conditions (2) and (3). Let $X_\bullet \colon I \to T$ be a diagram with colimit X. Descent identifies $T_{/X}$ with $\lim_{i \in I^{\mathrm{op}}} T_{/X_i}$, and it identifies their cores after applying $(-)^{\simeq}$. Since Σ is stable under base change, conditions (2) and (3) are therefore both equivalent to the following detection property:

$$f \colon Y \to X \text{ lies in } \Sigma \quad \Longleftrightarrow \quad f_i \colon Y \times_X X_i \to X_i \text{ lies in } \Sigma \text{ for every } i \in I. \tag{2.2}$$

Assume (1). The reverse implication in Equation (2.2) follows by forming the coproduct $\bigsqcup_i f_i$. This morphism lies in Σ, and it is the base change of f along the effective epimorphism $\bigsqcup_i X_i \twoheadrightarrow X$. The forward implication is simply stability under base change. Thus (1) implies both (2) and (3).

Conversely, assume the detection property. Applied to a coproduct diagram, it shows that Σ is closed under small coproducts. Now let $g \colon X' \twoheadrightarrow X$ be an effective epimorphism and suppose that the pullback f' of $f \colon Y \to X$ along g lies in Σ. The Čech nerve of g has colimit X, and every pullback of f to a term of this Čech nerve is a further base change of f'. The detection property therefore implies that f lies in Σ. Hence Σ is local, proving the equivalence of (1), (2), and (3).

We next compare this with (4). A diagram in $\mathrm{Ar}^{\mathrm{pb}}(T)$ is the same thing as a cartesian natural transformation $Y_\bullet \to X_\bullet$. By Proposition 2.16, its colimit in $\mathrm{Ar}^{\mathrm{pb}}(T)$ is represented by

$$\mathrm{colim}_i\, Y_i \longrightarrow \mathrm{colim}_i\, X_i,$$

and its pullback to each X_i recovers $Y_i \to X_i$. Consequently, closure of the arrows in Σ under colimits in $\mathrm{Ar}^{\mathrm{pb}}(T)$ is equivalent to the detection property Equation (2.2). This proves the equivalence with (4).

Finally, small colimits are generated by small coproducts and pushouts. A pushout diagram in $\mathrm{Ar}^{\mathrm{pb}}(T)$ is precisely a cube of the form displayed in (5): the top and bottom faces express the pushout in the source and target, while the remaining vertical faces express that the morphisms in the span lie in $\mathrm{Ar}^{\mathrm{pb}}(T)$. Thus closure under coproducts and the cube condition is equivalent to closure under all small colimits, proving the equivalence of (4) and (5). □

Definition 2.48. Let Σ be a local class in T. We say that a morphism $f_\Sigma \colon Y_\Sigma \to X_\Sigma$ *classifies* Σ if for every object $X \in T$ the map

$$\mathrm{Hom}_T(X, X_\Sigma) \to (T_{/X})^{\simeq}, \qquad g \mapsto g^*(f_\Sigma)$$

is a monomorphism of animae whose image is given by the full subanima spanned by the morphisms in Σ. In this situation, we also say that the object X_Σ *classifies* Σ.

By the Yoneda lemma, a classifying object is unique if it exists.

Corollary 2.49. *Let Σ be a local class in T. Then there exists a classifying object for Σ if and only if for every object $X \in T$ the full subanima of $(T_{/X})^{\simeq}$ spanned by the morphisms in Σ is small.*

Proof. If a classifying object exists, then for every object $X \in T$ the subanima of $(T_{/X})^{\simeq}$ spanned by the morphisms in Σ is equivalent to the small anima $\mathrm{Hom}_T(X, X_\Sigma)$, hence is itself small.

Conversely, if for every object $X \in T$ the subanima of $(T_{/X})^{\simeq}$ spanned by the morphisms in Σ is small, then these subanimae determine a functor $T^{\mathrm{op}} \to \mathrm{An}$. By Proposition 2.47, this functor preserves limits. The adjoint functor theorem then implies it is represented by some object X_Σ of T. □

The class of monomorphisms in a topos is local. Stability under base change is immediate, coproducts of monomorphisms are monomorphisms by disjointness and universality of coproducts, and locality follows by testing the diagonal after pullback along an effective epimorphism. Moreover, presentable categories are well-powered, so for every $X \in T$ the subanima of $(T_{/X})^{\simeq}$ spanned by the monomorphisms is small. The preceding corollary therefore applies.

Definition 2.50. For a topos T, we denote by Ω the classifying object for the class of monomorphisms, and call it the *subobject classifier*.

Lemma 2.51. *The subobject classifier is 0-truncated. That is, for every object $X \in T$, the anima $\mathrm{Hom}_T(X, \Omega)$ is 0-truncated.*

Proof. By definition, $\mathrm{Hom}_T(X, \Omega)$ is the full subanima of $(T_{/X})^{\simeq}$ spanned by the monomorphisms $U \hookrightarrow X$. If $U \hookrightarrow X$ is a monomorphism, then its anima of automorphisms in $T_{/X}$ is contractible. Thus this full subanima has no nontrivial self-identifications and is therefore 0-truncated. □

Lemma 2.52. *Let $Y \to \Omega$ be the universal monomorphism. Then Y is the terminal object.*

Proof. Write $p\colon Y \to \Omega$ for the universal monomorphism, and let $t\colon * \to \Omega$ be the map classifying the identity $\mathrm{id}_*\colon * \to *$, exhibited by a pullback square of the form

$$\begin{array}{ccc} * & \xrightarrow{s} & Y \\ {\scriptstyle \mathrm{id}_*}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ * & \xrightarrow[t]{} & \Omega. \end{array}$$

We claim s is an inverse to the canonical map $q\colon Y \to *$, showing Y is terminal. Since $q \circ s$ is homotopic to id_*, it remains to show that $s \circ q$ is homotopic to id_Y.

To this end, observe that we have two pullback diagrams of the following form:

$$\begin{array}{ccccc} Y & \xrightarrow{q} & * & \xrightarrow{s} & Y \\ {\scriptstyle \mathrm{id}_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{id}_*} & \lrcorner & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & * & \xrightarrow{t} & \Omega \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y & \xrightarrow{\mathrm{id}_Y} & Y \\ {\scriptstyle \mathrm{id}_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{p} & \Omega. \end{array}$$

In particular, both bottom maps $Y \to \Omega$ are classifying maps for id_Y, and it follows that p is homotopic to $t \circ q$. We then get the chain of homotopies

$$p \circ (s \circ q) \simeq (p \circ s) \circ q \simeq t \circ q \simeq p \circ \mathrm{id}_Y,$$

so $s \circ q$ defines an endomorphism of p in the slice category $T_{/\Omega}$. Since p is a monomorphism, the anima of endomorphisms of p in $T_{/\Omega}$ is (-1)-truncated and nonempty (it contains id_Y), hence contractible. Therefore $s \circ q$ is homotopic to id_Y, as desired. □

3 Truncation and connectivity

In Chapter 2, we saw that the effective epimorphisms and monomorphisms form a factorization system on every topos. This is the first stage of a hierarchy: effective epimorphisms are precisely the (-1)-connected maps, while monomorphisms are precisely the (-1)-truncated maps. In this chapter, we develop the corresponding notions of *n-connected* and *n-truncated* morphisms for every $n \geq -2$.

Truncation is defined recursively by diagonals: a morphism is n-truncated if its diagonal is $(n-1)$-truncated. In a topos, the n-truncated objects form a reflective subcategory, and truncation commutes with base change. Connectivity is then characterized by orthogonality to truncated maps. The main structural result of Sections 3.1 and 3.2 is that the n-connected and n-truncated morphisms form a factorization system for every n. These compatible factorizations provide the internal Postnikov calculus of a topos.

In Section 3.3, we attach homotopy group objects $\pi_n(X) \in T_{/X}$ to every object $X \in T$. They fit into long exact sequences and detect truncation and connectivity. In particular, a morphism is n-connected if and only if it is an effective epimorphism and its diagonal is $(n-1)$-connected. This recursive criterion is the connected analogue of the recursive definition of truncated maps.

Passing from finite n to ∞ leads to hypercompletion. A morphism is ∞-connected if it is n-connected for every n, and hypercompletion is the left exact localization which forces all such morphisms to become isomorphisms. Thus hypercompleteness asks whether a morphism which is invisible to every finite stage of the truncation hierarchy is already an isomorphism. We develop this in Section 3.4.

We conclude in Section 3.5 with objects lying simultaneously at adjacent levels of the two hierarchies. An n-gerbe is n-truncated and $(n-1)$-connected; a pointed n-gerbe is an Eilenberg–MacLane object. Iterated delooping identifies these objects with iterated group objects. For $n \geq 1$ and a static abelian group object A, the universal banded n-gerbe

$$* \longrightarrow \mathbf{B}^{n+1}A$$

classifies n-gerbes banded by an abelian group object A. This gives an intrinsic construction of cohomology with coefficients in A inside an arbitrary topos.

3.1 Truncation

Truncation organizes objects and morphisms according to the amount of homotopical information they contain. The definition is recursive: a morphism is n-truncated when its diagonal is $(n-1)$-truncated. We first recall this hierarchy and construct the truncation functors in any presentable

category. We then develop the relative truncation calculus needed later, with particular attention to its interaction with base change, monomorphisms, and effective epimorphisms.

Definition 3.1. Let C be a category with pullbacks. A morphism $f\colon X \to Y$ in C is called (-2)-*truncated* if it is an isomorphism. For $n \geq -1$, we say f is *n-truncated* if $\Delta_f\colon X \to X \times_Y X$ is $(n-1)$-truncated. We denote by $C_{\leq n} \subseteq C$ the full subcategory of n-truncated objects.

If the inclusion $C_{\leq n} \hookrightarrow C$ admits a left adjoint, we denote this adjoint by

$$\tau_n\colon C \to C_{\leq n},$$

and call it the *n-truncation functor*.

Remark 3.2. Being (-1)-truncated is the same as being a monomorphism.

Remark 3.3. The truncation functor $\tau_n\colon C \to C_{\leq n}$ always exists when C is presentable, hence in particular when T is a topos.

To see this, recall that C admits a tensoring and a cotensoring by the category An. There are unique functors

$$-\otimes -\colon \mathrm{An} \times C \to C \qquad \text{and} \qquad (-)^{(-)}\colon \mathrm{An}^{\mathrm{op}} \times C \to C$$

satisfying $* \otimes X \cong X \cong X^*$ for all $X \in C$ and preserving colimits (resp. limits) in the first variable. Here and below we use the convention $S^{-1} := \emptyset$. Thus, for $n = -2$, the map below is the terminal map $X \to X^{\emptyset} \cong *$. By an easy induction argument, an object $X \in C$ is n-truncated if and only if the map $X \to X^{S^{n+1}}$ induced by $S^{n+1} \to *$ is an isomorphism.

Since C is presentable, there is a small collection of objects $\{Y\}$ that detect equivalences. Since $\mathrm{Hom}_C(Y, X^{S^{n+1}}) \simeq \mathrm{Hom}_C(Y \otimes S^{n+1}, X)$, the n-truncated objects are precisely the local objects with respect to the small collection of morphisms $\{S^{n+1} \otimes Y \to * \otimes Y = Y\}$. In particular, the n-truncated objects form a reflective subcategory of C, admitting a reflector $\tau_n\colon C \to C_{\leq n}$.

Exercise 3.4. Show that a stable category does not admit any non-zero n-truncated objects.

We now prove some basic properties of truncated objects.

Lemma 3.5. *The n-truncated morphisms in C are closed under base change.*

Proof. Immediate from the definitions, details left to the reader. □

Lemma 3.6. *A morphism $f\colon X \to Y$ is n-truncated if and only if the induced map of animae $f_*\colon \mathrm{Hom}_C(Z, X) \to \mathrm{Hom}_C(Z, Y)$ is n-truncated for all $Z \in C$.*

Proof. Immediate from Yoneda lemma and the fact that $\mathrm{Hom}_C(Z, -)$ preserves pullbacks. □

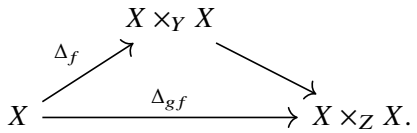
Lemma 3.7. *Consider morphisms $f\colon X \to Y$ and $g\colon Y \to Z$.*

(1) *If g is n-truncated, then f is n-truncated if and only if gf is n-truncated.*

(2) *If gf is n-truncated and g is $(n+1)$-truncated, then f is n-truncated.*

Proof. (1) We prove the claim by induction on n. The case $n = -2$ is clear by 2-out-of-3.

For $n \geq -1$, consider the following commutative triangle:



The right diagonal map is a base change of $\Delta_g : Y \to Y \times_Z Y$, hence is $(n-1)$-truncated. By the induction hypothesis, Δ_f is $(n-1)$-truncated if and only if Δ_{gf} is $(n-1)$-truncated. This completes the proof.

(2) We may factor f as the composite

$$X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

The first map is a base change of the n-truncated map $\Delta_g : Y \to Y \times_Z Y$. The second map is a base change of the n-truncated map $gf : X \to Z$. By (1) and Lemma 3.5, f is also n-truncated. □

Lemma 3.8. *Consider a functor $F : C \to D$.*

(1) *If F preserves pullbacks, then it preserves n-truncated objects for all $n \geq -2$.*

(2) *If F furthermore admits a right adjoint, and C and D admit n-truncation functors, then F commutes with n-truncation: for any $X \in C$ the canonical map*

$$\tau_n(F(X)) \to F(\tau_n X)$$

is an isomorphism.

Proof. Part (1) is clear, since n-truncatedness is formulated in terms of pullbacks.

For part (2), the claim is equivalent to the claim that the right adjoint $G : D \to C$ of F preserves n-truncated objects. This is an instance of (1). □

Lemma 3.9. *For every $n \geq -2$, the truncation functor $\tau_n : T \to T_{\leq n}$ of a topos preserves finite products.*

Proof. Choose a left exact localization $L : \mathrm{PSh}(C) \to T$ with fully faithful right adjoint R. In a presheaf topos, truncation is computed pointwise, and truncation of animae preserves finite products. It follows that τ_n^{pre} preserves finite products in $\mathrm{PSh}(C)$. By Lemma 3.8, the truncation functor on T is naturally isomorphic to the composite $L\tau_n^{\mathrm{pre}}R$. The functors L and R preserve finite limits, so this composite preserves finite products. □

Corollary 3.10. *For a morphism $f : X \to Y$ in a topos T, the pullback functor $f^* : T_{/Y} \to T_{/X}$ preserves n-truncated objects and commutes with n-truncation: for a map $Z \to Y$ we have $\tau_n(f^*Z/X) \cong f^*\tau_n(Z/Y)$.*

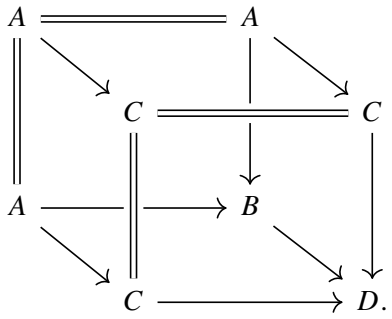
Proof. Lemma 3.8 applies: the functor f^* preserves limits (as it has a left adjoint) and preserves colimits (by universality of colimits), so that it admits a right adjoint $f_* : T_{/X} \to T_{/Y}$ by the adjoint functor theorem. □

Proposition 3.11 ([ABFJ20, Proposition 2.2.6])**.** *Consider a pushout square*

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ C & \longrightarrow & D \end{array}$$

in which the map $A \hookrightarrow B$ is a monomorphism. Then also $C \to D$ is a monomorphism and the square is a pullback square.

Proof. Consider the cube



The top and bottom faces are pushouts, and the left and back faces are pullbacks. By descent (Mather's first cube lemma), the front and right faces are also pullback squares. This says precisely that the starting square is a pullback square and that the map $C \to D$ is a monomorphism. □

Lemma 3.12. *Let $U \hookrightarrow X$ be a monomorphism. Then also $\tau_0 U \hookrightarrow \tau_0 X$ is a monomorphism, and the square*

$$\begin{array}{ccc} U & \twoheadrightarrow & \tau_0 U \\ \downarrow & \lrcorner & \downarrow \\ X & \twoheadrightarrow & \tau_0 X \end{array}$$

is a pullback square.

Proof. Choose a left exact localization $L\colon \mathrm{PSh}(C) \to T$ with fully faithful right adjoint $R\colon T \hookrightarrow \mathrm{PSh}(C)$, as in Theorem 2.42. Since R preserves limits, the map $R(U) \to R(X)$ is again a monomorphism. Truncation and limits in the presheaf topos are computed pointwise. For every $c \in C$, the monomorphism of animae

$$R(U)(c) \hookrightarrow R(X)(c)$$

is the inclusion of a union of connected components. Consequently, $\tau_0 R(U)(c) \to \tau_0 R(X)(c)$ is a monomorphism and the corresponding naturality square is a pullback. It follows pointwise that the square

$$\begin{array}{ccc} R(U) & \longrightarrow & \tau_0 R(U) \\ \downarrow & \lrcorner & \downarrow \\ R(X) & \longrightarrow & \tau_0 R(X) \end{array}$$

is a pullback in $\mathrm{PSh}(C)$. Applying the left exact functor L preserves this pullback square and the monomorphism on the right. Moreover, Lemma 3.8 identifies $L\tau_0 R(U)$ and $L\tau_0 R(X)$ with $\tau_0 U$ and $\tau_0 X$, respectively. This gives the asserted square in T. □

We now prove a crucial fact: effective epimorphisms in a topos can be tested on 0-truncations.

Lemma 3.13 (Key lemma)**.** *Let T be a topos.*

(1) *The map $X \to \tau_0 X$ is an effective epimorphism for every $X \in T$.*

(2) *A morphism $f\colon Y \to X$ is an effective epimorphism if and only if the map $\tau_0 Y \to \tau_0 X$ is an effective epimorphism.*

Remark 3.14. The corresponding argument in Lurie [Lur09] contains a circular dependency. Proposition 7.2.1.14 is the analogue of the lemma above. Its proof uses Lemma 7.2.1.13, which invokes Proposition 6.5.1.20; the proof of Proposition 6.5.1.20 in turn appeals to Proposition 7.2.1.14. Proposition 6.5.1.12 also uses Proposition 7.2.1.14.

There is a related gap in the proof of [Lur09, Proposition 6.5.1.18]: one still has to show that the diagonal of a 0-connected morphism is an effective epimorphism. The proof below establishes the effective-epimorphism criterion independently, and we will use it to supply this step in Theorem 3.22.

Proof. (1) Factor the map $X \to \tau_0 X$ into an effective epimorphism followed by a monomorphism:

$$X \twoheadrightarrow U \hookrightarrow \tau_0 X.$$

We must show that the second map is an isomorphism. Since it is a monomorphism ((-1)-truncated) and $\tau_0 X$ is 0-truncated, Lemma 3.7 implies that U is also 0-truncated. More explicitly, the terminal map $\tau_0 X \to *$ is 0-truncated, while the monomorphism $U \to \tau_0 X$ is (-1)-truncated and hence also 0-truncated. Part (1) of Lemma 3.7, applied to $U \to \tau_0 X \to *$, therefore shows that $U \to *$ is 0-truncated. By definition, $\tau_0 X$ is the initial 0-truncated object equipped with a map from X. Therefore the map $U \hookrightarrow \tau_0 X$ admits a section $\tau_0 X \to U$. Writing $\eta_X\colon X \to \tau_0 X$ for the unit, the effective epimorphism $X \to U$ factors uniquely through η_X because U is 0-truncated. This gives $s\colon \tau_0 X \to U$. The endomorphisms ms and $\mathrm{id}_{\tau_0 X}$ agree after precomposition with η_X, so the same universal property gives $ms = \mathrm{id}_{\tau_0 X}$. In particular, it is an isomorphism by Corollary 2.39.

(2) The "only if" direction is clear from the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \tau_0 Y & \xrightarrow{\tau_0 f} & \tau_0 X. \end{array}$$

Indeed, by part (1) the map $X \to \tau_0 X$ is an effective epimorphism. If f is also an effective epimorphism, then so is the composite $Y \to \tau_0 X$ by Lemma 2.37. Then $\tau_0 Y \to \tau_0 X$ is also an effective epimorphism by Lemma 2.40. Naturality of the unit gives $\eta_X f = (\tau_0 f)\eta_Y$. Thus the effective epimorphism $Y \to \tau_0 X$ factors as $Y \to \tau_0 Y \xrightarrow{\tau_0 f} \tau_0 X$, and cancellation applies to this factorization.

Conversely, assume that $\tau_0 f$ is an effective epimorphism. Consider the epi–mono factorization $Y \twoheadrightarrow U \hookrightarrow X$ of f and the induced diagram

$$\begin{array}{ccccc} Y & \twoheadrightarrow & U & \hookrightarrow & X \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \tau_0 Y & \longrightarrow & \tau_0 U & \longrightarrow & \tau_0 X \end{array}$$

By Lemma 3.12, the map $\tau_0 U \to \tau_0 X$ is again a monomorphism, and the right-hand square is a pullback square. Since the composite $\tau_0 Y \to \tau_0 U \to \tau_0 X$ is an effective epimorphism by assumption, Lemma 2.40 implies that $\tau_0 U \to \tau_0 X$ is an effective epimorphism. Here functoriality identifies the displayed composite with $\tau_0 f = (\tau_0 i)(\tau_0 e)$, where $i\colon U \hookrightarrow X$ is the monomorphism in the factorization of f. Cancellation is applied before the mono-plus-effective-epi criterion. Since it is also a monomorphism, it is an isomorphism by Corollary 2.39. Since the right-hand square is a pullback square, $U \to X$ is also an isomorphism, as desired. Indeed, pullbacks preserve isomorphisms, and $U \to X$ is the pullback of $\tau_0 U \to \tau_0 X$. □

Corollary 3.15. *A map $* \to X$ in a topos T is an effective epimorphism if and only if $\tau_0 X$ is the terminal object of T.* □

3.2 Connectivity

Complementary to the n-truncated morphisms are the n-connected morphisms. Truncated morphisms have no homotopy above degree n, while connected morphisms become terminal after discarding homotopy above degree n. We formulate this using relative truncation, identify the resulting orthogonality relation, and obtain the connected–truncated factorization system.

Definition 3.16. Let T be a topos, and let $n \geq -2$.

- We say that $X \in T$ is *n-connected* if $\tau_n X = *$.
- We say $f\colon X \to Y$ is *n-connected* if it is n-connected in $T_{/Y}$.

We denote by $T^{\geq n+1}$ the full subcategory of T spanned by the n-connected objects.

Warning 3.17. In the classical algebraic topology literature, what we call an n-connected morphism is often called $(n-1)$-connected. Our indexing is chosen so that n-connected and n-truncated morphisms form complementary classes with the same index, and so that the definition applies unchanged in every slice topos.

Note that every object is (-2)-connected. The universal property of relative truncation gives the complementary orthogonality description. Namely, $f\colon X \to Y$ is n-connected if and only if for every solid commutative diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & U \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ Y & = & Y \end{array}$$

where g is n-truncated, the anima of diagonal fillers is contractible. Indeed, this anima is the fiber of

$$\mathrm{Hom}_{/Y}(Y, U) \longrightarrow \mathrm{Hom}_{/Y}(X, U)$$

over the given map $X \to U$. Since $U \to Y$ is n-truncated, the universal property of $\tau_n(X/Y)$ identifies the target with $\mathrm{Hom}_{/Y}(\tau_n(X/Y), U)$. Thus the filler animae are contractible for all such U precisely when the map $\tau_n(X/Y) \to Y$ induces equivalences into every n-truncated object. Since both its source and target are n-truncated in $T_{/Y}$, Yoneda identifies this condition with $\tau_n(X/Y) \xrightarrow{\sim} Y$.

Proposition 3.18. *Let T be a topos.*

(1) *The pair* (n-connected, n-truncated) *forms a factorization system on T.*

(2) *The n-connected maps are stable under base change.*

(3) *A map $f\colon X \to Y$ is n-connected if and only if the functor $f^*\colon (T_{/Y})_{\leq n} \to (T_{/X})_{\leq n}$ is fully faithful.*

Proof. (1) We first show that any map $f\colon X \to Y$ factors as an n-connected map followed by an n-truncated map. Let $X_n := \tau_n(X/Y)$ be the n-truncation of $X \in T_{/Y}$. The map $X_n \to Y$ is n-truncated by definition.

To show that $X \to X_n$ is n-connected, consider any n-truncated morphism $U \to X_n$. We must show that the map

$$\mathrm{Hom}_{/X_n}(X_n, U) \to \mathrm{Hom}_{/X_n}(X, U)$$

is an equivalence. Since $T_{/X_n} \simeq (T_{/Y})_{/X_n}$, this map is induced on vertical fibers in the following diagram of Hom animae in $T_{/Y}$:

$$\begin{array}{ccc} \mathrm{Hom}_{/Y}(X_n, U) & \longrightarrow & \mathrm{Hom}_{/Y}(X, U) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{/Y}(X_n, X_n) & \longrightarrow & \mathrm{Hom}_{/Y}(X, X_n). \end{array}$$

Since U and X_n are n-truncated objects in $T_{/Y}$, the universal property of $X_n = \tau_n(X/Y) \in T_{/Y}$ guarantees that both horizontal maps are equivalences. Thus the induced map on fibers is also an equivalence.

Next, we show that any n-connected map $f\colon X \to Y$ is left orthogonal to any n-truncated map $g\colon W \to Z$. We must show that the anima of diagonal fillers in any commutative square

$$\begin{array}{ccc} X & \longrightarrow & W \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & Z \end{array}$$

is contractible. Since n-truncated maps are closed under base change, we may replace g by the projection $W \times_Z Y \to Y$. This reduces the problem to finding diagonal fillers in the commutative square:

$$\begin{array}{ccccc} X & \longrightarrow & W \times_Z Y & \longrightarrow & W \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle \mathrm{pr}} & \lrcorner & \downarrow{\scriptstyle g} \\ Y & = & Y & \longrightarrow & Z. \end{array}$$

The anima of such diagonal fillers is contractible by n-connectedness of f. We have therefore constructed the required factorization and proved orthogonality, which are exactly the two conditions in Definition A.2.

(2) Consider a pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

By Corollary 3.10, we have $\tau_n(X'/Y') \cong g^*\tau_n(X/Y)$. If f is n-connected, then $\tau_n(X/Y) = Y$, so $\tau_n(X'/Y') = Y'$. Hence f' is also n-connected.

(3) We need to show that f is n-connected if and only if for each two n-truncated maps $Z \to Y$ and $Z' \to Y$, the induced map

$$f^*\colon \operatorname{Hom}_{/Y}(Z, Z') \to \operatorname{Hom}_{/X}(X \times_Y Z, X \times_Y Z')$$

is an equivalence. By the universal property of $X \times_Y Z'$, the latter is equivalent to the condition that composition with the projection $X \times_Y Z \to Z$ induces an equivalence

$$\operatorname{Hom}_{/Y}(Z, Z') \xrightarrow{\sim} \operatorname{Hom}_{/Y}(X \times_Y Z, Z').$$

In other words, we have to show that f is n-connected if and only if the map $X \times_Y Z \to Z$ is left orthogonal to every n-truncated map $Z' \to Y$. If f is n-connected, this follows from parts (1) and (2), since $X \times_Y Z \to Z$ is a base change of f. Conversely, take $Z = Y$. The resulting orthogonality condition says that f is left orthogonal to every n-truncated map over Y, which is equivalent to n-connectedness by the characterization above. □

Lemma 3.19. *Consider a pullback square in a topos*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in which g is an effective epimorphism. Then, for every $n \geq -2$, the morphism f is n-truncated (resp. n-connected) if and only if f' is n-truncated (resp. n-connected).

Proof. Both properties are preserved by arbitrary base change, by Lemma 3.5 and proposition 3.18. We prove the converses.

For truncatedness, argue by induction on n. If $n = -2$, the claim says that an isomorphism can be detected after pullback along an effective epimorphism, which follows from Lemma 2.29. For $n \geq -1$, the diagonal of f' is obtained from the diagonal of f by base change along the effective epimorphism

$$X' \times_{Y'} X' \longrightarrow X \times_Y X.$$

The induction hypothesis therefore shows that Δ_f is $(n-1)$-truncated whenever $\Delta_{f'}$ is.

For connectedness, Corollary 3.10 gives an isomorphism

$$\tau_n(X'/Y') \cong g^*\tau_n(X/Y).$$

If f' is n-connected, the pullback of $\tau_n(X/Y) \to Y$ along g is therefore an isomorphism. Pullback along g is conservative by Lemma 2.29, so $\tau_n(X/Y) \to Y$ is itself an isomorphism. Thus f is n-connected. □

Corollary 3.20. *A morphism is (-1)-connected if and only if it is an effective epimorphism.*

Proof. Both the (-1)-connected maps as well as the effective epimorphisms are characterized as the morphisms that are left orthogonal to the monomorphisms/(-1)-truncated maps. □

Corollary 3.21. *Every pointed object $X \in T_*$ is (-1)-connected.*

Proof. A basepoint $x\colon * \to X$ determines a section of the map $X \to *$, which by Lemma 2.38 implies that $X \to *$ is an effective epimorphism, hence (-1)-connected. □

3.3 Homotopy group objects

The recursive definition of truncatedness has a complementary form for connectedness: an n-connected morphism is an effective epimorphism whose diagonal is $(n-1)$-connected. Our goal is to prove this characterization. It is not a formal consequence of the definitions, since one must first recognize connectedness from the vanishing of relative homotopy group objects. We therefore begin by constructing these objects and their long exact sequence, and then use them to establish the required detection results.

Theorem 3.22. *Let $n \geq 0$. Then a morphism $f\colon X \to Y$ is n-connected if and only if it is an effective epimorphism and $\Delta_f\colon X \to X \times_Y X$ is $(n-1)$-connected.*

3.3.1 Homotopy group objects and exact sequences

The theorem is rather non-trivial, and its proof requires the following internal version of the homotopy groups of a fiber.

Definition 3.23. Let $X \in T$ and let $n \geq 0$. Define the *n-th homotopy object* $\pi_n(X)$ as the following static (0-truncated) object of $T_{/X}$:

$$\pi_n(X) := \tau_0\big(X^{S^n} \to X\big) \quad \in \quad (T_{/X})_{\leq 0} \quad \subseteq \quad T_{/X}.$$

Here X^{S^n} denotes the cotensoring of X by S^n, cf. Remark 3.3. Since S^n is an E_n-cogroup in An and the map $* \to S^n$ is a map of E_n-cogroups, $\pi_n(X)$ is an E_n-group. In particular:

- $\pi_0(X)$ is a pointed object,
- $\pi_1(X)$ is a group,
- $\pi_n(X)$ is an abelian group for $n \geq 2$.

For this reason, we will often somewhat abusively speak of *homotopy group objects*, even though $\pi_0(X)$ is only a pointed object.

Example 3.24. Since $X^{S^0} = X \times X$, we see that $\pi_0(X) \cong \tau_0 X \times X \in T_{/X}$. It becomes a pointed object via the canonical section $X \xrightarrow{\Delta} X \times X \to \tau_0 X \times X$.

Lemma 3.25. *Let $F\colon T \to T'$ be a functor preserving colimits and finite limits. Then F preserves homotopy groups: for all $X \in T$ we have a natural isomorphism*

$$F(\pi_n(X)) \cong \pi_n(F(X)) \quad \in \quad T_{/F(X)}.$$

Proof. Since $F\colon T_{/X} \to T'_{/F(X)}$ preserves finite limits and X^{S^n} is a finite limit, we see that $F(X^{S^n}) \cong F(X)^{S^n}$. Moreover, a colimit-preserving functor between presentable categories is a left adjoint. Thus F admits a right adjoint, and Lemma 3.8 shows that it commutes with τ_0. □

3.3.2 The universal ∞-connected object and parametrized spectra

Lemma 3.26. *The functor $\pi_n : T \to T$ preserves finite products.*

Proof. Cotensoring by S^n preserves products, and Lemma 3.9 shows that 0-truncation preserves products in every slice topos. Applying the definition therefore gives

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y).$$

□

Notation 3.27. Given $n \geq 0$ and a morphism $f : X \to Y$, write

$$\pi_n(f) \in (T_{/Y})_{/f} \cong T_{/X}$$

for the n-th homotopy object of f regarded as an object of $T_{/Y}$. *(Note: We do not mean the induced map $\pi_n(X) \to \pi_n(Y)$.)*

Remark 3.28. Note that we have

$$\pi_n(f) \cong \pi_{n-1}(\Delta_f)$$

for $n \geq 1$.

Exercise 3.29. Show that every subgroup of $\pi_1(X)$ is normal.

Proposition 3.30. *Given a morphism $f : X \to Y$, there is a long exact sequence of pointed objects in $(T_{/X})_{\leq 0}$ of the form*

$$\cdots \to \pi_n(f) \to \pi_n(X) \to f^*\pi_n(Y) \to \pi_{n-1}(f) \to \dots.$$

Here, where three consecutive terms are group objects, exactness has its usual internal meaning: the image of one morphism is the kernel of the next. At degree zero, the group object $f^*\pi_1(Y)$ acts on the pointed object $\pi_0(f)$, and the fiber of $\pi_0(f) \to \pi_0(X)$ over the basepoint is the orbit of the basepoint under this action.

Proof. When $T = \mathrm{An}$, this is the usual long exact sequence of a homotopy fiber, including its group action on π_0 in the final nonabelian degree. The claim follows pointwise for any presheaf topos $\mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$. By Theorem 2.42, an arbitrary topos is a left exact localization of a presheaf topos. The localization preserves the finite limits, colimits, and homotopy group objects used in the construction, by Lemma 3.25, so it carries the pointwise long exact sequence to the asserted sequence in T. This is the argument of [Lur09, Remark 6.5.1.5]. □

Lemma 3.31. *Suppose that $p : X \to Y$ exhibits Y as the n-truncation of X. Then, for every $k \leq n$, the induced morphism*

$$\pi_k(X) \longrightarrow p^*\pi_k(Y)$$

is an isomorphism in $T_{/X}$.

Proof. Choose a left exact localization $L : \mathrm{PSh}(C) \to T$ with fully faithful right adjoint R. Form the n-truncation $R(X) \to \tau_n^{\mathrm{pre}} R(X)$ in $\mathrm{PSh}(C)$. Truncations and homotopy group objects in the presheaf topos are computed pointwise, so classical homotopy theory gives isomorphisms on π_k for $k \leq n$. Applying L preserves these homotopy group objects by Lemma 3.25. It also identifies $L\tau_n^{\mathrm{pre}} R(X)$ with $\tau_n X \cong Y$ by Lemma 3.8. The resulting isomorphisms are precisely those in the statement. See also [Lur09, Lemma 6.5.1.9]. □

3.3.3 Detection and the diagonal criterion

For a morphism known to be truncated in some finite degree, its remaining truncation level is detected by the vanishing of its homotopy group objects. Together with the preceding lemma, this also characterizes connectedness.

Proposition 3.32. *Let $n \geq -1$.*

(1) *If f is n-truncated, then $\pi_k(f) = *$ for $k > n$. Conversely, if f is m-truncated for some m and $\pi_k(f) = *$ for every $k > n$, then f is n-truncated.*

(2) *A morphism f is n-connected if and only if it is an effective epimorphism and $\pi_k(f) = *$ for all $k \leq n$.*

Proof. (1) We first prove, by induction on $r \geq -1$, that an r-truncated morphism has trivial π_k for $k > r$. If $r = -1$, then Δ_f is an isomorphism, so $\pi_k(f) \cong \pi_{k-1}(\Delta_f) = *$ for $k \geq 1$. Moreover, since f is 0-truncated, $\pi_0(f)$ is represented by the projection $X \times_Y X \to X$, which is an isomorphism because f is a monomorphism. For the induction step, Δ_f is $(r-1)$-truncated, and hence

$$\pi_k(f) \cong \pi_{k-1}(\Delta_f) = *$$

for $k > r$ by the induction hypothesis.

We next prove the sharper assertion that if $r \geq 0$, the morphism f is r-truncated, and $\pi_r(f) = *$, then f is $(r-1)$-truncated. For $r = 0$, the preceding description identifies $\pi_0(f)$ with $X \times_Y X \to X$. Its triviality says that the diagonal of f is an isomorphism, so f is (-1)-truncated. If $r \geq 1$, then Δ_f is $(r-1)$-truncated and

$$\pi_{r-1}(\Delta_f) \cong \pi_r(f) = *.$$

The induction hypothesis shows that Δ_f is $(r-2)$-truncated, and hence that f is $(r-1)$-truncated.

Now suppose that f is m-truncated for some m and that $\pi_k(f) = *$ for every $k > n$. If $m \leq n$, there is nothing to prove. If $m > n$, the sharper assertion successively lowers the truncation degree from m to $m-1$, and eventually to n.

(2) Since all assertions are relative, we may work in the slice $\mathcal{T}_{/Y}$ and assume that f is the terminal map $X \to *$. The case $n = -1$ is Corollary 3.20.

Suppose first that X is n-connected for $n \geq 0$. The map $X \to *$ is left orthogonal to every n-truncated morphism. In particular, it is left orthogonal to every monomorphism, and is therefore an effective epimorphism. Since $X \to * = \tau_n X$ is an n-truncation, Lemma 3.31 gives $\pi_k(X) = *$ for every $k \leq n$.

Conversely, suppose that $X \to *$ is an effective epimorphism and that $\pi_k(X) = *$ for $k \leq n$. Let $t\colon X \to \tau_n X$ be the truncation map. It is n-connected by the factorization constructed in Proposition 3.18, hence it is an effective epimorphism. For $k \leq n$, Lemma 3.31 gives

$$t^* \pi_k(\tau_n X) \cong \pi_k(X) = *.$$

Pullback along t is conservative by Lemma 2.29, so $\pi_k(\tau_n X) = *$ for every $k \leq n$. Since $\tau_n X$ is n-truncated, the sharper assertion proved in part (1) successively lowers its truncation degree and shows that it is (-1)-truncated. The composite

$$X \xrightarrow{t} \tau_n X \longrightarrow *$$

is an effective epimorphism by assumption. Hence $\tau_n X \to *$ is an effective epimorphism by Lemma 2.40. It is also a monomorphism, since $\tau_n X$ is (-1)-truncated, and is therefore an isomorphism. Thus X is n-connected. □

We are now ready to prove our characterization of n-connected maps:

Proof of Theorem 3.22. We must show that f is n-connected if and only if it is an effective epimorphism and Δ_f is $(n-1)$-connected.

By the proposition, f is n-connected if and only if it is an effective epimorphism and $\pi_k(f) = *$ for $0 \leq k \leq n$. Similarly, Δ_f is $(n-1)$-connected if and only if it is an effective epimorphism and $\pi_k(\Delta_f) = *$ for $0 \leq k \leq n-1$. By Remark 3.28, the latter vanishing condition is equivalent to $\pi_k(f) = *$ for $1 \leq k \leq n$. It remains to compare the two conditions in degree zero.

Work in the slice topos $T_{/Y}$, write Z for the object corresponding to f, and let $U = \tau_0 Z$. Suppose that f, or equivalently $Z \to *$, is an effective epimorphism. Then $U \to *$ is an effective epimorphism by Lemma 3.13. By the description of π_0 in Example 3.24, the morphism $\pi_0(f) \to Z$ is the pullback of $U \to *$ along $Z \to *$. Since pullback along the effective epimorphism $Z \to *$ is conservative by Lemma 2.29, we have $\pi_0(f) = *$ if and only if $U \to *$ is an isomorphism.

On the other hand, Lemmas 3.9 and 3.13 show that Δ_f is an effective epimorphism if and only if the diagonal $\Delta_U \colon U \to U \times U$ is an effective epimorphism. Since U is 0-truncated, Δ_U is a monomorphism. Consequently, if Δ_U is an effective epimorphism, it is an isomorphism, so $U \to *$ is a monomorphism. Together with the effective epimorphism $U \to *$, this makes U terminal. The converse is immediate. We have therefore proved, under the effective-epimorphism hypothesis on f, that

$$\pi_0(f) = * \quad \Longleftrightarrow \quad \Delta_f \text{ is an effective epimorphism.}$$

Combining this equivalence with the higher homotopy-group conditions proves the theorem. □

This theorem has various useful consequences.

Corollary 3.33. *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps. If gf is n-connected and g is $(n+1)$-connected, then f is n-connected.*

Proof. Dual to the proof of Lemma 3.7(2), we may factor f as $X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y$, where the first map is a base change of Δ_g, hence n-connected by the theorem, and the second map is a base change of gf. □

Corollary 3.34. *Let $n \geq 0$, and consider a 0-connected pointed object $X \in T_*^{\geq 1}$. Then X is n-connected if and only if ΩX is $(n-1)$-connected. Similarly, X is n-truncated if and only if ΩX is $(n-1)$-truncated.*

Proof. By the theorem, X is n-connected if and only if $\Delta \colon X \to X \times X$ is $(n-1)$-connected. Given a base point $x \colon * \to X$, it follows from 0-connectedness and Corollary 3.33 that x is (-1)-connected. Hence $(x, x) \colon * \to X \times X$ is an effective epimorphism. By Lemma 3.19, the diagonal Δ is $(n-1)$-connected if and only if its base change $\Omega X \to *$ is $(n-1)$-connected.

For truncatedness, X is n-truncated if and only if Δ is $(n-1)$-truncated. The same locality lemma identifies this with $(n-1)$-truncatedness of $\Omega X \to *$. □

Corollary 3.35. *Let $n \geq 0$ and let $f\colon X \to Y$ be n-connected. Then the functor $f^*\colon (T_{/Y})_{\leq n-1} \to (T_{/X})_{\leq n-1}$ is an equivalence.*

Proof. Full faithfulness of f^* was proved in Proposition 3.18.

For essential surjectivity, we may replace T by $T_{/Y}$ and assume $Y = *$. Given an $(n-1)$-truncated morphism $U \to X$, we must show that it is in the image of the functor $(-)\times X\colon T_{\leq n-1} \to (T_{/X})_{\leq n-1}$. We claim that the map $U \to \tau_{n-1}U \times X$ is an isomorphism. By the factorization system from Proposition 3.18, it suffices to show it is both $(n-1)$-truncated and $(n-1)$-connected.

- The map is $(n-1)$-truncated since it lives over X, and both maps to X are $(n-1)$-truncated. The claim follows by left cancellation.
- To see it is $(n-1)$-connected, note that it factors as

$$U \longrightarrow U \times X \longrightarrow \tau_{n-1}U \times X.$$

  The first map is $(n-1)$-connected, as it is a base change of the diagonal $X \to X \times X$, which is $(n-1)$-connected by the theorem. The second map is $(n-1)$-connected since it is a base change of the map $U \to \tau_{n-1}U$. □

3.4 Hypercompletion

Finite truncations need not detect all morphisms in a topos. A morphism may be n-connected for every finite n without being an isomorphism, so that it is invisible at every finite stage of the Postnikov hierarchy. Hypercompletion is the localization which forces all such morphisms to become isomorphisms. We now construct this localization and show that it is left exact, hence again a topos.

Definition 3.36. A morphism $f\colon X \to Y$ in T is called ∞-*connected* if it is n-connected for every n. It is called ∞-*truncated* (or *hypercomplete*) if it is right orthogonal to the class of ∞-connected maps.

Definition 3.37. Let $T_{\leq\infty}$ denote the full subcategory of ∞-truncated objects in T. An alternative notation for this subcategory is T^{hyp}. We say that T is *hypercomplete* if every object is ∞-truncated.

Remark 3.38. The term "hypercomplete" is historical but slightly misleading: hypercompleteness is a separation condition rather than a completeness condition. "Postnikov separated" might have been a better term. In [ABFJ25], the authors use the term "hyperreduced" instead.

Proposition 3.39. *Let T be a topos.*

(1) *The ∞-connected maps form a strongly saturated class of small generation, stable under base change and diagonals.*

(2) *The inclusion $T_{\leq\infty} \hookrightarrow T$ admits a left adjoint $\tau_\infty\colon T \to T_{\leq\infty}$.*

(3) *The pair* (∞-connected, ∞-truncated) *is a factorization system on T.*

Proof. (1) The n-connected maps form a saturated class (i.e. closed under pushouts and closed under colimits in $\mathrm{Ar}(T)$), by the factorization system from Proposition 3.18. In particular, the ∞-connected maps form a saturated class.

For the 2-out-of-3 property, it remains to check left cancellation. If gf and g are ∞-connected, then for every n the map gf is n-connected and g is $(n+1)$-connected, so Corollary 3.33 shows that f is n-connected. Thus f is ∞-connected. Closure under base change follows from part (2) of Proposition 3.18. Closure under diagonals follows from Theorem 3.22: if f is ∞-connected, then for every n it is $(n+1)$-connected, and hence Δ_f is n-connected.

Small generation is the additional accessibility input. The full subcategory of $\mathrm{Ar}(T)$ spanned by the ∞-connected morphisms is accessible, and therefore the resulting strongly saturated class is of small generation. We refer to [Lur09, Proposition 6.5.2.8] for this argument.

(2) The ∞-truncated objects are precisely the local objects with respect to this strongly saturated class of small generation. The existence and accessibility of the reflector τ_∞ therefore follow from Proposition A.15, or equivalently from [Lur09, Proposition 5.5.4.15].

For (3), it remains to show that any morphism $f\colon X \to Y$ can be factored into an ∞-connected morphism followed by an ∞-truncated one. The same argument as in part (1) of Proposition 3.18 shows that the map $X \to \tau_\infty(X/Y)$ provides such a factorization. □

Corollary 3.40. *The left adjoint $\tau_\infty\colon T \to T_{\leq\infty}$ is left exact. In particular, $T_{\leq\infty}$ is again a topos.*

Proof. We first explain why the ∞-connected maps are closed under finite limits in $\mathrm{Ar}(T)$. They contain the terminal object. For pullbacks, consider a morphism between cospans whose three component maps are ∞-connected. The induced map between their pullbacks factors as

$$X \times_Z Y \longrightarrow X \times_{Z'} Y \longrightarrow X' \times_{Z'} Y \longrightarrow X' \times_{Z'} Y'.$$

The last two maps are base changes of $X \to X'$ and $Y \to Y'$, while the first is a base change of the diagonal of $Z \to Z'$. Each factor is therefore ∞-connected by Proposition 3.39, and so is their composite. This is the same three-factor argument used in the proof of Lemma 4.15.

For a finite diagram $\{X_i\}_{i\in I}$ in T, it follows that the map $\lim_i X_i \to \lim_i \tau_\infty X_i$ is ∞-connected. Since ∞-truncated objects are local objects for a class of morphisms, they are closed under limits, so the target is ∞-truncated. The map therefore exhibits the target as the hypercompletion of $\lim_i X_i$, giving an isomorphism

$$\tau_\infty(\lim_i X_i) \xrightarrow{\sim} \lim_i \tau_\infty X_i.$$

□

3.5 Gerbes

3.5.1 Gerbes and Eilenberg–MacLane objects

Consider an object $X \in T$. We have the Postnikov tower

$$X \to \cdots \to \tau_n X \to \tau_{n-1} X \to \cdots \to \tau_0 X \to \tau_{-1} X \to \tau_{-2} X = *.$$

The object $\tau_n X \in T_{/\tau_{n-1}X}$ is n-truncated by Lemma 3.7 and $(n-1)$-connected by Proposition 3.18. Maps with this property are known as *gerbes*:

Definition 3.41. Given $n \geq 0$, we say an object $X \in T$ is an *n-gerbe* if it is n-truncated and $(n-1)$-connected. We define an *Eilenberg–MacLane object of degree n* to be a pointed n-gerbe. We denote by

$$\mathrm{Gerb}_n(T) \subseteq T \qquad \text{and} \qquad \mathrm{EM}_n(T) \subseteq T_*$$

the resulting full subcategories. More generally, we refer to a morphism $X \to Y$ as an *n-gerbe* if it is an n-gerbe in the slice topos $T_{/Y}$.

Example 3.42. Let K be an $(n+1)$-gerbe in T, and consider two maps $a, b \colon * \to K$. Define P as the following pullback:

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow b \\ * & \xrightarrow{a} & K. \end{array}$$

Then P is an n-gerbe. Indeed, the map $b \colon * \to K$ is an n-gerbe by the cancellation properties of truncated and connected maps from Lemma 3.7 and Corollary 3.33, hence the base change $P \to *$ is also an n-gerbe.

The Eilenberg–MacLane objects can in fact be completely classified in terms of abelian group objects in T, just as for the Eilenberg–MacLane animae in classical homotopy theory. To do this, we need an iterated version of the delooping principle from Proposition 2.35.

Definition 3.43. Given a category C with finite products, we inductively define for every $n \geq 0$ the category $E_n\mathrm{Grp}(C)$ of *E_n-groups in C* by setting

$$E_0\mathrm{Grp}(C) := C_* \qquad \text{and} \qquad E_{n+1}\mathrm{Grp}(C) := \mathrm{Grp}(E_n\mathrm{Grp}(C)).$$

Proposition 3.44 (Iterated delooping principle)**.** *For every $n \geq 0$, there is an equivalence*

$$\mathbf{B}^n \colon E_n\mathrm{Grp}(T) \;\rightleftarrows\; T_*^{\geq n} \;:\!\boldsymbol{\Omega}^n.$$

Proof. First observe that there is an equivalence $T_*^{\geq 0} \simeq T_*$. That is, every pointed object in T is automatically (-1)-connected. Indeed, any map $x \colon * \to X$ is a section of $X \to *$, so the latter is an effective epimorphism by Lemma 2.38.

Consider now the 'delooping equivalence'

$$\mathbf{B} \colon \mathrm{Grp}(T) \rightleftarrows T_*^{\geq 1} \;:\!\boldsymbol{\Omega}$$

from Proposition 2.35. By passing to E_{n-1}-groups, this results in an equivalence $E_n\mathrm{Grp}(T) \simeq E_{n-1}\mathrm{Grp}(T_*^{\geq 1})$. Since the equivalence is implemented by taking loops, Corollary 3.34 implies that it restricts to an equivalence

$$\mathbf{B} \colon E_n\mathrm{Grp}(T^{\geq k-1}) \;\rightleftarrows\; E_{n-1}\mathrm{Grp}(T_*^{\geq k}) \;:\!\boldsymbol{\Omega}$$

for every $k \geq 1$. Since $\mathrm{Grp}(C_*) \simeq \mathrm{Grp}(C)$ for any C, we may now paste together all these equivalences to obtain the result:

$$\mathbf{B}^n \colon E_n\mathrm{Grp}(T) \simeq E_{n-1}\mathrm{Grp}(T_*^{\geq 1}) \simeq \cdots \simeq E_1\mathrm{Grp}(T_*^{\geq n-1}) \simeq E_0\mathrm{Grp}(T_*^{\geq n}) = T_*^{\geq n} \;:\!\boldsymbol{\Omega}^n. \qquad \square$$

Corollary 3.45. *For $n \geq 0$ there are equivalences*

$$\mathrm{EM}_0(T) \simeq (T_{\leq 0})_*, \qquad \mathrm{EM}_1(T) \simeq \mathrm{Grp}(T_{\leq 0}), \qquad \textit{and} \qquad \mathrm{EM}_n(T) \simeq \mathrm{Ab}(T_{\leq 0}) \quad \text{for } n \geq 2.$$

Proof. By Corollary 3.34, the equivalence $\mathbf{\Omega}^n \colon T_*^{\geq n} \xrightarrow{\sim} E_n\mathrm{Grp}(T)$ from Proposition 3.44 restricts to an equivalence

$$\mathbf{\Omega}^n \colon \mathrm{EM}_n(T) \xrightarrow{\sim} E_n\mathrm{Grp}(T_{\leq 0})$$

for all n, with inverse given by $\mathbf{B}^n$. But since $T_{\leq 0}$ is a 1-category, we have $E_n\mathrm{Grp}(T_{\leq 0}) \simeq \mathrm{Ab}(T_{\leq 0})$ for all $n \geq 2$. □

3.5.2 Banded gerbes and cohomology

The n-gerbes play an important role in *obstruction theory*. Given objects $Y, X \in T$ and a map $Y \to \tau_0 X$, we could ask whether we can lift this map to X. Often, we can proceed inductively along the Postnikov tower of X. Assuming we have already found a lift $Y \to \tau_{n-1}X$, we may ask whether it lifts further to a map $Y \to \tau_n X$:

$$\begin{array}{ccc} & & \tau_n X \\ & \overset{?}{\nearrow} & \downarrow \\ Y & \longrightarrow & \tau_{n-1}X. \end{array}$$

In the remainder of this section we will explain how to reformulate this question in terms of the vanishing of a certain class in the cohomology of Y.

Lemma 3.46. *Let $n \geq 2$, and let X be an n-gerbe. Then there exists a unique $A \in \mathrm{Ab}(T_{\leq 0})$ such that $\pi_n X \cong X \times A$.*

Proof. Since X is $(n-1)$-connected, the functor $X \times - \colon T \to T_{/X}$ induces an equivalence $T_{\leq n-2} \xrightarrow{\sim} (T_{/X})_{\leq n-2}$ on $(n-2)$-truncated objects, by Corollary 3.35. In particular, the abelian group object $\pi_n(X) \in \mathrm{Ab}((T_{/X})_{\leq 0})$ corresponds to an object $A \in \mathrm{Ab}(T_{\leq 0})$ under this equivalence. □

Remark 3.47. If $n = 1$, then $A \in \mathrm{Grp}(T_{\leq 0})$ is unique if it exists, but it might not exist.

Definition 3.48. Let $n \geq 1$, let X be an n-gerbe, and consider $A \in \mathrm{Ab}(T_{\leq 0})$. We say that X is *banded by A* if it comes equipped with an isomorphism $\pi_n X \cong X \times A$ in $E_n\mathrm{Grp}((T_{/X})_{\leq 0})$. We denote by

$$\mathrm{Gerb}_n^A(T)$$

the anima of pairs consisting of an n-gerbe X together with an isomorphism $\pi_n X \cong X \times A$. Its morphisms are the band-preserving isomorphisms of n-gerbes.

Lemma 3.49. *Let $n \geq 1$. The trivial n-gerbe $\mathbf{B}^n A$ is banded by A. Moreover, the anima of pointed n-gerbes banded by A and band-preserving pointed isomorphisms is contractible, with distinguished object $\mathbf{B}^n A$.*

Proof. For the first claim, consider the two canonical maps $\mathbf{B}^n A \to (\mathbf{B}^n A)^{S^n}$ and $\Omega^n \mathbf{B}^n A \to (\mathbf{B}^n A)^{S^n}$. Using that $\mathbf{B}^n A \in \mathrm{CGrp}(T)$, this induces a map

$$\mathbf{B}^n A \times A \simeq \mathbf{B}^n A \times \Omega^n \mathbf{B}^n A \to (\mathbf{B}^n A)^{S^n}$$

in $T_{/\mathbf{B}^n A}$. We need to show that this induces an isomorphism on 0-truncations. Since $n \geq 1$, the map $* \twoheadrightarrow \mathbf{B}^n A$ is an effective epimorphism. Pullback along it is conservative by Lemma 2.29, so the claim may be checked after this base change. There the map becomes the identity on $\mathbf{\Omega}^n \mathbf{B}^n A$.

For the second claim, apply the equivalence $\mathbf{\Omega}^n \colon \mathrm{EM}_n(T) \xrightarrow{\sim} E_n\mathrm{Grp}(T_{\leq 0})$ from Corollary 3.45. Under this equivalence, a band on X is precisely an isomorphism $\mathbf{\Omega}^n X \cong A$. The anima of pairs consisting of an object A' and an isomorphism $A' \cong A$ is contractible. Transporting this statement across the equivalence proves the claim, including uniqueness of the band-preserving pointed isomorphism. □

Theorem 3.50. *Let $A \in \mathrm{Ab}(T_{\leq 0})$ and $n \geq 1$. Then for any object $X \in T$ there is an equivalence*

$$\mathrm{Hom}_T(X, \mathbf{B}^{n+1} A) \quad \simeq \quad \mathrm{Gerb}_n^A(T_{/X}), \qquad\qquad f \mapsto \mathrm{fib}_0(f).$$

Proof. First, note that the functor is well-defined. By Example 3.42, the map $0 \colon * \to \mathbf{B}^{n+1} A$ is an n-gerbe. Its fiber is $\mathbf{B}^n A$, so Lemma 3.49 supplies it with a canonical band by A, which is inherited by every pullback. The claim is that this is the *universal* n-gerbe banded by A.

To see this, consider an n-gerbe $\widetilde{X} \to X$ banded by A. There is a *unique* cartesian square of the form

$$\begin{array}{ccc} \widetilde{X} & \dashrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ X & \dashrightarrow & \mathbf{B}^{n+1} A. \end{array}$$

Here the cartesian square is required to identify the given band on $\widetilde{X}$ with the band pulled back from the universal gerbe. More formally, the homotopy fiber of the functor in the statement over $\widetilde{X} \to X$ is the subanima of

$$\mathrm{Hom}_{\mathrm{Ar}^{\mathrm{pb}}(T)}\big(\widetilde{X} \to X,\ * \to \mathbf{B}^{n+1} A\big)$$

spanned by the band-compatible cartesian squares. We show that this subanima is contractible.

Step 1: Suppose first that $\widetilde{X} \to X$ admits a section $s \colon X \to \widetilde{X}$. In the slice topos $T_{/X}$, the section makes $\widetilde{X}$ into a pointed n-gerbe. By Lemma 3.49, there is a unique band-preserving pointed isomorphism $\widetilde{X} \xrightarrow{\sim} X \times \mathbf{B}^n A$.

A band-compatible cartesian square with the universal gerbe is now the same as a map $u \colon X \to \mathbf{B}^{n+1} A$ together with a trivialization of its fiber $P_u := X \times_{\mathbf{B}^{n+1} A} * \to X$. Such a trivialization is equivalently a section of $P_u \to X$, or equivalently a nullhomotopy of u. The anima of pairs consisting of a map $u \colon X \to \mathbf{B}^{n+1} A$ and a nullhomotopy of u is $\mathrm{Hom}_T(X, *)$, which is contractible. The band compatibility is built into the unique pointed identification supplied by Lemma 3.49. This proves the claim in the pointed case.

Step 2: We now prove the claim in general. Since $n \geq 1$, the map $\widetilde{X} \to X$ is an effective epimorphism. We may therefore take $U = \widetilde{X}$; the pullback $\widetilde{X} \times_X U \to U$ admits the diagonal as a section. Let $\check{C}(U/X)$ denote the Čech nerve, and write $U_k := \check{C}(U/X)_k$ and $\widetilde{U}_k := \widetilde{X} \times_X U_k$. Each map $\widetilde{U}_k \to U_k$ is an n-gerbe banded by A admitting a section. Step 1 therefore shows that the anima of band-compatible cartesian squares

$$\widetilde{U}_k \to U_k \quad \longrightarrow \quad * \to \mathbf{B}^{n+1} A$$

is contractible for every k. Objects, isomorphisms, and identifications of their bands satisfy descent along effective epimorphisms. Hence the anima of band-compatible cartesian squares from $\widetilde{X} \to$

X to the universal gerbe is the limit of these contractible animae, and is therefore contractible. Thus every homotopy fiber of the functor in the statement is contractible, proving that it is an equivalence. □

Remark 3.51. Under the equivalence $T_{/\mathbf{B}G} \simeq \mathrm{Mod}_G(T)$, passing to fibers over T (with respect to the forgetful functor $T_{/\mathbf{B}G} \to T$ and the quotient functor $\mathrm{Mod}_G(T) \to T$) yields

$$\mathrm{Hom}_T(*, \mathbf{B}G) \;\simeq\; \{G\text{-torsors in } T\}^{\simeq}.$$

Let us now return to the question of obstruction theory. Given an object X and $n \geq 2$, the map $\tau_n X \to \tau_{n-1} X$ is an n-gerbe. By Lemma 3.46, it is banded by a static abelian group object

$$A \in \mathrm{Ab}\big((T_{/\tau_{n-1}X})_{\leq 0}\big).$$

Applying Theorem 3.50 in the slice topos $T_{/\tau_{n-1}X}$, we may write it as a pullback

$$\begin{array}{ccc} \tau_n X & \longrightarrow & \tau_{n-1} X \\ \downarrow & \lrcorner & \downarrow \\ \tau_{n-1} X & \longrightarrow & \mathbf{B}^{n+1} A. \end{array}$$

Here $\mathbf{B}^{n+1}A$ is formed in the slice over $\tau_{n-1}X$, and the upper-right object is the terminal object of this slice. In particular, finding a lift $Y \to \tau_n X$ of a map $Y \to \tau_{n-1}X$ is equivalent to nullhomotoping the induced map $Y \to \mathbf{B}^{n+1}A$ in the slice. We wish to think of this as the question of whether a certain cohomology class vanishes:

Definition 3.52. Let A be an abelian group object in $T_{\leq 0}$, and let $Y \in T$. We define the *n-th cohomology of Y with coefficients in A* as

$$H^n(Y, A) := \pi_0 \,\mathrm{Hom}_T(Y, \mathbf{B}^n A) \in \mathrm{Ab}.$$

4 The categories of topoi and logoi

A morphism between topoi can be viewed in two opposite directions. The *algebraic* direction records a left exact colimit-preserving functor $f^*\colon S \to T$, while the *geometric* direction records its right adjoint $f_*\colon T \to S$. We write Logos and Topos for the resulting categories. Although a morphism carries the same adjunction in either language, the two directions illuminate different constructions: generators, relations, and quotients are most naturally expressed in Logos, while slices, base change, and geometric morphisms are most naturally expressed in Topos.

The algebraic perspective allows topoi to be constructed by presentations. Given a small category C, we first form the free logos $\mathrm{An}[C]$ generated by C. Given a congruence K in a logos T, we then form a quotient T/K in which the morphisms of K become invertible. Combining these constructions produces presentations by generators and relations, and every logos admits such a presentation. In Section 4.2, we develop this formalism and its universal properties.

Presentations provide enough control to construct all small limits and colimits in Logos and Topos. Some of these constructions are computed in the underlying categories, while others require choosing presentations and combining their generators and relations. We carry this out in Section 4.3; among the consequences are that An is the terminal topos and that products of topoi are given by the tensor product of cocomplete categories.

The geometric part of the chapter concerns *étale morphisms*, the topos-theoretic analogues of local homeomorphisms. The fundamental classification says that the étale topoi over a fixed topos T are precisely its slice topoi:

$$T \simeq \mathrm{Topos}^{\text{ét}}_{/T}, \qquad X \longmapsto T_{/X}.$$

Thus the internal objects of T classify a distinguished class of geometric objects over T. The class of étale morphisms is stable under base change, and, more surprisingly, the wide subcategory $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$ admits colimits which are preserved by its inclusion into Topos. The proof of this colimit theorem occupies Section 4.5; its main ingredients are univalent families, dependent products, and a recognition criterion for étale morphisms.

4.1 Topoi and logoi

Following Anel and Joyal [AJ21], we use separate words for the two directions of the adjunction underlying a geometric morphism: *topos* for the geometric direction and *logos* for the algebraic direction. Their terminology emphasizes a duality analogous to that between affine schemes and commutative rings. It is not yet common, and Marc did not use it in his lectures, but it makes the variance of the constructions below more transparent.

Definition 4.1. Given two topoi S and T, a *morphism of topoi* (or *geometric morphism*) from S to T is a functor $\varphi_*\colon S \to T$ which admits a left exact colimit-preserving left adjoint $\varphi^*\colon T \to S$. We denote by

$$\mathrm{Topos} \quad \subseteq \quad \mathrm{Cat}$$

the subcategory spanned by the topoi and the morphisms of topoi.

Definition 4.2. A category T is called a *logos* if it is a topos, i.e. if it is presentable and satisfies descent for all colimits. Given two logoi S and T, a *morphism of logoi* (sometimes called *algebraic morphism*[1]) from S to T is a left exact colimit-preserving functor $\varphi^*\colon S \longrightarrow T$. We denote by

$$\mathrm{Logos} \quad \subseteq \quad \mathrm{Cat}$$

the subcategory spanned by the logoi and the morphisms of logoi.

Remark 4.3. By Lemma 3.8, every morphism of logoi preserves n-truncated objects and commutes with n-truncations for all n.

Remark 4.4. By the adjoint functor theorem, every morphism of logoi $\varphi^*\colon S \to T$ admits a right adjoint $\varphi_*\colon T \to S$, which is then a morphism of topoi. This assignment can be made functorial: there is an equivalence of categories

$$\mathrm{Topos} \quad \simeq \quad \mathrm{Logos}^{\mathrm{op}}, \qquad T \mapsto T, \qquad \varphi_* \mapsto \varphi^*.$$

More precisely, this equivalence arises as a restriction of the equivalence $\mathrm{Pr}^{\mathrm{R}} \simeq (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}}$ from [Lur09, Corollary 5.5.3.4].

We now upgrade Logos and Topos to 2-categories.

Construction 4.5. The category Cat is cartesian closed (the functor $C \times -\colon \mathrm{Cat} \to \mathrm{Cat}$ admits a right adjoint $\mathrm{Fun}(C, -)\colon \mathrm{Cat} \to \mathrm{Cat}$ for all C) and thus enriched over itself. This gives rise to a 2-category $\mathbb{C}\mathrm{at}$, whose objects are categories, whose morphisms are functors, and whose 2-morphisms are natural transformations between functors: $\mathrm{Hom}_{\mathbb{C}\mathrm{at}}(C, D) = \mathrm{Fun}(C, D)$.

This in particular allows us to upgrade the category Logos to a 2-category, by regarding it as a locally full subcategory of $\mathbb{C}\mathrm{at}$:

$$\mathbb{L}\mathrm{ogos} \quad \subseteq \quad \mathbb{C}\mathrm{at}.$$

The Hom-categories $\mathbb{L}\mathrm{ogos}$ are given by the full subcategories

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(S, T) \quad \subseteq \quad \mathrm{Fun}(S, T)$$

spanned by the morphisms of logoi. We may similarly upgrade Topos to a 2-category, by setting

$$\mathbb{T}\mathrm{opos} \quad := \quad \mathbb{L}\mathrm{ogos}^{\mathrm{op}} \quad \subseteq \quad \mathbb{C}\mathrm{at}^{\mathrm{op}}.$$

In other words, we define the Hom-categories of $\mathbb{T}\mathrm{opos}$ to be $\mathrm{Fun}_{\mathbb{T}\mathrm{opos}}(T, S) := \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(S, T)$.

Remark 4.6. The subcategory $\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(S, T)$ of $\mathrm{Fun}(S, T)$ is denoted by $\mathrm{Fun}^*(S, T)$ by Lurie. Marc in his lectures denoted $\mathrm{Fun}_{\mathbb{T}\mathrm{opos}}(T, S)$ by $\mathrm{Geom}(T, S)$.

[1] We warn the reader that this terminology clashes with the notion of 'algebraic morphism' used by Lurie [Lur09, Definition 6.3.6.1].

A point in a topological space X is the same as a continuous map $* \to X$ from the one-point space, i.e. the terminal object in the category of topological spaces. Given that the terminal object of Topos is An (see Example 4.12 below), we get the following analogous notion for topoi:

Definition 4.7. A *point* of a topos T is a geometric morphism $\mathrm{An} \to T$. We write

$$\mathrm{Pt}(T) \quad := \quad \mathrm{Fun}_{\mathbb{T}\mathrm{opos}}(\mathrm{An}, T)$$

for the category of points.

Example 4.8. Let $T = \mathrm{PSh}(C)[\Sigma^{-1}]$ be a left exact localization of a presheaf category. Then the inclusion

$$\mathrm{Pt}(T) \quad \simeq \quad \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T, \mathrm{An}) \quad \hookrightarrow \quad \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(C), \mathrm{An}) \quad \simeq \quad \mathrm{Pro}(C)^{\mathrm{op}}$$

identifies $\mathrm{Pt}(T)$ with the Σ-*local pro-objects in* C, i.e. those formal systems $(X_i)_{i \in I}$ for which the functor

$$\mathrm{PSh}(C) \to \mathrm{An}, \qquad F \longmapsto \mathrm{colim}_i\, F(X_i)$$

inverts all morphisms in Σ. Here we use the standard correspondence between pro-objects of C and left exact colimit-preserving functors $\mathrm{PSh}(C) \to \mathrm{An}$: a pro-object $(X_i)_i$ determines the displayed functor, and every such functor arises uniquely in this way.

4.2 Presentations of logoi

Algebraic objects can often be presented in terms of 'generators and relations'; think of presentations of groups or rings. In the case of commutative rings, one first forms the free commutative ring $\mathbb{Z}[S]$ on a given set of generators S (the polynomial ring), and then forms the quotient of $\mathbb{Z}[S]$ by an ideal generated by a given set of relations. There is an analogous notion of presentations for logoi:

- In Section 4.2.1 we construct the *free logos* $\mathrm{An}[C]$ on a given category of generators C.
- In Section 4.2.2, we define the *quotient logos* T/K of a logos T by a congruence K.
- In Section 4.2.3, we introduce *logos presentations* (C, Σ), consisting of a small category C and a small collection of morphisms Σ in $\mathrm{An}[C]$, and define the logos presented by (C, Σ) as $\langle C \mid \Sigma \rangle := \mathrm{An}[C]/\Sigma^c$, where Σ^c is the congruence generated by Σ.

4.2.1 Free logoi

We start by constructing the free logos on a small category.

Construction 4.9. Let C be a small category. We define the *free logos on* C as

$$\mathrm{An}[C] \quad := \quad \mathrm{PSh}(C^{\mathrm{lex}}),$$

where $C^{\mathrm{lex}} \subseteq \mathrm{PSh}(C^{\mathrm{op}})^{\mathrm{op}}$ is defined as the smallest subcategory containing C which is closed under finite limits. The category $\mathrm{An}[C]$ comes equipped with a canonical inclusion

$$C \hookrightarrow C^{\mathrm{lex}} \hookrightarrow \mathrm{PSh}(C^{\mathrm{lex}}) = \mathrm{An}[C].$$

Proposition 4.10. *For any small category C, the inclusion $C \hookrightarrow \mathrm{An}[C]$ exhibits $\mathrm{An}[C]$ as the free logos generated by C: for any other logos T, restriction along the inclusion induces an equivalence of categories*

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C], T) \quad \simeq \quad \mathrm{Fun}(C, T).$$

Proof. Recall from [Lur09, Proposition 5.3.6.2] that for a small category C, the subcategory $C^{\mathrm{rex}} \subseteq \mathrm{PSh}(C)$ generated by the representables under finite colimits is the free finite cocompletion of C. It follows that $C^{\mathrm{lex}} = ((C^{\mathrm{op}})^{\mathrm{rex}})^{\mathrm{op}}$ is the free finite limit completion: for any other category D with finite limits, restriction along $C \hookrightarrow C^{\mathrm{lex}}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(C^{\mathrm{lex}}, D) \quad \simeq \quad \mathrm{Fun}(C, D).$$

By the universal property of the presheaf category, we also see that restriction along $C^{\mathrm{lex}} \hookrightarrow \mathrm{PSh}(C^{\mathrm{lex}})$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(C^{\mathrm{lex}}), T) \quad \simeq \quad \mathrm{Fun}(C^{\mathrm{lex}}, T).$$

By Proposition 2.43, this functor restricts to the desired equivalence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C], T) \quad = \quad \mathrm{Fun}^{\mathrm{colim,lex}}(\mathrm{PSh}(C^{\mathrm{lex}}), T) \quad \simeq \quad \mathrm{Fun}^{\mathrm{lex}}(C^{\mathrm{lex}}, T). \qquad \square$$

Remark 4.11. We may think of this result as a categorification of the isomorphism $\mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[S], R) \cong \mathrm{Hom}_{\mathrm{Set}}(S, R)$ for every set S and every commutative ring R.

Example 4.12. For $C = \emptyset$, we get $\emptyset^{\mathrm{lex}} = *$ and so $\mathrm{An}[\emptyset] = \mathrm{PSh}(*) = \mathrm{An}$. In particular, An is the *initial logos*:

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}, T) \quad \simeq \quad \mathrm{Fun}(\emptyset, T) \quad \simeq \quad *.$$

By dualizing, it follows that $\mathrm{An} \in \mathrm{Topos}$ is the *terminal topos*.

Example 4.13. When $C = * = \{X\}$ is the terminal category, we get $C^{\mathrm{lex}} = (\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}}$, the opposite of the category of finite animae. In particular, the *free logos on one generator* is the logos

$$\mathrm{An}[X] \quad := \quad \mathrm{An}[\{X\}] \quad = \quad \mathrm{PSh}((\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}}) \quad = \quad \mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathrm{An}).$$

4.2.2 Congruences and quotients of logoi

In classical algebra, we can form the quotient of a ring R by an ideal I. As we discuss in this section, we may analogously form the quotient of a logos T by a class of morphisms K. This produces a new logos T/K in which the morphisms of K have been inverted. We begin by defining the precise conditions on K that ensure such a quotient behaves well.

Definition 4.14. Let T be a logos. A collection of morphisms K in T is called a *congruence of small generation* if it satisfies the following conditions:

(1) K is strongly saturated of small generation (see Section A.3).

(2) K is closed under base change.

Recall from Proposition A.15 that condition (1) ensures that the localization $T \to T[K^{-1}]$ exists and is an accessible Bousfield localization (i.e. the target is presentable and the localization functor admits a fully faithful right adjoint). Condition (2) ensures that the localization is compatible with the logos structure, i.e. that it is left exact:

Lemma 4.15 (cf. [Lur09, Proposition 6.2.1.1])**.** *Let $L\colon C \to D$ be a Bousfield localization, with fully faithful right adjoint $R\colon D \hookrightarrow C$. Assume that C admits finite limits. Then the following conditions are equivalent:*

(1) *The localization functor L is left exact.*

(2) *The class of morphisms inverted by L is closed under finite limits in* $\mathrm{Ar}(C)$.

(3) *The class of morphisms inverted by L is closed under base change.*

Proof. The implications (1) $\implies$ (2) $\implies$ (3) are clear. For (3) $\implies$ (1), let K be the class of morphisms inverted by L, and assume K is closed under base change. Observe that an object of C lies in the essential image of R if and only if it is local with respect to K. Since the terminal object 1 of C is clearly K-local, it follows that $L(1) \xrightarrow{\sim} 1$, and so L preserves the terminal object. It remains to show that L also preserves pullbacks, so consider two morphisms $X \to Y \leftarrow Z$. Since K-local objects are closed under limits in C, it follows that $RLX \times_{RLY} RLZ$ is K-local; in particular, it defines the pullback in D. It now remains to show that the canonical map $X \times_Y Z \to RLX \times_{RLY} RLZ$ lies in K. We may factor this map as a composite

$$X \times_Y Z \to X \times_{RLY} Z \to RLX \times_{RLY} Z \to RLX \times_{RLY} RLZ.$$

The second and third maps are base changes of morphisms in K, hence lie in K again by assumption. The first map is a base change of the diagonal of $Y \to RLY$, which is therefore a section of the projection $\mathrm{pr}_1 : Y \times_{RLY} Y \to Y$. Since K is closed under 2-out-of-3, it suffices to show that pr_1 is in K, which is true as it is a base change of $Y \to RLY$. □

We now combine these results with the properties of saturated classes discussed in the appendix to obtain the following characterization. Recall from Definition A.13 that the *kernel* of a cocontinuous functor φ between presentable categories is defined as the class of morphisms inverted by φ.

Corollary 4.16. *Let T be a logos and K a class of morphisms in T. The following are equivalent:*

(1) *K is a congruence of small generation.*

(2) *K is strongly saturated of small generation and is closed under finite limits in* $\mathrm{Ar}(T)$.

(3) *The localization functor $L\colon T \to T[K^{-1}]$ is left exact and admits a fully faithful right adjoint (i.e. it is a Bousfield localization).*

(4) *K is the kernel of some morphism of logoi $\varphi\colon T \to S$.*

Proof. The equivalence (1) $\iff$ (3) follows immediately from Definition 4.14 and Lemma 4.15, combined with Proposition A.15.

The implication (3) $\implies$ (4) follows since K is the kernel of $L\colon T \to T[K^{-1}]$, see Proposition A.15. For (4) $\implies$ (2), if K is the kernel of a morphism of logoi φ, then K is closed under finite limits because φ is left exact. Furthermore, since φ is a morphism of logoi, it preserves colimits, and thus its kernel is strongly saturated of small generation by [Lur09, Proposition 5.5.4.16] (see Proposition A.14).

Finally, for (2) $\implies$ (1), it suffices to note that a class closed under finite limits is in particular closed under base change. □

We can now define the quotient of a logos.

Definition 4.17. Let K be a congruence of small generation in a logos T. We define the *quotient of T by K* as the localization

$$T/K \quad := \quad T[K^{-1}].$$

By Corollary 4.16, T/K is a logos and the localization map $T \to T/K$ is a morphism of logoi.

This construction satisfies the expected universal property:

Proposition 4.18. *For every logos S, restriction along $T \to T/K$ induces a fully faithful functor*

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T/K, S) \quad \hookrightarrow \quad \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T, S).$$

Its essential image consists of those morphisms of logoi $\varphi\colon T \to S$ that invert K (i.e. $K \subseteq \ker(\varphi)$).

Proof. By definition of localization, restriction along $T \to T/K$ induces a fully faithful functor

$$\mathrm{Fun}(T/K, S) \hookrightarrow \mathrm{Fun}(T, S),$$

with essential image those functors $T \to S$ that invert K. It remains to show that a functor $T/K \to S$ preserves colimits and finite limits if and only if the composite $T \to T/K \to S$ does.

The 'only if' direction is clear. For the 'if' direction, note that every diagram in T/K may be regarded as a diagram in T via the inclusion. Since $T \to T/K$ preserves colimits and finite limits, colimits and finite limits in T/K may be computed by forming them first in T and then localizing. Thus if $T \to S$ is a logos morphism, then so is $T/K \to S$. □

Definition 4.19. A morphism of logoi $\varphi\colon T \to S$ is called a *quotient map* if it is a Bousfield localization, i.e. if its right adjoint is fully faithful.

Note that any quotient map $\varphi\colon T \to S$ induces an equivalence of logoi $T/\ker(\varphi) \simeq S$.

Lemma 4.20. *Any quotient map of logoi is an epimorphism in the category of logoi.*

Proof. Let $\varphi^*\colon T \to S$ be a quotient map and put $K = \ker(\varphi^*)$. The induced morphism $T/K \to S$ is an equivalence of logoi. By the universal property of the quotient from Proposition 4.18, for every logos T' the restriction functor

$$(\varphi^*)^*\colon \mathrm{Hom}_{\mathrm{Logos}}(S, T') = \mathrm{Fun}^{\mathrm{lex,colim}}(S, T')^{\simeq} \hookrightarrow \mathrm{Fun}^{\mathrm{lex,colim}}(T, T')^{\simeq} = \mathrm{Hom}_{\mathrm{Logos}}(T, T').$$

is fully faithful. This is precisely the assertion that φ^* is an epimorphism in Logos. □

Remark 4.21. The previous lemma implies that any quotient map of logoi is a *monomorphism* when regarded as a morphism in Topos. This is analogous to the fact that an epimorphism $R \to S$ of commutative rings induces a monomorphism $\mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$ of affine schemes. Recall that there are two common constructions for commutative rings that give rise to epimorphisms:

(1) Given a commutative ring R and an ideal I, we may form the quotient ring $R \to R/I$ by killing off all elements in I.

(2) Given a commutative ring R and a set S of elements of R, we may form the localization $R \to R[S^{-1}]$ by formally inverting all elements in S.

The quotient map $R \to R/I$ corresponds to a *closed immersion* of affine schemes. A localization map $R \to R[S^{-1}]$ induces a flat monomorphism $\mathrm{Spec}(R[S^{-1}]) \to \mathrm{Spec}(R)$; it is an *open immersion* when S is generated by finitely many elements, equivalently when the localization may be written as $R \to R[f^{-1}]$ for a single $f \in R$. We will discuss the analogous notions of open and closed immersions of topoi in Section 5.4 below.

We conclude with two fundamental examples of congruences.

Example 4.22. Let Σ be a small set of morphisms in a logos T. The *congruence generated by* Σ, denoted Σ^c, is defined as the intersection of all strongly saturated classes containing Σ that are closed under base change. It is a non-trivial fact that Σ^c is of small generation (and thus a congruence in the sense of Definition 4.14). This is proved by Lurie [Lur09, Proposition 6.2.1.2]. We will give an independent proof below, see Remark 5.42.

Example 4.23. Let $\varphi\colon S \to T$ be a morphism of logoi. Given a congruence of small generation K in T, we define its *preimage* as

$$\varphi^{-1}(K) \quad := \quad \{g \in \mathrm{Ar}(S) \mid \varphi(g) \in K\}.$$

This is again a congruence of small generation. Indeed, consider the composite morphism of logoi

$$S \xrightarrow{\varphi} T \xrightarrow{L} T/K.$$

As the kernel of L is K, the kernel of this composite is precisely $\varphi^{-1}(K)$. By Corollary 4.16, the kernel of any morphism of logoi is a congruence of small generation, which proves the claim.

4.2.3 Presentations of logoi

We now combine free logoi with quotients of logoi to arrive at the following definition:

Definition 4.24. A *logos presentation* is a pair (C, Σ) consisting of a small category C together with a small class of morphisms Σ in $\mathrm{An}[C]$. We define the *logos presented by* (C, Σ) as the quotient

$$\langle C \mid \Sigma \rangle \quad := \quad \mathrm{An}[C] \,/\, \Sigma^c$$

of the free logos on C by congruence Σ^c from Example 4.22. In particular, $\langle C \mid \Sigma \rangle$ is a logos and the quotient map $\mathrm{An}[C] \to \langle C \mid \Sigma \rangle$ is a logos morphism.

We wish to think of C as the 'generators' of $\langle C \mid \Sigma \rangle$, and of Σ as the 'relations'.

More explicitly, $\langle C \mid \Sigma \rangle$ may be identified with the full subcategory of $\mathrm{PSh}(C^{\mathrm{lex}})$ spanned by the Σ^c-local presheaves.

Lemma 4.25. *Let (C, Σ) be a logos presentation. Then the quotient functor $\mathrm{An}[C] \to \langle C \mid \Sigma \rangle$ is universal among logos morphisms $\mathrm{An}[C] \to T$ that invert Σ.*

Proof. By Proposition 4.18, $\mathrm{An}[C] \to \langle C \mid \Sigma \rangle$ is universal among logos morphisms $\varphi\colon \mathrm{An}[C] \to T$ satisfying $\Sigma^c \subseteq \ker(\varphi)$. But since $\ker(\varphi)$ is a congruence, this is equivalent to the condition that $\Sigma \subseteq \ker(\varphi)$, i.e. that φ inverts Σ. □

We now show that every logos admits a presentation.

Proposition 4.26. *Let T be a logos. Then there exists a logos presentation (C, Σ) and an equivalence of logoi*

$$\langle C \mid \Sigma \rangle \simeq T.$$

Proof. We saw in the proof of Theorem 2.42 that there is a small category C with finite limits and an accessible left exact localization

$$L\colon \mathrm{PSh}(C) \longrightarrow T.$$

Let $i\colon C \hookrightarrow C^{\mathrm{lex}}$ be the canonical functor, and let $F\colon C^{\mathrm{lex}} \to C$ be the left exact extension of id_C. The universal property of C^{lex} gives

$$\mathrm{Hom}_{C^{\mathrm{lex}}}(iX, Y) \simeq \mathrm{Hom}_C(X, FY)$$

for $X \in C$ and $Y \in C^{\mathrm{lex}}$. Thus $i \dashv F$, and the unit $\mathrm{id}_C \to Fi$ is an isomorphism.

The left Kan extension along F is therefore restriction along i:

$$F_! \simeq i^*\colon \mathrm{PSh}(C^{\mathrm{lex}}) \longrightarrow \mathrm{PSh}(C).$$

In particular, $F_!$ is left exact. Its right adjoint is the restriction functor $F^*\colon \mathrm{PSh}(C) \to \mathrm{PSh}(C^{\mathrm{lex}})$, which is fully faithful because $Fi \simeq \mathrm{id}_C$. Hence $F_!$ is a left exact Bousfield localization. It follows that the composite

$$Q\colon \mathrm{An}[C] = \mathrm{PSh}(C^{\mathrm{lex}}) \xrightarrow{F_!} \mathrm{PSh}(C) \xrightarrow{L} T$$

is again a left exact Bousfield localization.

Its kernel $K := \ker(Q)$ is a congruence of small generation by Corollary 4.16. Choose a small class $\Sigma \subseteq K$ which generates K as a strongly saturated class. Since K is a congruence, $\Sigma^c \subseteq K$; conversely, Σ^c is strongly saturated and contains Σ, so $K \subseteq \Sigma^c$. Hence $K = \Sigma^c$. The universal property of the quotient now identifies

$$\langle C \mid \Sigma \rangle = \mathrm{An}[C]/\Sigma^c \simeq T,$$

which is the desired presentation. □

Example 4.27. The following are some examples of presentations of logoi:

(1) As discussed, the initial logos (no generators and no relations) is An:

$$\langle \emptyset \mid \emptyset \rangle \quad = \quad \mathrm{An}.$$

(2) The free logos on a single generator X is $\mathrm{An}[X]$:

$$\langle \{X\} \mid \emptyset \rangle \quad = \quad \mathrm{An}[X] \quad = \quad \mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathrm{An}).$$

(3) More generally, the logos presented by $(C, \emptyset)$ is simply the free logos generated by C:

$$\langle C \mid \emptyset \rangle \quad = \quad \mathrm{An}[C] \quad = \quad \mathrm{PSh}(C^{\mathrm{lex}}).$$

(4) The free logos generated by some initial object X is simply An:

$$\langle \{X\} \mid \emptyset \to X \rangle \quad \simeq \quad \mathrm{An}.$$

Indeed, logos morphisms $F \colon \langle \{X\} \mid \emptyset \to X \rangle \to T$ correspond to logos morphisms $\mathrm{An}[X] = \langle \{X\} \mid \emptyset \rangle \to T$ sending the generator X to the initial object of T, but by definition of $\mathrm{An}[X]$ there is a unique such morphism.

(5) The free logos in which the initial object is terminal is the one-object logos:

$$\langle \{X\} \mid \emptyset \to * \rangle \quad \simeq \quad *.$$

Indeed, any logos in which the initial and terminal objects agree is trivial: for all Y we have $\emptyset \simeq \emptyset \times Y \simeq * \times Y \simeq Y$.

(6) The free logos generated by an n-truncated object is

$$\langle \{X\} \mid X \to X^{S^{n+1}} \rangle \quad \simeq \quad \mathrm{Fun}((\mathrm{An}_{\leq n})^{\mathrm{fin}}, \mathrm{An}),$$

where $(\mathrm{An}_{\leq n})^{\mathrm{fin}} \subseteq \mathrm{An}_{\leq n}$ is the subcategory generated under finite colimits by the point. The left-hand side classifies the assignment $T \mapsto T_{\leq n}$ of n-truncated objects. Indeed, an object of T corresponds to a left exact functor $(\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}} \to T$. This object is n-truncated precisely when the corresponding functor factors through the opposite of the truncation functor

$$\tau_n \colon \mathrm{An}^{\mathrm{fin}} \longrightarrow (\mathrm{An}_{\leq n})^{\mathrm{fin}}.$$

(7) The free logos generated by an n-connected object X is

$$\langle \{X\} \mid \tau_n(X) \to * \rangle.$$

It arises as a localization of $\mathrm{An}[X] = \langle * \mid \emptyset \rangle$.

(8) The *pointed object classifier* is the logos

$$\langle [1] \mid \mathrm{source} = * \rangle \quad \simeq \quad \mathrm{PSh}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}).$$

Note that a logos morphism $\langle [1] \mid \mathrm{source} = * \rangle \to T$ is just a morphism in T whose source is terminal, i.e. a pointed object. But the right-hand side also classifies pointed objects: we have

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}), T) \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, T) \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, T_*) \simeq T_*.$$

(9) Given a Lawvere theory L, there is the *L-algebra classifier* $\langle L \mid \text{finite products} \rangle$: for every logos T we have

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\langle L \mid \text{finite products} \rangle, T) \quad \simeq \quad \mathrm{Alg}_L(T) \quad := \quad \mathrm{Fun}^{\times}(L, T).$$

This logos may be explicitly described as

$$\langle L \mid \text{finite products} \rangle \quad \simeq \quad \mathrm{PSh}((\mathrm{PSh}_{\Sigma}(L)^{\mathrm{fin}})^{\mathrm{op}}).$$

Here $\mathrm{PSh}_{\Sigma}(L) \subseteq \mathrm{Fun}(L^{\mathrm{op}}, \mathrm{An})$ denotes the full subcategory of finite-product-preserving presheaves.

4.3 Limits and colimits of topoi

The variance between topoi and logoi makes the existence problem asymmetric. Colimits of topoi are limits of logoi, and these are computed in the underlying category of categories. Limits of topoi, equivalently colimits of logoi, require an explicit construction by generators and relations. Even when a general limit of topoi is not computed on underlying categories, a filtered limit is. Finally, finite products of topoi admit a particularly concrete description by the tensor product of cocomplete categories. We treat these four assertions in turn.

Theorem 4.28. *The categories* Topos *and* Logos *admit small limits and colimits. Furthermore, the following properties are satisfied:*

(1) *The inclusion* Logos $\hookrightarrow$ Cat*,* $\varphi \mapsto \varphi^*$*, preserves limits.*

(2) *The inclusion* Topos $\hookrightarrow$ Cat*,* $\varphi \mapsto \varphi_*$*, preserves filtered limits.*

(3) *The inclusion* Logos $\hookrightarrow \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{colim}})$ *preserves finite coproducts. In particular:*

- An *is the terminal topos.*
- *The product* $T \times S$ *of two topoi is computed as the Lurie tensor product* $T \otimes S$ *of cocomplete categories.*

Remark 4.29. This theorem does not give an explicit general description of all limits in Topos (e.g. pullbacks or cosimplicial limits); these are in general more subtle to describe.

4.3.1 Limits of logoi and colimits of topoi

Limits of logoi are computed on their underlying categories.

Proposition 4.30. *The category* Logos *admits small limits, and the inclusion* Logos $\hookrightarrow$ Cat *preserves them. Equivalently,* Topos *admits small colimits.*

Proof. Given a diagram $I \to \mathrm{Logos}$, $i \mapsto T_i$ of logoi, define $T := \lim_i T_i$, which is again a presentable category. Given an indexing diagram K, a functor $K^{\triangleright} \to T$ is a colimit diagram if and only if it becomes a colimit diagram in each T_i. The analogous statement holds for finite limit diagrams. In particular, colimits and finite limits in T are computed pointwise in the categories T_i. Note that we have

$$\mathrm{Ar}(T) \simeq \lim_i \mathrm{Ar}(T_i) \qquad \text{and} \qquad \mathrm{Ar}^{\mathrm{pb}}(T) \simeq \lim_i \mathrm{Ar}^{\mathrm{pb}}(T_i).$$

It then follows immediately from the characterization of logoi given in Proposition 2.16 that T is again a logos. Furthermore, we see that a functor $S \to T$ is a logos morphism if and only if each composite $S \to T_i$ is a logos morphism. It follows that the limit in Cat is in fact a limit in Logos, as desired. □

4.3.2 Colimits of logoi and limits of topoi

The reverse construction uses presentations by generators and relations.

Proposition 4.31. *The categories* Logos *and* Topos *admit small colimits and small limits, respectively.*

Proof. Let $T_\bullet \colon I \to \mathrm{Logos}$ be a small diagram. By uniformization for accessible categories and functors, we may choose a regular cardinal κ such that each T_i is κ-accessible, every transition morphism is strongly κ-accessible, and the full subcategory

$$C_i := (T_i)^\kappa$$

of κ-compact objects is closed under finite limits. The transition morphisms therefore restrict to a diagram $C_\bullet \colon I \to \mathrm{Cat}$ of small categories with finite limits. For the simultaneous choice of κ, see the uniformization argument in [Lur09, Remark 5.4.2.13]; closure of the compact objects under finite limits follows after a further enlargement from [Lur09, Proposition 5.4.7.4].

For each i, the colimit extension $\mathrm{PSh}(C_i) \to T_i$ of the inclusion $C_i \hookrightarrow T_i$ is left exact by Proposition 2.43. Its right adjoint is the restricted Yoneda functor, which is fully faithful because C_i is a collection of generators. Composing this localization with the canonical localization $\mathrm{An}[C_i] \to \mathrm{PSh}(C_i)$ from the proof of Proposition 4.26, we obtain a left exact Bousfield localization

$$q_i \colon \mathrm{An}[C_i] \longrightarrow T_i.$$

Choose a small class of morphisms $\Sigma_i \subseteq \mathrm{An}[C_i]$ whose generated congruence is $\ker(q_i)$. Thus q_i induces an equivalence $\langle C_i \mid \Sigma_i \rangle \simeq T_i$.

These presentations are compatible with the diagram. Indeed, for an arrow $a \colon i \to j$ in I, let $a^* \colon T_i \to T_j$ denote the transition morphism and also its restriction $C_i \to C_j$. The two logos morphisms

$$\mathrm{An}[C_i] \rightrightarrows T_j$$

given by $a^* q_i$ and by $q_j \mathrm{An}[a^*]$ agree on the generators C_i, hence are equivalent by the universal property of the free logos. It follows that $\mathrm{An}[a^*]$ sends $\ker(q_i)$ into $\ker(q_j)$ and therefore induces the original transition morphism between the presented logoi.

Now form the small category $C := \mathrm{colim}_i C_i$, and let Σ be the union of the images of the small classes Σ_i in $\mathrm{An}[C]$. Consider the logos

$$T := \langle C \mid \Sigma \rangle.$$

The functors $C_i \to C$ induce a cocone $T_i \to T$. To verify its universal property, let S be a logos. Then

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C], S) \simeq \mathrm{Fun}(C, S) \simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}(C_i, S) \simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C_i], S).$$

Under this equivalence, a logos morphism $\mathrm{An}[C] \to S$ inverts Σ if and only if each of its restrictions to $\mathrm{An}[C_i]$ inverts Σ_i. By Lemma 4.25, such data are precisely a compatible family of logos morphisms $T_i \to S$. Hence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T, S) \simeq \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T_i, S),$$

so T is the colimit of $T_\bullet$ in Logos. □

4.3.3 Filtered limits of topoi

Filtered limits are exceptional among limits of topoi: they are computed on the underlying categories.

Proposition 4.32. *The inclusion* Topos $\hookrightarrow$ Cat, $\varphi \mapsto \varphi_*$, *preserves filtered limits.*

Proof. We sketch the construction, following [Lur09, Section 6.3.3]. Given a functor $I \to$ Topos, consider the associated cartesian fibration $p\colon E \to I^{\mathrm{op}}$. The category of sections $\Gamma(p)$ is a topos by [Lur09, Lemma 6.3.3.2]; its colimits and finite limits are computed fiberwise, as also follows from Chapter B. Assume that I is filtered. The full subcategory $\Gamma_{\mathrm{cart}}(p)$ of cartesian sections is an accessible left exact localization of $\Gamma(p)$ by [Lur09, Proposition 6.3.3.3]. Consequently, $\Gamma_{\mathrm{cart}}(p)$ is again a topos. Moreover, the evaluation functors

$$\mathrm{ev}_i\colon \Gamma_{\mathrm{cart}}(p) \longrightarrow E_i$$

are morphisms of topoi by [Lur09, Proposition 6.3.3.5].

Let us recall why the left exact localization is the central point. After replacing the indexing category by a cofinal filtered partially ordered set, let $B \subseteq I$ be a cofinal subset. Write $\Gamma_B(p) \subseteq \Gamma(p)$ for the full subcategory of sections which are right Kan extensions of their restrictions to B. Restriction to B admits right Kan extension as a fully faithful right adjoint, so $\Gamma_B(p)$ is an accessible localization of $\Gamma(p)$. The reflector is left exact because both restriction and right Kan extension preserve finite limits.

The cartesian sections are precisely the intersection of the subcategories $\Gamma_B(p)$ as B ranges over all cofinal subsets. Indeed, a cartesian section is a right Kan extension from every cofinal subset. Conversely, given an arrow $i \to j$, take the cofinal subset of objects lying above j. If a section is a right Kan extension from this subset, its comparison morphism along $i \to j$ is cartesian. The intersection of a small collection of accessible left exact localizations is again an accessible left exact localization by [Lur09, Lemma 6.3.3.4]. This proves the asserted description of $\Gamma_{\mathrm{cart}}(p)$; see [Lur09, Proposition 6.3.3.3] for the complete argument.

The category $\Gamma_{\mathrm{cart}}(p)$ computes the limit of the underlying categories E_i. It remains to verify the universal property in Topos. A functor from a topos S to $\Gamma_{\mathrm{cart}}(p)$ is a morphism of topoi if and only if all its composites with the evaluation functors are morphisms of topoi. The nontrivial direction is [Lur09, Proposition 6.3.3.8]. Hence the limiting cone in Cat is also limiting in Topos. □

4.3.4 Products of topoi

Finite products have a concrete description in terms of the tensor product of cocomplete categories.

Proposition 4.33. *The inclusion* Logos $\hookrightarrow \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{colim}})$ *preserves finite coproducts. In particular,* An *is the terminal topos, and the product of two topoi T and S is their tensor product $T \otimes S$ as cocomplete categories.*

Proof. The initial object of Logos, equivalently the terminal object of Topos, is An by Example 4.12. It remains to compute binary coproducts in Logos.

Write $T = \mathrm{PSh}(C)[\Sigma^{-1}]$ and $S = \mathrm{PSh}(D)[\Gamma^{-1}]$ with $C, D \in \mathrm{Cat}^{\mathrm{lex}}$. We claim that a coproduct of C and D in $\mathrm{Cat}^{\mathrm{lex}}$ is given by the product category $C \times D$. We have left exact inclusions

$C \simeq C \times \{*\} \hookrightarrow C \times D$ and $D \simeq \{*\} \times D \hookrightarrow C \times D$, and restriction along them produces a functor

$$\mathrm{Fun}^{\mathrm{lex}}(C \times D, E) \quad \to \quad \mathrm{Fun}^{\mathrm{lex}}(C, E) \times \mathrm{Fun}^{\mathrm{lex}}(D, E).$$

We claim that this functor is an equivalence, with inverse sending a pair (f, g) to the functor $h\colon C \times D \to E$ defined by

$$h(x, y) \simeq f(x) \times g(y).$$

Indeed it is easy to see that $h(x, *) = f(x)$ and $h(*, y) = g(y)$, and conversely we always have $h(x, y) = h(x, *) \times h(*, y)$ for any h since we have the relation $(x, y) = (x, *) \times (*, y)$ in $C \times D$. Using Proposition 2.43, we conclude that for every topos T' we have an equivalence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(C \times D), T') \simeq \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(C), T') \times \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(D), T'),$$

and hence the logos $\mathrm{PSh}(C \times D) \simeq \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)$ is a coproduct of $\mathrm{PSh}(C)$ and $\mathrm{PSh}(D)$ in Logos, i.e. it is a product of topoi.

Now consider the following commutative diagram of tensor products in Pr^{L}:

$$\begin{array}{ccc} \mathrm{PSh}(C) \otimes \mathrm{PSh}(D) & \longrightarrow & \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)[\Gamma^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{PSh}(C)[\Sigma^{-1}] \otimes \mathrm{PSh}(D) & \longrightarrow & \mathrm{PSh}(C)[\Sigma^{-1}] \otimes \mathrm{PSh}(D)[\Gamma^{-1}] = T \otimes S \end{array}$$

Using the universal properties of localizations and tensor products, we observe that this is a pushout square in Pr^{L}. Passing to right adjoints then produces a pullback square in Pr^{R}, and we see that we may identify $T \otimes S$ with the intersection

$$(T \otimes \mathrm{PSh}(D)) \cap (\mathrm{PSh}(C) \otimes S)$$

as a full subcategory of $\mathrm{PSh}(C) \otimes \mathrm{PSh}(D) = \mathrm{PSh}(C \times D)$.

To conclude, we use that for every small category E and presentable category U there is a natural equivalence

$$\mathrm{PSh}(E) \otimes U \simeq \mathrm{Fun}(E^{\mathrm{op}}, U).$$

Hence if $L\colon U' \to U$ is a left exact localization with fully faithful right adjoint R, then the induced functor

$$\mathrm{PSh}(E) \otimes U' \to \mathrm{PSh}(E) \otimes U$$

identifies with postcomposition by L on $\mathrm{Fun}(E^{\mathrm{op}}, -)$. Note that this is again a left exact localization: it is left exact since finite limits in functor categories are computed pointwise, and its right adjoint is pointwise postcomposition by R, hence fully faithful. Applying this to $\mathrm{PSh}(D) \to S$ (with $E = C$) shows that $\mathrm{PSh}(C) \otimes S \subseteq \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)$ is a left exact localization; by symmetry, the same holds for $T \otimes \mathrm{PSh}(D) \subseteq \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)$. Intersections of accessible left exact localizations remain accessible left exact localizations by [Lur09, Lemma 6.3.3.4].

Thus the intersection

$$(T \otimes \mathrm{PSh}(D)) \cap (\mathrm{PSh}(C) \otimes S)$$

is the left exact localization of $\mathrm{PSh}(C \times D)$ at the union of the two induced localizing classes (equivalently, at the strongly saturated class they generate). In particular, $T \otimes S$ is a topos. By comparing universal properties, we now conclude that $T \otimes S$ is the coproduct of T and S in Logos, i.e. the product of T and S in Topos. □

Proof of Theorem 4.28. Combine Propositions 4.30 to 4.33. □

Exercise 4.34. The category Topos is tensored and cotensored over Cat: for topoi S, T and a category C, there exist topoi S^C and $C \otimes T$ such that

$$\mathrm{Geom}(T, S^C) \simeq \mathrm{Fun}(C, \mathrm{Geom}(T, S)) \simeq \mathrm{Geom}(C \otimes T, S).$$

Show that $C \otimes T \simeq \mathrm{Fun}(C, T)$, and describe S^C using a presentation of S.

4.4 Étale morphisms

Recall that a continuous map $f\colon X \to Y$ of topological spaces is called *étale* if it is a local homeomorphism: every point $x \in X$ admits an open neighborhood U such that the composite $U \hookrightarrow X \xrightarrow{f} Y$ is an open embedding. The topos-theoretic analogue is obtained from the projection $T_{/X} \to T$ associated to an object X of a topos T. We first recognize these morphisms by the existence of a conservative further left adjoint satisfying a projection formula. We then classify the étale morphisms over a fixed topos and record their compatibility with base change.

Definition 4.35. A morphism of logoi $\varphi^*\colon T \to S$ is called *étale* if there exists an object $X \in T$ and an equivalence of logoi $S \simeq T_{/X}$ such that the composite

$$T \xrightarrow{\varphi^*} S \simeq T_{/X}$$

is the functor $U \mapsto U \times X$. A morphism $\varphi_*\colon S \to T$ of topoi is étale if the corresponding morphism $\varphi^*\colon T \to S$ is étale. We denote by

$$\mathrm{Logos}^{\text{ét}} \subseteq \mathrm{Logos} \qquad \text{and} \qquad \mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$$

the wide subcategories spanned by étale morphisms.

Remark 4.36. As we will see below in Example 6.22, every continuous map of topological spaces $f\colon X \to Y$ gives rise to a morphism of topoi $f_*\colon \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ between their *sheaf categories*. One can show that f is an étale morphism of topological spaces (that is, a local homeomorphism) if and only if f_* is an étale morphism of topoi, justifying the terminology.

For an étale morphism φ^*, its further left adjoint is commonly denoted $\varphi_!$. We instead write $\varphi_\sharp$, in order to distinguish this adjoint from the lower shriek in a six-functor formalism.

Proposition 4.37. *A morphism of logoi $\varphi^*\colon T \to T'$ is étale if and only if:*

(1) *The functor φ^* admits a left adjoint $\varphi_\sharp$.*

(2) *The functor $\varphi_\sharp$ is conservative.*

(3) *For every morphism $X \to Y$ in T and every morphism $Z \to \varphi^* Y$ in T', the canonical map*

$$\varphi_\sharp\big(\varphi^* X \times_{\varphi^* Y} Z\big) \longrightarrow X \times_Y \varphi_\sharp(Z)$$

is an isomorphism.

Proof. If φ^* is étale, choose $A \in T$ and an identification $T' \simeq T_{/A}$ under which φ^* is the functor $U \mapsto U \times A$. Its left adjoint $\varphi_\sharp$ is the forgetful functor $T_{/A} \to T$. It is conservative, and the projection formula follows by composing pullback squares in T.

Conversely, suppose that φ^* admits a conservative left adjoint $\varphi_\sharp$ satisfying the projection formula, and set $A := \varphi_\sharp(*)$. Consider the functor

$$F \colon T' \longrightarrow T_{/A}, \qquad W \longmapsto \big(\varphi_\sharp(W) \to A\big).$$

It admits a right adjoint

$$G(Y \to A) := \varphi^*(Y) \times_{\varphi^* A} *.$$

For $Y \to A$, the projection formula identifies the underlying morphism of the counit $FG(Y) \to Y$ with an isomorphism, so the counit is an isomorphism in $T_{/A}$. Let $W \to GFW$ be the unit. By the triangle identity, its image under F is an isomorphism because the counit is. In particular, $\varphi_\sharp$ sends the unit to an isomorphism. Since $\varphi_\sharp$ is conservative, the unit itself is an isomorphism. Thus F is an equivalence. Under this equivalence, φ^* sends Y to the projection $Y \times A \to A$, as required. □

Proposition 4.38. *Let T be a topos.*

(1) *There is an equivalence of* 2*-categories*

$$T \;\simeq\; \mathrm{Topos}^{\text{ét}}_{/T}, \qquad X \longmapsto T_{/X}.$$

In particular, $\mathrm{Topos}^{\text{ét}}_{/T}$ *is a* 1*-category.*

(2) *For every morphism of topoi $\varphi \colon S \to T$ and every $X \in T$, the square*

$$\begin{array}{ccc} S_{/\varphi^* X} & \longrightarrow & T_{/X} \\ \downarrow & \lrcorner & \downarrow \\ S & \xrightarrow{\varphi} & T \end{array}$$

is a pullback in Topos*. In particular, the category* $\mathrm{Topos}^{\text{ét}}$ *admits pullbacks, and the inclusion* $\mathrm{Topos}^{\text{ét}} \hookrightarrow \mathrm{Topos}$ *preserves pullbacks.*

Proof. Given a morphism of topoi $\varphi \colon S \to T$ and an object $X \in T$, we claim there is a natural equivalence

$$\mathrm{Geom}_{/T}(S, T_{/X}) \;\simeq\; \mathrm{Hom}_S(*, \varphi^* X).$$

This immediately gives (2), and it gives (1) by taking $S = T_{/Y}$ and noticing that $\mathrm{Hom}_{T_{/Y}}(Y, X \times Y) \simeq \mathrm{Hom}_T(Y, X)$.

To prove the claim, we will start by producing a natural equivalence

$$\mathrm{Fun}^{\mathrm{lex}}_{T/}(T_{/X}, S) \;\simeq\; \mathrm{Hom}_S(*, \varphi^* X).$$

- Given $F \colon T_{/X} \to S$ in $\mathrm{Fun}^{\mathrm{lex}}_{T/}$, consider the morphism

$$f_F := F(\Delta) \colon F(X) \longrightarrow F(X \times X)$$

induced by the diagonal $\Delta \colon X \to X \times X$, viewed as a morphism in $T_{/X}$. Since (X, id_X) is terminal in $T_{/X}$, we have $F(X) \simeq *$. Moreover, because F is a functor under T, we identify $F(X \times X)$ with $\varphi^* X$. So f_F is a point in $\mathrm{Hom}_S(*, \varphi^* X)$.

- Conversely, given a morphism $f\colon * \to \varphi^* X$, define $F_f\colon T_{/X} \to S$ by sending $U \to X$ to the pullback

$$\begin{array}{ccc} F_f(U) & \longrightarrow & \varphi^*(U) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{f} & \varphi^*(X). \end{array}$$

  The functor F_f preserves finite limits because it is the composite of $\varphi^*\colon T_{/X} \to S_{/\varphi^* X}$ with pullback along f. It also preserves colimits: the first functor does so objectwise, while the second does so by descent in S. When $U = X \times Y$ for some $Y \in T$, we have $F_f(U) \simeq \varphi^*(Y)$, showing that F_f is a functor under T.

Now, applying F_f to the diagonal map $\Delta\colon X \to X \times X$ reproduces the original map f. Conversely, given any left exact functor $F\colon T_{/X} \to S$ under T and any $U \in T_{/X}$, we may always write U as the pullback in $T_{/X}$ of the diagram

$$\begin{array}{ccc} U & \longrightarrow & U \times X \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\Delta} & X \times X, \end{array}$$

and thus we get a pullback square

$$\begin{array}{ccc} F(U) & \longrightarrow & F(U \times X) \simeq \varphi^*(U) \\ \downarrow & \lrcorner & \downarrow \\ F(X) & \xrightarrow{F(\Delta)} & F(X \times X) \simeq \varphi^*(X), \end{array}$$

showing that F is naturally equivalent to F_{f_F}. This constructs the equivalence $\mathrm{Fun}^{\mathrm{lex}}_{T/}(T_{/X}, S) \simeq \mathrm{Hom}_S(*, \varphi^* X)$. The preceding description of F_f also shows that every such left exact functor preserves colimits. Hence the forgetful functor

$$\mathrm{Geom}_{/T}(S, T_{/X}) \to \mathrm{Fun}^{\mathrm{lex}}_{T/}(T_{/X}, S)$$

is an equivalence. This finishes the proof. □

By part (2) of the proposition, étale morphisms are closed under base change in Topos. In particular, given a morphism of topoi $\varphi\colon S \to T$, pullback along φ defines a functor $\varphi^*\colon \mathrm{Topos}^{\text{ét}}_{/T} \to \mathrm{Topos}^{\text{ét}}_{/S}$.

Corollary 4.39. *The functor* $\mathrm{Topos}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}, T \mapsto \mathrm{Topos}^{\text{ét}}_{/T}$ *is equivalent to the composite* $\mathrm{Topos}^{\mathrm{op}} \simeq \mathrm{Logos} \hookrightarrow \mathrm{Cat} \hookrightarrow \widehat{\mathrm{Cat}}$.

Proof sketch. For each topos T, consider the functors

$$\mathrm{Topos}^{\text{ét}}_{/T} \longrightarrow T, \qquad (\varphi\colon T' \to T) \longmapsto \varphi_\sharp(*)$$

and $T \to \mathrm{Topos}^{\text{ét}}_{/T}$, $X \mapsto T_{/X}$. The classification in Proposition 4.38 supplies inverse equivalences. These equivalences are natural in T: for a pullback square

$$\begin{array}{ccc} S' & \longrightarrow & T' \\ {\scriptstyle\varphi'}\downarrow & \lrcorner & \downarrow{\scriptstyle\varphi} \\ S & \xrightarrow{\psi} & T, \end{array}$$

we have a canonical isomorphism $\psi^*\varphi_\sharp(*) \simeq \varphi'_\sharp(*)$. These base-change isomorphisms are compatible with pasting, since both sides classify the same iterated pullback of the étale morphism φ. Thus the objectwise equivalences assemble into the asserted equivalence of functors. □

Corollary 4.40. *Consider morphisms of topoi $T \xrightarrow{\varphi} T' \xrightarrow{\psi} S$. If ψ and $\psi\varphi$ are étale morphisms, then so is φ.*

Proof. Choose equivalences $T \simeq S_{/X}$ and $T' \simeq S_{/Y}$ identifying $\psi\varphi$ and ψ with the slice projections. By Proposition 4.38, the morphism φ over S is classified by a morphism $f\colon X \to Y$. Regarding f as an object of $S_{/Y} \simeq T'$, there is an equivalence

$$T'_{/f} \simeq (S_{/Y})_{/f} \simeq S_{/X} \simeq T$$

under which the projection to T' identifies with φ. Hence φ is étale. □

4.5 Colimits of topoi along étale morphisms

Recall from Definition 4.35 the wide subcategory $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$ spanned by the étale morphisms of topoi. The goal in this section is the following somewhat surprising theorem:

Theorem 4.41 ([Uem25, Theorem 5.15])**.** *The category* $\mathrm{Topos}^{\text{ét}}$ *admits colimits, and these colimits are preserved by the inclusion* $\mathrm{Topos}^{\text{ét}} \hookrightarrow \mathrm{Topos}$*.*

There are two distinct issues in this theorem. For a fixed base topos T, the classification $\mathrm{Topos}^{\text{ét}}_{/T} \simeq T$ and descent in T make colimits of étale morphisms over T comparatively easy to control. The difficult point is global: if a diagram in Topos has étale transition morphisms, one must prove that its structure morphisms into the colimit are again étale.

We follow the proof of Uemura [Uem25, Theorem 5.15]. After passing to logoi, the problem becomes one about the projections from a limit of a diagram with étale transition morphisms. Uemura recognizes étale morphisms by two properties: preservation of dependent products and the existence of enough univalent families. The first property is stable under limits. For the second, univalent completion allows an initially unrelated tuple of families to be enlarged, by a fixed-point construction, to a compatible tuple in the limit logos. A general criterion for colimits in wide subcategories then completes the proof.

The theorem may also be viewed as a descent statement for the assignment $T \mapsto \mathrm{Topos}^{\text{ét}}_{/T} \simeq T$. This perspective connects with the small and large topoi associated to geometric objects, which will reappear in Sections 7.2 and 7.3; it is not needed for the proof here. We thank David Wärn for bringing Uemura's article to our attention.

4.5.1 Univalent families

A univalent family is one for which being obtained by pullback is a property rather than additional structure. The main result of this subsection is that every family admits a universal map to a univalent family. Besides avoiding the use of a larger universe of object classifiers, this has an important coherence consequence: maps between univalent families form propositions. Consequently, once

compatible maps have been constructed in both directions, all higher compatibility data are automatic. The material in this subsection follows [Uem25, Section 4].

Definition 4.42. Let T be a topos. We write

$$\mathrm{Fam}(T) := \mathrm{Ar}^{\mathrm{pb}}(T)$$

for the wide subcategory of $\mathrm{Ar}(T)$ whose morphisms are pullback squares. We call an object $u\colon E \to B$ of $\mathrm{Fam}(T)$ a *family in* T, and we call its codomain B the *base* of the family.

A family $u \in \mathrm{Fam}(T)$ is *univalent* if it is a (-1)-truncated object of the category $\mathrm{Fam}(T)$. We write

$$\mathrm{Fam}^{\mathrm{univ}}(T) \subseteq \mathrm{Fam}(T)$$

for the full subcategory spanned by the univalent families.

Informally, a morphism $u\colon E \to B$ is univalent if it is a *property* for an arbitrary morphism v in T to be a pullback of u, i.e. if the anima $\mathrm{Hom}_{\mathrm{Fam}(T)}(v, u)$ is (-1)-truncated. Once we know that $\mathrm{Fam}(T)$ admits binary products, this is equivalently the condition that the diagonal

$$u \longrightarrow u \times u$$

is an isomorphism in $\mathrm{Fam}(T)$.

Lemma 4.43 ([Uem25, Lemma 4.2])**.** *Let $u\colon E \to B$ be a family in a topos T. The functor*

$$\mathrm{Fam}(T)_{/u} \longrightarrow T_{/B}$$

sending a pullback square $v \to u$ to the induced morphism from the base of v to B is an equivalence.

Proof. The inverse sends a morphism $f\colon X \to B$ to the pullback family f^*u. Since morphisms in $\mathrm{Fam}(T)$ are by definition pullback squares, these two constructions are inverse to each other. □

Lemma 4.44 ([Uem25, Lemma 4.3])**.** *The wide subcategory $\mathrm{Fam}(T) \subseteq \mathrm{Ar}(T)$ is closed under small colimits and pullbacks.*

Proof. The assertion about colimits is precisely the topos direction of Proposition 2.16, after identifying $\mathrm{Fam}(T)$ with $\mathrm{Ar}^{\mathrm{pb}}(T)$.

For pullbacks, note that $\mathrm{Fam}(T)$ admits pullbacks if and only if for each $u \in \mathrm{Fam}(T)$ the slice category $\mathrm{Fam}(T)_{/u}$ admits binary products, which is immediate from Lemma 4.43. □

Proposition 4.45 ([Uem25, Lemma 4.4])**.** *The category $\mathrm{Fam}(T)$ has binary products, and these products preserve small colimits in each variable.*

Proof. Let $u_1\colon E_1 \to B_1$ and $u_2\colon E_2 \to B_2$ be families. For an object $X \to B_1 \times B_2$, write $f_i\colon X \to B_i$ for the two projections. Consider the presheaf on $T_{/B_1\times B_2}$ which sends X to

$$\mathrm{Iso}_{T_{/X}}\left(f_1^*u_1, f_2^*u_2\right),$$

the anima of isomorphisms in $T_{/X}$ between the pullbacks of u_1 and u_2 to X. This presheaf preserves limits: if $X \simeq \mathrm{colim}_i X_i$ in $T_{/B_1\times B_2}$, then descent identifies $T_{/X}$ with $\lim_i T_{/X_i}$, and under this

identification the two pulled-back families are the compatible systems of their pullbacks to the X_i. Thus isomorphisms between them are computed as the limit of the isomorphism animae over the X_i. By the adjoint functor theorem, the presheaf is therefore representable by some object $P \to B_1 \times B_2$. Pulling back either u_1 or u_2 along the universal isomorphism gives a family over P, and this family is the product $u_1 \times u_2$ in $\mathrm{Fam}(T)$.

It remains to prove preservation of colimits in each variable. Fix a family $u_1 \colon E_1 \to B_1$. To show that $u_1 \times - \colon \mathrm{Fam}(T) \to \mathrm{Fam}(T)$ preserves colimits, we may equivalently show that for any family $v \colon E' \to B'$ the functor

$$\mathrm{Fam}(T)_{/v} \longrightarrow \mathrm{Fam}(T)_{/u_1 \times v}$$

induced by $u_1 \times -$ preserves colimits, since slices detect colimits. Let P' denote the base of the product $u_1 \times v$. We claim that under the equivalences $\mathrm{Fam}(T)_{/v} \simeq T_{/B'}$ and $\mathrm{Fam}(T)_{/u_1 \times v} \simeq T_{/P'}$ from Lemma 4.43, the functor identifies with pullback along the projection map $P' \to B'$. Since this functor preserves colimits by descent in T, this will finish the proof. To show the claim, we must show that for every $f \colon u_2 \to v$ in $\mathrm{Fam}(T)$ with base $f_B \colon B_2 \to B'$, the square of bases

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \downarrow \\ B_1 \times B_2 & \xrightarrow{\mathrm{id} \times f_B} & B_1 \times B' \end{array}$$

is a pullback square in T. We compare the presheaves represented by the two objects over $B_1 \times B_2$. Let $X \to B_1 \times B_2$ be an object, and write $g_1 \colon X \to B_1$ and $g_2 \colon X \to B_2$ for the two projections. The object P represents the presheaf

$$X \longmapsto \mathrm{Iso}_{T_{/X}}\left(g_1^* u_1, g_2^* u_2\right).$$

On the other hand, the pullback $(B_1 \times B_2) \times_{B_1 \times B'} P'$ represents the presheaf

$$X \longmapsto \mathrm{Iso}_{T_{/X}}\left(g_1^* u_1, (f_B g_2)^* v\right),$$

because P' represents the analogous equivalence presheaf for u_1 and v over $B_1 \times B'$. Since $f \colon u_2 \to v$ is a morphism in $\mathrm{Fam}(T)$, the family u_2 is the pullback $f_B^* v$, and hence $g_2^* u_2 \simeq (f_B g_2)^* v$. These two presheaves are therefore naturally equivalent, proving that the square is a pullback. □

Corollary 4.46 ([Uem25, Corollary 4.6])**.** *Let $u_\bullet \colon [\omega] \to \mathrm{Fam}(T)$ be a sequence of univalent families. Then* $\mathrm{colim}_n u_n$ *is univalent.*

Proof. By Proposition 4.45, we have

$$(\mathrm{colim}_n u_n) \times (\mathrm{colim}_m u_m) \simeq \mathrm{colim}_{(n,m) \in [\omega] \times [\omega]} (u_n \times u_m). \tag{4.1}$$

The diagonal functor $[\omega] \to [\omega] \times [\omega]$ is final, so this is further equivalent to $\mathrm{colim}_n (u_n \times u_n)$. The diagonal of $\mathrm{colim}_n u_n$ is therefore the colimit of the diagonals $u_n \to u_n \times u_n$, which are isomorphisms by univalence. □

Proposition 4.47 ([Uem25, Proposition 4.7])**.** *The inclusion*

$$\mathrm{Fam}^{\mathrm{univ}}(T) \hookrightarrow \mathrm{Fam}(T)$$

admits a left adjoint. We call it the univalent completion functor.

Proof. Let u be a family. We construct a sequence $u_0 \to u_1 \to u_2 \to \cdots$ in $\mathrm{Fam}(T)$ by setting $u_0 = u$ and defining u_{n+1} by the pushout square

$$\begin{array}{ccc} u_n \times u_n & \xrightarrow{\mathrm{pr}_2} & u_n \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \ulcorner & \downarrow \\ u_n & \longrightarrow & u_{n+1}. \end{array}$$

Let $u_\infty := \mathrm{colim}_n\, u_n$. We claim that u_∞ is univalent. Indeed, the pushout square gives a canonical map $q_n \colon u_n \times u_n \to u_{n+1}$ for which both composites $u_n \rightrightarrows u_n \times u_n \to u_{n+1}$ agree with the structure map $u_n \to u_{n+1}$. In other words, q_n is a diagonal filler for

$$\begin{array}{ccc} u_n & \longrightarrow & u_{n+1} \\ \downarrow & & \downarrow \\ u_n \times u_n & \longrightarrow & u_{n+1} \times u_{n+1}. \end{array}$$

Moreover, the composites

$$u_n \longrightarrow u_n \times u_n \xrightarrow{q_n} u_{n+1} \qquad \text{and} \qquad u_n \times u_n \xrightarrow{q_n} u_{n+1} \longrightarrow u_{n+1} \times u_{n+1}$$

are the structure maps of the shifted sequence $(u_{n+1})_n$ and of the sequence $(u_n \times u_n)_n$, respectively. Since the shift $[\omega]_{\geq 1} \hookrightarrow [\omega]$ is final, and since (4.1) holds without assuming the u_n are univalent, passing to colimits identifies the resulting map with the diagonal

$$u_\infty \longrightarrow u_\infty \times u_\infty$$

and the maps q_n induce an inverse. Hence this diagonal is an isomorphism.

Now let v be a univalent family. We show that restriction along $u \to u_\infty$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{Fam}(T)}(u_\infty, v) \longrightarrow \mathrm{Hom}_{\mathrm{Fam}(T)}(u, v).$$

Given a map $u_n \to v$, the two induced maps $u_n \times u_n \rightrightarrows v$ agree because v is (-1)-truncated in $\mathrm{Fam}(T)$. Hence the map $u_n \to v$ extends uniquely across the pushout defining u_{n+1}. Iterating and passing to the limit gives the desired isomorphism. This proves the adjunction. □

Remark 4.48. This construction is the join construction for propositional truncation, transplanted from homotopy type theory to the category $\mathrm{Fam}(T)$; compare [Uem25, Proposition 4.7].

From this point on, we formulate the proof in the logos direction. Thus an étale morphism of logoi is, after choosing an object X, the canonical functor $T \to T_{/X}$ sending U to $U \times X$, dual to the corresponding geometric morphism of topoi.

4.5.2 Dependent products

If T is a logos and $p \colon X \to Y$ is a morphism in T, then the pullback functor

$$p^* \colon T_{/Y} \longrightarrow T_{/X}$$

preserves colimits by descent, hence admits a right adjoint Π_p, called *dependent product along* p.

Definition 4.49. Let $\varphi^*\colon T \to S$ be a morphism of logoi. We say that φ^* *preserves dependent products* if, for every morphism $p\colon X \to Y$ in T, the canonical comparison

$$\varphi^*(\Pi_p Z) \longrightarrow \Pi_{\varphi^*(p)}\varphi^*(Z)$$

is an isomorphism for every $Z \in T_{/X}$.

Proposition 4.50 ([Uem25, Proposition 3.13])**.** *The wide subcategory of* Logos *spanned by the morphisms which preserve dependent products is closed under small limits.*

Proof. Let $T_\bullet\colon I \to \mathrm{Logos}$ be a diagram whose transition morphisms preserve dependent products, and let $T := \lim_i T_i$. By Proposition 4.30, the underlying category of T is the limit of the categories T_i. Slices, pullback functors, and their right adjoints are therefore computed pointwise. Hence every projection $T \to T_i$ preserves dependent products. The same pointwise description shows that the universal cone remains a limit cone in the wide subcategory, which proves the claim. □

Definition 4.51. Let $\varphi^*\colon T \to S$ be a morphism of logoi and suppose that φ^* admits a left adjoint $\varphi_\sharp\colon S \to T$. We say that $\varphi_\sharp$ is T*-indexed* if, for every morphism $p\colon X' \to X$ in T and every pullback square in S of the form

$$\begin{array}{ccc} Y' & \longrightarrow & \varphi^*(X') \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi^*(p)} \\ Y & \longrightarrow & \varphi^*(X), \end{array}$$

the transposed square

$$\begin{array}{ccc} \varphi_\sharp(Y') & \longrightarrow & X' \\ {\scriptstyle \varphi_\sharp(q)}\downarrow & & \downarrow{\scriptstyle p} \\ \varphi_\sharp(Y) & \longrightarrow & X \end{array}$$

is a pullback in T.

Lemma 4.52 ([Uem25, Proposition 3.16])**.** *Let $\varphi^*\colon T \to S$ be a morphism of logoi. Then φ^* preserves dependent products if and only if it admits a T-indexed left adjoint.*

Proof. First suppose that φ^* admits a left adjoint $\varphi_\sharp\colon S \to T$. For every object $X \in T$, the induced functor

$$\varphi_X^*\colon T_{/X} \longrightarrow S_{/\varphi^*(X)}$$

has a left adjoint

$$\varphi_{\sharp,X}\colon S_{/\varphi^*(X)} \longrightarrow T_{/X},$$

which sends a map $Y \to \varphi^*(X)$ to its transpose $\varphi_\sharp(Y) \to X$.

Now let $p\colon X' \to X$ be a morphism in T. Since φ^* is left exact, we have a commutative square of pullback functors

$$\begin{array}{ccc} T_{/X} & \xrightarrow{\varphi_X^*} & S_{/\varphi^*(X)} \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle \varphi^*(p)^*} \\ T_{/X'} & \xrightarrow{\varphi_{X'}^*} & S_{/\varphi^*(X')}. \end{array}$$

The assertion that φ^* preserves dependent products along p is precisely that the corresponding right Beck–Chevalley comparison

$$\varphi_X^* \Pi_p \longrightarrow \Pi_{\varphi^*(p)} \varphi_{X'}^*$$

is an isomorphism. Taking mates under the adjunctions $\varphi_{\sharp,X} \dashv \varphi_X^*$, $\varphi_{\sharp,X'} \dashv \varphi_{X'}^*$, $p^* \dashv \Pi_p$, and $\varphi^*(p)^* \dashv \Pi_{\varphi^*(p)}$, this is equivalent to the left Beck–Chevalley comparison

$$\varphi_{\sharp,X'} \varphi^*(p)^* \longrightarrow p^* \varphi_{\sharp,X}$$

being an isomorphism. Evaluating this comparison on a map $Y \to \varphi^*(X)$ gives the canonical map from the transpose of the pullback square

$$\begin{array}{ccc} Y \times_{\varphi^*(X)} \varphi^*(X') & \longrightarrow & \varphi^*(X') \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \varphi^*(p)} \\ Y & \longrightarrow & \varphi^*(X) \end{array}$$

to the pullback of $\varphi_\sharp(Y) \to X$ along p. Thus the left Beck–Chevalley comparison is an isomorphism exactly when every such transposed square is a pullback. This is precisely the condition that $\varphi_\sharp$ is T-indexed.

It remains to note that preservation of dependent products implies that φ^* has a left adjoint. Let $\{X_i\}_{i \in I}$ be a small family of objects in T. The product $\prod_i X_i$ can be constructed as the dependent product of the object

$$\coprod_{i \in I} X_i \longrightarrow \coprod_{i \in I} *$$

along the fold map $\coprod_{i \in I} * \to *$. Since φ^* preserves colimits, the coproducts in this construction are carried to the corresponding coproducts in S, and by assumption φ^* preserves the dependent product. Hence φ^* preserves small products. Together with left exactness, this implies that φ^* preserves all small limits. Since φ^* is an accessible functor between presentable categories, the adjoint functor theorem gives a left adjoint to φ^*. The first part of the proof then shows that this left adjoint is T-indexed. □

Lemma 4.53 ([Uem25, Proposition 4.10])**.** *Let $\varphi^* \colon T \to S$ be a morphism of logoi which preserves dependent products. Then φ^* sends univalent families in T to univalent families in S.*

Proof. Let $\varphi_\sharp$ be the T-indexed left adjoint of φ^* from Lemma 4.52. The adjunction gives a natural isomorphism on arrow categories

$$\mathrm{Hom}_{\mathrm{Ar}(S)}(v, \varphi^*(u)) \simeq \mathrm{Hom}_{\mathrm{Ar}(T)}(\varphi_\sharp(v), u).$$

Under this isomorphism, the T-indexed condition says that any morphism $v \to \varphi^*(u)$ whose square in S is a pullback is sent to a morphism $\varphi_\sharp(v) \to u$ whose square in T is a pullback. Thus we get a map

$$\mathrm{Hom}_{\mathrm{Fam}(S)}(v, \varphi^*(u)) \longrightarrow \mathrm{Hom}_{\mathrm{Fam}(T)}(\varphi_\sharp(v), u).$$

This map is a monomorphism of animae: both sides are subanimae of the corresponding arrow-category mapping animae, and the ambient map is an isomorphism. If u is univalent, then $\mathrm{Hom}_{\mathrm{Fam}(T)}(\varphi_\sharp(v), u)$ is (-1)-truncated; hence its subanima $\mathrm{Hom}_{\mathrm{Fam}(S)}(v, \varphi^*(u))$ is also (-1)-truncated. Thus $\varphi^*(u)$ is univalent. □

4.5.3 Characterization of étale morphisms

We now combine dependent products with univalent families to recognize étale morphisms. Preservation of dependent products supplies an indexed left adjoint and hence a fully faithful factorization through a slice. The remaining question is whether that fully faithful factor is essentially surjective. The existence of enough families detects this, while univalence rigidifies the families sufficiently for the condition to survive the limit construction in the next subsection.

Definition 4.54. Let $\varphi^* : T \to S$ be a morphism of logoi.

(1) We say that φ^* *provides enough families* if, for every family v in S, there exists a family u in T and a morphism $v \to \varphi^*(u)$ in $\mathrm{Fam}(S)$.

(2) If φ^* preserves dependent products, we say that φ^* *provides enough univalent families* if the same condition holds after restricting to univalent families.

(3) We say that φ^* is *object-generating* if the closure of the image of φ^* under colimits and finite limits is all of S.

Lemma 4.55. *Let $\varphi^* : T \to S$ be a morphism of logoi with a T-indexed left adjoint $\varphi_\sharp$. Then φ^* factors as*

$$T \xrightarrow{(-)\times\varphi_\sharp(*)} T_{/\varphi_\sharp(*)} \xrightarrow{\psi^*} S,$$

where ψ^* is fully faithful.

Proof. Let $\eta : \mathrm{id}_S \to \varphi^*\varphi_\sharp$ be the unit of the adjunction $\varphi_\sharp \dashv \varphi^*$. Define

$$\psi^* : T_{/\varphi_\sharp(*)} \longrightarrow S$$

as the composite

$$T_{/\varphi_\sharp(*)} \xrightarrow{\varphi^*} S_{/\varphi^*\varphi_\sharp(*)} \xrightarrow{\eta_*^*} S,$$

where $\eta_* : * \to \varphi^*\varphi_\sharp(*)$ is the unit at the terminal object of S and the second functor is pullback along η_*. For $X \in T$, the object $\psi^*(X \times \varphi_\sharp(*))$ is the pullback of $\varphi^*(X) \times \varphi^*\varphi_\sharp(*) \to \varphi^*\varphi_\sharp(*)$ along η_*, hence is equivalent to $\varphi^*(X)$.

The left adjoint of ψ^* is $\varphi_\sharp$ followed by the evident map to $\varphi_\sharp(*)$. The counit is identified, by the T-indexed condition, with the transpose of a pullback square, and is therefore an isomorphism. Hence ψ^* is fully faithful. □

Proposition 4.56 (Uemura, [Uem25, Proposition 5.10])**.** *Let $\varphi^* : T \to S$ be a morphism of logoi which preserves dependent products. Then the following conditions are equivalent:*

(1) *φ^* is an étale morphism.*

(2) *The unit $\mathrm{id}_S \to \varphi^*\varphi_\sharp$ of the indexed adjunction is cartesian.*

(3) *φ^* provides enough families.*

(4) *φ^* provides enough univalent families.*

(5) *φ^* is object-generating.*

If these conditions hold, then φ^* is equivalent, under T, to the canonical étale morphism $T \to T_{/\varphi_\sharp(*)}$, $Y \mapsto Y \times \varphi_\sharp(*)$.

Proof. Assume first that φ^* is an étale morphism. Choose $X \in T$ and identify S with $T_{/X}$ so that φ^* is the functor $Y \mapsto Y \times X$. Then the left adjoint is the forgetful functor $T_{/X} \to T$, and the unit is cartesian because a morphism in a slice is recovered by pulling back its total space along the corresponding map to X. This gives (1) $\Rightarrow$ (2).

If the unit is cartesian and $v\colon Y' \to Y$ is a family in S, then the naturality square

$$\begin{array}{ccc} Y' & \longrightarrow & \varphi^*\varphi_\sharp(Y') \\ {\scriptstyle v}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi^*\varphi_\sharp(v)} \\ Y & \longrightarrow & \varphi^*\varphi_\sharp(Y) \end{array}$$

exhibits v as a pullback of the family $\varphi_\sharp(v)$ after applying φ^*. Thus (2) implies (3).

Condition (3) implies (5): applying (3) to the family $Y \to *$ gives a family $u\colon E \to B$ in T and a pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \varphi^*(E) \\ \downarrow & & \downarrow{\scriptstyle \varphi^*(u)} \\ * & \longrightarrow & \varphi^*(B). \end{array}$$

Since $* \simeq \varphi^*(*)$, the object Y is a finite limit of objects in the image of φ^*. Hence φ^* is object-generating.

Now assume (5). By Lemma 4.55, the morphism φ^* factors as an étale morphism followed by a fully faithful morphism ψ^*. Since φ^* is object-generating, so is ψ^*. A fully faithful morphism of logoi which is object-generating is an equivalence, because its essential image is already closed under colimits and finite limits. Hence φ^* is an étale morphism. This proves (5) $\Rightarrow$ (1).

It remains to compare (3) and (4). The implication (4) $\Rightarrow$ (3) follows by applying univalent completion from Proposition 4.47 to the target family. Conversely, if (3) holds and v is univalent in S, choose a family u in T and a map $v \to \varphi^*(u)$. Let u^{univ} be the univalent completion of u. By Lemma 4.53, $\varphi^*(u^{\mathrm{univ}})$ is univalent, and the composite

$$v \longrightarrow \varphi^*(u) \longrightarrow \varphi^*(u^{\mathrm{univ}})$$

shows that (4) holds. □

Proposition 4.57 ([Uem25, Corollary 5.11 and Lemma 5.12])**.** *For every logos C, the functor*

$$C^{\mathrm{op}} \longrightarrow \mathrm{Logos}^{\text{ét}}_{C/}, \qquad X \longmapsto (C \to C_{/X})$$

is an equivalence. Under this equivalence, the inclusion $\mathrm{Logos}^{\text{ét}}_{C/} \hookrightarrow \mathrm{Logos}_{C/}$ preserves limits.

Proof. Essential surjectivity and full faithfulness are the slice classification of étale morphisms from Proposition 4.38, translated to logoi. Explicitly, an étale morphism $\varphi^*\colon C \to S$ is equivalent to $C \to C_{/\varphi_\sharp(*)}$ by Proposition 4.56.

For preservation of limits, let $X_\bullet\colon I \to C$ be a diagram with colimit X. Descent gives

$$C_{/X} \simeq \lim_i C_{/X_i}$$

in $\mathrm{Logos}_{C/}$. Passing to C^{op}, this says exactly that $X \mapsto C_{/X}$ preserves the relevant limits. □

4.5.4 Proof of the main theorem

We are now in a position to prove Theorem 4.41, i.e. the fact that $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$ is closed under colimits, or equivalently that $\mathrm{Logos}^{\text{ét}} \subseteq \mathrm{Logos}$ is closed under limits.

Lemma 4.58 ([Uem25, Lemma 5.13])**.** *Let $\{C_i\}_{i\in I}$ be a small family of logoi. Then every projection*

$$\prod_{j\in I} C_j \longrightarrow C_i$$

is an étale morphism.

Proof. Let $D = \prod_{j\in I} C_j$. Define an object $x \in D$ by taking $x_i = *$ and $x_j = \emptyset$ for $j \neq i$. Then

$$D_{/x} \simeq \prod_{j\in I} (C_j)_{/x_j} \simeq C_i,$$

since the slice over the initial object is terminal. Under this equivalence, the canonical étale functor $D \to D_{/x}$ is the projection to C_i. □

Lemma 4.59 ([Uem25, Lemma 5.14])**.** *Let $C_\bullet \colon I \to \mathrm{Logos}^{\text{ét}}$ be a small diagram, and let*

$$C_{-\infty} := \lim_{i\in I} C_i$$

be its limit in Logos*. Then every projection $\pi_i \colon C_{-\infty} \to C_i$ is an étale morphism.*

Proof. Let $D := \prod_{i\in I} C_i$. By Lemma 4.58, it suffices to prove that the forgetful morphism

$$\pi \colon C_{-\infty} \longrightarrow D$$

is an étale morphism. By Proposition 4.56, it is enough to show that π preserves dependent products and provides enough univalent families.

Every étale morphism preserves dependent products by Proposition 4.37 and lemma 4.52. Hence Proposition 4.50 shows that π preserves dependent products.

Let $u = (u_i)_{i\in I}$ be a univalent family in D, i.e. a tuple of univalent families $u_i \in \mathrm{Fam}^{\mathrm{univ}}(C_i)$ without imposing the transition equivalences. Since limits of logoi are computed in Cat, we have $\mathrm{Fam}(C_{-\infty}) \simeq \lim_i \mathrm{Fam}(C_i)$. Hence $\mathrm{Fam}^{\mathrm{univ}}(C_{-\infty})$ is the full subposet of $\mathrm{Fam}^{\mathrm{univ}}(D)$ spanned by the tuples equipped with equivalences $s^*(u_i) \simeq u_j$ for all arrows $s \colon i \to j$ in I; the higher coherences are automatic because univalent families form a poset. We will enlarge u to such a compatible tuple.

For a morphism $s \colon i \to j$ in I, write $s^* \colon C_i \to C_j$ for the corresponding étale morphism of logoi, and write $s_\sharp \colon C_j \to C_i$ for its left adjoint. Both functors induce functors on families. For s^* this follows from left exactness. For $s_\sharp$ it follows from the indexed condition in Lemma 4.52; after identifying s^* with a pullback functor into a slice, $s_\sharp$ is the total-object functor and therefore preserves pullback squares. Both induced functors preserve colimits of families: s^* preserves colimits as a morphism of logoi, while $s_\sharp$ preserves them as a left adjoint, and colimits in $\mathrm{Fam}(C_i)$ and $\mathrm{Fam}(C_j)$ are computed in the corresponding arrow categories by Lemma 4.44.

Define an endofunctor Φ of $\mathrm{Fam}(D)$ by

$$\Phi(u)_i := \coprod_{(s\colon j\to i)} s^*(u_j) \sqcup \coprod_{(s\colon i\to j)} s_\sharp(u_j).$$

Let G be the endofunctor of $\mathrm{Fam}^{\mathrm{univ}}(D)$ obtained by applying univalent completion componentwise to Φ. The summands indexed by the identity morphisms give a natural map $u \to G(u)$. Moreover, for every $s\colon i \to j$ there are natural maps

$$s^*(u_i) \longrightarrow G(u)_j \qquad \text{and} \qquad u_j \longrightarrow s^*G(u)_i,$$

the second obtained by adjunction from $s_\sharp(u_j) \to G(u)_i$.

Now form the chain

$$u \longrightarrow G(u) \longrightarrow G^2(u) \longrightarrow \cdots$$

and set $u^\infty := \mathrm{colim}_n\, G^n(u)$. This colimit is computed in $\mathrm{Fam}(D)$ and remains univalent by Corollary 4.46. The functor G preserves sequential colimits: the functor Φ does so by the preceding paragraph, and univalent completion does so because it is a left adjoint. Consequently,

$$G(u^\infty) \simeq \mathrm{colim}_n\, G^{n+1}(u).$$

Here it is essential that univalent families form a poset. In particular, the two morphisms $G(u) \to G^2(u)$ obtained by applying G to $u \to G(u)$ and by evaluating the natural transformation $\mathrm{id} \to G$ at $G(u)$ agree, and all higher compatibilities are unique. Thus applying G to the displayed chain gives its tail. Under the resulting identification, the natural map

$$u^\infty \longrightarrow G(u^\infty)$$

agrees with the map induced by the structure maps $G^n(u) \to G^{n+1}(u)$. It is an isomorphism because the inclusion of the tail of $[\omega]$ is final. This is the usual Adámek fixed-point argument [Ada74].

For every arrow $s\colon i \to j$, the displayed maps for G pass to the colimit and give morphisms

$$s^*(u_i^\infty) \longrightarrow u_j^\infty \qquad \text{and} \qquad u_j^\infty \longrightarrow s^*(u_i^\infty)$$

between univalent families. Here the fixed-point equivalence identifies the shifted colimits appearing in the construction with u^∞. Since univalent families form a poset inside $\mathrm{Fam}(C_j)$, these two maps exhibit $s^*(u_i^\infty)$ and u_j^∞ as equivalent. The higher compatibility data are unique for the same reason. Thus u^∞ is a univalent family in the limit logos $C_{-\infty}$, and the map $u \to u^\infty$ shows that π provides enough univalent families. □

Proposition 4.60 ([Uem25, Proposition 2.6])**.** *Let C be a category with small colimits, and let $D \subseteq C$ be a wide subcategory. Suppose that:*

(1) *For every object $X \in C$, the wide subcategory $D_{/X} \subseteq C_{/X}$ is closed under small colimits.*

(2) *For every small diagram $X_\bullet\colon I \to D$, if $X = \mathrm{colim}_i\, X_i$ is computed in C, then all structure maps $X_i \to X$ lie in D.*

Then D is closed under small colimits in C.

Proof. Let $X_\bullet\colon I \to D$ be a diagram and let $X = \mathrm{colim}_i\, X_i$ in C. We need to show that X has the universal property of a colimit in D. For any object Y, a morphism $f\colon X \to Y$ belongs to D if and only if all composites $X_i \to X \to Y$ belong to D: the forward direction follows from (2), and the reverse direction follows from (1), since f is the colimit in $C_{/Y}$ of the diagram of the composites $X_i \to Y$. This identifies the mapping anima from X to Y in D with the limit of the mapping animae from the X_i to Y in D, as desired. □

We finally get to the main result.

Proof of Theorem 4.41. We apply Proposition 4.60 to the wide subcategory $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$.

First fix a topos S. We claim that $(\mathrm{Topos}^{\text{ét}})_{/S} \subseteq \mathrm{Topos}_{/S}$ is closed under colimits. Under the equivalence

$$S \simeq \mathrm{Topos}^{\text{ét}}_{/S}, \qquad X \longmapsto S_{/X}$$

from Proposition 4.38, a diagram of étale morphisms over S corresponds to a diagram $X_\bullet$ in S. If $X = \mathrm{colim}_i X_i$, descent in S gives

$$S_{/X} \simeq \lim_i S_{/X_i},$$

which says that the corresponding colimit in $\mathrm{Topos}_{/S}$ is again étale over S.

It remains to verify the structure-map condition. Let $T_\bullet \colon I \to \mathrm{Topos}^{\text{ét}}$ be a diagram, and let $T = \mathrm{colim}_i T_i$ be its colimit in Topos. Passing to logoi, this is the limit of the opposite diagram in $\mathrm{Logos}^{\text{ét}}$. By Lemma 4.59, each projection $T \to T_i$ on the logos side is an étale morphism. Equivalently, each structure morphism $T_i \to T$ on the topos side is étale.

The criterion therefore shows that $\mathrm{Topos}^{\text{ét}}$ is closed under colimits in Topos, and that the colimit computed in Topos has the universal property in $\mathrm{Topos}^{\text{ét}}$. Hence the inclusion preserves these colimits. □

5 Modalities

The factorization systems constructed in Chapter 3 are not isolated constructions. For every $-2 \leq n \leq \infty$, the factorization into an n-connected map followed by an n-truncated map is preserved by base change and therefore behaves uniformly in every slice of the topos. This raises a general question: which factorization systems on a topos have this fiberwise stability, and what additional structure follows from it?

A *modality* is a factorization system (L, R) whose left class is stable under base change. Modalities were developed systematically in a series of articles by Anel, Biedermann, Finster, and Joyal [ABFJ18; ABFJ20; ABFJ22; ABFJ24; ABFJ25]. They provide a common language for factorization, descent, localization, and excision phenomena in topoi.

We begin with a basic but less familiar example of a modality. In Section 5.1, we show that the epimorphisms form a modality and identify the epimorphisms in An with the acyclic maps; the associated reflection recovers Quillen's plus construction. We then turn to excision. In Section 5.2, we prove a generalized Blakers–Massey theorem for an arbitrary modality. It gives a condition on the pushout product of the diagonals of two maps which ensures that the gap map of their pushout square lies in the left class of the modality. Its proof rests on a modal form of descent.

The next part of the chapter connects fiberwise closure properties to localizations of topoi. An *acyclic class* is a saturated class stable under base change, while a *congruence* is an acyclic class satisfying 2-out-of-3. Small generation upgrades the former to a modality and identifies the latter with the class inverted by a left exact localization. The decisive relation, proved in Section 5.3, is that the congruence generated by a set Σ is the acyclic class generated by all iterated diagonals of maps in Σ:

$$\Sigma^c = (\Sigma^{\Delta})^m.$$

Equivalently, congruences are precisely the acyclic classes closed under diagonals. This formula gives explicit control over left exact localization and will later yield a short proof that sheafification on a Grothendieck site is left exact.

These ideas also lead to structure theory. We construct closed complements of open immersions in Section 5.4, and in Section 5.5 we factor every morphism of logoi into a monogenic quotient, an epigenic quotient, and a conservative morphism. Finally, products of acyclic classes restrict to products of congruences and give rise to adic filtrations of topoi. Applied to the congruence of ∞-connected maps, this construction produces the Goodwillie tower of a topos; for functor topoi it recovers Goodwillie's polynomial approximations and their stable layers. This is developed in Sections 5.6 and 5.7.

Definition 5.1. A *modality in T* is a factorization system (L, R) in T such that L is stable under base change.

Definition 5.2. An *acyclic class* in T is a saturated class of morphisms that is stable under base change. We write $\mathrm{Acyc}(T)$ for the partially ordered set of acyclic classes in T.

Thus the left class of every modality is an acyclic class. Conversely, an acyclic class of small generation is the left class of a modality by Proposition A.11. The distinction is useful: closure arguments can often be carried out for arbitrary acyclic classes, whereas orthogonal factorizations require a modality.

Terminology 5.3. Recall from Proposition A.3 that the morphisms in R are precisely those that are right orthogonal to the morphisms in L. In particular, a modality is completely determined by the class L. In what follows, we may sometimes refer to a class of morphisms L as a *modality* if the pair $(L, L^{\perp})$ is a modality.

Lemma 5.4. *Every acyclic class in T is local, in the sense of Definition 2.46. In a modality (L, R) on T, the right class R is local as well.*

Proof. Let L be an acyclic class, let $f\colon X \to Z$ be a morphism, and let $p\colon Z' \to Z$ be an effective epimorphism such that the base change $f'\colon X \times_Z Z' \to Z'$ lies in L. Form the Čech nerve $Z'_\bullet \to Z$ of p. Every level of the induced morphism

$$X \times_Z Z'_\bullet \longrightarrow Z'_\bullet$$

is a base change of f', hence lies in L. Since p is an effective epimorphism, the colimit of this morphism in $\mathrm{Ar}(T)$ is f. The class L is saturated and therefore closed under colimits, so $f \in L$.

Now let (L, R) be a modality and assume instead that $f' \in R$. Factor f as $X \xrightarrow{l} Y \xrightarrow{r} Z$ with $l \in L$ and $r \in R$. After base change along p, this is an (L, R)-factorization of f'. Hence the base change of l is an isomorphism. Since pullback along an effective epimorphism is conservative, l is an isomorphism and $f \in R$. □

5.1 Epimorphisms and acyclic maps

In this section, we discuss in detail the example of a modality given by the *epimorphisms* in a topos. For An, this recovers the acyclic maps: those that induce isomorphisms on cohomology with local coefficients.

5.1.1 Cotruncated maps and epimorphisms

We first construct the modality of cotruncated maps and establish the perhaps surprising fact that every epimorphism of a topos is already 0-connected.

Definition 5.5 (Cotruncation)**.** We say that a map $f\colon X \to Y$ is *n-cotruncated* if it is n-truncated in T^{op}. Inductively, this equivalently means that the (-2)-cotruncated maps are precisely the isomorphisms, and that a map f is $(n+1)$-cotruncated if and only if its codiagonal $\nabla_f\colon Y \sqcup_X Y \to Y$ is n-cotruncated.

Lemma 5.6. *Let L be the collection of n-cotruncated morphisms in T. Then the pair $(L, L^{\perp})$ forms a modality on T.*

Proof. We start by showing that $(L, L^{\perp})$ is a factorization system. In light of Proposition A.11, it suffices to show that L is saturated and of small generation.

Let $\nabla\colon \mathrm{Ar}(T) \to \mathrm{Ar}(T)$ be the codiagonal functor, which sends $f\colon X \to Y$ to $\nabla_f\colon Y \sqcup_X Y \to Y$. This is an accessible functor, since it is constructed from finite colimits. By Definition 5.5, a morphism f is n-cotruncated if and only if $\nabla^{n+2}(f)$ is an isomorphism. Consequently, L is the inverse image under the accessible functor ∇^{n+2} of the accessible subcategory of isomorphisms in $\mathrm{Ar}(T)$. It follows that $L \subseteq \mathrm{Ar}(T)$ is accessible and accessibly embedded.

The class L is saturated. Indeed, this is immediate for $n = -2$, and the inductive step follows because the codiagonal construction preserves colimits in $\mathrm{Ar}(T)$. Since L is accessible, accessibly embedded, and closed under all colimits, it is generated under colimits by a small collection of its objects. In particular, it is the saturation of a small set of morphisms.

To show that the n-cotruncated maps are stable under base change, consider a pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y, \end{array}$$

and assume that f is n-cotruncated. We will show that also f' is n-cotruncated. By induction, it will suffice to show that the following square is a pullback square:

$$\begin{array}{ccc} Y' \sqcup_{X'} Y' & \longrightarrow & Y \sqcup_X Y \\ {\scriptstyle \nabla_{f'}}\downarrow & & \downarrow{\scriptstyle \nabla_f} \\ Y' & \longrightarrow & Y. \end{array}$$

But this is clear from universality of colimits. □

Definition 5.7. A morphism $f\colon X \to Y$ in T is called an *epimorphism* if it is (-1)-cotruncated, i.e. if the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \| \\ Y & = & Y \end{array}$$

is a pushout square. Let $T^{+} \subseteq T$ be the full subcategory of objects Z for which $Z \to *$ is right orthogonal to all epimorphisms. The factorization system of Lemma 5.6 gives a reflection $(-)^{+}\colon T \to T^{+}$ and hence a factorization

$$X \to X^{+} \to *$$

into an epimorphism followed by a map which is right orthogonal to the epimorphisms.

As already mentioned in Warning 2.28, the terminology regarding 'epimorphism' and 'effective epimorphism' is a rather unfortunate historical accident: effective epimorphisms are not generally epimorphisms.

Example 5.8. For $T = \mathrm{An}$, the map $S^0 \to *$ is an effective epimorphism (as it is a surjection on path components), but it is not an epimorphism: the pushout of $S^0 \to *$ along itself is S^1, which is not isomorphic to $*$.

The situation is made even more confusing by the fact that the converse *does* hold.

Proposition 5.9. *Every epimorphism in T is 0-connected, hence in particular an effective epimorphism.*

Proof. We may assume that $Y = *$ by passing to the slice topos: the forgetful functor $T_{/Y} \to T$ preserves all colimits, hence preserves and detects epimorphisms. So we need to show: if $X \to *$ is an epimorphism, then $\tau_0 X$ is terminal.

Since the suspension functor $\Sigma^\infty \colon T \to \mathrm{Sp}(T)$ preserves colimits, it follows that the commutative square

$$\begin{array}{ccc} \Sigma^\infty(X) & \longrightarrow & \Sigma^\infty(*) \\ \downarrow & & \| \\ \Sigma^\infty(*) & = & \Sigma^\infty(*) \end{array}$$

is a pushout square, hence a pullback square by stability. It follows that the map $\Sigma^\infty(X) \to \Sigma^\infty(*)$ is an isomorphism. The suspension functor factors as

$$T \xrightarrow{F^{\mathrm{CGrp}}} \mathrm{CGrp}(T) \xhookrightarrow{\mathbb{B}^\infty} \mathrm{Sp}(T),$$

where the first functor is the free commutative group functor and the second functor is the classifying spectrum functor which identifies commutative groups with connective spectra. By conservativity of $\mathbb{B}^\infty$ we conclude that $F^{\mathrm{CGrp}}(X) \xrightarrow{\sim} F^{\mathrm{CGrp}}(*)$. Now, since the truncation functor $\tau_0 \colon T \to T_{\leq 0}$ commutes with finite products, it induces a functor $\tau_0 \colon \mathrm{CGrp}(T) \to \mathrm{CGrp}(T_{\leq 0})$. Since also the inclusion $T_{\leq 0} \hookrightarrow T$ preserves finite products, it follows that τ_0 also commutes with the free commutative group construction, giving a commutative diagram as follows:

$$\begin{array}{ccc} T & \xrightarrow{F^{\mathrm{CGrp}}} & \mathrm{CGrp}(T) \\ {\scriptstyle\tau_0}\downarrow & & \downarrow{\scriptstyle\tau_0} \\ T_{\leq 0} & \xrightarrow{F^{\mathrm{CGrp}}} & \mathrm{CGrp}(T_{\leq 0}). \end{array}$$

In particular, we conclude that $F^{\mathrm{CGrp}}(\tau_0 X) \xrightarrow{\sim} F^{\mathrm{CGrp}}(*)$. To deduce that $\tau_0 X = *$, note that for any 0-truncated object X' the unit map $X' \to F^{\mathrm{CGrp}}(X')$ is a monomorphism. Indeed, this is clear for $T = \mathrm{An}$ (where $F^{\mathrm{CGrp}}(X') = \mathbb{Z}[X']$ is the free abelian group generated by the set X'), thus also for any presheaf topos, and then finally for any left exact localization $f^* \colon \mathrm{PSh}(C) \to T$ of a presheaf topos as f^* preserves monomorphisms and commutes with the adjunction $T \rightleftarrows \mathrm{CGrp}(T)$. From the commutative diagram

$$\begin{array}{ccc} \tau_0 X & \hookrightarrow & F^{\mathrm{CGrp}}(\tau_0 X) \\ \downarrow & & \downarrow{\scriptstyle\sim} \\ * & \hookrightarrow & F^{\mathrm{CGrp}}(*), \end{array}$$

it follows that $\tau_0 X \to *$ is a monomorphism in $T_{\leq 0}$. Since $\tau_0(-)$ preserves colimits, it is also an epimorphism. But then we are done, as in a classical topos, every monomorphism which is also an epimorphism is an isomorphism, see Lemma 5.10. □

Lemma 5.10. *Let T be a topos, and let $f\colon X \to Y$ be a morphism in $T_{\leq 0}$ which is both an epimorphism and a monomorphism. Then f is an isomorphism.*

Proof. Since f is a monomorphism, it is contained in $(T_{/Y})_{\leq -1}$. The fact that it is an epimorphism in $T_{\leq 0}$ means that the map $\tau_0(Y \sqcup_X Y) \to Y$ induced by the codiagonal is an isomorphism. This 0-truncation is a priori computed in T, but since Y is 0-truncated it may as well be computed in $T_{/Y}$, so that the map $f\colon X \to Y$ is also an epimorphism in $(T_{/Y})_{\leq 0}$. This reduces the claim to $Y = *$.

In this case, the map $f\colon X \to *$ to the terminal object is a monomorphism. By definition of the subobject classifier Ω from Definition 2.50, this means that there is a pullback square in T of the following form:

$$\begin{array}{ccc} X & \xrightarrow{f} & * \\ {\scriptstyle f}\big\downarrow & \lrcorner & \big\downarrow \\ * & \xrightarrow{g} & \Omega. \end{array}$$

Recall from Lemma 2.51 that Ω is 0-truncated, hence this is a pullback square in $T_{\leq 0}$. Since f is assumed to be an epimorphism in $T_{\leq 0}$, it follows that the bottom map g must agree with the universal monomorphism $* \hookrightarrow \Omega$. But then X is the pullback of the map $* \hookrightarrow \Omega$ along itself, which implies that $X \xrightarrow{\sim} *$. This finishes the proof. □

5.1.2 Acyclic maps and the plus construction

We now specialize to animae. Cohomology with all local coefficient systems detects precisely the epimorphisms, and the corresponding reflection is Quillen's plus construction. We first record the two detection arguments used in the proof.

Lemma 5.11 (Detection by local coefficients)**.** *Let $f\colon X \to Y$ be a map of animae. If*

$$H^0(Y, A) \longrightarrow H^0(X, f^*A)$$

is an isomorphism for every local system A of abelian groups on Y, then $\pi_0(f)$ is an isomorphism and $\pi_1(X, x) \to \pi_1(Y, f(x))$ is surjective for every $x \in X$.

Proof. Taking A to be the constant local system $\mathbb{Z}/2$ shows that pullback along $\pi_0(f)$ induces a bijection on functions to $\mathbb{Z}/2$. Hence $\pi_0(f)$ is a bijection.

We may therefore work on one connected component. Fix $x \in X$, write $G = \pi_1(Y, f(x))$, and let $H \subseteq G$ be the image of $\pi_1(X, x)$. If $H \neq G$, consider the G-module

$$M := (\mathbb{Z}/2)^{G/H}$$

with the permutation action. The characteristic function of the coset H is fixed by H but not by G. Thus the inclusion $M^G \to M^H$ is not surjective. This contradicts the assumed isomorphism on H^0, since $H^0(Y, M) = M^G$ and $H^0(X, f^*M) = M^H$. Therefore $H = G$. □

Lemma 5.12 (Whitehead theorem with local coefficients)**.** *Let $f\colon X \to Y$ be a map of animae which induces isomorphisms on π_0 and π_1. If f induces an isomorphism on cohomology with every local system of abelian groups on Y, then f is an isomorphism.*

Proof. The claim may be checked componentwise, so suppose X and Y are connected and put $G = \pi_1(Y) \xrightarrow{\sim} \pi_1(X)$. Choose CW models and lift f to a G-equivariant map of universal covers. Let Q_* be the mapping cone of the induced map of cellular chain complexes over $\mathbb{Z}[G]$. The hypothesis says that

$$H^*\mathrm{Hom}_{\mathbb{Z}[G]}(Q_*, M) = 0$$

for every $\mathbb{Z}[G]$-module M.

We claim that Q_* is acyclic. Otherwise, let q be the least degree for which $H_q(Q_*)$ is nonzero, and choose an injective $\mathbb{Z}[G]$-module I receiving a nonzero map from $H_q(Q_*)$. The universal coefficient spectral sequence for the complex of projective $\mathbb{Z}[G]$-modules Q_* degenerates for the coefficient module I and gives

$$H^q\mathrm{Hom}_{\mathbb{Z}[G]}(Q_*, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}[G]}(H_q(Q_*), I) \neq 0,$$

contradicting the hypothesis. Thus the map of universal covers is a homology isomorphism. Since both universal covers are simply connected, the relative Hurewicz theorem implies that it is a homotopy equivalence. The original map f is therefore an isomorphism of animae. □

We now identify the epimorphisms in the topos of animae.

Proposition 5.13. *Consider the topos $T = \mathrm{An}$.*

(1) *A map of animae $f\colon X \to Y$ is an epimorphism if and only if f is* acyclic*, meaning that for any local system A of abelian groups on Y, the induced map*

$$H^*(Y, A) \xrightarrow{\cong} H^*(X, f^*A)$$

is an isomorphism.

(2) *A map of animae $f\colon X \to Y$ is in the right orthogonal class to the epimorphisms if and only if the group $\pi_1(f) \in \mathrm{Grp}((T_{/Y})_{\leq 0})$ is* hypoabelian*.*

(3) *For any anima X, the unique factorization $X \to *$ as*

$$X \xrightarrow{\text{epi}} X^+ \xrightarrow{\pi_1 \text{ hypoabelian}} *$$

is Quillen's $+$-construction.

In (2), we say that a group G is *perfect* if it is left orthogonal to abelian groups, i.e. if every map $G \to A$ into an abelian group is trivial, or equivalently if the abelianization G^{ab} is trivial. A group A is *hypoabelian* if it is right orthogonal to perfect groups, i.e. if every map $G \to A$ from a perfect group is trivial, or equivalently if A has no nontrivial perfect subgroup.

Proof. (1) Suppose first that f is an epimorphism. For any local system A on Y, the Mayer–Vietoris sequence associated to the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & \ulcorner & \| \\ Y & = & Y. \end{array}$$

identifies the restriction map $H^*(Y, A) \to H^*(X, f^*A)$ with an isomorphism. Thus f is acyclic.

Conversely, suppose that f is acyclic and set $C := Y \sqcup_X Y$. Let $i_1, i_2 \colon Y \to C$ be the two coprojections. For a local system A on C, the Mayer–Vietoris sequence contains

$$\cdots \longrightarrow H^q(C, A) \longrightarrow H^q(Y, i_1^*A) \oplus H^q(Y, i_2^*A) \longrightarrow H^q(X, f^*i_1^*A) \longrightarrow \cdots .$$

The two restrictions of A to X are canonically identified, and both restriction maps from the two copies of $H^q(Y)$ to $H^q(X)$ are isomorphisms by acyclicity of f. Exactness therefore identifies $H^q(C, A)$ with the diagonal copy of $H^q(Y, i_1^*A)$. In particular, i_1 is itself acyclic.

By Lemma 5.11, the map f induces an isomorphism on π_0 and a surjection on every fundamental group. We may consequently apply the Seifert–van Kampen theorem componentwise to the pushout defining C. It gives a pushout square

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{f} & \pi_1(Y, f(x)) \\ \downarrow{\scriptstyle f} & & \downarrow \\ \pi_1(Y, f(x)) & \longrightarrow & \pi_1(C, i_1(f(x))) \end{array}$$

in groups. Since the two maps out of $\pi_1(X, x)$ agree and are surjective, the coprojection i_1 induces an isomorphism on fundamental groups. The same van Kampen calculation on components shows that i_1 induces an isomorphism on π_0. Hence Lemma 5.12 applies and shows that i_1 is an isomorphism. Since $\nabla_f i_1 = \mathrm{id}_Y$, the codiagonal $\nabla_f \colon C \to Y$ is its inverse. Thus f is an epimorphism.

(2) See [Hoy19a, Corollary 10].

(3) By definition, Quillen's +-construction is the reflection of X into the subcategory of animae with hypoabelian fundamental group. By (2), this subcategory precisely consists of the objects Y for which $Y \to *$ is right orthogonal to the epimorphisms, and the claim follows. □

5.2 The Blakers–Massey theorem

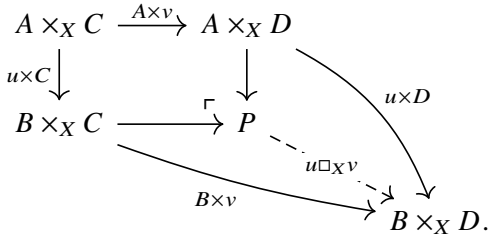
Recall from classical algebraic topology the important *Blakers–Massey theorem*:

Theorem 5.14 (Classical Blakers–Massey theorem)**.** *Let X be a topological space and let $A, B \subseteq X$ be two subspaces whose interiors cover X. Assume that A and B are connected and that their intersection $C := A \cap B$ is (connected and) simply connected. If the pair (A, C) is $(m-1)$-connected and the pair (B, C) is $(n-1)$-connected, $m, n \geq 3$, then the triple $(X; A, B)$ is $(m+n-2)$-connected.*

In homotopy-theoretic language, we may think of X, A, B and C as animae, where $X = A \sqcup_C B$ is the (homotopy) pushout of two maps $f \colon C \to A$ and $g \colon C \to B$. The assumptions then require f to be n-connected and g to be m-connected (recall from Warning 3.17 the shift in terminology) and the conclusion translates to the condition that the map $C \to A \times_X B$ is $(m + n)$-connected.

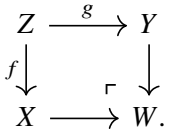
It turns out that this statement holds in an arbitrary topos, and in fact admits a purely topos-theoretic proof. It will be a special case of a more general version of the Blakers–Massey theorem, due to Anel, Biedermann, Finster, and Joyal [ABFJ20], which can be formulated for an arbitrary modality L.

Notation 5.15 (Day convolution on slices)**.** Let $\square_X$ denote Day convolution in $\mathrm{Ar}(T_{/X})$, with respect to the minimum operation on $[1]$, where $\mathrm{Ar}(T_{/X}) = \mathrm{Fun}([1], T_{/X})$. Given maps $u\colon A \to B$ and $v\colon C \to D$ in $T_{/X}$, the map $u \square_X v$ is the dashed arrow in the pushout square:



This operation is also known as the *pushout product of u and v over X*.

Theorem 5.16 (Generalized Blakers–Massey, Anel, Biedermann, Finster, and Joyal [ABFJ20])**.** *Let (L, R) be a modality in T, and consider a pushout square*



If $\Delta_f \square_Z \Delta_g$ lies in L, then also the gap map $Z \to X \times_W Y$ lies in L.

In the statement we regard Δ_f and Δ_g as maps in $T_{/Z}$ via the projections $p_1\colon Z \times_X Z \to Z$ and $p_1\colon Z \times_Y Z \to Z$.

We prove this theorem below. We first deduce the ordinary Blakers–Massey theorem.

Lemma 5.17. *Let u and v be maps in $T_{/Z}$. If u is m-connected and v is n-connected, then $u \square_Z v$ is $(m + n + 2)$-connected.*

Proof. We work in the topos $T_{/Z}$. Choose a small collection $\mathcal{G}$ of generators. The class of m-connected maps is the saturation of the maps

$$S^{m+1} \otimes U \longrightarrow U, \qquad U \in \mathcal{G}.$$

Indeed, a map is right orthogonal to these generators precisely when its fibers are m-truncated, by the characterization of truncated objects in Section 3.1. The same description holds for the n-connected maps.

The pushout product preserves colimits separately in both variables. It therefore suffices to take u and v of the displayed form, allowing arbitrary objects U and V. Their pushout product is

$$(S^{m+1} * S^{n+1}) \otimes (U \times_Z V) \longrightarrow U \times_Z V,$$

where $*$ denotes the join of animae. Since $S^{m+1} * S^{n+1} \xrightarrow{\sim} S^{m+n+3}$, this map is $(m+n+2)$-connected. The $(m+n+2)$-connected maps form a saturated class, so the same follows for arbitrary u and v. □

Corollary 5.18 (Blakers–Massey)**.** *Assume f is m-connected and g is n-connected. Then the map $Z \to X \times_W Y$ is $(m + n)$-connected.*

Proof. The map Δ_f is $(m-1)$-connected while Δ_g is $(n-1)$-connected. By Lemma 5.17, this means that $\Delta_f \square_Z \Delta_g$ is $(m-1+(n-1)+2)$-connected, and thus the theorem applies. □

Remark 5.19. While the proof of the Blakers–Massey theorem is quite involved, the *dual Blakers–Massey theorem* is easy to prove. It states that for a pullback square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z, \end{array}$$

if the absolute pushout product $f \square g$ lies in L, then the cogap map $X \sqcup_W Y \to Z$ lies in L. Indeed, the cogap map is the base change of

$$f \square g \colon (X \times Z) \sqcup_{X \times Y} (Z \times Y) \longrightarrow Z \times Z$$

along the diagonal $\Delta_Z \colon Z \to Z \times Z$. The claim therefore follows from stability of L under base change.

5.2.1 Modal descent

We first give a proof of Theorem 5.16 using modal descent, which is the proof given by Anel, Biedermann, Finster, and Joyal [ABFJ20]. We start by introducing some terminology.

Definition 5.20. A commutative square in T is called *L-cartesian* if its gap map lies in L.

Proposition 5.21 (Properties of L-cartesian squares)**.** *Consider two composable squares*

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X'' \\ \downarrow & A & \downarrow & B & \downarrow \\ Y & \xrightarrow[f]{} & Y' & \longrightarrow & Y''. \end{array}$$

(1) *If A and B are L-cartesian, then their composite $A + B$ is L-cartesian.*

(2) *If A and $A + B$ are L-cartesian and f is an effective epimorphism, then B is L-cartesian.*

(3) *If B is cartesian and $A + B$ is L-cartesian, then A is L-cartesian.*

Proof. The gap map $X \to Y \times_{Y''} X''$ of $A + B$ may be factored as

$$X \to Y \times_{Y'} X' \to Y \times_{Y''} X'',$$

where the first map is the gap map of A and the second map is the base change along $Y \to Y'$ of the gap map of B. This immediately implies (1). For (2), it follows from right cancellation that the base change along $Y \to Y'$ of the gap map of B lies in L, hence so does the gap map itself by Lemma 5.4. Part (3) is clear: if B is cartesian, then the gap map of A agrees with that of $A + B$. □

Lemma 5.22. *Consider a commutative diagram of the form*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle \in L}\downarrow & & \downarrow{\scriptstyle \in L} \\ Z & \longrightarrow & Z' \\ {\scriptstyle \in R}\downarrow & & \downarrow{\scriptstyle \in R} \\ Y & \longrightarrow & Y', \end{array}$$

where the four maps lie in L and R as indicated. If the outer square is L-cartesian, then the bottom square is cartesian.

Proof. By left cancellation, the gap map $Z \to Y \times_{Y'} Z'$ of the bottom square lies in R. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\in L} & Y \times_{Y'} X' \\ {\scriptstyle \in L}\downarrow & & \downarrow{\scriptstyle \in L} \\ Z & \longrightarrow & Y \times_{Y'} Z'. \end{array}$$

Then the three maps lie in L as indicated, hence by right cancellation so does the map $Z \to Y \times_{Y'} Z'$. Since (L, R) is a factorization system, it follows that this map is an isomorphism, as desired. □

Proposition 5.23 (Modal descent)**.** *Let $X_\bullet, Y_\bullet \in \mathrm{Fun}(I, T)$ be two diagrams with colimits X and Y. If we are given an L-cartesian transformation $X_\bullet \to Y_\bullet$, then the square*

$$\begin{array}{ccc} X_\bullet & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_\bullet & \longrightarrow & Y \end{array}$$

is L-cartesian. Equivalently: the subcategory $\mathrm{Ar}^{L\text{-cart}}(T) \subseteq \mathrm{Ar}(T)$ is closed under colimits.

Proof. Let $P_\bullet := Y_\bullet \times_Y X$. The transformation $X_\bullet \to P_\bullet$ has the same gap maps as $X_\bullet \to Y_\bullet$ and is therefore L-cartesian, while $P_\bullet \to Y_\bullet$ is cartesian. Since colimits in a topos are universal, $\operatorname{colim} P_\bullet \xrightarrow{\sim} X$. By Proposition 5.21, it is therefore enough to treat the transformation $X_\bullet \to P_\bullet$. Replacing $Y_\bullet$ by $P_\bullet$, we may assume that the induced map $X \to Y$ on colimits is an isomorphism.

Factor the transformation pointwise as

$$X_\bullet \xrightarrow{\in L} Z_\bullet \xrightarrow{\in R} Y_\bullet,$$

and consider the resulting commutative diagram:

$$\begin{array}{ccc} X_\bullet & \longrightarrow & X \\ {\scriptstyle \in L}\downarrow & & \downarrow \\ Z_\bullet & \longrightarrow & Z \\ {\scriptstyle \in R}\downarrow & & \downarrow \\ Y_\bullet & \longrightarrow & Y. \end{array}$$

By Lemma 5.22, the transformation $Z_\bullet \to Y_\bullet$ is cartesian. Descent then implies that the bottom square is a pullback. Since L is closed under colimits in $\mathrm{Ar}(T)$, the map $X \to Z$ lies in L. Its composite $X \to Z \to Y$ is an isomorphism and hence lies in L, so right cancellation gives $Z \to Y \in L$. Each map $Z_i \to Y_i$ is a base change of $Z \to Y$, because the bottom square is cartesian, and hence also belongs to L. But it belongs to R by construction, so it is an isomorphism. Thus $Z_\bullet \to Y_\bullet$ and $Z \to Y$ are isomorphisms. Under these identifications, the gap maps $X_i \to Y_i \times_Y X$ are precisely the maps $X_i \to Z_i$, which lie in L by construction. □

Proof of the Generalized Blakers–Massey theorem. The proof has three steps. We first replace one leg by its effective-epimorphism part. We then encode the assumed pushout product as the gap map of one face of a cube. Modal descent propagates this information across the cube, after which cancellation identifies the original square as L-cartesian.

Consider a pushout square

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ X & \longrightarrow & W, \end{array}$$

where $\Delta_f \square_Z \Delta_g \in L$. We must show that also $Z \to X \times_W Y \in L$.

We first reduce to the case where f is an effective epimorphism. Indeed, letting X' be the image of f, we get a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f'}\downarrow\!\!\downarrow & \ulcorner & \downarrow\!\!\downarrow \\ X' & \longrightarrow & W' \\ \downarrow & \ulcorner & \downarrow \\ X & \longrightarrow & W, \end{array}$$

where the two displayed squares are still pushouts. By Proposition 3.11, the lower square is a pullback square. We then observe that the diagonal Δ_f is isomorphic to the diagonal $\Delta_{f'}$, and that the gap map $Z \to X \times_W Y$ agrees with the gap map $Z \to X' \times_{W'} Y$ of the top square. Since $X' \to X$ is a monomorphism and f factors through X', the canonical map $Z \times_{X'} Z \to Z \times_X Z$ is an isomorphism. Under this isomorphism the two diagonal maps from Z agree. Moreover, the lower pullback square identifies X' with $X \times_W W'$. Since $Y \to W$ factors through W', we obtain

$$X \times_W Y \xrightarrow{\sim} (X \times_W W') \times_{W'} Y \xrightarrow{\sim} X' \times_{W'} Y.$$

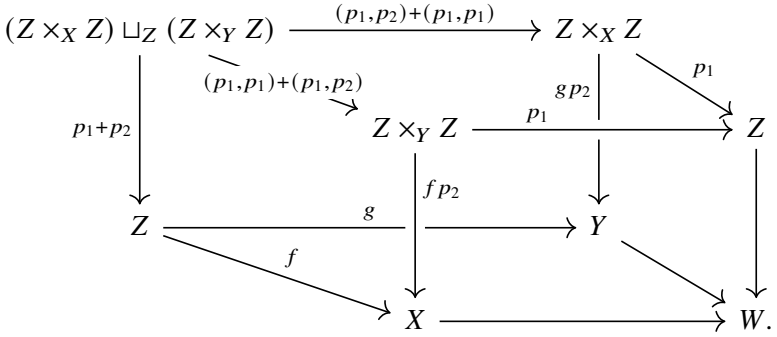
These identifications commute with the maps from Z. It thus suffices to prove the statement for the top square, or equivalently in the case where f is an effective epimorphism.

Unwinding definitions, we see that the map

$$\Delta_f \square_Z \Delta_g \colon (Z \times_X Z) \sqcup_Z (Z \times_Y Z) \longrightarrow (Z \times_X Z) \times_Z (Z \times_Y Z)$$

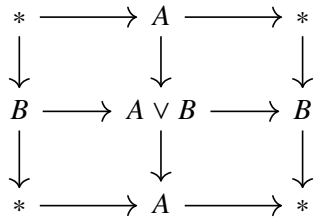
is given on the first component by (p_1, p_2, p_1, p_1) and on the second by (p_1, p_1, p_1, p_2). Regard $Z \times_X Z$ and $Z \times_Y Z$ as pointed objects of $T_{/Z}$, with structure map p_1 and basepoint given by the corresponding diagonal. The pushout product of the two basepoints is the canonical map from their wedge to their product. Its source is therefore $(Z \times_X Z) \sqcup_Z (Z \times_Y Z)$, while its target is their fiber

product over Z. In particular, this map is the gap map of the top face of the following commutative cube:



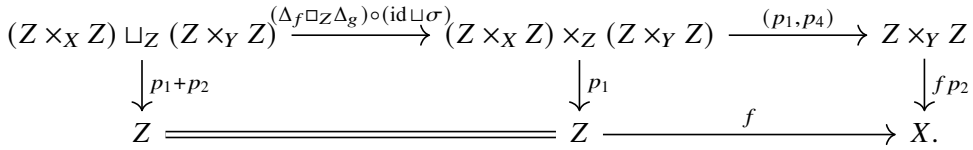
We now make the following series of claims:

- The top face is a pushout. More generally, for objects A, B in some pointed category C (here $C = T_{Z//Z}$) the lower-right corner of



  is a pushout, by the pasting property of pushout squares. In $C = T_{Z//Z}$ the zero object is the identity of Z. Take $A = Z \times_X Z$ and $B = Z \times_Y Z$, with their diagonal basepoints and first projections as structure maps. Then $A \vee B$ is exactly $(Z \times_X Z) \sqcup_Z (Z \times_Y Z)$, and the lower-right square in the displayed 3-by-3 diagram is the top face of the cube.

- The left and back faces are L-cartesian. By symmetry it suffices to do this for the left face. Writing $\sigma\colon Z \times_Y Z \xrightarrow{\cong} Z \times_Y Z$ for the swap map, this face decomposes as



  The left square is L-cartesian by hypothesis. The right square is a pullback square. It follows that the outer rectangle is L-cartesian. The gap map of the left square is the assumed map $\Delta_f \square_Z \Delta_g$, precomposed with the isomorphism $\mathrm{id} \sqcup \sigma$. Hence it lies in L. The gap map of the right square is an isomorphism because that square is cartesian, so it also lies in L. Part (1) of Proposition 5.21 then shows that their composite, which is the left face of the cube, is L-cartesian.

- By modal descent applied when I is the span diagram $\ulcorner$, the front and right faces are L-cartesian. The cube is a pushout square in the arrow category: its top and bottom faces are pushouts, while its left and back faces are the two input morphisms. Those input faces are L-cartesian by the preceding claims. Since Proposition 5.23 says that L-cartesian arrows are closed under colimits, the remaining two structure maps in this pushout square, namely the front and right faces, are also L-cartesian.

- Finally, we may decompose the right face of the diagram as the composite of two squares:

$$\begin{array}{ccccc} Z \times_X Z & \xrightarrow{p_2} & Z & \xrightarrow{g} & Y \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f} & & \downarrow \\ Z & \xrightarrow{f} & X & \longrightarrow & W. \end{array}$$

  Here the left square is cartesian, f was assumed to be an effective epimorphism, and the outer rectangle is L-cartesian by the previous point. It follows from Proposition 5.21 that the right square is L-cartesian. Apply part (2) of Proposition 5.21. Its square A is the cartesian left square displayed above, its composite $A + B$ is the L-cartesian outer rectangle, and the map called f there is the effective epimorphism $Z \to X$ along their common lower edge. The conclusion is that B, the right square, is L-cartesian.

This gives the claim. The right square just proved L-cartesian is the original pushout square with vertices Z, Y, X, W. Its gap map is, by definition, the canonical map $Z \to X \times_W Y$. Thus its being L-cartesian is exactly the desired conclusion. □

5.2.2 Wärn's alternative proof

We now explain a second proof of Theorem 5.16, following the description of path spaces of pushouts due to Wärn [Wär25]. The main point of this approach is that the pullback $B \times_D C$ of a pushout

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & D \end{array}$$

can be built as a sequential colimit of objects of zigzags. The first non-trivial step in this filtration is the relative pushout product $\Delta_f \square_A \Delta_g$. So if we can show that all later maps in the filtration lie in the modality L, then so does the gap map $A \to B \times_D C$.

Construction 5.24 (Making one leg cartesian)**.** Fix a span $B \leftarrow A \to C$. A *span over* this span is a diagram of the form

$$\begin{array}{ccccc} Q & \longleftarrow & P & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ B & \longleftarrow & A & \longrightarrow & C. \end{array}$$

Given such a span, we define two new spans over $B \leftarrow A \to C$ as follows.

(1) The span obtained by *making the left leg cartesian* is

$$Q^\ell := Q, \qquad P^\ell := Q \times_B A, \qquad R^\ell := P^\ell \sqcup_P R.$$

(2) The span obtained by *making the right leg cartesian* is

$$R^r := R, \qquad P^r := R \times_C A, \qquad Q^r := P^r \sqcup_P Q.$$

Lemma 5.25. *In both cases, the evident map from the original span to the new span induces an equivalence on pushouts.*

Proof. For the left construction, the new pushout is

$$Q^\ell \sqcup_{P^\ell} R^\ell \simeq Q \sqcup_{P^\ell} (P^\ell \sqcup_P R) \simeq Q \sqcup_P R,$$

by pasting of pushout squares. The right construction is symmetric. □

Construction 5.26 (Zigzag construction, Wärn [Wär25])**.** Starting with a span $Q_0 \leftarrow P_0 \rightarrow R_0$ over $B \leftarrow A \rightarrow C$, define a sequence of spans

$$Q_n \leftarrow P_n \rightarrow R_n$$

by alternately making the left and right legs cartesian: for odd n we apply the left construction to the previous span, and for even $n \geq 2$ we apply the right construction. Let

$$Q_\infty := \operatorname{colim}_n Q_n, \qquad P_\infty := \operatorname{colim}_n P_n, \qquad R_\infty := \operatorname{colim}_n R_n.$$

The zigzag construction computes pullbacks of pushouts in the following precise sense.

Theorem 5.27 (Wärn [Wär25, Theorem 3.6])**.** *Let*

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & D \end{array}$$

be a pushout square in T. Let $Q_0 \leftarrow P_0 \rightarrow R_0$ be a span over $B \leftarrow A \rightarrow C$, and write

$$S := Q_0 \sqcup_{P_0} R_0.$$

Then the colimit span $Q_\infty \leftarrow P_\infty \rightarrow R_\infty$ from Construction 5.26 has pushout S, and the canonical maps

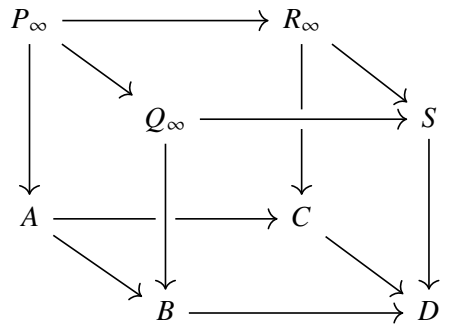
$$Q_\infty \rightarrow S \times_D B, \qquad P_\infty \rightarrow S \times_D A, \qquad R_\infty \rightarrow S \times_D C$$

are isomorphisms.

Proof. By Construction 5.24, each step in the zigzag construction preserves the pushout of the span. Since colimits commute with colimits, the pushout of $Q_\infty \leftarrow P_\infty \rightarrow R_\infty$ is again S.

For odd n, the map $P_n \rightarrow Q_n$ is cartesian over $A \rightarrow B$ by construction. Since the odd natural numbers are cofinal in $\mathbb{N}$, and since sequential colimits in a topos are universal, it follows that $P_\infty \rightarrow Q_\infty$ is cartesian over $A \rightarrow B$. Similarly, using the even stages, $P_\infty \rightarrow R_\infty$ is cartesian over $A \rightarrow C$.

Now compare the pushout square $Q_\infty \sqcup_{P_\infty} R_\infty \simeq S$ with the original pushout square $B \sqcup_A C \simeq D$:



The left and back faces of the cube are cartesian by the previous paragraph, while the top and bottom faces are pushouts. Descent for pushouts therefore implies that the front and right faces are cartesian. This gives $Q_\infty \xrightarrow{\sim} S \times_D B$ and $R_\infty \xrightarrow{\sim} S \times_D C$, and then also $P_\infty \xrightarrow{\sim} S \times_D A$. □

We wish to apply this result to the Blakers–Massey problem, where we start with $B \leftarrow \emptyset \rightarrow \emptyset$. The following is the starting observation:

Lemma 5.28. *For the zigzag construction associated to the initial span $B \leftarrow \emptyset \rightarrow \emptyset$ over $B \xleftarrow{f} A \xrightarrow{g} C$, the map $P_2 \rightarrow P_3$ is a pushout of $\Delta_f \square_A \Delta_g$.*

Proof. The first step gives

$$Q_1 \simeq B, \qquad P_1 \simeq A, \qquad R_1 \simeq A.$$

The second step makes the right leg cartesian, hence gives

$$R_2 \simeq A, \qquad P_2 \simeq A \times_C A, \qquad Q_2 \simeq (A \times_C A) \sqcup_A B,$$

where the map $A \rightarrow A \times_C A$ is Δ_g and $A \rightarrow B$ is f. The third step makes the left leg cartesian, so

$$P_3 \simeq Q_2 \times_B A \simeq ((A \times_C A) \times_B A) \sqcup_{A \times_B A} A,$$

where we used universality of pushouts. Observe that we may rewrite $(A \times_C A) \times_B A$ as $(A \times_C A) \times_A (A \times_B A)$. Write j_C and j_B for the two coprojections into $(A \times_C A) \sqcup_A (A \times_B A)$, and write q for the canonical map from this pushout to the corresponding product. Consider now the following commutative diagram:

$$\begin{array}{ccccc}
A & \xrightarrow{\Delta_g} & (A \times_C A) & & \\
\downarrow{\scriptstyle \Delta_f} & & \downarrow{\scriptstyle j_C} & & \\
(A \times_B A) & \xrightarrow[j_B]{} & (A \times_C A) \sqcup_A (A \times_B A) & \xrightarrow{q = \Delta_g \square_A \Delta_f} & (A \times_C A) \times_A (A \times_B A) \\
\downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & (A \times_C A) & \longrightarrow & P_3
\end{array}$$

The bottom rectangle is the pushout square defining P_3. The upper-left square is the pushout square defining the source of q, and the left rectangle is a pushout by construction. The pasting law therefore shows that the lower-right square is a pushout. Hence $P_2 = A \times_C A \rightarrow P_3$ is the cobase change of $q = \Delta_g \square_A \Delta_f$. By symmetry of the pushout product, this is isomorphic to $\Delta_f \square_A \Delta_g$, as desired. □

The next observation is that passing to the next stage of the zigzag construction preserves any existing connectivity estimates of the current stage.

Lemma 5.29. *Let L be a modality, and consider the zigzag construction associated to a span $Q_0 \leftarrow P_0 \rightarrow R_0$ over $B \leftarrow A \rightarrow C$. If $P_{n-1} \rightarrow P_n$ lies in L for some $n \geq 2$, then also $P_n \rightarrow P_{n+1}$ lies in L.*

Proof. Suppose first that P_n is obtained from P_{n-1} by making the left leg cartesian, so that the next step makes the right leg cartesian. Since $n \geq 2$, the preceding step made the right leg cartesian, so

$$P_{n-1} \xrightarrow{\sim} R_{n-1} \times_C A.$$

Unwinding the definitions of these two consecutive steps gives

$$R_n \xrightarrow{\sim} P_n \sqcup_{P_{n-1}} R_{n-1} \qquad \text{and} \qquad P_{n+1} \xrightarrow{\sim} R_n \times_C A.$$

Since pushouts are universal, it follows that

$$P_{n+1} \xrightarrow{\sim} (P_n \times_C A) \sqcup_{P_{n-1} \times_C A} (R_{n-1} \times_C A) \xrightarrow{\sim} (P_n \times_C A) \sqcup_{P_{n-1} \times_C A} P_{n-1}.$$

Under this identification, the composite $P_{n-1} \to P_n \to P_{n+1}$ is the canonical map from the second summand. It is therefore the pushout of

$$P_{n-1} \times_C A \longrightarrow P_n \times_C A$$

along $P_{n-1} \times_C A \to P_{n-1}$. The displayed map is a base change of $P_{n-1} \to P_n$, hence lies in L. It follows that the composite $P_{n-1} \to P_{n+1}$ lies in L. Since $P_{n-1} \to P_n$ lies in L by assumption, right cancellation for the class L now implies that $P_n \to P_{n+1}$ lies in L.

The case where P_n is obtained by making the right leg cartesian is symmetric, with B in place of C. □

Combining the previous two lemmas, we obtain a neat simple proof of the generalized Blakers–Massey theorem.

Wärn's zigzag proof of Theorem 5.16. Consider a pushout square

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & D \end{array}$$

and let L be a modality such that $\Delta_f \square_A \Delta_g$ lies in L. We must show that the gap map $A \to B \times_D C$ lies in L.

Apply the zigzag construction to the span $B \leftarrow \emptyset \to \emptyset$. The first step produces the span

$$B \leftarrow A \xrightarrow{=} A,$$

so that $R_1 \simeq A$. Note that $S = B \sqcup_\emptyset \emptyset = B$, so by Theorem 5.27 we have $R_\infty \xrightarrow{\sim} B \times_D C$. The gap map $A \to B \times_D C$ identifies with

$$R_1 \longrightarrow R_\infty.$$

Since morphisms in L are closed under transfinite composites, it thus remains to show that each map $R_m \to R_{m+1}$ is in L for $m \geq 1$. For m odd, these maps are isomorphisms. For m even, these maps are pushouts of $P_m \to P_{m+1}$, so it remains to show that all these maps are in L. By Lemma 5.29, it suffices to check this for $m = 2$. But by Lemma 5.28, the map $P_2 \to P_3$ is a pushout of $\Delta_f \square_A \Delta_g$, hence lies in L by assumption. □

5.3 Congruences, modalities, and saturated classes

Four closure notions meet in this section. Their definitions, and the smallness hypotheses needed to turn them into categorical constructions, are summarized as follows:

Class	Closure properties	Small-generation consequence
Saturated	Isomorphisms, composition, colimits	Left class of a factorization system.
Acyclic	Saturated and stable under base change	Left class of a modality.
Strongly saturated	Saturated and 2-out-of-3	Kernel of an accessible localization.
Congruence	Strongly saturated and stable under base change	Kernel of a logos morphism.

In particular, neither "acyclic class" nor "congruence" includes a smallness condition. Small generation is stated separately whenever it is needed to produce a modality or a quotient logos.

We will establish three relations among these notions. First, the acyclic class Σ^m generated by a set Σ is of small generation and hence defines a modality. Second, congruences are precisely the acyclic classes closed under diagonals. Third, closing a set under iterated diagonals before taking its acyclic closure gives the generated congruence:

$$\Sigma^c = (\Sigma^\Delta)^m.$$

These results will be our main tool for checking that localizations of topoi are again topoi. As an application, they give a short proof that the category of sheaves on a Grothendieck site is a topos; we spell this out in Section 6.1. This application seems to be a new observation by Marc Hoyois which, to the note-taker's knowledge, does not appear in the literature.

Notation 5.30. We denote by

$$\mathrm{Sat}(T), \qquad \mathrm{SSat}(T), \qquad \mathrm{Acyc}(T), \qquad \mathrm{Mdl}(T), \qquad \text{and} \qquad \mathrm{Cong}(T)$$

the partially ordered sets of saturated classes, strongly saturated classes, acyclic classes, modalities, and congruences in T, with partial order given by inclusion. We regard a modality as its left class, giving inclusions $\mathrm{Mdl}(T) \subseteq \mathrm{Acyc}(T) \subseteq \mathrm{Sat}(T)$, while $\mathrm{Cong}(T) \subseteq \mathrm{Acyc}(T) \cap \mathrm{SSat}(T)$.

Notation 5.31. Given a class of morphisms $\Sigma \subseteq \mathrm{Ar}(T)$, we define the following closures:

- Σ^s: the *saturation* (closure under composition, identities, and colimits).
- Σ^{ss}: the *strong saturation* (closure under composition, colimits, and 2-out-of-3).
- Σ^m: the smallest acyclic class containing Σ.
- Σ^c: the smallest strongly saturated class closed under base change containing Σ.

5.3.1 Generated acyclic classes and modalities

The notation Σ^m refers a priori only to an acyclic class. We now show that if Σ is a set, this acyclic class is of small generation and therefore is the left class of a modality.

Proposition 5.32 ([ABFJ22, Lemma 3.2.13, Proposition 3.2.18, Corollary 3.2.19])**.** *Let T be a topos, let Σ be a set of morphisms, and let $\mathcal{G}$ be a set of generators of T. Let Σ^{bc} denote the set of base changes of morphisms in Σ to objects in $\mathcal{G}$. Then we have*

$$\Sigma^m \quad = \quad (\Sigma^{bc})^s.$$

In particular, Σ^m is a modality of small generation.

Proof. Write $\Sigma' := (\Sigma^{bc})^s$, the saturation of Σ^{bc}. It is clear that $\Sigma' \subseteq \Sigma^m$, so it remains to show that also $\Sigma^m \subseteq \Sigma'$. To this end, consider the following class of morphisms

$$\Sigma'' := \{\, u \in \Sigma' \mid \text{every base change of } u \text{ lies in } \Sigma' \,\}.$$

We have $\Sigma'' \subseteq \Sigma'$, so it remains to show that in fact $\Sigma^m \subseteq \Sigma''$. We will do this by showing that Σ'' is a saturated class closed under base change which contains Σ.

It is clear that Σ'' is closed under base change. We claim that it is also saturated. It is clear that Σ'' is closed under composition and contains all isomorphisms, so it remains to show that it is closed under colimits. Consider a diagram of maps $X_\bullet \to Y_\bullet$ in Σ'', and let $Z \to \operatorname{colim}_i Y_i$ be a map. We need to show that the base change $Z \times_{\operatorname{colim}_i Y_i} \operatorname{colim}_i X_i \to Z$ lies in Σ'. By descent, we may write Z as $\operatorname{colim}_i Z_i$, and hence this base change map may be written as the colimit of the maps $Z_i \times_{Y_i} X_i \to Z_i$. Since each of these base changes lies in Σ' by definition of Σ'', the claim follows from the fact that Σ' is closed under colimits.

It remains to show that Σ'' contains Σ. In other words, given a morphism $u\colon A \to B$ in Σ and an arbitrary map $Z \to B$, we need to show that the base change $Z \times_B A \to Z$ lies in Σ'. To this end, let $\mathcal{E} \subseteq T_{/B}$ be the full subcategory spanned by those $Z \to B$ for which this condition holds. By definition of Σ', we have $\Sigma^{bc} \subseteq \Sigma'$, and in particular $\mathcal{E}$ contains all maps $Z \to B$ with $Z \in \mathcal{G}$. Since Σ' is closed under colimits, it follows from descent that $\mathcal{E} \subseteq T_{/B}$ is closed under colimits. But since $\mathcal{G}$ is a set of generators, it follows that $\mathcal{E} = T_{/B}$, showing the claim. □

Remark 5.33. On classes of small generation, the assignments $\Sigma \mapsto \Sigma^m$ and $\Sigma \mapsto \Sigma^c$ define the respective left adjoints to the inclusions of modalities and congruences into saturated classes. Without a smallness restriction, the same closure operations are left adjoint to the inclusions $\mathrm{Acyc}(T) \hookrightarrow \mathrm{Sat}(T)$ and $\mathrm{Cong}(T) \hookrightarrow \mathrm{Sat}(T)$ at the level of partially ordered classes.

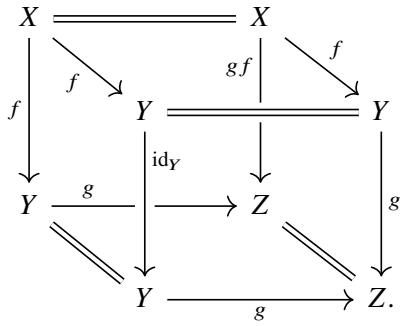
5.3.2 Characterizations of congruences

The definition of Σ^c involves closure under 2-out-of-3, an operation over which we have little explicit control. The following characterizations replace it by finite-limit or diagonal closure.

Lemma 5.34. *Let $\mathcal{E}$ be a category with finite limits and let K be a class of morphisms in $\mathcal{E}$ which contains all identities and is closed under finite limits. Then K satisfies the left cancellation property. Dually, a class containing all identities and closed under finite colimits satisfies the right cancellation property.*

Proof. It suffices to prove the first claim, as the second one is dual. Given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, the following commutative square in $\mathrm{Ar}(T)$ exhibits f as a limit in $\mathrm{Ar}(T)$ of the

morphisms id_Y, gf and g:



Since K contains identities and is closed under finite limits, if $g, gf \in K$, then $f \in K$. □

Proposition 5.35. *The following conditions are equivalent for a class of morphisms K:*

(1) *K is a congruence;*

(2) *K is strongly saturated and stable under base change;*

(3) *K is strongly saturated and closed under finite limits;*

(4) *K is saturated and closed under finite limits;*

(5) *K is an acyclic class and is closed under diagonals: if $f\colon X \to Y$ is in K, then so is $\Delta_f\colon X \to X \times_Y X$.*

Proof. The equivalence between (1) and (2) is the definition. A congruence is closed under finite limits in $\mathrm{Ar}(T)$, see [ABFJ22, Proposition 4.2.3], which gives (2)⇒(3). Clearly (3) implies (4). Conversely, finite-limit closure implies stability under base change, since a base-change square is a pullback in $\mathrm{Ar}(T)$. Moreover, Lemma 5.34 gives left cancellation. Saturation already gives composition and right cancellation, so K satisfies 2-out-of-3. This proves (4)⇒(2).

Condition (2) implies (5): the projection $X \times_Y X \to X$ is a base change of f, and its section Δ_f belongs to K by 2-out-of-3. Conversely, assume (5). Since K is saturated, it is closed under composition and has right cancellation. It remains to check left cancellation. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g, gf \in K$, factor f as

$$X \xrightarrow{(\mathrm{id}, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

The second map is a base change of $gf \in K$, hence in K. The first is a base change of Δ_g, which is in K by assumption. Thus $f \in K$. □

5.3.3 Décalage and generated congruences

We now extract the largest congruence contained in an acyclic class. Iterating this construction will turn diagonal closure into the explicit formula for a generated congruence.

Definition 5.36 ([ABFJ24, Section 2.2.7])**.** Let L be an acyclic class in T. We define the *décalage* of L by

$$D(L) := \{\, u \in L \mid \Delta_u \in L \,\} \subseteq L.$$

We inductively define $D^n(L)$ by $D^0(L) := L$ and $D^{n+1}(L) := D(D^n(L))$. Finally, we set $D^\infty(L) := \cap_n D^n(L)$.

Proposition 5.37 ([ABFJ24, Section 2.2.7])**.** *If L is an acyclic class, then $D(L)$ is again an acyclic class. If L is of small generation, then so is $D(L)$; in this case both classes are left classes of modalities.*

Proof. Stability under base change and the inclusion of all isomorphisms are immediate. For composition, let $A \xrightarrow{f} B \xrightarrow{g} C$ lie in $D(L)$. Then $gf \in L$, and Δ_{gf} factors as

$$A \xrightarrow{\Delta_f} A \times_B A \longrightarrow A \times_C A,$$

where the second map is a base change of Δ_g. Since $\Delta_f, \Delta_g \in L$ and L is an acyclic class, $\Delta_{gf} \in L$.

For colimits, let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a diagram in $\mathrm{Ar}(T)$ with every $f_i \in D(L)$, and write $f \colon X \to Y$ for its colimit. We already know $f \in L$. The acyclic-descent argument of [ABFJ24, Section 2.2.7] says that colimits of L-cartesian transformations remain L-cartesian. We spell out its application here, since it also fixes the variance of the maps involved.

For a morphism $i \to j$ in the indexing category, the gap map of the naturality square is

$$X_i \longrightarrow Y_i \times_{Y_j} X_j.$$

It factors as

$$X_i \longrightarrow X_i \times_{Y_j} X_j \longrightarrow Y_i \times_{Y_j} X_j.$$

The first map is a base change of Δ_{f_j}, and the second is a base change of f_i. Both lie in L, so the transformation $f_\bullet$ is L-cartesian. Acyclic descent now shows that for every i the square

$$\begin{array}{ccc} X_i & \longrightarrow & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ Y_i & \longrightarrow & Y \end{array}$$

is L-cartesian. Equivalently, the gap map $h_i \colon X_i \to Y_i \times_Y X$ lies in L.

By universality of colimits, the diagonal Δ_f is the colimit of the maps $X_i \to X_i \times_Y X$. Each of these factors as

$$X_i \xrightarrow{\Delta_{f_i}} X_i \times_{Y_i} X_i \longrightarrow X_i \times_Y X,$$

where the second map is a base change of h_i. Thus both maps lie in L. Closure under colimits gives $\Delta_f \in L$, proving that $D(L)$ is saturated. The small-generation assertion is the accessibility part of the cited décalage result. □

Corollary 5.38. *For an acyclic class L in T, the class $D^\infty(L)$ is a congruence. Moreover, it is the largest congruence contained in L. Thus $L \mapsto D^\infty(L)$ defines a right adjoint* $\mathrm{Acyc}(T) \to \mathrm{Cong}(T)$ *to the inclusion* $\mathrm{Cong}(T) \hookrightarrow \mathrm{Acyc}(T)$.

Proof. By Proposition 5.37, each $D^n(L)$ is an acyclic class. Intersections of acyclic classes are acyclic, so $D^\infty(L) = \bigcap_{n \geq 0} D^n(L)$ is acyclic. It is closed under diagonals by construction and hence is a congruence by Proposition 5.35.

If $K \subseteq L$ is any congruence, then K is closed under diagonals. Inductively, every $f \in K$ belongs to $D^n(L)$ for all n, so $K \subseteq D^\infty(L)$. This proves maximality and the adjunction. □

We now state and prove the main result of this section. Given a class of morphisms Σ, write Σ^{Δ} for the closure of Σ under diagonals $f \mapsto \Delta_f$. Thus Σ^{Δ} consists of the maps in Σ and all their iterated diagonals.

Theorem 5.39 (Anel–Biedermann–Finster–Joyal, [ABFJ22, Proposition 4.2.12])**.** *Let T be a topos and let $\Sigma \subseteq \mathrm{Ar}(T)$ be a set of maps. Then the congruence Σ^c is given by*

$$\Sigma^c = (\Sigma^{\Delta})^m,$$

the acyclic class generated by the iterated diagonals of Σ. This class is of small generation and hence is the left class of a modality.

Proof. Set $L := (\Sigma^{\Delta})^m$. Since Σ^{Δ} is again a set, Proposition 5.32 shows that L is of small generation. Let $K := D^{\infty}(L)$, the largest congruence contained in L.

For $f \in \Sigma$, every iterated diagonal $\Delta^n f$ belongs to $\Sigma^{\Delta} \subseteq L$. Hence $f \in D^n(L)$ for every n, and therefore $f \in K$. Thus K is a congruence containing Σ, which gives $\Sigma^c \subseteq K$. Conversely, Σ^c is an acyclic class containing Σ^{Δ}, since it is closed under diagonals by Proposition 5.35. It therefore contains the acyclic closure L. We obtain

$$\Sigma^c \subseteq K \subseteq L \subseteq \Sigma^c,$$

so all three classes agree. □

Remark 5.40. The same formula holds for an arbitrary class Σ, see [ABFJ22, Proposition 4.2.12]. One writes Σ as the union of a filtered system of sets closed under taking the diagonals that have already appeared and then uses that both acyclic and congruence closure preserve these filtered unions. The set case above is the form needed in these notes, and it has the additional advantage that $(\Sigma^{\Delta})^m$ automatically defines a modality.

Warning 5.41. The formula $\Sigma^m = (\Sigma^{bc})^s$ for the acyclic class generated by a set Σ does *not* have the naive congruence analogue: we generally have $\Sigma^c \neq (\Sigma^{bc})^{ss}$. Indeed, the right-hand side generally has no reason to be closed under base change.

Remark 5.42. For a set Σ, Theorem 5.39 and Proposition 5.32 show that the congruence $\Sigma^c = (\Sigma^{\Delta})^m$ is of small generation. Thus the quotient by the generated congruence exists as a quotient logos.

Corollary 5.43. *If $\Sigma \subseteq \mathrm{Ar}(T)$ consists of monomorphisms, then $\Sigma^c = \Sigma^m$.*

Proof. If f is a monomorphism, Δ_f is an isomorphism, hence automatically contained in Σ^m. The class version of the ABFJ formula from Remark 5.40 therefore gives $\Sigma^c = (\Sigma^{\Delta})^m = \Sigma^m$. □

5.4 Open and closed immersions

An open subset of a topological space determines both an open subtopos and a complementary closed subtopos, and the ambient topos can be reconstructed by gluing these two pieces. We now develop the corresponding picture for arbitrary topoi. We first characterize open immersions as slices over (-1)-truncated objects, then construct their closed complements using the theory of congruences, and finally describe the resulting recollement.

Definition 5.44. A morphism $\mathcal{U} \to \mathcal{X}$ is an *open immersion* if it is both étale and a monomorphism. A morphism in Logos is an *open localization* if its corresponding morphism in Topos is an open immersion.

Lemma 5.45. *Let $j_* \colon \mathcal{X}_{/U} \to \mathcal{X}$ be an étale morphism of topoi, $U \in \mathcal{X}$. Then j_* is an open immersion if and only if U is (-1)-truncated.*

Proof. We saw in Proposition 4.38 that the base change of j_* along itself is given by the functor $(\mathcal{X}_{/U})_{/U\times U} \to \mathcal{X}_{/U}$. This is an equivalence if and only if the object $(U \times U) \in \mathcal{X}_{/U}$ is terminal, i.e. if the projection map $\mathrm{pr}_1 \colon U \times U \to U$ is an isomorphism. This in turn is equivalent to the diagonal $\Delta \colon U \to U \times U$ being an isomorphism, which by definition means that U is (-1)-truncated. □

If $U \hookrightarrow X$ is an open embedding of topological spaces, then by definition its complement $Z := X \setminus U$ is a closed subspace. Its toposic analogue is as follows:

Definition 5.46. Let $j_* \colon \mathcal{U} \hookrightarrow \mathcal{X}$ be an open immersion of topoi, and let $j^* \colon \mathcal{X} \to \mathcal{U}$ be its left adjoint. We define the *closed subtopos complementary to $\mathcal{U}$* as

$$\mathcal{X}/\mathcal{U} \quad := \quad \{\, X \in \mathcal{X} \mid j^*(X) = * \,\} \quad \subseteq \quad \mathcal{X}.$$

More generally, we say that a morphism of topoi $i \colon \mathcal{Z} \to \mathcal{X}$ is a *closed immersion* if there exists a (-1)-truncated object $U \in \mathcal{X}$ such that $i_* \colon \mathcal{Z} \to \mathcal{X}$ induces an equivalence $\mathcal{Z} \xrightarrow{\sim} \mathcal{X}/\mathcal{U}$.

Lemma 5.47. *Given an open immersion $j \colon \mathcal{U} \hookrightarrow \mathcal{X}$, the closed complement $\mathcal{Z} := \mathcal{X}/\mathcal{U}$ is a topos, and the inclusion $i \colon \mathcal{Z} \hookrightarrow \mathcal{X}$ is a morphism of topoi.*

Proof. By Lemma 5.45, we may identify $\mathcal{U}$ with the slice $\mathcal{X}_{/U}$ for some (-1)-truncated object U of $\mathcal{X}$. Given $X \in \mathcal{X}$, we then have $X \in \mathcal{Z}$ if and only if the projection map $X \times U \to U$ is an isomorphism. Equivalently, given an object $Y \in \mathcal{X}$, every morphism $Y \to U$ admits a unique lift $Y \to U \times X$. We conclude that the subcategory $\mathcal{Z} = \mathcal{X}/\mathcal{U}$ consists precisely of those objects that are local with respect to the following class of morphisms:

$$S_U := \{\, \emptyset \to V \mid \text{there exists a map } V \to U \,\}.$$

Choose a small set of generators $\mathcal{G}$ of $\mathcal{X}$. The objects $g \colon G \to U$, with $G \in \mathcal{G}$ and g ranging through all morphisms to U, form a small set of generators of the slice $\mathcal{X}_{/U}$. It therefore suffices to impose locality with respect to the corresponding set

$$S_U^0 := \{\, \emptyset \to G \mid G \in \mathcal{G},\ g \colon G \to U \,\},$$

Indeed, every object of $\mathcal{X}_{/U}$ is a colimit of these generators, and mapping out of this colimit identifies locality with respect to S_U^0 with locality with respect to the entire class S_U. Consequently, Proposition A.15 gives a left adjoint $\mathcal{X} \to \mathcal{Z}$ to the inclusion.

It remains to show that this left adjoint preserves finite limits, or equivalently that the strong saturation of S_U^0, which agrees with the strong saturation of S_U, is a congruence. The class S_U is closed under base change, so $(S_U)^s = (S_U)^m$ by Proposition 5.32. Moreover, its morphisms are monomorphisms, whence $(S_U)^m = (S_U)^c$ by Corollary 5.43. We deduce that the localization $\mathcal{X} \to \mathcal{Z}$ is left exact, exhibiting $\mathcal{Z}$ as a topos and its inclusion into $\mathcal{X}$ as a morphism of topoi. □

If $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ is an open immersion with closed complement $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$, we may recover the topos $\mathcal{X}$ from the topoi $\mathcal{U}$ and $\mathcal{Z}$ together with the composite $i^* j_*\colon \mathcal{U} \to \mathcal{Z}$.

Construction 5.48. Let $\mathcal{U}$ and $\mathcal{Z}$ be categories with finite limits, and let $F\colon \mathcal{U} \to \mathcal{Z}$ preserve terminal objects. We define the category $\mathcal{X}$ as the comma category of F:

$$\mathcal{X} \quad := \quad \mathcal{U} \times_{\mathcal{Z},\, F} \mathrm{Ar}(\mathcal{Z}),$$

where the functor $\mathrm{Ar}(\mathcal{Z}) \to \mathcal{Z}$ is evaluation at the target. In other words, the objects of $\mathcal{X}$ are triples (U, Z, φ) where $U \in \mathcal{U}$, $Z \in \mathcal{Z}$ and $\varphi\colon Z \to F(U)$.

Note that the two forgetful functors $j^*\colon \mathcal{X} \to \mathcal{U}$ and $i^*\colon \mathcal{X} \to \mathcal{Z}$ admit canonical right adjoints

$$j_*\colon \mathcal{U} \hookrightarrow \mathcal{X}, U \mapsto (U, F(U), \mathrm{id}_{F(U)}), \qquad \text{and} \qquad i_*\colon \mathcal{Z} \hookrightarrow \mathcal{X}, Z \mapsto (*, Z, Z \to F(*) \xrightarrow{\sim} *).$$

Lemma 5.49. *Consider topoi $\mathcal{U}$ and $\mathcal{Z}$ and an accessible left exact functor $F\colon \mathcal{U} \to \mathcal{Z}$.*

(1) *The associated recollement $\mathcal{X}$ is a topos,*

(2) *The functor $j_*\colon \mathcal{U} \to \mathcal{X}$ is an open immersion of topoi,*

(3) *The functor $i_*\colon \mathcal{Z} \to \mathcal{X}$ is a closed immersion of topoi.*

Proof. The comma category $\mathcal{X}$ is accessible, and the results of Proposition B.1 show that it admits all small colimits and finite limits. Hence $\mathcal{X}$ is presentable. Moreover, the forgetful functor

$$(j^*, i^*)\colon \mathcal{X} \longrightarrow \mathcal{U} \times \mathcal{Z}$$

is cocontinuous, left exact, and conservative. The Giraud conditions for $\mathcal{X}$ may therefore be checked after applying this functor, where they follow from the corresponding conditions in $\mathcal{U}$ and $\mathcal{Z}$. Thus $\mathcal{X}$ is a topos.

Let

$$V := (*, \emptyset, \emptyset \to F(*)) \in \mathcal{X}.$$

This object is (-1)-truncated. Base change along $V \to *$ identifies $\mathcal{X}_{/V}$ with $\mathcal{U}$: a morphism $(U, Z, \varphi) \to V$ forces $Z \xrightarrow{\sim} \emptyset$, and is then determined by U. Under this equivalence, the inverse-image functor of the étale morphism $\mathcal{X}_{/V} \to \mathcal{X}$ identifies with j^*, and its right adjoint identifies with j_*. Thus j_* is an open immersion by Lemma 5.45.

Finally, an object (U, Z, φ) belongs to the closed complement of this open immersion precisely when

$$j^*(U, Z, \varphi) = U \xrightarrow{\sim} *.$$

Since $F(*) \xrightarrow{\sim} *$, such an object is uniquely of the form $(*, Z, Z \to *) = i_*(Z)$. Hence i_* identifies $\mathcal{Z}$ with the closed complement of $\mathcal{U}$, and is therefore a closed immersion. □

5.5 Structure theory for topoi

The epi–mono factorization of individual morphisms induces a parallel structure theory for acyclic classes and morphisms of logoi. We first decompose an acyclic class into its monogenic and epigenic

parts. Applied to the kernel of a morphism of logoi, this produces a quotient triple factorization into a monogenic quotient, an epigenic quotient, and a conservative morphism. We then compare monogenic congruences with hypercomplete congruences and refine the conservative factor through the image logos.

More explicitly, a morphism of logoi $\varphi\colon T \to S$ admits a factorization

$$T \xrightarrow{\varphi^{\mathrm{mono}}} T/K^{\mathrm{mono}} \xrightarrow{\varphi^{\mathrm{epi}}} T/K \xrightarrow{\varphi^{\mathrm{cons}}} S.$$

The morphism φ^{cons} in fact factors further as $T/K \to \langle \varphi(T) \rangle \hookrightarrow S$, where the second morphism is fully faithful.

5.5.1 Monogenic and epigenic acyclic classes

Definition 5.50. For an acyclic class L, we define

$$L^{\mathrm{epi}} := L \cap \mathrm{EffEpi}, \qquad \text{and} \qquad L^{\mathrm{mono}} := (L \cap \mathrm{Mono})^m = (L \cap \mathrm{Mono})^c,$$

where the last equality holds by Corollary 5.43. We say L is *monogenic* if $L = L^{\mathrm{mono}}$ (i.e. generated by its monomorphisms), and *epigenic* if $L = L^{\mathrm{epi}}$ (i.e. it is contained in the effective epimorphisms).

Remark 5.51. Monogenic/epigenic congruences are also called *topological/cotopological* in the literature. We will avoid that terminology, as it is less descriptive.

Both L^{epi} and L^{mono} are again acyclic classes. The following lemma explains their relation to epi–mono factorizations and will also determine when they define modalities.

Lemma 5.52. *Let L be an acyclic class, and let $f\colon A \to B$ be a morphism with epi–mono factorization*

$$A \overset{\mathrm{coim}(f)}{\twoheadrightarrow} \mathrm{Im}(f) \xhookrightarrow{\mathrm{im}(f)} B.$$

Then f is contained in L if and only if both $\mathrm{im}(f)$ *and* $\mathrm{coim}(f)$ *are contained in L.*

Proof. If both $\mathrm{coim}(f)$ and $\mathrm{im}(f)$ lie in L, then so does their composite f. Conversely, assume $f \in L$. Since L is local by Lemma 5.4 and $\mathrm{coim}(f)$ is an effective epimorphism, we may test whether $\mathrm{coim}(f)$ lies in L after base change along itself. The resulting projection $A \times_{\mathrm{Im}(f)} A \to A$ is isomorphic to $A \times_B A \to A$, because $\mathrm{Im}(f) \to B$ is a monomorphism. This projection is a base change of f and hence lies in L. Thus $\mathrm{coim}(f) \in L$, and right cancellation applied to $f = \mathrm{im}(f)\,\mathrm{coim}(f)$ gives $\mathrm{im}(f) \in L$. □

Corollary 5.53. *Given a modality L, also L^{epi} is a modality.*

Proof. Let $R := L^{\perp}$ and define $R^{\mathrm{epi}} := (L^{\mathrm{epi}})^{\perp}$. We need to show that the pair $(L^{\mathrm{epi}}, R^{\mathrm{epi}})$ is a factorization system. Orthogonality holds by construction, so it remains to check that any morphism $f\colon A \to B$ admits an $(L^{\mathrm{epi}}, R^{\mathrm{epi}})$-factorization. Since $L^{\mathrm{epi}} \subset L$, we get $R \subseteq R^{\mathrm{epi}}$. Similarly, from $L^{\mathrm{epi}} \subseteq \mathrm{EffEpi}$ it follows that $\mathrm{Mono} \subseteq R^{\mathrm{epi}}$. Consider the (L, R)-factorization $A \xrightarrow{l} C \xrightarrow{r} B$ of f, and consider the epi-mono factorization $A \overset{\mathrm{coim}(l)}{\twoheadrightarrow} \mathrm{Im}(l) \xhookrightarrow{\mathrm{im}(l)} C$ of l. By Lemma 5.52 we have that $\mathrm{coim}(l) \in L \cap \mathrm{EffEpi} = L^{\mathrm{epi}}$, while $r \circ \mathrm{im}(l) \in R \circ \mathrm{Mono} \subseteq R^{\mathrm{epi}}$ because the right class R^{epi} is closed under composition. □

The corollary uses the factorization system and is therefore genuinely a statement about modalities. For L^{mono}, small generation will provide the required factorization system.

Notation 5.54. Given a morphism $f\colon A \to B$ in T, we denote by $A \overset{\mathrm{coim}(f)}{\twoheadrightarrow} \mathrm{Im}(f) \overset{\mathrm{im}(f)}{\hookrightarrow} B$ its epi-mono factorization. If Σ is a class of morphisms in T, we define

$$\mathrm{im}(\Sigma) := \{\mathrm{im}(f) \mid f \in \Sigma\} \qquad \text{and} \qquad \mathrm{coim}(\Sigma) := \{\mathrm{coim}(f) \mid f \in \Sigma\}.$$

Remark 5.55. If L is an acyclic class, it follows from Lemma 5.52 that $\mathrm{im}(L) = L \cap \mathrm{Mono}$ and $\mathrm{coim}(L) = L \cap \mathrm{EffEpi}$.

Lemma 5.56 ([ABFJ24, Proposition 4.1.14])**.** *For a set of morphisms Σ in T, we have*

$$(\Sigma^m)^{\mathrm{mono}} = (\mathrm{im}(\Sigma))^m \qquad \textit{and} \qquad (\Sigma^c)^{\mathrm{mono}} = (\mathrm{im}(\Sigma^\Delta))^m = (\mathrm{im}(\Sigma^\Delta))^c.$$

In particular, if L is an acyclic class of small generation, then L^{mono} is an acyclic class of small generation and hence defines a modality. If L is a congruence of small generation, then so is L^{mono}.

Proof. The second relation follows from the first, since $\Sigma^c = (\Sigma^\Delta)^m$ by Theorem 5.39. The last two claims immediately follow by writing $L = \Sigma^m$ and noting that $\mathrm{im}(\Sigma)$ and $\mathrm{im}(\Sigma^\Delta)$ are small. We need to show the first relation.

Let $L := \Sigma^m$. By Lemma 5.52, we have $\mathrm{im}(\Sigma) \subseteq L^{\mathrm{mono}}$. Since L^{mono} is an acyclic class, it follows that $\mathrm{im}(\Sigma)^m \subseteq L^{\mathrm{mono}}$. For the converse, consider the class L' of morphisms $f\colon A \to B$ such that $\mathrm{im}(f) \in \mathrm{im}(\Sigma)^m$. We are done if we can show that $L \subseteq L'$, since then $L \cap \mathrm{Mono} = \mathrm{im}(L) \subseteq \mathrm{im}(\Sigma)^m$ and thus also $L^{\mathrm{mono}} = (L \cap \mathrm{Mono})^m \subseteq \mathrm{im}(\Sigma)^m$. By construction we have $\Sigma \subseteq L'$, so it remains to show that L' is an acyclic class.

To this end, note that $\mathrm{im}(\Sigma)^m = \mathrm{im}(\Sigma)^c$ by Corollary 5.43, so in particular $\mathrm{im}(\Sigma)^m$ is a congruence of small generation. Consider now the quotient map $T \to T/\mathrm{im}(\Sigma)^m$. This localization preserves effective epimorphisms because it preserves colimits, and it preserves monomorphisms because it preserves finite limits. It therefore preserves epi–mono factorizations. It follows that $f \in L'$ if and only if the morphism f is sent to an effective epimorphism under this localization functor. Since the effective epimorphisms in $T/\mathrm{im}(\Sigma)^c$ form an acyclic class, the same follows for L', finishing the proof. □

Corollary 5.57. *Let Σ be a small class of monomorphisms in T. Then the congruence Σ^c generated by Σ is monogenic.*

Proof. For a monomorphism, every positive iterated diagonal is an isomorphism. Thus $\mathrm{im}(\Sigma^\Delta)$ consists of the maps in Σ together with isomorphisms, which do not change the acyclic class they generate. The claim is therefore immediate from the lemma. □

Remark 5.58. By definition, a monogenic congruence K is completely determined by the class of monomorphisms $K \cap \mathrm{Mono}$. The classes of monomorphisms in T arising this way are precisely the *Grothendieck topologies on T*, see Proposition 6.9.

Corollary 5.59. *For every acyclic class L, we have*

$$L \;=\; L^{\mathrm{epi}} \vee L^{\mathrm{mono}},$$

i.e. L is the smallest acyclic class containing both L^{epi} and L^{mono}.

Proof. It is clear that L contains both L^{epi} and L^{mono}. Conversely, let L' be an acyclic class containing both parts. For every $f \in L$, Lemma 5.52 places $\mathrm{coim}(f)$ in L^{epi} and $\mathrm{im}(f)$ in L^{mono}. Hence $f = \mathrm{im}(f)\,\mathrm{coim}(f)$ lies in L', proving $L \subseteq L'$. □

Lemma 5.60. *If L is a monogenic acyclic class, then L is a congruence. In particular, L^{mono} is a congruence for every acyclic class L.*

Proof. By assumption we have $L = L^{\mathrm{mono}} = (L \cap \mathrm{Mono})^m$. Since $L \cap \mathrm{Mono}$ consists of monomorphisms, it follows from Corollary 5.43 that $(L \cap \mathrm{Mono})^m$ agrees with $(L \cap \mathrm{Mono})^c$, hence is a congruence. □

Lemma 5.61. *A congruence K is epigenic if and only if it is contained in the class of ∞-connected maps.*

Proof. If K is contained in Conn_∞, then it is in particular contained in EffEpi, so that K is epigenic. Conversely, assume that $K \subseteq \mathrm{EffEpi}$. For any morphism f in K, the iterated diagonals Δ_f^n of f also lie in K, hence are effective epimorphisms by assumption. By Theorem 3.22 it follows inductively that f is n-connected for all $n \geq 0$, hence ∞-connected. □

5.5.2 The quotient triple factorization

We now discuss the promised triple factorization of morphisms of logoi. The terminology and perspective is due to [ABFJ25].

Definition 5.62. Let $\varphi\colon T \to S$ be a morphism of logoi.

(1) We call φ a *monogenic quotient* if it is a quotient map (Definition 4.19) whose kernel is a monogenic congruence.

(2) We call φ an *epigenic quotient* if it is a quotient map whose kernel is an epigenic congruence.

(3) We call φ *conservative* if it only inverts isomorphisms.

(4) We call φ *weakly conservative* if it only inverts effective epimorphisms.

Notation 5.63. Let $\varphi\colon T \to S$ be a morphism of logoi with kernel $K := \ker(\varphi)$. Then we have an inclusion $K^{\mathrm{mono}} \subseteq K$, giving rise to three morphisms of logoi as follows:

$$T \xrightarrow{\varphi^{\mathrm{mono}}} T/K^{\mathrm{mono}} \xrightarrow{\varphi^{\mathrm{epi}}} T/K \xrightarrow{\varphi^{\mathrm{cons}}} S.$$

We refer to this as the *quotient triple factorization* of φ. We further write

$$\varphi^{\mathrm{quot}} := \varphi^{\mathrm{epi}} \circ \varphi^{\mathrm{mono}} \qquad \text{and} \qquad \varphi^{\mathrm{wcons}} := \varphi^{\mathrm{cons}} \circ \varphi^{\mathrm{epi}}.$$

Proposition 5.64. *Let $\varphi\colon T \to S$ be a morphism of logoi.*

(1) *The morphisms φ^{mono}, φ^{epi}, φ^{cons}, φ^{quot} and φ^{wcons} are, respectively, a monogenic quotient, an epigenic quotient, conservative, a quotient map, and weakly conservative;*

(2) *The decomposition* $\varphi = \varphi^{\mathrm{cons}} \circ \varphi^{\mathrm{quot}}$ *is the unique decomposition of* φ *into a quotient map followed by a conservative map;*

(3) *The decomposition* $\varphi = \varphi^{\mathrm{wcons}} \circ \varphi^{\mathrm{mono}}$ *is the unique decomposition of* φ *into a monogenic quotient followed by a weakly conservative map;*

(4) *The decomposition* $\varphi = \varphi^{\mathrm{cons}} \circ \varphi^{\mathrm{epi}} \circ \varphi^{\mathrm{mono}}$ *is the unique decomposition of* φ *into a monogenic quotient followed by an epigenic quotient followed by a conservative map.*

Proof. (1) It is clear that φ^{mono} is a monogenic quotient, that φ^{quot} is a quotient map, and that φ^{cons} is conservative. Let $q\colon T \to T/K^{\mathrm{mono}}$ denote the first quotient. The kernel of the induced quotient $\varphi^{\mathrm{epi}}\colon T/K^{\mathrm{mono}} \to T/K$ is the congruence generated by the image $q(K)$. By [ABFJ24, Proposition 4.2.5], this image congruence is contained in the ∞-connected maps. It is therefore epigenic by Lemma 5.61. Consequently, φ^{wcons} is weakly conservative.

(2) Given a decomposition $T \xrightarrow{\psi} T' \to S$ into a quotient map and a conservative map, we see that $\ker(\psi) = \ker(\varphi) = K$. Since ψ is a quotient map, it follows that $T' \xrightarrow{\sim} T/K$.

(3) Given a decomposition $T \xrightarrow{\psi} T' \to S$ into a monogenic quotient map and a weakly conservative map, the fact that the second map only inverts effective epimorphisms implies

$$\ker(\psi) \cap \mathrm{Mono} = \ker(\varphi) \cap \mathrm{Mono} = K \cap \mathrm{Mono}\,.$$

Since ψ is a monogenic quotient, we have $\ker(\psi) = \ker(\psi)^{\mathrm{mono}} = K^{\mathrm{mono}}$, and thus $T' \xrightarrow{\sim} T/K^{\mathrm{mono}}$.

(4) This follows by first factoring φ as in (2) and then applying (3) to the quotient map $T \to T/K$. These identifications also determine the comparison maps between any two such factorizations, proving uniqueness. □

Remark 5.65. In [ABFJ25, Remark 2.1.34], the authors make the following analogy with ring theory. Given a morphism $\varphi\colon A \to B$ of commutative rings with kernel I, its usual image factorization is

$$A \longrightarrow A/I \hookrightarrow B.$$

The injectivity of the second map is analogous to the conservativity of $\varphi^{\mathrm{cons}}\colon T/K \to S$. Let $W \subseteq A$ be the multiplicative subset of elements whose images in A/I are units. Then the quotient map $A \to A/I$ factors further as

$$A \longrightarrow A[W^{-1}] \longrightarrow A/I.$$

Altogether, this gives the triple factorization

$$A \longrightarrow A[W^{-1}] \longrightarrow A/I \hookrightarrow B,$$

which the authors compare with the factorization of $T \to S$ through T/K^{mono} and T/K.

5.5.3 Hypercomplete congruences

We now discuss the hypercompletion of a congruence.

Definition 5.66. Given a congruence K of small generation, we define its *hypercompletion* as the kernel of the composite:

$$K^{\mathrm{hyp}} \quad := \quad \ker\big(T \to T/K \to (T/K)^{\mathrm{hyp}}\big),$$

where $(-)^{\mathrm{hyp}}$ denotes the hypercompletion of a topos (localization at ∞-connected maps). Note that we always have $K \subseteq K^{\mathrm{hyp}}$. We say a congruence K is *hypercomplete* if $K = K^{\mathrm{hyp}}$.

Lemma 5.67. *Let K be a congruence of small generation. Hypercompletion does not affect its monogenic part:*

$$K^{\mathrm{hyp}} \cap \mathrm{Mono} \;=\; K \cap \mathrm{Mono}\,.$$

Furthermore, the hypercompletion only depends on the monogenic part of a congruence: we have

$$(K^{\mathrm{mono}})^{\mathrm{hyp}} = K^{\mathrm{hyp}}.$$

Proof. For the first part, consider the sequence of localizations $T \to T/K \to (T/K)^{\mathrm{hyp}}$. Let f be a monomorphism in T contained in K^{hyp}; we need to show that f is in fact contained in K. Since $T \to T/K$ is left exact, the image of f in T/K is a monomorphism that becomes an equivalence in the hypercompletion $(T/K)^{\mathrm{hyp}}$. Thus, in T/K, the map f is ∞-connected. But a map that is both a monomorphism and ∞-connected is an equivalence. Therefore, f is already inverted in T/K, so $f \in K$.

For the second part, we always have $(K^{\mathrm{mono}})^{\mathrm{hyp}} \subseteq K^{\mathrm{hyp}}$. For the converse, we may consider the monogenic-epigenic factorization of the quotient $T \to T/K$:

$$T \to T/K^{\mathrm{mono}} \to T/K.$$

The second map is an epigenic quotient by Proposition 5.64, hence its kernel is contained in the ∞-connected maps. An epigenic quotient induces an equivalence on hypercompletions, see [ABFJ25, Lemma 2.1.33]. We therefore obtain

$$(T/K^{\mathrm{mono}})^{\mathrm{hyp}} \xrightarrow{\sim} (T/K)^{\mathrm{hyp}}.$$

It follows that $(K^{\mathrm{mono}})^{\mathrm{hyp}} = K^{\mathrm{hyp}}$, as desired. □

Let $\mathrm{MonoCong}(T)$ and $\mathrm{HypCong}(T)$ denote the posets of monogenic and hypercomplete congruences of small generation, respectively. It follows from the lemma that hypercompletion defines an equivalence

$$(-)^{\mathrm{hyp}} \colon\ \mathrm{MonoCong}(T) \xrightarrow{\sim} \mathrm{HypCong}(T),$$

with inverse $K \mapsto K^{\mathrm{mono}}$. The inclusion $\mathrm{MonoCong}(T) \hookrightarrow \mathrm{Cong}(T)$ has right adjoint $K \mapsto K^{\mathrm{mono}}$, while the inclusion of the epigenic congruences has right adjoint $K \mapsto K^{\mathrm{epi}}$. Under the preceding equivalence, hypercompletion is the fully faithful further right adjoint to $K \mapsto K^{\mathrm{mono}}$, as proved in [ABFJ25, Theorem 2.1.26]. Thus the three constructions record, respectively, the largest monogenic congruence below K, the largest epigenic congruence below K, and the largest hypercomplete congruence with the same monomorphisms as K.

5.5.4 Images

In Remark 5.65, we compared the factorization $T \to T/K \to S$ of a logos morphism $\varphi\colon T \to S$ with the image factorization of a ring morphism. The analogy is not yet exact, since $T/K \to S$ is merely conservative and need not be a monomorphism. We now factor it further through an actual sublogos $\langle \varphi(T) \rangle \subseteq S$.

Construction 5.68. Let $\varphi\colon T \to S$ be a morphism of logoi. We define its *image* $\langle \varphi(T) \rangle$ as the smallest full subcategory of S that contains the objects $\varphi(X)$ for all $X \in T$ and is closed under colimits and finite limits.

Proposition 5.69. *Let $\varphi\colon T \to S$ be a morphism of logoi. Then $\langle \varphi(T) \rangle$ is a logos, and the inclusion $\langle \varphi(T) \rangle \hookrightarrow S$ is a morphism of logoi. In particular, φ admits a factorization in* Logos *as*

$$T \longrightarrow \langle \varphi(T) \rangle \hookrightarrow S,$$

where the second map is a monomorphism in Logos*.*

Proof. Presentability is the substantive point and is proved in [Lur09, Proposition 6.3.6.2]. By construction, the inclusion $\langle \varphi(T) \rangle \hookrightarrow S$ preserves all colimits and finite limits. Consequently, colimits in the image are universal, coproducts are disjoint, and groupoid objects are effective because the corresponding statements hold in S and the inclusion preserves and reflects the diagrams involved. Thus the Giraud conditions of Theorem 2.42 hold in $\langle \varphi(T) \rangle$.

The inclusion is therefore a morphism of logoi. Since it is fully faithful, it is a monomorphism in Logos. □

Remark 5.70. Let $\varphi\colon T \to S$ be a morphism of logoi with kernel $K = \ker(\varphi)$. Combining the quotient triple factorization with the image factorization gives

$$T \xrightarrow{(1)} T/K^{\mathrm{mono}} \xrightarrow{(2)} T/K \xrightarrow{(3)} \langle \varphi(T) \rangle \xhookrightarrow{(4)} S.$$

The four factors isolate the following properties:

factor	property of the logos morphism	kernel or image condition
(1)	monogenic quotient	$\ker = K^{\mathrm{mono}}$
(2)	epigenic quotient	$\ker \subseteq \mathrm{Conn}_\infty$
(3)	conservative and algebraic	its image generates the target
(4)	fully faithful	inclusion of the image logos.

The composite (1)(2) is the quotient part of φ, while (3)(4) is its conservative part. The composite (1)(2)(3) is algebraic in Lurie's terminology. Further terminology for composites, especially variants of "surjective", depends on whether one works with classical or higher topoi, so we will use the explicit properties above.

5.6 Products of acyclic classes

Given two ideals I and J in a commutative ring R, we may form their *product* $IJ \subseteq R$, the ideal generated by the products ij for $i \in I$ and $j \in J$. There is an analogous construction for classes of morphisms in a topos. Its natural domain is the partially ordered set $\mathrm{Acyc}(T)$ of acyclic classes, rather than only the smaller collection of modalities.

5.6.1 The acyclic product

Definition 5.71. For $u\colon A \to B$ and $v\colon C \to D$, the *pushout product*

$$u \square v\colon (B \times C) \sqcup_{A\times C} (A \times D) \longrightarrow B \times D$$

is the cogap map of the commutative square

$$\begin{array}{ccc} A \times C & \xrightarrow{A\times v} & A \times D \\ {\scriptstyle u\times C}\downarrow & & \downarrow{\scriptstyle u\times D} \\ B \times C & \xrightarrow{B\times v} & B \times D. \end{array}$$

We already encountered this operation before in Notation 5.15.

Definition 5.72. For acyclic classes K and L, define their *product* as the acyclic class

$$KL \quad := \quad (K \square L)^m,$$

where $K \square L$ is the class of morphisms of the form $u \square v$ for $u \in K$ and $v \in L$.

Lemma 5.73. *We have an inclusion $KL \subseteq K \cap L$.*

Proof. By symmetry, it suffices to show $KL \subseteq K$. If $u \in K$ and $v \in L$, then also the maps $u \times C$ and $u \times D$ are in K, hence so is the cobase change $A \times D \to (B \times C) \sqcup_{A\times C} (A \times D)$. By right cancellation, it follows that the cogap map $u \square v$ is in K as well. □

By definition, KL is again an acyclic class. If K and L are of small generation, the following lemma shows that their product is of small generation as well and hence defines a modality.

Lemma 5.74. *Given two classes of maps Σ, Σ' in T, we have $\Sigma^m\, \Sigma'^m = (\Sigma \square \Sigma')^m$. In particular, if K and L are acyclic classes of small generation, then so is KL.*

Proof. The inclusion $(\Sigma \square \Sigma')^m \subseteq \Sigma^m\, \Sigma'^m$ is clear. For the converse, it suffices to show that $(\Sigma \square \Sigma')^m$ contains $\Sigma^m \square \Sigma'^m$. Let us first show that it contains $\Sigma \square \Sigma'^m$. For a fixed morphism $u \in \Sigma$, consider the collection of morphisms $v \in \mathrm{Ar}(T)$ such that $u \square v \in (\Sigma \square \Sigma')^m$. This collection clearly contains Σ', hence to show it contains all of Σ'^m it remains to show that it is a saturated class closed under base change.

Closure under base change follows from universality of colimits: if v' is a base change of v, then $u \square v'$ is the corresponding base change of $u \square v$. The class clearly contains all isomorphisms. For closure under composition, let $C \xrightarrow{v} D \xrightarrow{w} E$ be composable. The map $u \square (wv)$ factors as

$$(B \times C) \sqcup_{A\times C} (A \times E) \longrightarrow (B \times D) \sqcup_{A\times D} (A \times E) \longrightarrow B \times E.$$

The first map is a cobase change of $u \square v$, and the second is $u \square w$. Thus it belongs to $(\Sigma \square \Sigma')^m$ whenever both $u \square v$ and $u \square w$ do. Finally, closure under colimits follows because $u \square -\colon \mathrm{Ar}(T) \to \mathrm{Ar}(T)$ preserves colimits.

We may now repeat the argument to show that for fixed $v \in \Sigma'^m$ the collection of $u \in \mathrm{Ar}(T)$ such that $u \square v \in (\Sigma \square \Sigma')^m$ contains Σ and is a saturated class closed under base change, hence contains all of Σ^m. This finishes the proof. □

Proposition 5.75 ([ABFJ25, Theorem 3.2.2])**.** *This product turns* $\mathrm{Acyc}(T)$ *into a commutative algebra object in the category* $\mathrm{Pos}^{\mathrm{cocompl}}$ *of cocomplete posets. Its unit is the acyclic class* All *of all morphisms.*

Corollary 5.76. *Let K and L be monogenic acyclic classes. Then KL is monogenic. If K and L are of small generation, then so is KL.*

Proof. By monogenicity, we may write $K = \Sigma^m$ and $L = \Sigma'^m$ with $\Sigma = K \cap \mathrm{Mono}$ and $\Sigma' = L \cap \mathrm{Mono}$. The class $\Sigma \,\square\, \Sigma'$ consists of monomorphisms, so Lemma 5.74 shows that KL is monogenic. If K and L are of small generation, Lemma 5.56 allows us to choose Σ and Σ' to be sets, which also proves small generation of KL. □

Corollary 5.77. *Let K be a monogenic acyclic class. Then $K^2 = K$.*

Proof. The inclusion $K^2 \subseteq K$ is immediate from Lemma 5.73. For the converse, we may assume that $K = \Sigma^m$ for a class of monomorphisms Σ, and it will suffice to show that $u \in K^2$ for every morphism $u\colon A \hookrightarrow B$ in Σ. By construction, the morphism $u \,\square\, u\colon (B \times A) \sqcup_{A\times A} (A \times B) \to B \times B$ is in K^2, hence so is its base change along $\Delta\colon B \to B \times B$. Universality of pushouts identifies this base change with $A \sqcup_{A\times_B A} A \to B$. Since u is a monomorphism, we have $A \xrightarrow{\sim} A \times_B A$, so this map identifies with $u\colon A \to B$. □

5.6.2 Examples and division

We now discuss various examples of products of acyclic classes.

Example 5.78. Let $L = \mathrm{Conn}_n$ be the n-connected maps. As a modality this is generated by the constant map $S^{n+1} \to *$ in T. The pushout product of $S^{n+1} \to *$ with $S^{m+1} \to *$ is $S^{n+m+3} \to *$, so by Lemma 5.74 we get

$$(\mathrm{Conn}_n)(\mathrm{Conn}_m) \;=\; \mathrm{Conn}_{n+m+2}\,.$$

As a special case, for $\mathrm{EffEpi} = (S^0 \to *)^m$ we get $\mathrm{Conn}_n = \mathrm{EffEpi}^{\,n+2}$.

More generally, note that for any morphism $u\colon X \to Y$, the pushout product $(S^0 \to *) \,\square\, u$ is the codiagonal $\nabla_u\colon Y \sqcup_X Y \to Y$. It follows that for every acyclic class L we have

$$\mathrm{EffEpi} \cdot L \;=\; \nabla(L)^m.$$

Example 5.79. For any topos T, one has $\mathrm{Epi}^{\,2} \subseteq \mathrm{Conn}_\infty$. Indeed, recall from Proposition 5.9 that every epimorphism is 0-connected, so that

$$\mathrm{Epi}^{\,2} \subseteq \mathrm{Conn}_0^2 = \mathrm{Conn}_2 \subseteq \mathrm{Conn}_1\,.$$

We claim that we also have an inclusion $\mathrm{Epi} \cap \mathrm{Conn}_1 \subseteq \mathrm{Conn}_\infty$. Indeed, if $f\colon X \to Y$ is an epimorphism, then we get a pushout square

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle f}\downarrow & \ulcorner & \| \\ Y & = & Y. \end{array}$$

If $f \in \mathrm{Conn}_n$ for some n, then $\Delta_f \in \mathrm{Conn}_{n-1}$. The relative pushout product $\Delta_f \square_X \Delta_f$ is a base change of the ordinary pushout product, hence belongs to

$$\mathrm{Conn}_{n-1}\,\mathrm{Conn}_{n-1} = \mathrm{Conn}_{2n}\,.$$

Since the gap map of this square is f, Blakers–Massey implies that $f \in \mathrm{Conn}_{2n}$. Assuming $f \in \mathrm{Conn}_1$, it follows inductively that $f \in \mathrm{Conn}_{2^k}$ for all k, so $f \in \mathrm{Conn}_\infty$.

Construction 5.80. Given acyclic classes K and L, we define

$$K\backslash L := \{\, f \mid K \square f \subseteq L \,\}.$$

This is again an acyclic class by [ABFJ25, Lemma 3.1.2 and Theorem 3.2.2]. If M is a third acyclic class, then we have

$$KM \subseteq L \iff M \subseteq K\backslash L,$$

so that $K\backslash L$ functions as an internal hom in $\mathrm{Acyc}(T)$.

Example 5.81. Since $(S^0 \to *) \square u = \nabla(u)$, we see that $u \in \mathrm{EffEpi} \setminus \mathrm{Iso}$ if and only if u is an epimorphism. More generally, we inductively get

$$\mathrm{Cotr}_{n+1} = \mathrm{EffEpi} \setminus \mathrm{Cotr}_n\,.$$

5.6.3 Products of congruences

We now show that the acyclic product of two congruences is again a congruence. The following is the key input.

Lemma 5.82 (Key lemma, [ABFJ25, Lemmas 2.3.27 and 2.3.28])**.** *Let L be an acyclic class.*

(1) $\Delta\nabla(L) \subseteq L$.

(2) $\Delta^{-1}(L) \cap \mathrm{EffEpi} \subseteq \mathrm{EffEpi}\cdot L$.

Proof. (1) For $u\colon A \to B$ in L, the morphism $\Delta\nabla(u)$ takes the form

$$\Delta\nabla(u)\colon B \sqcup_A B \longrightarrow (B \sqcup_A B) \times_B (B \sqcup_A B).$$

The two summand inclusions $i_k\colon B \to B \sqcup_A B$ induce two maps $i_k \times \mathrm{id}\colon B \times_B (B \sqcup_A B) \to (B \sqcup_A B) \times_B (B \sqcup_A B)$ which are jointly an effective epimorphism. Since L is local, it suffices to show that the left vertical map in the following pullback square is in L:

$$\begin{array}{ccc} B & \xrightarrow{\;i_k\;} & B \sqcup_A B \\ {\scriptstyle i_k}\downarrow & \lrcorner & \downarrow{\scriptstyle \Delta\nabla(u)} \\ B \sqcup_A B & \xrightarrow{\;i_k \times \mathrm{id}\;} & (B \sqcup_A B) \times_B (B \sqcup_A B). \end{array}$$

This holds since i_k is a cobase change of u.

(2) This is the substantive direction of the suspension theorem for acyclic classes, proved in [ABFJ25, Lemma 2.3.28]. The assumption that u is an effective epimorphism identifies B with the colimit of

the Čech nerve of u. The condition $\Delta_u \in L$, together with base-change stability and composition, controls all higher maps in this Čech nerve. The simplicial orthogonality argument of the cited lemma then shows that u belongs to the acyclic class generated by the codiagonals of maps in L. By Example 5.78, this class is

$$\nabla(L)^m = \mathrm{EffEpi} \cdot L,$$

as required. □

Corollary 5.83 ([ABFJ25, Theorem 2.3.22])**.** *For any acyclic class L, we have*

$$D(L) \cap \mathrm{EffEpi} \;=\; \mathrm{EffEpi} \cdot L.$$

In particular, if K is a congruence we get

$$K^{\mathrm{epi}} = K \cap \mathrm{EffEpi} = \mathrm{EffEpi} \cdot K.$$

Proof. Since $D(L) = L \cap \Delta^{-1}(L)$, the inclusion "$\subseteq$" holds by (2) of the previous lemma. Conversely, Lemma 5.73 gives $\mathrm{EffEpi} \cdot L \subseteq L$. We also have $\mathrm{EffEpi} \cdot L = \nabla(L)^m$, so it remains to show that $\nabla(L) \subseteq \Delta^{-1}(L)$. This is precisely part (1) of the previous lemma.

The last statement holds because $D(K) = K$ for any congruence K. □

We now prove that the product of two congruences is again a congruence.

Theorem 5.84. *Let $K, L \in \mathrm{Cong}(T)$ be congruences. Then their product KL is also a congruence.*

Proof. By (5) of Proposition 5.35, it suffices to show that it is closed under diagonals, i.e. $KL \subseteq D(KL)$. Recall that any acyclic class M decomposes as $M = M^{\mathrm{mono}} \vee M^{\mathrm{epi}}$. Applying this to K and L, we can distribute the product:

$$KL = (K^{\mathrm{epi}} \vee K^{\mathrm{mono}})(L^{\mathrm{epi}} \vee L^{\mathrm{mono}}) = K^{\mathrm{epi}}L^{\mathrm{epi}} \vee K^{\mathrm{epi}}L^{\mathrm{mono}} \vee K^{\mathrm{mono}}L^{\mathrm{epi}} \vee K^{\mathrm{mono}}L^{\mathrm{mono}}.$$

The product distributes over joins by Proposition 5.75. Using that $K^{\mathrm{epi}} = \mathrm{EffEpi} \cdot K$ and $L^{\mathrm{epi}} = \mathrm{EffEpi} \cdot L$, we see that the first three terms are contained in $\mathrm{EffEpi} \cdot K \cdot L$, which by Corollary 5.83 is contained in $D(KL)$. It thus remains to show that $K^{\mathrm{mono}}L^{\mathrm{mono}} \subseteq D(KL)$. The product $K^{\mathrm{mono}}L^{\mathrm{mono}}$ is monogenic by Corollary 5.76, and it is contained in KL. Hence it is contained in the largest monogenic acyclic class below KL, namely $(KL)^{\mathrm{mono}}$. By Lemma 5.60, the latter is a congruence and therefore closed under diagonals. Thus $(KL)^{\mathrm{mono}} \subseteq D(KL)$, finishing the proof. □

5.7 The Goodwillie tower of a topos

The acyclic product allows us to form powers of a congruence, just as one forms powers of an ideal in a commutative ring. The corresponding tower of quotients is a completion tower. Its successive layers already exhibit a stable phenomenon, and the special case associated with the ∞-connected maps recovers Goodwillie's tower of excisive approximations. The results of this section are due to Anel, Biedermann, Finster, and Joyal, principally [ABFJ18; ABFJ25].

5.7.1 Adic filtrations and their layers

Construction 5.85. Let $K \in \mathrm{Cong}(T)$ be a congruence. By Theorem 5.84, congruences form a subalgebra of $\mathrm{Acyc}(T)$. We can therefore form powers K^n, giving congruences for all $n \geq 0$ sitting in a chain

$$\cdots \subseteq K^3 \subseteq K^2 \subseteq K \subseteq \mathrm{All}\,.$$

We call this the *K-adic filtration*. If K is of small generation, so are its powers, and the filtration gives a tower of quotient logoi

$$T \to \cdots \to T/K^3 \to T/K^2 \to T/K \to T/\mathrm{All} \xrightarrow{\sim} *.$$

Let $(K^{n+1})^\perp$ denote the right class of the modality determined by K^{n+1}. We define the *$(n+1)$-st layer* of the filtration by

$$K^n/K^{n+1} := K^n \cap (K^{n+1})^\perp.$$

For $X \in T$, let $(K^n/K^{n+1})_X \subseteq T_{/X}$ be the full subcategory spanned by the maps $Y \to X$ in this class. At the terminal object, it can equivalently be described as the fiber

$$\begin{array}{ccc} (K^n/K^{n+1})_1 & \longrightarrow & T/K^{n+1} \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{*} & T/K^n. \end{array}$$

Thus an object of $(K^n/K^{n+1})_1$ is K^{n+1}-local and becomes terminal after localization at K^n. The same description in the slice $T_{/X}$ gives the layer at an arbitrary object X.

Definition 5.86. A category is *pre-stable* if it admits pullbacks and pushouts and a commutative square is cartesian if and only if it is cocartesian.

Proposition 5.87 ([ABFJ25, Theorem 4.2.11])**.** *Let $n \geq 1$ and $X \in T$. The layer $(K^n/K^{n+1})_X$ is pre-stable and has a terminal object. Its category of pointed objects is stable.*

Proof. Replacing T by $T_{/X}$ reduces us to $X = *$. Put $T' := T/K^{n+1}$, and let J be the congruence in T' generated by the image of K^n. Then $T/K^n \xrightarrow{\sim} T'/J$. Since the quotient functor preserves acyclic products and $2n \geq n+1$, we have

$$J^2 = \mathrm{Iso}$$

in T'. The layer $(K^n/K^{n+1})_*$ therefore identifies with $(J/J^2)_*$, the full subcategory of T' spanned by objects E for which $E \to *$ belongs to J.

This subcategory is closed under pullbacks and pushouts in T'. For pullbacks, use base-change stability of J and then compose with the terminal map of one of the factors; for pushouts, use closure of J under colimits in the arrow category. Consider a commutative square in the subcategory. Every morphism in the square belongs to J by the three-for-two property. If the square is cocartesian, the generalized Blakers–Massey theorem for the modality $(J, J^\perp)$ shows that its cartesian gap map belongs to $J^2 = \mathrm{Iso}$, so the square is cartesian. The dual Blakers–Massey theorem proves the converse. Thus the layer is pre-stable. The terminal object of T' belongs to it, and adjoining a point makes it pointed; a pointed pre-stable category is stable. □

We can apply the K-adic filtration to the specific case where K is the congruence of ∞-connected maps.

Definition 5.88. Let T be a topos. We define the *n-th Goodwillie approximation* of T as

$$T^{(n)} := T/(\mathrm{Conn}_\infty)^{n+1}.$$

This gives rise to a tower of logoi

$$T \to \cdots \to T^{(2)} \to T^{(1)} \to T^{(0)} \to T^{(-1)} = *.$$

By passing to right adjoints (geometric morphisms), this may alternatively be regarded as a filtration of T by subtopoi:

$$* = T^{(-1)} \hookrightarrow T^{(0)} \hookrightarrow T^{(1)} \hookrightarrow \cdots \hookrightarrow T.$$

The terminology is justified by the comparison with excisive functors developed next.

5.7.2 Excisive functors and the Goodwillie comparison

We show that the topos-theoretic filtration recovers classical Goodwillie calculus when applied to functor categories.

Construction 5.89. Let C be a small category with finite colimits, and let T be a topos. The category C is filtered, so the colimit functor

$$\mathrm{colim}\colon \mathrm{Fun}(C,T) \to T$$

preserves colimits and finite limits, since filtered colimits commute with finite limits in a topos. It is therefore a morphism of logoi. Its right adjoint is the constant-diagram functor $X \mapsto \mathrm{const}_X$. This functor is fully faithful because C is weakly contractible, as follows already from the initial object of C. Thus colim is a quotient map.

Choose a small set $\mathcal{G}$ of generators of T, closed under finite products. For $G \in \mathcal{G}$, write

$$y_G\colon C^{\mathrm{op}} \longrightarrow \mathrm{Fun}(C,T), \qquad X \longmapsto y(X) \otimes G.$$

A functor $F\colon C \to T$ is constant if and only if it is local with respect to every map $y_G(f)\colon y_G(Y) \to y_G(X)$, where $f\colon X \to Y$ ranges through the morphisms of C and G ranges through $\mathcal{G}$. Indeed, the corresponding locality map is obtained by applying $\mathrm{Hom}_T(G,-)$ to $F(X) \to F(Y)$, and the generators detect isomorphisms. Consequently, if

$$\Sigma_T := \{\, y_G(f) \mid f \in \mathrm{Ar}(C),\ G \in \mathcal{G} \,\},$$

then the kernel K of colim is the strong saturation Σ_T^s.

We denote the monogenic-epigenic factorization of colim as follows:

$$\mathrm{Fun}(C,T) \to \mathrm{Shv}(C^{\mathrm{op}};T) \to T.$$

The middle term is the localization at the monogenic part K^{mono}. We use the notation $\mathrm{Shv}(C^{\mathrm{op}};T)$ because, for $T = \mathrm{An}$, it is the logos of sheaves for the Grothendieck topology in which every morphism of C^{op} generates a covering sieve. The notation in general denotes the corresponding T-valued sheaf logos.

We claim that the tower associated to the logos $\mathrm{Shv}(C^{\mathrm{op}}; T)$ is precisely the tower of excisive functors. Recall the definition:

Definition 5.90. A functor $F\colon C \to T$ is called *excisive* if it sends pushout squares in C to pullback squares in T.

For $n \geq 0$, we say F is *n-excisive* if it sends strongly cocartesian $(n+1)$-cubes in C to cartesian $(n+1)$-cubes in T. Recall that an $(n+1)$-cube $X\colon [1]^{n+1} \to C$ is called *strongly cocartesian* if it is left Kan extended from its restriction to the $n+1$ edges $\{0\}^k \times [1] \times \{0\}^{n-k}$, and it is *cartesian* if it is a limit cone.

We denote by $\mathrm{Exc}^n(C,T) \subseteq \mathrm{Fun}(C,T)$ the full subcategory of n-excisive functors.

Remark 5.91. For $n = 0$, we see that a 0-excisive functor F is one that sends any map $X \to Y$ in C to a diagram $F(X) \to F(Y)$ exhibiting $F(X)$ as the limit of $\{F(Y)\}$. This is precisely saying that F is a constant functor.

Theorem 5.92. *Let K be the kernel of the colimit functor* $\mathrm{colim}\colon \mathrm{Fun}(C,T) \to T$. *Then there is an equivalence*

$$\mathrm{Fun}(C,T)/K^{n+1} \simeq \mathrm{Exc}^n(C,T).$$

In particular, if T is hypercomplete, we have an equivalence $\mathrm{Shv}(C^{\mathrm{op}}; T)^{(n)} \simeq \mathrm{Exc}^n(C,T)$.

Proof. We use the set Σ_T from Construction 5.89. Since $K = \Sigma_T^s$ is the kernel of a morphism of logoi, it is a congruence. It follows from the two universal properties that

$$K = \Sigma_T^c :$$

the inclusion $\Sigma_T^c \subseteq K$ holds because K is a congruence containing Σ_T, while $K \subseteq \Sigma_T^c$ holds because Σ_T^c is strongly saturated. Moreover, Σ_T is closed under diagonals up to isomorphisms. Indeed, the diagonal of $y_G(f)$ is obtained by applying y_G to the codiagonal of f, since C has pushouts; the same description applies to every iterated diagonal. The ABFJ formula now gives

$$K = \Sigma_T^c = \Sigma_T^m.$$

By Lemma 5.74, the power K^{n+1} is generated as an acyclic class by the $(n+1)$-fold pushout products of maps in Σ_T.

For maps $f_i\colon A_i \to B_i$ in C, let

$$f_0 \boxplus \cdots \boxplus f_n\colon [1]^{n+1} \longrightarrow C$$

be their external coproduct cube. It is strongly cocartesian. The iterated pushout product of the corresponding opposite Yoneda maps is the cocartesian gap map of the Yoneda image of this cube. Every strongly cocartesian cube in C is a cobase change of such a free cocartesian cube, and base changes of its Yoneda gap map correspond to such cobase changes. Consequently, locality with respect to the $(n+1)$-fold pushout products is equivalent to carrying every strongly cocartesian $(n+1)$-cube to a cartesian cube. This is the content of [ABFJ25, Lemmas 4.4.2–4.4.4]; tensoring with the generators $G \in \mathcal{G}$ makes the same argument detect cartesian cubes in T. Thus the K^{n+1}-local objects are precisely the n-excisive functors, proving the first equivalence.

Now suppose that T is hypercomplete, and put $E := \mathrm{Shv}(C^{\mathrm{op}}; T) = \mathrm{Fun}(C, T)/K^{\mathrm{mono}}$. Since $\mathrm{Fun}(C, T)/K \xrightarrow{\sim} T$ is hypercomplete, Lemma 5.67 gives

$$K = (K^{\mathrm{mono}})^{\mathrm{hyp}}.$$

It follows that the image of K in E is precisely the congruence $\mathrm{Conn}_\infty(E)$. Quotient functors preserve acyclic products, see [ABFJ25, Proposition 3.6.1], so the image of K^{n+1} is $\mathrm{Conn}_\infty(E)^{n+1}$. The first part now identifies

$$E^{(n)} = E/\mathrm{Conn}_\infty(E)^{n+1} \xrightarrow{\sim} \mathrm{Exc}^n(C, T).$$

□

Lemma 5.93. *Let $C = \mathrm{An}^{\mathrm{fin}}$ be the category of finite animae. Then the inclusion $i\colon \mathrm{An}^{\mathrm{fin}} \hookrightarrow \mathrm{An}$ defines an ∞-connected object in the topos $\mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}})$. Moreover, it exhibits $\mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}})$ as the classifying topos for ∞-connected objects: for every topos T, evaluation at i induces an equivalence of categories*

$$\mathrm{Geom}(T, \mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}})) \xrightarrow{\sim} T^{\geq\infty}, \qquad \varphi \mapsto \varphi^*(i).$$

Proof. Every object $X \in T$ determines a morphism of logoi $\mathrm{PSh}(\mathrm{An}^{\mathrm{fin,op}}) \to T$ given on generators by $A \mapsto X^A$. The object X is ∞-connected if and only if, for every morphism $f\colon A \to B$ of finite animae, the induced map

$$X^B \longrightarrow X^A$$

is an effective epimorphism. One direction follows by building finite animae from finitely many cells and using stability of effective epimorphisms under base change and composition. Conversely, the maps between finite spheres include the tests asserting that every iterated diagonal of $X \to *$ is an effective epimorphism. This criterion is also recorded in [ABFJ25, Example 2.1.16(g)].

Let K be the kernel of $\mathrm{colim}\colon \mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathrm{An}) \to \mathrm{An}$. By Lemma 5.56, its monogenic part is generated by the monomorphisms $\mathrm{im}(y(f))$ for maps f of finite animae. A morphism of logoi preserves epi–mono factorizations, so the morphism classified by X sends $\mathrm{im}(y(f))$ to an isomorphism if and only if it sends $y(f)$ to an effective epimorphism, which is precisely the condition above. Thus the free morphism classified by X factors through $\mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathrm{An})/K^{\mathrm{mono}}$ exactly when X is ∞-connected. The universal property of the free logos therefore restricts from

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(\mathrm{An}^{\mathrm{fin,op}}), T) \xrightarrow{\sim} T, \qquad \varphi \mapsto \varphi(i)$$

to the desired equivalence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}}), T) \xrightarrow{\sim} T^{\geq\infty}.$$

□

Remark 5.94. In the same way, one can show that $\mathrm{Shv}(\mathrm{An}_*^{\mathrm{fin,op}})$ classifies *pointed* ∞-connected objects.

This classification allows us to identify the layers of the Goodwillie tower with known categories of spectra.

Proposition 5.95. *The category* $\mathrm{Exc}^1(\mathrm{An}_*^{\mathrm{fin}}, T)$ *is equivalent to the total category of* T*-parametrized spectra:*

$$\int_{X \in T} \mathrm{Sp}(T_{/X}).$$

Proof idea. A 1-excisive functor on finite pointed animae determines its base object X by evaluation at the zero object. Its reduced part is an excisive reduced functor in the slice $T_{/X}$, hence a spectrum object of that slice. Conversely, a spectrum object in $T_{/X}$ determines such a reduced excisive functor, and adjoining the base X recovers the original functor. These constructions are inverse and natural in X. This is the internal version of Goodwillie's classification of 1-excisive functors; see [Goo03, Section 5] and [ABFJ25, Section 4.3]. □

Proposition 5.96. *The category* $\mathrm{Exc}^1(\mathrm{An}^{\mathrm{fin}}, T)$ *is equivalent to the total category of* T*-parametrized spectral torsors:*

$$\{(X, E, s) \mid X \in T, E \in \mathrm{Sp}(T_{/X}), s \in \Gamma(X, E)\}.$$

Here the right-hand side denotes the category whose morphisms consist of a map of base objects together with a compatible map of the pulled-back spectra carrying one section to the other.

Proof idea. Adjoining a disjoint basepoint gives a functor $\mathrm{An}^{\mathrm{fin}} \to \mathrm{An}_*^{\mathrm{fin}}$. Under the preceding proposition, the additional datum required to descend a pointed excisive functor along this construction is a section of the underlying parametrized spectrum. This identifies unpointed 1-excisive functors with parametrized spectral torsors. □

5.7.3 Goodwillie equivalences and homogeneous layers

We conclude by translating the multiplicative properties of the adic filtration into familiar results about the Goodwillie tower. Let

$$P_n \colon \mathrm{Fun}(C, T) \longrightarrow \mathrm{Exc}^n(C, T)$$

be the left adjoint to the inclusion. By Theorem 5.92, this is localization at K^{n+1}, where $K := \ker(\mathrm{colim} \colon \mathrm{Fun}(C, T) \to T)$. Thus a morphism is a P_n-equivalence precisely when it belongs to K^{n+1}. Notice that this is a statement about the class inverted by P_n, not about the right class of P_n-local morphisms.

Corollary 5.97. *We have*

$$(P_n\text{-equiv})(P_m\text{-equiv}) = (P_{n+m+1}\text{-equiv}).$$

Proof. Under the identification above, this is the equality

$$K^{n+1}K^{m+1} = K^{n+m+2}.$$

See also [ABFJ18, Theorem 3.3.4(4)]. □

The generalized Blakers–Massey theorem now gives the corresponding excision estimate.

Corollary 5.98. *Consider a pushout square in* $\mathrm{Fun}(C,T)$ *of the form*

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ {\scriptstyle g}\downarrow & & \downarrow \\ H & \longrightarrow & K, \end{array}$$

where f is a P_n-equivalence and g is a P_m-equivalence. Then the gap map $F \to G \times_K H$ is a P_{n+m+1}-equivalence.

Proof. The two maps belong to K^{n+1} and K^{m+1}, respectively. The generalized Blakers–Massey theorem places the gap map in their acyclic product, which is K^{n+m+2}. This is the class of P_{n+m+1}-equivalences. Compare [ABFJ18, Theorem 3.4.1]. □

Definition 5.99. Let $n \geq 1$. A functor $F\colon C \to T$ is called *n-reduced* if the map $F \to *$ is a P_{n-1}-equivalence, or equivalently if $P_{n-1}F \xrightarrow{\sim} *$. More generally, a map $F \to B$ is called *n-reduced relative to B* if it is a P_{n-1}-equivalence.

An n-excisive and n-reduced functor is called *n-homogeneous*.

The shift in this definition is essential: n-reduced means that all polynomial information of degree at most $n-1$ vanishes.

Corollary 5.100 (Goodwillie's structure theorem)**.** *Consider a functor $F\colon C \to T$, and define G via the following pushout square:*

$$\begin{array}{ccc} P_nF & \longrightarrow & P_{n-1}F \\ \downarrow & & \downarrow \\ P_0F & \longrightarrow & G. \end{array}$$

Then the canonical map $G \to P_0F$ is n-reduced relative to P_0F, and the square is P_n-cartesian. Equivalently, applying P_n turns it into a pullback square.

Proof. The map $P_nF \to P_{n-1}F$ is a P_{n-1}-equivalence. Its cobase change $P_0F \to G$ is therefore a P_{n-1}-equivalence, and so is the induced map $G \to P_0F$, by three-for-two. This proves the first assertion.

The maps $P_nF \to P_{n-1}F$ and $P_nF \to P_0F$ are a P_{n-1}-equivalence and a P_0-equivalence, respectively. By Corollary 5.98, the gap map

$$P_nF \longrightarrow P_{n-1}F \times_G P_0F$$

is a P_n-equivalence. After applying P_n, it becomes an isomorphism, which is precisely the second assertion. This argument is the cartesian part of [ABFJ18, Theorem 3.5.2]. □

Suppose now that C is pointed. The zero object then gives natural maps $P_0F \to F \to P_nF$. Define the *n-th homogeneous layer D_nF* by the pullback square

$$\begin{array}{ccc} D_nF & \longrightarrow & P_nF \\ \downarrow & & \downarrow \\ P_0F & \longrightarrow & P_{n-1}F. \end{array}$$

Corollary 5.101. *There is a canonical isomorphism*

$$D_n F \xrightarrow{\sim} \Omega_{P_0 F} P_n G$$

in the pointed slice category $\mathrm{Fun}(C,T)_{/P_0F,*}$. *Thus* P_nG *is a delooping of the* n*-th homogeneous layer of* F.

Proof. By Corollary 5.100, applying P_n to the defining pushout square for G gives a pullback square

$$\begin{array}{ccc} P_nF & \longrightarrow & P_{n-1}F \\ \downarrow & \lrcorner & \downarrow \\ P_0F & \longrightarrow & P_nG. \end{array}$$

The lower horizontal map points P_nG in the slice over P_0F. Pulling this square back along the canonical point $P_0F \to P_{n-1}F$ identifies the defining pullback for D_nF with the relative loop object of P_nG. See also [ABFJ18, Corollary 3.5.3]. □

Corollary 5.102. *If* C *is pointed, the category of* n*-homogeneous functors* $C \to T$ *is stable.*

Proof. An n-homogeneous functor is a K^{n+1}-local object whose map to the terminal object belongs to K^n. Hence these functors form the layer $(K^n/K^{n+1})_*$. Since C is pointed, an n-reduced functor has a canonical point: the map $P_0F \to F$ identifies with $* \to F$. Thus the category of n-homogeneous functors identifies with the category of pointed objects in this layer. It is stable by Proposition 5.87; compare [ABFJ25, Corollary 4.3.5]. □

6 Topics in topos theory

This chapter develops several further parts of topos theory which build on the common foundations of descent, truncation, and geometric morphisms, but which do not form a single linear theory. They fall into four broad groups: concrete forms of descent, convergence and homotopical invariants, finiteness and completeness, and global constructions in the category of topoi. Many of the sections can therefore be read independently.

The first group gives concrete models for the abstract descent condition. In Section 6.1, we construct the topos $\mathrm{Shv}_\tau(C)$ associated with a Grothendieck site and prove left exactness of sheafification using the theory of generated congruences. In Section 6.2, we study actions and principal bundles for a groupoid object $\mathcal{G}$ and their classification by the object $\mathbf{B}\mathcal{G}$ obtained as the colimit of $\mathcal{G}$. Hypercovers, treated in Section 6.3, refine Čech nerves at every simplicial degree. They provide a bridge from descent to completion theory: their effectiveness characterizes hypercomplete topoi.

The second group concerns convergence and homotopical invariants. In Section 6.4, we compare localic, hypercomplete, Postnikov-complete, and bounded topoi. These notions need not agree in general. Dimension provides useful sufficient conditions: a topos which is locally of finite dimension is hypercomplete, while a uniform local dimension bound implies Postnikov-completeness. Finally, Section 6.6 studies a different kind of homotopical information. After passage to pro-categories, the inverse image functor of every geometric morphism admits a further left adjoint; its value on the terminal object defines the relative shape, and in particular the shape of a topos as a pro-anima.

The third group introduces finiteness conditions. Coherent objects provide a higher-categorical analogue of quasi-compact and quasi-separated objects and satisfy strong compactness properties after truncation. The completeness theorem shows that every locally coherent topos admits a surjection $\mathrm{An}^I \to T$ for some set I. In Section 6.8, we relate coherent topoi to pretopoi, their finitary counterparts, by means of sheafification for the effective epimorphism topology. Makkai completeness then reconstructs an appropriate coherent topos from its category of points together with its ultraproduct structure.

The final group studies constructions involving topoi as objects of Topos. A topos is exponentiable precisely when its underlying presentable category is compactly assembled; this is proved in Section 6.10. In Section 6.11, we ask when the category $\int_T C$ of T-parametrized objects in a category C is itself a topos. The answer is expressed by the notion of a *locus*: an accessible category with pullbacks and van Kampen weakly contractible colimits. The parametrized-object material is not used later and may be skipped on a first reading.

6.1 Sheaf topoi

Sheaves on Grothendieck sites provide one of the fundamental constructions of topoi. We begin by showing that localization at any pullback-stable set of sieves is left exact. We then relate Grothendieck topologies on a small category to intrinsic Grothendieck topologies and monogenic congruences on its presheaf topos. Finally, we describe the resulting sheaves by Čech descent, characterize effective epimorphisms by local sections, and discuss the functoriality induced by continuous and cocontinuous morphisms of sites.

Definition 6.1. Let C be a category. A *sieve* on an object $X \in C$ is a subfunctor $U \hookrightarrow y(X)$ of the representable presheaf $y(X) = \mathrm{Hom}_C(-, X)$. Equivalently, it is a collection of morphisms with codomain X that is closed under precomposition.

Theorem 6.2 ([Lur09, Proposition 6.2.2.7])**.** *Let C be a small category, and let τ be a collection of sieves on C stable under pullbacks. Let*

$$\mathrm{Shv}_\tau(C) \quad \subseteq \quad \mathrm{PSh}(C)$$

be the full subcategory spanned by the τ-local objects. Then $\mathrm{Shv}_\tau(C)$ is a left exact localization of $\mathrm{PSh}(C)$. In particular, $\mathrm{Shv}_\tau(C)$ is a topos.

The standard proof of this result (as found for example in [Lur09]) involves a detailed analysis of the "plus-construction", which is fairly intricate. However, using the machinery of modalities and congruences established in Section 5.3, we can give a much simpler and more conceptual proof. We will freely use the notations introduced in that section.

Proof. We know from Proposition A.15 that the inclusion admits an accessible left adjoint $L\colon \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C)$. We need to show that L is left exact. By Corollary 4.16, it suffices to show that the class of morphisms K inverted by L is a congruence.

The class K is precisely the strong saturation τ^{ss} of τ. We claim that it agrees with τ^c. Since τ consists of monomorphisms, it follows from Corollary 5.43 that $\tau^c = \tau^m$. Choose the representable presheaves as a set of generators of $\mathrm{PSh}(C)$. In the notation of Proposition 5.32, the set τ^{bc} then consists of the pullbacks of the sieves in τ along maps from representable presheaves. It agrees with τ: one inclusion follows from pullback stability, while the other follows by taking the identity map of the codomain of each sieve. Hence Proposition 5.32 gives $\tau^m = (\tau^{bc})^s = \tau^s$. All in all, we see that $\tau^{ss} \subseteq \tau^c = \tau^m = \tau^s \subseteq \tau^{ss}$, and thus all four classes agree. In particular τ^{ss} is a congruence, finishing the proof. □

6.1.1 Grothendieck sites

While not necessary for the definition of $\mathrm{Shv}_\tau(C)$, the collection τ in the previous theorem is usually taken to be a *Grothendieck topology*. We first formulate an intrinsic, accessible version of this notion in an arbitrary topos:

Definition 6.3 ([ABFJ24, Definition 3.1.2])**.** An *accessible Grothendieck topology* on a topos T is a class τ of monomorphisms in T, called the *covering monomorphisms*, such that:

(a) The class τ contains all isomorphisms, and its saturation τ^s is of small generation;

(b) The class τ is a local class, in the sense of Definition 2.46;

(c) Covering monomorphisms are closed under composition;

(d) Given monomorphisms $f\colon X \hookrightarrow Y$ and $g\colon Y \hookrightarrow Z$ in T, if $g \circ f$ is a covering monomorphism, then so is g.

We refer to a τ-local object as a τ-*sheaf* and denote the full subcategory of τ-sheaves by

$$\mathrm{Shv}_\tau(T) \quad \subseteq \quad T.$$

Remark 6.4. The small-generation condition is not part of the unrestricted notion of an extended Grothendieck topology in [ABFJ24, Definition 3.1.2]. We impose it here so that localization at the covering monomorphisms is accessible. In the remainder of these notes, "Grothendieck topology on a topos" will always mean an accessible Grothendieck topology in this sense.

The argument from Theorem 6.2 goes through, showing that the localization functor $L_\tau\colon T \to \mathrm{Shv}_\tau(T)$ is left exact.

Remark 6.5. Let (T, τ) and (S, τ') be topoi equipped with Grothendieck topologies. A morphism of topoi $\varphi_*\colon T \to S$ restricts to a functor $\varphi^{\mathrm{Shv}}\colon \mathrm{Shv}_\tau(T) \to \mathrm{Shv}_{\tau'}(S)$ if and only if its associated logos morphism $\varphi^*\colon S \to T$ preserves covering monomorphisms. Indeed, by adjunction the restriction exists precisely when φ^* sends every τ'-covering monomorphism to a τ-local equivalence. Since φ^* is left exact, these images are monomorphisms, and Proposition 6.9 identifies the τ-local equivalences which are monic with the τ-covering monomorphisms. In this case, the functor φ^{Shv} is automatically a morphism of topoi. By passing to left adjoints, we see that φ^* commutes with sheafification:

$$\begin{array}{ccc} S & \xrightarrow{\varphi^*} & T \\ {\scriptstyle L_{\tau'}}\downarrow & & \downarrow{\scriptstyle L_\tau} \\ \mathrm{Shv}_{\tau'}(S) & \xrightarrow{(\varphi^{\mathrm{Shv}})^*} & \mathrm{Shv}_\tau(T). \end{array}$$

When specializing Definition 6.3 to presheaf topoi, we recover the usual notion of a Grothendieck site.

Notation 6.6. Let $f\colon X \to Y$ be a morphism in some category C. For every sieve $U \hookrightarrow y(Y)$, we denote by $f^*U \hookrightarrow y(X)$ the sieve obtained by pullback along $y(f)\colon y(X) \to y(Y)$. Note that it consists of those morphisms $g\colon Z \to X$ such that the composite $f \circ g$ is in the sieve U.

Definition 6.7 (Grothendieck topology, [Lur09, Definition 6.2.2.1])**.** A *Grothendieck topology* on a category C consists of a specification, for each object X of C, of a collection of sieves on C which we will refer to as *covering sieves*. The collections of covering sieves are required to satisfy the following properties:

(1) For every object X of C, the identity $y(X) \to y(X)$ is a covering sieve;

(2) For every morphism $f\colon X \to Y$ in C and every covering sieve $U \hookrightarrow y(Y)$ on Y, the pullback sieve $f^*U \hookrightarrow y(X)$ is a covering sieve on X;

(3) Let X be an object of C, $U \hookrightarrow y(X)$ a covering sieve on X, and $V \hookrightarrow y(X)$ an arbitrary sieve on X. Suppose that, for every morphism $f\colon Y \to X$ belonging to U, the pullback sieve f^*V is a covering sieve on Y. Then V is a covering sieve on X.

A *Grothendieck site* is a category C equipped with a Grothendieck topology.

Lemma 6.8. *Let C be a small category. Then there is a bijection between Grothendieck topologies on C in the sense of Definition 6.7 and Grothendieck topologies on* $\mathrm{PSh}(C)$ *in the sense of Definition 6.3.*

Proof. If τ is a Grothendieck topology on $\mathrm{PSh}(C)$, we declare a sieve $U \hookrightarrow y(X)$ to be covering if it is a covering monomorphism in $\mathrm{PSh}(C)$, i.e. if it belongs to τ. Conditions (1) and (2) follow from (a) and base-change stability. For (3), let $i\colon U \hookrightarrow y(X)$ be a covering sieve and let $j\colon V \hookrightarrow y(X)$ be a sieve such that for every map $f\colon y(Y) \to U$ from a representable the projection $y(Y)\times_{y(X)} V \hookrightarrow y(Y)$ is a covering monomorphism. Since τ is local, it follows by descent that $U\times_{y(X)} V \to U$ is a covering monomorphism. Its composite with $U \hookrightarrow y(X)$ is covering by (c), and hence $V \hookrightarrow y(X)$ is covering by (d).

Conversely, given a Grothendieck topology on C, we define a monomorphism $U \hookrightarrow X$ in $\mathrm{PSh}(C)$ to be covering if its pullback along every map $y(Y) \to X$ from a representable is a covering sieve. We verify properties (a)–(d).

For (a), all isomorphisms are covering. Moreover, the saturation τ^s is generated by the set of covering sieves on representables. Indeed, every monomorphism $U \hookrightarrow X$ is the colimit in $\mathrm{Ar}(\mathrm{PSh}(C))$ of its pullbacks along maps $y(Y) \to X$, by the density of the Yoneda embedding. Hence every covering monomorphism belongs to the saturation of the covering sieves.

For (b), stability under base change follows directly from the definition, and closure under coproducts follows because every map from a representable into a coproduct factors through one of its summands. For descent, consider a pullback square as in Definition 2.46 in which the bottom map is an effective epimorphism and the upper vertical map is covering. After pulling back along a map from a representable, the bottom effective epimorphism admits a section: this follows by evaluating it at the object representing its target. The corresponding pullback of the lower vertical map is therefore a base change of the upper vertical map and hence is covering.

For (c), let $V \hookrightarrow U$ and $U \hookrightarrow X$ be covering monomorphisms. To show that $V \hookrightarrow X$ is covering, we may pull back along a map from a representable and thus assume that $X = y(X')$. Then $U \hookrightarrow y(X')$ is a covering sieve. By axiom (3), it suffices to show that for every $f\colon Y \to X'$ in U, the pullback sieve $f^*V \hookrightarrow y(Y)$ is covering. This map is a base change of $V \hookrightarrow U$, so it is covering by assumption.

For (d), consider monomorphisms $V \overset{f}{\hookrightarrow} U \overset{g}{\hookrightarrow} X$ such that $gf \in \tau$. We may again assume that $X = y(X')$ is representable. Then $V \hookrightarrow y(X')$ is a covering sieve. By axiom (3), it suffices to show that for every $h\colon Y \to X'$ in V, the pullback $h^*U \hookrightarrow y(Y)$ is covering. Since h factors through $V \hookrightarrow U$, this pullback is the identity of $y(Y)$ and hence is covering.

It is immediate from the construction that these two assignments are inverse to each other, finishing the proof. □

Proposition 6.9 ([ABFJ24, Proposition 3.1.10])**.** *Let T be a topos. There is a bijection*

$$\{\text{Grothendieck topologies on } T\} \underset{K\cap\mathrm{Mono}\hookleftarrow K}{\overset{\tau\mapsto\tau^c}{\longleftrightarrow}} \{\text{monogenic congruences of small generation on } T\}\,.$$

Proof. First let τ be a Grothendieck topology on T. The congruence τ^c is monogenic, since $\tau \subseteq \mathrm{Mono}$ gives $\tau^c = \tau^m = (\tau \cap \mathrm{Mono})^m$, where the first equality uses Corollary 5.43. Moreover, the saturation τ^s is stable under base change: base-change functors preserve colimits, and a base change of a composite of covering monomorphisms is again such a composite. Hence $\tau^m = \tau^s$, and therefore $\tau^c = \tau^s$. In particular, τ^c is of small generation. We claim that $\tau^c \cap \mathrm{Mono} = \tau$. Since τ consists of monomorphisms, it suffices to show $\tau^c \cap \mathrm{Mono} \subseteq \tau$. Let

$$E := \{f \in \mathrm{Ar}(T) \mid \mathrm{im}(f) \in \tau\}$$

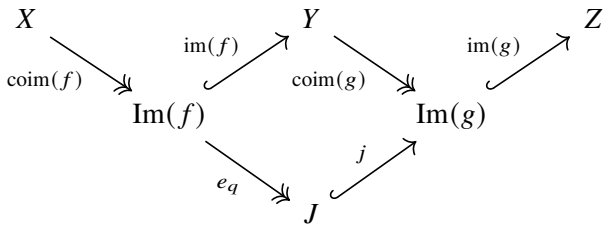
be the class of *τ-coverings* (cf. [ABFJ24, Proposition 3.1.10]). We claim that E is saturated and contains τ, hence $\tau^s \subseteq E$. Clearly E contains all isomorphisms, and $\tau \subseteq E$ since $\mathrm{im}(m) = m$ for every monomorphism m. Let us verify the remaining conditions.

Base change. Since the epi-mono factorization is stable under base change and τ is closed under base change, E is closed under base change.

Composition. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in E, with epi-mono factorizations

$$X \overset{\mathrm{coim}(f)}{\twoheadrightarrow} \mathrm{Im}(f) \overset{\mathrm{im}(f)}{\hookrightarrow} Y \overset{\mathrm{coim}(g)}{\twoheadrightarrow} \mathrm{Im}(g) \overset{\mathrm{im}(g)}{\hookrightarrow} Z,$$

so that $\mathrm{im}(f), \mathrm{im}(g) \in \tau$. Let $q\colon \mathrm{Im}(f) \to \mathrm{Im}(g)$ be the composite $\mathrm{coim}(g) \circ \mathrm{im}(f)$, with epi-mono factorization $q = j \circ e_q$ through some J:



Then $\mathrm{im}(gf) = \mathrm{im}(g) \circ j$, so it suffices to show $j \in \tau$. Pull back j along the effective epimorphism $\mathrm{coim}(g)\colon Y \twoheadrightarrow \mathrm{Im}(g)$ to obtain $\pi\colon M := Y \times_{\mathrm{Im}(g)} J \hookrightarrow Y$. Since $\mathrm{im}(f)$ factors through π (via the map $\mathrm{Im}(f) \to J$ and the inclusion $\mathrm{Im}(f) \hookrightarrow Y$), the composite $\mathrm{Im}(f) \hookrightarrow M \overset{\pi}{\hookrightarrow} Y$ equals $\mathrm{im}(f) \in \tau$. By axiom (d), it follows that $\pi \in \tau$. Since π is the pullback of j along an effective epimorphism and τ is a local class, we conclude $j \in \tau$. Finally, $\mathrm{im}(gf) = \mathrm{im}(g) \circ j \in \tau$ by axiom (c).

Colimits. First, E is closed under coproducts: since images commute with coproducts (as both the class of monomorphisms and the class of effective epimorphisms are closed under coproducts), we have $\mathrm{im}(\coprod f_i) = \coprod \mathrm{im}(f_i)$, and τ is closed under coproducts by locality. For a general colimit $f'\colon A' \to B'$ of a diagram $\{f_i\colon A_i \to B_i\}_{i \in I}$ in $\mathrm{Ar}(T)$, let $f\colon A \to B$ be the coproduct $\coprod_{i \in I} f_i$, so that $f \in E$. The canonical maps $a\colon A \twoheadrightarrow A'$ and $b\colon B \twoheadrightarrow B'$ are effective epimorphisms, and $f'a = bf$. Since b is an effective epimorphism (hence in E) and $f \in E$, the composition $bf = f'a$ lies in E by the above. Since a is an effective epimorphism, $\mathrm{im}(f'a) = \mathrm{im}(f')$, so $f' \in E$.

This establishes that E is saturated. If $m \in \tau^s \cap \mathrm{Mono}$, then $m \in E$, so $\mathrm{im}(m) = m$ lies in τ. Thus $\tau^c \cap \mathrm{Mono} = \tau^s \cap \mathrm{Mono} \subseteq \tau$.

Conversely, let K be a monogenic congruence of small generation and set $\tau_K := K \cap \mathrm{Mono}$. The congruence generated by τ_K is

$$\tau_K^c = \tau_K^m = (K \cap \mathrm{Mono})^m = K,$$

where the first equality uses Corollary 5.43, and the last is the monogenicity of K. We now verify that τ_K is a Grothendieck topology.

(a) The class τ_K contains all isomorphisms. Since K is monogenic and of small generation, Lemma 5.56 provides a set $\Sigma \subseteq \tau_K$ of monomorphisms which generates K as an acyclic class. Choose a set of generators of T and let Σ^{bc} be the corresponding set of base changes. Since τ_K is closed under base change, we have $\Sigma^{bc} \subseteq \tau_K$. Hence

$$K = \Sigma^m = (\Sigma^{bc})^s \subseteq \tau_K^s \subseteq K,$$

where the second equality is Proposition 5.32. Thus $\tau_K^s = K$, so τ_K^s is of small generation.

(b) The class τ_K is local:

- Closure under base change is immediate from base-change stability of K and Mono.
- Closure under small coproducts follows since K is saturated (hence closed under coproducts in $\mathrm{Ar}(T)$), and coproducts of monomorphisms are monomorphisms in a topos.
- For descent along effective epimorphisms, consider a pullback square as in Definition 2.46 with $f' \in \tau_K$. Then $f' \in K$, so $f \in K$ by locality of K. Also, since pullback along an effective epimorphism is conservative and preserves monomorphisms, f is monic. Hence $f \in K \cap \mathrm{Mono} = \tau_K$.

(c) If $f, g \in \tau_K$, then $f, g \in K$ and f, g are monomorphisms. Since K is saturated, $gf \in K$, and since monomorphisms are closed under composition, $gf \in \mathrm{Mono}$. Thus $gf \in \tau_K$.

(d) Let $V \overset{f}{\hookrightarrow} U \overset{g}{\hookrightarrow} X$ be monomorphisms with $gf \in \tau_K$. Since $gf \in K$, the image of gf in the quotient $q\colon T \to T/K$ is an isomorphism, and thus $q(g)$ admits a section. As q is left exact, $q(g)$ is a monomorphism, hence an isomorphism, so $g \in K$. Since g is monic, $g \in K \cap \mathrm{Mono} = \tau_K$.

Hence τ_K is a Grothendieck topology. Together with $\tau_K^c = K$ and $\tau^c \cap \mathrm{Mono} = \tau$, this shows the two assignments are inverse. □

6.1.2 Sheaves and Čech descent

We now introduce the category of sheaves on a Grothendieck site, and formulate its objects in terms of Čech descent.

Definition 6.10 (Sheaf category, [Lur09, Definition 6.2.2.6])**.** Let C be a category equipped with a Grothendieck topology τ. We denote by

$$\mathrm{Shv}_\tau(C) \quad \subseteq \quad \mathrm{PSh}(C)$$

the full subcategory of τ-sheaves. We denote the left exact left adjoint by

$$L_\tau\colon\ \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C).$$

The functor L_τ is known as the *sheafification functor*.

Example 6.11. Let X be a topological space. Then the poset $\mathrm{Open}(X)$ of open subsets admits a Grothendieck topology, called the *open covering topology*, for which the covering sieves of some $U \in \mathrm{Open}(X)$ are those sieves generated by open coverings $U = \bigcup_{i \in I} U_i$. We denote the resulting topos by

$$\mathrm{Shv}(X) \quad := \quad \mathrm{Shv}_{\mathrm{open}}(\mathrm{Open}(X)).$$

Definition 6.12. The *sheafified Yoneda functor* y_τ is the composite

$$C \overset{y}{\hookrightarrow} \mathrm{PSh}(C) \xrightarrow{L_\tau} \mathrm{Shv}_\tau(C).$$

Notation 6.13. Let $\mathcal{U} = \{U_i \to X\}_{i \in I}$ be a collection of morphisms in a category C. The morphism $\bigsqcup_{i \in I} y(U_i) \to y(X)$ in $\mathrm{PSh}(C)$ factors as an effective epimorphism followed by a monomorphism:

$$\bigsqcup_{i \in I} y(U_i) \twoheadrightarrow U \hookrightarrow y(X).$$

We refer to the sieve $U \hookrightarrow y(X)$ as the *sieve generated by* $\mathcal{U}$.

Since limits and colimits in $\mathrm{PSh}(C)$ are computed pointwise, we see that this epi-mono factorization is also computed pointwise: for every $Z \in C$, the subanima

$$U(Z) \subseteq y(X)(Z) = \mathrm{Hom}_C(Z, X)$$

consists of those morphisms $Z \to X$ in C that factor through $U_i \to X$ for some $i \in I$.

If C carries a Grothendieck topology τ, we say that $\mathcal{U}$ is a *covering family* if the sieve it generates is a covering sieve.

Sheaves on C may equivalently be described in terms of *Čech descent*.

Definition 6.14 (Čech descent)**.** Let C be a category and let $\mathcal{U} = \{U_i \to X\}_{i \in I}$ be a collection of morphisms. We will denote by $\check{C}_\bullet(\mathcal{U})$ the Čech nerve of the morphism

$$\bigsqcup_{i \in I} y(U_i) \to y(X)$$

in $\mathrm{PSh}(C)$. A presheaf $\mathcal{F} \in \mathrm{PSh}(C)$ is said to *satisfy Čech descent with respect to* $\mathcal{U}$ if the map

$$\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{PSh}(C)}(y(X), \mathcal{F}) \to \lim_{[n] \in \mathbf{\Delta}} \mathrm{Hom}_{\mathrm{PSh}(C)}(\check{C}_n(\mathcal{U}), \mathcal{F})$$

is an equivalence. More concretely, if the relevant iterated fiber products of the U_i over X exist in C, then $\mathcal{F}$ satisfies Čech descent with respect to $\mathcal{U}$ if the diagram

$$\mathcal{F}(X) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_X U_j) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

is a limit diagram.

Proposition 6.15 (cf. [Lur09, Lemma 6.2.3.18])**.** *Let $\mathcal{U} = \{U_i \to X\}_{i \in I}$ be a collection of morphisms in a category C and let $U \hookrightarrow y(X)$ be the sieve generated by $\mathcal{U}$. Then a presheaf $\mathcal{F}$ on C satisfies Čech descent with respect to $\mathcal{U}$ if and only if it is local with respect to $U \hookrightarrow y(X)$.*

Proof. By definition, $U \hookrightarrow y(X)$ is obtained from the epi-mono factorization $\bigsqcup_{i \in I} y(U_i) \twoheadrightarrow U \hookrightarrow y(X)$. It follows that U is equivalent to the colimit of the Čech nerve $\check{C}(\mathcal{U})$ of $\mathcal{U}$, and we obtain for every $\mathcal{F} \in \mathrm{PSh}(C)$ an equivalence

$$\mathrm{Hom}_{\mathrm{PSh}(C)}(U, \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{PSh}(C)}(\mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \check{C}_n(\mathcal{U}), \mathcal{F}) \simeq \lim_{[n] \in \mathbf{\Delta}} \mathrm{Hom}_{\mathrm{PSh}(C)}(\check{C}_n(\mathcal{U}), \mathcal{F}).$$

The claim now follows immediately. □

6.1.3 Effective epimorphisms in sheaf topoi

The effective epimorphisms in a sheaf topos $\mathrm{Shv}_\tau(C)$ can be characterized as those morphisms which *admit local sections*, in the following sense:

Definition 6.16. Let C be a category equipped with a Grothendieck topology τ. Let $f\colon X \to Y$ be a morphism in $\mathrm{Shv}_\tau(C)$ and assume that $Y = y_\tau(Y')$ lies in the image of the sheafified Yoneda functor. We say that f *admits local sections* if there exists a covering family $\{U_i \to Y'\}_{i\in I}$ of Y' such that the base change

$$X \times_{y_\tau(Y')} y_\tau(U_i) \to y_\tau(U_i)$$

admits a section for every $i \in I$.

If $f\colon X \to Y$ is an arbitrary morphism in $\mathrm{Shv}_\tau(C)$, we say that f *admits local sections* if its base change along every map $y_\tau(Y') \to Y$ from a sheafified representable admits local sections.

Lemma 6.17. *A morphism $f\colon X \to Y$ in $\mathrm{Shv}_\tau(C)$ is an effective epimorphism if and only if it admits local sections.*

Proof. Suppose first that f admits local sections. If $Y = y_\tau(Y')$, choose a covering family $\{U_i \to Y'\}_{i\in I}$ over which f admits sections. The induced map

$$\coprod_{i\in I} y_\tau(U_i) \longrightarrow y_\tau(Y')$$

is an effective epimorphism. After base change along this map, f is a coproduct of morphisms admitting sections and hence is an effective epimorphism by Lemma 2.38. Since effective epimorphisms form a local class, f is an effective epimorphism. For arbitrary Y, the same argument applies after choosing an effective epimorphism from a coproduct of sheafified representables to Y.

Conversely, assume that $f\colon X \to Y$ is an effective epimorphism. Since effective epimorphisms are stable under base change, it suffices to treat the case $Y = y_\tau(Y')$. Let $i\colon \mathrm{Shv}_\tau(C) \hookrightarrow \mathrm{PSh}(C)$ denote the inclusion and form the pullback

$$\widetilde{X} := i(X) \times_{i(y_\tau(Y'))} y(Y')$$

along the unit $y(Y') \to i(y_\tau(Y'))$. Consider the epi–mono factorization in $\mathrm{PSh}(C)$:

$$\widetilde{X} \twoheadrightarrow U_f \hookrightarrow y(Y').$$

Applying sheafification $L_\tau\colon \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C)$ gives a factorization

$$X \twoheadrightarrow L_\tau(U_f) \longrightarrow y_\tau(Y'),$$

where we have used left exactness of L_τ to identify $L_\tau(\widetilde{X})$ with X. The composite is f, so right cancellation for effective epimorphisms shows that the map $L_\tau(U_f) \to y_\tau(Y')$ is an effective epimorphism. It is also a monomorphism, since L_τ is left exact, and hence it is an isomorphism.

Therefore the monomorphism $U_f \hookrightarrow y(Y')$ is inverted by L_τ. By Theorem 6.2, the class of morphisms inverted by L_τ is the congruence τ^c. Thus $U_f \hookrightarrow y(Y')$ lies in $\tau^c \cap \mathrm{Mono} = \tau$ by Proposition 6.9, i.e. it is a covering sieve on Y'. By construction, a morphism $U \to Y'$ belongs to U_f precisely when the corresponding map $y_\tau(U) \to y_\tau(Y')$ admits a lift to X. Taking all morphisms in U_f as a covering family therefore shows that f admits local sections. □

6.1.4 Morphisms of sites

Given a functor $u\colon C \to D$ between two Grothendieck sites, we may ask under what conditions u induces a topos morphism between the sheaf categories of C and D. There are two natural candidates for such a notion, called *continuous* and *cocontinuous* morphisms of sites.

Definition 6.18 (Continuous morphism)**.** Let (C, τ) and (D, τ') be Grothendieck sites. A functor $u\colon C \to D$ is called *continuous* if the restriction functor $u^*\colon \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ preserves sheaves, i.e. restricts to a functor

$$u^*\colon \mathrm{Shv}_{\tau'}(D) \to \mathrm{Shv}_{\tau}(C).$$

We say that u is a *continuous morphism of sites* if this restriction is a morphism of topoi, i.e. its left adjoint is left exact. We denote the category of Grothendieck sites and continuous morphisms of sites by $\mathrm{Site}^{\mathrm{cont}}$. By definition, the assignment $(C, \tau) \mapsto \mathrm{Shv}_\tau(C) \subseteq \mathrm{PSh}(C)$ determines a contravariant functor

$$\mathrm{Shv}(-)\colon (\mathrm{Site}^{\mathrm{cont}})^{\mathrm{op}} \to \mathrm{Topos},$$

obtained by restricting the precomposition functoriality of the presheaf construction $\mathrm{PSh}(-)\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$.

Remark 6.19. The left adjoint of the restriction of u^* is given by the composite

$$\mathrm{Shv}_\tau(C) \hookrightarrow \mathrm{PSh}(C) \xrightarrow{u_!} \mathrm{PSh}(D) \xrightarrow{L_{\tau'}} \mathrm{Shv}_{\tau'}(D),$$

where $u_!$ is left Kan extension along u. In particular, any continuous functor u for which the left Kan extension functor $u_!$ is left exact is a continuous morphism of sites.

Note that $u_!$ is left exact whenever u admits a left adjoint, since then $u_!$ is given by restriction along that left adjoint.

The following criterion is often useful for checking continuity:

Lemma 6.20. *Let (C, τ) and (D, τ') be Grothendieck sites, and let $u\colon C \to D$ be a functor. Assume that every covering sieve $U \hookrightarrow y(X)$ of $X \in C$ is generated by a collection of morphisms $\{f_i\colon U_i \to X\}_{i\in I}$ in C satisfying the following conditions:*

- *Pullbacks along f_i exist in C and are preserved by $u\colon C \to D$;*
- *The sieve on $y(u(X))$ in D generated by the maps $u(f_i)$ is a covering sieve.*

Then u is a continuous functor.

Proof. Let $\mathcal{F}$ be a τ'-sheaf on D. We need to show that $u^*\mathcal{F}$ is a τ-sheaf on C. For a covering sieve $U \hookrightarrow y(X)$, let $\mathcal{U} = \{f_i\colon U_i \to X\}_{i\in I}$ be a collection of morphisms in C generating it satisfying the two conditions. Using Proposition 6.15 and the adjunction $u_! \dashv u^*$, we may equivalently show that the map

$$(u^*\mathcal{F})(X) \simeq \mathrm{Hom}_{\mathrm{PSh}(D)}(y(u(X)), \mathcal{F}) \to \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathrm{PSh}(D)}(u_!(\check{C}_n(\mathcal{U})), \mathcal{F})$$

is an isomorphism. The first assumption on $\mathcal{U}$ guarantees that the simplicial object $u_!(\check{C}_n(\mathcal{U}))$ in $\mathrm{PSh}(D)$ agrees with the Čech nerve of the map $\bigsqcup_{i\in I} y(u(U_i)) \to y(u(X))$. The second assumption says that the associated sieve is a covering sieve. So a second application of Proposition 6.15 implies the claim. □

Corollary 6.21. *Let (C, τ) and (D, τ') be Grothendieck sites, and assume that the following two conditions are satisfied:*

- *The category C admits finite limits, and $u\colon C \to D$ preserves finite limits;*
- *The functor u preserves covering families.*

Then u is a continuous morphism of sites.

Proof. The left Kan extension functor $u_!\colon \mathrm{PSh}(C) \to \mathrm{PSh}(D)$ is left exact by Proposition 2.43, as it is the colimit extension of the left exact functor $C \xrightarrow{u} D \hookrightarrow \mathrm{PSh}(D)$. Since u is a continuous functor by Lemma 6.20, the claim follows from Remark 6.19. □

Example 6.22. Let $f\colon X \to Y$ be a continuous map of topological spaces. Then the preimage functor $f^{-1}\colon \mathrm{Open}(Y) \to \mathrm{Open}(X)$ preserves finite limits and preserves coverings, hence is a continuous morphism of sites. This gives rise to a functor $\mathrm{Open}(-)\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Site}^{\mathrm{cont}}$. Composing it with the sheaf functor $(\mathrm{Site}^{\mathrm{cont}})^{\mathrm{op}} \to \mathrm{Topos}$, we obtain a functor

$$\mathrm{Shv}\colon \mathrm{Top} \to \mathrm{Topos}.$$

We now discuss a second mechanism by which a functor between Grothendieck sites induces a morphism of topoi, namely cocontinuity.

Definition 6.23 (Cocontinuous morphism)**.** Let (C, τ) and (D, τ') be Grothendieck sites. A functor $u\colon C \to D$ is called a *cocontinuous morphism of sites* if the right Kan extension functor restricts to a functor

$$u_*\colon \mathrm{Shv}_\tau(C) \to \mathrm{Shv}_{\tau'}(D).$$

In particular, u_* is a morphism of topoi, with left exact left adjoint given by the composite

$$\mathrm{Shv}_{\tau'}(D) \hookrightarrow \mathrm{PSh}(D) \xrightarrow{u^*} \mathrm{PSh}(C) \xrightarrow{L_\tau} \mathrm{Shv}_\tau(C).$$

If we denote by $\mathrm{Site}^{\mathrm{cocont}}$ the category of Grothendieck sites and cocontinuous morphisms, then the assignment $(C, \tau) \mapsto \mathrm{Shv}_\tau(C)$ defines a functor

$$\mathrm{Site}^{\mathrm{cocont}} \to \mathrm{Topos},$$

obtained by restricting the right Kan extension functoriality of the presheaf construction $\mathrm{PSh}\colon \mathrm{Cat} \to \mathrm{Pr}^{\mathrm{R}}$.

Let us make the cocontinuity condition on u more concrete.

Notation 6.24. Given an object $X \in C$ and a sieve $U \hookrightarrow y(u(X))$ of $u(X)$ in D, we define its *pullback along u* as the sieve $u^{-1}(U) \hookrightarrow y(X)$ of X obtained by forming the following pullback square in $\mathrm{PSh}(C)$:

$$\begin{array}{ccc} u^{-1}(U) & \longrightarrow & u^*(U) \\ \downarrow & \lrcorner & \downarrow \\ y(X) & \longrightarrow & u^*(y(u(X))); \end{array}$$

here the bottom map is the unit map $y(X) \to u^* u_! y(X) = u^*(y(u(X)))$. Note that $u^{-1}(U)$ consists of those morphisms $Z \to X$ in C for which the map $u(Z) \to u(X)$ lies in the sieve U.

Lemma 6.25. *A functor $u\colon C \to D$ is a cocontinuous morphism of sites if and only if for every $X \in C$ and every covering sieve $U \hookrightarrow y(u(X))$ of $u(X)$ in D, the pullback sieve $u^{-1}(U) \hookrightarrow y(X)$ is a covering sieve in C.*

Proof. By definition, u is a cocontinuous morphism if and only if the restriction functor $u^*\colon \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ preserves the covering monomorphisms, or equivalently if for every covering sieve $V \hookrightarrow y(Y)$ in D the map $u^*(V) \hookrightarrow u^*y(Y)$ is a covering monomorphism in $\mathrm{PSh}(C)$. The latter condition means that for any map $f\colon y(X) \to u^*y(Y)$ in $\mathrm{PSh}(C)$ from a representable, the base change $u^*(V) \times_{u^*y(Y)} y(X) \hookrightarrow y(X)$ is a covering sieve. The map f corresponds to a map $u(X) \to Y$ in D, and pulling back the covering sieve $V \hookrightarrow y(Y)$ gives a covering sieve $U \hookrightarrow y(u(X))$. The claim now follows, since we have a pullback square

$$\begin{array}{ccccc} u^{-1}(U) & \longrightarrow & u^*(U) & \longrightarrow & u^*(V) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ y(X) & \longrightarrow & u^*(y(u(X))) & \longrightarrow & u^*(y(Y)). \end{array}$$

□

Corollary 6.26. *Let $v\colon (C,\tau) \to (D,\tau')$ be a functor between sites and assume that v admits a right adjoint $u\colon D \to C$. Then v is a cocontinuous morphism of sites if and only if u is a continuous morphism of sites. In this case, we have $v_* \simeq u^*$.*

Proof. The adjunction $v \dashv u$ induces an adjunction $v^* \dashv u^*$ on presheaf categories, showing that $v_* \simeq u^*$. By definition, v is a cocontinuous morphism of sites if and only if v_* preserves sheaves, while it follows from Remark 6.19 that u is a continuous morphism of sites if and only if u^* preserves sheaves, showing the claim. □

6.2 Groupoid actions and principal bundles

In this section, we discuss actions of groupoid objects in a topos T and prove that the category of such actions is equivalent to the slice over the classifying object. We then characterize principal bundles as regular actions with effective-epimorphic atlas and show that they are classified by maps into the classifying object. Our treatment is based on [NSS15] and [SS21, Section 3.2], where the analogous situation for *group* actions is discussed.

Throughout this section, we fix a topos T and a groupoid object $\mathcal{G}\colon \Delta^{\mathrm{op}} \to T$.

6.2.1 Actions by groupoid objects

We introduce the notion of an action of a groupoid object, generalizing the treatment of group actions in [SS21].

Definition 6.27. Let $\mathcal{G}$ be a groupoid object in T, and let $X \in T$ be an object. An *action* of $\mathcal{G}$ on X consists of the following data:

(1) A groupoid object $\mathcal{G} \ltimes X$ in T;

(2) An isomorphism $(\mathcal{G} \ltimes X)_0 \simeq X$;

(3) A cartesian natural transformation $c\colon \mathcal{G} \ltimes X \to \mathcal{G}$ of simplicial objects in T.

We let $\mathrm{Act}_{\mathcal{G}}(T) \subseteq \mathrm{Grpd}(T)_{/\mathcal{G}}$ denote the full subcategory spanned by the $\mathcal{G}$-actions in T.

Remark 6.28. Let $\mathcal{G} \ltimes X$ be an action of $\mathcal{G}$ on X. Since $(\mathcal{G} \ltimes X)_0 \simeq X$, the map $c\colon \mathcal{G} \ltimes X \to \mathcal{G}$ induces a structure map $c_0\colon X \to \mathcal{G}_0$. Since c is a cartesian natural transformation, there are cartesian squares

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & (\mathcal{G} \ltimes X)_2 & \xrightarrow{d_0} & (\mathcal{G} \ltimes X)_1 & \xrightarrow{d_0} & X \\
 & & \downarrow{\scriptstyle c_2} & \lrcorner & \downarrow{\scriptstyle c_1} & \lrcorner & \downarrow{\scriptstyle c_0} \\
\cdots & \longrightarrow & \mathcal{G}_2 & \xrightarrow{d_0} & \mathcal{G}_1 & \xrightarrow{d_0} & \mathcal{G}_0,
\end{array}$$

and in particular we obtain isomorphisms

$$\begin{aligned}
(\mathcal{G} \ltimes X)_1 &\simeq \mathcal{G}_1 \times_{\mathcal{G}_0} X, \\
(\mathcal{G} \ltimes X)_2 &\simeq \mathcal{G}_2 \times_{\mathcal{G}_0} X \simeq \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} X,
\end{aligned}$$

and so on. Under these isomorphisms, the map $d_1\colon (\mathcal{G} \ltimes X)_1 \to (\mathcal{G} \ltimes X)_0$ corresponds to a map $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X$, which we think of as the *action map*. The rest of the simplicial diagram $\mathcal{G} \ltimes X$ encodes the data witnessing that this action is unital and associative up to coherent homotopy.

Definition 6.29. For a groupoid object $\mathcal{G}$, we define its *classifying object* as the geometric realization $\mathbf{B}\mathcal{G} := |\mathcal{G}|$.

If $\mathcal{G} \ltimes X$ is an action of a groupoid object $\mathcal{G}$ on an object $X \in T$, we define the *quotient* $X/\!\!/\mathcal{G}$ as the geometric realization of $\mathcal{G} \ltimes X$:

$$X/\!\!/\mathcal{G} \quad := \quad |\mathcal{G} \ltimes X| \quad \simeq \quad \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}_n \times_{\mathcal{G}_0} X.$$

The cartesian transformation $\mathcal{G} \ltimes X \to \mathcal{G}$ induces a map $X/\!\!/\mathcal{G} \to |\mathcal{G}| = \mathbf{B}\mathcal{G}$ on geometric realizations, and this defines a functor

$$-/\!\!/\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(T) \to T_{/\mathbf{B}\mathcal{G}}.$$

The following result shows that groupoid actions are classified by the classifying object:

Proposition 6.30 (Classification of groupoid actions)**.** *The quotient functor is an equivalence of categories:*

$$-/\!\!/\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(T) \xrightarrow{\sim} T_{/\mathbf{B}\mathcal{G}}.$$

Proof. Recall from Lemma 2.33 that sending a groupoid $\mathcal{G}$ to the map $\mathcal{G}_0 \to \mathbf{B}\mathcal{G}$ defines an equivalence

$$\mathrm{Grpd}(T) \xrightarrow{\sim} \mathrm{EffEpi}(T)$$

between the categories of groupoid objects and effective epimorphisms in T. Moreover, it follows from descent that a morphism of groupoids $f\colon \mathcal{H} \to \mathcal{G}$ is cartesian if and only if the induced square

$$\begin{array}{ccc}
\mathcal{H}_0 & \longrightarrow & \mathbf{B}\mathcal{H} \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \mathbf{B}f} \\
\mathcal{G}_0 & \longrightarrow & \mathbf{B}\mathcal{G}
\end{array}$$

is a pullback square, giving an equivalence of categories $\mathrm{Grpd}(T)^{\mathrm{cart}} \xrightarrow{\sim} \mathrm{EffEpi}(T)^{\mathrm{cart}}$. Passing to slices over $\mathcal{G}$, this results in an equivalence

$$\mathrm{Act}_{\mathcal{G}}(T) \quad = \quad (\mathrm{Grpd}(T)^{\mathrm{cart}})_{/\mathcal{G}} \quad \xrightarrow{\sim} \quad (\mathrm{EffEpi}(T)^{\mathrm{cart}})_{/(\mathcal{G}_0 \to \mathbf{B}\mathcal{G})} \quad \xrightarrow{\sim} \quad T_{/\mathbf{B}\mathcal{G}}.$$

Here the last equivalence is induced by the target map. Restricted to cartesian morphisms over the fixed effective epimorphism $\mathcal{G}_0 \to \mathbf{B}\mathcal{G}$, this functor is an equivalence: its inverse sends a morphism $B \to \mathbf{B}\mathcal{G}$ to the pullback square

$$\begin{array}{ccc} B \times_{\mathbf{B}\mathcal{G}} \mathcal{G}_0 & \longrightarrow & \mathcal{G}_0 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ B & \longrightarrow & \mathbf{B}\mathcal{G}. \end{array}$$

□

6.2.2 Free and transitive actions

Given a $\mathcal{G}$-action in T, one can define when the action is *free* or *transitive*. Actions which are both free and transitive will correspond to principal $\mathcal{G}$-bundles.

Definition 6.31 (Shear maps)**.** Let $\mathcal{G} \ltimes X$ be an action of a groupoid object $\mathcal{G}$ on an object X in T. We define the *shear map* of $\mathcal{G} \ltimes X$ as the map

$$\mathrm{shear}_1 = (a, \mathrm{pr}_X) \colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X \times X,$$

where $a \colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X$ is the action map and pr_X is the projection onto X.

More generally, we define the *higher shear maps* as follows. The atlas $X \to X/\!\!/\mathcal{G}$ is an effective epimorphism, and the equivalence of Lemma 2.33 identifies the action groupoid $\mathcal{G} \ltimes X$ with its Čech nerve $\check{C}(X \to X/\!\!/\mathcal{G})$. Consider now the commutative square

$$\begin{array}{ccc} X & = & X \\ \downarrow\!\!\downarrow & & \downarrow \\ X/\!\!/\mathcal{G} & \longrightarrow & *. \end{array}$$

Regarding this as a morphism from $X \twoheadrightarrow X/\!\!/\mathcal{G}$ to $X \to *$ in $\mathrm{Ar}(T)$ and passing to Čech nerves, we obtain a morphism of groupoid objects

$$\mathrm{shear} \colon (\mathcal{G} \ltimes X) \to \check{C}(X \to *).$$

Under the identifications of Remark 6.28, this may be displayed as follows:

$$\begin{array}{ccccc} \mathcal{G} \ltimes X & = & \cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{G}_2 \times_{\mathcal{G}_0} X \;\substack{\longrightarrow\\ \longrightarrow}\; \mathcal{G}_1 \times_{\mathcal{G}_0} X \;\rightrightarrows\; X \\ \downarrow \mathrm{shear} & & \qquad\qquad \downarrow \mathrm{shear}_2 \qquad\quad \downarrow \mathrm{shear}_1 \qquad\quad \| \mathrm{shear}_0 \\ \check{C}(X \to *) & = & \cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X \times X \times X \;\substack{\longrightarrow\\ \longrightarrow}\; X \times X \;\rightrightarrows\; X. \end{array}$$

Definition 6.32 (Free, transitive and regular actions)**.** Let $\mathcal{G} \ltimes X$ be an action of a groupoid object $\mathcal{G}$ on an object $X \in T$. We say that this action is

(1) *Free* if its shear map shear_1 is a monomorphism;

(2) *Transitive* if its shear map shear_1 is an effective epimorphism;

(3) *Regular* if its shear map shear_1 is an isomorphism.

Note that $\mathcal{G} \ltimes X$ is regular if and only if it is both free and transitive.

In case the morphism $X \to *$ is an effective epimorphism in T, regularity of $\mathcal{G} \ltimes X$ can be expressed as the triviality of the quotient:

Lemma 6.33. *Let $X \in T$ be an object such that $X \to *$ is an effective epimorphism. Let $\mathcal{G} \ltimes X$ be an action of a groupoid object on X. Then the action is regular if and only if the quotient $X /\!\!/ \mathcal{G}$ is the terminal object of T.*

Proof. First assume that $X /\!\!/ \mathcal{G} \simeq *$. Then the map $X \twoheadrightarrow X /\!\!/ \mathcal{G}$ is equivalent to the map $X \to *$. It follows that the shear map $\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to *)$ is an isomorphism, being defined by applying Čech nerves to these two equivalent morphisms.

Conversely, assume that $\mathcal{G} \ltimes X$ is regular and that $X \twoheadrightarrow *$ is an effective epimorphism. The map $X /\!\!/ \mathcal{G} \to *$ is obtained from the higher shear map $\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to *)$ by passing to colimits. It thus suffices to show that each higher shear map $\mathrm{shear}_n\colon \mathcal{G}_n \times_{\mathcal{G}_0} X \to X^{n+1}$ is an isomorphism. Both its source and target are groupoid objects with object of objects X. By the Segal condition, their terms in degree n are the n-fold iterated fiber products of their terms in degree 1 over X, and shear_n is the corresponding iterated pullback of shear_1. Since shear_1 is an isomorphism by regularity, every shear_n is an isomorphism. □

6.2.3 Principal bundles

Given a groupoid object $\mathcal{G}$ in T and an object $B \in T$, we obtain a notion of *principal $\mathcal{G}$-bundles over B* by applying the definition of regular actions to the slice topos $T_{/B}$. Observe that $\mathcal{G}$ gives rise to a groupoid $\mathcal{G} \times B$ in the slice $T_{/B}$ by applying the pullback functor $B \times -\colon T \to T_{/B}$. For every object $P \in T_{/B}$, there is an isomorphism

$$(\mathcal{G}_n \times B) \times_{\mathcal{G}_0 \times B} P \;\simeq\; \mathcal{G}_n \times_{\mathcal{G}_0} P \qquad \text{in } T_{/B},$$

and thus a $(\mathcal{G} \times B)$-action on P is the same as a $\mathcal{G}$-action on the underlying object of P for which the map $P \to B$ is invariant. For this reason, we will refer to a $(\mathcal{G} \times B)$-action in the slice $T_{/B}$ simply as a *$\mathcal{G}$-action over B*.

Definition 6.34 (Principal bundles)**.** Let $B \in T$ be an object and let $\mathcal{G}$ be a groupoid object in T.

(1) Let $p\colon P \to B$ be an object of $T_{/B}$ equipped with a $\mathcal{G}$-action over B. We say that p is a *formally principal $\mathcal{G}$-bundle over B* if this action is regular as a $(\mathcal{G} \times B)$-action in the slice $T_{/B}$, i.e. if the shear map

$$\mathrm{shear}_1\colon \mathcal{G}_1 \times_{\mathcal{G}_0} P \xrightarrow{\sim} P \times_B P$$

is an isomorphism.

(2) A formally principal $\mathcal{G}$-bundle $p\colon P \to B$ is called a *principal $\mathcal{G}$-bundle* if p is an effective epimorphism.

(3) We write

$$\mathrm{Bun}_{\mathcal{G}}(B) \quad \subseteq \quad \mathrm{Act}_{\mathcal{G}\times B}(T_{/B})$$

for the full subcategory of principal $\mathcal{G}$-bundles over B.

Remark 6.35. For any morphism $f\colon B' \to B$ in T, the pullback functor $f^*\colon T_{/B} \to T_{/B'}$ preserves principal $\mathcal{G}$-bundles.

Lemma 6.36. *Let $p\colon P \twoheadrightarrow B$ be an effective epimorphism equipped with a $\mathcal{G}$-action over B, and let $P/\!\!/\mathcal{G} \in T_{/B}$ denote the quotient. Then p is a principal $\mathcal{G}$-bundle if and only if the canonical map $P/\!\!/\mathcal{G} \to B$ is an isomorphism.*

Proof. This is an instance of Lemma 6.33 applied in the slice topos $T_{/B}$. □

It follows from Lemma 6.36 that the forgetful functor from principal $\mathcal{G}$-bundles over B to $\mathcal{G}$-actions in T lands in the fiber over B of the composite

$$\mathrm{Act}_{\mathcal{G}}(T) \xrightarrow{-/\!\!/\mathcal{G}} T_{/\mathbf{B}\mathcal{G}} \longrightarrow T.$$

Proposition 6.37. *The forgetful functor induces an equivalence of categories*

$$\mathrm{Bun}_{\mathcal{G}}(B) \xrightarrow{\sim} \mathrm{Act}_{\mathcal{G}}(T) \times_T \{B\}.$$

Proof. We construct an explicit inverse. Consider a $\mathcal{G}$-action $c\colon \mathcal{G} \ltimes P \to \mathcal{G}$ in T equipped with an isomorphism $B \xrightarrow{\sim} P/\!\!/\mathcal{G}$. The colimit cocone exhibiting B as the realization of $\mathcal{G} \ltimes P$ supplies compatible maps $(\mathcal{G} \ltimes P)_n \to B$ and hence lifts the action groupoid to a groupoid object in $T_{/B}$. Pairing these structure maps with c gives a morphism of groupoid objects

$$\mathcal{G} \ltimes P \longrightarrow \mathcal{G} \times B$$

in $T_{/B}$. Its underlying morphism in T is c, so it is cartesian and therefore defines a $(\mathcal{G} \times B)$-action in the slice. This construction defines a functor

$$\mathrm{Act}_{\mathcal{G}}(T) \times_T \{B\} \to \mathrm{Act}_{\mathcal{G}\times B}(T_{/B}).$$

The map $P \to B$ is an effective epimorphism because it is the atlas of the realization of the groupoid object $\mathcal{G} \ltimes P$, and its quotient in $T_{/B}$ is the terminal object $B \to B$. Thus Lemma 6.36 shows that the resulting action is principal. Conversely, starting from a principal bundle, the same lemma identifies its quotient with B, and the colimit cocone recovers its original structure maps to B. Hence the two constructions are inverse. □

We are now ready to state the main classification result:

Theorem 6.38 (Classification of principal bundles)**.** *Let $\mathcal{G}$ be a groupoid object in T and let $B \in T$ be an object. There is an equivalence of categories*

$$\mathrm{Hom}_T(B, \mathbf{B}\mathcal{G}) \xrightarrow{\sim} \mathrm{Bun}_{\mathcal{G}}(B),$$

given on objects by sending a morphism $c\colon B \to \mathbf{B}\mathcal{G}$ to the principal $\mathcal{G}$-bundle $P \to B$ defined by the pullback square

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{G}_0 \\ \downarrow & & \downarrow \\ B & \xrightarrow{c} & \mathbf{B}\mathcal{G}. \end{array}$$

Proof. Recall from Proposition 6.30 the equivalence $-/\!\!/\mathcal{G}\colon \mathrm{Act}_{\mathcal{G}}(T) \xrightarrow{\sim} T_{/\mathbf{B}\mathcal{G}}$, which induces an equivalence on fibers over T:

$$\mathrm{Act}_{\mathcal{G}}(T) \times_T \{B\} \;\simeq\; T_{/\mathbf{B}\mathcal{G}} \times_T \{B\} \;\simeq\; \mathrm{Hom}_T(B, \mathbf{B}\mathcal{G}).$$

The inverse sends a morphism $c\colon B \to \mathbf{B}\mathcal{G}$ to the base change $P = B \times_{\mathbf{B}\mathcal{G}} \mathcal{G}_0 \twoheadrightarrow B$ of the atlas $\mathcal{G}_0 \twoheadrightarrow \mathbf{B}\mathcal{G}$ along c. Combining with the equivalence from Proposition 6.37 finishes the proof. □

Corollary 6.39. *For every groupoid object $\mathcal{G}$ in T and every object $B \in T$, the category $\mathrm{Bun}_{\mathcal{G}}(B)$ is an anima.* □

6.3 Hypercovers

In Section 3.4, we defined a topos T to be *hypercomplete* if every ∞-connected morphism is an isomorphism, and showed that the ∞-connected and ∞-truncated morphisms form a factorization system. In this section, we give an alternative characterization of hypercompleteness in terms of *hypercovers*.

Recall that an effective epimorphism $f\colon X \twoheadrightarrow Y$ in a topos T may be thought of as a *cover* of Y. The Čech nerve $\check{C}(f)$ encodes all the iterated self-intersections $X^{\times^n_Y}$ of X over Y, and since f is an effective epimorphism this expresses Y as a colimit of these iterated self-intersections.

A *hypercover* allows these intersections to be refined coherently in every simplicial degree. It begins with an effective epimorphism $U_0 \twoheadrightarrow Y$; the object U_1 covers the pairwise intersections, while in degree n the object U_n covers the object of compatible boundary data determined by the preceding degrees. We will show that the realization of any hypercover of Y maps ∞-connectedly to Y. Thus every hypercover is effective in a hypercomplete topos, and the converse holds as well:

Theorem 6.40 ([Lur09, Theorem 6.5.3.12])**.** *Let T be a topos. The following conditions are equivalent:*

(1) *The topos T is hypercomplete.*

(2) *For every object $X \in T$, every hypercover of X is effective.*

We start by defining hypercovers and establishing their basic formal properties. We then prove the connectivity statement which relates their effectiveness to hypercompleteness.

6.3.1 Hypercovers and matching objects

Notation 6.41. For each natural number $n \geq 0$, let $\mathbf{\Delta}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the objects $\{[0], [1], \ldots, [n]\}$, and set $\mathbf{\Delta}^{\leq -1} := \emptyset$. If T is a topos, the restriction functor

$$(-)|_{\mathbf{\Delta}^{\leq n}}\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T) \to \mathrm{Fun}((\mathbf{\Delta}^{\leq n})^{\mathrm{op}}, T)$$

admits a right adjoint given by right Kan extension along the inclusion $(\mathbf{\Delta}^{\leq n})^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$. We let

$$\mathrm{cosk}_n\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T)$$

denote the composition of the restriction functor with its right adjoint and refer to cosk_n as the *n-coskeleton functor*. In particular, cosk_{-1} is the constant simplicial object at the terminal object of T. A simplicial object $U_\bullet$ is called *n-coskeletal* if the unit map $U_\bullet \to \mathrm{cosk}_n(U_\bullet)$ is an isomorphism, or equivalently if $U_\bullet$ is a right Kan extension of its restriction to $(\mathbf{\Delta}^{\leq n})^{\mathrm{op}}$.

Definition 6.42 (Hypercover)**.** Let T be a topos. A simplicial object $U_\bullet \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T)$ is called a *hypercover in T* if, for each $n \geq 0$, the canonical map

$$U_n \to (\mathrm{cosk}_{n-1} U_\bullet)_n$$

is an effective epimorphism. We say that $U_\bullet$ is an *effective hypercover* if the colimit of $U_\bullet$ is a terminal object of T.

For an object $X \in T$, a *hypercover of X* is a hypercover in the slice topos $T_{/X}$.

Remark 6.43. The object $(\mathrm{cosk}_{n-1} U_\bullet)_n$ is known as the *n-th matching object* of $U_\bullet$. By the pointwise formula for right Kan extensions, it may be computed as a limit over the category $(\mathbf{\Delta}^{\leq n-1})_{/[n]}$. Since this category is finite, the object $(\mathrm{cosk}_{n-1} U_\bullet)_n$ is a finite limit of the objects $U_0, \ldots, U_{n-1}$.

Let us spell out the hypercover condition for small n:

- For $n = 0$, the (-1)-coskeleton of any simplicial object is constant at the terminal object, so $(\mathrm{cosk}_{-1} U_\bullet)_0 = *$. The condition thus says that $U_0 \to *$ is an effective epimorphism.
- For $n = 1$, the 0-coskeleton is determined by U_0, and one computes $(\mathrm{cosk}_0 U_\bullet)_1 = U_0 \times U_0$. The condition thus says that the map $(d_1, d_0) \colon U_1 \to U_0 \times U_0$ is an effective epimorphism.
- For $n = 2$, the matching object is the finite limit parametrizing compatible triples of edges which form the boundary of a 2-simplex. More explicitly, it can be written as

  $$(\mathrm{cosk}_1 U_\bullet)_2 \xrightarrow{\sim} \big(U_1 \underset{d_0, U_0, d_1}{\times} U_1\big) \underset{U_0 \times U_0}{\times} U_1,$$

  where the first factor maps to $U_0 \times U_0$ by $(d_1 \,\mathrm{pr}_1, d_0 \,\mathrm{pr}_2)$ and the second by (d_1, d_0). The hypercover condition says that U_2 covers this object of compatible boundary data.

Example 6.44. Let $f \colon X \twoheadrightarrow Y$ be an effective epimorphism in T. Then the Čech nerve $\check{C}(f)$, viewed as a simplicial object in $T_{/Y}$, is a hypercover of Y. Its matching map in degree zero is f, hence is an effective epimorphism. By Lemma 2.20, the simplicial object $\check{C}(f)$ is 0-coskeletal, so its matching maps in every positive degree are isomorphisms.

In preparation of the proof of Theorem 6.40, we establish various basic properties of hypercovers.

Lemma 6.45. *Let $\varphi^* \colon S \to T$ be a morphism of logoi and let $U_\bullet$ be a hypercover in S. Then $\varphi^*(U_\bullet)$ is a hypercover in T.*

Proof. Since φ^* is left exact and the matching object $(\mathrm{cosk}_{n-1} U_\bullet)_n$ is a finite limit (Remark 6.43), we have $\varphi^*((\mathrm{cosk}_{n-1} U_\bullet)_n) \simeq (\mathrm{cosk}_{n-1} \varphi^*(U_\bullet))_n$. Since φ^* preserves effective epimorphisms, the result follows. □

Corollary 6.46. *Let T be a topos with hypercompletion $L\colon T \to T^{\mathrm{hyp}}$. A simplicial object $U_\bullet$ is a hypercover in T if and only if $L(U_\bullet)$ is a hypercover in T^{hyp}.*

Proof. The "only if" direction is Lemma 6.45. Conversely, suppose that $L(U_\bullet)$ is a hypercover. Since L is left exact, it preserves the matching objects, so it sends every matching map

$$f_n\colon U_n \longrightarrow (\mathrm{cosk}_{n-1} U_\bullet)_n$$

to an effective epimorphism. Factor f_n as an effective epimorphism followed by its image monomorphism m_n. The functor L preserves this factorization. Since $L(f_n)$ is an effective epimorphism, right cancellation shows that $L(m_n)$ is an effective epimorphism; being monic, it is an isomorphism. Hence m_n belongs to the kernel of hypercompletion and is therefore ∞-connected. Since it is also a monomorphism, it is an isomorphism. Thus f_n is an effective epimorphism for every n, proving that $U_\bullet$ is a hypercover. □

Lemma 6.47. *Let $U \in T$ be an ∞-connected object. Then the constant simplicial object with value U is a hypercover in T.*

Proof. By Corollary 6.46, we may assume that T is hypercomplete. Then $U \simeq *$, so the constant simplicial object with value U is the terminal object of $\mathrm{Fun}(\Delta^{\mathrm{op}}, T)$. Since the coskeleton functors preserve limits, the matching maps are all isomorphisms. □

6.3.2 Effectiveness and hypercompleteness

The key point is that an arbitrary hypercover can be approximated by coskeletal ones. The latter are effective, and comparison with these approximations shows that the realization of any hypercover is ∞-connected.

Lemma 6.48 ([Lur09, Lemma 6.5.3.9])**.** *Let T be a topos and let $U_\bullet$ be an n-coskeletal hypercover in T. Then $U_\bullet$ is effective.*

Proof sketch. The details of the proof are intricate; we refer to [Lur09, Lemma 6.5.3.9] for a detailed argument. We only sketch the idea of Lurie's proof.

We proceed by induction on n. If $n = 0$, then $U_\bullet$ can be identified with (the underlying groupoid object of) the Čech nerve of the map $U_0 \to *$. Since $U_\bullet$ is a hypercover, this map is an effective epimorphism, so the Čech nerve is a colimit diagram and $|U_\bullet| \simeq *$.

Now assume $n > 0$ and let $V_\bullet = \mathrm{cosk}_{n-1} U_\bullet$. This is an $(n-1)$-coskeletal hypercover, so $|V_\bullet| \simeq *$ by induction. The unit $f_\bullet\colon U_\bullet \to V_\bullet$ is an isomorphism below degree n. Moreover, each f_m is a finite composite of pullbacks of the matching map f_n, and hence is an effective epimorphism.

Form the degreewise Čech nerve W^+ of $f_\bullet$, regarded as an augmented bisimplicial object. Realizing in the Čech direction gives $V_\bullet$, and then realizing in the other direction gives $|V_\bullet| \simeq *$. By cofinality of the diagonal, the diagonal simplicial object $D_\bullet$ of the underlying bisimplicial object therefore has terminal realization. Lurie constructs a retract of the underlying semisimplicial object of $D_\bullet$ onto that of $U_\bullet$. Forgetting degeneracies does not change geometric realizations, so $|U_\bullet|$ is a retract of $|D_\bullet| \simeq *$ and is therefore terminal. □

Lemma 6.49 ([Lur09, Lemma 6.5.3.10])**.** *Let $f_\bullet \colon U_\bullet \to V_\bullet$ be a morphism of simplicial objects in T. Suppose that $f_k \colon U_k \to V_k$ is an isomorphism for all $k \leq n$. Then the induced map $|f_\bullet| \colon |U_\bullet| \to |V_\bullet|$ is n-connected.*

Proof. This is [Lur09, Lemma 6.5.3.10]. Lurie reduces first to animae and then proves the claim using projectively cofibrant simplicial Kan-complex models. □

Lemma 6.50. *Let T be a topos and let $U_\bullet$ be a hypercover in T. Then the canonical map $|U_\bullet| \to *$ is ∞-connected.*

Proof. We show that $|U_\bullet|$ is n-connected for every $n \geq 0$. Let $V_\bullet = \mathrm{cosk}_{n+1} U_\bullet$ and let $u_\bullet \colon U_\bullet \to V_\bullet$ be the unit map. The matching maps of $V_\bullet$ agree with those of $U_\bullet$ through degree $n+1$ and are isomorphisms thereafter, so $V_\bullet$ is again a hypercover. Moreover, u_m is an isomorphism for $m \leq n+1$. By Lemma 6.49, the induced map $|U_\bullet| \to |V_\bullet|$ is n-connected. By Lemma 6.48, $|V_\bullet| \simeq *$. It follows that $|U_\bullet|$ is n-connected. □

We are now ready to prove the main result of this section.

Proof of Theorem 6.40. (1) $\Rightarrow$ (2): Let $U_\bullet$ be a hypercover in $T_{/X}$, or equivalently a hypercover of X. By Lemma 6.50, the canonical map $|U_\bullet| \to X$ is ∞-connected. Since T is hypercomplete, so is $T_{/X}$, hence this map is an isomorphism. Thus $U_\bullet$ is effective.

(2) $\Rightarrow$ (1): Let $f \colon U \to X$ be an ∞-connected morphism in T. Consider f as an object of $T_{/X}$ and let $f_\bullet$ be the constant simplicial object with value f. By Lemma 6.47, $f_\bullet$ is a hypercover in $T_{/X}$. By assumption, $f_\bullet$ is effective, so its realization is the terminal object id_X of $T_{/X}$. On the other hand, the realization of a constant simplicial object is its constant value, so $|f_\bullet| \xrightarrow{\sim} f$. Hence f is an isomorphism, showing that T is hypercomplete. □

Remark 6.51. Theorem 6.40 is often useful in practice: to show that a topos T is hypercomplete, it suffices to verify that all hypercovers are effective. For sheaf topoi $\mathrm{Shv}(X)$ on a topological space X, one may work with open hypercovers when these are sufficiently plentiful. Ordinary open covers alone do not detect hypercompleteness, since their Čech nerves are effective in every topos. The dimension criteria of Section 6.5 provide more concrete sufficient conditions.

6.4 Localic topoi, Postnikov-completeness, and boundedness

Given an anima X, we may form the truncations $\tau_n X \in \mathrm{An}_{\leq n}$ for every $n \geq 0$. These truncations fit into the *Postnikov tower*

$$X \to \cdots \to \tau_{n+1} X \to \tau_n X \to \cdots \to \tau_0 X.$$

Moreover, every anima is canonically the limit of its Postnikov tower; in fact, the functor $\mathrm{An} \to \lim_n \mathrm{An}_{\leq n}$ sending X to the system $(\tau_n X)_n$ is an equivalence of categories.

The goal of this section is to study a closely related phenomenon in the world of topoi. Recall that every anima X gives rise to the slice topos $\mathrm{An}_{/X}$. We saw in Proposition 4.38 that the assignment $X \mapsto \mathrm{An}_{/X}$ defines a fully faithful functor

$$\mathrm{An} \hookrightarrow \mathrm{Topos}.$$

As we will see in this section, there are subcategories $\mathrm{Topos}_n \subseteq \mathrm{Topos}$ of 'n-localic topoi' for all n, and left adjoints $L_n\colon \mathrm{Topos} \to \mathrm{Topos}_n$. In the toposic situation, the comparison functor $\mathrm{Topos} \to \lim_n \mathrm{Topos}_n$ will no longer be an equivalence. Instead, it admits fully faithful left and right adjoints:

- The image of the fully faithful left adjoint $\lim_n \mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$ is given by the *Postnikov-complete* topoi.
- The image of the fully faithful right adjoint $\lim_n \mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$ is given by the *bounded* topoi.

We now properly define these notions. A great reference for this material is [BGH18, Section 3.2].

6.4.1 Localic topoi

The n-localic reflection of a topos retains precisely its $(n-1)$-truncated objects. Thus localic reflection plays, for topoi, the role played by truncation for animae. We first characterize the topoi which are already determined by this truncated information.

Definition 6.52. Let $n \geq -1$. A topos T is called *n-localic* if for every topos S the induced functor

$$\mathrm{Geom}(S,T) \longrightarrow \mathrm{Geom}(S_{\leq n-1}, T_{\leq n-1}) := \mathrm{Fun}^{\mathrm{lex},\mathrm{colim}}(T_{\leq n-1}, S_{\leq n-1})$$

is an equivalence.

Notation 6.53. Let $\mathrm{Topos}_n \subseteq \mathrm{Topos}$ denote the full subcategory spanned by the n-localic topoi.

Proposition 6.54. (1) *For any topos T, there exists an $(n,1)$-category C with finite limits and a Grothendieck topology on C such that $T_{\leq n-1} \simeq \mathrm{Shv}_\tau(C)_{\leq n-1}$.*

(2) *The inclusion $\mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$ admits a left adjoint $L_n\colon \mathrm{Topos} \to \mathrm{Topos}_n$ such that $L_nT = \mathrm{Shv}_\tau(C)$ for all (C,τ) as in (1).*

Proof. The category $T_{\leq n-1}$ is an n-topos. The presentation theorem for n-topoi provides a small $(n,1)$-category C with finite limits and a Grothendieck topology τ for which

$$T_{\leq n-1} \xrightarrow{\sim} \mathrm{Shv}_\tau(C)_{\leq n-1}.$$

One may obtain such a site by choosing a sufficiently large regular cardinal κ, taking a small finite-limit-closed subcategory of κ-compact objects which generates $T_{\leq n-1}$ as an n-topos, and equipping it with the effective epimorphism topology. This proves (1); see [Lur09, Theorem 6.4.1.5 and Proposition 6.4.5.9] for the presentation theorem and its localic refinement.

Set $L_nT := \mathrm{Shv}_\tau(C)$. By [Lur09, Proposition 6.4.5.9], this topos is n-localic. Its $(n-1)$-truncated objects are identified with those of T, independently of the chosen presentation. If S is any n-localic topos, then the defining property of n-localic topoi gives natural equivalences

$$\begin{aligned}\mathrm{Geom}(L_nT, S) &\xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex},\mathrm{colim}}(S_{\leq n-1}, (L_nT)_{\leq n-1}) \\ &\xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex},\mathrm{colim}}(S_{\leq n-1}, T_{\leq n-1}) \\ &\xrightarrow{\sim} \mathrm{Geom}(T, S).\end{aligned}$$

Thus L_n is left adjoint to the inclusion $\mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$, proving (2). □

Example 6.55. For $T = \mathrm{An}_{/X}$ we have $L_n(\mathrm{An}_{/X}) = \mathrm{An}_{/\tau_n X}$, providing a commutative diagram as follows:

$$\begin{array}{ccc} \mathrm{An} & \xrightarrow{\tau_n} & \mathrm{An}_{\leq n} \\ {\scriptstyle X\mapsto \mathrm{An}_{/X}}\downarrow & & \downarrow{\scriptstyle X\mapsto \mathrm{An}_{/X}} \\ \mathrm{Topos} & \xrightarrow{L_n} & \mathrm{Topos}_n. \end{array}$$

So this is the sense in which L_n "generalizes" the truncation functor $\tau_n \colon \mathrm{An} \to \mathrm{An}_{\leq n}$.

Warning 6.56. The finite-limit hypothesis in the site presentation is essential. If (C, τ) is an $(n, 1)$-site whose underlying category does not have finite limits, then $\mathrm{Shv}_\tau(C)$ need not be N-localic for any N. For example, [Lur18a, Counterexample 20.4.0.1] constructs a basis C for the topology of the Hilbert cube

$$Q := \prod_{k\in\mathbb{N}} [0, 1]$$

such that the topos of sheaves on C for the induced topology is not N-localic for any N. Its hypercompletion nevertheless agrees with the hypercompletion of the 0-localic topos $\mathrm{Shv}(Q)$.

Despite the previous warning, the presheaf category of an $(n, 1)$-category is still n-localic *after hypercompletion*. The proof requires the following auxiliary result:

Lemma 6.57. *Let T be a topos and let $u \colon C \to T$ be a functor from a small category. Consider the following four statements:*

(1) *The restriction functor*

$$u^* \colon T \to \mathrm{PSh}(C), \qquad X \mapsto \mathrm{Hom}_T(u(-), X)$$

is fully faithful.

(2) *T is generated under colimits by the image of u.*

(3) *Every $X \in T$ is covered by the image of u, i.e. there exist objects $c_i \in C$ and an effective epimorphism $\bigsqcup_i u(c_i) \twoheadrightarrow X$.*

(4) *The counit $u_! u^*(X) \to X$ is ∞-connected for every $X \in T$.*

Then we have implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, if T is hypercomplete then all four statements are equivalent.

Proof. To see that (1) implies (2), let $u_! \colon \mathrm{PSh}(C) \to T$ be the left Kan extension of u. Since u^* is fully faithful, the counit $u_! u^* \to \mathrm{id}_T$ is an equivalence. As every presheaf is a colimit of representables and $u_!(y(c)) = u(c)$, we conclude that every object of T is a colimit of objects in the image of u.

For (2) $\Rightarrow$ (3), observe that the collection of objects receiving such an effective epimorphism is closed under colimits.

For (3) $\Rightarrow$ (4), the pointwise formula for left Kan extensions gives

$$u_! u^*(X) \simeq \mathrm{colim}_{(c,\alpha)\in C_{/X}} u(c),$$

where $C_{/X}$ is the category of pairs (c, α) with $c \in C$ and $\alpha\colon u(c) \to X$. Write the counit as

$$X' := \mathrm{colim}_{(c,\alpha)\in C_{/X}}\, u(c) \to X$$

induced by the maps α. The chosen cover of X by objects in the image of u factors through this counit, so the counit is an effective epimorphism.

We now prove inductively that it is n-connected for every $n \geq 0$. Assume that the counit is $(n-1)$-connected for every object of T. Universality of colimits identifies

$$X' \times_X X' \xrightarrow{\sim} \mathrm{colim}_{(c,\alpha),(d,\beta)\in C_{/X}}\, u(c) \times_X u(d).$$

Under this identification, descent for the colimit defining X' identifies the diagonal $X' \to X' \times_X X'$ with the colimit of the counits

$$\mathrm{colim}_{(e,\gamma)\in C_{/u(c)\times_X u(d)}}\, u(e) \longrightarrow u(c) \times_X u(d)$$

as (c, α) and (d, β) vary. Each of these maps is $(n-1)$-connected by the induction hypothesis. Since $(n-1)$-connected morphisms are closed under colimits, the diagonal is $(n-1)$-connected. Together with effective epimorphy, Theorem 3.22 shows that the counit is n-connected. Thus it is ∞-connected.

Finally, if T is hypercomplete and (4) is satisfied, then the counit map $u_! u^* \to \mathrm{id}$ is a natural isomorphism, and so u^* is fully faithful, giving (1). □

Lemma 6.58. *Let $n \geq -1$, and let T be a topos such that every object is covered by $(n-1)$-truncated objects. Then the map of topoi $T \to L_n(T)$ induces an equivalence on hypercompletions:*

$$T^{\mathrm{hyp}} \xrightarrow{\sim} (L_n T)^{\mathrm{hyp}}.$$

Proof. By Proposition 6.54, there is an $(n,1)$-category C with finite limits and a Grothendieck topology τ such that $L_n T \simeq \mathrm{Shv}_\tau(C)$. Using the hypothesis on T, we may choose $C \subseteq T_{\leq n-1}$ so that every object of T is covered by objects of C. Let $u\colon C \hookrightarrow T$ denote the inclusion. Since u preserves finite limits, its colimit extension $u_!\colon \mathrm{PSh}(C) \to T$ is left exact by Proposition 2.43. By Lemma 6.57, the counit of $u_! \dashv u^*$ is ∞-connected. It follows that the composite

$$\mathrm{PSh}(C) \xrightarrow{u_!} T \twoheadrightarrow T^{\mathrm{hyp}}$$

is a left exact localization.

Let K be its kernel. We claim that the monogenic part of K is τ^c. By Proposition 6.9, it suffices to test monomorphisms $R \hookrightarrow y(X)$ which are sieves on representables. Such a sieve is inverted by the displayed localization if and only if $u_!(R) \to u(X)$ becomes an isomorphism after hypercompletion. Since this map is a monomorphism, this happens if and only if it is already an isomorphism in T, or equivalently if $u_!(R) \to u(X)$ is an effective epimorphism. By the definition of the effective epimorphism topology τ, this is precisely the condition that R be a covering sieve. Hence $K^{\mathrm{mono}} = \tau^c$.

The congruence K is hypercomplete because its quotient is T^{hyp}. By Lemma 5.67, a hypercomplete congruence is determined by its monogenic part, so

$$K = (K^{\mathrm{mono}})^{\mathrm{hyp}} = (\tau^c)^{\mathrm{hyp}}.$$

But $(\tau^c)^{\mathrm{hyp}}$ is also the kernel of the localization $\mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C)^{\mathrm{hyp}} = (L_n T)^{\mathrm{hyp}}$. The two hypercomplete localizations therefore have the same kernel, which proves the asserted equivalence. □

6.4.2 Postnikov-completeness

Localic reflection records the truncated layers of a topos. Postnikov-completion asks instead whether compatible truncated objects can be assembled into an actual object. This is stronger than convergence of the Postnikov tower of each object: convergence gives full faithfulness of the comparison below, while Postnikov-completeness also requires essential surjectivity.

Definition 6.59. The *Postnikov-completion* of a presentable category C is defined as the limit

$$\widehat{C} \quad := \quad \lim_n C_{\leq n}.$$

Here the transition functor $C_{\leq n+1} \to C_{\leq n}$ is τ_n, and the limit is formed in presentable categories and colimit-preserving functors. We say that C is *Postnikov-complete* if the canonical functor $\tau_* \colon C \to \widehat{C}, X \mapsto (\tau_n X)_n$ is an equivalence.

Remark 6.60. The functor τ_* admits a right adjoint $\lim_n C_{\leq n} \to C$ given by sending a system (X_n) of n-truncated objects to their limit $X = \lim_n X_n$. We see:

- τ_* is fully faithful if and only if the unit map $X \to \lim_n \tau_n X$ is an isomorphism for all $X \in C$, i.e. every object is the limit of its Postnikov tower. Note that this is necessary but not sufficient for τ_* to be an equivalence.
- τ_* is essentially surjective if and only if every system (X_n) satisfying the compatibility conditions arises as $(\tau_n X)$ for some $X \in C$.

Proposition 6.61 (cf. [Lur18a, Corollary A.7.2.8])**.** *If T is a topos, then $\widehat{T}$ is a topos. For any Postnikov-complete topos S, the morphism of topoi $\widehat{T} \to T$ induces an equivalence*

$$\mathrm{Geom}(S, \widehat{T}) \xrightarrow{\sim} \mathrm{Geom}(S, T).$$

Proof. The category $\widehat{T}$ is presentable because it is a limit of presentable categories along accessible colimit-preserving functors, and its colimits are computed levelwise. We verify descent. For $X = (X_n)_n \in \widehat{T}$, slicing commutes with limits of categories and gives

$$\widehat{T}_{/X} \xrightarrow{\sim} \lim_n (T_{\leq n})_{/X_n}.$$

Each $T_{\leq n}$ is an n-topos, so its slice functor sends colimits to limits. Consequently, if $X = \mathrm{colim}_i X_i$ in $\widehat{T}$, then

$$\begin{aligned} \widehat{T}_{/X} &\xrightarrow{\sim} \lim_n (T_{\leq n})_{/\mathrm{colim}_i X_{i,n}} \\ &\xrightarrow{\sim} \lim_n \lim_i (T_{\leq n})_{/X_{i,n}} \\ &\xrightarrow{\sim} \lim_i \widehat{T}_{/X_i}. \end{aligned}$$

Thus every colimit in $\widehat{T}$ is van Kampen, and $\widehat{T}$ is a topos by Theorem 2.42.

Projection onto the n-th component identifies $(\widehat{T})_{\leq n}$ with $T_{\leq n}$. Now let S be Postnikov-complete. A logos morphism into $S \xrightarrow{\sim} \lim_n S_{\leq n}$ is the same as a compatible collection of left exact colimit-preserving functors into the $S_{\leq n}$. Moreover, a left exact colimit-preserving functor with n-truncated

target factors uniquely through n-truncation. We therefore obtain natural equivalences

$$\begin{aligned}\mathrm{Fun}_{\mathrm{Logos}}(\widehat{T}, S) &\xrightarrow{\sim} \lim_n \mathrm{Fun}^{\mathrm{lex,colim}}((\widehat{T})_{\leq n}, S_{\leq n}) \\ &\xrightarrow{\sim} \lim_n \mathrm{Fun}^{\mathrm{lex,colim}}(T_{\leq n}, S_{\leq n}) \\ &\xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Logos}}(T, S).\end{aligned}$$

Passing to the opposite category of topoi gives the asserted universal property. □

Lemma 6.62. *Every Postnikov-complete topos is hypercomplete.*

Proof. Let $f : X \to Y$ be an ∞-connected map in T. Since T is Postnikov-complete, the map f is the limit of the maps $\tau_n X \to \tau_n Y$, so it suffices to show that each of these maps is an isomorphism. The maps $X \to \tau_n X$ and $Y \to \tau_n Y$ are n-connected, and so is $f : X \to Y$. By right cancellation, it follows that $\tau_n X \to \tau_n Y$ is also n-connected. On the other hand, since $\tau_n X$ and $\tau_n Y$ are n-truncated, the maps $\tau_n X \to *$ and $\tau_n Y \to *$ are n-truncated. By left cancellation, it follows that $\tau_n X \to \tau_n Y$ is n-truncated as well. Thus $\tau_n X \to \tau_n Y$ is both n-connected and n-truncated, hence an isomorphism. □

Notation 6.63. We denote by $\mathrm{Topos}^{\mathrm{post}} \subseteq \mathrm{Topos}^{\mathrm{hyp}} \subseteq \mathrm{Topos}$ the full subcategories of Postnikov-complete and hypercomplete topoi. These two inclusions have right adjoints sending T to $T^{\mathrm{hyp}} = T_{\leq \infty}$ and $T \mapsto \widehat{T}$.

6.4.3 Bounded topoi

Postnikov-completion is one reconstruction of a topos from its truncated layers. Bounded topoi arise from the opposite reconstruction: one first takes every localic reflection and then forms their inverse limit in Topos.

Definition 6.64. A topos T is called *bounded* if the canonical map $T \to \lim_n L_n T$ is an equivalence. We write $\mathrm{Topos}^b \subseteq \mathrm{Topos}$ for the full subcategory of bounded topoi.

Lemma 6.65 ([Lur18a, Lemma A.7.1.4])**.** *The functor $L_n : \mathrm{Topos} \to \mathrm{Topos}_n$ preserves colimits and small limits.*

Proof. Preservation of colimits follows because L_n is a left adjoint. The limit-preservation statement is [Lur18a, Lemma A.7.1.4]; it is the additional input needed to commute localic reflection with the inverse tower of localic approximations. □

Proposition 6.66 (cf. [Lur18a, Proposition A.7.1.5 and Corollary A.7.2.8])**.** *The functor $L_* : \mathrm{Topos} \to \lim_n \mathrm{Topos}_n$ admits fully faithful adjoints on both sides.*

- *A topos T lies in the image of the left adjoint if and only if it is Postnikov complete.*
- *A topos T lies in the image of the right adjoint if and only if it is bounded.*

Proof. The right adjoint sends a compatible system $(T_n)_n$ to $\lim_n T_n$. By Lemma 6.65,

$$L_m\big(\lim_n T_n\big) \xrightarrow{\sim} \lim_n L_m T_n \xrightarrow{\sim} T_m.$$

Thus the right adjoint is fully faithful, and its essential image consists precisely of the bounded topoi. In particular, the bounded reflection of an arbitrary topos is

$$T^b := \lim_n L_n T.$$

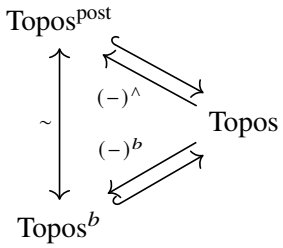
We next identify the left adjoint. For arbitrary topoi T and S, the universal properties of truncation and localic reflection give

$$\begin{aligned} \mathrm{Geom}(\widehat{T}, S) &\xrightarrow{\sim} \lim_n \mathrm{Fun}^{\mathrm{lex,colim}}(S_{\leq n}, T_{\leq n}), \\ \mathrm{Geom}(T, S^b) &\xrightarrow{\sim} \lim_n \mathrm{Geom}(T, L_n S) \\ &\xrightarrow{\sim} \lim_n \mathrm{Fun}^{\mathrm{lex,colim}}(S_{\leq n-1}, T_{\leq n-1}). \end{aligned}$$

Reindexing gives a natural equivalence $\mathrm{Geom}(\widehat{T}, S) \xrightarrow{\sim} \mathrm{Geom}(T, S^b)$. Thus Postnikov-completion is left adjoint to bounded reflection.

The Postnikov-completion $\widehat{T}$ is Postnikov-complete and has the same truncated objects as T. Conversely, if T is Postnikov-complete, then $T \xrightarrow{\sim} \widehat{T} \xrightarrow{\sim} \widehat{T^b}$. Hence the essential image of the fully faithful left adjoint is exactly $\mathrm{Topos}^{\mathrm{post}}$, while the essential image of the right adjoint is Topos^b. □

We may summarize this in the following diagram:



Let us emphasize the distinctions among the completion conditions. Hypercompletion forces every ∞-connected morphism to be an isomorphism. Convergence of Postnikov towers says that each existing object is recovered from its truncations. Postnikov-completeness additionally says that every compatible system of truncated objects is realized by an object. Boundedness reconstructs the topos from the same truncated data through the opposite adjoint, by taking the inverse limit of its localic reflections.

6.5 Dimension theory

The homotopy dimension of a topos measures how far its global sections functor can lower connectivity. For a slice topos $\mathrm{An}_{/X}$, this recovers a classical finiteness invariant of the anima X: it asks whether X is a retract of a finite-dimensional CW-complex. Dimension also controls the convergence problem from the preceding section. A local finite-dimensionality hypothesis forces Postnikov towers to converge, and a uniform local bound also ensures that every compatible system of truncations is realized by an object.

6.5.1 Homotopy dimension and examples

Definition 6.67. Let $\varphi_* \colon S \to T$ be a morphism of topoi. We say that φ_* *has dimension* $\leq d$ if, for every $n \geq d-1$, the functor φ_* sends n-connected maps to $(n-d)$-connected maps.

We say that T *has dimension* $\leq d$ if the terminal morphism $\Gamma_* = \mathrm{Hom}_T(*, -) \colon T \to \mathrm{An}$ has dimension $\leq d$.

Lemma 6.68. *A topos T has dimension $\leq d$ if and only if every $(d-1)$-connected object of T has a global section.*

Proof. For the 'only if'-direction, suppose T has dimension $\leq d$, and let X be a $(d-1)$-connected object. Then the unique map $X \to *$ is $(d-1)$-connected. By the dimension condition, the induced map $\Gamma_*(X) \to \Gamma_*(*) = *$ is (-1)-connected. A map to the terminal object being (-1)-connected means that its domain is nonempty, so X has a global section.

For the 'if'-direction, suppose every $(d-1)$-connected object of T has a global section. We prove by induction on $n \geq d-1$ that Γ_* sends n-connected maps to $(n-d)$-connected maps.

For the initial case, let $f \colon X \to Y$ be $(d-1)$-connected. An effective epimorphism of animae is characterized by having nonempty fibers over every point. Given $y \colon * \to Y$, the fiber of $\Gamma_*(f)$ over y is

$$\Gamma_*(X) \times_{\Gamma_*(Y)} \{y\} \;\simeq\; \mathrm{Hom}_T(*, X \times_Y *) \;=\; \Gamma_*(X_y),$$

where $X_y := X \times_Y *$. Since X_y is $(d-1)$-connected, it has a global section by assumption. Hence the displayed fiber is nonempty, so $\Gamma_*(f)$ is an effective epimorphism, i.e. it is (-1)-connected.

Now let $n \geq d$ and let f be n-connected. It is in particular $(d-1)$-connected, so $\Gamma_*(f)$ is an effective epimorphism. Since Γ_* preserves limits, the diagonal of $\Gamma_*(f)$ is $\Gamma_*(\Delta_f)$. The map Δ_f is $(n-1)$-connected, so the induction hypothesis shows that its image is $(n-d-1)$-connected. The characterization of connected maps by their diagonals in Theorem 3.22 now shows that $\Gamma_*(f)$ is $(n-d)$-connected. □

Definition 6.69. We say that a topos T is *locally of dimension* $\leq d$ (resp. *locally of finite dimension*) if it is generated under colimits by objects U such that the slice topos $T_{/U}$ has dimension $\leq d$ (resp. has some finite dimension).

Here are some basic sources of dimension bounds.

Example 6.70. Let $X \in \mathrm{An}$. If X is represented by a d-dimensional CW-complex, or by a d-dimensional simplicial set, then $\dim(\mathrm{An}_{/X}) \leq d$.

More precisely, $\dim(\mathrm{An}_{/X}) \leq d$ if and only if X is a retract, in the homotopy category of animae, of an anima represented by a d-dimensional CW-complex. If $d \neq 2$, this is further equivalent to X itself admitting a d-dimensional CW-model. For $d = 2$, the retract condition implies that X admits a 3-dimensional CW-model, but it need not admit a 2-dimensional one. See [Lur09, Example 7.2.1.4].

Example 6.71. If C admits a terminal object 1, then $\mathrm{PSh}(C)$ has dimension ≤ 0. Indeed, global sections are given by evaluation at 1, and evaluation preserves effective epimorphisms. Equivalently, if $\mathcal{F}$ is (-1)-connected, then $\Gamma_*(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}(1)$ is nonempty. See [Lur09, Example 7.2.1.3].

Example 6.72. The presheaf category $\mathrm{PSh}(\mathbb{N})$ on the poset of natural numbers has dimension ≤ 1. Indeed, a 0-connected presheaf is a tower

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

of connected animae. Its limit is nonempty: after replacing the tower by a tower of Kan fibrations, one may choose a point of X_0 and lift it successively through the tower. Thus every 0-connected object has a global section, and Lemma 6.68 applies.

Example 6.73. Let X be a paracompact Hausdorff space. We say that X has *covering dimension* $\leq d$ if every open covering is refined by one with empty $(d+2)$-fold intersections. Then

$$\dim \mathrm{Shv}(X) \leq d \quad \Longleftrightarrow \quad X \text{ has covering dimension} \leq d.$$

This applies, for example, to every CW-complex of dimension $\leq d$. See [Lur09, Theorem 7.2.3.6 and Corollary 7.2.3.7].

Example 6.74. Let X be a spectral space of Krull dimension $\leq d$. Then $\mathrm{Shv}(X)$ has dimension $\leq d$; see [CM21, Theorem 3.12].

Example 6.75. If X is a qcqs scheme, then the following bounds hold:

- $\dim(X_{\mathrm{Nis}}) \leq \dim_{\mathrm{Krull}}(X)$; see [CM21, Theorem 3.18].
- $\dim(X_{\mathrm{cdh}}) \leq \dim_{\mathrm{val}}(X)$; see [EHIK21, Theorem 2.4.15].

6.5.2 Convergence and Postnikov-completeness

The following result explains why dimension enters the comparison among the completion conditions in Section 6.4. Its first part is local and only concerns Postnikov towers of existing objects. Essential surjectivity of $T \to \widehat{T}$ requires a uniform local bound.

Proposition 6.76 ([Lur09, Proposition 7.2.1.10 and Corollary 7.2.1.12])**.** *Let T be a topos.*

(1) *If T is locally of finite dimension, then for all $X \in T$, the map $X \xrightarrow{\sim} \lim_n \tau_n X$ is an isomorphism. In particular, T is hypercomplete.*

(2) *If T is locally of dimension $\leq d$ for some d, then T is Postnikov-complete.*

Proof. (1) Let $\{U_\alpha\}$ be a collection of generators such that every slice $T_{/U_\alpha}$ has finite dimension, say at most d_α. For every $X \in T$, the dimension bound shows that

$$\mathrm{Hom}_T(U_\alpha, X) \longrightarrow \mathrm{Hom}_T(U_\alpha, \tau_n X)$$

is $(n - d_\alpha)$-connected. The target is n-truncated. The standard convergence criterion for towers of animae therefore identifies $\mathrm{Hom}_T(U_\alpha, X)$ with $\lim_n \mathrm{Hom}_T(U_\alpha, \tau_n X)$: in each fixed degree, the homotopy groups and the transition maps stabilize. This is also the mapping-anima argument in the proof of [Lur09, Proposition 7.2.1.10]. Since the U_α generate T, it follows that $X \to \lim_n \tau_n X$ is an isomorphism.

If $f\colon X \to Y$ is ∞-connected, then $\tau_n f$ is an isomorphism for every n. Applying the convergence statement to X and Y shows that f itself is an isomorphism. Hence T is hypercomplete, in agreement with [Lur09, Corollary 7.2.1.12].

(2) By (1), the comparison $T \to \widehat{T}$ is fully faithful. Let $(X_n)_n \in \widehat{T}$ and put $X := \lim_n X_n$. The transition map $X_{m+1} \to X_m \xrightarrow{\sim} \tau_m X_{m+1}$ is m-connected. The highly connected tower criterion proved in [Lur09, Proposition 7.2.1.10], applied with the uniform bound d, shows that, for every n, the map $X \to X_m$ is n-connected for all sufficiently large m. For $m \geq n$, the composite

$$X \longrightarrow X_m \longrightarrow X_n$$

is therefore n-connected. Since X_n is n-truncated, this identifies X_n with $\tau_n X$. Thus every object of $\widehat{T}$ is the Postnikov tower of an object of T, so $T \to \widehat{T}$ is essentially surjective. □

6.6 Essential morphisms and shape theory

Shape theory extracts a pro-object which records how a topos sees constant objects. For an arbitrary morphism of topoi, this construction naturally takes values in a pro-category. It is represented by an ordinary object precisely when the inverse image functor admits a further left adjoint. We first discuss these *essential* morphisms, then construct relative shape using pro-left adjoints, and finally characterize essentiality by the local representability of relative shape. We closely follow Aizenbud and Carmeli [AC21, Section 3.1].

6.6.1 Essential morphisms

Definition 6.77 (Essential morphism)**.** A morphism of topoi $\varphi_*\colon T \to S$ is called *essential* if the inverse image functor $\varphi^*\colon S \to T$ admits a further left adjoint $\varphi_\sharp\colon T \to S$. We thus have an adjoint triple

$$\varphi_\sharp \quad \dashv \quad \varphi^* \quad \dashv \quad \varphi_*.$$

We call a topos T *essential* if the terminal morphism $\Gamma_*\colon T \to \mathrm{An}$ is essential.

Example 6.78. Every étale morphism of topoi is essential: given a topos T and an object $U \in T$, the functor $j^*\colon T \to T_{/U}$ given by $X \mapsto X \times U$ admits a left adjoint $j_\sharp\colon T_{/U} \to T$ given by the forgetful functor.

Example 6.79. For any small category C, the presheaf topos $\mathrm{PSh}(C)$ is essential. Indeed, the terminal morphism $\Gamma_*\colon \mathrm{PSh}(C) \to \mathrm{An}$ is given by taking the limit $\Gamma_*(F) = \lim_{c \in C^{\mathrm{op}}} F(c)$, and its left adjoint $\Gamma^*\colon \mathrm{An} \to \mathrm{PSh}(C)$ sends an anima A to the constant presheaf $\underline{A}$. The functor Γ^* admits a further left adjoint $\Gamma_\sharp\colon \mathrm{PSh}(C) \to \mathrm{An}$ given by the colimit:

$$\Gamma_\sharp(F) \quad = \quad \mathrm{colim}_{c \in C^{\mathrm{op}}} F(c).$$

6.6.2 Pro-left adjoints and relative shape

For a general morphism of topoi φ_*, the inverse image functor φ^* need not admit a left adjoint. It does acquire one after passage to pro-categories. For a presentable category C which admits finite

limits, we use the large pro-category

$$\mathrm{Pro}(C) \quad := \quad \mathrm{Fun}^{\mathrm{lex,acc}}(C, \mathrm{An})^{\mathrm{op}},$$

where the superscript indicates the accessible left exact functors. For a small category C, this agrees with $\mathrm{Ind}(C^{\mathrm{op}})^{\mathrm{op}}$. There is a fully faithful embedding $C \hookrightarrow \mathrm{Pro}(C)$ given by $X \mapsto \mathrm{Hom}_C(X, -)$. An object of $\mathrm{Pro}(C)$ is called *pro-constant*, or *corepresentable*, if it lies in the essential image of this embedding.

Proposition 6.80 (Pro-left adjoint)**.** *Let $\varphi_* \colon T \to S$ be a morphism of topoi. The induced functor*

$$\mathrm{Pro}(\varphi^*) \colon \mathrm{Pro}(S) \longrightarrow \mathrm{Pro}(T)$$

admits a left adjoint

$$\varphi_\sharp \colon \mathrm{Pro}(T) \longrightarrow \mathrm{Pro}(S).$$

On corepresentable objects, it is characterized by

$$\varphi_\sharp(X) = \mathrm{Hom}_T(X, \varphi^*(-)) \quad \in \quad \mathrm{Fun}^{\mathrm{lex,acc}}(S, \mathrm{An})^{\mathrm{op}}.$$

Its restriction $\varphi_\sharp \colon T \to \mathrm{Pro}(S)$ preserves colimits. Moreover, φ_* is essential if and only if this restriction factors through $S \hookrightarrow \mathrm{Pro}(S)$. In that case, the factorization is a left adjoint to φ^*.

Proof. Since φ^* is accessible and left exact, the universal property of pro-categories gives the asserted left adjoint. For $X \in T$ and $Y \in S$, its adjunction isomorphism reads

$$\mathrm{Hom}_{\mathrm{Pro}(S)}(\varphi_\sharp(X), Y) \;\simeq\; \mathrm{Hom}_{\mathrm{Pro}(T)}(X, \mathrm{Pro}(\varphi^*)(Y)) \;\simeq\; \mathrm{Hom}_T(X, \varphi^*(Y)),$$

which gives the displayed formula. To see directly that the restriction to T preserves colimits, let $(X_i)_i$ be a diagram in T. For every $Y \in S$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Pro}(S)}\big(\varphi_\sharp(\mathrm{colim}_i\, X_i), Y\big) &\xrightarrow{\sim} \mathrm{Hom}_T(\mathrm{colim}_i\, X_i, \varphi^* Y) \\ &\xrightarrow{\sim} \lim_i \mathrm{Hom}_T(X_i, \varphi^* Y) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Pro}(S)}\big(\mathrm{colim}_i\, \varphi_\sharp(X_i), Y\big). \end{aligned}$$

Since corepresentable objects detect isomorphisms in $\mathrm{Pro}(S)$, the claim follows.

Finally, the restriction factors through S if and only if every functor $\mathrm{Hom}_T(X, \varphi^*(-))$ is corepresentable by an object of S. This says precisely that φ^* admits a left adjoint. □

The pro-left adjoint contains more information than we need for shape. Evaluating it on the terminal object records the effect of φ^* on global sections.

Definition 6.81 (Relative shape)**.** Let $\varphi_* \colon T \to S$ be a morphism of topoi. We define the *relative shape* of φ_* as the pro-object

$$|\varphi| \quad := \quad \varphi_\sharp(*) \qquad \in \mathrm{Pro}(S).$$

Note that this pro-object corepresents the composite functor $\Gamma_T \circ \varphi^* \colon S \to \mathrm{An}$, where $\Gamma_T \colon T \to \mathrm{An}$ is the global sections functor of T.

We say that φ_* is of *constant shape* if $|\varphi|$ is pro-constant, i.e., if it is corepresentable by an object of S.

Definition 6.82 (Shape)**.** The *shape* $|T|$ of a topos T is the relative shape of the terminal morphism $\Gamma_* \colon T \to \mathrm{An}$:

$$|T| \quad := \quad |\Gamma_*| \quad \in \quad \mathrm{Pro}(\mathrm{An}).$$

We say that T *has constant shape* if the pro-anima $|T|$ is pro-constant. We say that an object $X \in T$ *has constant shape* if the slice topos $T_{/X}$ does.

Example 6.83. For a presheaf topos $\mathrm{PSh}(C)$, the terminal morphism is essential by Example 6.79, and its shape is represented by

$$|\mathrm{PSh}(C)| \quad \simeq \quad \mathrm{colim}_{c \in C^{\mathrm{op}}} * \ \simeq \ |C|,$$

the classifying anima of C. In particular, $\mathrm{PSh}(C)$ has constant shape.

Example 6.84. Let X be a paracompact, locally contractible topological space. Then $\mathrm{Shv}(X)$ is essential and its shape is represented by the singular anima $\Pi_\infty X$. Without a local contractibility hypothesis, the shape need not be represented by the weak homotopy type of X; the pro-object can retain information which the singular anima loses. See [Lur17, Appendix A].

6.6.3 Local constancy and essentiality

A morphism need not have globally constant relative shape. Essentialness is instead detected after passing to a generating collection of slices.

Definition 6.85 (Locally of constant shape)**.** Let $\varphi_* \colon T \to S$ be a morphism of topoi. Given an object $U \in T$, we say that U *has constant shape relative to* φ_* if the composite morphism

$$T_{/U} \xrightarrow{j_*} T \xrightarrow{\varphi_*} S$$

is of constant shape.

We say that φ_* is *locally of constant shape* if there exists a collection of objects $\{U_i\}_{i \in I}$ which generates T under colimits such that each U_i has constant shape relative to φ_*.

A topos T is *locally of constant shape* if the terminal morphism $\Gamma_* \colon T \to \mathrm{An}$ is locally of constant shape.

The following proposition provides a key characterization of essential morphisms:

Proposition 6.86 (Aizenbud and Carmeli [AC21, Proposition 3.1.5])**.** *Let $\varphi_* \colon T \to S$ be a morphism of topoi and let $\{U_i\}_{i \in I}$ be a collection of objects which generates T under colimits. Then the following conditions are equivalent:*

(1) *The morphism φ_* is essential;*

(2) *Every object U_i has constant shape relative to φ_*.*

Proof. Recall from Example 6.78 that the étale morphism $j_* \colon T_{/U} \to T$ is essential. Pro-left adjoints compose, so the relative shape of the composite $\varphi_* \circ j_* \colon T_{/U} \to S$ is the image of the terminal object of $T_{/U}$ under the composite

$$T_{/U} \xrightarrow{j_\sharp} T \xrightarrow{\varphi_\sharp} \mathrm{Pro}(S).$$

Since $j_\sharp$ is the forgetful functor sending $(V \to U)$ to V, we see that the relative shape of $\varphi_* \circ j_*$ is equivalent to $\varphi_\sharp(U) \in \mathrm{Pro}(S)$.

If φ_* is essential, then $\varphi_\sharp$ factors through S, so $\varphi_\sharp(U)$ is pro-constant for all $U \in T$, and in particular for all U_i.

Conversely, assume that $\varphi_\sharp(U_i)$ is pro-constant for all $i \in I$. The embedding $S \hookrightarrow \mathrm{Pro}(S)$ and the functor $\varphi_\sharp \colon T \to \mathrm{Pro}(S)$ both preserve colimits. Since the collection $\{U_i\}$ generates T under colimits, it follows that $\varphi_\sharp$ factors through S. By Proposition 6.80, this means that φ_* is essential. □

Corollary 6.87 ([Lur17, Proposition A.1.8])**.** *A topos T is essential if and only if it is locally of constant shape.* □

Essential morphisms interact well with objects of constant shape:

Lemma 6.88. *Let $\varphi_* \colon T_1 \to T_2$ and $\psi_* \colon T_2 \to T_3$ be morphisms of topoi and assume that φ_* is essential. Let $U \in T_1$ be an object which has constant shape relative to the composite $\psi_* \circ \varphi_* \colon T_1 \to T_3$. Then the object $\varphi_\sharp(U) \in T_2$ has constant shape relative to ψ_*.*

Proof. The shape of $\varphi_\sharp(U) \in T_2$ relative to ψ_* is $\psi_\sharp(\varphi_\sharp(U)) \in \mathrm{Pro}(T_3)$. Since φ_* is essential, we have $\varphi_\sharp(U) \in T_2$, and the shape of U relative to $\psi_* \circ \varphi_*$ is

$$(\psi \circ \varphi)_\sharp(U) \;\simeq\; \psi_\sharp(\varphi_\sharp(U)).$$

By assumption, this is pro-constant, so $\varphi_\sharp(U)$ has constant shape relative to ψ_*. □

Corollary 6.89. *Let $\varphi_* \colon T_1 \to T_2$ be an essential morphism of topoi. Then $\varphi_\sharp \colon T_1 \to T_2$ sends objects of constant shape in T_1 to objects of constant shape in T_2.* □

6.7 Coherence

We will now discuss the notion of *coherent topoi*. While this is a somewhat technical topic, it plays an important role in topos theory, and also shows up prominently in algebraic geometry, so it seems worth at least briefly discussing it. An alternative detailed treatment may be found in [BGH18, Chapter 3].

6.7.1 An overview

Before diving into the technical details, let us provide an overview of the results of this section.

The main concept introduced in this section is that of *coherent objects*. For intuition, the example to keep in mind is the coherent objects in An, which as we will see are precisely the animae X for which $\pi_0(X)$ is finite and every homotopy group $\pi_i(X, x)$ with $i \geq 1$ is finite. We may think of coherence as a strong finiteness condition on an object of a topos. For example, we will prove the following compactness result below in Theorem 6.114:

Theorem (Compactness theorem)**.** Let $X \in T$ be a coherent object. Then $\tau_n X \in T_{\leq n}$ is compact for all $n \geq -1$.

Corollary 6.90. *For a coherent object $X \in T$, the functor $\mathrm{Ab}(T_{\leq 0}) \to \mathrm{Ab}^{\mathbb{Z}}, A \mapsto H^*(X; A)$ preserves filtered colimits.*

This theorem can be thought of as the underlying reason for various compactness results in the literature. For example, it is the reason that étale cohomology groups preserve filtered colimits: the étale topos is coherent.

While the theorem says that all truncations of a coherent object X are compact, the object X itself need not be compact! In particular, this notion of "finiteness" behaves quite differently from the other usual notion of finiteness that we are used to from homotopy theory, namely the subcategory $\mathrm{An}^{\mathrm{fin}} \subseteq \mathrm{An}$ of finite animae, obtained from the point by closing up under pushouts and finite coproducts.

There is another, quite unrelated, result that we will discuss, called the *completeness theorem*. In the higher categorical setting this is due to Lurie, but it follows a classical result for classical topoi due to Deligne.

Definition 6.91. A topos T is called *locally coherent* if every object $X \in T$ admits a cover $\bigsqcup_\alpha U_\alpha \twoheadrightarrow X$ by coherent objects.

Theorem (Completeness theorem, see Theorem 6.118)**.** Every locally coherent topos admits a surjection[1] from a topos of the form $\mathrm{An}_{/S}$ for some set[2] S.

Applying this result to an anima A, regarded as a topos by forming the slice $\mathrm{An}_{/A}$, this recovers the basic fact that there exists a surjection $S \twoheadrightarrow A$ from a set S.

A generic way to apply this theorem is to reduce suitable types of statements in locally coherent topoi to statements in sets/animae.

Example 6.92. If T is a hypercomplete locally coherent topos, we get a surjection $\mathrm{An}_{/S} \twoheadrightarrow T$, for which the pullback functor is conservative.

Example 6.93. A Kan fibration in any hypercomplete topos is a "realization-fibration", i.e. pullback along a Kan fibration commutes with geometric realizations: if $X_\bullet \to Z_\bullet$ is a Kan fibration and $Y_\bullet \to Z_\bullet$ is arbitrary, we have

$$\mathrm{colim}_n X_n \times_{Z_n} Y_n \xrightarrow{\sim} (\mathrm{colim}_n X_n) \times_{\mathrm{colim}_n Z_n} (\mathrm{colim}_n Y_n).$$

In the next couple of subsections, we will provide proper definitions and discuss the proofs of these theorems. We start in Section 6.7.2 with the definition of n-coherent objects for all n, and prove criteria for n-coherence. In Section 6.7.3 we define what it means for a site to be *finitary*, and show that the hypercomplete locally coherent topoi are precisely the hypercompletions of sheaves on finitary sites with pullbacks. In Section 6.7.4 we prove the compactness theorem and in Theorem 6.118 we prove the completeness theorem.

[1] The notion of "surjection" of topoi was defined in Remark 5.70.

[2] Recall that by "set" we mean a 0-truncated anima.

6.7.2 Coherent objects

The notion of coherence is closely related to the notion of quasi-compact objects in a topos:

Definition 6.94. An object $X \in T$ is called *quasi-compact* if any cover $\bigsqcup_i U_i \twoheadrightarrow X$ can be refined to a finite cover, i.e. there is a finite subset $J \subseteq I$ such that the map $\bigsqcup_{j \in J} U_j \to X$ is still an effective epimorphism.

We say a topos T is *quasi-compact* if $* \in T$ is quasi-compact.

One problem with the notion of quasi-compactness is that it is not closed under the formation of fiber products. We may think of the collection of coherent objects in a topos as the universal approximation of the collection of quasi-compact objects by a collection of objects closed under fiber products. As a first approximation, we may consider the following class of objects:

Definition 6.95. An object $X \in T$ is called *quasi-separated* if for any two maps $Y \to X$ and $Z \to X$ from quasi-compact objects Y and Z, also the fiber product $Y \times_X Z$ is quasi-compact.

The quasi-separatedness condition is very familiar for people working in algebraic geometry. However, it does not yet achieve our goal: there is in general no reason that quasi-separated objects are closed under base change. Now, in principle one could iterate this condition, and consider objects X such that the quasi-separated objects are closed under pullbacks, etcetera. However, what we would like to argue here is that this is not the correct way to think about things. In fact, this notion of quasi-separated objects is not really well-behaved in an arbitrary topos; one might argue it is somewhat of a "coincidence" that it works in algebraic geometry. For example, in algebraic geometry it is true that $Y \times_X Z$ is quasi-compact for all Y, Z if and only if Δ_X is relatively quasi-compact, but this fails to hold in an arbitrary topos; the "if"-direction always holds, but the "only if"-direction fails: Δ_X being relatively quasi-compact is generally stronger. The reason this works in algebraic geometry is that you can cover any scheme by affine schemes, which are both quasi-compact and quasi-separated. After unwinding, you will find that the question reduces to the property that the product of two qcqs schemes over $\mathrm{Spec}(\mathbb{Z})$ is again qcqs. Note that this condition is precisely the next one in the hierarchy we hinted at before. More generally, this fact about Δ_X being relatively quasi-compact will hold in a topos T whenever the following conditions are satisfied:

- Every object in T is covered by qcqs objects.
- The terminal object of T is "2-coherent", meaning that products of qcqs objects are again qcqs.

Warning 6.96. The terminology on "coherence" can be somewhat confusing in the literature. For example, in SGA the word "coherent" is used in the following three different ways:

- There is a notion for an *object* $X \in T$ to be coherent; it just means that it is qcqs.
- There is a notion for a *morphism* $X \to Y$ to be coherent. Confusingly, this is *not* equivalent to the condition that the morphism $X \to Y$ is qcqs when regarded as an object in $T_{/Y}$. Since we would like to stick to the general convention that notions for maps are simply those for the associated object in the slice topos, we will follow Lurie to refer to this other definition by adding the adverb *relatively*, giving notions like "relatively quasi-compact" and "relatively qcqs".

- There is a notion for a *topos* to be coherent. This is the "correct" notion from the viewpoint of general topos theory. Marc mentioned he finds it quite impressive that Grothendieck managed to find the correct definition of coherence already so early on.

After all this preamble, let us now give a formal definition of coherent objects.

Definition 6.97. Let T be a topos. We will inductively define what it means for objects to be *n-coherent* for all $n \geq 0$:

- We say that an object $X \in T$ is 0-*coherent* if it is quasi-compact. (This is really a condition on the slice topos $T_{/X}$.)
- We say that T is *locally n-coherent* if any $X \in T$ has a cover by n-coherent objects.
- An object $X \in T$ is said to be *n-coherent* if $T_{/X}$ is $(n-1)$-coherent, locally $(n-1)$-coherent, and for all maps $Y \to X \leftarrow Z$ such that Y and Z are $(n-1)$-coherent, the pullback $Y \times_X Z$ is $(n-1)$-coherent.
- We say T is *n-coherent* if the terminal object is n-coherent.

Example 6.98. A topos is 1-coherent if it is locally quasi-compact, and qcqs. A topos T is 2-coherent if and only if it is qcqs, locally qcqs, and for all qcqs $X, Y \in T$ also $X \times Y$ is qcqs.

Definition 6.99. We say that $X \in T$ is *coherent* if it is n-coherent for all n. We say T is *coherent* if $* \in T$ is coherent and *locally coherent* if every object $X \in T$ admits a cover by coherent objects.

We denote the resulting subcategories of T by

$$T \supseteq \cdots \supseteq T^{n-\mathrm{coh}} \supseteq T^{(n+1)-\mathrm{coh}} \supseteq \cdots \supseteq \bigcap_n T^{n-\mathrm{coh}} =: T^{\mathrm{coh}}.$$

Example 6.100. One can show that an anima $X \in \mathrm{An}$ is n-coherent if and only if $\pi_0(X)$ is finite and $\pi_i(X, x)$ is finite for every base point x and every $1 \leq i \leq n$; this can for example be proved using Proposition 6.105 below. Any anima is locally coherent, since it may be covered by points, which are coherent.

Example 6.101. Let X be a locale. An element of X is *compact* if every expression of it as a join admits a finite subcover; equivalently, the corresponding subterminal object of $\mathrm{Shv}(X)$ is quasi-compact. The topos $\mathrm{Shv}(X)$ is locally coherent if and only if the compact elements form a basis which is closed under finite meets. In this case X is called a *locally coherent locale*. It is coherent precisely when, in addition, the top element is compact.

For locales, local coherence already implies local n-coherence for every n. Consequently, the preceding conditions characterize locally coherent and coherent localic topoi, respectively. The completeness theorem below implies that a coherent locale is spatial, and its spatial realization is a spectral space. Conversely, the locale of open subsets of a spectral space is coherent. The corresponding local statement identifies locally coherent spatial locales with locally spectral spaces.

Definition 6.102. A map $f : Y \to X$ is called *relatively n-coherent* if for every $Z \to X$ with Z n-coherent, also $Z \times_X Y$ is n-coherent.

Warning 6.103. This is *not* the same as the condition that $f \in T_{/X}$ is n-coherent.

Lemma 6.104. *Let $f : Y \to X$ be a map and assume that X is $(n+1)$-coherent. Then f is relatively n-coherent if and only if Y is n-coherent.*

Proposition 6.105. *Let T be a topos, and suppose $T_0 \subseteq T$ is a full subcategory of n-coherent objects such that every $X \in T$ is covered by objects of T_0. Then the following conditions hold:*

- *A morphism $f : Y \to X$ is relatively n-coherent if and only if for every map $U \to X$ with $U \in T_0$ the pullback $U \times_X Y$ is n-coherent.*
- *An object $X \in T$ is $(n+1)$-coherent if and only if X is quasi-compact and for every pair of maps $U \to X \leftarrow V$ with $U, V \in T_0$, the pullback $U \times_X V$ is n-coherent.*

Corollary 6.106. *A topos T is locally coherent if and only if there exists a full subcategory $T_0 \subseteq T$ covering T which is closed under fiber products and such that every $X \in T_0$ is quasi-compact.*

Proof. The "only if" is clear: if T is locally coherent, we may take $T_0 := T^{\mathrm{coh}}$ the subcategory of coherent objects, which by design are closed under fiber products.

Conversely, we inductively use the proposition to show that $T_0 \subseteq T^{n\text{-coh}}$ for all n. In particular, T_0 consists of coherent objects. Since they cover T, we see that T is locally coherent by definition. □

Remark 6.107. If T is not just locally coherent, but also coherent, then the collection of objects T_0 provided by the corollary may be assumed to contain the final object.

Proposition 6.108 (Coherence and connectivity)**.** *Consider a morphism $f : X \to Y$.*

(1) *If X is n-coherent and f is $(n-1)$-connected, then Y is n-coherent.*

(2) *If Y is n-coherent and f is n-connected, then X is n-coherent.*

(3) *If Y is n-coherent, f is $(n-2)$-truncated, and X is $(n-1)$-coherent, then X is n-coherent.*

Proof. We argue simultaneously for the three assertions by induction on n. For $n = 0$, the first assertion says that effective-epimorphic images of quasi-compact objects are quasi-compact, while the second and third reduce to the stability of quasi-compactness under pullback along monomorphisms. Suppose now that the assertions are known in degree $n-1$. The characterization in Proposition 6.105 reduces n-coherence to quasi-compactness together with the $(n-1)$-coherence of pullbacks against a covering family of $(n-1)$-coherent objects. After pulling back f along such a family, its connectivity or truncation bound is unchanged. The three required statements about these pullbacks are then exactly the induction hypotheses in the relevant slice topoi. This proves all three assertions. □

Corollary 6.109. *Consider the unique factorization of $f : X \to Y$ as*

$$X \to \tau_{n-1}(X/Y) \to Y$$

into an $(n-1)$-connected map followed by an $(n-1)$-truncated map. If X is n-coherent and Y is m-coherent, then $\tau_{n-1}(X/Y)$ is $\max(n, m)$-coherent.

Proof. By Proposition 6.108(1), the object $\tau_{n-1}(X/Y)$ is n-coherent, so we are done if $n \geq m$. If $m > n$, we may inductively show that $\tau_{n-1}(X/Y)$ is $(n+k)$-coherent for all $k = 0, \ldots, m-n$. If we already know it is $(n+k-1)$-coherent, then using that Y is m-coherent and $\tau_{n-1}(X/Y) \to Y$ is $(n-1)$-truncated, we get by Proposition 6.108(3) that $\tau_{n-1}(X/Y)$ is $(n+k)$-coherent. □

Corollary 6.110. *Let T be an n-localic topos. If T is $(n+1)$-coherent, then T is coherent and locally coherent.*

Proof. By assumption, every object $X \in T$ is covered by objects from $T_{\leq n-1}$. Let $X \in T_{\leq n-1}$. By assumption on T, there exists an effective epimorphism $\bigsqcup_i U_i \twoheadrightarrow X$ where each U_i is n-coherent. Because X is $(n-1)$-truncated, this map descends to a map $\bigsqcup_i \tau_{n-1}(U_i) \twoheadrightarrow X$, which is still an effective epimorphism. By Proposition 6.108(1), each $\tau_{n-1}(U_i)$ is again n-coherent. We conclude that every $X \in T$ is covered by objects from $T^{n-\mathrm{coh}}_{\leq n-1}$.

Since T is $(n+1)$-coherent, it follows that $T^{n-\mathrm{coh}}_{\leq n-1}$ is closed under finite limits. By Corollary 6.106, it follows that T is both coherent and locally coherent. □

Corollary 6.111. *Assume that T is both bounded and coherent. Then T is locally coherent.*

Proof. Since T is bounded, it is generated under colimits by truncated objects. Hence every object of T admits an effective epimorphism from a coproduct of truncated objects, and it is enough to cover a truncated object $X \in T_{\leq n}$ by coherent objects. Since T is coherent, we may find an effective epimorphism $\bigsqcup_i U_i \twoheadrightarrow X$ where each U_i is $(n+1)$-coherent. Because X is n-truncated, this map factors through an effective epimorphism $\bigsqcup_i \tau_n(U_i) \twoheadrightarrow X$. By Proposition 6.108(1) and (3), each $\tau_n(U_i)$ is coherent. □

6.7.3 Finitary sites

Definition 6.112. A Grothendieck topology τ on a category C is called *finitary* if every τ-covering sieve contains a τ-covering sieve which is generated by a finite number of morphisms in C.

Proposition 6.113.

(1) *If (C, τ) is a finitary Grothendieck site and C admits fiber products, then $\mathrm{Shv}_\tau(C)$ is locally coherent. If C also admits a terminal object, then $\mathrm{Shv}_\tau(C)$ is also coherent.*

(2) *If T is hypercomplete and locally coherent, then there exists a finitary site (C, τ) with fiber products such that $T \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. If T is also coherent, then we can choose C to have a terminal object.*

Proof. For (1), let $a \colon C \to \mathrm{Shv}_\tau(C)$ denote sheafified Yoneda. Every sheaf is covered by objects in the image of a. Finitariness says precisely that each $a(c)$ is quasi-compact: a cover of $a(c)$ corresponds to a covering sieve on c, which admits a finite covering refinement. Since C admits fiber products and a preserves them, the image of a is closed under fiber products. The claim now follows from Corollary 6.106. If C has a terminal object, its image is terminal, so the resulting topos is coherent.

For (2), choose a small full subcategory $C \subseteq T^{\mathrm{coh}}$ which covers T and is closed under fiber products; in the coherent case we also arrange that C contains the terminal object, using Remark 6.107. Give

C the topology in which a family $(U_i \to U)$ covers if $\bigsqcup_i U_i \to U$ is an effective epimorphism in T. Since U is quasi-compact, every covering family admits a finite subfamily which still covers, so this topology is finitary.

Let $u\colon C \hookrightarrow T$ denote the inclusion. Since C is closed under fiber products, Proposition 2.43 shows that the left Kan extension

$$u_!\colon \mathrm{PSh}(C) \to T$$

is left exact. By definition of τ, all generating τ-covering sieves are sent to effective epimorphisms in T, so $u_!$ factors as

$$\mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C) \to T.$$

On the other hand, Lemma 6.57 shows that the restriction functor $T \to \mathrm{PSh}(C)$ is fully faithful. Since its essential image consists of τ-sheaves, the induced functor $T \to \mathrm{Shv}_\tau(C)$ is also fully faithful. Thus $\mathrm{Shv}_\tau(C) \to T$ is a left exact localization. It is epigenic because its kernel is generated by maps from covering sieves to representables, all of which become effective epimorphisms in T. An epigenic quotient induces an equivalence on hypercompletions by [ABFJ25, Lemma 2.1.33]. Since T is hypercomplete, we conclude that $T \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. □

6.7.4 Compactness of coherent objects

We will now show the compactness result for coherent objects mentioned before.

Theorem 6.114. *If $X \in T$ is n-coherent, then $\mathrm{Hom}(X, -)\colon T_{\leq n-1} \to \mathrm{An}$ preserves filtered colimits, i.e. the object $\tau_{n-1}(X) \in T_{\leq n-1}$ is compact.*

Corollary 6.115. *If T is locally n-coherent, then $T_{\leq n-1}$ is compactly generated.*

Warning 6.116. An object X being coherent does *not* imply that it is compact.

Remark 6.117. The property of coherence is analogous to the property of "almost perfect" in the context of modules.

Proof of Theorem 6.114. Pulling a filtered diagram back to the slice $T_{/X}$ identifies the required mapping anima with the anima of sections. We may therefore replace T by $T_{/X}$ and assume that $X = *$; the hypothesis becomes the corresponding coherence condition on the terminal object of the slice. We prove the claim by induction on n. For $n = 0$, the assertion is exactly the definition of quasi-compactness, expressed in terms of filtered unions of subterminal objects.

Now consider a filtered category I and an object $Y \in \mathrm{Fun}(I, T_{\leq n-1})$. We have to show that the canonical comparison map

$$\beta_Y\colon \mathrm{colim}_i \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \mathrm{colim}_i Y_i)$$

is an isomorphism. We will show the following claim:

Claim. The map β_Y is an effective epimorphism.

Assuming this for a moment, we may conclude. It remains to check that β_Y induces isomorphisms on path animae. Consider two points $x, y \in \mathrm{colim}_i \mathrm{Hom}(X, Y_i)$. We may represent both at a common

stage and, after replacing I by the corresponding undercategory, assume that this stage is an initial object $0 \in I$. For each i, let

$$E_i := X \times_{(x_i, y_i), Y_i \times Y_i} Y_i,$$

where the second map to $Y_i \times Y_i$ is the diagonal. The anima of sections of $E_i \to X$ is the path anima from x_i to y_i. Since Y_i is $(n-1)$-truncated, $E_i \to X$ is $(n-2)$-truncated. Filtered colimits commute with finite limits in a topos, so the analogous object for $\operatorname{colim}_i Y_i$ is $\operatorname{colim}_i E_i$. The induction hypothesis in $T_{/X}$ therefore identifies the two path animae.

Let us now prove the claim. Given a map $X \to \operatorname{colim}_i Y_i$, we need to show that it factors through some finite stage $Y_i \to \operatorname{colim}_i Y_i$. Since X is quasi-compact and locally $(n-1)$-coherent, there is an effective epimorphism $U \twoheadrightarrow X$ where U is $(n-1)$-coherent, satisfying the additional property that the composite $U \twoheadrightarrow X \to \operatorname{colim}_i Y_i$ factors through some Y_0, where $0 \in I$. (This uses that I is filtered.) Letting $U_\bullet := \check{C}_\bullet(U \to X)$ and $V_{i,\bullet} := \check{C}_\bullet(Y_0 \to Y_i)$, then the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{colim}_i Y_i \end{array}$$

induces a map on Čech nerves of the form

$$U_\bullet \to \operatorname{colim}_i V_{i,\bullet}.$$

Since U is $(n-1)$-coherent and X is n-coherent, each U_m is $(n-1)$-coherent.

By induction on $m \leq n$, we show that the map $\mathrm{sk}_m(U_\bullet) \to \operatorname{colim}_i \mathrm{sk}_m(V_{i,\bullet})$ factors through some $\mathrm{sk}_m(V_{i,\bullet})$, as semisimplicial objects. Here sk_m denotes the m-skeleton. At the successor step, the boundary of the desired m-simplex has already been chosen at a finite stage. The remaining extension problem is a problem of sections for the pullback of

$$V_{i,m} \longrightarrow V_{i,\bullet}[\partial \Delta^m].$$

This map is $(n-m-1)$-truncated, while U_m is $(n-1)$-coherent; the induction hypothesis in the relevant slice therefore moves the extension to a finite stage. Filteredness lets us choose one stage for the finitely many faces and degeneracies. With $m = n$, we obtain

$$\tau_{n-1}|\mathrm{sk}_n(U_\bullet)| \xrightarrow{\sim} \tau_{n-1}(X)$$

and since $\mathrm{Hom}_T(X, Y_i) \cong \mathrm{Hom}_T(\tau_{n-1}(X), Y_i)$ we may conclude.

This is the skeletal lifting argument of [Lur18a, Proposition A.2.3.1]; the preceding discussion records the coherence and truncation estimates needed in its application here. □

6.7.5 The completeness theorem

Recall that a morphism of logoi $\varphi\colon S \to T$ is called *surjective* if it detects effective epimorphisms (or equivalently if it detects ∞-connected maps). The goal of this section is to prove the following theorem:

Theorem 6.118 (Completeness theorem)**.** *If T is locally coherent, then there exists a surjection $S \to T$, where $S = \mathrm{An}^I$ for some (small) set I.*

Remark 6.119. For locales, this is exactly the result that says that coherent locales are spatial, i.e. the locale is a spectral space. For locally coherent classical topoi (in the classical sense), this recovers Deligne's completeness theorem.

There are several steps in the proof of this theorem which are interesting. Here is an outline of the proof:

(1) First, we reduce to the case where T is coherent. In this case, we saw that $T^{\mathrm{hyp}} \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$, where (C, τ) is a finitary site with finite limits.

(2) For every site (C, τ), there exists a surjection $\mathrm{Shv}_\rho(D) \to \mathrm{Shv}_\tau(C)$, where D is a poset. (This is called the *Diaconescu cover*. It is highly non-canonical.)

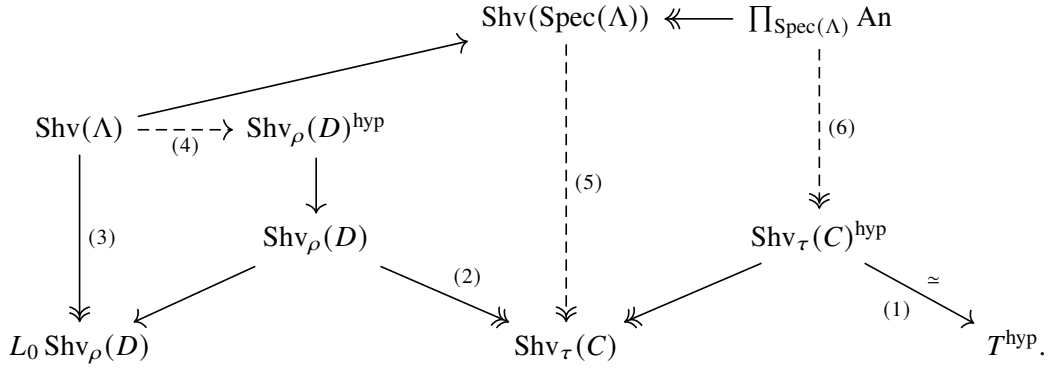
(3) If L is a 0-localic topos (i.e. a *locale*), then there exists a surjection $\mathrm{Shv}(\Lambda) \to L$ where Λ is a Boolean locale (i.e. a complete Boolean algebra).

(4) The locale $\mathrm{Shv}(\Lambda)$ is hypercomplete, so the functor $\mathrm{Shv}(\Lambda) \to L_0 \mathrm{Shv}_\rho(D)$ factors through the hypercompletion $(L_0 \mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \simeq \mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$.

(5) If (C, τ) is a finitary site with finite limits, then any map $\mathrm{Shv}(\Lambda) \to \mathrm{Shv}_\tau(C)$ factors through $\mathrm{Shv}(\mathrm{Spec}(\Lambda))$, where $\mathrm{Spec}(\Lambda)$ is the spectrum of Λ, seen as a commutative ring.

(6) For any topological space X, the topos $\mathrm{Shv}(X)$ is covered by $\prod_{x \in X} \mathrm{An}$.

We illustrate the proof strategy with the following diagram:



Assuming all these claims, we may give a proof of the theorem. We first consider $\mathrm{Shv}_\tau(C)^{\mathrm{hyp}} \to T$, reducing us to $\mathrm{Shv}_\tau(C)$. Then we have some surjection $\mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C)$, where D is a poset. Consider the 0-localic reflection $L_0 \mathrm{Shv}_\rho(D)$. There is a surjection $\mathrm{Shv}(\Lambda) \twoheadrightarrow L_0 \mathrm{Shv}_\rho(D)$. Now, since $\mathrm{Shv}(\Lambda)$ is hypercomplete, this surjection will factor through $(L_0 \mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \simeq \mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$. All in all, we obtain the composite

$$\mathrm{Shv}(\Lambda) \to (\mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \to \mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C).$$

This then factors through a map $\mathrm{Shv}(\mathrm{Spec}(\Lambda)) \twoheadrightarrow \mathrm{Shv}_\tau(C)$, which is surjective by cancellation properties. There is a covering $\prod_{x \in \mathrm{Spec}(\Lambda)} \mathrm{An} \twoheadrightarrow \mathrm{Shv}(\mathrm{Spec}(\Lambda))$, and since its source is hypercomplete, the map to $\mathrm{Shv}_\tau(C)$ factors through $\mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. All in all, we obtain the desired surjection

$$\prod_{x \in \mathrm{Spec}(\Lambda)} \mathrm{An} \twoheadrightarrow \mathrm{Shv}_\tau(C)^{\mathrm{hyp}} \simeq T^{\mathrm{hyp}} \twoheadrightarrow T.$$

In the remainder of this section, we will go through the details of this proof.

Boolean algebras

We recall some basic definitions regarding Boolean algebras.

Definition 6.120. A *Boolean algebra* Λ is a poset with finite joins $x \vee y$ and meets $x \wedge y$, where $\wedge$ distributes over $\vee$, and where every $x \in \Lambda$ has a complement x^c, i.e. an element satisfying $x \wedge x^c = \bot$ and $x \vee x^c = \top$.

Proposition 6.121. *Let Λ be a complete Boolean algebra. Then* $\mathrm{Shv}(\Lambda)$ *has dimension* ≤ 0*. In particular, it is Postnikov complete.*

Proof. It suffices to show that every (-1)-connected object $X \in \mathrm{Shv}(\Lambda)$ admits a global section. We construct, by transfinite induction, a strictly increasing family of elements $U_\alpha \in \Lambda$, together with compatible sections $s_\alpha \in X(U_\alpha)$, for as long as $U_\alpha \neq \top$. Start with $U_0 = \bot$ and its unique section. At a limit ordinal λ, set $U_\lambda = \bigvee_{\alpha<\lambda} U_\alpha$. The compatible sections glue uniquely to a section over U_λ by the sheaf condition.

Suppose that $U_\alpha < \top$, and let $V = U_\alpha^c$. Since $X \to *$ is an effective epimorphism, there is a covering $V = \bigvee_i V_i$ such that $X(V_i)$ is nonempty for every i. Choose a nonzero V_i and a section $t_i \in X(V_i)$. The intersection $U_\alpha \wedge V_i$ is initial, so s_α and t_i agree there. They therefore glue to a section over $U_{\alpha+1} := U_\alpha \vee V_i$, and this element strictly contains U_α.

If this process never reached $\top$, it would define a strictly increasing map from the class of ordinals to the underlying set of Λ, which is impossible. Hence some U_α is $\top$, and the corresponding section is a global section of X. □

To construct the functor (2) in the above diagram, we use the following:

Proposition 6.122. *For every* 0*-localic topos L, there exists a complete Boolean algebra Λ and a surjection* $\mathrm{Shv}(\Lambda) \twoheadrightarrow L$*.*

Proof. Write $L = \mathrm{Shv}(\mathcal{U})$ for a frame $\mathcal{U}$. For every strict inequality $u < v$ in $\mathcal{U}$, consider the interval frame

$$[u, v] = \{x \in \mathcal{U} \mid u \leq x \leq v\}.$$

The map $q_{u,v} \colon \mathcal{U} \to [u, v]$ given by $x \mapsto (x \vee u) \wedge v$ preserves arbitrary joins and finite meets, and sends u and v to the bottom and top elements of the interval, respectively.

For any frame A, let

$$\mathrm{Reg}(A) = \{a \in A \mid \neg\neg a = a\}.$$

This is a complete Boolean algebra: finite meets are inherited from A, joins are given by $\neg\neg(\bigvee_i a_i)$, and complementation is given by $a \mapsto \neg a$. The double-negation map $A \to \mathrm{Reg}(A)$, $a \mapsto \neg\neg a$, is a frame homomorphism. Applied to the nontrivial frame $[u, v]$, it still distinguishes its bottom and top elements. Hence the composite

$$h_{u,v} \colon \mathcal{U} \xrightarrow{q_{u,v}} [u, v] \xrightarrow{\neg\neg} \mathrm{Reg}([u, v])$$

distinguishes u from v.

Now set

$$\Lambda := \prod_{u<v} \mathrm{Reg}([u,v])$$

and let $h\colon \mathcal{U} \to \Lambda$ have components $h_{u,v}$. Products of complete Boolean algebras are complete Boolean algebras, and h is a frame homomorphism. It is injective, since the component indexed by (u,v) distinguishes any prescribed strict inequality $u < v$. The corresponding map of locales therefore has conservative inverse-image functor, which is to say that the induced morphism $\mathrm{Shv}(\Lambda) \to \mathrm{Shv}(\mathcal{U})$ is surjective. □

The Diaconescu cover

Proposition 6.123 (Diaconescu cover)**.** *Let C be a category with topology τ. Then there exists a surjection $\mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C)$, where D is a poset.*

We will prove this below. First, we state a corollary:

Corollary 6.124. *Let T be any topos. Then there exists a surjection $\mathrm{Shv}(\Lambda) \twoheadrightarrow T$, where Λ is a complete Boolean algebra.*

Proof. We may write $T^{\mathrm{hyp}} \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$ for some C. In light of the surjection $T^{\mathrm{hyp}} \twoheadrightarrow T$, we may assume $T = \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. By Proposition 6.123, there is a surjection $\mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C)$, which induces a surjection $\mathrm{Shv}_\rho(D)^{\mathrm{hyp}} \twoheadrightarrow \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. Consider now the 0-localic reflection $L_0 \mathrm{Shv}_\rho(D)$ of this sheaf category. By Proposition 6.122, this receives a surjection $\mathrm{Shv}(\Lambda) \twoheadrightarrow L_0 \mathrm{Shv}_\rho(D)$. Since $\mathrm{Shv}(\Lambda)$ is hypercomplete, this functor factors through the hypercompletion $(L_0 \mathrm{Shv}_\rho(D))^{\mathrm{hyp}}$, which is equivalent to $\mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$ by Lemma 6.58. It follows that the resulting functor $\mathrm{Shv}(\Lambda) \twoheadrightarrow \mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$ is surjective. All in all, we obtain surjections

$$\mathrm{Shv}(\Lambda) \twoheadrightarrow (\mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \twoheadrightarrow \mathrm{Shv}_\tau(C)^{\mathrm{hyp}} \simeq T^{\mathrm{hyp}} \twoheadrightarrow T$$

as desired. □

Proof of Proposition 6.123. We claim that there exists a poset D with a functor $f\colon D \to C$, such that:

(1) The functor f is essentially surjective;

(2) For all $x \in D$, the induced functor $D_{/x} \to C_{/f(x)}$ is essentially surjective.

To this end, let $\hat{C}$ be a simplicial set whose associated complete Segal anima is C. Define a poset P as follows:

$$P = \{(n,\sigma) \mid n \in \mathbb{N}, \sigma\colon \Delta^n \to \hat{C}^{\mathrm{op}}\},$$

where $(n,\sigma) \leq (n',\sigma')$ if and only if $n \leq n'$ and $\sigma = \sigma'|_{\Delta^{\{0,1,\dots,n\}}}$. We may define a functor $N(P) \to \hat{C}^{\mathrm{op}}$, which sends a sequence $(n_0,\sigma_0) \leq \cdots \leq (n_k,\sigma_k)$ to the composite $\Delta^k \xrightarrow{n_0,\dots,n_k} \Delta^{n_k} \xrightarrow{\sigma_k} \hat{C}^{\mathrm{op}}$. By passing to realizations, this gives a functor

$$D := P^{\mathrm{op}} \to C.$$

We prove that this satisfies (1) and (2). Essential surjectivity is clear from the 0-simplices. For (2), consider $P_{(n,\sigma)/} \to (C^{\mathrm{op}})_{\sigma(n)/}$ and an arrow $\alpha\colon \sigma(n) \to c$ in C^{op}. Together, σ and α define a map

$$\Delta^n \cup_{\{n\}} \Delta^1 \longrightarrow \widehat{C}^{\mathrm{op}}.$$

The inclusion $\Delta^n \cup_{\{n\}} \Delta^1 \hookrightarrow \Delta^{n+1}$ is inner anodyne. Since $\widehat{C}^{\mathrm{op}}$ is a category, we may extend this map to a simplex $\widetilde{\sigma}\colon \Delta^{n+1} \to \widehat{C}^{\mathrm{op}}$. The object $(n+1, \widetilde{\sigma})$ lies above (n, σ) and its image represents α. Thus the indicated functor on undercategories is essentially surjective. Passing to opposites gives (2).

From now on, fix such an f. For a sieve $R \hookrightarrow y(x)$ and a map $\alpha\colon x' \to x$ in D, let $\langle f(\alpha^* R)\rangle$ denote the sieve on $f(x')$ generated by the image of $\alpha^* R$. Declare R to be a ρ-cover if $\langle f(\alpha^* R)\rangle$ is a τ-cover of $f(x')$ for every α. This defines a Grothendieck topology. The maximal sieve is covering and stability under pullback is built into the definition. For transitivity, suppose that R covers x and that a sieve S on x pulls back to a cover along every map belonging to R. After an arbitrary base change $x' \to x$, the images under f of the arrows in the pullback of R cover $f(x')$. Over each of them, the image of the corresponding pullback of S covers. The local character axiom for τ then shows that $\langle f(S|_{x'})\rangle$ covers $f(x')$.

Claim. The composite

$$\mathrm{PSh}(C) \xrightarrow{f^*} \mathrm{PSh}(D) \to \mathrm{Shv}_\rho(D)$$

factors through $\mathrm{Shv}_\tau(C)$. Let $R \hookrightarrow y(c)$ be a τ-covering sieve. It suffices to show that $f^*R \to f^*y(c)$ becomes an effective epimorphism after ρ-sheafification. Pull it back along a map $y(d) \to f^*y(c)$, corresponding to a map $f(d) \to c$. The resulting sieve S on d consists of those arrows $e \to d$ whose composite $f(e) \to c$ belongs to R. After any base change $d' \to d$, the pullback of R along $f(d') \to c$ is a τ-cover. By (2), every object of its indexing slice is, up to equivalence, the image of an object of $D_{/d'}$. It follows that the sieve generated by $f(S|_{d'})$ is this pullback sieve, and hence is a τ-cover. Thus S is a ρ-cover. This proves the claim and gives a morphism of logoi

$$L_\rho f^*\colon\ \mathrm{Shv}_\tau(C) \longrightarrow \mathrm{Shv}_\rho(D).$$

It remains to prove that $L_\rho f^*$ detects effective epimorphisms. Let $X \to Y$ be a map of τ-sheaves whose image is an effective epimorphism. To test that $X \to Y$ is an effective epimorphism, fix $y(c) \to Y$. By (1), after replacing c by an equivalent object we may write $c = f(d)$. The corresponding section of f^*Y over d lifts locally through f^*X after a ρ-covering sieve S of d, because the sheafification of $f^*X \to f^*Y$ is an effective epimorphism. By the definition of ρ, the sieve generated by $f(S)$ is a τ-cover of c. Over this cover the original section lifts through X, so $X \to Y$ is an effective epimorphism. Hence $L_\rho f^*$ is conservative on effective epimorphisms, and the corresponding morphism of topoi is surjective. □

Factorization through the spectrum

Now, let Λ be a complete Boolean algebra. Then $\mathrm{Spec}(\Lambda)$ is a 0-dimensional spectral space. By some abstract theory we will treat as a black-box, there is an inclusion

$$u\colon \Lambda \hookrightarrow \mathrm{Open}(\mathrm{Spec}(\Lambda)),$$

which is the inclusion of the quasi-compact open (=clopen) subsets. This inclusion admits a left exact left adjoint

$$v\colon \operatorname{Open}(\operatorname{Spec}(\Lambda)) \to \Lambda,$$

which is given by sending U to $\bigvee_{u(\lambda)\subseteq U} \lambda$. Thus $v(U)$ is the join in Λ of all clopen subsets contained in U; it need not itself correspond to a clopen subset of $\operatorname{Spec}(\Lambda)$. This results in a left exact localization

$$\varphi_*\colon \operatorname{Shv}(\Lambda) \rightleftarrows \operatorname{Shv}(\operatorname{Spec}(\Lambda)) : \varphi^*,$$

where φ^* is the left Kan extension along v, while φ_* is left Kan extension along u, or equivalently precomposition with v. Note that φ_* is fully faithful.

Lemma 6.125. *For every effective epimorphism $\bigsqcup_{i=1}^n Y_i \to X$ in $\operatorname{Shv}(\Lambda)$, the map $\bigsqcup_{i=1}^n \varphi_*(Y_i) \to \varphi_*(X)$ is an effective epimorphism in $\operatorname{Shv}(\operatorname{Spec}(\Lambda))$.*

Proof. The category $\operatorname{Shv}(\operatorname{Spec}(\Lambda))$ is generated under colimits by the image of u: as $\operatorname{Spec}(\Lambda)$ is a spectral space, it is generated by the quasi-compact opens. Since we have

$$\varphi_*(Y_i) \times_{\varphi_*(X)} u(\lambda) = \varphi_*(Y_i \times_X \lambda),$$

we may assume that $X = \lambda \in \Lambda$. Now, consider the epi-mono factorization

$$Y_i \overset{u_i}{\twoheadrightarrow} \lambda_i \hookrightarrow \lambda.$$

Note that the map $\lambda_i \hookrightarrow \lambda$ is representable, because any subobject of a representable is representable (since Λ is precisely the subobjects of the terminal sheaf).

Since $\Lambda_{/\lambda_i}$ has dimension at most 0, the map u_i has a section s_i. So, we then have the composite

$$\bigsqcup_{i=1}^n u(\lambda_i) \xrightarrow{(s_i)} \bigsqcup_{i=1}^n \varphi_*(Y_i) \to u(\lambda),$$

which is an effective epimorphism in $\operatorname{Shv}(\operatorname{Spec}(\Lambda))$, because u preserves *finite* joins. But then it follows that also the second map is an effective epimorphism. □

Proof of the completeness theorem

We can now finally deduce the proof of the main result.

Proof of Theorem 6.118. First suppose that T is coherent. By Proposition 6.113, there is a finitary site (C, τ) with finite limits and an equivalence $T^{\mathrm{hyp}} \simeq \operatorname{Shv}_\tau(C)^{\mathrm{hyp}}$. The composite $T^{\mathrm{hyp}} \to T$ is surjective, so it is enough to construct a surjection onto $\operatorname{Shv}_\tau(C)^{\mathrm{hyp}}$.

By Corollary 6.124, there exists a complete Boolean algebra Λ and a surjection $\operatorname{Shv}(\Lambda) \twoheadrightarrow \operatorname{Shv}_\tau(C)$. We claim that this factors as follows:

$$\operatorname{Shv}(\Lambda) \xrightarrow{\varphi_*} \operatorname{Shv}(\operatorname{Spec}(\Lambda)) \to \operatorname{Shv}_\tau(C).$$

Since C has finite limits, the given morphism of topoi $\operatorname{Shv}(\Lambda) \to \operatorname{Shv}_\tau(C)$ corresponds to a morphism of logoi $\operatorname{Shv}_\tau(C) \to \operatorname{Shv}(\Lambda)$, which is obtained by left Kan extending some left exact functor $C \to \operatorname{Shv}(\Lambda)$. We may then consider the following composite:

$$C \to \operatorname{Shv}(\Lambda) \xrightarrow{\varphi_*} \operatorname{Shv}(\operatorname{Spec}(\Lambda)).$$

This is still left exact. Since τ is finitary, it suffices for descent that finite covering families be carried to effective epimorphisms, and this follows from Lemma 6.125. We therefore obtain a morphism of topoi $\mathrm{Shv}(\mathrm{Spec}(\Lambda)) \to \mathrm{Shv}_\tau(C)$. Since $\varphi^*\varphi_* \simeq \mathrm{id}$ by full faithfulness of φ_*, its composite with $\mathrm{Shv}(\Lambda) \to \mathrm{Shv}(\mathrm{Spec}(\Lambda))$ is the original surjection. Right cancellation shows that $\mathrm{Shv}(\mathrm{Spec}(\Lambda)) \to \mathrm{Shv}_\tau(C)$ is itself surjective.

For every point $x \in \mathrm{Spec}(\Lambda)$, the stalk functor gives a point $\mathrm{An} \to \mathrm{Shv}(\mathrm{Spec}(\Lambda))$. Their coproduct defines a surjection

$$\coprod_{x \in \mathrm{Spec}(\Lambda)} \mathrm{An} \longrightarrow \mathrm{Shv}(\mathrm{Spec}(\Lambda)),$$

because stalks at all points jointly detect isomorphisms of sheaves on a topological space. The source is hypercomplete, so the composite with $\mathrm{Shv}(\mathrm{Spec}(\Lambda)) \to \mathrm{Shv}_\tau(C)$ factors through the hypercompletion $\mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. The factor remains surjective, and composing with $T^{\mathrm{hyp}} \to T$ proves the theorem for coherent T.

Finally, suppose that T is only locally coherent. Choose a set of coherent objects $(U_i)_{i \in I}$ covering the terminal object. Then $\prod_{i \in I} T_{/U_i} \to T$ is surjective, and every slice $T_{/U_i}$ is coherent. Applying the coherent case to each slice and taking the product of the resulting sets gives a surjection $\mathrm{An}^S \to T$ for a set S. □

6.8 Pretopoi

Pretopoi isolate the finitary operations which preserve coherent objects: fiber products, finite coproducts, and effective quotients of groupoid objects. Thus the coherent objects of a coherent topos form its finitary or syntactic core, while sheafification for the effective epimorphism topology reconstructs the ambient topos. The main result of this section makes this relationship precise in the bounded and hypercomplete settings. It also prepares for Makkai completeness: points are models of the pretopos of coherent objects, and ultraproducts retain enough structure to recover that syntax.

Definition 6.126. A category C is a *local pretopos* if the following conditions are satisfied:

(1) C admits fiber products;

(2) C admits finite coproducts, and they are van Kampen;

(3) C admits colimits of groupoid objects, which are effective and universal (or equivalently van Kampen).

A *pretopos* is a local pretopos with a final object.

Remark 6.127. In a local pretopos, we have the factorization system (effective epi, mono). Also the factorization system (n-connected, n-truncated) exists, but this is less obvious. It amounts to showing that n-truncation exists.

Definition 6.128. Let C be a local pretopos. The *effective epi topology* on C is defined as follows: a sieve on $X \in C$ is declared to be a covering sieve if it contains a *finite* family $(Y_i \to X)_i$ such that the map $\bigsqcup_i Y_i \to X$ is an effective epimorphism.

If C is small, we obtain a topos $\mathrm{Shv}(C)$ with respect to this topology.

Proposition 6.129. *If T is a topos, then T^{coh} is a local pretopos. In particular, if T is coherent, then T^{coh} is a pretopos.*

Proof. Since T satisfies conditions (1)–(3), it suffices to show that T^{coh} is closed under the following operations:

- Fiber products: This is clear from the definition of coherent objects.
- Finite coproducts: This follows by induction, reducing in degree zero to the stability of quasi-compactness under finite coproducts.
- Colimits of groupoids: Let $X_\bullet$ be a groupoid object with realization X. The square

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X. \end{array}$$

  is a pullback and $X_0 \to X$ is an effective epimorphism. The descent criterion for relative coherence, applied successively in every degree, therefore shows that X is coherent whenever X_0 and X_1 are coherent. This is [Lur18a, Proposition A.6.1.6]. □

Definition 6.130. A *prelogos* is just a pretopos. A *morphism of prelogoi* from S to T is a functor $f^*\colon S \to T$ which preserves finite limits, finite coproducts, and colimits of groupoids. This defines a category $\mathrm{Logos}^{\mathrm{pre}}$ of prelogoi. The category of *pretopoi* is defined as $\mathrm{Topos}^{\mathrm{pre}} := (\mathrm{Logos}^{\mathrm{pre}})^{\mathrm{op}}$.

Definition 6.131. We denote by $\mathrm{Topos}^{\mathrm{coh}} \subseteq \mathrm{Topos}$ the non-full subcategory spanned by the coherent topoi and the *coherent morphisms*: those morphisms of topoi $f\colon T \to S$ such that $f^*\colon S \to T$ preserves coherent objects.

There are functors

$$\mathrm{Topos}^{\mathrm{coh}} \to \mathrm{Topos}^{\mathrm{pre}}, \qquad T \mapsto T^{\mathrm{coh}}$$

and

$$\mathrm{Topos}^{\mathrm{pre}} \to \mathrm{Topos}^{\mathrm{coh}}, \qquad C \mapsto \mathrm{Shv}(C).$$

Definition 6.132. A pretopos C is called *bounded* if it is small and every $X \in C$ is truncated.

A pretopos C is *hypercomplete* if the following conditions are satisfied:

- It admits colimits of Kan simplicial objects, meaning simplicial objects whose matching maps satisfy the internal Kan lifting condition.
- If $X_\bullet \to X$ is a trivial Kan fibration, meaning a Kan fibration whose relative homotopy objects are all trivial, then $|X_\bullet| \simeq X$.

Theorem 6.133. *There are equivalences*

$$\mathrm{Topos}^{\mathrm{coh},b} \simeq \mathrm{Topos}^{\mathrm{pre},b}, \qquad T \mapsto T^{\mathrm{coh}}_{<\infty}, \qquad C \mapsto \mathrm{Shv}(C)$$

and

$$\mathrm{Topos}^{\mathrm{coh},\textit{loc coh},\mathrm{hyp}} \simeq \mathrm{Topos}^{\mathrm{pre},\mathrm{hyp}}, \qquad T \mapsto T^{\mathrm{coh}}, \qquad C \mapsto \mathrm{Shv}(C)^{\mathrm{hyp}}.$$

These equivalences are induced by the effective epimorphism topology and are proved in [Lur18a, Sections A.6.5–A.7.4]. In the second equivalence, local coherence ensures that T^{coh} is essentially small; hypercompleteness is exactly the additional descent condition needed to recover T from this pretopos.

6.9 Makkai completeness

The preceding section recovers a coherent topos from its pretopos of coherent objects. Makkai completeness addresses the converse logical question: can this finitary syntax be recovered from its models? For a coherent topos T, the models are its points, and the additional structure needed for reconstruction is given by ultraproducts.

Question 6.134. Can we recover T from its category of points?

The category $\mathrm{Pt}(T)$ alone does not contain enough information. For example, if T classifies local rings, then $\mathrm{Pt}(T)$ is the category of local rings, but the purely categorical structure does not remember which constructions are first-order. Ultraproducts record this extra structure. The resulting completeness statement says that the theory, equivalently its coherent topos or pretopos, can be recovered from its category of models equipped with these operations.

Definition 6.135. An *ultracategory* is a category equipped with coherent ultraproduct operations: for every ultrafilter $\mathcal{U}$ on a set I, there is a functor

$$P^{\mathcal{U}} \colon C^I \to C.$$

More precisely, let $\mathrm{Stone}^{\mathrm{free}}$ denote the category of Stone–Čech compactifications of sets. The category $\mathrm{Cat}^{\mathrm{ultra}}$ of ultracategories is the subcategory of $\mathrm{LaxFun}((\mathrm{Stone}^{\mathrm{free}})^{\mathrm{op}}, \mathrm{Cat})$ spanned by the lax functors F satisfying:

- We have $F(\beta I) \simeq \prod_I F(*)$.
- Given $f : \beta I \to \beta J$ and $g : \beta J \to \beta K$, if f is induced by a map $I \to J$, then $f^* \circ g^* \simeq (g \circ f)^*$.

Theorem 6.136 (Makkai completeness)**.** *The 2-functor*

$$\mathrm{Topos}^{\mathrm{coh,b}} \longrightarrow \mathrm{Cat}^{\mathrm{ultra}}, \qquad T \longmapsto \mathrm{Pt}(T),$$

is fully faithful.

Warning 6.137. The higher-categorical formulation in Theorem 6.136 is the version stated in the lectures. The available reference [Lur18b, Theorem 2.3.1 and Corollary 2.3.3] proves strong conceptual completeness and Makkai duality for small classical pretopoi, classical coherent topoi, and ordinary categories of set-valued models. In particular, that reference proves the restriction of the theorem to classical topoi and classical categories, but it does not by itself supply a proof of the displayed higher-categorical extension. We do not know a counterexample to the extension.

Remark 6.138. There is a pullback square

$$\begin{array}{ccc} \mathrm{CondAn} & \longrightarrow & \mathrm{Cat}^{\mathrm{ultra}} \\ \downarrow & & \downarrow \\ \mathrm{An} & \longrightarrow & \mathrm{Cat}. \end{array}$$

For An, every lax functor occurring here is automatically strict.

6.10 Compactly assembled categories and exponentiability

In this section we characterize the exponentiable topoi, i.e. those topoi T for which the exponential S^T exists for all topoi S. The answer is provided by the notion of compactly assembled categories: a topos is exponentiable if and only if it is compactly assembled as a presentable category. We begin by studying finitary functors as an auxiliary notion, then introduce compactly assembled categories and establish their connection to classifying topoi, before proving the characterization of exponentiable topoi.

6.10.1 Finitary functor topoi

We write $\mathrm{Fun}^{\kappa}(C,T) \subseteq \mathrm{Fun}(C,T)$ for the full subcategory of κ-filtered-colimit-preserving functors. We write $\mathrm{Cat}^{\mathrm{acc},\kappa}$ for the category of accessible categories admitting κ-filtered colimits and functors preserving those colimits.

Proposition 6.139. *Let C be an accessible category with κ-filtered colimits, let T be a topos. Then $\mathrm{Fun}^{\kappa}(C,T)$ is a topos.*

Proof. We may write $C = \mathrm{Ind}_{\lambda}(C_0)$ for some small category C_0, and some regular cardinal $\lambda \gg \kappa$. It then follows that

$$\mathrm{Fun}^{\kappa}(C,T) \subseteq \mathrm{Fun}^{\lambda}(C,T) \simeq \mathrm{Fun}(C_0,T).$$

Note that $\mathrm{Fun}(C_0,T)$ is a topos. Moreover, $\mathrm{Fun}^{\kappa}(C,T)$ is closed under colimits and finite limits. It remains to show that $\mathrm{Fun}^{\kappa}(C,T)$ is accessible. We will do this by cutting it out of $\mathrm{Fun}(C_0,T)$ by imposing a small set of conditions.

Note that a functor $F \in \mathrm{Fun}^{\lambda}(C,T)$ preserves κ-filtered colimits if and only if F preserves colimits indexed by κ-filtered posets. Using $\lambda \gg \kappa$, we may write every κ-filtered poset as a λ-filtered colimit of λ-small κ-filtered colimits, so the condition is equivalent to F preserving λ-small κ-filtered colimits.

There is only a set of isomorphism classes of λ-small κ-filtered posets. For each such poset A, the category $\mathrm{Fun}(A,C)$ is accessible, so choose a small generating family of diagrams $\alpha\colon A \to C$. Preservation of the colimit of these generators implies preservation for every A-diagram, because both sides of the comparison commute with sufficiently filtered colimits of diagrams. We therefore obtain a pullback square

$$\begin{array}{ccc} \mathrm{Fun}^{\kappa}(C,T) & \longrightarrow & \mathrm{Fun}^{\lambda}(C,T) \\ \downarrow & & \downarrow \\ \prod_{\alpha} T & \longrightarrow & \prod_{\alpha} \mathrm{Ar}(T), \end{array}$$

where the right vertical functor sends F to the collection of comparison maps $\mathrm{colim}_A F\alpha \to F(\mathrm{colim}_A \alpha)$, and the bottom map sends a family of objects to the corresponding family of identity arrows. All three other corners and all functors in this square are accessible. Hence $\mathrm{Fun}^\kappa(C,T)$ is accessible. Since it is a full subcategory of the topos $\mathrm{Fun}(C_0,T)$ closed under colimits and finite limits, it is itself a topos. □

Remark 6.140. The category $\mathrm{Fun}^\kappa(C,T)$ has a universal property: for every topos S, there is a natural equivalence

$$\mathrm{Geom}(\mathrm{Fun}^\kappa(C,T),S) \simeq \mathrm{Fun}^\kappa(C,\mathrm{Geom}(T,S)).$$

Both sides are naturally a full subcategory of $\mathrm{Fun}(C\times S,T)$, and one can check that in both cases we precisely get those functors $C\times S\to T$ that preserve κ-filtered colimits in the left variable, and preserve finite limits and colimits in the second variable.

Corollary 6.141. *The functor* $\mathrm{Pt}\colon \mathrm{Topos}\to \mathrm{Cat}^{\mathrm{acc},\kappa}$ *admits a (2-categorical) left adjoint*

$$\mathrm{Cat}^{\mathrm{acc},\kappa}\to \mathrm{Topos},\qquad C\mapsto \mathrm{Fun}^\kappa(C,\mathrm{An}).$$

6.10.2 Compactly assembled categories

We now introduce the central notion of this section. A category is called compactly assembled if it satisfies certain retract conditions with respect to ind-categories. The following proposition provides three equivalent characterizations.

Proposition 6.142. *Let C be a category with filtered colimits. Then the following are equivalent:*

(1) *There exists a small category C_0 such that C is a retract of* $\mathrm{Ind}(C_0)$ *in* Cat^ω *(the category of categories with filtered colimits).*

(2) *There exists a small category C_0 such that there exists an adjunction $C\rightleftarrows \mathrm{Ind}(C_0)$ in* Cat^ω *such that $C\hookrightarrow \mathrm{Ind}(C_0)$ is fully faithful.*

(3) *C is accessible, and the "colimit functor"* $\mathrm{colim}\colon \mathrm{Ind}(C)\to C$ *(defined as the left adjoint to $y\colon C\hookrightarrow \mathrm{Ind}(C)$) admits a further left adjoint $y'\colon C\hookrightarrow \mathrm{Ind}(C)$.*

Definition 6.143. A category satisfying the equivalent conditions of Proposition 6.142 is called *compactly assembled*. We write $\mathrm{Cat}^{\mathrm{ca}}$ for the category of compactly assembled categories and filtered-colimit-preserving functors.

Proof of Proposition 6.142. Note that (2) clearly implies (1), since the unit of the adjunction exhibits the map $\mathrm{Ind}(C_0)\to C$ as a retraction of $C\hookrightarrow \mathrm{Ind}(C_0)$.

For (3) $\implies$ (2), we write $\mathrm{Ind}(C)=\bigcup_{\lambda>\kappa}\mathrm{Ind}(C^\lambda)$, where C is κ-accessible. Since C^κ is small, we have $y'(C^\kappa)\subseteq \mathrm{Ind}(C^\lambda)$ for some λ. Since C is generated under colimits by C^κ and y' preserves colimits, this implies that $y'(C)\subseteq \mathrm{Ind}(C^\lambda)$. But then the functor $y'\colon C\to \mathrm{Ind}(C^\lambda)$ is a left adjoint to $\mathrm{colim}\colon \mathrm{Ind}(C^\lambda)\to C$, giving (2).

Finally, we prove that (1) $\implies$ (3). The assumption gives us functors

$$C\xrightarrow{u}\mathrm{Ind}(C_0)\xrightarrow{v}C$$

satisfying $vu = \mathrm{id}_C$. We may now consider the idempotent $uv\colon \mathrm{Ind}(C_0) \to \mathrm{Ind}(C_0)$. We may then write

$$C = \lim(\cdots \to \mathrm{Ind}(C_0) \xrightarrow{uv} \mathrm{Ind}(C_0) \xrightarrow{uv} \mathrm{Ind}(C_0)),$$

and in particular C is a limit of accessible categories, hence itself accessible. Consider the following diagram

$$\begin{array}{ccccc} \mathrm{Ind}(C) & \xrightarrow{\mathrm{Ind}(u)} & \mathrm{Ind}(\mathrm{Ind}(C_0)) & \xrightarrow{\mathrm{Ind}(v)} & \mathrm{Ind}(C) \\ \downarrow{\scriptstyle \mathrm{colim}} & & \downarrow{\scriptstyle \mathrm{colim}} & & \downarrow{\scriptstyle \mathrm{colim}} \\ C & \xrightarrow{u} & \mathrm{Ind}(C_0) & \xrightarrow{v} & C. \end{array}$$

The middle vertical map admits a left adjoint $y'\colon \mathrm{Ind}(C_0) \hookrightarrow \mathrm{Ind}(\mathrm{Ind}(C_0))$.

Now, we show that the left adjoint $y'\colon C \hookrightarrow \mathrm{Ind}(C)$ exists. We may do this pointwise, so consider some $X \in C$. The map $\mathrm{Hom}(X, \mathrm{colim}(-))$ is a retract of the representable copresheaf $\mathrm{Hom}(\mathrm{Ind}(v)(y'(u(X))), -)$ in $\mathrm{Fun}(\mathrm{Ind}(C), \mathrm{An})$. Since $\mathrm{Ind}(C)$ is idempotent complete, this means that $\mathrm{Hom}(X, \mathrm{colim}(-))$ is representable. Hence there exists some $y'(X) \in \mathrm{Ind}(C)$, which then provides the desired left adjoint object to X. □

Corollary 6.144. *Let C be a presentable category. Then C is compactly assembled if and only if it is a retract in Pr^{L} of a compactly generated category.*

6.10.3 Classifying topoi

The key property of compactly assembled categories is that they admit classifying topoi for sheaves valued in them. We establish this through an equivalence relating functors out of $\mathrm{Fun}^{\omega}(C, \mathrm{An})$ to functors from C^{op}.

Proposition 6.145. *Let C be compactly assembled, and let D be presentable. Then there is an equivalence*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), D) \simeq \mathrm{Fun}_{\omega}(C^{\mathrm{op}}, D),$$

where the RHS means those functors that preserve cofiltered limits.

Proof sketch. If $C = \mathrm{Ind}(C_0)$ is compactly generated, we have $\mathrm{Fun}^{\omega}(C, \mathrm{An}) = \mathrm{Fun}(C_0, \mathrm{An}) = \mathrm{PSh}(C_0^{\mathrm{op}})$, whereas $\mathrm{Fun}_{\omega}(C^{\mathrm{op}}, D) = \mathrm{Fun}(C_0^{\mathrm{op}}, D)$. We would like to say: the claim now follows as every C is a retract of a compactly generated one. However, for this reasoning to work, we need to find a way to write down the comparison map for the compactly generated case in a way that does not rely on picking an identification $C = \mathrm{Ind}(C_0)$.

To this end, consider the category

$$\mathrm{Fun}^{+}(C, \mathrm{An}) := \mathrm{Fun}^{\omega}(C, \mathrm{An}) \cup \mathrm{Fun}^{\mathrm{rep}}(C, \mathrm{An}) \subseteq \mathrm{Fun}(C, \mathrm{An}).$$

Consider a functor $F\colon \mathrm{Fun}^{+}(C, \mathrm{An}) \to D$. Then, if $C = \mathrm{Ind}(C_0)$, one can show that:

- F is left Kan extended from $\mathrm{Fun}^{\mathrm{rep}}$ if and only if $F|_{\mathrm{Fun}^{\omega}}$ preserves colimits.
- F is right Kan extended from Fun^{ω} if and only if $F|_{\mathrm{Fun}^{\mathrm{rep}}}$ preserves cofiltered limits.

Using the adjunction $C \rightleftarrows \mathrm{Ind}(C_0)$, we then deduce that the following are equivalent:

- F is left Kan extended from $\mathrm{Fun}^{\mathrm{rep}}$ and $F|_{\mathrm{Fun}^{\mathrm{rep}}}$ preserves cofiltered limits.
- F is right Kan extended from Fun^{ω} and $F|_{\mathrm{Fun}^{\omega}}$ preserves colimits.

It follows that both sides $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), D)$ and $\mathrm{Fun}_{\omega}(C^{\mathrm{op}}, D)$ agree with the full subcategory of $\mathrm{Fun}(\mathrm{Fun}^{+}(C, \mathrm{An}), D)$ satisfying these two equivalent conditions. □

Corollary 6.146. *Let T be a topos and C a presentable category which is compactly assembled. Then the equivalence*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), T) \simeq \mathrm{Fun}_{\omega}(C^{\mathrm{op}}, T)$$

restricts to an equivalence

$$\mathrm{Fun}_{\mathrm{Logos}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), T) \simeq \mathrm{Fun}^{\mathrm{R}}(C^{\mathrm{op}}, T).$$

Remark 6.147. Recall that the category $\mathrm{Fun}^{\mathrm{R}}(C^{\mathrm{op}}, T)$ is the Lurie tensor product $C \otimes T$ in Pr^{L}, which in turn may be written as $\mathrm{Fun}^{\mathrm{R}}(T^{\mathrm{op}}, C)$. We denote this category by $\mathrm{Shv}_C(T)$ and call it the category of *C-valued sheaves on T*. Keeping track of the functoriality in T shows that the topos $\mathrm{Fun}^{\omega}(C, \mathrm{An})$ classifies C-valued sheaves.

We have thus shown that if C is compactly assembled, then a classifying topos for C-valued sheaves exists. It turns out that the converse is also true:

Proposition 6.148. *Let C be a presentable category. Then a classifying topos for C-valued sheaves exists if and only if C is compactly assembled.*

Proof. It remains to show the "only if" part. Let E be such a classifying topos, and write it as a left exact localization $i_*\colon E \hookrightarrow \mathrm{PSh}(D)$ of a presheaf category. This induces a geometric morphism $i_*\colon \mathrm{Shv}_C(E) \hookrightarrow \mathrm{Shv}_C(\mathrm{PSh}(D))$. Let $F_{\mathrm{univ}} \in \mathrm{Shv}_C(E)$ be the universal C-valued sheaf on E. By universality, there exists a morphism of topoi $f\colon \mathrm{PSh}(D) \to E$ satisfying $i_*(F_{\mathrm{univ}}) \simeq f^*(F_{\mathrm{univ}})$. By full faithfulness of i_*, we have $F_{\mathrm{univ}} \simeq i^* i_*(F_{\mathrm{univ}}) \simeq (f \circ i)^*(F_{\mathrm{univ}})$. Universality again gives $f \circ i \simeq \mathrm{id}_E$, so E is a retract of $\mathrm{PSh}(D)$ in Topos.

Applying $\mathrm{Pt}(-)$ shows that $C \simeq \mathrm{Shv}_C(\mathrm{An}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathrm{An}, E)$ is a retract of $\mathrm{Pt}(\mathrm{PSh}(D))$ in Cat^{ω}. Points of $\mathrm{PSh}(D)$ are flat functors $D \to \mathrm{An}$, and their category is the filtered-colimit completion $\mathrm{Ind}(D^{\mathrm{op}})$. It is therefore compactly generated. Hence C is a retract of a compactly generated category in Cat^{ω}, and is compactly assembled by Proposition 6.142. □

Remark 6.149. Let C be a compactly assembled category. Then there is an equivalence

$$\mathrm{Pt}(\mathrm{Fun}^{\omega}(C, \mathrm{An})) \simeq C,$$

under which $X \in C$ corresponds to the evaluation functor $\mathrm{ev}_X\colon \mathrm{Fun}^{\omega}(C, \mathrm{An}) \to \mathrm{An}$.

Lemma 6.150. *The functor* $\mathrm{Cat}^{\mathrm{ca}} \to \mathrm{Topos}$ *given by* $C \mapsto \mathrm{Fun}^{\omega}(C, \mathrm{An})$ *is a fully faithful* 2*-functor.*

Proof. By Corollary 6.141, the functor $\mathrm{Pt}\colon \mathrm{Topos} \to \mathrm{Cat}^{\mathrm{acc},\omega}$ admits a left adjoint given by $C \mapsto \mathrm{Fun}^{\omega}(C, \mathrm{An})$. The unit of this adjunction provides a map $C \to \mathrm{Pt}(\mathrm{Fun}^{\omega}(C, \mathrm{An}))$, which by Remark 6.149 is an equivalence when C is compactly assembled. Therefore the restriction of this adjunction to $\mathrm{Cat}^{\mathrm{ca}}$ exhibits $\mathrm{Cat}^{\mathrm{ca}} \to \mathrm{Topos}$ as fully faithful. □

6.10.4 Exponentiable topoi

Having established that compactly assembled categories have classifying topoi, we now turn to the main application: characterizing exponentiable topoi. Recall that the exponential S^T in Topos, if it exists, represents the functor $U \mapsto \mathrm{Hom}_{\mathrm{Topos}}(U \otimes T, S)$.

Let S and T be topoi. If the *exponential* (internal hom) in Topos exists, it will be denoted S^T. In other words, this is a topos S^T equipped with a morphism of topoi $S^T \otimes T \to S$ which induces an equivalence

$$\mathrm{Hom}_{\mathrm{Topos}}(U, S^T) \simeq \mathrm{Hom}_{\mathrm{Topos}}(U \otimes T, S).$$

Observation 6.151. If S^T exists, it is automatically a 2-categorical exponential. Indeed, for every topos U, the arrow topos $\mathrm{Ar}(U)$ represents arrows between geometric morphisms out of U, and

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Geom}(U, S^T)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Ar}(U), S^T)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Ar}(U) \otimes T, S)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Ar}(U \otimes T), S)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Geom}(U \otimes T, S)).\end{aligned}$$

Together with the original universal property on objects, this identifies the entire Hom categories. Equivalently, one may repeat the same argument with the topos of $[n]$-diagrams for every n and use the complete Segal description of a category.

Remark 6.152. We have $\mathrm{Pt}(S^T) \simeq \mathrm{Geom}(T, S)$.

Definition 6.153. We say that a topos T is *exponentiable* if S^T exists for all S.

Lemma 6.154. *If C is compactly assembled and D is compactly generated, then $C \otimes D$ is compactly assembled.*

Proof. Choose a retraction of C from a compactly generated presentable category C'. Tensoring with D gives a retraction of $C \otimes D$ from $C' \otimes D$. The tensor product of compactly generated presentable categories is compactly generated: if $C' \simeq \mathrm{Ind}(C'_0)$ and $D \simeq \mathrm{Ind}(D_0)$, it is generated under filtered colimits by the objects $c \otimes d$ with $c \in C'_0$ and $d \in D_0$. The claim follows from Corollary 6.144. □

Lemma 6.155. *Every topos is a pullback in* Topos *of a diagram of presheaf topoi* $\mathrm{PSh}(D)$ *with D admitting finite limits.*

Proof sketch. Present the topos as a left exact localization of a presheaf topos. The localization is the equalizer of the identity and its idempotent localization functor. Replacing this equalizer by the usual pullback involving an arrow category expresses the original topos as a pullback of three presheaf topoi. Closing the small indexing categories under finite limits gives the stated form. See [AL18, Section 4, in particular the construction preceding Theorem 4.33]. □

Theorem 6.156. *A topos T is exponentiable if and only if it is compactly assembled.*

Proof. For the "only if" direction, assume that T is exponentiable. To show it is compactly assembled, consider some compactly assembled category C and consider $S := \mathrm{Fun}^{\omega}(C, \mathrm{An})$. By assumption, the exponential S^T exists. Moreover, we have

$$\mathrm{Geom}(U, S^T) \simeq \mathrm{Geom}(U \otimes T, S) \simeq \mathrm{Shv}_C(U \otimes T) \simeq \mathrm{Shv}_{C \otimes T}(U).$$

In particular, taking $C = \mathrm{An}$, we see that S^T classifies T-valued sheaves. By Proposition 6.148, it follows that T is compactly assembled.

For the "if" direction, assume T is compactly assembled. By Lemma 6.155, we may write S as a pullback in Topos of a square of the form

$$\begin{array}{ccc} S & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & C_0, \end{array}$$

where A_0, B_0 and C_0 are presheaf topoi. Since exponentiation by T, whenever defined, is right adjoint to $(-) \otimes T$, it preserves limits. It is therefore enough to construct S^T when S is a presheaf topos.

Assume then that $S = \mathrm{PSh}(D)$, where D has finite limits. We have

$$S = \mathrm{PSh}(D) \xrightarrow{\sim} \mathrm{Fun}^{\omega}(\mathrm{Ind}(D^{\mathrm{op}}), \mathrm{An}).$$

Set $K := \mathrm{Ind}(D^{\mathrm{op}})$. The category K is compactly generated, so $K \otimes T$ is compactly assembled by Lemma 6.154. Its classifying topos E exists by Proposition 6.148, and for every topos U we have

$$\mathrm{Geom}(U, E) \simeq \mathrm{Shv}_{K \otimes T}(U) \simeq \mathrm{Shv}_K(U \otimes T) \simeq \mathrm{Geom}(U \otimes T, \mathrm{PSh}(D)).$$

Thus $E \simeq S^T$. This proves the theorem; compare [AL18, Theorem 4.33]. □

6.11 Topoi of parametrized objects

The Grothendieck construction $\int_{\mathrm{An}} C$ organizes families of objects of C parametrized by animae. Its universal property separates arbitrary colimits into anima-indexed colimits and weakly contractible colimits, and this makes the van Kampen condition particularly transparent. The main result characterizes those accessible categories C for which $\int_T C$ is a topos for every topos T: these are the *loci*, the categories whose weakly contractible colimits are van Kampen. This also identifies stable categories and categories of ∞-connected objects as two manifestations of the same construction. The section is logically independent of the later chapters, but the results provide a useful application of descent and may also be found in [Hoy19b].

6.11.1 Categories of An-parametrized objects

Definition 6.157. Let C be a category, not necessarily small. We define the functor $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ as the cartesian unstraightening of the functor

$$\mathrm{Fun}(-, C)\colon \mathrm{An}^{\mathrm{op}} \to \mathrm{Cat}, \qquad A \mapsto \mathrm{Fun}(A, C).$$

We refer to $\int_{\mathrm{An}} C$ as the category of An-*parametrized objects of* C. Note that an object of $\int_{\mathrm{An}} C$ is a pair (A, X), where $A \in \mathrm{An}$ is an anima and $X\colon A \to C$ is an A-indexed family of objects of C. A morphism $(A, X) \to (B, Y)$ consists of a morphism of animae $f\colon A \to B$ and a map $X \to f^*Y$ in $\mathrm{Fun}(A, C)$, where $f^*Y := Y \circ f\colon A \to C$.

The fiber of $\int_{\mathrm{An}} C$ over $A \in \mathrm{An}$ is $\mathrm{Fun}(A, C)$. By taking $A = *$, we in particular obtain an inclusion $C \hookrightarrow \int_{\mathrm{An}} C$. Note that this is fully faithful, since $\{*\} \hookrightarrow \mathrm{An}$ is fully faithful.

While we gave an explicit construction of $\int_{\mathrm{An}} C$, we will show next that it can also be characterized via the following two universal properties:

(a) For arbitrary C, the category $\int_{\mathrm{An}} C$ is obtained from C by freely adding anima-indexed colimits;

(b) If C admits weakly contractible colimits, then $\int_{\mathrm{An}} C$ admits all small colimits, the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ preserves weakly contractible colimits, and $\int_{\mathrm{An}} C$ is the free category with a functor from C satisfying these two properties.

We start with property (a), which is relatively straightforward to prove.

Lemma 6.158. *The category $\int_{\mathrm{An}} C$ admits anima-indexed colimits, and the functor $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ preserves anima-indexed colimits.*

Proof. Consider a functor $F = (A_\bullet, X_\bullet)\colon I \to \int_{\mathrm{An}} C$ for which we wish to construct a colimit, where I is an anima. Consider the colimit $A := \mathrm{colim}_{i \in I} A_i$ of the underlying animae, thus extending $A_\bullet$ to a colimit cocone $\overline{A}_\bullet\colon I^\triangleright \to \mathrm{An}$. Denote by $f_i\colon A_i \to A$ the maps in the colimit cocone. The diagram F is a lift of $A_\bullet\colon I \to \mathrm{An}$, corresponding to a section of the cartesian fibration $I \times_{\mathrm{An}} \int_{\mathrm{An}} C \to I$. Since I is an anima, this section is automatically a cartesian section, and it thus defines an object of the category

$$\Gamma^{\mathrm{cart}}_I(I \times_{\mathrm{An}} \int_{\mathrm{An}} C \to I) \quad \simeq \quad \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}(A_i, C) \quad \simeq \quad \mathrm{Fun}(\mathrm{colim}_i A_i, C) \quad \simeq \quad \mathrm{Fun}(A, C).$$

This defines an object $X \in \mathrm{Fun}(A, C)$, hence an object $(A, X) \in \int_{\mathrm{An}} C$. Applying a similar reasoning to the diagram $\overline{A}_\bullet\colon I^\triangleright \to \mathrm{An}$, we see that

$$\Gamma^{\mathrm{cart}}_{I^\triangleright}(I^\triangleright \times_{\mathrm{An}} \int_{\mathrm{An}} C \to I^\triangleright) \quad \simeq \quad \lim_{i \in (I^\triangleright)^{\mathrm{op}}} \mathrm{Fun}(A_i, C) = \mathrm{Fun}(A, C),$$

so that the object X gives rise to a cartesian section $(\overline{A}_\bullet, \overline{X}_\bullet)\colon I^\triangleright \to \int_{\mathrm{An}} C$ lifting $\overline{A}_\bullet$, which essentially by construction extends $X_\bullet$. We will show that this is a colimit cocone in $\int_{\mathrm{An}} C$, which simultaneously establishes both claims of the lemma.

To this end, consider another object $(B, Y) \in \int_{\mathrm{An}} C$. We need to show that top map in the following commutative diagram is an equivalence:

$$\begin{array}{ccc} \mathrm{Hom}_{\int_{\mathrm{An}} C}((A, X), (B, Y)) & \longrightarrow & \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\int_{\mathrm{An}} C}((A_i, X_i), (B, Y)) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Hom}_{\mathrm{An}}(A, B) & \xrightarrow{\simeq} & \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{An}}(A_i, B). \end{array}$$

The bottom map in this diagram is an equivalence by definition of A as a colimit. Therefore, it suffices to check that the square induces equivalences on vertical fibers over every map of animae $f\colon A \to B$. Letting $f_i\colon A_i \to A \xrightarrow{f} B$ denote the composite maps for all i, this map on fibers takes the following form:

$$\mathrm{Hom}_{\mathrm{Fun}(A,C)}(X, f^*Y) \to \lim_{i\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Fun}(A_i,C)}(X_i, f_i^*Y).$$

But this map is an isomorphism, since the pullback functors along the maps $A_i \to A$ induce an equivalence $\mathrm{Fun}(A, C) \xrightarrow{\sim} \lim_{i\in I^{\mathrm{op}}} \mathrm{Fun}(A_i, C)$, and the restriction of X to A_i is X_i by construction. This finishes the proof. □

Lemma 6.159. *Let D be a category admitting anima-indexed colimits. Then the inclusion $D \hookrightarrow \int_{\mathrm{An}} D$ admits a left adjoint* $\mathrm{colim}\colon \int_{\mathrm{An}} D \to D$*, given on objects by sending (A, X) to* $\mathrm{colim}_{a\in A} X_a$.

Proof. Since left adjoints may be constructed objectwise, it will suffice to show that for every $Y \in D$ there is a natural equivalence

$$\mathrm{Hom}_{\int_{\mathrm{An}} D}((A, X), (*, Y)) \xrightarrow{\sim} \mathrm{Hom}_D(\mathrm{colim}_{a\in A} X_a, Y).$$

But this is clear: since there is a unique map $A \to *$ in An, the left-hand side simplifies to $\mathrm{Hom}_{\mathrm{Fun}(A,D)}(X, \mathrm{const}_Y)$, which by the very definition of colimits in D is also the right-hand side. □

Lemma 6.160. *Let* $\mathrm{Cat}^{\mathrm{An\text{-}colim}}$ *denote the subcategory of* Cat *consisting of the categories with anima-indexed colimits and functors preserving anima-indexed colimits. Then the construction $C \mapsto \int_{\mathrm{An}} C$ defines a left adjoint to the inclusion* $\mathrm{Cat}^{\mathrm{An\text{-}colim}} \hookrightarrow \mathrm{Cat}$*, with unit and counit given by the functors*

$$i\colon C \hookrightarrow \int_{\mathrm{An}} C \qquad \textit{and} \qquad \mathrm{colim}\colon \int_{\mathrm{An}} D \to D,$$

respectively.

Proof. We check that the two triangle identities are satisfied. The first one takes the form

$$\begin{array}{ccc} D & \xhookrightarrow{i} & \int_{\mathrm{An}} D \\ & \searrow^{=} & \downarrow{\scriptstyle \mathrm{colim}} \\ & & D. \end{array}$$

This commutes via the counit map $\mathrm{colim} \circ i \to \mathrm{id}_D$, which is an isomorphism as i is fully faithful. The second triangle identity amounts to the claim that for every object $(A, X) \in \int_{\mathrm{An}} C$, the canonical map $\mathrm{colim}_{a\in A}(\{a\}, X_a) \to (A, X)$ in $\int_{\mathrm{An}} C$ is an isomorphism, which as observed before is an immediate consequence of the computation of colimits in $\int_{\mathrm{An}} C$. □

We may now deduce the first claimed universal property of $\int_{\mathrm{An}} C$:

Proposition 6.161. *Let C and D be categories and assume that D admits anima-indexed colimits. Then restriction along $i\colon C \hookrightarrow \int_{\mathrm{An}} C$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{An\text{-}colim}}(\int_{\mathrm{An}} C, D) \xrightarrow{\sim} \mathrm{Fun}(C, D).$$

Proof. By the Yoneda lemma, we may check this after applying $\mathrm{Hom}_{\mathrm{Cat}}(E, -)$ for all $E \in \mathrm{Cat}$. Observe that $\mathrm{Fun}(E, D)$ still admits anima-indexed colimits, which are computed pointwise in D. Under currying-uncurrying, the map in question then translates to the map

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{An\text{-}colim}}}(\int_{\mathrm{An}} C, \mathrm{Fun}(E, D)) \to \mathrm{Hom}_{\mathrm{Cat}}(C, \mathrm{Fun}(E, D))$$

given by restriction along $i\colon C \hookrightarrow \int_{\mathrm{An}} C$. This is an isomorphism by the adjunction from the previous lemma. □

6.11.2 Weakly contractible colimits and free cocompletion

We now move on to the second universal property of $\int_{\mathrm{An}} C$, which is more subtle; Marc said that he has not seen this formulation in the literature. For the proof, we use the category $\mathrm{PSh}^{\mathrm{small}}(C)$ of *small presheaves* on C, i.e. the full subcategory of $\mathrm{PSh}(C)$ generated by the representable presheaves under small colimits. The inclusion $C \hookrightarrow \mathrm{PSh}^{\mathrm{small}}(C)$ is the universal functor from C into a category with small colimits. Recall that $\mathrm{PSh}^{\mathrm{small}}(C) = \mathrm{PSh}(C)$ whenever C is small, but not for arbitrary C. The small presheaves may be characterized as those $\mathcal{F}$ for which the category of elements $\mathrm{El}(\mathcal{F}) = C_{/\mathcal{F}} \subseteq \mathrm{PSh}(C)_{/\mathcal{F}}$ admits a final functor from a small category.

Proposition 6.162. *Let C be a category admitting small weakly contractible colimits.*

(1) *The inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ preserves weakly contractible colimits.*

(2) *The inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ uniquely extends to a left adjoint $L\colon \mathrm{PSh}^{\mathrm{small}}(C) \to \int_{\mathrm{An}} C$.*

(3) *The right adjoint $R\colon \int_{\mathrm{An}} C \hookrightarrow \mathrm{PSh}^{\mathrm{small}}(C)$ is fully faithful, exhibiting $\int_{\mathrm{An}} C$ as a Bousfield localization of $\mathrm{PSh}^{\mathrm{small}}(C)$. In particular, $\int_{\mathrm{An}} C$ admits small colimits.*

(4) *For a category D with small colimits, a colimit-preserving functor $F\colon \mathrm{PSh}^{\mathrm{small}}(C) \to D$ inverts L-local maps if and only if its restriction $C \to D$ preserves weakly contractible colimits.*

Proof. (1) Consider a functor $X_\bullet\colon I \to C$, where I is weakly contractible, and set $X := \mathrm{colim}_{i \in I} X_i$. We have to show that the colimit of this diagram in $\int_{\mathrm{An}} C$ is $(*, x)$. This amounts to the claim that for every object $(B, Y) \in \int_{\mathrm{An}} C$, the canonical map

$$\mathrm{Hom}_{\int_{\mathrm{An}} C}((*, X), (B, Y)) \to \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\int_{\mathrm{An}} C}((*, X_i), (B, Y))$$

is an isomorphism. Since this map lives over $\mathrm{Hom}_{\mathrm{An}}(*, B) = B$, it suffices to show that the induced map on fibers over each $b \in B$ is an isomorphism. But this map on fibers is the map $\mathrm{Hom}_C(X, Y_b) \to \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_C(X_i, Y_b)$, which is an isomorphism by construction of X.

(2) We will show that the left Kan extension of the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ along the Yoneda embedding $C \hookrightarrow \mathrm{PSh}^{\mathrm{small}}(C)$ exists. Given a small presheaf $\mathcal{F}$, the pointwise formula for left Kan extensions tells us that it suffices to show that in $\int_{\mathrm{An}} C$ we may form the colimit of the functor

$$\mathrm{El}(\mathcal{F}) = C_{/\mathcal{F}} \xrightarrow{\mathrm{fgt}} C \hookrightarrow \int_{\mathrm{An}} C.$$

By assumption on $\mathcal{F}$, the category $\mathrm{El}(\mathcal{F})$ comes with a final functor $I \to \mathrm{El}(\mathcal{F})$ from some small category I. It will thus suffice to show that for any small category I and any functor $X\colon I \to C$ the colimit of $I \to \int_{\mathrm{An}} C$ exists. We do this in two steps:

- Consider the map $p\colon I \to |I|$ to the geometric realization of I. We will show that the Kan extension of X along p exists in $\int_{\mathrm{An}} C$. Since p is a cofinal functor, its relative slices are weakly contractible by Quillen's Theorem A. By part (1), this means that the Kan extension of X along p exists in C, and that it is preserved by the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$.
- The colimit $\mathrm{colim}_i X_i$ in $\int_{\mathrm{An}} C$ is now computed as the colimit of the left Kan extension $p_! X\colon |I| \to \int_{\mathrm{An}} C$, which exists since $|I|$ is an anima and we established the existence of anima-indexed colimits in Lemma 6.158.

By general nonsense, it follows that the left Kan extension $L\colon \mathrm{PSh}^{\mathrm{small}}(C) \to \int_{\mathrm{An}} C$ preserves all colimits. A right adjoint R to L is given explicitly as follows:

$$R\colon \int_{\mathrm{An}} C \to \mathrm{PSh}^{\mathrm{small}}(C), \qquad R((A,X))(Y) := \mathrm{Hom}_{\int_{\mathrm{An}} C}((*,Y),(A,X)).$$

The functor $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ induces a map $\mathrm{Hom}_{\int_{\mathrm{An}} C}((*,Y),(A,X)) \to \mathrm{Hom}_{\mathrm{An}}(*,A) \cong A$, whose fiber over $a \in A$ is $\mathrm{Hom}_C(Y, X_a)$, so we obtain isomorphisms

$$R((A,X))(Y) \;\cong\; \mathrm{colim}_{a\in A} \mathrm{Hom}_C(Y, X_a) \;\cong\; \mathrm{colim}_{a\in A}\, y(X_a)(Y).$$

In particular, we see that $R((A,X)) \simeq \mathrm{colim}_{a\in A}\, y(X_a)$ is a small colimit of representables, so that it is indeed contained in $\mathrm{PSh}^{\mathrm{small}}(C)$.

(3) To show that R is fully faithful, we must show that for every object $(A,X) \in \int_{\mathrm{An}} C$, the counit $LR(A,X) \to (A,X)$ is an isomorphism. But this is clear from the computation $R(A,X) \cong \mathrm{colim}_{a\in A}\, y(X_a)$ from before, and the fact that $(A,X) \cong \mathrm{colim}_{a\in A}(*, X_a)$ by the explicit description of anima indexed colimits from Lemma 6.158.

(4) If F inverts all L-local maps, then F in particular inverts the canonical map $\mathrm{colim}_{i\in I}\, y(X_i) \to y(\mathrm{colim}_{i\in I} X_i)$ for every weakly contractible diagram $X\colon I \to C$, so that $F(y(\mathrm{colim}_i X_i)) \simeq \mathrm{colim}_i F(y(X_i))$. Conversely, assume $F \circ y\colon C \to D$ preserves weakly contractible colimits. It will suffice to show that F inverts the unit map $\mathcal{F} \to RL(\mathcal{F})$ for all $\mathcal{F} \in \mathrm{PSh}^{\mathrm{small}}(C)$. Choose a final map $I \to \mathrm{El}(\mathcal{F})$ from a small category. Writing $\mathcal{F}$ as the associated colimit of representables and left Kan extending along $p\colon I \to |I|$ expresses $\mathcal{F}$ as an $|I|$-indexed colimit of objects $\mathrm{colim}_{i\in I_x}\, y(X_i)$, where $I_x = \mathrm{fib}_x(I \to |I|)$ is weakly contractible. The same construction in C expresses $RL(\mathcal{F})$ as the corresponding $|I|$-indexed colimit of $y(\mathrm{colim}_{i\in I_x} X_i)$. Thus the unit $\mathcal{F} \to RL(\mathcal{F})$ is an $|I|$-indexed colimit of the canonical maps which F inverts by assumption. Since F preserves colimits, it inverts the unit. □

Lemma 6.163. *A category C admits small colimits if and only if it admits anima-indexed colimits and weakly contractible colimits. Similarly, if C and D are categories with small colimits, then a functor $F\colon C \to D$ preserves small colimits if and only if it preserves both anima-indexed colimits and weakly contractible colimits.*

Proof. Assume that C admits anima-indexed colimits and weakly contractible colimits. We need to show that it admits a colimit for an arbitrary diagram $X\colon I \to C$, where I is a small category. To

this end, consider the localization functor $p\colon I \to |I|$ inverting all morphisms in I. As discussed before, the finality of p implies that the relative slices of p are weakly contractible, so that the left Kan extension $p_!X\colon |I| \to C$ of X along p exists in C. Moreover, the colimit of $p_!X$ exists in C since $|I|$ is an anima. Since Kan extensions compose, we conclude that the colimit of X exists in C.

The argument for preservation of colimits is entirely analogous. □

We are now ready to establish the second universal property of $\int_{\mathrm{An}} C$. To formulate this precisely, let us denote by $\mathrm{Cat}^{\mathrm{w.c.colim}}$ the subcategory of Cat spanned by the categories admitting weakly contractible colimits and those functors that preserve weakly contractible colimits. Note that it contains the category $\mathrm{Cat}^{\mathrm{colim}}$ of categories with colimits and colimit-preserving functors.

Proposition 6.164. *The adjunction* $\mathrm{Cat} \rightleftarrows \mathrm{Cat}^{\mathrm{An\text{-}colim}}$ *from Lemma 6.160 restricts to an adjunction*

$$\mathrm{Cat}^{\mathrm{w.c.colim}} \rightleftarrows \mathrm{Cat}^{\mathrm{colim}}.$$

In particular, if C is a category with small weakly contractible colimits and D is a category with small colimits, then restriction along the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\int_{\mathrm{An}} C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{w.c.colim}}(C, D).$$

Proof. We saw in Proposition 6.161 that restriction along the inclusion induces an equivalence

$$\mathrm{Fun}^{\mathrm{An\text{-}colim}}(\int_{\mathrm{An}} C, D) \xrightarrow{\sim} \mathrm{Fun}(C, D).$$

By part (1) of Proposition 6.162, this restriction functor restricts to a fully faithful functor

$$\mathrm{Fun}^{\mathrm{colim}}(\int_{\mathrm{An}} C, D) \hookrightarrow \mathrm{Fun}^{\mathrm{w.c.colim}}(C, D).$$

For essential surjectivity, let $F\colon C \to D$ be a functor preserving weakly contractible colimits, and consider its colimit-preserving extension $\mathrm{PSh}^{\mathrm{small}}(C) \to D$. By part (4) of Proposition 6.162, this inverts the L-local maps in $\mathrm{PSh}^{\mathrm{small}}(C)$, and so descends to a colimit-preserving functor $\int_{\mathrm{An}} C \to D$ extending F. This finishes the proof. □

6.11.3 Categories of objects parametrized by topoi

The construction of $\int_{\mathrm{An}} C$ may be generalized, replacing An by an arbitrary topos.

Definition 6.165. Let T be a cocomplete category and let C be a category admitting weakly contractible colimits. We define the *category of T-parametrized objects of C* as the tensor product

$$\int_T C \quad := \quad T \otimes \int_{\mathrm{An}} C$$

in the category $\mathrm{Cat}^{\mathrm{colim}}$ of cocomplete categories (using that $\int_{\mathrm{An}} C$ admits colimits by Proposition 6.162).

Corollary 6.166. *The functor $T \times C \hookrightarrow T \times \int_{\mathrm{An}} C \to \int_T C$ is universal among functors $F\colon T \times C \to D$ into a cocomplete category D which preserve all colimits in the first variable and weakly contractible colimits in the second variable.*

Proof. By definition, the tensor product $\int_T C = T \otimes \int_{\mathrm{An}} C$ comes equipped with a functor from $T \times \int_{\mathrm{An}} C$ satisfying the universal property that for every other cocomplete category D, precomposition with this functor defines an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\int_T C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{colim},\mathrm{colim}}(T \times \int_{\mathrm{An}} C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{colim}}(T, \mathrm{Fun}^{\mathrm{colim}}(\int_{\mathrm{An}} C, D)).$$

The claimed universal property for $\int_T C$ thus follows immediately from the universal property of $\int_{\mathrm{An}} C$ established in Proposition 6.164. □

The projection $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ preserves all colimits. Indeed, its restriction to C is the constant functor with value $*$, which preserves weakly contractible colimits, and its colimit-preserving extension under Proposition 6.164 sends (A, X) to $\mathrm{colim}_A * = A$. Hence it is precisely π. Tensoring with T in $\mathrm{Cat}^{\mathrm{colim}}$ gives an analogous functor $\pi_T\colon \int_T C \to T$. We will now identify the fiber of π_T over $A \in T$ with the category of A-parametrized objects of C.

Proposition 6.167. *Let T and C be presentable categories. Then the functor $\pi_T\colon \int_T C \to T$ is a bicartesian fibration, which is classified by both of the following functors:*

$$T \to \mathrm{Cat}, \qquad A \mapsto T_{/A} \otimes C,$$

and

$$T^{\mathrm{op}} \to \mathrm{Cat}, \qquad A \mapsto \mathrm{Fun}^{\mathrm{lim}}(T^{\mathrm{op}}_{/A}, C).$$

Proof sketch. Note that the first functor takes the form $T \to \mathrm{Pr}^{\mathrm{L}}$, hence its cocartesian unstraightening is always a bicartesian fibration whose cartesian straightening is the corresponding functor $T^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}$ obtained by passing to right adjoints, which by Lurie's explicit formula for tensor products in Pr^{L} is given by $A \mapsto \mathrm{Fun}^{\mathrm{lim}}(T^{\mathrm{op}}_{/A}, C)$. It thus suffices to prove the first claim.

Since T is presentable, write it as an accessible localization of a presheaf category $\mathrm{PSh}(D)$. Both constructions in the statement commute with such localizations, so it suffices to treat $T = \mathrm{PSh}(D)$. In this case

$$\mathrm{PSh}(D) \otimes \int_{\mathrm{An}} C \simeq \mathrm{Fun}(D^{\mathrm{op}}, \int_{\mathrm{An}} C),$$

and the projection to $\mathrm{PSh}(D)$ is computed pointwise. Its fiber over a presheaf $A\colon D^{\mathrm{op}} \to \mathrm{An}$ is therefore the category of compatible C-valued families indexed contravariantly by the category of elements $\mathrm{El}(A)$, namely $\mathrm{Fun}(\mathrm{El}(A)^{\mathrm{op}}, C)$. On the other hand, $\mathrm{PSh}(D)_{/A} \simeq \mathrm{PSh}(\mathrm{El}(A))$, so

$$\mathrm{PSh}(D)_{/A} \otimes C \simeq \mathrm{Fun}(\mathrm{El}(A)^{\mathrm{op}}, C).$$

These identifications are natural in A and identify the cocartesian transport. Passing to right adjoints identifies the cartesian straightening with $A \mapsto \mathrm{Fun}^{\mathrm{lim}}(T^{\mathrm{op}}_{/A}, C)$, as claimed. □

6.11.4 Loci

We now turn to the problem of characterizing when parametrized objects form a topos.

Question 6.168. Given a topos T and a category C, when is the T-parametrization $\int_T C$ a topos?

This is interesting because of examples like $\int_T \mathrm{Sp} = \mathrm{Exc}^1(\mathrm{An}_*^{\mathrm{fin}}, T)$, which is a topos. To answer this, we introduce the concept of a *locus*.

Definition 6.169. A *locus* is an accessible category with pullbacks and van Kampen weakly contractible colimits.

Lemma 6.170. *Let C and C' be accessible categories with pullbacks and weakly contractible colimits, and let $G\colon C' \to C$ be a conservative functor which preserves pullbacks and weakly contractible colimits. If C is a locus, then also C' is a locus.*

Proof. Consider a diagram $X_\bullet\colon I \to C'$, with I weakly contractible, and set $X := \mathrm{colim}_{i\in I} X_i$. We need to show that the top adjunction in the following diagram is an adjoint equivalence:

$$\begin{array}{ccc} C'_{/X} & \rightleftarrows & \lim_{i\in I^{\mathrm{op}}} C'_{/X_i} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\ C_{/G(X)} & \rightleftarrows & \lim_{i\in I^{\mathrm{op}}} C_{/G(X_i)}. \end{array}$$

By assumption, the bottom adjunction is an adjoint equivalence. Moreover, since G preserves pullbacks and weakly contractible colimits, the vertical maps (are defined and) commute with both of the adjoint functors. It then follows from conservativity of G that both the unit and counit of the top adjunction are isomorphisms, proving the claim. □

Corollary 6.171. *If C and C' are loci, and $F\colon C' \to C$ is accessible and preserves pullbacks and weakly contractible colimits, then also the fibers of F are loci.*

Proof. For an object $c \in C$, the fiber $F^{-1}(c)$ is the pullback of $C' \xrightarrow{F} C \leftarrow *$. Since F is accessible, this pullback is an accessible category. Pullbacks and weakly contractible colimits in the fiber are computed in C', since both F and the inclusion $c\colon * \to C$ preserve them. The functor $F^{-1}(c) \to C'$ is conservative, so the claim follows from the previous lemma. □

Corollary 6.172. *If C is a locus, then all slices $C_{/X}$ and $C_{X/}$ are loci.*

Proof. Note that $\mathrm{Ar}(C)$ is a locus, so the claim follows from the previous corollary. □

Example 6.173. Let C be a locus with a final object, and let A be a small category with finite colimits and a final object. Then the subcategory

$$\mathrm{Exc}^{[m,n]}(A, C) \subseteq \mathrm{Fun}(A, C)$$

of n-reduced m-excisive functors is a locus. Indeed, it is the fiber of the functor

$$P_{n-1}\colon\ \mathrm{Exc}^m(A, C) \longrightarrow \mathrm{Exc}^{n-1}(A, C).$$

For example, we have $\mathrm{Exc}^{[n,1]}(A, C) = \mathrm{Exc}^n_*(A, C)$.

Lemma 6.174. *Every topos is a locus. Conversely, a locus T is a topos if and only if it admits a strictly initial object.*

Proof. It is clear that every topos is a locus admitting a strictly initial object. Conversely, if T is a locus admitting a strictly initial object, then it in particular admits colimits, so T is presentable. Moreover, the functor $T^{\mathrm{op}} \to \mathrm{Cat}$, $X \mapsto T_{/X}$ preserves weakly contractible limits (T a locus) and the terminal object (as $\emptyset \in T$ strictly initial, see Example 2.11), hence all limits. □

Lemma 6.175. *Let C be an accessible category with weakly contractible colimits. If C is stable, then it is a locus.*

Proof. Let C be a stable accessible category with weakly contractible colimits. It has all finite limits, so in particular pullbacks. First, weakly contractible colimits are universal. Given a map $Y \to X = \mathrm{colim}_{i \in I} X_i$, form the pullback squares

$$\begin{array}{ccc} X_i \times_X Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X_i & \longrightarrow & X. \end{array}$$

Since pullback squares agree with pushout squares, they are closed under colimits, so the induced square

$$\begin{array}{ccc} \mathrm{colim}_{i \in I} X_i \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{colim}_{i \in I} X_i & \xrightarrow{\sim} & X \end{array}$$

is still a pullback square. In particular, the top map is an isomorphism.

It remains to verify the full van Kampen condition. Let $X \colon I \to C$ with I weakly contractible. A cartesian transformation $Y \to X$ determines cofiber sequences

$$Y_i \longrightarrow X_i \longrightarrow K_i.$$

Since every naturality square is a pullback and hence also a pushout, every map $K_i \to K_j$ is an isomorphism. Thus $K \colon I \to C$ factors through the maximal anima of C. Since $|I|$ is contractible, K is uniquely equivalent to a constant diagram. Consequently, cartesian transformations $Y \to X$ are equivalently pairs consisting of an object $K \in C$ and a map $X \to \mathrm{const}_K$. Taking colimits identifies these with pairs $(K, \mathrm{colim}_I X \to K)$, which in turn are equivalent, by taking fibers, to objects of $C_{/\mathrm{colim}_I X}$. Hence

$$C_{/\mathrm{colim}_I X} \simeq \lim_{i \in I^{\mathrm{op}}} C_{/X_i},$$

so the colimit is van Kampen. Compare [Hoy19b, Example 7]. □

Lemma 6.176. *Let T be a topos. Then $T^{\geq \infty}$ is a locus in which every map is an effective epimorphism.*

Proof. The subcategory $T^{\geq \infty}$ of T spanned by ∞-connected objects is closed under colimits and finite limits in T by Proposition 3.39. Since weakly contractible colimits in T are van Kampen, $T^{\geq \infty}$ inherits this condition.

For any $X \in T^{\geq \infty}$, the map $X \to *$ is ∞-connected by definition. It follows from left cancellation that any map $X \to Y$ is ∞-connected, so in particular an effective epimorphism in T. Since $T^{\geq \infty}$ is closed under finite limits and geometric realizations, it is then also an effective epimorphism in $T^{\geq \infty}$. □

The following is our main theorem of this section:

Theorem 6.177. *Let C be a category with pullbacks and weakly contractible colimits. Then the following conditions are equivalent:*

(1) *The category C is a locus;*

(2) *The category $\int_{\mathrm{An}} C$ is a topos;*

(3) *For every topos T, the category $\int_T C$ is a topos.*

Moreover, the following conditions are equivalent to each other:

(4) *The category C is a locus in which every map is an effective epimorphism;*

(5) *There exists a topos T such that $C = T^{\geq \infty}$.*

(6) *There exists a topos T such that $T_{\leq \infty} = \mathrm{An}$ and $T^{\geq \infty} = C$.*

Remark 6.178. We may think of the categories in (4-6) as "nilpotent thickenings of the point". They include all stable categories.

Proof. We momentarily assume the equivalence between (1)–(3) and deduce the equivalence between (4)–(6). It is clear that (6) implies (5), while (5) implies (4) by Lemma 6.176. Assume (4), and set $T = \int_{\mathrm{An}} C$, which is a topos by (2). The terminal object of T lies over $* \in \mathrm{An}$, and therefore determines a terminal object $* \in C$. Since every map in C is an effective epimorphism, every truncated object of C is terminal. Indeed, for a 0-truncated object the diagonal is both a monomorphism and an effective epimorphism, hence an isomorphism, and the claim follows; the general case follows inductively by applying the same argument to iterated diagonals.

The projection $\pi : T \to \mathrm{An}$ has a fully faithful right adjoint $r(A) = (A, \mathrm{const}_*)$. For every $(A, X) \in T$, the map $(A, X) \to r(A)$ is ∞-connected: its iterated diagonals are computed in C, where every map is an effective epimorphism. Hence $r(A)$ is the hypercompletion of (A, X). It follows that $T_{\leq \infty} \simeq \mathrm{An}$. Moreover, (A, X) is ∞-connected precisely when $A \simeq *$, so the fiber C over $*$ identifies with $T^{\geq \infty}$. This proves (6).

For the equivalence of (1)–(3), first note that (2) is equivalent to (3), since $\int_T C = T \otimes \int_{\mathrm{An}} C$, where the right-hand side is the product of topoi if $\int_{\mathrm{An}} C$ is a topos. For (2) $\implies$ (1), the projection $\int_{\mathrm{An}} C \to \mathrm{An}$ is an accessible functor between presentable categories and C is its fiber over $*$. Thus Corollary 6.171 applies. It remains to show that (1) implies (2).

Choose a regular cardinal κ for which C is κ-accessible and its weakly contractible colimit operations are accessible. The usual accessibility argument for Grothendieck constructions shows that $\int_{\mathrm{An}} C$ is generated under κ-filtered colimits by pairs (A, X) with A κ-compact and X taking values in a small subcategory of κ-compact objects of C. Hence $\int_{\mathrm{An}} C$ is accessible. It has all colimits by Proposition 6.162, so it is presentable. It remains to show that its colimits are van Kampen.

Consider a diagram $(X_\bullet, E_\bullet) : I \to \int_{\mathrm{An}} C$, where $X_i \in \mathrm{An}$ and $E_i \in \mathrm{Fun}(X_i, C)$. We need to show that the functor $\mathrm{colim} : \mathrm{Fun}^{\mathrm{cart}}(I, \int_{\mathrm{An}} C)_{/(X_\bullet, E_\bullet)} \to (\int_{\mathrm{An}} C)_{/(X,E)}$ is an equivalence. Note that this

functor fits in a commutative diagram as follows:

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{cart}}(I, \int_{\mathrm{An}} C)_{/(X_\bullet, E_\bullet)} & \longrightarrow & (\int_{\mathrm{An}} C)_{/(X,E)} \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\mathrm{cart}}(I, \mathrm{An})_{/X_\bullet} & \longrightarrow & \mathrm{An}_{/X}. \end{array}$$

The vertical two maps are cartesian fibrations and the bottom map is an equivalence, so it will suffice to show that the top map induces equivalences on fibers.

Note that $(\int_{\mathrm{An}} C)_{/(X,E)} \simeq \lim_{x \in X} (\int_{\mathrm{An}} C)_{/(*,E_x)}$, so it will suffice to prove the claim when $X = *$. In that case $\mathrm{An}_{/X} \simeq \mathrm{An}$. Given $Y \in \mathrm{An}$, the map on fibers over Y is identified with a limit over $y \in Y$ of copies of the following map

$$\lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}(X_i, C)_{/E_i} \to C_{/E},$$

and it remains to show this is an equivalence. Rewriting the left-hand side as a limit over the category of elements $\mathrm{El}(X_\bullet)$, the map becomes

$$\lim_{x \in \mathrm{El}(X_\bullet)^{\mathrm{op}}} C_{/E_x} \longrightarrow C_{/E}.$$

The realization of $\mathrm{El}(X_\bullet)$ is $X \simeq *$, so this indexing category is weakly contractible. The map is therefore an equivalence by the van Kampen condition in C. This proves (1)–(3); compare [Hoy19b, Corollary 5]. □

6.11.5 Further questions on relative cocompletions

The category $\int_{\mathrm{An}} C$ is a special case of a relative free-cocompletion construction $\mathcal{P}_K^{K'}(C)$. This viewpoint explains why anima-indexed and weakly contractible colimits occur together, and suggests a broader question about when relative cocompletions are topoi. We first record the formal statements that follow from the universal properties, then separate the cases presently known from the open ones. Some of this discussion is related to [Rez25].

Notation 6.179. Let $K \subseteq K'$ be two classes of small categories. Given a category C with K-indexed colimits, we may form a functor

$$C \to \mathcal{P}_K^{K'}(C)$$

which is universal among K-colimit preserving functors into a category with K'-indexed colimits.

Remark 6.180. Assume for simplicity that C is small. If K' consists of all small categories and K of none, $\mathcal{P}^{\mathrm{all}}(C) = \mathrm{PSh}(C)$ is simply the presheaf category. For arbitrary K (but still $K' = \mathrm{all}$), we may identify $\mathcal{P}_K^{\mathrm{all}}(C)$ with the localization of $\mathrm{PSh}(C)$ at the class of maps

$$\{\alpha_X \colon \mathrm{colim}_{i \in I}\, y(X_i) \to y(\mathrm{colim}_{i \in I}\, X_i) \mid I \in K, X \colon I \to C\} :$$

for every cocomplete category D, a colimit-preserving functor $\mathrm{PSh}(C) \to D$ is determined by its restriction to C, and it factors through the localization if and only if it inverts each α_X, which is exactly the condition that the restriction preserve K-indexed colimits.

It follows from Yoneda that we may identify $\mathcal{P}_K^{\mathrm{all}}(C)$ with the full subcategory of $\mathrm{PSh}(C)$ consisting of those presheaves $F \colon C^{\mathrm{op}} \to \mathrm{An}$ which send K-indexed colimits in C to limits in An.

Example 6.181. Let K' be the class of all small categories and let K be the class of the weakly contractible categories. If C is a category with weakly contractible colimits, the category $\mathcal{P}^{\mathrm{all}}_{\mathrm{wc}}(C)$ is universal among functors $C \to C'$ into a cocomplete category C' which preserve weakly contractible colimits. Since this is also precisely the universal property of the category $\int_{\mathrm{An}} C$ established in Proposition 6.164, we obtain an equivalence

$$\int_{\mathrm{An}} C \xrightarrow{\sim} \mathcal{P}^{\mathrm{all}}_{\mathrm{wc}}(C).$$

Together with Proposition 6.161, the preceding example gives

$$\mathcal{P}^{\mathrm{animae}}(C) \simeq \mathcal{P}^{\mathrm{all}}_{\mathrm{wc}}(C).$$

Thus freely adjoining anima-indexed colimits is the same as freely adjoining all colimits while preserving the weakly contractible colimits already present in C. This motivates the following terminology.

Notation 6.182. Let K and K' be two classes of small categories. We write $K \perp K'$, and say that *K is complementary to K'*, if for every category C with K'-indexed colimits, the categories $\mathcal{P}^K(C)$ and $\mathcal{P}^{\mathrm{all}}_{K'}(C)$ define the same full subcategory of $\mathrm{PSh}(C)$ (or of the category of small presheaves, if C is large), under the identifications from Remark 6.180.

Equivalently, the unique K-colimit-preserving extension

$$\mathcal{P}^K(C) \to \mathcal{P}^{\mathrm{all}}_{K'}(C)$$

of $C \to \mathcal{P}^{\mathrm{all}}_{K'}(C)$ is an equivalence.

Warning 6.183. This relation is *not* symmetric: K being complementary to K' does not imply that K' is complementary to K.

Lemma 6.184. *If $K \perp K'$, then the following two conditions are satisfied:*

(1) *For every category D, the unique $(K \cup K')$-colimit-preserving extension $\mathcal{P}^{K \cup K'}(D) \to \mathcal{P}^{\mathrm{all}}(D)$ of $D \to \mathcal{P}^{\mathrm{all}}(D)$ is an equivalence. Equivalently,* $\mathrm{PSh}(D)$ *is generated by representables under K-colimits and K'-colimits.*

(2) *In* An*, K-indexed colimits commute with K'-indexed limits.*

Proof. For (1), take $C = \mathcal{P}^{K'}(D)$ in the definition of complementarity. The relative completion $\mathcal{P}^{\mathrm{all}}_{K'}(\mathcal{P}^{K'}(D))$ is $\mathcal{P}^{\mathrm{all}}(D)$ by the two universal properties. Hence every presheaf on D is obtained from representables by first forming K'-indexed colimits and then K-indexed colimits. In particular, the subcategory generated under both classes is all of $\mathrm{PSh}(D)$.

For (2), let $I \in K'$, $J \in K$, and let $X \colon I \times J \to \mathrm{An}$ be a diagram. We must show that the canonical map $\mathrm{colim}_{j \in J} \lim_{i \in I} X_{i,j} \to \lim_{i \in I} \mathrm{colim}_{j \in J} X_{i,j}$ is an isomorphism. Let $C := \mathrm{PSh}(I^{\mathrm{op}})$, and for each $j \in J$ let $d_j \in C$ be the presheaf $i \mapsto X_{i,j}$. Also let

$$u := \mathrm{colim}_{i \in I^{\mathrm{op}}} y(i) \quad \in \quad C \qquad \text{and} \qquad F := \mathrm{colim}_{j \in J} y(d_j) \quad \in \quad \mathrm{PSh}(C).$$

By the large-C variant of Remark 6.180 (using small presheaves), $\mathcal{P}^{\mathrm{all}}_{K'}(C)$ is a full subcategory of $\mathrm{PSh}(C)$. Using the assumption $K \perp K'$, this full subcategory agrees with $\mathcal{P}^K(C) \subseteq \mathrm{PSh}(C)$, hence

is closed under K-indexed colimits. Since each $y(d_j)$ belongs to it, it follows that $F \in \mathcal{P}^{\mathrm{all}}_{K'}(C)$, and thus F sends the I^{op}-indexed colimit $u = \mathrm{colim}_{i \in I^{\mathrm{op}}} y(i)$ in C to an I-indexed limit in An:

$$F(u) \simeq \lim_{i \in I} F(y(i)).$$

Now compute both sides using pointwise colimits and Yoneda:

$$F(u) \simeq \mathrm{colim}_{j \in J} \mathrm{Hom}_C(u, d_j) \simeq \mathrm{colim}_{j \in J} \lim_{i \in I} \mathrm{Hom}_C(y(i), d_j) \simeq \mathrm{colim}_{j \in J} \lim_{i \in I} X_{i,j},$$

while

$$\lim_{i \in I} F(y(i)) \simeq \lim_{i \in I} \mathrm{colim}_{j \in J} \mathrm{Hom}_C(y(i), d_j) \simeq \lim_{i \in I} \mathrm{colim}_{j \in J} X_{i,j}.$$

This proves (2). □

Example 6.185. The following are examples of complementary classes:

- Filtered categories are complementary to finite categories.
- Sifted categories are complementary to finite sets.
- Weakly contractible categories are complementary to the empty category.
- All small categories are complementary to the empty class.
- Animae are complementary to weakly contractible categories, by Proposition 6.164.

The question about $\int_{\mathrm{An}} C$ being a topos now has the following natural generalization:

Question 6.186. Let $K \perp K'$ be complementary and let C be a category which has K'-indexed colimits which are van Kampen. Is $\mathcal{P}^K(C) = \mathcal{P}^{\mathrm{all}}_{K'}(C)$ necessarily a topos?

Marc says he doesn't know whether this is true in general. But it is true when K is one of the following classes of categories:

- Weakly contractible categories. In this case, $\mathcal{P}^{\mathrm{w.c.}}(C)$ is also known as $\mathcal{P}_\emptyset(C)$, the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ spanned by those functors F satisfying $F(\emptyset) = *$. It is readily verified that both weakly contractible colimits and the initial object are van Kampen, so that $\mathcal{P}_\emptyset(C)$ is a topos.
- Sifted categories. In this case, $\mathcal{P}^{\mathrm{sifted}}(C)$ is also known as $\mathcal{P}_\Sigma(C)$, the subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ spanned by the finite-product-preserving functors. This is in fact a sheaf category, hence a topos; see [BH21, Lemma 2.4] for a proof.
- All categories. In this case, $\mathcal{P}^{\mathrm{all}}(C)$ is also known as the presheaf category $\mathrm{PSh}(C)$, which is a topos.
- All animae. In this case, $\mathcal{P}^{\mathrm{animae}}(C) = \int_{\mathrm{An}} C$, so this is a topos by Theorem 6.177.

It is natural to ask whether the same conclusion holds for n-groupoids paired with weakly n-connected indexing categories. We do not record this as an established case. However, it seems to Marc that this is *not* true when $K = \{\text{Filtered categories}\}$. You would need to show that if C is a category with finite colimits which are van Kampen, then $\mathrm{Ind}(C)$ is a topos. Although Marc does not know an explicit counterexample, he expects this to fail in general.

7 Geometry

Topoi provide a notion of generalized geometric object, but a bare topos does not remember the additional structure which distinguishes schemes, analytic spaces, or other geometric objects. The purpose of this final chapter is to explain several formalisms for adding such structure. The recurring pattern is that a geometric object is locally modeled on a chosen category of affine objects, with a distinguished class of admissible morphisms and a topology describing how the local models glue.

We begin in Section 7.1 with the classical example. A scheme can be described as a ringed topos which is locally isomorphic to the Zariski topos of a ring. In this language, classical schemes are the 0-localic Zariski schemes, while classical Deligne–Mumford stacks are the 1-localic étale schemes. Moreover, the category of Zariski schemes and étale morphisms is itself, up to size issues, a topos. These observations isolate the structures that the subsequent theory abstracts.

Lurie's notion of a *geometry* packages three ingredients: a category C of affine test objects, a class C^{ad} of admissible morphisms, and a topology generated by admissible covers. A structure on a topos is then a left exact functor from C satisfying descent for these covers. Associated with a geometry are two related sheaf topoi, obtained from all morphisms in C and from only the admissible morphisms. In Section 7.3, we study the resulting notion of a *fractured topos*. We show that every geometric site produces a fracture and that, on a fixed topos, fractured structures are equivalent to suitable local admissibility structures. The fracture in turn remembers which structured topoi and morphisms should be regarded as schemes and étale maps.

Six-functor formalisms provide a different source of this structure. A presentable six-functor formalism on a category with admissible morphisms determines the D-topology, generated by families satisfying universal !-descent. This produces a geometric site and hence a fractured topos, and the original formalism extends to the resulting categories of sheaves. Thus a cohomological formalism can itself determine the geometry on which it satisfies descent. We describe this construction in Section 7.4.

The final two sections indicate how this perspective enters current geometric frameworks. In Section 7.5, we outline the construction of analytic stacks from analytic rings in the sense of Clausen–Scholze and the six-functor formalism available for them. In Section 7.6, we discuss Scholze–Stefanich's proposed category of Gestalten and the way a six-functor formalism should produce Gestalt-valued invariants satisfying descent.

This is an outlook chapter. The structural results on geometries and fractured topoi are stated precisely, although several proofs are quoted from the literature or omitted. The sections on analytic stacks and especially Gestalten are more schematic: some statements are included only as a roadmap to developing work and are explicitly marked informal where necessary.

7.1 Schemes

We begin with the topos-theoretic formulation of schemes. The point is to isolate the features that will later be abstracted by Lurie's notions of geometry and fractured topos.

Definition 7.1. A *ringed topos* is a pair $(T, \mathcal{O})$ where T is a topos and $\mathcal{O}$ is a commutative ring object of T whose underlying object is 0-truncated. In other words, $\mathcal{O}$ is a sheaf of static commutative rings on T.

Definition 7.2. A *Zariski scheme* is a ringed topos $(T, \mathcal{O})$ which is locally isomorphic to $\mathrm{Spec}(R)_{\mathrm{Zar}}$, i.e. there exists a covering family $\{U_i \to *\}_{i \in I}$ in T such that for every $i \in I$ the induced ringed topos $(T_{/U_i}, \mathcal{O}|_{U_i})$ is isomorphic to $\mathrm{Spec}(R_i)_{\mathrm{Zar}}$ for some ring R_i. A *morphism of Zariski schemes* $(T, \mathcal{O}_T) \to (S, \mathcal{O}_S)$ is a pair $(f, f^\sharp)$ consisting of a morphism of topoi $f : T \to S$ together with a morphism of sheaves of static rings $f^\sharp : \mathcal{O}_S \to f_* \mathcal{O}_T$. We require the adjoint morphism

$$f^* \mathcal{O}_S \longrightarrow \mathcal{O}_T$$

to be *local*, meaning that the following square of objects of T is a pullback:

$$\begin{array}{ccc} (f^* \mathcal{O}_S)^\times & \longrightarrow & \mathcal{O}_T^\times \\ \downarrow & & \downarrow \\ f^* \mathcal{O}_S & \longrightarrow & \mathcal{O}_T. \end{array}$$

Equivalently, this condition may be tested on sections over every object of T: a section of $f^* \mathcal{O}_S$ is invertible if and only if its image in $\mathcal{O}_T$ is invertible. This gives a category $\mathrm{Sch}_{\mathrm{Zar}}$.

Definition 7.3. An *étale scheme* is a ringed topos $(T, \mathcal{O})$ that is locally isomorphic to $\mathrm{Spec}(R)_{\text{ét}}$. We similarly obtain a category $\mathrm{Sch}_{\text{ét}}$.

Example 7.4.

- A classical scheme is exactly a 0-localic Zariski scheme.
- A classical Deligne–Mumford stack is exactly a 1-localic étale scheme.
- If $(T, \mathcal{O})$ is a Zariski scheme, and $X \in T$, then $(T_{/X}, \mathcal{O}|_{T_{/X}})$ is again a Zariski scheme.

Remark 7.5. Étale schemes are also called *Deligne–Mumford ∞-stacks*.

The following are standard facts from scheme theory, translated into the language of structured topoi; see [Lur11, Sections 2 and 3] for this formalism.

- The global section functor

  $$\mathrm{LocRingTopos} \to \mathrm{CRing}^{\mathrm{op}}, \qquad (T, \mathcal{O}) \mapsto \mathcal{O}(T)$$

  admits a fully faithful right adjoint

  $$\mathrm{CRing}^{\mathrm{op}} \to \mathrm{LocRingTopos}, \qquad R \mapsto \mathrm{Spec}(R)_{\mathrm{Zar}}.$$

- The functor

$$\mathrm{Sch}_{\mathrm{Zar}} \to \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{op}}), \qquad (T, \mathcal{O}) \mapsto \mathrm{Hom}(\mathrm{Spec}(-)_{\mathrm{Zar}}, (T, \mathcal{O}))$$

  is fully faithful. Moreover, a Zariski sheaf X is in the image if and only if there exists an effective epimorphism $\bigsqcup_i U_i \xrightarrow{f_i} X$ where U_i is representable and f_i is representable by an étale map (of ringed topoi).

- The category $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}})$ is the classifying topos for local rings.

We have made a number of different definitions here. These constructions fit into a common pattern, which we will now discuss. We first list some more facts about scheme theory which are a bit less standard.

Definition 7.6. A morphism $(f, f^\sharp)\colon (T, \mathcal{O}_T) \to (S, \mathcal{O}_S)$ of Zariski schemes is called *étale* if the underlying morphism of topoi f is étale and the adjoint structural morphism

$$f^* \mathcal{O}_S \longrightarrow \mathcal{O}_T$$

is an isomorphism. This use of the word étale is unrelated to the notion of an étale scheme from Definition 7.3. For example, every morphism which is locally on source and target an open immersion of schemes is étale in this sense.

We denote the resulting wide subcategory of $\mathrm{Sch}_{\mathrm{Zar}}$ by

$$\mathrm{Sch}^{\text{ét}}_{\mathrm{Zar}} := \mathrm{Sch}_{\mathrm{Zar}} \times_{\mathrm{Topos}} \mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Sch}_{\mathrm{Zar}}\,.$$

Proposition 7.7. *The category* $\mathrm{Sch}^{\text{ét}}_{\mathrm{Zar}}$ *is (up to size issues) a topos. In fact, there is an equivalence*

$$\mathrm{Sch}^{\text{ét}}_{\mathrm{Zar}} \simeq \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{ad,op}}),$$

where $\mathrm{CRing}^{\mathrm{ad}}$ *is the wide subcategory of* CRing *spanned by the localization maps of the form* $R \to R[s^{-1}]$.

The equivalence identifies a Zariski scheme with its functor of points restricted to open affine morphisms. The structural isomorphism in Definition 7.6 is essential here: requiring only that the underlying morphism of topoi be étale would give a larger category.

Observation 7.8. For every topos T, the category $\mathrm{Shv}_{\mathrm{CRing}}(T)$ admits a factorization system: every map $A \to B$ factors uniquely as

$$A \to A[S^{-1}] \to B,$$

where the second map is a local map. This factorization is functorial in T.

Based on these results and observations, we see that the topos $E = \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}})$ has the following additional structure:

(1) Sheafified left Kan extension along the inclusion $\mathrm{CRing}^{\mathrm{fp,ad,op}} \hookrightarrow \mathrm{CRing}^{\mathrm{fp,op}}$ induces a faithful functor

$$E^{\mathrm{corp}} := \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,ad,op}}) \hookrightarrow \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}}) = E$$

whose image is also a topos. We refer to this subcategory E^{corp} as the subcategory of *corporeal objects*.

(2) There exists a wide subcategory E^{ad} of E consisting of the "representable étale maps".

(3) The functor $\mathrm{Geom}(-, E)\colon \mathrm{Topos}^{\mathrm{op}} \to \mathrm{Cat}$ factors as

$$\mathrm{Topos}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{fact}} \to \mathrm{Cat},$$

where $\mathrm{Cat}^{\mathrm{fact}}$ is the category of categories equipped with a factorization system.

(4) The category $\mathrm{Topos}_{//E}$ of ringed topoi admits a wide subcategory

$$(\mathrm{Topos}_{//E})^{\mathrm{loc}} \subseteq \mathrm{Topos}_{//E}$$

consisting of the *local* morphisms of ringed topoi.

The guiding point is that these four structures are different manifestations of the same geometry. The theory below makes precise the relation between corporeal objects and admissible morphisms, and then explains how local morphisms recover the corresponding notion of scheme.

In the next two sections we will consider abstract versions of this story: Lurie's notion of a *geometry* which encapsulates the idea of having affine test objects with morphisms playing the role of open inclusions and the covers they generate, and the resulting notion of a *fractured topos*.

7.2 Geometries

Classical algebraic geometry is built from affine schemes, basic open immersions, and covers by basic opens. Lurie's notion of a geometry abstracts exactly this pattern; see [Lur11]:

$$\begin{aligned} \text{affine test objects} &\rightsquigarrow \text{objects of a category } C, \\ \text{basic open immersions} &\rightsquigarrow \text{admissible morphisms}, \\ \text{covers by basic opens} &\rightsquigarrow \text{admissible covers}. \end{aligned}$$

For the applications below, we will use the following slightly site-theoretic formulation.

Definition 7.9. Let C be a category with finite limits. An *admissibility structure* is a wide subcategory $C^{\mathrm{ad}} \subseteq C$ of *admissible morphisms* satisfying the following properties:

- C^{ad} is closed under base change in C;
- C^{ad} is left cancellable: given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, if g and gf are admissible morphisms, then so is f;
- admissible morphisms are closed under retracts.

Definition 7.10. Let (C, C^{ad}) be a category with an admissibility structure. A topology τ is said to be *compatible with* C^{ad} if every τ-covering sieve contains a τ-covering sieve generated by admissible maps. We refer to such triples

$$\mathcal{G} = (C, C^{\mathrm{ad}}, \tau)$$

as *geometric sites*. Covering families generated by admissible morphisms will be called *admissible covers*. If C is small and idempotent complete, we will also call such a triple a *geometry*.

7.2.1 Zariski geometries

The basic example is the *classical Zariski geometry*. Its underlying category is

$$\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}} := (\mathrm{CRing}^{\mathrm{fp}})^{\mathrm{op}},$$

where $\mathrm{CRing}^{\mathrm{fp}}$ denotes the category of finitely presented ordinary commutative rings. Since $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}}$ is the opposite of a category of rings, we will describe its morphisms in terms of the corresponding maps of rings in the opposite direction.

The admissible morphisms in $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}}$ correspond to principal localizations

$$R \longrightarrow R[f^{-1}], \qquad f \in R.$$

Geometrically, this is the inclusion of the principal open

$$D(f) = \mathrm{Spec}(R[f^{-1}]) \hookrightarrow \mathrm{Spec}(R).$$

A finite family

$$\{R \to R[f_i^{-1}]\}_{i \in I}$$

generates a covering sieve precisely when the elements f_i generate the unit ideal in R.

Lemma 7.11. *The preceding data define a geometry* $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}}$.

Proof. Finite limits in $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}}$ are finite colimits of finitely presented rings. The topology is generated by admissible covers by definition. Base change of admissible morphisms follows from the formula

$$S \otimes_R R[f^{-1}] \cong S[\varphi(f)^{-1}]$$

for a map $\varphi\colon R \to S$.

For left cancellability, suppose that a principal localization $R \to R[g^{-1}]$ factors through another principal localization $R \to R[f^{-1}]$. Then f is already invertible in $R[g^{-1}]$, and hence

$$R[g^{-1}] \cong R[g^{-1}][f^{-1}] \cong R[f^{-1}][g^{-1}].$$

It follows that the induced map $R[f^{-1}] \to R[g^{-1}]$ is again a principal localization.

It remains to check closure under retracts. Consider a retract diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & R & \xrightarrow{\rho} & A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\ell} & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[\beta]{} & R[f^{-1}] & \xrightarrow[\sigma]{} & B \end{array}$$

in $\mathrm{CRing}^{\mathrm{fp}}$, where ℓ is localization at f. For every ring C, the restriction map

$$\mathrm{Hom}_{\mathrm{CRing}}(B, C) \longrightarrow \mathrm{Hom}_{\mathrm{CRing}}(A, C)$$

is a retract of

$$\mathrm{Hom}_{\mathrm{CRing}}(R[f^{-1}], C) \longrightarrow \mathrm{Hom}_{\mathrm{CRing}}(R, C).$$

The latter map is a monomorphism whose image consists of the maps that send f to a unit. Consequently, restriction along φ is also a monomorphism. If $a := \rho(f) \in A$, the retract identities show that a map $A \to C$ extends across φ if and only if it sends a to a unit. The universal property of localization therefore gives an isomorphism $B \cong A[a^{-1}]$ under A. □

There is also a *spectral Zariski geometry*

$$\mathcal{G}_{\mathrm{Zar}}^{\mathrm{sp}} := (\mathrm{CAlg}^{\omega})^{\mathrm{op}},$$

where CAlg^{ω} denotes the category of compact commutative $\mathbb{E}_\infty$-rings. The admissible morphisms are again principal localizations

$$R \longrightarrow R[f^{-1}], \qquad f \in \pi_0(R),$$

and a finite family $\{R \to R[f_i^{-1}]\}_{i \in I}$ is a cover precisely when the elements f_i generate the unit ideal in $\pi_0(R)$. The verification is the same as in the classical case, using the corresponding localization formula for commutative $\mathbb{E}_\infty$-rings.

7.2.2 Other examples

Here are some other geometries obtained by changing the category of affine test objects, the admissible morphisms, or the topology:

- *Derived Zariski geometry*: the analogous Zariski geometry based on compact animated rings.
- *Classical étale geometry*: take $C = (\mathrm{CRing}^{\mathrm{fp}})^{\mathrm{op}}$, take τ to be the étale topology, and let C^{ad} be the class of morphisms corresponding to étale maps of rings.
- *Derived étale geometry*: the analogous version based on compact animated rings.
- *Spectral étale geometry*: replace ordinary rings by compact commutative $\mathbb{E}_\infty$-rings and use the spectral étale topology.
- *Differential geometry*: use finitely presented C^∞-rings, with admissible maps encoding open immersions.

7.2.3 Structured topoi

Let $\mathcal{G} = (C, C^{\mathrm{ad}}, \tau)$ be a geometry and let T be a topos. A *$\mathcal{G}$-structure* on T is a left exact functor

$$\mathcal{O} \colon C \to T$$

such that for every admissible cover $\{U_i \to X\}_{i \in I}$, the induced map

$$\coprod_{i \in I} \mathcal{O}(U_i) \longrightarrow \mathcal{O}(X)$$

is an effective epimorphism in T. We denote by $\mathrm{Str}_{\mathcal{G}}(T)$ the full subcategory of $\mathrm{Fun}^{\mathrm{lex}}(C, T)$ spanned by the $\mathcal{G}$-structures.

In the classical Zariski case, the equivalence

$$\mathrm{Ind}((\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}})^{\mathrm{op}}) \simeq \mathrm{CRing}$$

identifies a left exact functor $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}} \to \mathrm{An}$ with an ordinary commutative ring A. Indeed, every ordinary ring is a filtered colimit of finitely presented rings, and filtered colimits of discrete animae are discrete. Under this identification, the functor associated to A sends a finitely presented ring S to

$$\mathrm{Hom}_{\mathrm{CRing}}(S, A).$$

The Zariski cover condition says that, whenever $f_1, \ldots, f_n \in S$ generate the unit ideal, the map

$$\coprod_i \mathrm{Hom}_{\mathrm{CRing}}(S[f_i^{-1}], A) \longrightarrow \mathrm{Hom}_{\mathrm{CRing}}(S, A)$$

is surjective. In concrete terms: for every map $\varphi\colon S \to A$, at least one of the elements $\varphi(f_i) \in A$ must be invertible.

This condition is equivalent to A being a local ring. Indeed, if A is local, then not all of the elements $\varphi(f_i)$ can lie in the maximal ideal, so one must be invertible. Conversely, suppose that the displayed condition holds. If $a + b = 1$, apply it to the map

$$\mathbb{Z}[x, y]/(x + y - 1) \longrightarrow A, \qquad x \longmapsto a, \quad y \longmapsto b,$$

and to the cover by $D(x)$ and $D(y)$. It follows that either a or b is invertible. Replacing (a, b) by $(-a, a + b)$ shows that the sum of two nonunits is again a nonunit. Since the nonunits are also closed under multiplication by arbitrary elements, they form an ideal, necessarily the unique maximal ideal. All in all, we see that

$$\mathrm{Str}_{\mathcal{G}_{\mathrm{Zar}}^{\mathrm{cl}}}(\mathrm{An}) \simeq \{\text{ordinary local rings}\}.$$

In the spectral Zariski case, the same argument starts from

$$\mathrm{Ind}((\mathcal{G}_{\mathrm{Zar}}^{\mathrm{sp}})^{\mathrm{op}}) \simeq \mathrm{CAlg}$$

and identifies a left exact functor $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{sp}} \to \mathrm{An}$ with a commutative ring spectrum R. The cover condition says that if $f_1, \ldots, f_n \in \pi_0(S)$ generate the unit ideal, then for every map $\varphi\colon S \to R$ at least one of the elements $\varphi(f_i) \in \pi_0(R)$ is invertible. Equivalently, $\pi_0(R)$ is a local ring. Thus

$$\mathrm{Str}_{\mathcal{G}_{\mathrm{Zar}}^{\mathrm{sp}}}(\mathrm{An}) \simeq \{\text{local commutative ring spectra}\}.$$

The upshot of this discussion is that Zariski structures provide a point-free version of the usual locality condition on stalks. Left exact functors from the Zariski geometry to a topos T correspond to internal commutative ring objects of T. Pulling such a functor back along a point $x^*\colon T \to \mathrm{An}$ produces the corresponding stalk. If T admits enough points, then these inverse-image functors jointly detect effective epimorphisms, so the admissible-cover condition is equivalent to requiring every stalk to be a local ring. If T does not admit enough points, the definition in terms of admissible covers remains meaningful and supplies the correct point-free replacement.

7.2.4 Local morphisms of structures

Let $\mathcal{O}, \mathcal{O}' \in \mathrm{Str}_{\mathcal{G}}(T)$ be two $\mathcal{G}$-structures. A natural transformation

$$\alpha\colon \mathcal{O} \to \mathcal{O}'$$

is called *local* if for every admissible morphism $U \to X$ in C, the square

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{O}'(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(X) & \longrightarrow & \mathcal{O}'(X) \end{array}$$

is a pullback square in T.

For the classical Zariski geometry and $T = \mathrm{An}$, this recovers the usual notion of a local homomorphism. A map of Zariski structures corresponds to a map of local rings $A \to B$. Testing locality on an admissible map $S \to S[f^{-1}]$ says that the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{CRing}}(S[f^{-1}], A) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(S[f^{-1}], B) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{CRing}}(S, A) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(S, B) \end{array}$$

is a pullback. In other words, an element of A is invertible if and only if its image in B is invertible. For local rings this is equivalent to the classical condition that the preimage of the maximal ideal of B is the maximal ideal of A. The spectral case is identical after passing to π_0: a map $R \to R'$ of local commutative $\mathbb{E}_\infty$-rings is local precisely when $\pi_0(R) \to \pi_0(R')$ is local in the classical sense.

7.3 Fractured topoi

A geometry produces two closely related sheaf topoi: the topos of all sheaves on the affine test objects, and the topos obtained by restricting to admissible morphisms. Lurie's notion of *fractured topos* abstracts the structure retained by such a pair. In the Zariski case, this is the structure behind the four equivalent descriptions encountered above: corporeal objects, admissible morphisms, factorization systems, and local morphisms of structured topoi.

Definition 7.12. A *fractured topos* is a topos E equipped with a functor $j_! \colon E^{\mathrm{corp}} \to E$ satisfying the following properties:

(1) The functor $j_!$ is faithful and induces a fully faithful functor on groupoid cores.

(2) The category E^{corp} admits fiber products and $j_!$ preserves them.

(3) The functor $j_!$ admits a right adjoint j^* which is conservative and preserves colimits.

(4) For every morphism $U \to V$ in E^{corp}, the commutative square

$$\begin{array}{ccc} j_! j^* j_! U & \longrightarrow & j_! j^* j_! V \\ \downarrow & & \downarrow \\ j_! U & \longrightarrow & j_! V \end{array}$$

is a pullback square.

We refer to objects of E^{corp}, and to their images in E, as *corporeal objects*. The functor $j_!$ is generally not fully faithful. The fractured topos is called *complete* if every retract in E of a corporeal object is corporeal.

Proposition 7.13. *Let (E, E^{corp}) be a fractured topos. Then:*

(1) *The category E^{corp} is a topos.*

(2) *For every $X \in E^{\mathrm{corp}}$, the functor*

$$j_! \colon E^{\mathrm{corp}}_{/X} \longrightarrow E_{/j_! X}$$

is fully faithful.

(3) *The topos E is generated under colimits by corporeal objects.*

(4) *The functor j^* admits a right adjoint $j_* \colon E^{\mathrm{corp}} \to E$, and the adjunction $j^* \dashv j_*$ is a morphism of topoi.*

Proof. See [Lur18a, Propositions 20.1.3.1 and 20.1.3.3, Corollary 20.1.3.4, and Remark 20.1.3.6]. □

We will now discuss how to construct fractured topoi.

7.3.1 Fractures from geometries

Construction 7.14. Let $(C, C^{\mathrm{ad}}, \tau)$ be a geometric site. The category C^{ad} inherits a topology τ^{ad}: a sieve in $C^{\mathrm{ad}}_{/X}$ is covering if its image generates a τ-covering sieve in $C_{/X}$. Restriction along $j \colon C^{\mathrm{ad}} \hookrightarrow C$ defines a functor

$$j^* \colon \mathrm{Shv}_\tau(C) \longrightarrow \mathrm{Shv}_{\tau^{\mathrm{ad}}}(C^{\mathrm{ad}}).$$

Its left adjoint $j_!$ is obtained by left Kan extension followed by sheafification.

Theorem 7.15. *The functor*

$$j_! \colon \mathrm{Shv}_{\tau^{\mathrm{ad}}}(C^{\mathrm{ad}}) \longrightarrow \mathrm{Shv}_\tau(C)$$

exhibits $\mathrm{Shv}_\tau(C)$ *as a complete fractured topos.*

Proof. See [Lur18a, Theorem 20.6.3.4]; completeness follows from the associated geometric admissibility structure, by [Lur18a, Theorem 20.3.4.4]. □

Example 7.16. Every geometry listed in Section 7.2 gives a fractured topos by this construction. For instance, the classical Zariski geometry gives the pair

$$\mathrm{Shv}_{\mathrm{Zar}}((\mathrm{CRing}^{\mathrm{fp,ad}})^{\mathrm{op}}) \longrightarrow \mathrm{Shv}_{\mathrm{Zar}}((\mathrm{CRing}^{\mathrm{fp}})^{\mathrm{op}}),$$

which is the fractured topos underlying the topos-theoretic description of ordinary schemes above.

Definition 7.17. Let (E, E^{corp}) be a fractured topos. A morphism $f : Y \to X$ in E is *admissible* if, for every map $j_! U \to X$ with $U \in E^{\mathrm{corp}}$, the pullback

$$Y \times_X j_! U \longrightarrow j_! U$$

belongs to the image of $E^{\mathrm{corp}}_{/U} \to E_{/j_! U}$. This determines a wide subcategory $E^{\mathrm{ad}} \subseteq E$.

Definition 7.18. Let E be a topos. An admissibility structure E^{ad} is called *local* if the following conditions are satisfied:

- The class E^{ad} is a local class of maps.
- For every $X \in E$, the slice $E^{\mathrm{ad}}_{/X} := (E^{\mathrm{ad}})_{/X}$ is presentable, and the inclusion $E^{\mathrm{ad}}_{/X} \to E_{/X}$ preserves colimits. In particular, it admits a right adjoint $\rho_X : E_{/X} \to E^{\mathrm{ad}}_{/X}$.

Given a local admissibility structure on E, we say that an object $X \in E$ is *corporeal* if the functor $\rho_X : E_{/X} \to E^{\mathrm{ad}}_{/X}$ preserves colimits. We denote by

$$E^{\mathrm{corp}} \subseteq E^{\mathrm{ad}}$$

the full subcategory spanned by the corporeal objects. We say that E^{ad} is *geometric* if, moreover, E is generated under colimits by corporeal objects.

Theorem 7.19. *Let E be a topos. The assignments $(E, E^{\mathrm{corp}}) \mapsto (E, E^{\mathrm{ad}})$ and $(E, E^{\mathrm{ad}}) \mapsto (E, E^{\mathrm{corp}})$ determine an equivalence of posets*

$$\{\textit{Complete fractured structures on } E\} \simeq \{\textit{Geometric admissibility structures on } E\}.$$

Proof. This is [Lur18a, Theorem 20.3.4.4 and Remark 20.3.4.6]. The completeness hypothesis is essential: without it, the admissible morphisms recover the closure of the corporeal objects under retracts, rather than necessarily recovering the original fracture. □

7.3.2 Local maps

Consider a fractured topos (E, E^{corp}). Consider two morphisms of topoi $f, g : T \to E$, and let $\alpha : f^* \to g^*$ be a natural transformation. We say that α is *local* if for every morphism $h : U \to V$ in E^{corp}, the following square is a pullback square:

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha_U} & g^*U \\ \downarrow{\scriptstyle f^*h} & & \downarrow{\scriptstyle g^*h} \\ f^*V & \xrightarrow{\alpha_V} & g^*V. \end{array}$$

This determines a collection of maps in $\mathrm{Geom}(T, E)$ which is the right class of a factorization system on $\mathrm{Geom}(T, E)$. We denote by

$$(\mathrm{Topos}_{//E})^{\mathrm{loc}} \subseteq \mathrm{Topos}_{//E}$$

the wide subcategory consisting of those morphisms corresponding to maps $S \to T$ with a local natural transformation α of morphisms $S \to E$.

Example 7.20. For $E = \mathrm{PSh}(\mathrm{CRing}^{\mathrm{fp,op}})$, the lax slice $\mathrm{Topos}_{//E}$ is the category of ringed topoi. For $E = \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}})$, the lax slice $\mathrm{Topos}_{//E}$ is the full subcategory of the previous one spanned by the locally ringed topoi. Here, the fracture $E^{\mathrm{corp}} = \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,ad,op}}) \to E$ determines the subcategory $(\mathrm{Topos}_{//E})^{\mathrm{loc}}$ consisting of the *local* morphisms of locally ringed topoi.

We now explain how to recover the notion of schemes from the notion of local morphisms.

Theorem 7.21. *Let (E, E^{corp}) be a complete fractured topos. Consider the functor*

$$E \to \mathrm{Fun}(\mathrm{Topos}^{\mathrm{loc}}_{//E}, \mathrm{An}), \qquad X \mapsto \Gamma^X : (f : T \to E) \mapsto \Gamma(f^*(X)).$$

(1) *If X is corporeal, then Γ^X is representable by the pair $(E^{\mathrm{ad}}_{/X}, E^{\mathrm{ad}}_{/X} \hookrightarrow E_{/X} \to E)$.*

(2) *The resulting functor*

$$E^{\mathrm{corp}} \xrightarrow{\mathrm{Re}} \mathrm{Topos}^{\mathrm{loc}}_{//E}$$

is fully faithful.

(3) *A morphism $f : Y \to X$ in E is admissible if and only if, after every base change $f' : Y' \to X'$ for which $\Gamma^{X'}$ is represented by an object of $\mathrm{Topos}^{\mathrm{loc}}_{//E}$, the functor $\Gamma^{Y'}$ is represented by an object étale over it.*

Combining these statements, it follows that the subcategory $\mathrm{Topos}^{\mathrm{loc}}_{//E} \subseteq \mathrm{Topos}_{//E}$ determines the fracture structure on E.

Proof. These assertions summarize the reconstruction results of [Lur18a, Chapter 20]. The third assertion uses the usual identification of étale objects over a topos with objects of that topos. □

7.3.3 Schemes

In Section 7.1, we provided a topos-theoretic perspective on the classical notion of a scheme. We may extend this definition to an arbitrary geometric site $\mathcal{G} = (C, C^{\mathrm{ad}}, \tau)$.

Construction 7.22. Consider the fractured topos $E = \mathrm{Shv}_\tau(C)$, $E^{\mathrm{corp}} := \mathrm{Shv}_\tau(C^{\mathrm{ad}})$. By Theorem 7.21, we obtain a functor

$$\Gamma : \mathrm{Topos}^{\mathrm{loc}}_{//E} \to \mathrm{Pt}(\mathrm{PSh}(C))^{\mathrm{op}} \simeq \mathrm{Pro}(C),$$

An object $f : T \to E$ is sent to the pro-object classified by the left exact functor

$$C \hookrightarrow \mathrm{PSh}(C) \twoheadrightarrow \mathrm{Shv}_\tau(C) = E \xrightarrow{f^*} T \xrightarrow{\Gamma} \mathrm{An}.$$

The functor Γ admits a right adjoint

$$\mathrm{Spec}_{\mathcal{G}} : \mathrm{Pro}(C) \to \mathrm{Topos}^{\mathrm{loc}}_{//E}$$

called the *spectrum functor*. If τ is subcanonical, then $\mathrm{Spec}_{\mathcal{G}}$ is fully faithful. See [Lur11, Chapter 2] for this structured-topos formulation of the spectrum construction.

Definition 7.23. We define the subcategory

$$\mathrm{Sch}_{\mathcal{G}} \quad \subseteq \quad \mathrm{Topos}^{\mathrm{loc}}_{//E}$$

of *$\mathcal{G}$-schemes* as the full subcategory on those objects which are locally in the image of $\mathrm{Spec}_{\mathcal{G}}$. Concretely, a $\mathcal{G}$-scheme admits a jointly surjective family of étale morphisms from affine objects $\mathrm{Spec}_{\mathcal{G}}(A)$ with $A \in \mathrm{Pro}(C)$.

Question 7.24. Does this category depend only on the fractured topos (E, E^{corp}), or is the choice of geometric site important?

7.4 Geometries from six-functor formalisms

Six-functor formalisms, developed systematically in [Sch25], provide a categorification of classical cohomology theories. The key insight relevant to our discussion is that every six-functor formalism on a category with an admissibility structure canonically determines a topology, the D-topology, and hence gives rise to a geometric site and a fractured topos.

7.4.1 The D-topology

Definition 7.25. Let C be a category with finite limits and let $C^{\mathrm{ad}} \subseteq C$ be an admissibility structure. A *presentable six-functor formalism* is a lax symmetric monoidal functor

$$D\colon \mathrm{Span}(C, C^{\mathrm{ad}}) \longrightarrow \mathrm{Pr}\,.$$

Definition 7.26. A family $(X_i \to X)_i$ in C^{ad} is a *D-cover* if it is a covering family for the canonical topology on C and satisfies universal !-descent: after every base change $Y \to X$, the resulting family $(X_i \times_X Y \to Y)_i$ satisfies descent for $D^!$. The *D-topology* τ_D is the topology whose covering families are the D-covers.

Proposition 7.27. *The triple $(C, C^{\mathrm{ad}}, \tau_D)$ is a geometric site.*

Proof. The canonical-cover condition is stable under base change and composition, and universal !-descent has the same stability properties. Since every D-cover consists of admissible morphisms, the resulting topology is compatible with C^{ad}. See [Sch25, Definition 5.13]. □

Corollary 7.28. *The functor*

$$\mathrm{Shv}_{\tau_D^{\mathrm{ad}}}(C^{\mathrm{ad}}) \longrightarrow \mathrm{Shv}_{\tau_D}(C)$$

exhibits $\mathrm{Shv}_{\tau_D}(C)$ *as a complete fractured topos. In particular, it determines a geometric admissibility structure* $\mathrm{Shv}_{\tau_D}(C)^{\mathrm{ad}}$.

7.4.2 Passage to sheaves

Set $\widetilde{C} := \mathrm{Shv}_{\tau_D}(C)$. There are several natural candidates for the class of admissible morphisms in $\widetilde{C}$, and they should not be identified without argument. We begin with the class directly controlled by the extension theorem.

Definition 7.29. A map $f\colon Y \to X$ in $\widetilde{C}$ is *representably admissible* if, for every map $h_U \to X$ from a representable sheaf, the pullback $Y \times_X h_U$ is representable and the induced morphism to h_U comes from a morphism in C^{ad}. We write $\widetilde{C}_0^{\mathrm{ad}}$ for the resulting admissibility structure.

Proposition 7.30 (Mann–Scholze)**.** *The six-functor formalism D extends uniquely from (C, C^{ad}) to $(\widetilde{C}, \widetilde{C}_0^{\mathrm{ad}})$.*

Proof. See [Man22, Proposition A.5.16] and [Sch25, Proposition 5.16]. □

Remark 7.31. Marc stated in the lectures that D extends to the entire fracture-derived admissibility structure $\mathrm{Shv}_{\tau_D}(C)^{\mathrm{ad}}$ of Corollary 7.28. No proof was given. The cited results establish the extension first for the representably admissible class $\widetilde{C}_0^{\mathrm{ad}}$ and then for Scholze's minimal local enlargement $\widetilde{C}_D^{\mathrm{ad}}$; they do not identify either class with $\mathrm{Shv}_{\tau_D}(C)^{\mathrm{ad}}$.

Scholze constructs the further enlargement mentioned in the remark. More precisely, there is a minimal class $\widetilde{C}_D^{\mathrm{ad}}$ containing $\widetilde{C}_0^{\mathrm{ad}}$ which is stable under disjoint unions, local on the target, local on the source, and tame. The formalism extends to $(\widetilde{C}, \widetilde{C}_D^{\mathrm{ad}})$; see [Sch25, Theorem 5.19].

7.4.3 A further congruence localization

The D-topology imposes descent along covering families. One may localize further by all morphisms which are universally invisible to $D^!$. This second step need not arise from a Grothendieck topology.

Construction 7.32. Consider the functor

$$D^!\colon \mathrm{Shv}_{\tau_D^{\mathrm{ad}}}(C^{\mathrm{ad}})^{\mathrm{op}} \longrightarrow \mathrm{Cat}.$$

Let Σ_D be the largest congruence contained in the inverse image of the isomorphisms under this functor. More explicitly,

$$\Sigma_D = \{f \mid \text{every base change of every } \Delta^n f \text{ is a } D^!\text{-isomorphism}\}.$$

Every morphism of Σ_D is ∞-connected: the condition for $n = 0$ implies that every base change is a D-cover and hence an effective epimorphism, and the same applies to every diagonal.

Assume that Σ_D has small generation, so that it defines an accessible left exact localization. We define $\mathrm{Shv}^+_{\tau_D}(C)$ by the pullback square

$$\begin{array}{ccc} \mathrm{Shv}^+_{\tau_D}(C) & \longrightarrow & \mathrm{Shv}_{\tau_D}(C) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle j^*} \\ \mathrm{Shv}_{\tau_D^{\mathrm{ad}}}(C^{\mathrm{ad}})[\Sigma_D^{-1}] & \longrightarrow & \mathrm{Shv}_{\tau_D^{\mathrm{ad}}}(C^{\mathrm{ad}}), \end{array}$$

where the bottom functor is the fully faithful right adjoint of the localization. The top functor is likewise the fully faithful right adjoint of a left exact localization. The left vertical functor admits a left adjoint, and the pair

$$\left(\mathrm{Shv}^+_{\tau_D}(C), \mathrm{Shv}_{\tau_D^{\mathrm{ad}}}(C^{\mathrm{ad}})[\Sigma_D^{-1}]\right)$$

is a fractured topos.

Remark 7.33. There is a general formalism of *fractured localization*, in which a compatible localization of the corporeal topos induces a localization of the ambient topos and a new fracture. The preceding construction is an instance of this formalism.

Lemma 7.34. *The six-functor formalism descends further to a functor*

$$D\colon \operatorname{Span}\left(\operatorname{Shv}^{+}_{\tau_D}(C), \operatorname{Shv}_{\tau_D^{\mathrm{ad}}}(C^{\mathrm{ad}})[\Sigma_D^{-1}]\right) \to \mathrm{Pr}^{\mathrm{L}}.$$

Proof. By construction, $D^!$ inverts Σ_D. The span formalism is self-dual along admissible morphisms: the duality exchanges the pullback and lower-shriek operations. Hence every universal $D^!$-isomorphism is also a universal D^*-isomorphism. The four operations therefore descend through the two localizations, and the Beck–Chevalley and projection-formula transformations descend with them. □

Remark 7.35. The self-duality used in the proof is also the reason that universal !-descent implies universal $*$-descent. It is important here that descent is required after every base change: this makes the class compatible with composition of spans.

7.5 Analytic stacks

In this section, we discuss analytic stacks in the sense of Clausen–Scholze as an example of the preceding formalism.

Definition 7.36. A *condensed anima* is a hypercomplete sheaf on the site CHaus of compact Hausdorff spaces equipped with the topology in which a finite collection of maps $\{X_i \to X\}_i$ is declared to be a covering if and only if the map $\bigsqcup_i X_i \twoheadrightarrow X$ is surjective. Equivalently, it is a hypercomplete sheaf on the site ProFin of profinite sets, equipped with the topology inherited from CHaus.

In practice, we will work with *light condensed animae*: hypercomplete sheaves on the site of *countably* profinite sets (i.e. profinite sets presented by countable cofiltered diagrams). We will be denoting this by

$$\mathrm{CondAn} := \operatorname{Shv}(\{\text{Countably profinite sets}\})^{\mathrm{hyp}}.$$

In this setting, we take "ring" to mean "light condensed animated commutative ring".

Definition 7.37. An *analytic ring* is a pair $A = (A^\triangleright, \mathrm{Mod}_A)$ consisting of a light condensed animated commutative ring $A^\triangleright$ together with a reflective subcategory $\iota\colon \mathrm{Mod}_A \hookrightarrow \mathrm{Mod}_{A^\triangleright}$, with left adjoint $L\colon \mathrm{Mod}_{A^\triangleright} \to \mathrm{Mod}_A$, satisfying the following conditions:

- Mod_A is closed under colimits (also limits, but this is automatic).
- Mod_A is closed under the internal hom $\underline{\mathrm{Hom}}(M, -)$ for every $M \in \mathrm{Mod}_{A^\triangleright}$.
- The functor $\iota L\colon \mathrm{Mod}_{A^\triangleright} \to \mathrm{Mod}_{A^\triangleright}$ preserves connective objects with respect to the canonical t-structure.
- We have $A^\triangleright \in \mathrm{Mod}_A$.

Example 7.38. The basic examples arise from complete topological rings, in particular from Huber pairs. The analytic ring structure remembers which condensed modules are complete for the chosen topology; it is therefore strictly more information than the underlying condensed ring $A^{\triangleright}$. See [Sch25, Lecture IX] for the constructions used below.

Remark 7.39. In the original definition of analytic rings, this was defined in the "non-light" setting. In this case, there is an additional condition one needs to put: the Frobenius map $A \to A/p$ is a map of analytic rings for every prime p. In the light setting, this condition turns out to be automatic. (It is not known whether this condition is automatic in the "non-light" setting.)

Definition 7.40. A map $f\colon A \to B$ of analytic rings is a map $A^{\triangleright} \to B^{\triangleright}$ satisfying the condition that the restriction-of-scalars functor $\mathrm{Mod}_{B^{\triangleright}} \to \mathrm{Mod}_{A^{\triangleright}}$ restricts to a functor

$$f_*\colon \mathrm{Mod}_B \to \mathrm{Mod}_A.$$

We denote its left adjoint by

$$f^*\colon \mathrm{Mod}_A \to \mathrm{Mod}_B,$$

which is given by first including into $\mathrm{Mod}_{A^{\triangleright}}$, then applying base change to $\mathrm{Mod}_{B^{\triangleright}}$, and then reflecting back into Mod_B. We denote the resulting category of analytic rings by AnRing.

Definition 7.41. A map $f\colon A \to B$ of analytic rings is said to be:

- *Proper* if the functor f_* induces an equivalence

$$\mathrm{Mod}_B \xrightarrow{\sim} \mathrm{Mod}_{B^{\triangleright}} \times_{\mathrm{Mod}_{A^{\triangleright}}} \mathrm{Mod}_A.$$

- An *open immersion* if the functor f^* admits a further left adjoint

$$f_!\colon \mathrm{Mod}_B \to \mathrm{Mod}_A$$

  which is fully faithful and Mod_A-linear. Equivalently, it is fully faithful and satisfies the projection formula

$$f_!(f^*(M) \otimes_B N) \cong M \otimes_A f_!(N).$$

- !*-able* if the canonical map of analytic rings

$$(B^{\triangleright}, \mathrm{Mod}_{B^{\triangleright}} \times_{\mathrm{Mod}_{A^{\triangleright}}} \mathrm{Mod}_A) \longrightarrow (B, \mathrm{Mod}_B)$$

  is an open immersion.

Lemma 7.42. *A map f is !-able if and only if it factors as a proper map followed by an open immersion.*

Proof. For a map $A \to B$, equip $B^{\triangleright}$ first with the analytic structure induced from A. The map from A to this induced analytic ring is proper by definition, while the comparison from the induced analytic ring to B is an open immersion precisely when $A \to B$ is !-able. Conversely, any factorization of this form satisfies the defining condition. See [Sch25, Definition 9.16]. □

Proposition 7.43. *There is a presentable six-functor formalism*

$$D\colon \mathrm{Span}(\mathrm{AnRing}^{\mathrm{op}}, \mathrm{AnRing}^{!,\mathrm{op}}) \to \mathrm{Pr}, \qquad A \mapsto \mathrm{Mod}_A.$$

Proof. The proper and open classes above satisfy the compactification, base-change, and projection-formula hypotheses, and every !-able map has the factorization of Lemma 7.42. The resulting analytic six-functor formalism is constructed in [Sch25, Lecture IX]. The formal assembly of the coherent span-valued functor may also be expressed using [CLL25]. □

Definition 7.44. The category of *analytic stacks* is defined to be

$$\mathrm{AnStck} := \mathrm{Shv}^{+}_{\tau_D}(\mathrm{AnRing}^{\mathrm{op}}).$$

Here the superscript + denotes the further congruence localization of Construction 7.32, not merely sheafification for the D-topology. In the light setting all sites and colimit constructions are understood with the corresponding countability restrictions.

7.6 Gestalten

We finish the lecture course with the recent framework of *Gestalten*, developed by Scholze and Stefanich [Sch26]. Its purpose is to regard all the categorical layers attached to a geometric object as a single ring-like invariant. The construction from a six-functor formalism and its finite-limit and descent properties are precise. The ambient category of all Gestalten, however, is too large to be a topos. The actual topos theorem applies after fixing a base and a finite étaleness level, and after restricting to countable colimits.

Construction 7.45. There are presentably symmetric monoidal categories $n\,\mathrm{Pr}$ for all $n \geq 0$, with

$$0\,\mathrm{Pr} = \mathrm{An}, \qquad 1\,\mathrm{Pr} = \mathrm{Pr}\,.$$

The construction is iterated by passing to presentable module categories. Throughout this section the relevant presentability cardinal is $\aleph_1$, so that "presentable" means compatible with countable colimits. We suppress this cardinal from the notation.

Definition 7.46. The category of *Stefanich rings* is

$$\mathrm{StRing} := \mathrm{colim}_{1\,\mathrm{Pr}}\big(\mathrm{CAlg}(\mathrm{An}) \longrightarrow \mathrm{CAlg}(1\,\mathrm{Pr}) \longrightarrow \mathrm{CAlg}(2\,\mathrm{Pr}) \longrightarrow \cdots\big).$$

Equivalently,

$$\mathrm{StRing} \simeq \lim_{\mathrm{CAT}}\big(\mathrm{CAlg}(\mathrm{An}) \longleftarrow \mathrm{CAlg}(1\,\mathrm{Pr}) \longleftarrow \mathrm{CAlg}(2\,\mathrm{Pr}) \longleftarrow \cdots\big),$$

where the transition functor sends a symmetric monoidal category to the endomorphism object of its unit. Thus a Stefanich ring may be written as a sequence $A = (A_0, A_1, A_2, \ldots)$ with $A_n \in \mathrm{CAlg}(n\,\mathrm{Pr})$ and compatible isomorphisms

$$A_n \cong \mathrm{End}_{A_{n+1}}(\mathbb{1}).$$

Definition 7.47. The category of *Gestalten* is defined as

$$\mathrm{Gest} := \mathrm{StRing}^{\mathrm{op}}.$$

Theorem 7.48 (Scholze–Stefanich)**.** *Let A be a Stefanich ring and let $n \geq 0$. The opposite of the category* $\mathrm{StRing}_{A/}^{n\text{-ét}}$ *of n-étale Stefanich A-algebras is a countable topos. More precisely,*

$$\mathrm{Ind}_{\aleph_1}\left(\left(\mathrm{StRing}_{A/}^{n\text{-ét}}\right)^{\mathrm{op}}\right)$$

is a topos.

Proof. This is [Sch26, Theorem 7.1]. □

Thus one should not say that the entire category Gest is a topos. In applications one works in a countable topos of n-étale Gestalten over a fixed base. The restriction to a finite n is substantive, although many geometric examples are already 1- or 2-étale.

Construction 7.49. For every n, there is a fully faithful functor

$$\mathrm{CAlg}(n\,\mathrm{Pr})^{\mathrm{op}} \hookrightarrow \mathrm{Gest}.$$

Indeed, an object $A_n \in \mathrm{CAlg}(n\,\mathrm{Pr})$ admits a canonical delooping $A_{n+1} := \mathrm{Mod}_{A_n}(n\,\mathrm{Pr})$. Iterating this construction supplies the higher terms of a Stefanich ring, while the lower terms are obtained by taking endomorphisms of the unit.

Definition 7.50. A Gestalt A is called *n-affine* if it lies in the image of this fully faithful functor. Equivalently, its corresponding Stefanich ring satisfies $A_{k+1} \cong \mathrm{Mod}_{A_k}(k\,\mathrm{Pr})$ for all $k \geq n$.

Example 7.51. The 0-affine Gestalten are precisely the commutative monoids:

$$\mathrm{CMon}(\mathrm{An})^{\mathrm{op}} = \mathrm{CAlg}(\mathrm{An})^{\mathrm{op}} \hookrightarrow \mathrm{Gest}.$$

Since $\mathrm{Pr}_{\mathrm{st}} \subseteq \mathrm{Pr}$ is fully faithful, we similarly get a fully faithful functor

$$\mathrm{CAlg}(\mathrm{Sp})^{\mathrm{op}} \hookrightarrow \mathrm{Gest}.$$

Remark 7.52. Objects in the image of $\mathrm{CAlg}(\mathrm{Sp})$ are 1-affine, but not 0-affine in the absolute sense. One may instead work relative to the sphere, in the slice

$$\mathrm{Gest}_{\mathbb{S}} := \mathrm{Gest}_{/\mathrm{Gest}(\mathrm{Sp})}.$$

For every ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$, the induced object over $\mathrm{Gest}(\mathrm{Sp})$ is 0-affine relative to this base.

7.6.1 Gestalten from six-functor formalisms

Consider a presentable six-functor formalism

$$D\colon\ \mathrm{Span}(C, C^{\mathrm{ad}}) \to \mathrm{Pr}.$$

For every $X \in C$, restriction to the admissible slice gives a lax symmetric monoidal functor

$$D_X\colon\ \mathrm{Span}(C^{\mathrm{ad}}_{/X}) \longrightarrow \mathrm{Pr}.$$

The functor category

$$\mathrm{Fun}\big(\mathrm{Span}(C^{\mathrm{ad}}_{/X}), \mathrm{Pr}\big)$$

carries its Day convolution symmetric monoidal structure, with respect to which D_X is a commutative algebra object. The *category of kernels* at X is

$$K_D(X) := \mathrm{Mod}_{D_X}\Big(\mathrm{Fun}\big(\mathrm{Span}(C^{\mathrm{ad}}_{/X}), \mathrm{Pr}\big)\Big) \quad \in \quad 2\,\mathrm{Pr}\,.$$

It is functorial in X, and its pullback functors admit compatible left and right adjoints. It deloops D in the sense that

$$\mathrm{End}_{K_D(X)}(\mathbb{1}) \cong D(X)$$

for all $X \in C$.

Iterating the kernel construction produces a Stefanich ring

$$A_{D,X} := (D(X), K_D(X), K_D^2(X), \ldots).$$

For the final object $*$ of C, write $A_D := A_{D,*}$. Passing to opposites gives the *transmutation functor*

$$[-]_D \colon C \longrightarrow \mathrm{Gest}_{/\mathrm{Gest}(A_D)}, \qquad X \longmapsto [X]_D := \mathrm{Gest}(A_{D,X}).$$

Definition 7.53. We say that D *satisfies Künneth* if, for all $X, Y \in C$, the canonical tensor comparison

$$D(X) \otimes_{D(*)} D(Y) \longrightarrow D(X \times Y)$$

is an isomorphism.

The comparison controls affineness at the first categorified level. In particular, D satisfies Künneth if and only if there is an equivalence

$$K_D(*) \simeq \mathrm{Mod}_{D(*)}(1\,\mathrm{Pr}).$$

Analogous relative comparisons in every slice identify $K_D(X)$ with $\mathrm{Mod}_{D(X)}(1\,\mathrm{Pr})$ and make $[X]_D$ 1-affine. No identification of all higher kernel categories is automatic from the first comparison alone.

Example 7.54. The six-functor formalism D on $\mathrm{AnRing}^{\mathrm{op}}$ constructed above satisfies Künneth; see [Sch26, Lecture IX].

Proposition 7.55. *The functor*

$$[-]_D \colon C \longrightarrow \mathrm{Gest}_{/\mathrm{Gest}(A_D)}$$

preserves finite limits. Moreover, every map $[Y]_D \to [X]_D$ induced by a morphism $Y \to X$ in C is 1*-étale and* 1*-proper.*

Proof. See [Sch26, Proposition 9.5]. □

Proposition 7.56. *If a family $(X_i \to X)_i$ satisfies universal $!$-descent, then the map*

$$\bigsqcup_i [X_i]_D \longrightarrow [X]_D$$

is an effective epimorphism in the relevant countable topos of Gestalten.

Proof. This is [Sch26, Proposition 9.7]. The converse is not asserted there. □

Combining finite-limit preservation with the preceding descent statement gives the following extension. Let $\mathrm{Shv}_D(C)_{\mathrm{ctbl}}$ denote the countable-colimit completion of the representables inside the D-sheaf topos. Then $[-]_D$ extends to a functor

$$F\colon \mathrm{Shv}_D(C)_{\mathrm{ctbl}} \longrightarrow \mathrm{Gest}^{1\text{-ét}}_{/\mathrm{Gest}(A_D)}$$

which preserves countable colimits and finite limits; see [Sch26, Corollary 9.8].

Applying the same construction to the admissible site gives a functor F^{ad} on the countable part of $\mathrm{Shv}_{\tau_D^{\mathrm{ad}}}(C^{\mathrm{ad}})$. Its kernel congruence satisfies

$$(F^{\mathrm{ad}})^{-1}(\mathrm{Iso}) \subseteq \Sigma_D.$$

This inclusion does *not* by itself imply that F^{ad} factors through the localization at Σ_D: such a factorization requires the opposite inclusion $\Sigma_D \subseteq (F^{\mathrm{ad}})^{-1}(\mathrm{Iso})$.

Remark 7.57. For analytic rings, Scholze states that the two congruences are equal; see [Sch26, Lecture IX]. Under this additional assertion, the Gestalt-valued functor does factor through the further localization used to define analytic stacks. Marc expects the equality to hold more generally, but the preceding argument proves only the displayed inclusion.

The chapter has therefore passed through three increasingly global encodings of geometry. A geometry specifies affine tests and admissible covers; a fractured topos records their local behavior intrinsically; a six-functor formalism can generate such a geometry and, after transmutation, a Gestalt-valued invariant. The last step is promising precisely because it remembers every categorical level at once, but its strongest descent and comparison statements remain part of an active theory.

A Factorization systems and saturated classes

Factorization systems turn a class of morphisms into a canonical two-step decomposition of every map. Orthogonality supplies the uniqueness of this decomposition and gives reflective subcategories in every slice. These formal properties underlie the effective epimorphism–monomorphism and connected–truncated factorizations used throughout the text.

In a presentable category, factorization systems can be generated from classes of morphisms. A saturated class of small generation is the left class of a factorization system, while a strongly saturated class of small generation is the kernel of an accessible localization. The distinction between these two closure operations is essential later: modalities arise from saturated classes stable under base change, whereas congruences arise from strongly saturated classes stable under base change. This appendix collects the required existence and closure results.

A.1 Factorization systems

We begin with the definition of orthogonality.

Definition A.1. Let C be a category. A morphism $l\colon A \to B$ is said to be *left orthogonal* to a morphism $r\colon X \to Y$, written $l \perp r$, if the following square is a pullback square:

$$\begin{array}{ccc} \mathrm{Hom}_C(B,X) & \xrightarrow{-\circ l} & \mathrm{Hom}_C(A,X) \\ {\scriptstyle r\circ -}\downarrow & & \downarrow{\scriptstyle r\circ -} \\ \mathrm{Hom}_C(B,Y) & \xrightarrow{-\circ l} & \mathrm{Hom}_C(A,Y). \end{array}$$

Equivalently, for every commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle l}\downarrow & \nearrow & \downarrow{\scriptstyle r} \\ B & \longrightarrow & Y, \end{array}$$

the anima of dashed fillers making both triangles commute is contractible. In this case, we also say that r is *right orthogonal* to l.

Given a collection of morphisms S, we denote by $S^{\perp}$ the collection of all morphisms that are right orthogonal to every morphism in S, and by ${}^{\perp}S$ the collection of those that are left orthogonal to every morphism in S.

Definition A.2. A *factorization system* on a category C consists of two classes of morphisms L and R of C satisfying the following two conditions:

(1) *Existence of factorization*: Every morphism $f\colon A \to B$ in C admits a factorization

$$A \xrightarrow{l} X \xrightarrow{r} B$$

with $l \in L$ and $r \in R$.

(2) *Orthogonality*: Every morphism $l \in L$ is left orthogonal to every morphism $r \in R$.

The classes of maps in a factorization system satisfy various useful closure properties.

Proposition A.3. *Let (L, R) be a factorization system on C.*

(1) *We have $L = {}^{\perp}R$ and $R = L^{\perp}$.*

(2) *The classes L and R are closed under composition and contain all isomorphisms.*

(3) *The classes L and R are closed under retracts.*

(4) *The class R is closed under base change (pullbacks).*

(5) *The class L is closed under cobase change (pushouts).*

(6) *The class R has the left cancellation property: if $g \circ f \in R$ and $g \in R$, then $f \in R$.*

(7) *The class L has the right cancellation property: if $g \circ f \in L$ and $f \in L$, then $g \in L$.*

(8) *If C admits small limits, the class R is closed under small limits in* $\mathrm{Ar}(C)$.

(9) *If C admits small colimits, the class L is closed under small colimits in* $\mathrm{Ar}(C)$.

Proof. The inclusions $L \subseteq {}^{\perp}R$ and $R \subseteq L^{\perp}$ are part of the definition. Conversely, suppose that $f \in {}^{\perp}R$ and choose a factorization of f into $l \in L$ followed by $r \in R$. Orthogonality of f and r gives a section s of r. Comparing the endomorphism sr of the middle object with its identity, using $l \perp r$, shows that s is also a retraction. Thus r is an isomorphism and $f \in L$. This proves $L = {}^{\perp}R$; the proof that $R = L^{\perp}$ is dual.

Orthogonality classes contain the isomorphisms and are closed under composition and retracts, which proves (2) and (3). A lifting problem against a base change of a morphism in R is equivalently a lifting problem against the original morphism; hence R is closed under base change. Dually, L is closed under cobase change. The cancellation properties follow by applying orthogonality successively to the two composable morphisms.

Finally, if $(r_i)_i$ is a small diagram of morphisms in R, then for every $l \in L$ the anima of fillers from l to $\lim_i r_i$ is the limit of the contractible animae of fillers from l to the r_i. It is therefore contractible, so $\lim_i r_i \in R$. The colimit statement for L is dual. Compare [Lur09, Propositions 5.2.8.6 and 5.2.8.11]. □

Lemma A.4. *Let (L, R) be a factorization system on C. Any morphism $f\colon X \to Y$ which is both in L and in R is an isomorphism.*

Proof. Consider the commutative square:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle f}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle f} \\ Y & = & Y \end{array}$$

Since $f \in L$ as the left vertical map and $f \in R$ as the right vertical map, orthogonality provides a unique filler g. The two triangles say precisely that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$, so g is an inverse to f. □

Lemma A.5. *Let (L, R) be a factorization system on C, and consider morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.*

(1) *Assume C admits pullbacks. If $gf \in R$ and $\Delta_g \in R$, then $f \in R$.*

(2) *Assume C admits pushouts. If $gf \in L$ and $\nabla_f \in L$, then $g \in L$.*

Proof. We prove (1); the proof for (2) is dual. We may factor f as the composite

$$X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

The second map is a base change of $gf\colon X \to Z$ along g, hence lies in R because $gf \in R$. The first map is a base change of the diagonal $\Delta_g\colon Y \to Y \times_Z Y$ along the map $f \times \mathrm{id}_Y\colon X \times_Z Y \to Y \times_Z Y$. Since $\Delta_g \in R$, it follows that $(\mathrm{id}_X, f) \in R$. Since R is closed under composition, we conclude $f \in R$. □

Lemma A.6 ([ABFJ22, Lemma 3.1.8])**.** *Let (L, R) be a factorization system on a category C. Then for every object $X \in C$ the slice $C_{/X}$ admits a factorization system in which a morphism lies in the left, resp. right, class if and only if its underlying morphism in C lies in L, resp. R.*

Proof. Given a morphism $A \to B$ in $C_{/X}$, factor its underlying morphism in C as $A \xrightarrow{l} M \xrightarrow{r} B$. Giving M the structure map $M \xrightarrow{r} B \to X$ makes this a factorization in $C_{/X}$. Orthogonality in the slice follows from orthogonality in C: the unique filler of the underlying square automatically commutes with the structure maps to X. □

Lemma A.7. *Let (L, R) be a factorization system on C.*

(1) *Let $\mathrm{Ar}^R(C) \subseteq \mathrm{Ar}(C)$ denote the full subcategory spanned by the morphisms in R. Then the inclusion $\mathrm{Ar}^R(C) \hookrightarrow \mathrm{Ar}(C)$ admits a left adjoint. Given a morphism $f\colon A \to B$ with factorization $A \xrightarrow{l} C \xrightarrow{r} B$, this left adjoint sends f to r.*

(2) *Similarly, the inclusion $\mathrm{Ar}^L(C) \hookrightarrow \mathrm{Ar}(C)$ of the morphisms in L admits a right adjoint, sending f to l.*

(3) *For every object $X \in C$, let $R[X] \subseteq C_{/X}$ be the full subcategory spanned by the morphisms in R. Then $R[X]$ is reflective in $C_{/X}$, and the reflection sends a morphism $f\colon A \to X$ to its right factor $r\colon C \to X$.*

Proof. By symmetry, it suffices to prove statement (1). The morphism

$$\begin{array}{ccc} A & \xrightarrow{l} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r} \\ B & = & B \end{array}$$

is the unit of the reflection. Indeed, for any $r' \in \mathrm{Ar}^R(C)$, the map

$$\mathrm{Hom}_{\mathrm{Ar}(C)}(r, r') \to \mathrm{Hom}_{\mathrm{Ar}(C)}(f, r')$$

is an isomorphism as a consequence of the fact that $l \perp r'$. This proves (1), and (2) follows dually. Applying the same argument in the slice factorization system of Lemma A.6 proves (3). □

A.2 Saturated classes

In practice, factorization systems are often constructed by specifying a set of generators for the left class.

Definition A.8. Let L be a class of morphisms of a presentable category C. We say that L is *saturated* if it contains all isomorphisms, is closed under composition, and the subcategory of $\mathrm{Ar}(C)$ spanned by the morphisms in L is closed under colimits.

Remark A.9. Given a factorization system (L, R) on C, it immediately follows from Proposition A.3 that L is saturated.

Note that the intersection of any collection of saturated classes is again saturated. This allows us to make the following definition:

Definition A.10. Let Σ be a class of morphisms in C. The *saturation* of Σ, denoted Σ^s, is the smallest saturated class containing Σ. We say that a saturated class L is *of small generation* if $L = \Sigma^s$ for some *set* of morphisms Σ.

The following proposition provides the bridge between saturated classes and factorization systems.

Proposition A.11 ([Lur09, Proposition 5.5.5.7])**.** *Let C be a presentable category and let L be a saturated class of small generation. Then the pair $(L, L^{\perp})$ forms a factorization system on C.*

A.3 Strongly saturated classes

When studying localizations of presentable categories, we require a stronger stability property.

Definition A.12. Let L be a class of morphisms of a presentable category C. We say L is *strongly saturated* if it is saturated and closed under 2-out-of-3.

As with saturated classes, we say a strongly saturated class L is *of small generation* if it is the smallest strongly saturated class containing some *set* of morphisms Σ. The following two results identify these classes with kernels of accessible Bousfield localizations.

Definition A.13. Let $\varphi\colon C \to D$ be a cocontinuous functor between presentable categories. We define its *kernel* as the collection of morphisms in C that is inverted by φ:

$$\ker(\varphi) \quad := \quad \{f\colon X \to Y \in \mathrm{Ar}(C) \mid \varphi(f) \text{ is an isomorphism}\}.$$

Proposition A.14 ([Lur09, Proposition 5.5.4.16])**.** *Let $\varphi\colon C \to D$ be a cocontinuous functor between presentable categories. Then $\ker(\varphi)$ is strongly saturated and of small generation.*

Proposition A.15 ([Lur09, Propositions 5.2.7.12 and 5.5.4.15])**.** *Let C be a presentable category and let Σ be a strongly saturated class of small generation.*

(1) *The inclusion $\mathrm{Loc}_\Sigma(C) \hookrightarrow C$ of the Σ-local objects in C admits an accessible left adjoint $L\colon C \to \mathrm{Loc}_\Sigma(C)$.*

(2) *The category $\mathrm{Loc}_\Sigma(C)$ is presentable.*

(3) *We have $\ker(L) = \Sigma$, i.e. a morphism in C is inverted by L if and only if it lies in Σ.*

(4) *For every category E, precomposition with L induces a fully faithful functor*

$$\mathrm{Fun}(\mathrm{Loc}_\Sigma(C), E) \longrightarrow \mathrm{Fun}(C, E)$$

whose essential image consists of the functors that invert every morphism in Σ.

B Limits and colimits in categories of sections

A category of sections is assembled from the fibers of a cartesian or cocartesian fibration. Its limits and colimits are therefore often computed fiberwise, but the required hypotheses depend on the variance. Colimits of sections of a cocartesian fibration require cocartesian transport to preserve the colimits in question, whereas limits require no preservation hypothesis. Dually, limits of sections of a cartesian fibration require preservation by cartesian transport, whereas colimits do not. The following proposition records all four cases. It is used in the analysis of filtered limits of topoi in Section 4.3.

Proposition B.1. *Let K and I be categories, let $F\colon I \to \mathrm{Cat}$, and let $G\colon I^{\mathrm{op}} \to \mathrm{Cat}$. Write*

$$p\colon \int_I^{\mathrm{cc}} F \longrightarrow I \qquad \text{and} \qquad q\colon \int_I^{\mathrm{ct}} G \longrightarrow I$$

for the associated cocartesian and cartesian fibrations, respectively.

(1) *Suppose that every $F(i)$ admits K-indexed colimits and that every cocartesian transport functor $\alpha_!\colon F(i) \to F(j)$ preserves them. Then $\Gamma_I(p)$ admits K-indexed colimits.*

(2) *Suppose that every $F(i)$ admits K-indexed limits. Then $\Gamma_I(p)$ admits K-indexed limits.*

(3) *Suppose that every $G(i)$ admits K-indexed limits and that every cartesian transport functor $\alpha^*\colon G(j) \to G(i)$ preserves them. Then $\Gamma_I(q)$ admits K-indexed limits.*

(4) *Suppose that every $G(i)$ admits K-indexed colimits. Then $\Gamma_I(q)$ admits K-indexed colimits.*

In every case, the indicated limits or colimits are computed pointwise. In particular, all evaluation functors preserve them.

Proof. We first prove (1). Consider the full subcategory of $\mathrm{Cat}_{/I}$ spanned by those functors $u\colon J \to I$ for which the category of sections of u^*p admits K-indexed colimits computed pointwise. This subcategory is closed under colimits. Indeed, if $J \simeq \mathrm{colim}_a J_a$ in $\mathrm{Cat}_{/I}$, then

$$\mathrm{Fun}_{/I}\left(J, \int_I^{\mathrm{cc}} F\right) \simeq \lim_a \mathrm{Fun}_{/I}\left(J_a, \int_I^{\mathrm{cc}} F\right).$$

The restriction functors in this limit preserve pointwise colimits, so if all the categories on the right admit K-indexed colimits computed pointwise, then so does the category on the left.

Every category is a colimit of simplices, and each $[n]$ is an iterated pushout of copies of $[1]$ along $[0]$. It therefore suffices to consider maps $[0] \to I$ and $[1] \to I$. The first case is precisely the assumption that each fiber admits K-indexed colimits. A map $[1] \to I$ classifies a morphism $\alpha\colon i \to j$. The category of sections of the resulting cocartesian fibration sits in a pullback square

$$\begin{array}{ccc} \Gamma_{[1]}([1] \times_I \int_I^{\mathrm{cc}} F) & \longrightarrow & \mathrm{Ar}(F(j)) \\ {\scriptstyle \mathrm{ev}_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ F(i) & \xrightarrow[\alpha_!]{} & F(j). \end{array}$$

The categories in the other three corners admit K-indexed colimits, the source evaluation s preserves them, and $\alpha_!$ preserves them by assumption. Hence the pullback admits K-indexed colimits computed by the two projections. This proves (1).

Statement (4) is [Lur09, Proposition 5.1.2.2], applied to the cartesian fibration q. Passing to opposite categories and using K^{op} as the indexing category gives (2): the functor $p^{\mathrm{op}}\colon (\int_I^{\mathrm{cc}} F)^{\mathrm{op}} \to I^{\mathrm{op}}$ is a cartesian fibration with fibers $F(i)^{\mathrm{op}}$, and

$$\Gamma_I(p)^{\mathrm{op}} \simeq \Gamma_{I^{\mathrm{op}}}(p^{\mathrm{op}}).$$

Finally, applying (1), with indexing category K^{op}, to the cocartesian fibration $q^{\mathrm{op}}\colon (\int_I^{\mathrm{ct}} G)^{\mathrm{op}} \to I^{\mathrm{op}}$ proves (3), because the opposite of a cartesian transport functor $\alpha^*\colon G(j) \to G(i)$ preserves K^{op}-indexed colimits precisely when α^* preserves K-indexed limits. The pointwise descriptions in these arguments also show that all evaluation functors preserve the indicated limits or colimits. □

C Dense subcategories closed under finite limits

The implication (3) ⇒ (4) of Theorem 2.42 requires a presentation of a presentable category T as a localization of a presheaf category $\mathrm{PSh}(C)$, where C is a small category with finite limits and the inclusion $C \hookrightarrow T$ is left exact. Lurie [Lur09, Proposition 6.1.5.3] obtains such a presentation from [Lur09, Proposition 5.4.7.4]: if T is κ-accessible and $\tau \gg \kappa$ is an uncountable regular cardinal such that T^κ is essentially τ-small, then the full subcategory T^τ of τ-compact objects is closed under all κ-small limits which exist in T. In particular, T^τ is closed under finite limits.

For the application, compactness of the objects of C is irrelevant. What is needed is that the restricted Yoneda functor $T \to \mathrm{PSh}(C)$ is fully faithful, i.e. that C is *dense*. Density, unlike compactness, is inherited by larger full subcategories. It therefore suffices to close a small dense subcategory under finite limits, and this closure remains essentially small. We record this elementary alternative, which avoids the cardinal arithmetic of [Lur09, Section 5.4.7].

C.1 Dense subcategories

Definition C.1. Let T be a locally small category and let $i\colon C \hookrightarrow T$ be the inclusion of an essentially small full subcategory. We denote by

$$i^*\colon T \longrightarrow \mathrm{PSh}(C), \qquad X \longmapsto \mathrm{Hom}_T(i(-), X)$$

the *restricted Yoneda functor*. We say that C is *dense* in T if i^* is fully faithful.

Remark C.2. If T admits small colimits, then i^* admits a left adjoint $i_!\colon \mathrm{PSh}(C) \to T$, the colimit-preserving extension of i. In this case C is dense if and only if the counit $i_! i^* X \to X$ is an isomorphism for every $X \in T$, i.e. if and only if every object X is the colimit of the canonical diagram $C_{/X} \to C \hookrightarrow T$. Density is condition (1) of Lemma 6.57.

Lemma C.3. *Let T be a locally small category, and let $D \subseteq C \subseteq T$ be essentially small full subcategories. If D is dense in T, then so is C.*

Proof. Write $i_D\colon D \hookrightarrow T$, $i_C\colon C \hookrightarrow T$ and $j\colon D \hookrightarrow C$ for the inclusions, so that $i_D^* \simeq j^* \circ i_C^*$. Since An admits small limits, the restriction functor $j^*\colon \mathrm{PSh}(C) \to \mathrm{PSh}(D)$ admits a right adjoint j_*, given by right Kan extension. Let $\eta\colon \mathrm{id} \to j_* j^*$ denote the unit.

Claim. For every $Y \in T$, the map $\eta_{i_C^* Y}\colon i_C^* Y \to j_* i_D^* Y$ is an isomorphism.

It suffices to check this after evaluation at every object $c \in C$. By the Yoneda lemma and the adjunction $j^* \dashv j_*$, the evaluation at c is identified with the map

$$\mathrm{Hom}_{\mathrm{PSh}(C)}\big(h_C(c), i_C^* Y\big) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(D)}\big(j^* h_C(c), j^* i_C^* Y\big)$$

induced by j^*. Since C is a full subcategory of T, we have $h_C(c) \simeq i_C^*(c)$, so $j^* h_C(c) \simeq i_D^*(c)$ and $j^* i_C^* Y \simeq i_D^* Y$. Precomposing with the Yoneda isomorphism $\mathrm{Hom}_T(c, Y) \simeq \mathrm{Hom}_{\mathrm{PSh}(C)}(h_C(c), i_C^* Y)$, the displayed map becomes the map

$$\mathrm{Hom}_T(c, Y) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(D)}\big(i_D^* c, i_D^* Y\big)$$

induced by i_D^*. It is an isomorphism because D is dense. This proves the claim.

Now let $X, Y \in T$. Consider the composite

$$\mathrm{Hom}_T(X,Y) \xrightarrow{i_C^*} \mathrm{Hom}_{\mathrm{PSh}(C)}\big(i_C^* X, i_C^* Y\big) \xrightarrow{(\eta_{i_C^* Y})_*} \mathrm{Hom}_{\mathrm{PSh}(C)}\big(i_C^* X, j_* i_D^* Y\big) \simeq \mathrm{Hom}_{\mathrm{PSh}(D)}\big(i_D^* X, i_D^* Y\big).$$

By construction of the adjunction isomorphism, the composite of the last two maps is induced by j^*. The total composite is therefore the map induced by $i_D^* \simeq j^* i_C^*$, which is an isomorphism because D is dense. The second map is an isomorphism by the claim. Hence the first map is an isomorphism, so C is dense. □

C.2 Finite-limit closures

Lemma C.4. *Let T be a locally small category which admits finite limits, and let $A \subseteq T$ be an essentially small full subcategory. Then the smallest full subcategory of T which contains A and is closed under finite limits is essentially small.*

Proof. Define full subcategories $A_0 \subseteq A_1 \subseteq \cdots$ of T inductively: $A_0 := A$, and A_{n+1} is spanned by the objects of A_n, a terminal object of T, and the pullbacks in T of all cospans in A_n. Since T is locally small and A_n is essentially small, there is only a small set of equivalence classes of cospans in A_n. Hence every A_n is essentially small, and so is $A_\omega := \bigcup_n A_n$. Every cospan in A_ω lies in some A_n, so A_ω is closed under pullbacks, and it contains a terminal object. By the inductive argument in the proof of (the dual of) [Lur09, Corollary 4.4.2.4], every finite limit in T of a diagram in A_ω may be built from iterated pullbacks and the terminal object, hence lies in A_ω. Thus A_ω is the smallest full subcategory containing A and closed under finite limits. □

C.3 Presentable categories

Proposition C.5. *Let T be a presentable category, and let $A \subseteq T$ be an essentially small full subcategory. Then there exists an essentially small full subcategory $C \subseteq T$ which contains A, is closed under finite limits, and is dense in T.*

Proof. Choose a regular cardinal κ such that T is κ-accessible. By [Lur09, Proposition 5.4.2.2], the full subcategory T^κ of κ-compact objects is essentially small and generates T under κ-filtered colimits, so that $T \simeq \mathrm{Ind}_\kappa(T^\kappa)$ by [Lur09, Proposition 5.3.5.11]. Under this equivalence, the

restricted Yoneda functor $T \to \mathrm{PSh}(T^\kappa)$ corresponds to the inclusion $\mathrm{Ind}_\kappa(T^\kappa) \subseteq \mathrm{PSh}(T^\kappa)$: for $F \in \mathrm{Ind}_\kappa(T^\kappa)$ with image $X \in T$ and $c \in T^\kappa$, we have $\mathrm{Hom}_T(c, X) \simeq \mathrm{Hom}_{\mathrm{Ind}_\kappa(T^\kappa)}(h(c), F) \simeq F(c)$ by the Yoneda lemma. Hence T^κ is dense in T.

A presentable category admits all small limits [Lur09, Corollary 5.5.2.4]. Let C be the smallest full subcategory of T which contains $T^\kappa \cup A$ and is closed under finite limits. It is essentially small by Lemma C.4, and dense by Lemma C.3. □

Corollary C.6. *Let T be a presentable category, and let $C \subseteq T$ be as in Proposition C.5. Let $i\colon C \hookrightarrow T$ denote the inclusion. Then C admits finite limits, i is left exact, and the colimit-preserving extension $i_!\colon \mathrm{PSh}(C) \to T$ is a localization whose right adjoint i^* is fully faithful. If T satisfies the Giraud axioms, then $i_!$ is left exact.*

Proof. Since C is a full subcategory closed under finite limits, it admits finite limits and i preserves them. The right adjoint of $i_!$ is the restricted Yoneda functor i^*, which is fully faithful because C is dense. The final assertion is Proposition 2.43. □

Remark C.7. The subcategory C produced by Proposition C.5 need not consist of κ-compact objects. Since C is essentially small, there nevertheless exists a regular cardinal λ such that every object of C is λ-compact. The restricted Yoneda functor i^* then commutes with λ-filtered colimits, so the localization of Corollary C.6 is accessible.

For suitable $\tau \gg \kappa$, one may take $C = T^\tau$. The subcategory T^τ is closed under finite limits by [Lur09, Proposition 5.4.7.4], and T is τ-accessible by [Lur09, Proposition 5.4.2.11], so $T \simeq \mathrm{Ind}_\tau(C)$. This stronger form is used in Proposition 4.31, where presentations must be chosen compatibly along a diagram of topoi.

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